Stochastic flows associated to coalescent processes III: Limit theorems

Dedicated to the memory of Joseph Leo DOOB

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Summary. We prove several limit theorems that relate coalescent processes to continuous-state branching processes. Some of these theorems are stated in terms of the so-called generalized Fleming-Viot processes, which describe the evolution of a population with fixed size, and are duals to the coalescents with multiple collisions studied by Pitman and others. We first discuss asymptotics when the initial size of the population tends to infinity. In that setting, under appropriate hypotheses, we show that a rescaled version of the generalized Fleming-Viot process converges weakly to a continuous-state branching process. As a corollary, we get a hydrodynamic limit for certain sequences of coalescents with multiple collisions: Under an appropriate scaling, the empirical measure associated with sizes of the blocks converges to a (deterministic) limit which solves a generalized form of Smoluchowski's coagulation equation. We also study the behavior in small time of a fixed coalescent with multiple collisions, under a regular variation assumption on the tail of the measure ν governing the coalescence events. Precisely, we prove that the number of blocks with size less than εx at time $(\varepsilon\nu([\varepsilon,1]))^{-1}$ behaves like $\varepsilon^{-1}\lambda_1(]0,x[)$ as $\varepsilon \to 0$, where λ_1 is the distribution of the size of one cluster at time 1 in a continuous-state branching process with stable branching mechanism. This generalizes a classical result for the Kingman coalescent.

Key words. Flow, coalescence, Fleming-Viot process, continuous state branching process, Smoluchowski's coagulation equation.

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1 Introduction

J. L. Doob was a pioneer in the development of the theory of martingales and its applications to probability theory, potential theory or functional analysis. The fundamental contributions that he made in this field form the cornerstones of one of the richest veins explored in mathematics during the last half-century. In particular, martingales and stochastic calculus provide nowadays key tools for studying the asymptotic behavior of random processes; see the classical books by Ethier and Kurtz [10] and Jacod and Shiryaev [12]. In the present work, we shall apply such techniques to investigate a class of stochastic flows related to certain population dynamics.

The general motivation for the present work is to get a better understanding of the relations between the so-called coalescents with multiple collisions, which were introduced independently by Pitman [16] and Sagitov [17], and continuous-state branching processes. Note from [17] that coalescents with multiple collisions can be viewed as asymptotic models for the genealogy of a discrete population with a fixed size, and so the existence of connections with branching processes should not come as a surprise. Such connections were already derived in [3], where the Bolthausen-Sznitman coalescent was shown to describe the genealogical structure of a particular continuous-state branching process introduced by Neveu, and in [6], where similar relations were obtained between the so-called beta-coalescents and continuous-state branching processes with stable branching mechanism. Here, we do not focus on exact distributional identities, but rather on asymptotics for functionals of coalescent processes, where the limiting objects are given in terms of branching processes. In order to get such asymptotics, we apply the machinery of limit theorems for semimartingales [12] to the so-called generalized Fleming-Viot processes, which where shown in [4] to be duals to the coalescents with multiple collisions.

Generalized Fleming-Viot processes, which model the evolution of a continuous population with fixed size 1, have appeared in articles by Donnelly and Kurtz [7, 8], and were studied more recently in our work [4, 5]. It is convenient to view a generalized Fleming-Viot process as a stochastic flow $(F_t, t \geq 0)$ on $[0,1]$, such that for each $t \geq 0$, $F_t : [0,1] \to [0,1]$ is a (random) right-continuous increasing map with $F_t(0) = 0$ and $F_t(1) = 1$. We should think of the unit interval as a population, and then of F_t as the distribution function of a (random) probability measure $dF_t(x)$ on $[0,1]$. The evolution of the latter is related to the dynamics of the population as follows : For every $0 \leq r_1 < r_2 \leq 1$, the interval $]F_t(r_1), F_t(r_2)]$ represents the sub-population at time t which consists of descendants of the sub-population $]r_1, r_2]$ at the initial time. The transitions of the flow are Markovian, and more precisely, for every $s, t \geq 0$, we have $F_{t+s} = \tilde{F}_s \circ F_t$, where $\tilde{F}_s$ is a copy of F_s independent of $(F_r, 0 \leq r \leq t)$. The distribution of the flow is then characterized by a measure ν on $]0,1]$ such that $\int_{]0,1]} x^2 \nu(dx) < \infty$. To explain this, consider the simple case where ν is a finite measure. Let $((T_i, U_i, \xi_i), i \in \mathbb{N})$ denote the sequence of atoms of a Poisson random measure on $[0,\infty[\times[0,1] \times [0,1]$ with intensity $dt \otimes du \otimes \nu(dx)$, ranked in the increasing order of the first coordinate. The process $(F_t, t \geq 0)$ starts from $F_0 = \mathrm{Id}$, remains constant on the intervals $[T_{i-1}, T_i[$ (with the usual convention that $T_0 = 0$), and for every $i \in \mathbb{N}$

$$F_{T_i} \;=\; \Delta_i \circ F_{T_{i-1}}$$

where

$$\Delta_i(r) = \xi_i \mathbf{1}_{\{U_i \leq r\}} + r(1 - \xi_i)\,, \qquad r \in [0,1].$$

