Escape probabilities for branching Brownian motion among mild obstacles

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Abstract

We derive asymptotics for the probability that a critical branching Brownian motion killed at a small rate ε in Poissonian obstacles exits a large domain. Results are formulated in terms of the solution to a nonlinear partial differential equation with singular boundary conditions. The proofs depend on a quenched homogenization theorem for branching Brownian motion among mild obstacles.

1 Introduction

In the present work, we are interested in the long-term behaviour of branching Brownian motion killed in Poissonian obstacles. Let us start by describing a simple special case of our results. We consider a critical branching Brownian motion in $\mathbb{R}^d$ ($d \geq 1$), where all initial particles start from the origin. We assume that particles are killed at a (small) rate $\varepsilon > 0$ within random balls of fixed radius, whose centers are distributed according to a homogeneous Poisson point process on $\mathbb{R}^d$. Then, how many initial particles do we need so that, with high probability, one of their descendants reaches distance R from the origin ? Let $p_\varepsilon(R)$ be the (quenched) probability for our randomly killed branching Brownian motion starting with a single particle at 0 to visit the complement of a large ball of radius R centered at the origin. The preceding question is equivalent to determining the limiting behaviour of $p_\varepsilon(R)$ when ε tends to 0 and simultaneously R tends to infinity.

The answer involves several regimes depending on the respective values of ε and R. If ε is small in comparison with $1/R^2$, the killing phenomenon does not matter and the result is the same as if there were no killing: $p_\varepsilon(R)$ behaves like a constant times $1/R^2$ (informally, the branching process must survive up to a time of order R^2 so that at least one of the particles travels a distance R, and well-known estimates for critical branching processes then lead to the correct asymptotics). On the other hand, if ε is large in comparison with $1/R^2$, then the probability $p_\varepsilon(R)$ decreases exponentially fast as a function of $R\sqrt{\varepsilon}$: See Proposition 1 below.

Our main results focus on the critical regime where εR^2 converges to a constant $a > 0$. We show that the probability $p_\varepsilon(R)$ behaves like R^{-2}, as in the case without killing, but with a multiplicative constant which depends on a and can be identified as the value at the origin of the solution of a semilinear partial differential equation with singular boundary conditions. A key tool to derive these asymptotics is a quenched homogenization theorem which shows that our branching Brownian motions among obstacles, suitably rescaled, are close to super-Brownian motion killed at a certain rate depending on a.

Let us formulate our assumptions more precisely in order to state our results. First, let us define the collection of obstacles. We denote the set of all compact subsets of $\mathbb{R}^d$ by $\mathcal{K}$. This set is equipped with the usual Hausdorff metric d_H. Recall that $(\mathcal{K}, d_H)$ is a Polish space. For every $r > 0$, $\mathcal{K}_r$ denotes the subset of $\mathcal{K}$ which consists of all compact sets that are contained in the closed ball of radius r centered at the origin. Let Θ be a finite measure on $\mathcal{K}$, and assume that Θ is supported on $\mathcal{K}_{r_0}$ for some $r_0 > 0$. Let

$$\mathcal{N} = \sum_{i\in I} \delta_{(x_i,K_i)}$$

be a Poisson point measure on $\mathbb{R}^d \times \mathcal{K}$ with intensity $\lambda_d \otimes \Theta$, where λ_d stands for Lebesgue measure on $\mathbb{R}^d$. We assume that this point measure is defined on a probability space $(\mathbf{\Omega}, \mathbf{P})$ and we denote the generic element of $\mathbf{\Omega}$ by ϖ. Our set of obstacles is then defined by

$$\Gamma_\varpi = \bigcup_{i\in I} (x_i + K_i), \tag{1}$$

where obviously $x_i + K_i = \{z = x_i + y : y \in K_i\}$. Note that we use the notation Γ_ϖ to emphasize that the set of obstacles depends on the variable ϖ representing the environment. Let us also define a constant κ by

$$\kappa = \mathbf{P}(0 \in \Gamma_\varpi) = 1 - \exp\Big(- \int_{\mathcal{K}} \Theta(dK)\, \lambda_d(K)\Big).$$

To avoid trivial cases, we assume that $\kappa > 0$, or equivalently $\Theta(\lambda_d(K) > 0) > 0$. By translation invariance, we also have $\mathbf{P}(x \in \Gamma_\varpi) = \kappa$ for every $x \in \mathbb{R}^d$.

Let us now introduce the sequence of branching Brownian motions of interest. Given $\varpi \in \mathbf{\Omega}$ and a parameter $\varepsilon \geq 0$, we consider a branching Brownian motion on $\mathbb{R}^d$ such that

- each particle moves around in $\mathbb{R}^d$ according to the law of Brownian motion killed at rate ε within Γ_ϖ ;
- each particle branches at rate 1. During a branching event, the particle generates a random number of offspring, according to an offspring distribution ν which has mean one and finite variance $\sigma^2 \in (0, \infty)$.

This branching Brownian motion is denoted by $Z^{\varpi,\varepsilon} = (Z^{\varpi,\varepsilon}_t)_{t\geq 0}$, where $Z^{\varpi,\varepsilon}_t$ stands for the sum of the Dirac point masses at the particles alive at time t. The processes $Z^{\varpi,\varepsilon}$ are defined on a probability space Ω. For every finite point measure μ on $\mathbb{R}^d$, we use the notation $\mathbb{P}_\mu$ for the probability measure on Ω under which each of the processes $Z^{\varpi,\varepsilon}$ starts from μ.

Let A be a bounded domain of class C^2 in $\mathbb{R}^d$ containing 0. We say that the branching Brownian motion $Z^{\varpi,\varepsilon}$ hits A^c if there exists $t > 0$ such that $Z^{\varpi,\varepsilon}_t(A^c) > 0$. We are interested in asymptotics for the quantity

$$\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } (RA)^c)$$

when $R \to \infty$ and $\varepsilon \to 0$. Here we use the obvious notation $RA = \{z = Ry : y \in A\}$.

Theorem 1. *For every $a \geq 0$, let $u_{(a)} = (u_{(a)}(x), x \in A)$ be the unique nonnegative solution of the singular boundary value problem*

$$\begin{cases} \dfrac{1}{2}\Delta u = \dfrac{\sigma^2}{2}\, u^2 + a\, u & \text{in } A, \\ u_{|\partial A} = +\infty\ . \end{cases} \tag{2}$$

Then,

$$\lim_{R\to\infty} \Big(\sup_{\varepsilon\geq 0} \Big| R^2 \mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } (RA)^c) - u_{(\kappa\varepsilon R^2)}(0) \Big| \Big) = 0\ , \qquad \mathbf{P}(d\varpi) \text{ a.s.}$$

Let us state a corollary of the theorem, which is motivated by the simple question we asked at the beginning of this introduction. For every $a \geq 0$, we denote by $u^\circ_{(a)}$ the solution of the boundary problem (2) when A is the open unit ball of $\mathbb{R}^d$.

Corollary 1. *For every $\varepsilon \in (0,1)$, let n_ε be a positive integer. Assume that $\varepsilon n_\varepsilon \longrightarrow b$ as $\varepsilon \to 0$, for some $b > 0$. Denote by $R^{\varpi,\varepsilon}$ the maximal distance from the origin attained by a particle of the branching Brownian motion $Z^{\varpi,\varepsilon}$. Then, $\mathbf{P}(d\varpi)$ a.s., the law of $\sqrt{\varepsilon}\, R^{\varpi,\varepsilon}$ under $\mathbb{P}_{n_\varepsilon\delta_0}$ converges as $\varepsilon \to 0$ towards the probability measure π_b on $\mathbb{R}_+$ defined by*

$$\pi_b([0,r]) = \exp(-\frac{b}{r^2}\, u^\circ_{(\kappa r)}(0))$$

for every $r > 0$.

In the setting of Theorem 1, it is not hard to see that $u_{(a)}(0)$ tends to 0 as $a \to \infty$ (see Lemma 2 below) and thus this theorem does not give much information when $R \to \infty$ and $\varepsilon \to 0$ in such a way that εR^2 tends to ∞. In that case, the next proposition provides an exponential decay, which contrasts with the preceding theorem. Since our bounds are clearly not optimal, we consider only the case when A is a ball. We denote the open ball of radius r centered at the origin by $B(0,r)$. In the general case we may apply the bounds (i) and (ii) of the proposition after replacing A by a ball $B(0,r)$ such that $B(0,r) \supset A$ or $B(0,r) \subset A$ respectively.

Proposition 1. (i) *There exists a positive constant $C_0 = C_0(\nu)$ such that, for every $\varpi \in \mathbf{\Omega}$, $R \geq 1$ and $\varepsilon \in [1/R^2, 1]$,*

$$\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,R)^c) \geq C_0\varepsilon \exp(-R\sqrt{2\varepsilon}).$$

(ii) *There exists a positive constant $C_1 = C_1(\Theta)$ such that for every $R \geq 1$ and $\varepsilon \in [1/R^2, 1]$,*

$$\mathbf{P} \otimes \mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,R)^c) \leq \exp(-C_1 R\sqrt{\varepsilon}).$$

Consequently, we can find two positive constants $C_2 = C_2(\Theta)$ and $C_3 = C_3(\Theta)$ such that $\mathbf{P}(d\varpi)$ a.s., for every sufficiently large R and every $\varepsilon \in [C_2(\log\log R)^2/R^2, 1]$,

$$\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,R)^c) \leq \exp(-C_3 R\sqrt{\varepsilon}).$$

Part (i) of the proposition is derived from an estimate about branching Brownian motion killed homogeneously at rate ε, which explains why this bound holds for every $\varpi \in \mathbf{\Omega}$ and does not depend on the measure Θ. The bounds in (ii) follow from an estimate for Brownian motion killed in mild obstacles and therefore do not depend on the offspring distribution ν. The first assertion in (ii) may be compared to Proposition 5.2.8 in [Szn98].

Proposition 1 only gives rather crude estimates, and it would be of interest to obtain more precise information on the decay of the quenched probabilities $\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon}$ hits $(RA)^c)$ in the case when εR^2 tends to ∞. This leads to large deviations problems in the spirit of the work of Sznitman [Szn98], which we do not address here.

A major ingredient of the proof of Theorem 1 is the following quenched homogenization result. We need to introduce a rescaled version of the process $Z^{\varpi,\varepsilon}$. For every $\varepsilon > 0$ and every $t \geq 0$, let us define a random measure $X^{\varpi,\varepsilon}_t$ on $\mathbb{R}^d$ by setting, for any nonnegative measurable function φ on $\mathbb{R}^d$,

$$\langle X^{\varpi,\varepsilon}_t, \varphi\rangle = \varepsilon \int Z^{\varpi,\varepsilon}_{\varepsilon^{-1}t}(dx)\, \varphi(\varepsilon^{1/2}x). \tag{3}$$

Here and later, the notation $\langle \mu, \varphi\rangle$ stands for the integral of the function φ against the measure μ, whenever this integral makes sense.

For every real $x \geq 0$, $[x]$ denotes the integer part of x.

Theorem 2. *Except for a $\mathbf{P}$-negligible set of values of ϖ, the law of $(X^{\varpi,\varepsilon}_t)_{t\geq 0}$ under $\mathbb{P}_{[\varepsilon^{-1}]\delta_0}$ converges weakly as $\varepsilon \to 0$, in the Skorokhod sense, to that of a super-Brownian motion with branching mechanism $\psi_{(\kappa)}(u) := \frac{\sigma^2}{2}u^2 + \kappa u$ started at δ_0.*

The definition of super-Brownian motion with branching mechanism $\psi_{(\kappa)}$ is recalled in Section 2 below.

As a hint of why Theorem 2 should be true, notice that for a given realization of the obstacles, the probability that a single Brownian motion starting from 0 and killed at rate ε within Γ_ϖ is still alive by time $t > 0$ is given by

$$\mathrm{E}\Big[\exp\Big\{-\varepsilon\int_0^t \mathbb{I}_{\Gamma_\varpi}(\xi_s)ds\Big\}\Big], \tag{4}$$

where ξ denotes standard d-dimensional Brownian motion. Let us focus on the integral within the exponential in (4). Averaging over the law of the obstacles and using Fubini's theorem, we obtain for each $t \geq 0$

$$\mathbf{E}\Big[\mathrm{E}\Big[\varepsilon\int_0^t \mathbb{I}_{\Gamma_\varpi}(\xi_s)ds\Big]\Big] = \varepsilon\ \mathrm{E}\Big[\int_0^t \mathbf{P}[\xi_s \in \Gamma_\varpi]ds\Big] = \varepsilon\kappa t.$$

We can thus guess, and easily prove, that the rescaled Brownian motion $(\sqrt{\varepsilon}\ \xi_{\varepsilon^{-1}t},\ t \geq 0)$, which is killed at rate 1 within $\sqrt{\varepsilon}\ \Gamma_\varpi$, converges to Brownian motion killed at homogeneous rate κ as $\varepsilon \to 0$. Theorem 2 shows that an analogous convergence indeed holds in our more general framework of branching Brownian motions, for any fixed ϖ contained in a set of $\mathbf{P}$-probability one.