In terms of the population model, this means that at each time T_i, an individual in the population at time T_{i-1} is picked uniformly at random and gives birth to a sub-population of size ξ_i. Simultaneously, the rest of the population shrinks by factor $1-\xi_i$, so the total size of the population remains 1. The previous description does not apply when ν is infinite, since then the Poisson measure will have infinitely many atoms on a finite time interval. Still, the Fleming-Viot flow can be constructed via a suitable limiting procedure ([4] Theorem 2).

Our first motivation for studying generalized Fleming-Viot processes came from their remarkable connection [4] with the class of coalescents with multiple collisions considered by Pitman [16] and Sagitov [17]. To describe this connection, fix some time $T>0$ viewed as the present date at which the population is observed, and pick a sequence of individuals labelled $1,2,\dots$ independently and uniformly over $[0,1]$. For every $t\le T$, we obtain a partition $\Pi(t)$ of $\mathbb{N}$ by gathering individuals having the same ancestor at time $T-t$. The process $(\Pi(t), 0\le t\le T)$ is then a Markovian coalescent process on the space of partitions of $\mathbb{N}$. In the terminology of [16], it is a Λ-coalescent, with $\Lambda(dx)=x^2\nu(dx)$, started from the partition of $\mathbb{N}$ into singletons. As a consequence of Kingman's theory of exchangeable partitions, for every $t\ge 0$, each block of $\Pi(t)$ has an asymptotic frequency, also called the size of the block, and the ranked sequence of these frequencies yields a Markov process called the mass-coalescent. As a consequence of the preceding construction, the mass-coalescent at time t has the same distribution as the ranked sequence of jump sizes of F_t.

The first purpose of the present work is to investigate the asymptotic behavior of a rescaled version of the preceding population model. Specifically, we consider a family $(\tilde\nu^{(a)}, a>0)$ of measures on $]0,1]$ such that $\int_{]0,1]} x^2\tilde\nu^{(a)}(dx)<\infty$ for every $a>0$, and the associated generalized Fleming-Viot processes $\tilde F^{(a)}$. For each $a>0$, we rescale $\tilde F^{(a)}$ by a factor a in space and time, i.e. we set

$$F_t^{(a)}(r) := a\tilde F_{at}^{(a)}(r/a)\,, \qquad r\in[0,a], t\ge 0\,.$$

So the process $F^{(a)}$ describes the evolution of a population with fixed size a. Roughly speaking, considering $F^{(a)}$ in place of $\tilde F^{(a)}$ enables us to focus on the dynamics of a sub-population having size of order $1/a$. Denote by $\nu^{(a)}$ the image of $\tilde\nu^{(a)}$ under the dilation $x\to ax$, and assume that the measures $(x^2\wedge x)\nu^{(a)}(dx)$ converge weakly as $a\to\infty$ to a finite measure on $]0,\infty[$, which we may write in the form $(x^2\wedge x)\pi(dx)$. Then Theorem 1 shows that $F^{(a)}$ converges in distribution to the critical continuous-state branching process Z with branching mechanism

$$\Psi(q) = \int_{]0,\infty[} (\mathrm{e}^{-qx}-1+qx)\pi(dx)\,, \qquad q\ge 0\,.$$

As a consequence of this limit theorem, we derive a hydrodynamic limit for the associated coalescent processes (Theorem 2). Precisely, we show that under the same assumptions as above, for every $t\ge 0$, the empirical measure corresponding to the jumps of $\tilde F_t^{(a)}$ (or equivalently to the block sizes in the associated coalescent) converges, modulo a suitable rescaling, towards a deterministic measure λ_t. Informally, λ_t is the distribution of a cluster at time t, that is a collection of individuals sharing the same ancestor at the initial time, in the continuous-state branching process with branching mechanism Ψ. In a way analogous to the derivation of Smoluchovski's coagulation equation from stochastic models (see Aldous [1], Norris [15] and the references therein for background) we prove that the family $(\lambda_t, t>0)$ solves a generalized coagulation equation of the form

$$\frac{d\langle \lambda_t, f\rangle}{dt} = \sum_{k=2}^{\infty} \frac{(-1)^k \Psi^{(k)}(\langle \lambda_t, 1\rangle)}{k!} \int_{]0,\infty[^k} (f(x_1+\cdots+x_k) - (f(x_1)+\cdots+f(x_k)))\, \lambda_t(dx_1)\ldots\lambda_t(dx_k)$$

where f can be any continuous function with compact support on $]0,\infty[$ (Proposition 3).

In the last part of this work, we study the small time behavior of generalized Fleming-Viot processes and Λ-coalescents, under a regular variation assumption on the measure ν (recall that $\Lambda(dx) = x^2\nu(dx)$). Precisely, we assume that the tail $\nu([\varepsilon, 1])$ is regularly varying with index $-\gamma$ when ε goes to 0. We are interested in the case when the Λ-coalescent comes down from infinity (i.e. for every $t > 0$, Π_t has finitely many blocks), which forces $1 \leq \gamma \leq 2$. Leaving aside the boundary cases we suppose that $1 < \gamma < 2$. As a consequence of Theorem 1, we prove that the rescaled Fleming-Viot process

$$F_t^{\varepsilon}(x) := \frac{1}{\varepsilon} F_{t/(\varepsilon\nu([\varepsilon,1]))}(\varepsilon x)$$

converges in distribution to the continuous-state branching process with stable branching mechanism:

$$\Psi_\gamma(q) = \frac{\Gamma(2-\gamma)}{\gamma-1}\, q^\gamma.$$

We then use this result to investigate the small time behavior of the size of blocks in the Λ-coalescent. Write $N_t(]0,x[)$ for the number of blocks with size less than x in the Λ-coalescent at time t. If $g(\varepsilon) = (\varepsilon\nu([\varepsilon,1]))^{-1}$, Theorem 4 states that

$$\sup_{x\in]0,\infty[} \left| \varepsilon N_{g(\varepsilon)}(]0,\varepsilon x[) - \lambda_1(]0,x[) \right| \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

in probability. Furthermore, the measure λ_1 can be characterized by its Laplace transform

$$\int (1 - \mathrm{e}^{-qr})\lambda_1(dr) = (\Gamma(2-\gamma) + q^{1-\gamma})^{1/(1-\gamma)}.$$

Theorem 4 is analogous to a classical result for the sizes of blocks in the Kingman coalescent in small time (see Aldous [1]). The proof uses an intermediate estimate for the total number of blocks in a Λ-coalescent, which is closely related to the recent paper [2] dealing with beta-coalescents.

The paper is organized as follows. Section 2 gives a few preliminary results about continuous-state branching processes. In particular, the Poisson representation (Proposition 2) may have other applications. Section 3 states our first limit theorem for generalized Fleming-Viot processes. The derivation of the hydrodynamic limit is developed in Section 4, which also discusses the generalized coagulation equation for the family $(\lambda_t, t \geq 0)$. Finally Section 5 is devoted to the behavior in small time of generalized Fleming-Viot processes and Λ-coalescents.

Notation. We use the notation $\langle \mu, f\rangle$ for the integral of the function f with respect to the measure μ. We denote by $\mathcal{M}_{\mathrm{F}}$ the space of all finite measures on $]0,\infty[$, which is equipped with the usual weak topology. We also denote by $\mathcal{M}_{\mathrm{R}}$ the space of all Radon measures on $]0,\infty[$. The set $\mathcal{M}_{\mathrm{R}}$ is equipped with the vague topology: A sequence $(\mu_n, n \in \mathbb{N})$ in $\mathcal{M}_{\mathrm{R}}$ converges to $\mu \in \mathcal{M}_{\mathrm{R}}$ if and only if for every continuous function $f :]0,\infty[\to \mathbb{R}$ with compact support, $\lim_{n\to\infty}\langle \mu_n, f\rangle = \langle \mu, f\rangle$.