Let us briefly explain how Theorem 1 is derived from Theorem 2. Consider a sequence (ε_n, R_n) such that $R_n \to \infty$ and $\varepsilon_n R_n^2$ converges to a positive constant a. By a simple scaling transformation, the probability $\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon_n}$ hits $(R_nA)^c)$ coincides with $\mathbb{P}_{\delta_0}(X^{\varpi,\varepsilon_n}$ hits $(b_nA)^c)$, where $b_n = \varepsilon_n^{1/2}R_n$ converges to $\sqrt{a}$. We can then use Theorem 2 to investigate the asymptotic behaviour of the latter hitting probabilities. This limiting behaviour involves the corresponding hitting probabilities for super-Brownian motion, which are known to be related to solutions of semilinear partial differential equations from the work of Dynkin [Dy91, Dy93]. One difficulty in implementing the preceding idea comes from the fact that the convergence in Theorem 2 is not strong enough to ensure that hitting probabilities for the processes $(X_t^{\varpi,\varepsilon})_{t\geq 0}$ converge to hitting probabilities for the limiting process. Much of the proof of Theorem 1 in Section 4 is devoted to a precise justification of this property (Lemma 7).

To complete this introduction, let us mention that branching Brownian motion and superprocesses among random obstacles have been studied recently in several papers, including Engländer and den Hollander [EdH03] and Engländer [Eng08]. These papers concentrate on the case of supercritical branching, in contrast with critical branching which is considered here. See also the survey [Eng07]. A homogenization theorem related to Theorem 2 has been proved in [Véb09] for super-Brownian motion among hard obstacles, in the case when the intensity of the obstacles grows to infinity but their diameters shrink to 0. There is a huge literature about Brownian motion and random walks among (hard or mild) obstacles, and the reader may look at the book of Sznitman [Szn98] for additional references.

The rest of this paper is laid out as follows. In Section 2, we introduce the basic notation and objects, and state several results about hitting probabilities for spatial branching processes we shall need in the sequel. Theorem 2 and Proposition 1 are proved in Section 3. Theorem 1 and Corollary 1 are then derived in Section 4.

2 Preliminaries

2.1 Notation

We denote the set of all finite measures on $\mathbb{R}^d$ by $\mathcal{M}_f(\mathbb{R}^d)$. This set is equipped with the weak topology. We write $\mathcal{M}_p(\mathbb{R}^d)$ for the subset of $\mathcal{M}_f(\mathbb{R}^d)$ which consists of all finite point measures on $\mathbb{R}^d$.

If E is a metric space, we denote the set of all bounded continuous functions on the space E by $\bar{C}(E)$ and we let $\|f\|$ stand for the supremum norm of $f \in \bar{C}(E)$. We write $C^2(\mathbb{R}^d)$ for the set of all twice continuously differentiable functions on $\mathbb{R}^d$, and $\bar{C}^2(\mathbb{R}^d)$ for that of all bounded functions in

$C^2(\mathbb{R}^d)$ whose first and second derivatives are also bounded. An index + added to this notation means that we require the functions to be nonnegative. We equip $\bar{C}^2(\mathbb{R}^d)$ with the topology induced by the seminorms $\|f\|_{(R)}$, where for every $R > 0$

$$\|f\|_{(R)} := \sup_{|x|\leq R} \Big\{ |f(x)| + \sum_{i=1}^{d} \Big|\frac{\partial f}{\partial x_i}(x)\Big| + \sum_{i,j=1}^{d} \Big|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\Big| \Big\}.$$

If E is a Polish space, we let $D_E[0,\infty)$ be the set of all càdlàg paths with values in E, equipped with the Skorokhod topology.

If $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the open ball of radius r centered at x, and $\bar{B}(x,r)$ stands for the corresponding closed ball. More generally, the closure of a subset F of $\mathbb{R}^d$ is denoted by $\bar{F}$. Lebesgue measure on $\mathbb{R}^d$ is denoted by λ_d.

Finally, the notation $\xi = (\xi_t)_{t\geq 0}$ will stand for a standard Brownian motion in $\mathbb{R}^d$, which starts from x under the probability measure P_x. It will also be convenient to use the notation $\xi^{\varpi,\varepsilon}$ for Brownian motion in $\mathbb{R}^d$ killed at rate ε in the set Γ_ϖ. As usual, the value of $\xi^{\varpi,\varepsilon}$ after its killing time is a cemetery point Δ added to $\mathbb{R}^d$, and we agree that all functions vanish at Δ.

2.2 Super-Brownian motion

Let $a \geq 0$ and set $\psi_{(a)}(u) = \frac{\sigma^2}{2}u^2 + a\,u$, for every $u \geq 0$ (the offspring distribution ν, and thus the parameter $\sigma > 0$ are fixed throughout this work). Super-Brownian motion with branching mechanism $\psi_{(a)}$ is the continuous strong Markov process with values in $\mathcal{M}_f(\mathbb{R}^d)$, whose transition kernels $(Q_t)_{t\geq 0}$ are characterized as follows: For every $g \in \bar{C}_+(\mathbb{R}^d)$ and every $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, we have for every $t \geq 0$

$$\int Q_t(\mu, d\mu')\,\exp(-\langle \mu', g\rangle) = \exp(-\langle \mu, V_t g\rangle), \tag{5}$$

where the function $u_t(x) = V_t g(x)$, $t \geq 0$, $x \in \mathbb{R}^d$, is the unique nonnegative solution of the semilinear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \psi_{(a)}(u) & \text{in } (0,\infty)\times\mathbb{R}^d\,, \\ u_0 = g\,. \end{cases}$$

Let $Y = (Y_t)_{t\geq 0}$ be a super-Brownian motion with branching mechanism $\psi_{(a)}$, started at $\mu \in \mathcal{M}_f(\mathbb{R}^d)$. Then, for every $g \in \bar{C}^2_+(\mathbb{R}^d)$

$$e^{-\langle Y_t, g\rangle} - e^{-\langle Y_0, g\rangle} - \int_0^t \Big\langle Y_s, -\frac{1}{2}\Delta g + \psi_{(a)}(g)\Big\rangle\, e^{-\langle Y_s, g\rangle} ds \tag{6}$$

is a martingale. It is well known that this martingale problem and the initial value μ characterize the law of Y. This is indeed an application of the classical "duality method" (see in particular Chapter 4 in [EK86]). The nonlinear semigroup $g \to V_t g$ provides a deterministic dual to super-Brownian motion, and the duality argument then shows that if a measure-valued process started from μ satisfies the preceding martingale problem, the Laplace functional of its value at time t must be given by the right-hand side of (5). See Section 1.6 of [Eth00] for more details.

2.3 Branching Brownian motion among random obstacles

In view of our applications (and in particular because we want to refer to some results of [Ch91]), it will be convenient to give a more formal description of the branching Brownian motions that were already introduced in Section 1 above. Recall that our offspring distribution ν is assumed to be critical and that $\mathrm{var}(\nu) = \sigma^2 \in (0,\infty)$. The probability generating function of ν will be denoted by Υ.

Let $\mathcal{T}$ be a Galton-Watson tree with offspring distribution ν (see e.g. [LG05]). As usual, we view $\mathcal{T}$ as a random finite subset of

$$U := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{N}^0 = \{\varnothing\}$. If $v = (v_1, \ldots, v_n) \in U \backslash \{\varnothing\}$, the parent of v is denoted by $\hat{v} = (v_1, \ldots, v_{n-1})$ and we also use the notation $v \prec v'$ to mean that v' is a descendant of v distinct from v. Consider a collection $(e_v, v \in U)$ of independent exponential random variables with parameter 1, which is also independent of $\mathcal{T}$. We define for every $v \in U$ its birth time α_v and its death time β_v recursively by setting $\alpha_\varnothing = 0$ and $\beta_\varnothing = e_\varnothing$, and for every $v \in U \backslash \{\varnothing\}$,

$$\alpha_v = \beta_{\hat{v}} \ , \ \beta_v = \alpha_v + e_v.$$

Let us now construct the spatial motions. Fix a starting point $x \in \mathbb{R}^d$, and consider a collection $(B^v, v \in U)$ of independent standard Brownian motions in $\mathbb{R}^d$ (started from 0), independent of $\mathcal{T}$ and of $(e_v, v \in U)$. For every $v \in U$, define the *historical path* $\omega^v = (\omega^v_t, 0 \le t \le \beta_v)$ associated with v in the following way. First $\omega^\varnothing_t = x + B^\varnothing_t$ for $0 \le t \le \beta_\varnothing$. Then, if $v \in U \backslash \{\varnothing\}$, set $\omega^v_t = \omega^{\hat{v}}_t$ for all $0 \le t \le \alpha_v$ and

$$\omega^v_t = \omega^{\hat{v}}_{\alpha_v} + B^v_{t-\alpha_v} \qquad \text{for } \alpha_v \le t \le \beta_v.$$

A branching Brownian motion (without killing in obstacles) starting from δ_x is obtained by setting for every $t \ge 0$,

$$Z_t = \sum_{v \in \mathcal{T}, v \sim t} \delta_{\omega^v_t},$$

where the notation $v \sim t$ means that $\alpha_v \le t < \beta_v$.

In this formalism, it is now easy to introduce killing in obstacles. Consider yet another independent collection $(\gamma_v)_{v \in U}$ of independent exponential random variables with parameter 1, and define for every $\varpi \in \mathbf{\Omega}$, $\varepsilon \ge 0$, and for every $v \in U$

$$\zeta^{\varpi,\varepsilon}_v := \inf \left\{ s \in [\alpha_v, \beta_v) : \int_{\alpha_v}^{s} dr \, \mathbb{I}_{\Gamma_\varpi}(\omega^v_r) > \varepsilon^{-1} \gamma_v \right\},$$

where $\inf \varnothing = \infty$. By setting

$$Z^{\varpi,\varepsilon}_t = \sum_{v \in \mathcal{T}, v \sim t} \mathbb{I}_{\{t < \zeta^{\varpi,\varepsilon}_v \text{ and } \zeta^{\varpi,\varepsilon}_{v'} = \infty, \text{ for every } v' \prec v\}} \delta_{\omega^v_t},$$

we obtain a branching Brownian motion killed at rate ε in the obstacle set Γ_ϖ, starting from δ_x. An obvious extension of the preceding construction allows us to obtain branching Brownian motions starting from any point measure $\mu \in \mathcal{M}_p(\mathbb{R}^d)$.

We now recall a special case of the classical convergence of rescaled branching Brownian motions towards super-Brownian motion. For our applications, we state the case where particles are killed at a constant rate homogeneously over $\mathbb{R}^d$ (this case is obtained from the preceding construction of $Z^{\varpi,\varepsilon}$ by replacing Γ_ϖ by $\mathbb{R}^d$). In the next two statements, for every $\varepsilon > 0$, $Z^{(\varepsilon)} = (Z^{(\varepsilon)}_t)_{t \ge 0}$ denotes a branching Brownian motion with offspring distribution ν, where particles are killed homogeneously over $\mathbb{R}^d$ at rate ε. As previously $Z^{(\varepsilon)}$ starts from μ under the probability measure $\mathbb{P}_\mu$, for every $\mu \in \mathcal{M}_p(\mathbb{R}^d)$.

Proposition 2. *Let $a \ge 0$. For every $\varepsilon > 0$, define a measure-valued process $(X^{(\varepsilon)}_t)_{t \ge 0}$ by setting*

$$\langle X^{(\varepsilon)}_t, \varphi \rangle = \varepsilon \int Z^{(a\varepsilon)}_{\varepsilon^{-1}t}(dx) \, \varphi(\varepsilon^{1/2} x),$$

for every $\varphi \in \bar{C}(\mathbb{R}^d)$. For every fixed $\eta > 0$, the law of $X^{(\varepsilon)}$ under $\mathbb{P}_{[\eta\varepsilon^{-1}]\delta_0}$ converges as $\varepsilon \to 0$, in the Skorokhod sense, towards the law of super-Brownian motion with branching mechanism $\psi_{(a)}$ starting from $\eta\delta_0$.

A proof of Proposition 2 can be found in Chapter 1 of [Eth00] in the case $a = 0$, and arguments are easily adapted to cover the general case.

Finally, we shall use an estimate for the probability that a branching Brownian motion starting from δ_0 exits a large ball centered at the origin. Similar estimates can be found in Sawyer and Fleischman [SF79], but we provide a short proof for the sake of completeness.

Lemma 1. *Suppose that $d = 1$. There exist two positive constants $C_0' = C_0'(\nu)$ and $C_1' = C_1'(\nu)$ such that, for every $\varepsilon \in [0,1]$ and $r \geq 1$,*

$$C_0'(r^{-2}\,\mathbb{I}_{\{r\leq\frac{1}{\sqrt{\varepsilon}}\}} + \varepsilon e^{-r\sqrt{2\varepsilon}}\,\mathbb{I}_{\{r>\frac{1}{\sqrt{\varepsilon}}\}}) \leq \mathbb{P}_{\delta_0}(Z^{(\varepsilon)} \text{ hits } (-r,r)^c) \leq C_1'(r^{-2}\,\mathbb{I}_{\{r\leq\frac{1}{\sqrt{\varepsilon}}\}} + \varepsilon e^{-r\sqrt{2\varepsilon}}\,\mathbb{I}_{\{r>\frac{1}{\sqrt{\varepsilon}}\}}).$$

Remark. As an immediate consequence of the upper bound of the lemma, we have in dimension d, for every $r > 0$,

$$\mathbb{P}_{\delta_0}(Z^{(0)} \text{ hits } B(0,r)^c) \leq C_1''\,(r+1)^{-2} \tag{7}$$

with a constant $C_1'' = C_1''(d,\nu)$.