2 Stochastic flows of branching processes

In this section, we give a few properties of continuous-state branching processes that will be needed in the proof of our limit theorems. A *critical branching mechanism* is a function $\Psi : [0,\infty[\to [0,\infty[$ of the type

$$\Psi(q) \;=\; \beta q^2 + \int_{]0,\infty[} \left(\mathrm{e}^{-rq} - 1 + rq\right) \pi(dr) \tag{1}$$

where $\beta \geq 0$ is the so-called Gaussian coefficient and π is a measure on $]0,\infty[$ such that $\int (r \wedge r^2)\pi(dr) < \infty$. The continuous-state branching process with branching mechanism Ψ (in short the Ψ-CSBP) is the Markov process with values in $\mathbb{R}_+$, whose transition kernels $Q_t(x,dy)$ are determined by the Laplace transform

$$\int Q_t(x,dy)\, \mathrm{e}^{-qy} = \exp(-x\, u_t(q)) \;,\;\; x,t \geq 0,\;\; q \geq 0\,, \tag{2}$$

where the function $u_t(q)$ solves

$$\frac{\partial u_t(q)}{\partial t} \;=\; -\Psi(u_t(q)) \quad , \quad u_0(q) \;=\; q\,. \tag{3}$$

The criticality of Ψ implies that a Ψ-CSBP is a nonnegative martingale. If Z^1 and Z^2 are two independent Ψ-CSBP's started respectively at x_1 and x_2, then $Z^1 + Z^2$ is also a Ψ-CSBP, obviously with initial value $x_1 + x_2$. From this additivity or branching property, we may construct a two-parameter process $Z = (Z(t,x), t, x \geq 0)$, such that:

- For each fixed $x \geq 0$, $(Z(t,x), t \geq 0)$ is a Ψ-CSBP with càdlàg paths and initial value $Z(0,x) = x$.
- If $x_1, x_2 \geq 0$, $Z(\cdot, x_1 + x_2) - Z(\cdot, x_1)$ is independent of the processes $(Z(\cdot, x), 0 \leq x \leq x_1)$ and has the same law as $Z(\cdot, x_2)$.

These properties entail that for each fixed $t \geq 0$, $Z(t,\cdot)$ is an increasing process with independent and stationary increments. Its right-continuous version is a subordinator with Laplace exponent u_t determined by (2) and (3). By the Lévy-Khintchin formula, there exists a unique drift coefficient $d_t \geq 0$ and a unique measure λ_t on $]0,\infty[$ with $\int_{]0,\infty[}(1 \wedge x)\lambda_t(dx) < \infty$ such that

$$u_t(q) \;=\; q d_t + \int_{]0,\infty[} (1 - \mathrm{e}^{-qx})\lambda_t(dx)\,, \qquad q \geq 0\,. \tag{4}$$

One refers to λ_t as the Lévy measure of $Z(t,\cdot)$. Measures λ_t play an important role in this work. Informally, we may say that λ_t is the 'distribution' of the size of the set of descendants at time t of a single individual at time 0. This assertion is informal since λ_t is not a probability distribution (it may even be an infinite measure). A correct way of stating the above (in the case $d_t = 0$) is as follows: $Z(t,x)$ is the sum of the atoms of a Poisson measure with intensity $x\lambda_t(\cdot)$. Moreover, the study of the genealogical structure of the Ψ-CSBP (see e.g. [9]) allows one to interpret each of these atoms as the size of a family of individuals at time t that have the same ancestor at the initial time.

From now on, we assume that $\beta = 0$ and we exclude the trivial case $\pi = 0$.

We start by recalling in our special case an important connection between continuous-state branching processes and Lévy processes due to Lamperti [14]. Let $x > 0$ be fixed, and let $\xi = (\xi_t, t \geq 0)$ denote a real-valued Lévy process with no negative jumps, started from $\xi_0 = x$, and whose Laplace exponent is specified by

$$\mathbb{E}\left[\exp(-q(\xi_t - \xi_0))\right] \,=\, \exp t\Psi(q)\,, \qquad q \geq 0\,.$$

In particular π is the Lévy measure of ξ. The criticality of the branching mechanism Ψ ensures that the Lévy process ξ has centered increments and thus oscillates. In particular the first passage time $\zeta := \inf\{t \geq 0 : \xi_t = 0\}$ is finite a.s. Next, introduce for every $t \geq 0$

$$\gamma(t) \,=\, \int_0^{t\wedge\zeta} \frac{ds}{\xi_s} \quad , \quad C_t \,=\, \inf\{s \geq 0 : \gamma(s) > t\} \wedge \zeta\,.$$

Then the time-changed process $(\xi \circ C_t, t \geq 0)$ has the same distribution as $(Z(t,x), t \geq 0)$.

It follows from this representation that $(Z(t,x), t \geq 0)$ is a purely discontinuous martingale. We can also use the Lamperti transformation to calculate the compensator of the jump measure of this martingale. By the Lévy-Itô decomposition, the compensator of the jump measure of ξ,

$$\sum_{\{t:\Delta\xi_t\neq 0\}} \delta_{(t,\Delta\xi_t)},$$

is $dt \otimes \pi(dx)$. By a time change argument, we can then deduce that the compensator of the measure

$$\sum_{\{t:\Delta Z(t,x)\neq 0\}} \delta_{(t,\Delta Z(t,x))}$$

is $Z(t,x)dt \otimes \pi(dr)$.

Since $(Z(t,x), t \geq 0)$ is a purely discontinuous martingale, the knowledge of the compensator of its jump measure completely determines the characteristics of this semimartingale, in the sense of [12] Chapter II. We will need the fact that the distribution of $(Z(t,x), t \geq 0)$, and more generally of the multidimensional process $((Z(t,x_1), Z(t,x_2), \ldots, Z(t,x_p)); t \geq 0)$ for any choice of p and $x_1, \ldots, x_p$, is uniquely determined by its characteristics.

Fix an integer $p \geq 1$ and define

$$\mathcal{D}_p := \{x = (x_1, \ldots, x_p) : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_p\}. \tag{5}$$

For every $(y_1, \ldots, y_p) \in \mathcal{D}_p$, define a σ-finite measure $U(y_1, \ldots, y_p; dz_1, \ldots, dz_p)$ on $\mathbb{R}_+^p \backslash \{0\}$ by setting, for any measurable function $\varphi : \mathbb{R}_+^p \to \mathbb{R}_+$ that vanishes at 0,

$$\int U(y_1, \ldots, y_p; dz_1, \ldots, dz_p)\, \varphi(z_1, \ldots, z_p) = \int \pi(dr) \int_0^\infty du\, \varphi(r\mathbf{1}_{\{u\leq y_1\}}, \ldots, r\mathbf{1}_{\{u\leq y_p\}}). \tag{6}$$

Proposition 1 *Let $(x_1, \ldots, x_p) \in \mathcal{D}_p$ and let $(Z^1, \ldots, Z^p)$ be a p-dimensional semimartingale taking values in $\mathcal{D}_p$, such that $(Z_0^1, \ldots, Z_0^p) = (x_1, \ldots, x_p)$. The following two properties are equivalent:*

(i) *The processes* $((Z^1_t,\ldots,Z^p_t);t\geq 0)$ *and* $((Z(t,x_1),\ldots,Z(t,x_p));t\geq 0)$ *have the same distribution.*

(ii) *The process* $((Z^1_t,\ldots,Z^p_t);t\geq 0)$ *is a purely discontinuous local martingale, and the compensator of its jump measure is the measure*

$$\theta(dt,dz_1\ldots dz_p)=dt\,U(Z^1_t,\ldots,Z^p_t;dz_1,\ldots,dz_p).$$

Proof: The implication (i)⇒(ii) is a straightforward consequence of the remarks preceding the statement and the branching property of continuous-state branching processes. We concentrate on the proof of the converse implication (ii)⇒(i). Let $q=(q_1,\ldots,q_p)\in]0,\infty[^p$, and let $Y_t=(Y^1_t,\ldots,Y^p_t)$ be defined by $Y^i_t=Z^i_t-Z^{i-1}_t$ if $i\geq 2$ and $Y^1_t=Z^1_t$. Notice that $Y^i_t\geq 0$. Using property (ii), an application of Itô's formula (cf Theorem II.2.42 in [12]) yields that the process

$$\begin{aligned}&\exp(-q\cdot Y_t)-\exp(-q\cdot Y_0)\\&-\sum_{i=1}^p\int_{[0,t]\times[0,\infty[\times]0,\infty[}\exp(-q\cdot Y_s)\left(\mathrm{e}^{-q_ir}-1+q_ir\right)\mathbf{1}_{\{u\leq Y^i_s\}}ds\,du\,\pi(dr)\end{aligned}$$

is a local martingale. This local martingale is bounded over the time interval $[0,t]$ for any $t\geq 0$, hence is a martingale. Taking expectations leads to

$$\mathbb{E}[\mathrm{e}^{-q\cdot Y_t}]=\mathbb{E}[\mathrm{e}^{-q\cdot Y_0}]+\sum_{i=1}^p\Psi(q_i)\int_0^t ds\,\mathbb{E}[Y^i_s\,\mathrm{e}^{-q\cdot Y_s}].\tag{7}$$