Proof. It clearly suffices to prove that the stated bounds hold for the quantity $\mathbb{P}_{\delta_r}(Z^{(\varepsilon)}$ hits $(-\infty,0])$ instead of $\mathbb{P}_{\delta_0}(Z^{(\varepsilon)}$ hits $(-r,r)^c)$. We fix $\varepsilon \geq 0$, and for every $x > 0$ and $t \geq 0$, we set

$$q_\varepsilon(x,t) = \mathbb{P}_{\delta_x}\left(Z^{(\varepsilon)} \text{ does not hit } (-\infty,0] \text{ before time } t\right).$$

and

$$p_\varepsilon(x) = \mathbb{P}_{\delta_x}\left(Z^{(\varepsilon)} \text{ hits } (-\infty,0]\right) = \lim_{t\uparrow\infty} \uparrow (1 - q(x,t)).$$

In this proof only, we write P_x^ε for the probability under which ξ is a Brownian motion starting from x and killed at rate ε (upon killing, ξ is sent to the cemetery point Δ and we recall that all functions vanish at Δ). Write $S := \inf\{t \geq 0 : \xi_t \in (-\infty,0]\}$. By standard arguments (see e.g. the proof of Proposition II.3 in [LG99]), the function q_ε solves the integral equation

$$q_\varepsilon(x,t) = \mathrm{P}_x^\varepsilon(S > t) + \mathrm{E}_x^\varepsilon\Big[\int_0^{t\wedge S} (\Upsilon(q_\varepsilon(\xi_s,t-s)) - q_\varepsilon(\xi_s,t-s))\,ds\Big],$$

where we recall that Υ denotes the generating function of the offspring distribution ν. For every $a \in [0,1]$, set $\Phi(a) = \Upsilon(1-a) - (1-a)$. Note that $\Phi(0) = 0$ and the function Φ is monotone increasing under our assumptions. Furthermore, $\Phi(a) = \frac{\sigma^2}{2}a^2 + o(a^2)$ when $a \to 0$. By a monotone passage to the limit we get that, for every $x > 0$,

$$p_\varepsilon(x) + \mathrm{E}_x^\varepsilon\Big[\int_0^S \Phi(p_\varepsilon(\xi_s))\,ds\Big] = \mathrm{P}_x^\varepsilon(S < \infty). \tag{8}$$

It follows that the function p_ε satisfies the differential equation

$$\frac{1}{2}p_\varepsilon'' = \varepsilon\,p_\varepsilon + \Phi(p_\varepsilon)$$

on $(0,\infty)$ with boundary conditions $p_\varepsilon(0) = 1, p_\varepsilon(\infty) = 0$. By solving this differential equation, we get, for every $x > 0$,

$$\int_{p_\varepsilon(x)}^1 \frac{du}{\sqrt{2\varepsilon u^2 + 4\Gamma(u)}} = x$$

where $\Gamma(u) = \int_0^u \Phi(v)\,dv$. Note that there exist positive constants c, c' such that $cu^3 \leq \Gamma(u) \leq c'u^3$ for every $u \in [0,1]$. The desired bounds then follow from easy analytic arguments. □

2.4 Hitting probabilities for super-Brownian motion

Let $Y^{(a)} = (Y_t^{(a)})_{t\geq 0}$ be a super-Brownian motion with branching mechanism $\psi_{(a)}$ for some $a \geq 0$. Suppose that $Y^{(a)}$ starts from μ under the probability measure P_μ, for every $\mu \in \mathcal{M}_f(\mathbb{R}^d)$.

The range of $Y^{(a)}$ is by definition

$$\mathcal{R}(Y^{(a)}) = \bigcup_{\varepsilon>0} \left(\overline{\bigcup_{t=\varepsilon}^{\infty} \operatorname{supp}(Y_t^{(a)})} \right),$$

where for every $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, $\operatorname{supp}(\mu)$ denotes the topological support of μ.

Let D be a domain in $\mathbb{R}^d$ and let $x \in D$. Consider the process $Y^{(a)}$ started from δ_x. We say that $Y^{(a)}$ hits D^c if the range $\mathcal{R}(Y^{(a)})$ intersects D^c. By a famous result of Dynkin [Dy91, Dy93] the function

$$u_{(a)}^D(x) = -\log\left(1 - P_{\delta_x}(\mathcal{R}(Y^{(a)}) \cap D^c \neq \emptyset)\right), \quad x \in D,$$

is the maximal nonnegative solution of the semilinear partial differential equation $\frac{1}{2}\Delta u = \psi_{(a)}(u)$ in D.

Under mild regularity assumptions on D (which hold e.g. when D satisfies an exterior cone condition at every point of ∂D), the function $u_{(a)}^D$ has boundary value $+\infty$ at every point of ∂D and is the unique nonnegative solution of the equation $\frac{1}{2}\Delta u = \psi_{(a)}(u)$ in D with boundary value $+\infty$ everywhere on ∂D. A discussion of this result and related ones can be found in Chapter VI of the book [LG99]. This reference considers only the case $a = 0$, but the same results can be obtained for any $a \geq 0$ by similar arguments: Note that the Brownian snake approach can be extended from the case $a = 0$ considered in [LG99] to $a \geq 0$, simply by replacing the reflecting Brownian motion driving the snake by a reflecting Brownian motion with negative drift (see Chapter 4 of [DLG02] for a discussion of the snake approach to superprocesses with a general branching mechanism).

We shall be interested in the special case $D = A$. Recall that $0 \in A$ and that we assume A is a domain of class C^2, meaning that the boundary of A can be represented locally as the graph of a twice continuously differentiable function, in a suitable system of coordinates. We write $u_{(a)}(x) = u_{(a)}^A(x)$ to simplify notation. From the analytic viewpoint, the function $u_{(a)}$ may be constructed as follows. For every integer $n \geq 1$, let $u_{(a),n}$ be the unique nonnegative solution of the nonlinear Dirichlet problem

$$\begin{cases} \frac{1}{2}\Delta u = \psi_{(a)}(u) & \text{in } A, \\ u_{|\partial A} = n. \end{cases}$$

Then $u_{(a)} = \lim \uparrow u_{(a),n}$ as $n \to \infty$.

The following lemma records certain analytic properties which will be useful in the forthcoming proofs.

Lemma 2. (i) *Let $x \in A$. The function $a \longrightarrow u_{(a)}(x)$ is continuous and nonincreasing on $[0, \infty)$, and tends to 0 as $a \to \infty$.*
(ii) *For every $\delta \in (0, \operatorname{dist}(0, A^c))$, let A_δ be the subdomain of A defined as the connected component of the open set $\{x \in A : \operatorname{dist}(x, A^c) > \delta\}$ that contains 0. Then, for every $a \geq 0$, $u_{(a)}^{A_\delta}(0)$ tends to $u_{(a)}(0)$ as $\delta \to 0$.*

Proof. (i) Let us first verify that the function $a \longrightarrow u_{(a)}(x)$ is monotone nonincreasing, for every $x \in A$. To see this, we apply a standard comparison principle (see e.g. Lemma V.7 in [LG99]) to obtain that $u_{(a'),n} \leq u_{(a),n}$ if $a \leq a'$, for every $n \geq 1$. It then suffices to let $n \to \infty$.

Let $(a_k)_{k\geq 1}$ be a sequence of nonnegative reals increasing to $a \in (0, \infty)$. We can set for every $x \in A$

$$v(x) = \lim_{k\uparrow\infty} \downarrow u_{(a_k)}(x),$$

and we have $v \geq u_{(a)}$. In order to verify that $v \leq u_{(a)}$, we only need to check that v solves $\frac{1}{2}\Delta v = \psi_{(a)}(v)$ in A (recall that $u_{(a)}$ is the maximal nonnegative solution of this equation). To do so, let B be an open ball whose closure $\bar{B}$ is contained in A. For every $k \geq 1$, the restriction of $u_{(a_k)}$ to B solves the equation $\frac{1}{2}\Delta u = \psi_{(a_k)}(u)$ in B. By the probabilistic interpretation of the integral equation associated with this PDE (see e.g. Chapter V in [LG99]), this implies that, for every $x \in B$,

$$u_{(a_k)}(x) + \mathrm{E}_x\left[\int_0^{\tau_B} \psi_{(a_k)}(u_{(a_k)}(\xi_s))\, ds\right] = \mathrm{E}_x\big[u_{(a_k)}(\xi_{\tau_B})\big],$$

where we recall our notation ξ for a Brownian motion starting from x under the probability measure P_x, and $\tau_B := \inf\{t \geq 0 : \xi_t \notin B\}$. By passing to the limit $k \to \infty$ in the previous display, we can write

$$v(x) + \mathrm{E}_x\left[\int_0^{\tau_B} \psi_{(a)}(v(\xi_s))\, ds\right] = \mathrm{E}_x\big[v(\xi_{\tau_B})\big],$$

which is enough to obtain that v solves $\frac{1}{2}\Delta v = \psi_{(a)}(v)$ in B, and therefore in A since B was arbitrary.

Similar arguments show that, if $(a_k)_{k\geq 1}$ is a decreasing sequence of nonnegative reals converging to $a \in [0,\infty)$, then $u_{(a_k)}(x)$ converges to $u_{(a)}(x)$ for every $x \in A$. Finally, the fact that $u_{(a)}(x)$ tends to 0 as $a \to \infty$ can be obtained from the comparison principle: If B is a ball such that $\bar{B} \subset A$, the restriction of $u_{(a)}$ to B is bounded above by the solution $v_{(a)}$ of the linear equation $\frac{1}{2}\Delta v_{(a)} = a\, v_{(a)}$ in B, with boundary value equal to the restriction of $u_{(0)}$ to B. It is easily seen that $v_{(a)}(x) \longrightarrow 0$ as $a \to \infty$, for instance by using the Feynman-Kac formula.

(ii) Fix $a \geq 0$. If $0 < \delta < \delta'$, the closure of $A_{\delta'}$ is contained in A_δ. The restriction of $u^{A_\delta}_{(a)}$ to $A_{\delta'}$ is a nonnegative solution of $\frac{1}{2}\Delta u = \psi_{(a)}(u)$ in $A_{\delta'}$ and is thus bounded above by $u^{A_{\delta'}}_{(a)}$. Hence, for every fixed $x \in A$ the function $\delta \to u^{A_\delta}_{(a)}(x)$, which is defined for $\delta > 0$ small enough, is nondecreasing and we can set

$$v(x) = \lim_{\delta \downarrow 0} u^{A_\delta}_{(a)}(x).$$

By the same argument we used to obtain the monotonicity of the mapping $\delta \to u^{A_\delta}_{(a)}(x)$, we also have $v(x) \geq u_{(a)}(x)$ for every $x \in A$. To obtain the reverse inequality $v \leq u_{(a)}$, it is enough to verify that v solves $\frac{1}{2}\Delta v = \psi_{(a)}(v)$ in B. But this follows by arguments similar to those we used in the proof of part (i) of the lemma. □

Lemma 3. *For every $a \geq 0$ and $x \in A$,*

$$\{\mathcal{R}(Y^{(a)}) \cap A^c \neq \emptyset\} = \{\mathcal{R}(Y^{(a)}) \cap \bar{A}^c \neq \emptyset\}\,, \quad P_{\delta_x} \text{ a.s.}$$

Proof. The inclusion

$$\{\mathcal{R}(Y^{(a)}) \cap A^c \neq \emptyset\} \supset \{\mathcal{R}(Y^{(a)}) \cap \bar{A}^c \neq \emptyset\}$$

is trivial. To show the reverse inclusion, we may argue as follows. By Theorem IV.9 in [LG99] (which holds under much less stringent assumptions on A), the event $\{\mathcal{R}(Y^{(a)}) \cap A^c \neq \emptyset\}$ holds if and only if the exit measure of the super-Brownian motion $Y^{(a)}$ from A is nonzero. Applying the special Markov property of superprocesses [Dy93, Theorem 1.3], we see that it is enough to prove that for super-Brownian motion starting from a nonzero initial measure supported on ∂A, the range immediately hits $(\bar{A})^c$. This is however easy under our regularity assumptions on A. We leave the details to the reader. □

From now on, we write $\{Y^{(a)} \text{ hits } F\}$ for the event $\{\mathcal{R}(Y^{(a)}) \cap F \neq \emptyset\}$.

3 Quenched convergence to super-Brownian motion

The main goal of this section is to prove Theorem 2. At the end of the section, we also establish Proposition 1, using certain arguments related to the proof of Theorem 2. To simplify notation, we set for every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$,

$$\Gamma_\varpi^\varepsilon = \sqrt{\varepsilon}\,\Gamma_\varpi = \bigcup_{i\in I} \sqrt{\varepsilon}(x_i + K_i).$$

The following lemma identifies a martingale problem solved by our branching Brownian motion $Z^{\varpi,\varepsilon}$. It can be proved by standard arguments (see e.g. Section 9.4 in [EK86]).

Lemma 4. *Let $\varpi \in \mathbf{\Omega}$ and $\varepsilon \geq 0$. Under each probability $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p(\mathbb{R}^d)$, the process $Z^{\varpi,\varepsilon}$ solves the following martingale problem: For every $f \in \bar{C}^2_+(\mathbb{R}^d)$ such that $0 < \inf f \leq f \leq 1$, the process*

$$e^{\langle Z_t^{\varpi,\varepsilon},\log f\rangle} - e^{\langle Z_0^{\varpi,\varepsilon},\log f\rangle} - \int_0^t \left\langle Z_s^{\varpi,\varepsilon}, \frac{\frac{1}{2}\Delta f + \varepsilon \mathbb{I}_{\Gamma_\varpi}(1-f) + \Upsilon(f) - f}{f}\right\rangle e^{\langle Z_s^{\varpi,\varepsilon},\log f\rangle} ds$$

is a martingale.

We can derive from Lemma 4 (or from a direct argument) that for every $g \in \bar{C}^2(\mathbb{R}^d)$,

$$M_t(g) := \langle Z_t^{\varpi,\varepsilon}, g\rangle - \langle Z_0^{\varpi,\varepsilon}, g\rangle - \int_0^t \left\langle Z_s^{\varpi,\varepsilon}, \frac{1}{2}\Delta g - \varepsilon \mathbb{I}_{\Gamma_\varpi} g\right\rangle ds \tag{9}$$

is a martingale. An easy computation gives that the square bracket of this martingale is

$$[M(g), M(g)]_t = \int_0^t \langle Z_s^{\varpi,\varepsilon}, \nabla g.\nabla g\rangle\, ds + \sum_{0\leq s\leq t} \langle Z_s^{\varpi,\varepsilon} - Z_{s-}^{\varpi,\varepsilon}, g\rangle^2. \tag{10}$$

The last sum in the right-hand side is an increasing process with compensator

$$\int_0^t \langle Z_s^{\varpi,\varepsilon}, (\sigma^2 + \varepsilon \mathbb{I}_{\Gamma_\varpi}) g^2\rangle\, ds. \tag{11}$$

The proof of Theorem 2 relies on the following two results, in which we use the notation $(\mathcal{Y}_t)_{t\geq 0}$ for the canonical process on $D_{\mathcal{M}_f(\mathbb{R}^d)}[0,\infty)$. Recall from (3) the definition of the process $X^{\varpi,\varepsilon}$ in terms of $Z^{\varpi,\varepsilon}$.

Lemma 5. *For every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$, let $\Pi^{\varpi,\varepsilon}$ be the law of the process $X^{\varpi,\varepsilon}$ under $\mathbb{P}_{[\varepsilon^{-1}]\delta_0}$. Then,*

(i) *For every $\delta > 0$ and $T > 0$, there is a compact subset $K_{\delta,T}$ of $\mathbb{R}^d$ such that, for every $\varpi \in \mathbf{\Omega}$,*

$$\sup_{\varepsilon\in(0,1)} \Pi^{\varpi,\varepsilon}\Big(\sup_{0\leq t\leq T} \mathcal{Y}_t(K^c_{\delta,T})\Big) < \delta$$

(ii) *For every $g \in \bar{C}^2(\mathbb{R}^d)$ and $\varpi \in \mathbf{\Omega}$, the collection of the laws of the process $(\langle \mathcal{Y}_t, g\rangle)_{t\geq 0}$ under $\Pi^{\varpi,\varepsilon}$, $\varepsilon \in (0,1)$, is relatively compact in the space of all probability measures on $D_{\mathbb{R}}[0,\infty)$.*

Consequently, for every $\varpi \in \mathbf{\Omega}$, the collection $(\Pi^{\varpi,\varepsilon})_{\varepsilon\in(0,1)}$ is relatively compact in the space of all probability measures on $D_{\mathcal{M}_f(\mathbb{R}^d)}[0,\infty)$.

The last assertion of the lemma is an immediate consequence of (i) and (ii) using Theorem II.4.1 in [Per02].

Proposition 3. *Let $g \in \bar{C}^2_+(\mathbb{R}^d)$. There exists a measurable subset $\mathbf{\Omega}_g$ of $\mathbf{\Omega}$ such that $\mathbf{P}(\mathbf{\Omega}_g) = 1$ and the following holds for every $\varpi \in \mathbf{\Omega}_g$. For every $s, t \geq 0$, for every integer $p \in \mathbb{N}$ and every choice of $t_1, \ldots, t_p \in [0, t]$ and $f_1, \ldots, f_p \in \bar{C}\big(\mathcal{M}_f(\mathbb{R}^d)\big)$, we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\left[\left\{e^{-\langle X^{\varpi,\varepsilon}_{t+s},g\rangle} - e^{-\langle X^{\varpi,\varepsilon}_t,g\rangle} - \int_t^{t+s} \left\langle X^{\varpi,\varepsilon}_u, -\frac{1}{2}\Delta g + \psi_{(\kappa)}(g)\right\rangle e^{-\langle X^{\varpi,\varepsilon}_u,g\rangle} du\right\} \prod_{i=1}^p f_i(X^{\varpi,\varepsilon}_{t_i})\right] = 0.$$

We postpone the proof of Lemma 5 and Proposition 3, and explain how Theorem 2 follows from these two statements. We choose a countable dense subset G of $\bar{C}^2_+(\mathbb{R}^d)$ and set

$$\mathbf{\Omega}' = \bigcap_{g\in G} \mathbf{\Omega}_g.$$

Fix $\varpi \in \mathbf{\Omega}'$. By Lemma 5, the collection $(\Pi^{\varpi,\varepsilon})_{\varepsilon\in(0,1)}$ is relatively compact. Let Π^* be a sequential limit of this collection as ε tends to 0. We deduce from Proposition 3 that, for every $g \in G$, for every $s, t \geq 0$ and every choice of $t_1, \ldots, t_p \in [0, t]$ and $f_1, \ldots, f_p \in \bar{C}\big(\mathcal{M}_f(\mathbb{R}^d)\big)$, we have

$$\Pi^*\left[\left\{e^{-\langle \mathcal{Y}_{t+s},g\rangle} - e^{-\langle \mathcal{Y}_t,g\rangle} - \int_t^{t+s} \left\langle \mathcal{Y}_u, -\frac{1}{2}\Delta g + \psi_{(\kappa)}(g)\right\rangle e^{-\langle \mathcal{Y}_u,g\rangle} du\right\} \prod_{i=1}^p f_i(\mathcal{Y}_{t_i})\right] = 0. \qquad (12)$$

The required passage to the limit under the expectation sign is easily justified: Note that, by comparing with the case when there is no killing and using standard results of the theory of critical branching processes, we have for every $T > 0$

$$\sup_{\varepsilon\in(0,1)} \left(\mathbb{E}_{[\varepsilon^{-1}\delta_0]}\left[\sup_{0\leq r\leq T} \langle X^{\varpi,\varepsilon}_r, 1\rangle^2\right]\right) < \infty. \qquad (13)$$

Since G is dense in $\bar{C}^2_+(\mathbb{R}^d)$, another easily justified passage to the limit shows that (12) holds for every $g \in \bar{C}^2_+(\mathbb{R}^d)$. Thus, Π^* satisfies the martingale problem for super-Brownian motion with branching mechanism $\psi_{(\kappa)}$ as stated in Section 2. Since it is also clear that $\Pi^*(Y_0 = \delta_0) = 1$, Π^* must be the law of super-Brownian motion with branching mechanism $\psi_{(\kappa)}$ started from δ_0. This completes the proof of Theorem 2, but we still need to establish Lemma 5 and Proposition 3. □

Proof of Lemma 5. The compact containment condition (i) in the lemma is immediately obtained by observing that $X^{\varpi,\varepsilon}$ is dominated by $X^0 = X^{\varpi,0}$ and by using the convergence of rescaled branching Brownian motions (without killing in the obstacles) towards super-Brownian motion. So we only need to verify (ii). In the remaining part of the proof, we fix $\varpi \in \mathbf{\Omega}$ and $g \in \bar{C}^2_+(\mathbb{R}^d)$. To simplify notation, we set

$$\mathcal{X}^\varepsilon_t = \langle X^{\varpi,\varepsilon}_t, g\rangle.$$

By the remarks following Lemma 4 and an elementary scaling transformation, we have

$$\mathcal{X}^\varepsilon_t = \mathcal{X}^\varepsilon_0 + M^\varepsilon_t + V^\varepsilon_t,$$

where

$$V^\varepsilon_t = \int_0^t \left\langle X^{\varpi,\varepsilon}_s, \frac{1}{2}\Delta g - \mathbb{I}_{\Gamma^\varepsilon_\varpi} g\right\rangle ds$$

and M^ε is a martingale, whose square bracket is given by

$$[M^\varepsilon, M^\varepsilon]_t = \int_0^t \langle X^{\varpi,\varepsilon}_s, \varepsilon\nabla g.\nabla g\rangle\, ds + \sum_{0\leq s\leq t} \langle X^{\varpi,\varepsilon}_s - X^{\varpi,\varepsilon}_{s-}, g\rangle^2.$$

Furthermore, the oblique bracket of M^ε is equal to

$$\langle M^\varepsilon, M^\varepsilon\rangle_t = \int_0^t \big\langle X_s^{\varpi,\varepsilon}, \sigma^2 g^2 + \varepsilon\big(\nabla g.\nabla g + \mathbb{I}_{\Gamma_\varpi^\varepsilon} g^2\big)\big\rangle\; ds.$$

By standard criteria (see in particular Theorem VI.4.13 in Jacod and Shiryaev [JS87]), the tightness of the laws of the processes $\mathcal{X}^\varepsilon$, $\varepsilon \in (0,1)$, will follow if we can verify that the laws of the processes V^ε and $\langle M^\varepsilon, M^\varepsilon\rangle$, for $\varepsilon \in (0,1)$, are C-tight. But this is immediate from the preceding explicit formulas and (13). □

Proof of Proposition 3. Let us fix $s,t \geq 0$, $t_1,\dots,t_p \in [0,t]$ and $f_1,\dots,f_p \in \bar{C}\big(\mathcal{M}_f(\mathbb{R}^d)\big)$. Let $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$. Using the facts that $\Upsilon(1) = \Upsilon'(1) = 1$, $\Upsilon''(1) = \sigma^2$ and Lemma 4 applied to the function $f(x) = \exp\big\{-\varepsilon g(x\sqrt{\varepsilon})\big\}$, we can write

$$\begin{aligned} 0 \;&=\; \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Bigg[\Bigg\{e^{-\langle X_{t+s}^{\varpi,\varepsilon},g\rangle} - e^{-\langle X_t^{\varpi,\varepsilon},g\rangle} - \varepsilon^{-2}\int_t^{t+s}\Big\langle X_u^{\varpi,\varepsilon}, \frac{1}{2}\big(-\varepsilon^2\Delta g + \varepsilon^3\nabla g.\nabla g\big) \\ &\quad + \varepsilon\mathbb{I}_{\Gamma_\varpi^\varepsilon}\big(1-e^{-\varepsilon g}\big)e^{\varepsilon g} + e^{\varepsilon g}\big(\Upsilon\big(e^{-\varepsilon g}\big) - e^{-\varepsilon g}\big)\Big\rangle\; e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg] \\ &=\; \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Bigg[\Bigg\{e^{-\langle X_{t+s}^{\varpi,\varepsilon},g\rangle} - e^{-\langle X_t^{\varpi,\varepsilon},g\rangle} - \int_t^{t+s}\Big\langle X_u^{\varpi,\varepsilon}, -\frac{1}{2}\Delta g + \frac{\varepsilon}{2}\nabla g.\nabla g \\ &\quad + \mathbb{I}_{\Gamma_\varpi^\varepsilon}(g+\eta_\varepsilon) + \frac{\sigma^2}{2}(g^2+\tilde{\eta}_\varepsilon)\Big\rangle\; e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg], \end{aligned} \tag{14}$$

where $\|\eta_\varepsilon\| \leq C_1(g)\varepsilon$ for some constant $C_1(g)$ depending only on g, and $\|\tilde{\eta}_\varepsilon\| \to 0$ as $\varepsilon \to 0$. On the one hand, using Fubini's theorem, the fact that $(\langle X_r^{\varpi,\varepsilon}, 1\rangle)_{r\geq 0}$ is a supermartingale and the inequality $\mathbb{E}_{[\varepsilon^{-1}]\delta_0}[\langle X_0^{\varpi,\varepsilon}, 1\rangle] \leq 1$, we have

$$\begin{aligned} &\Bigg|\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\;\Bigg[\;\Bigg\{\int_t^{t+s}\Big\langle X_u^{\varpi,\varepsilon}, \frac{\varepsilon}{2}\nabla g.\nabla g + \mathbb{I}_{\Gamma_\varpi^\varepsilon}\eta_\varepsilon + \frac{\sigma^2}{2}\tilde{\eta}_\varepsilon\Big\rangle\; e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg]\Bigg| \\ &\quad\leq \Bigg(\frac{\varepsilon}{2}\big(\|\nabla g.\nabla g\| + 2C_1(g)\big) + \frac{\sigma^2}{2}\|\tilde{\eta}_\varepsilon\|\Bigg)\Bigg(\prod_{i=1}^p\|f_i\|\Bigg)\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Bigg[\int_t^{t+s}\langle X_u^{\varpi,\varepsilon}, 1\rangle\; du\Bigg] \\ &\quad\leq \big(C_2(g)\,\varepsilon + (\sigma^2/2)\|\tilde{\eta}_\varepsilon\|\big)\;\Bigg(\prod_{i=1}^p\|f_i\|\Bigg)\, s \end{aligned}$$

with a constant $C_2(g)$ depending only on g. On the other hand,

$$\begin{aligned} &\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\;\Bigg[\;\Bigg\{\int_t^{t+s}\langle X_u^{\varpi,\varepsilon},\; \mathbb{I}_{\Gamma_\varpi^\varepsilon}g\rangle e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg] \\ &\quad= \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Bigg[\Bigg\{\int_t^{t+s}\langle X_u^{\varpi,\varepsilon}, \kappa g\rangle\; e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg] \\ &\quad\quad + \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Bigg[\Bigg\{\int_t^{t+s}\big\langle X_u^{\varpi,\varepsilon}, \big(\mathbb{I}_{\Gamma_\varpi^\varepsilon} - \kappa\big)g\big\rangle\; e^{-\langle X_u^{\varpi,\varepsilon},g\rangle}du\Bigg\}\prod_{i=1}^p f_i\big(X_{t_i}^{\varpi,\varepsilon}\big)\Bigg]. \end{aligned} \tag{15}$$