It is immediate to verify from (ii) that each Y^i is also a nonnegative local martingale, and so $\mathbb{E}[Y^i_s]\leq\mathbb{E}[Y^i_0]=x_i-x_{i-1}$ (by convention $x_0=0$). If we set $f_t(q)=\mathbb{E}[\mathrm{e}^{-q\cdot Y_t}]$ we have

$$\frac{\partial f_t(q)}{\partial q_i}=-\mathbb{E}[Y^i_t\,\mathrm{e}^{-q\cdot Y_t}]$$

and so we deduce from (7) that

$$\frac{\partial f_t(q)}{\partial t}+\Psi(q)\cdot\nabla f_t(q)\;=\;0\,,\tag{8}$$

where we write $\Psi(q)=(\Psi(q_1),\ldots,\Psi(q_p))$. In order to solve (8), fix $t_1>0$, and consider the function $g(t)=(u_{t_1-t}(q_1),\ldots,u_{t_1-t}(q_p))$ for $t\in[0,t_1]$, where $u_t(q)$ is as in (3). Since

$$g'(t)=(\Psi(u_{t_1-t}(q_1)),\ldots,\Psi(u_{t_1-t}(q_p))),$$

it follows that

$$\frac{\partial f_t\circ g}{\partial t}=\frac{\partial f_t}{\partial t}\circ g+g'(t)\cdot\nabla f_t(g(t))=0$$

by (8). Hence $f_t\circ g(t)$ is constant over $[0,t_1]$, and

$$f_{t_1}(q)=f_{t_1}(g(t_1))=f_0(g(0))=\exp(-\sum_{i=1}^p(x_i-x_{i-1})u_{t_1}(q_i)).$$

This shows that

$$(Y^1_{t_1},\dots,Y^p_{t_1}) \overset{\rm (d)}{=} (Z(t_1,x_1),Z(t_1,x_2)-Z(t_1,x_1),\dots,Z(t_1,x_p)-Z(t_1,x_{p-1}))$$

and so

$$(Z^1_{t_1},\dots,Z^p_{t_1}) \overset{\rm (d)}{=} (Z(t_1,x_1),Z(t_1,x_2),\dots,Z(t_1,x_p)).$$

It is easy to iterate this argument to obtain that the processes $((Z^1_t,\dots,Z^p_t);t\geq 0)$ and $((Z(t,x_1),\dots,Z(t,x_p));t\geq 0)$ have the same finite-dimensional marginal distributions. The desired result follows since both processes have càdlàg paths. □

We now turn our attention to the representation of critical CSBP as stochastic flows on $[0,\infty[$ solving simple stochastic differential equations. On a suitable filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, we consider :

• an $(\mathcal{F}_t)$-Poisson random measure

$$M \;=\; \sum_{i=1}^{\infty}\delta_{(t_i,u_i,r_i)}\,,$$

on $\mathbb{R}_+\times[0,\infty[\times]0,\infty[$, with intensity $dt\otimes du\otimes\pi(dr)$.

• a collection $(X_t(x),t\geq 0)$, $x\in\mathbb{R}_+$ of càdlàg $(\mathcal{F}_t)$-martingales with values in $\mathbb{R}_+$,

• the stochastic differential equation

$$X_t(x) \;=\; x+\int_{[0,t]\times[0,\infty[\times]0,\infty[} M(ds,du,dr)\,r\,\mathbf{1}_{\{u\leq X_{s-}(x)\}}\,. \tag{9}$$

The Poissonian stochastic integral in the right-hand side should be understood with respect to the compensated Poisson measure M (see e.g. Section II.1 of [12]). This stochastic integral is well defined according to Definition II.1.37 of [12], since the increasing process

$$t\longrightarrow\Big(\int_{[0,t]\times[0,\infty[\times[0,\infty[} M(ds,du,dr)\,r^2\,\mathbf{1}_{\{u\leq X_{s-}(x)\}}\Big)^{1/2}$$

is locally integrable under our assumption on π.

A pair $(M,(X_\cdot(a),a\geq 0))$ satisfying the above conditions will be called a *weak solution* of (9).

Proposition 2 *The equation* (9) *has a weak solution which satisfies the additional property that $X_t(x_1)\leq X_t(x_2)$ for every $t\geq 0$, a.s. whenever $0\leq x_1\leq x_2$. Moreover, for every such solution (M,X), for every $p\in\mathbb{N}$ and $0\leq x_1\leq\ldots\leq x_p$, the process $((X_t(x_1),\dots,X_t(x_p)),t\geq 0)$ has the same distribution as $((Z(t,x_1),\dots,Z(t,x_p)),t\geq 0)$.*

Proof: The second part of the statement is immediate from the implication (ii)⇒(i) in Proposition 1. The first part can be deduced from Theorem 14.80 in [11] by the same arguments that were used in the proof of Theorem 2 in [5]. We leave details to the reader as this result is not really needed below except for motivation. □

3 Generalized Fleming-Viot flows and their limits

We now recall some results from [4, 5] on generalized Fleming-Viot processes and related stochastic flows. Let ν denote a σ-finite measure on $]0,1]$ such that $\int_{]0,1]} x^2\nu(dx) < \infty$. According to Section 5.1 in [4], one can associate with ν a Feller process $(F_t, t \geq 0)$ with values in the space of distribution functions of probability measures on $]0,1]$ (i.e. for each $t \geq 0$, F_t is a càdlag increasing map from $[0,1]$ to $[0,1]$ with $F_t(0) = 0$ and $F_t(1) = 1$), whose evolution is characterized by ν and has been described in Section 1 in the special case when ν is finite.