The absolute value of the second term in the right-hand side of (15) is bounded above by

$$\Bigg(\prod_{i=1}^p\|f_i\|\Bigg)\int_t^{t+s}\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Big[\big|\big\langle X_u^{\varpi,\varepsilon}, \big(\mathbb{I}_{\Gamma_\varpi^\varepsilon} - \kappa\big)g\big\rangle\big|\Big]du.$$

Consequently, going back to (14) and using the preceding estimates, we obtain for every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$,

$$\left| \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\left[\left\{ e^{-\langle X^{\varpi,\varepsilon}_{t+s}, g\rangle} - e^{-\langle X^{\varpi,\varepsilon}_{t}, g\rangle} - \int_t^{t+s} \left\langle X^{\varpi,\varepsilon}_u, -\frac{1}{2}\Delta g + \kappa g + \frac{\sigma^2}{2} g^2 \right\rangle e^{-\langle X^{\varpi,\varepsilon}_u, g\rangle} du \right\} \prod_{i=1}^p f_i\big(X^{\varpi,\varepsilon}_{t_i}\big) \right] \right|$$
$$\leq \left(\prod_{i=1}^p \|f_i\| \right) \Big(\big\{ C_2(g)\,\varepsilon + (\sigma^2/2)\|\tilde{\eta}_\varepsilon\| \big\} s + r_\varepsilon(\varpi, g, t, t+s) \Big), \tag{16}$$

where $r_\varepsilon(\varpi, g, t_1, t_2)$ is defined for $0 \leq t_1 \leq t_2$ by

$$r_\varepsilon(\varpi, g, t_1, t_2) = \int_{t_1}^{t_2} \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Big[\big| \big\langle X^{\varpi,\varepsilon}_u, (\mathbb{I}_{\Gamma^\varepsilon_\varpi} - \kappa) g \big\rangle \big| \Big] du.$$

Lemma 6. *Let $u > 0$. Then, $\mathbf{P}$-a.s.*

$$\lim_{\varepsilon \to \infty} \mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Big[\big| \big\langle X^{\varpi,\varepsilon}_u,\ \big(\mathbb{I}_{\Gamma^\varepsilon_\varpi} - \kappa\big) g \big\rangle \big| \Big] = 0. \tag{17}$$

Assuming that the lemma is proved, we can readily complete the proof of Proposition 3. Using Fubini's theorem, we may find a set $\mathbf{\Omega}_g$ with $\mathbf{P}\big[\mathbf{\Omega}_g\big] = 1$, such that, for every $\varpi \in \mathbf{\Omega}_g$, the convergence in (17) holds simultaneously for all $u > 0$, except possibly on a set of zero Lebesgue measure (depending on ϖ). Since for every $\varpi \in \mathbf{\Omega}$, $\varepsilon \in (0,1)$ and $u \geq 0$

$$\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Big[\big| \big\langle X^{\varpi,\varepsilon}_u,\ \big(\mathbb{I}_{\Gamma^\varepsilon_\varpi} - \kappa\big) g \big\rangle \big| \Big] \leq \|g\|\, \mathbb{E}_{[\varepsilon^{-1}]\delta_0}[\langle X^{\varpi,\varepsilon}_u, 1\rangle] \leq \|g\|,$$

dominated convergence guarantees that for each $\varpi \in \mathbf{\Omega}_g$, $\lim_{\varepsilon\to 0} r_\varepsilon(\varpi, g, t, t+s) = 0$ for all $t, s \geq 0$. It follows that the right-hand side of (16) tends to 0 as $\varepsilon \to 0$ when $\varpi \in \mathbf{\Omega}_g$, which completes the proof of Proposition 3. □

Proof of Lemma 6. Recall that $\xi^{\varpi,\varepsilon}$ denotes standard d-dimensional Brownian motion killed at rate ε within Γ_ϖ. We also use the notation $\chi^{\varpi,\varepsilon}$ for Brownian motion killed at rate 1 within $\Gamma^\varepsilon_\varpi$. Both processes $\xi^{\varpi,\varepsilon}$ and $\chi^{\varpi,\varepsilon}$ start from x under the probability measure P_x.

We first recall classical moment formulas for branching Brownian motion. For every $x \in \mathbb{R}^d$, $k \in \mathbb{N}$, and every bounded measurable function h on $\mathbb{R}^d$, we have

$$\mathbb{E}_{k\delta_x}[\langle Z^{\varpi,\varepsilon}_t, h\rangle] = k\, \mathrm{E}_x[h(\xi^{\varpi,\varepsilon}_t)]$$

and

$$\mathbb{E}_{k\delta_x}[\langle Z^{\varpi,\varepsilon}_t, h\rangle^2] = k\, \mathrm{E}_x[h(\xi^{\varpi,\varepsilon}_t)^2] + k(k-1)\mathrm{E}_x[h(\xi^{\varpi,\varepsilon}_t)]^2 + k\sigma^2\, \mathrm{E}_x\left[\int_0^t ds\, \mathrm{E}_{\xi^{\varpi,\varepsilon}_s}[h(\xi^{\varpi,\varepsilon}_{t-s})]^2 \right].$$

These formulas are easily derived, first for $k = 1$, from the well-known formula for the Laplace functional of $Z^{\varpi,\varepsilon}_t$ under $\mathbb{P}_{\delta_x}$: See e.g. Proposition II.3 in [LG99].

Recalling the definition of $X^{\varpi,\varepsilon}$ in terms of $Z^{\varpi,\varepsilon}$, we get similar formulas for the first and second moment of $\langle X^{\varpi,\varepsilon}_s, h\rangle$. In particular, for $s \geq 0$ and for every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$, we have

$$\mathbb{E}_{[\varepsilon^{-1}]\delta_0}\Big[\big\langle X^{\varpi,\varepsilon}_s, \big(\mathbb{I}_{\Gamma^\varepsilon_\varpi} - \kappa\big) g \big\rangle^2 \Big] = \varepsilon^2 \big([\varepsilon^{-1}]^2 - [\varepsilon^{-1}]\big) A^{\varpi,\varepsilon}_1(0, s, g)^2 + \varepsilon[\varepsilon^{-1}] A^{\varpi,\varepsilon}_2(0, s, g), \tag{18}$$

where, for every $x \in \mathbb{R}^d$,

$$A^{\varpi,\varepsilon}_1(x, s, g) := \mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\chi^{\varpi,\varepsilon}_s) - \kappa\big) g(\chi^{\varpi,\varepsilon}_s) \Big],$$

and

$$A_2^{\varpi,\varepsilon}(x,s,g) := \sigma^2 \ \mathrm{E}_x\Big[\int_0^s A_1^{\varpi,\varepsilon}(\chi_v^{\varpi,\varepsilon}, s-v, g)^2 dv\Big] + \varepsilon \ \mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\chi_s^{\varpi,\varepsilon}) - \kappa\big)^2 g(\chi_s^{\varpi,\varepsilon})^2\Big].$$

We claim that, for every $x \in \mathbb{R}^d$ and $s > 0$,

$$\mathbf{P}\text{−a.s.,} \qquad \lim_{\varepsilon\to 0} \mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\chi_s^{\varpi,\varepsilon}) - \kappa\big) g\big(\chi_s^{\varpi,\varepsilon}\big)\Big] = 0. \tag{19}$$

Assume for the moment that the claim holds. By taking $x = 0$ in (19), we obtain that $A_1^{\varpi,\varepsilon}(0,s,g)$ tends to 0 as $\varepsilon \to 0$, **P**-a.s. Consider next $A_2^{\varpi,\varepsilon}(0,s,g)$. The second term in the formula for $A_2^{\varpi,\varepsilon}(x,s,g)$ obviously tends to 0 uniformly in x, s and independently of the obstacles, as $\varepsilon \to 0$. To handle the first term, use Fubini's theorem to obtain that, for every ϖ belonging to a set $\widetilde{\mathbf{\Omega}}$ of full probability, there exists a set $\mathcal{N}_\varpi \subset \mathbb{R}^d \times \mathbb{R}_+$ of zero Lebesgue measure such that the convergence in (19) holds simultaneously for all $(x,s) \notin \mathcal{N}_\varpi$. We can then write the first term in the formula for $A_2^{\varpi,\varepsilon}(0,s,g)$ as follows:

$$\sigma^2 \int_0^s dv \int_{\mathbb{R}^d} dy \ p_v^{\varpi,\varepsilon}(0,y) \ \mathrm{E}_y\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\chi_{s-v}^{\varpi,\varepsilon}) - \kappa\big) g(\chi_{s-v}^{\varpi,\varepsilon})\Big]^2, \tag{20}$$

where $p_v^{\varpi,\varepsilon}(\cdot,\cdot)$ is the transition kernel of $\chi^{\varpi,\varepsilon}$ at time v. Plainly, we have $p_v^{\varpi,\varepsilon}(x,y) \le p_v(x,y)$ for each $v \ge 0$ and $\varepsilon \in (0,1)$, where $p_v(\cdot,\cdot)$ is the transition kernel of standard Brownian motion at time v. Using this bound and dominated convergence gives us that the quantity in (20) tends to 0 as $\varepsilon \to 0$, for every $\varpi \in \widetilde{\mathbf{\Omega}}$. Hence $A_2^{\varpi,\varepsilon}(0,s,g)$ tends to 0 as $\varepsilon \to 0$, **P**-a.s. The result of Lemma 6 now follows from (18) and the Cauchy-Schwarz inequality.

We still have to prove our claim (19). We fix $x \in \mathbb{R}^d$ and $s > 0$. Let $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$. From the definition of Brownian motion killed at rate 1 in $\Gamma_\varpi^\varepsilon$, we have

$$\mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\chi_s^{\varpi,\varepsilon}) - \kappa\big) g(\chi_s^{\varpi,\varepsilon})\Big] = \mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_s) - \kappa\big) g(\xi_s) \exp\Big\{-\int_0^s \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_u) du\Big\}\Big]. \tag{21}$$

Let $\eta > 0$ and choose $\theta \in (0,s)$ such that $e^{-\theta} \ge 1 - \eta$. By the Markov property applied at time $s - \theta$, the quantities in (21) are equal to

$$\begin{aligned}
&\mathrm{E}_x\Big[\exp\Big\{-\int_0^{s-\theta} \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_u)du\Big\} \ \mathrm{E}_{\xi_{s-\theta}}\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_\theta) - \kappa\big) g(\xi_\theta) \exp\Big\{-\int_0^\theta \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_v)dv\Big\}\Big]\Big] \\
&\quad = \mathrm{E}_x\Big[\exp\Big\{-\int_0^{s-\theta} \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_u)du\Big\} \ \mathrm{E}_{\xi_{s-\theta}}\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_\theta) - \kappa\big) g(\xi_\theta)\Big]\Big] \\
&\quad\quad + \mathrm{E}_x\Big[\exp\Big\{-\int_0^{s-\theta} \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_u)du\Big\} \ \mathrm{E}_{\xi_{s-\theta}}\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_\theta) - \kappa\big) g(\xi_\theta) \Big(\exp\Big\{-\int_0^\theta \mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_v)dv\Big\} - 1\Big)\Big]\Big].
\end{aligned} \tag{22}$$

The condition $1 - e^{-\theta} \le \eta$ entails that the absolute value of the second term in the right-hand side of (22) is bounded above by $\|g\|\eta$. Suppose we know that for every $y \in \mathbb{R}^d$ we have

$$\mathbf{P}\text{−a.s.,} \qquad \lim_{\varepsilon\to 0} \mathrm{E}_y\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\xi_\theta) - \kappa\big) g(\xi_\theta)\Big] = 0. \tag{23}$$

Then, using the fact that the law of $\xi_{s-\theta}$ is absolutely continuous with respect to Lebesgue measure (and Fubini's theorem to see that the convergence in (23) holds simultaneously for almost all $y \in \mathbb{R}^d$, **P**-a.s.), we conclude that the first term in the right-hand side of (22) converges to 0 with **P**-probability 1 as $\varepsilon \to 0$. Consequently, we have **P**-a.s.

$$\limsup_{\varepsilon\to 0} \Big| \mathrm{E}_x\Big[\big(\mathbb{I}_{\Gamma_\varpi^\varepsilon}(\chi_s^{\varpi,\varepsilon}) - \kappa\big) g(\chi_s^{\varpi,\varepsilon})\Big]\Big| \le \|g\|\eta.$$

Since η was arbitrary, our claim (19) follows.