In [5], we have shown that such generalized Fleming-Viot processes can be described as the solution to a certain system of Poissonian SDE's. More precisely, on a suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, one can construct the following processes:

- an $(\mathcal{F}_t)$-Poisson point process N on $\mathbb{R}_+\times]0,1[\times]0,1]$ with intensity $dt \otimes du \otimes \nu(dr)$,
- a collection $(Y_t(x), t \geq 0)$, $x \in [0,1]$, of adapted càdlàg processes with values in $[0,1]$ with $Y_t(x_1) \leq Y_t(x_2)$ for all $t \geq 0$ a.s. when $0 \leq x_1 \leq x_2 \leq 1$,

in such a way that for every $r \in [0,1]$, a.s.

$$Y_t(x) \;=\; x + \int_{[0,t]\times]0,1[\times]0,1]} N(ds,du,dr)\, r\, \Big(\mathbf{1}_{\{u\leq Y_{s-}(x)\}} - Y_{s-}(x)\Big)\;. \tag{10}$$

The Poissonian stochastic integral in the right-hand side should again be understood with respect to the compensated Poisson measure N.

Weak uniqueness holds for this system of SDE's (Theorem 2 in [5]). Furthermore, for every integer $p \geq 1$ and every $0 \leq x_1 \leq \ldots \leq x_p \leq 1$, the processes $((Y_t(x_1),\ldots,Y_t(x_p)), t \geq 0)$ and $((F_t(x_1),\ldots,F_t(x_p)), t \geq 0)$ have the same distribution. Note the similarity with Proposition 2: Compare (9) and (10). This strongly suggests to look for asymptotic results relating the processes $Z(t,x)$ and $F_t(x)$.

For every integer $p \geq 1$ and every $a > 0$, set

$$\mathcal{D}_p^a = \mathcal{D}_p \cap [0,a]^p = \{(x_1,\ldots,x_p) \in \mathbb{R}_+^p : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_p \leq a\}.$$

From (10) we see that for every $(x_1,\ldots,x_p) \in \mathcal{D}_p^1$, the process $(F_t(x_1),\ldots,F_t(x_p))$ is a purely discontinuous martingale, and the compensator of its jump measure is

$$dt\; R(F_t(x_1),\ldots,F_t(x_p); dz_1,\ldots,dz_p),$$

where for every $(y_1,\ldots,y_p) \in \mathcal{D}_p^1$, the measure $R(y_1,\ldots,y_p;dz_1,\ldots,dz_p)$ on $\mathbb{R}^p\backslash\{0\}$ is determined by

$$\int R(y_1,\ldots,y_p;dz_1,\ldots,dz_p)\,\varphi(z_1,\ldots,z_p) = \int \nu(dr)\int_0^1 du\,\varphi(r(\mathbf{1}_{\{u\leq y_1\}}-y_1),\ldots,r(\mathbf{1}_{\{u\leq y_p\}}-y_p)).$$

Consider now a family $(\tilde\nu^{(a)}, a > 0)$ of measures on $]0,1]$ with $\int_{]0,1]} r^2\tilde\nu^{(a)}(dr) < \infty$, and for each $a > 0$, let $\tilde F^{(a)}$ be the associated Fleming-Viot process. We then write

$$F_t^{(a)}(x) := a\tilde F_{at}^{(a)}(x/a)\,, \qquad x \in [0,a], t \geq 0$$

for the rescaled version of the Fleming-Viot flow. So, for each $t \geq 0$, $F_t^{(a)}$ is the distribution function of a measure on $]0, a]$ with total mass a. For every fixed real number $a > 0$, we also denote by $\nu^{(a)}$ the measure on $]0, \infty[$ which is 0 on $]a, \infty[$ and whose restriction to $]0, a]$ is given by the image of $\tilde{\nu}^{(a)}$ under the dilation $r \to ar$ from $]0, 1]$ to $]0, a]$. In particular $r^2 \nu^{(a)}(dr)$ is a finite measure on $]0, \infty[$.