It thus remains to prove that (23) holds. We fix $y \in \mathbb{R}^d$ and $\theta > 0$. In the following, ξ' stands for another Brownian motion independent of ξ, which also starts from y under the probability measure P_y. For each $\varepsilon \in (0,1)$, we have

$$\begin{aligned}\mathbf{E}\Big[\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta) - \kappa\big)g(\xi_\theta)\big]^2\Big] &= \mathbf{E}\Big[\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta) - \kappa\big)\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi'_\theta) - \kappa\big)g(\xi_\theta)g(\xi'_\theta)\big]\Big] \\ &= \mathrm{E}_y\Big[g(\xi_\theta)g(\xi'_\theta)\Big\{\mathbf{P}\big[\xi_\theta \in \Gamma^\varepsilon_\varpi;\ \xi'_\theta \in \Gamma^\varepsilon_\varpi\big] - \kappa^2\Big\}\Big], \end{aligned} \tag{24}$$

where the last line uses Fubini's theorem and the definition of κ. Recall that the measure Θ is supported on compact sets which are contained in the fixed ball $\overline{B}(0,r_0)$. If $|x - x'| > 2r_0$, the sets $\{(z,K) : x \in z + K \text{ and } K \subset \overline{B}(0,r_0)\}$ and $\{(z,K) : x' \in z + K \text{ and } K \subset \overline{B}(0,r_0)\}$ are disjoint, and so the events $\{x \in \Gamma_\varpi\}$ and $\{x' \in \Gamma_\varpi\}$ are independent under $\mathbf{P}$. Recalling that $\Gamma^\varepsilon_\varpi = \sqrt{\varepsilon}\,\Gamma_\varpi$, we see that if $|x - x'| > 2r_0\sqrt{\varepsilon}$ we have $\mathbf{P}[x \in \Gamma^\varepsilon_\varpi;\ x' \in \Gamma^\varepsilon_\varpi] = \mathbf{P}[x \in \Gamma^\varepsilon_\varpi]\mathbf{P}[x' \in \Gamma^\varepsilon_\varpi] = \kappa^2$, which enables us to write

$$\mathbf{P}\big[\xi_\theta \in \Gamma^\varepsilon_\varpi;\ \xi'_\theta \in \Gamma^\varepsilon_\varpi\big] = \kappa^2\ \mathbb{I}_{\{|\xi_\theta - \xi'_\theta| > 2r_0\sqrt{\varepsilon}\}} + \mathbf{P}\big[\xi_\theta \in \Gamma^\varepsilon_\varpi;\ \xi'_\theta \in \Gamma^\varepsilon_\varpi\big]\ \mathbb{I}_{\{|\xi_\theta - \xi'_\theta| \le 2r_0\sqrt{\varepsilon}\}}.$$

Going back to (24), we obtain

$$\begin{aligned}\mathbf{E}\Big[\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta) - \kappa\big)g(\xi_\theta)\big]^2\Big] &= \mathrm{E}_y\Big[g(\xi_\theta)g(\xi'_\theta)\mathbb{I}_{\{|\xi_\theta - \xi'_\theta| \le 2r_0\sqrt{\varepsilon}\}}\Big\{\mathbf{P}\big[\xi_\theta \in \Gamma^\varepsilon_\varpi;\ \xi'_\theta \in \Gamma^\varepsilon_\varpi\big] - \kappa^2\Big\}\Big] \\ &\le \|g\|^2\mathrm{P}_y\big[|\xi_\theta - \xi'_\theta| \le 2r_0\sqrt{\varepsilon}\big] \\ &= \|g\|^2\mathrm{P}_0\big[|\xi_{2\theta}| \le 2r_0\sqrt{\varepsilon}\big] \\ &\le C\frac{\|g\|^2}{\theta^{d/2}}\ \varepsilon^{d/2}, \end{aligned} \tag{25}$$

where the constant $C > 0$ depends only on r_0. Let $\eta > 0$. By the Markov inequality, we can write

$$\mathbf{P}\Big[\big|\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta) - \kappa\big)g(\xi_\theta)\big]\big| > \eta\Big] \le \frac{1}{\eta^2}\ \mathbf{E}\Big[\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta) - \kappa\big)g(\xi_\theta)\big]^2\Big] \le C\frac{\|g\|^2}{\eta^2\theta^{d/2}}\ \varepsilon^{d/2}.$$

Applying the last bound with $\varepsilon = \varepsilon_k = k^{-3}$, for every $k \in \mathbb{N}$, yields a convergent series, and so by the Borel-Cantelli lemma we have $\mathbf{P}$-a.s.,

$$\limsup_{k\to\infty}\ \Big|\mathrm{E}_y\big[\big(\mathbb{I}_{\Gamma^{\varepsilon_k}_\varpi}(\xi_\theta) - \kappa\big)g(\xi_\theta)\big]\Big| \le \eta.$$

Since $\eta > 0$ was arbitrary, the convergence (23) holds along the sequence $(\varepsilon_k)_{k\ge1}$.

To complete the proof, we set for every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$

$$U_{\varpi,\varepsilon} = \mathrm{E}_y\big[\mathbb{I}_{\Gamma^\varepsilon_\varpi}(\xi_\theta)\, g(\xi_\theta)\big]$$

(recall that y and θ are fixed). We shall prove the following result: For every $\varpi \in \mathbf{\Omega}$,

$$\lim_{k\to\infty}\ \sup_{(k+1)^{-3}\le\varepsilon\le k^{-3}} \big|U_{\varpi,\varepsilon} - U_{\varpi,\varepsilon_k}\big| = 0. \tag{26}$$

Combining (26) with the fact that the convergence (23) is true along the sequence $(\varepsilon_k)_{k\ge1}$ will then lead to the desired conclusion. In order to prove (26), we first note that for every $\varepsilon \in (0,1)$,

$$U_{\varpi,\varepsilon} = \frac{1}{(2\pi\theta)^{d/2}}\int_{\mathbb{R}^d} \mathbb{I}_{\Gamma^\varepsilon_\varpi}(x)\, g(x) e^{-\frac{|x-y|^2}{2\theta^2}}\, dx = \int_{\mathbb{R}^d} \mathbb{I}_{\Gamma^\varepsilon_\varpi}(x)\, h(x) dx,$$

with a function $h \in \bar{C}^2_+(\mathbb{R}^d)$ which depends on y and θ, but not on ϖ. Furthermore, for any fixed $\eta > 0$ we can find a large closed ball B centered at the origin and such that $\int_{B^c} h(x)dx < \eta$. Hence, if we set

$$U'_{\varpi,\varepsilon} = \int_B h(x)\ \mathbb{I}_{\Gamma^\varepsilon_\varpi}(x) dx,$$

we have $|U_{\varpi,\varepsilon} - U'_{\varpi,\varepsilon}| \le \eta$ for every $\varepsilon \in (0,1)$. Thanks to this remark, it is enough to prove that (26) holds when $U_{\varpi,\varepsilon}$ and U_{ϖ,ε_k} are replaced by $U'_{\varpi,\varepsilon}$ and $U'_{\varpi,\varepsilon_k}$ respectively.

Let $k \in \mathbb{N}$ and $\varepsilon \in [(k+1)^{-3}, k^{-3})$. We have

$$\begin{aligned} U'_{\varpi,\varepsilon} &= \varepsilon^{d/2} \int_{\varepsilon^{-1/2}B} h\big(x\sqrt{\varepsilon}\big) \mathbb{I}_{\Gamma_\varpi}(x)dx \\ &= \varepsilon^{d/2} \int_{k^{3/2}B} h\big(x\sqrt{\varepsilon}\big) \mathbb{I}_{\Gamma_\varpi}(x)dx + \varepsilon^{d/2} \int_{(\varepsilon^{-1/2}B)\setminus(k^{3/2}B)} h\big(x\sqrt{\varepsilon}\big) \mathbb{I}_{\Gamma_\varpi}(x)dx. \end{aligned} \tag{27}$$

Now, the first term in the right-hand side of (27) is equal to

$$\begin{aligned} &\big(\varepsilon k^3\big)^{d/2} \frac{1}{k^{3d/2}} \int_{k^{3/2}B} h\Big(\frac{x}{k^{3/2}}\Big) \mathbb{I}_{\Gamma_\varpi}(x)dx + \varepsilon^{d/2} \int_{k^{3/2}B} \Big[h\Big(x\sqrt{\varepsilon}\Big) - h\Big(\frac{x}{k^{3/2}}\Big)\Big] \mathbb{I}_{\Gamma_\varpi}(x)dx \\ &= \big(\varepsilon k^3\big)^{d/2} U'_{\varpi,\varepsilon_k} + \iota(\varpi,\varepsilon,k), \end{aligned}$$

where

$$\iota(\varpi,\varepsilon,k) = (\varepsilon k^3)^{d/2} \int_B \Big[h((k^{3/2}\sqrt{\varepsilon})y) - h(y)\Big] \mathbb{I}_{\Gamma_\varpi}(k^{3/2}y)dy.$$

Note that $0 \le 1 - k^{3/2}\sqrt{\varepsilon} \le \frac{C}{k}$ with a constant C independent of ε and k, from which it easily follows that

$$\sup_{(k+1)^{-3} \le \varepsilon \le k^{-3}} |\iota(\varpi,\varepsilon,k)| \le \frac{C'}{k}\, \|\nabla h\|,$$

with a constant C' depending only on B.

Similarly, the second term in the right-hand side of (27) is bounded above by

$$\varepsilon^{d/2}\, \|h\| \big[\varepsilon^{-d/2} - k^{3d/2}\big] \lambda_d(B) \le \frac{C''\|h\|}{k}.$$

Finally, from (27) and the preceding estimates, we have

$$\big|U'_{\varpi,\varepsilon} - U'_{\varpi,\varepsilon_k}\big| \le (1 - (\varepsilon k^3)^{d/2})\, U'_{\varpi,\varepsilon_k} + \frac{C'\|\nabla h\|}{k} + \frac{C''\|h\|}{k},$$

and the convergence (26) follows. This completes the proof. □

Proof of Proposition 1. Clearly we get a lower bound for the quantity $\mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon}$ hits $B(0,R)^c)$ if we replace $Z^{\varpi,\varepsilon}$ by branching Brownian motion killed homogeneously over $\mathbb{R}^d$ at rate ε. Part (i) of the proposition thus follows from the lower bound in Lemma 1.

Let us turn to the proof of the first assertion in (ii). We have the following inequality: For every $\varpi \in \mathbf{\Omega}$ and $\varepsilon \in (0,1)$,

$$\mathbb{P}_{\delta_0}\big[Z^{\varpi,\varepsilon} \text{ hits } B(0,R)^c\big] \le \mathrm{P}_0\big[\xi^{\varpi,\varepsilon} \text{ hits } B(0,R)^c\big]. \tag{28}$$

Indeed, using the formalism of Subsection 2.3, the criticality of the offspring distribution can be used to check that the right-hand side of (28) is just the expected value of the number of those historical paths ω^v that first exit $B(0,R)$ during the interval $[\alpha_v, \beta_v \wedge \zeta_v^{\varpi,\varepsilon})$. Alternatively, it is easy to derive an integral equation similar to (8) for the function $x \to \mathbb{P}_{\delta_x}\big[Z^{\varpi,\varepsilon}$ hits $B(0,R)^c\big]$, and the bound (28) then trivially follows from this integral equation.

Then, let us bound $\mathbf{P} \otimes \mathrm{P}_0\big[\xi^{\varpi,\varepsilon}$ hits $B(0,R)^c\big]$. We write $\xi^{\varpi,\varepsilon,k}$ for the k-th coordinate of $\xi^{\varpi,\varepsilon}$, for every $k = 1,\ldots,d$. First, observe that

$$\mathbf{P} \otimes \mathrm{P}_0\big[\xi^{\varpi,\varepsilon} \text{ hits } B(0,R)^c\big] \le \sum_{k=1}^{d} \mathbf{P} \otimes \mathrm{P}_0\big[\xi^{\varpi,\varepsilon,k} \text{ hits } (-R/\sqrt{d}, R/\sqrt{d})^c\big]. \tag{29}$$

Clearly, we can restrict our attention to the first term in the sum. Define $N_{\varepsilon,R} := [R\sqrt{\varepsilon}/\sqrt{d}]$ and for every $j \in \mathbb{Z}$, set $T_j^\varepsilon = \inf\{t \geq 0 : \xi_t^1 = j\varepsilon^{-1/2}\}$, where ξ^1 stands for the first coordinate of ξ. In this notation, we can write

$$
\begin{aligned}
&\mathbf{P} \otimes \mathrm{P}_0\big[\xi^{\varpi,\varepsilon,1} \text{ hits } (-R/\sqrt{d}, R/\sqrt{d})^c\big] \\
&\leq \mathbf{P} \otimes \mathrm{P}_0\Big[\xi^{\varpi,\varepsilon,1} \text{ hits } \big(-N_{\varepsilon,R}\varepsilon^{-1/2}, N_{\varepsilon,R}\varepsilon^{-1/2}\big)^c\Big] \\
&\leq \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\int_0^{T^\varepsilon_{N_{\varepsilon,R}}} \mathbb{I}_{\Gamma_\varpi}(\xi_s)ds\bigg\}\bigg] + \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\int_0^{T^\varepsilon_{-N_{\varepsilon,R}}} \mathbb{I}_{\Gamma_\varpi}(\xi_s)ds\bigg\}\bigg]. \qquad (30)
\end{aligned}
$$

Consider the first term in the right-hand side of (30). Recall the definition (1) of the set of obstacles, and set for every $j \in \mathbb{N}$ and $\varepsilon > 0$,

$$
I_j^\varepsilon := \big\{i \in I : \ x_i \in ((j-1)\varepsilon^{-1/2}, j\varepsilon^{-1/2}) \times \mathbb{R}^{d-1}\big\},
$$

$$
\Gamma_\varpi(j,\varepsilon) := \bigcup_{i \in I_j^\varepsilon} (x_i + K_i) \qquad \text{and} \qquad \Gamma_\varpi^\varepsilon(j) := \sqrt{\varepsilon}\,\Gamma_\varpi(j,\varepsilon).
$$

Note that the random sets $\Gamma_\varpi(j,\varepsilon)$, $j \in \mathbb{N}$, are independent under $\mathbf{P}$, by properties of Poisson measures. We have then

$$
\begin{aligned}
\mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\int_0^{T^\varepsilon_{N_{\varepsilon,R}}} \mathbb{I}_{\Gamma_\varpi}(\xi_s)ds\bigg\}\bigg] &\leq \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\sum_{j=1}^{N_{\varepsilon,R}}\int_{T^\varepsilon_{j-1}}^{T^\varepsilon_j} \mathbb{I}_{\Gamma_\varpi(j,\varepsilon)}(\xi_s)ds\bigg\}\bigg] \\
&= \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\int_0^{T^\varepsilon_1} \mathbb{I}_{\Gamma_\varpi(1,\varepsilon)}(\xi_s)ds\bigg\}\bigg]^{N_{\varepsilon,R}}, \qquad (31)
\end{aligned}
$$

where the equality comes from an application of the strong Markov property of ξ, together with the independence of the random sets $\Gamma_\varpi(j,\varepsilon)$ and the fact that the distribution of each of these random sets is invariant under translations by elements of $\{0\} \times \mathbb{R}^{d-1}$. By scaling, if T_1 denotes the entrance time of ξ into $[1,\infty) \times \mathbb{R}^{d-1}$, we can write

$$
\alpha_\varepsilon := \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\varepsilon\int_0^{T^\varepsilon_1} \mathbb{I}_{\Gamma_\varpi(1,\varepsilon)}(\xi_s)ds\bigg\}\bigg] = \mathbf{E} \otimes \mathrm{E}_0\bigg[\exp\bigg\{-\int_0^{T_1} \mathbb{I}_{\Gamma^\varepsilon_\varpi(1)}(\xi_s)ds\bigg\}\bigg].
$$

We then observe that

$$
\int_0^{T_1} \mathbb{I}_{\Gamma^\varepsilon_\varpi(1)}(\xi_s)\mathbb{I}_{(0,1)}(\xi^1_s)\,ds \ \xrightarrow[\varepsilon\to 0]{(P)} \ \kappa\int_0^{T_1} \mathbb{I}_{(0,1)}(\xi^1_s)ds \qquad (32)
$$

where the notation $\xrightarrow{(P)}$ refers to convergence in probability under $\mathbf{P}\otimes\mathrm{P}_0$. To see this, we use arguments similar to the proof of Lemma 6. Notice that $\mathbf{P}[y \in \Gamma^\varepsilon_\varpi(1)] \leq \kappa$ if $y \in (0,1) \times \mathbb{R}^{d-1}$, with equality if $y \in (r_0\sqrt{\varepsilon}, 1 - r_0\sqrt{\varepsilon}) \times \mathbb{R}^{d-1}$. Using this remark, and the same argument as in the derivation of (25), we can write for every fixed $u > 0$,

$$
\begin{aligned}
&\mathbf{E} \otimes \mathrm{E}_0\bigg[\bigg\{\int_0^{T_1\wedge u} \big(\mathbb{I}_{\Gamma^\varepsilon_\varpi(1)}(\xi_s) - \kappa\big)\mathbb{I}_{(0,1)}(\xi^1_s)ds\bigg\}^2\bigg] \\
&= \mathbf{E} \otimes \mathrm{E}_0\bigg[\int_0^{T_1\wedge u}\int_0^{T_1\wedge u} \big(\mathbb{I}_{\Gamma^\varepsilon_\varpi(1)}(\xi_s) - \kappa\big)\big(\mathbb{I}_{\Gamma^\varepsilon_\varpi(1)}(\xi_t) - \kappa\big)\,\mathbb{I}_{(0,1)}(\xi^1_s)\,\mathbb{I}_{(0,1)}(\xi^1_t)\,dsdt\bigg] \\
&= \mathrm{E}_0\bigg[\int_0^{T_1\wedge u}\int_0^{T_1\wedge u} \big\{\mathbf{P}\big[\xi_s, \xi_t \in \Gamma^\varepsilon_\varpi(1)\big] - \kappa^2\big\}\,\mathbb{I}_{(0,1)}(\xi^1_s)\,\mathbb{I}_{(0,1)}(\xi^1_t)\,dsdt\bigg] + O(\varepsilon^{1/2}) \\
&= \mathrm{E}_0\bigg[\int_0^{T_1\wedge u}\int_0^{T_1\wedge u} \big\{\mathbf{P}\big[\xi_s, \xi_t \in \Gamma^\varepsilon_\varpi(1)\big] - \kappa^2\big\}\,\mathbb{I}_{\{|\xi_s-\xi_t|\leq 2r_0\sqrt{\varepsilon}\}}\,\mathbb{I}_{(0,1)}(\xi^1_s)\,\mathbb{I}_{(0,1)}(\xi^1_t)\,dsdt\bigg] + O(\varepsilon^{1/2}) \\
&\leq \mathrm{E}_0\bigg[\int_0^u\int_0^u \mathbb{I}_{\{|\xi_s-\xi_t|\leq 2r_0\sqrt{\varepsilon}\}}\,dsdt\bigg] + O(\varepsilon^{1/2})
\end{aligned}
$$

where the error term $O(\varepsilon^{1/2})$ corresponds to the contribution of times s,t such that ξ^1_s or ξ^1_t belongs to the set $(0, r_0\sqrt{\varepsilon}] \cup [1-r_0\sqrt{\varepsilon}, 1)$. The preceding quantity tends to 0 as $\varepsilon \to 0$, which yields the convergence (32). Since the limiting variable in (32) is (strictly) positive a.s., we can find $\varepsilon_1 \in (0,1)$ and $c_1 < 1$ such that $\alpha_\varepsilon \leq c_1$ for every $\varepsilon \in (0, \varepsilon_1)$. Using (29), (30) and (31), we arrive at

$$\mathbf{P} \otimes \mathrm{P}_0\big[\xi^{\varpi,\varepsilon} \text{ hits } B(0,R)^c\big] \leq 2d\, c_1^{N_{\varepsilon,R}},$$

for $\varepsilon \in (0, \varepsilon_1)$. This completes the proof of the first assertion in (ii).

Let us turn to the second assertion. Let $C_2 > 0$ be a positive constant whose choice will be specified later. By simple comparison arguments, it is enough to prove the desired estimate when R is of the form $R = 2^k$, for $k \in \mathbb{N}$ large enough, and ε is of the form $\varepsilon = 2^{-j}$, with $j \in \{0, 1, \ldots, 2k\}$ such that $\varepsilon \geq C_2(\log\log R)^2/R^2$.

By the first assertion in (ii) and the Markov inequality,

$$\mathbf{P}\Big[\mathbb{P}_{\delta_0}\big[Z^{\varpi,\varepsilon} \text{ hits } B(0,R)^c\big] \geq \exp(-C_1 R\sqrt{\varepsilon}/2)\Big] \leq \exp(-C_1 R\sqrt{\varepsilon}/2). \tag{33}$$

However, if $R = 2^k$ and $\varepsilon \geq C_2(\log\log R)^2/R^2$, we have

$$R\sqrt{\varepsilon} \geq \sqrt{C_2}\,\log\log R = \sqrt{C_2}(\log k + \log\log 2).$$

Using this bound, we can choose the constant C_2 sufficiently large so that we get a convergence series when we sum the right-hand side of (33) over all $R = 2^k$ and $\varepsilon = 2^{-j}$ for $j \in \{0, 1, \ldots, 2k\}$ such that $\varepsilon \geq C_2(\log\log R)^2/R^2$. The Borel-Cantelli lemma now yields the desired result.

□

4 Proof of the main result

In this section, we prove Theorem 1. We fix the environment ϖ such that the weak convergence of Theorem 2 holds, and derive the convergence in Theorem 1 for this fixed value of the environment. For the sake of simplicity, we shall omit ϖ in the notation and write Z^ε instead of $Z^{\varpi,\varepsilon}$, and X^ε instead of $X^{\varpi,\varepsilon}$.

We shall verify that for any increasing sequence $(R_n)_{n\geq 1}$ of positive reals converging to $+\infty$ and any sequence $(\varepsilon_n)_{n\geq 1}$ of nonnegative reals such that $\varepsilon_n R_n^2 \to a \in [0,\infty]$, we have

$$\lim_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) = u_{(\kappa a)}(0), \tag{34}$$

where $u_{(\infty)}(0) = 0$ by convention.

The statement of Theorem 1 follows from this convergence. Indeed, if the conclusion of the theorem fails, then we can find a sequence $R_n \uparrow \infty$ and a sequence (ε_n) of nonnegative reals such that, for every $n \geq 1$,

$$|R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) - u_{(\kappa\varepsilon_n R_n^2)}(0)| \geq \delta$$

for some constant $\delta > 0$. By extracting a subsequence, we may assume that $\varepsilon_n R_n^2 \longrightarrow a \in [0,\infty]$ and thus obtain a contradiction with (34) since we know from Lemma 2 that the mapping $b \to u_{(b)}(0)$ is continuous on $[0,\infty]$.

In proving (34), we may assume that $\varepsilon_n \to 0$ as $n \to \infty$. Indeed, suppose that (34) holds in this particular case and let (ε'_n) be a sequence that does not converge to 0. If the sequence $\varepsilon'_n R_n^2$ converges then necessarily its limit is $+\infty$, and we can find another sequence ε''_n such that $0 \leq \varepsilon''_n \leq \varepsilon'_n$, $\varepsilon''_n \to 0$ and $\varepsilon''_n R_n^2 \to \infty$. So, if we know that (34) holds in the case when the sequence (ε_n) tends to 0, we obtain

$$\lim_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon''_n} \text{ hits } R_n A^c) = 0.$$

However, from the inequality $\varepsilon''_n \leq \varepsilon'_n$ and a coupling argument (obvious if one uses the construction described in Subsection 2.3), we get the same result for the sequence (ε'_n).

A similar comparison argument shows that it is enough to prove (34) in the case when $a < \infty$. Otherwise, it suffices to replace ε_n by $\varepsilon_n \wedge bR_n^{-2}$ and let $b \to \infty$, using the fact that $u_{(b)}(0) \to 0$ as $b \to \infty$.

Let us now proceed to the proof of (34). We fix the sequences $R_n \uparrow \infty$ and $\varepsilon_n \to 0$ such that $\varepsilon_n R_n^2 \longrightarrow a \in [0, \infty)$. We first assume that $a > 0$. The case $a = 0$ will be discussed at the end of the section.

Let B be a closed subset of $\mathbb{R}^d$. For every $\varepsilon > 0$, we have by the definition of X^ε

$$\begin{aligned}\mathbb{P}_{[\varepsilon^{-1}]\delta_0}(X^\varepsilon \text{ hits } B) &= \mathbb{P}_{[\varepsilon^{-1}]\delta_0}(\exists t \geq 0 : X_t^\varepsilon(B) > 0)\\ &= \mathbb{P}_{[\varepsilon^{-1}]\delta_0}\left(\exists t \geq 0 : \int Z^\varepsilon_{\varepsilon^{-1}t}(dx)\, \mathbb{I}_B(x\sqrt{\varepsilon}) > 0\right)\\ &= \mathbb{P}_{[\varepsilon^{-1}]\delta_0}(Z^\varepsilon \text{ hits } \varepsilon^{-1/2}B)\\ &= 1 - \mathbb{P}_{\delta_0}(Z^\varepsilon \text{ does not hit } \varepsilon^{-1/2}B)^{[\varepsilon^{-1}]},\end{aligned}$$

since (for a fixed environment) the law of Z^ε under $\mathbb{P}_{[\varepsilon^{-1}]\delta_0}$ is obtained by adding $[\varepsilon^{-1}]$ independent copies of Z^ε under $\mathbb{P}_{\delta_0}$. Applying the preceding identity with $\varepsilon = \varepsilon_n$ and $B = b_n A^c$, where $b_n = \varepsilon_n^{1/2} R_n$, gives us that

$$1 - \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ does not hit } R_n A^c)^{[\varepsilon_n^{-1}]} = \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) \tag{35}$$

By Theorem 2, we know that the law of X^{ε_n} under $\mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}$ converges as $n \to \infty$ to the law of super-Brownian motion with branching mechanism $\psi_{(\kappa)}$ started from δ_0. The next lemma is essentially a consequence of this convergence. We use the notation of Subsection 2.4.

Lemma 7. *We have*

$$\lim_{n\to\infty} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) = P_{\delta_0}(Y^{(\kappa)} \text{ hits } bA^c),$$

where $b = \sqrt{a} = \lim b_n$.

We postpone the proof of Lemma 7 and proceed to the proof of (34), in the case when $a > 0$. By the results recalled in Subsection 2.4, we know that

$$P_{\delta_0}(Y^{(\kappa)} \text{ hits } bA^c) = 1 - \exp(-v(0))\ , \tag{36}$$

where the function $(v(x), x \in bA)$ is the unique nonnegative solution of the singular boundary value problem

$$\begin{cases} \frac{1}{2}\Delta v = \psi_{(\kappa)}(u) & \text{in } bA\ , \\ u_{|\partial(bA)} = +\infty\ . \end{cases}$$

It is immediate to verify that $u_{(\kappa a)}(x) = a\, v(bx)$ for every $x \in A$, and in particular $u_{(\kappa a)}(0) = a\, v(0)$.

From (35), (36) and Lemma 7, we obtain

$$\lim_{n\to\infty} (1 - \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c))^{[\varepsilon_n^{-1}]} = \exp(-v(0))$$

and thus

$$\lim_{n\to\infty} \varepsilon_n^{-1}\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) = v(0),$$

or equivalently, since $\varepsilon_n R_n^2 \longrightarrow a$,

$$\lim_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) = av(0) = u_{(\kappa a)}(0).$$

This completes the proof of (34), in the case $a > 0$. $\square$

Proof of Lemma 7. By replacing A with bA, we may and shall assume in this proof that $b = 1$. We thus have $b_n \longrightarrow 1$ as $n \to \infty$. We first prove that

$$\liminf_{n\to\infty} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) \geq P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c). \tag{37}$$

By Lemma 3, the events $\{Y^{(\kappa)} \text{ hits } A^c\}$ and $\{Y^{(\kappa)} \text{ hits } \bar{A}^c\}$ coincide a.s. We can then find a countable collection $(\varphi_i)_{i\geq 1}$ of continuous functions with compact support contained in $(\bar{A})^c$, such that

$$\{Y^{(\kappa)} \text{ hits } (\bar{A})^c\} = \Big\{ \sup_{i\geq 1} \Big(\sup_{t>0} \langle Y_t^{(\kappa)}, \varphi_i \rangle \Big) > 0 \Big\}, \qquad P_{\delta_0} \text{ a.s..}$$

Hence, if $(t_j)_{j\geq 1}$ is a sequence dense in $[0,\infty)$, we have

$$P_{\delta_0}(Y^{(\kappa)} \text{ hits } (\bar{A})^c) = \lim_{N\to\infty} \uparrow P_{\delta_0}\Big(\sup_{1\leq i\leq N} \Big(\sup_{1\leq j\leq N} \langle Y_{t_j}^{(\kappa)}, \varphi_i \rangle \Big) > 0 \Big). \tag{38}$$

However, Theorem 2 implies that, for every $N \geq 1$,

$$\liminf_{n\to\infty} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}\Big(\sup_{1\leq i\leq N} \Big(\sup_{1\leq j\leq N} \langle X_{t_j}^{\varepsilon_n}, \varphi_i \rangle \Big) > 0 \Big) \geq P_{\delta_0}\Big(\sup_{1\leq i\leq N} \Big(\sup_{1\leq j\leq N} \langle Y_{t_j}^{(\kappa)}, \varphi_i \rangle \Big) > 0 \Big). \tag{39}$$

Recall that $b_n \to 1$, and note that the support of each function φ_i is at a strictly positive distance of the set A. As a consequence, for every fixed N, the support of φ_i will be contained in $b_n \bar{A}^c$ for every $i = 1, \ldots, N$, as soon as n is large enough. Hence, for all large enough n,

$$\Big\{ \sup_{1\leq i\leq N} \Big(\sup_{1\leq j\leq N} \langle X_{t_j}^{\varepsilon_n}, \varphi_i \rangle \Big) > 0 \Big\} \subset \{X^{\varepsilon_n} \text{ hits } b_n A^c\}.$$

Using this inclusion and then (39) and (38), we immediately obtain (37).

We next turn to the more difficult upper bound

$$\limsup_{n\to\infty} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) \leq P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c). \tag{40}$$

We fix $\delta > 0$ small enough so that the closed ball of radius 4δ centered at 0 is contained in A. As in Lemma 2, we let A_δ be the connected component of the open set

$$\{x \in A : \mathrm{dist}(x, A^c) > \delta\}$$

that contains 0. We denote the exit measure from A_δ for the rescaled branching Brownian motion X^{ε_n} by $\mathcal{E}_\delta^n$. In other words, the measure $\mathcal{E}_\delta^n$ is equal to ε_n times the sum of the Dirac point masses at all points of ∂A_δ which are first exit points from A_δ for one of the historical paths associated with X^{ε_n} (these historical paths are defined in Subsection 2.3 for the branching Brownian motion Z^{ε_n}, and this definition is extended to X^{ε_n} by an obvious scaling transformation).

Let Φ be a continuous function on $\mathbb{R}^d$ such that $0 \leq \Phi \leq 1$, $\Phi = 0$ on $A_{3\delta}$ and $\Phi = 1$ on $A_{2\delta}^c$. Then, for every $\eta > 0$ and $\rho > 0$,

$$\begin{aligned} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) = {} & \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c,\ \langle \mathcal{E}_\delta^n, 1 \rangle < \eta) \\ & + \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}\Big(X^{\varepsilon_n} \text{ hits } b_n A^c,\ \langle \mathcal{E}_\delta^n, 1 \rangle \geq \eta,\ \int_0^\infty \langle X_s^{\varepsilon_n}, \Phi \rangle ds \leq \rho \Big) \\ & + \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}\Big(X^{\varepsilon_n} \text{ hits } b_n A^c,\ \langle \mathcal{E}_\delta^n, 1 \rangle \geq \eta,\ \int_0^\infty \langle X_s^{\varepsilon_n}, \Phi \rangle ds > \rho \Big). \end{aligned} \tag{41}$$

Let $\alpha_n(\eta)$, $\beta_n(\eta, \rho)$ and $\gamma_n(\eta, \rho)$ be the three terms appearing in the right-hand side of (41) in this order.

We first bound $\alpha_n(\eta)$. Provided n is sufficiently large, $b_n A^c$ is contained in $A^c_{\delta/2}$ and thus

$$\alpha_n(\eta) \leq \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } A^c_{\delta/2},\ \langle \mathcal{E}^n_\delta, 1\rangle < \eta).$$

Note that the times at which the historical paths of X^{ε_n} exit A_δ form a stopping line in the sense of [Ch91]. We can thus apply the strong Markov property at a stopping line (Proposition 2.1 in [Ch91]) to see that $\alpha_n(\delta)$ is bounded above by the probability for a branching Brownian motion (without killing) starting initially with less than $\eta\varepsilon_n^{-1}$ particles, that one of the historical paths reaches a distance greater than $\delta/(2\sqrt{\varepsilon_n})$ from its starting point (to be precise we need a slight extension of the results in [Ch91], since our spatial motion is not standard Brownian motion, but Brownian motion killed inside Γ_ϖ). The estimate (7) now gives

$$\alpha_n(\eta) \leq C_1''(d,\nu)\,\frac{4\eta}{\delta^2}. \tag{42}$$

Then, we have

$$\beta_n(\eta,\rho) \leq \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}\Big(\langle \mathcal{E}^n_\delta, 1\rangle \geq \eta,\ \int_0^\infty \langle X^{\varepsilon_n}_s, \Phi\rangle ds \leq \rho\Big).$$

Recall that $\Phi = 1$ on $A^c_{2\delta}$ and in particular $\Phi = 1$ on $\overline{B}(x,\delta)$ for every $x \in \partial A_\delta$. We use the strong Markov property at the same stopping line as in the previous argument, together with a simple coupling argument, to write that

$$\beta_n(\eta,\rho) \leq \mathbb{P}_{[\eta\varepsilon_n^{-1}]\delta_0}\Big(\int_0^\infty \langle \widetilde{X}^{\varepsilon_n}_s, \mathbb{I}_{\overline{B}(0,\delta)}\rangle ds \leq \rho\Big),$$

where $\widetilde{X}^{\varepsilon_n}$ is defined in terms of a branching Brownian motion $\widetilde{Z}^{\varepsilon_n}$ in the same way as X^{ε_n} was defined from Z^{ε_n}. This branching Brownian motion $\widetilde{Z}^{\varepsilon_n}$ has the same offspring distribution as Z^{ε_n}, but particles are now killed at rate ε_n homogeneously over $\mathbb{R}^d$. Furthermore, $\widetilde{Z}^{\varepsilon_n}$ also starts from $k\delta_0$ under the probability measure $\mathbb{P}_{k\delta_0}$.

By Proposition 2, the law of $(\widetilde{X}^{\varepsilon_n}_t)_{t\geq 0}$ under $\mathbb{P}_{[\eta\varepsilon_n^{-1}]\delta_0}$ converges as $n\to\infty$ to the law of $Y^{(1)}$ under $P_{\eta\delta_0}$ (in the notation of Subsection 2.4). Noting that, for every fixed $s>0$, $Y^{(1)}_s$ a.s. does not charge the boundary of the ball $\overline{B}(0,\delta)$, it follows that

$$\limsup_{n\to\infty} \mathbb{P}_{[\eta\varepsilon_n^{-1}]\delta_0}\Big(\int_0^\infty \langle \widetilde{X}^{\varepsilon_n}_s, \mathbb{I}_{\overline{B}(0,\delta)}\rangle ds \leq \rho\Big) \leq P_{\eta\delta_0}\Big(\int_0^\infty \langle Y^{(1)}_s, \mathbb{I}_{\overline{B}(0,\delta)}\rangle ds \leq \rho\Big) =: \beta_\infty(\eta,\rho).$$

The continuity of sample paths of $Y^{(1)}$ ensures that $\beta_\infty(\eta,\rho) \longrightarrow 0$ as $\rho\to 0$, for every fixed $\eta>0$.

For the term $\gamma_n(\eta,\rho)$, we simply use the bound

$$\gamma_n(\eta,\rho) \leq \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}\Big(\int_0^\infty \langle X^{\varepsilon_n}_s, \Phi\rangle ds > \rho\Big).$$

This bound and the weak convergence of Theorem 2 imply that

$$\limsup_{n\to\infty} \gamma_n(\eta,\rho) \leq P_{\delta_0}\Big(\int_0^\infty \langle Y^{(\kappa)}_s, \Phi\rangle ds \geq \rho\Big) \leq P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c_{3\delta}),$$

since $\Phi = 0$ on $A_{3\delta}$. (To justify the first inequality in the last display, we also use the fact that the extinction times of X^{ε_n} under $\mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}$ are stochastically bounded, which follows from a standard result in the case without killing.)

To complete the argument, fix $\vartheta > 0$. By Lemma 2 (ii), we can choose $\delta > 0$ sufficiently small so that

$$P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c_{3\delta}) \leq P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c) + \frac{\vartheta}{3}.$$

From (42), we can then choose $\eta > 0$ sufficiently small so that for all large n,

$$\alpha_n(\eta) \leq \frac{\vartheta}{3}.$$

Finally we choose $\rho > 0$ such that $\beta_\infty(\eta, \rho) \leq \frac{\vartheta}{3}$. From (41) and the previous estimates, we obtain

$$\limsup_{n\to\infty} \mathbb{P}_{[\varepsilon_n^{-1}]\delta_0}(X^{\varepsilon_n} \text{ hits } b_n A^c) \leq P_{\delta_0}(Y^{(\kappa)} \text{ hits } A^c) + \vartheta,$$

and since ϑ was arbitrary this completes the proof of (40) and Lemma 7. □

We still have to discuss the case $a = 0$ in (34). So, let us consider two sequences $(\varepsilon_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ such that $\varepsilon_n R_n^2 \to 0$. Let $a_0 > 0$ and $\varepsilon'_n = \varepsilon_n \vee (a_0 R_n^{-2})$. Since $\varepsilon_n \leq \varepsilon'_n$, we have

$$\liminf_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) \geq \liminf_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon'_n} \text{ hits } R_n A^c) = u_{(\kappa a_0)}(0),$$

by the case $a > 0$. By Lemma 2 (i), $u_{(\kappa a_0)}(0)$ can be made arbitrarily close to $u_{(0)}(0)$ when a_0 is small, and so

$$\liminf_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) \geq u_{(0)}(0).$$

To obtain the corresponding upper bound, a similar coupling argument shows that it suffices to consider the case when $\varepsilon_n = 0$ for every n, that is when there is no killing inside the obstacles. Hence, consider the branching Brownian motion $Z^0 = Z^{\varpi,0}$ (the notation is even more legitimate since $Z^{\varpi,0}$ does not depend on ϖ). For every $\rho > 0$, define a rescaled version of Z^0 by setting

$$\langle \overline{X}_t^{(\rho)}, \varphi \rangle = \rho \int Z^0_{\rho^{-1}t}(dx)\, \varphi(\rho^{1/2}x).$$

By Proposition 2, the law of $(\overline{X}_t^{(\rho)})_{t\geq 0}$ under $\mathbb{P}_{[\rho^{-1}]\delta_0}$ converges to the law of $Y^{(0)}$ under P_{δ_0} as ρ tends to 0. Set $\rho_n = R_n^{-2}$, in such a way that

$$\{Z^0 \text{ hits } R_n A^c\} = \{\overline{X}^{(\rho_n)} \text{ hits } A^c\}. \tag{43}$$

A simplified version of the arguments of the proof of Lemma 7 shows that

$$\limsup_{n\to\infty} \mathbb{P}_{[\rho_n^{-1}]\delta_0}(\overline{X}^{(\rho_n)} \text{ hits } A^c) \leq P_{\delta_0}(Y^{(0)} \text{ hits } A^c) = 1 - \exp(-u_{(0)}(0)).$$

Arguing as in the first part of the proof of the theorem and using (43) yields

$$\limsup_{n\to\infty} R_n^2\, \mathbb{P}_{\delta_0}(Z^{\varepsilon_n} \text{ hits } R_n A^c) \leq u_{(0)}(0),$$

which completes the proof of Theorem 1. □

Proof of Corollary 1. For every $r > 0$,

$$\mathbb{P}_{n_\varepsilon\delta_0}(\sqrt{\varepsilon}\, R^{\varpi,\varepsilon} < r) = \Big(1 - \mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,\varepsilon^{-1/2}r)^c)\Big)^{n_\varepsilon}.$$

However, Theorem 1 shows that, $\mathbf{P}(d\varpi)$ a.s.,

$$n_\varepsilon\, \mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,\varepsilon^{-1/2}r)^c) = \frac{\varepsilon n_\varepsilon}{r^2} \times \Big(\frac{r^2}{\varepsilon}\, \mathbb{P}_{\delta_0}(Z^{\varpi,\varepsilon} \text{ hits } B(0,\varepsilon^{-1/2}r)^c)\Big)$$

converges to $\frac{b}{r^2}\, u^\circ_{\kappa r^2}(0)$ as $\varepsilon \to 0$. The desired result follows. □

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