By a scaling argument, we see that, for every $(x_1, \ldots, x_p) \in \mathcal{D}_p^a$, $(F_t^{(a)}(x_1), \ldots, F_t^{(a)}(x_p))$ is a purely discontinuous martingale, with values in $\mathcal{D}_p^a$, and the compensator of its jump measure is

$$\mu_{(a)}(dt, dz_1 \ldots dz_p) = dt\, R^{(a)}(F_t^{(a)}(x_1), \ldots, F_t^{(a)}(x_p); dz_1, \ldots, dz_p) \tag{11}$$

where

$$\begin{aligned} &\int R^{(a)}(y_1, \ldots, y_p; dz_1, \ldots, dz_p)\, \varphi(z_1, \ldots, z_p) \\ &\quad = \int \nu^{(a)}(dr) \int_0^a du\, \varphi(r(\mathbf{1}_{\{u \leq y_1\}} - a^{-1} y_1), \ldots, r(\mathbf{1}_{\{u \leq y_p\}} - a^{-1} y_p)). \end{aligned} \tag{12}$$

Let π be as in Section 2 a nontrivial measure on $]0, \infty[$ such that $\int (r \wedge r^2) \pi(dr) < \infty$, and let Ψ be as in (1). Denote by $(Z(t, x), t \geq 0, x \geq 0)$ the associated flow of continuous-state branching processes constructed in Section 2.

Assumption (H) *The measures $(r \wedge r^2)\nu^{(a)}(dr)$ converge to $(r \wedge r^2)\pi(dr)$ as $a \to \infty$, in the sense of weak convergence in $\mathcal{M}_{\mathrm{F}}$.*

Theorem 1 *Under Assumption* (H), *for every $(x_1, \ldots, x_p) \in \mathcal{D}_p$,*

$$((F_t^{(a)}(x_1), \ldots, F_t^{(a)}(x_p)); t \geq 0) \xrightarrow[a \to \infty]{\text{(d)}} ((Z(t, x_1), \ldots, Z(t, x_p)); t \geq 0)$$

in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^p)$.

Proof: The proof only uses the facts that $(F_t^{(a)}(x_1), \ldots, F_t^{(a)}(x_p))$ is a purely discontinuous martingale and that the compensator of its jump measure is given by (11) and (12). The latter properties indeed characterize the law of the process $(F_t^{(a)}(x_1), \ldots, F_t^{(a)}(x_p))$ (cf Lemma 1 in [5]), but we do not use this uniqueness property in the proof. We fix a sequence (a_n) tending to $+\infty$, and $(x_1, \ldots, x_p) \in \mathcal{D}_p$. To simplify notation we write

$$Y_t^n = (Y_t^{n,1}, \ldots, Y_t^{n,p}) = (F_t^{(a_n)}(x_1), \ldots, F_t^{(a_n)}(x_p))$$

which makes sense as soon as $a_n \geq x_p$, hence for all n sufficiently large. We also set

$$Z_t = (Z_t^1, \ldots, Z_t^p) = (Z(t, x_1), \ldots, Z(t, x_p)).$$

We rely on general limit theorems for semimartingales with jumps which can be found in the book [12]. To this end, we first need to introduce a truncation function $h : \mathbb{R} \to \mathbb{R}$, that is a bounded continuous function such that $h(x) = x$ for every $x \in [-\delta, \delta]$, for some $\delta > 0$. We may and will assume that h is nondecreasing, $|h(x)| \leq |x| \wedge 1$ for every $x \in \mathbb{R}$ and that h is

Lipschitz continuous with Lipschitz constant 1, that is $|h(x)-h(y)| \leq |x-y|$ for every $x, y \in \mathbb{R}$. We can then consider the associated (modified) triplet of characteristics of the p-dimensional semimartingale Y^n:

$$(B^n, \tilde{C}^n, \mu_{(a_n)}).$$

See Definition II.2.16 in [12]. To be specific, $\mu_{(a_n)}$ is defined in (11). Then, since Y_t^n is a purely discontinuous martingale, we have $B_t^n = (B_t^{n,i})_{1 \leq i \leq p}$, with

$$B_t^{n,i} = -\int_{[0,t]\times\mathbb{R}^p} \mu_{(a_n)}(dt, dz_1 \ldots dz_p)\,(z_i - h(z_i))$$

Similarly, $\tilde{C}_t^n = (\tilde{C}_t^{i,j,n})_{1 \leq i,j \leq p}$, with

$$\tilde{C}_t^{i,j,n} = \int_{[0,t]\times\mathbb{R}^p} \mu_{(a_n)}(dt, dz_1 \ldots dz_p) h(z_i) h(z_j).$$

Write $C_*(\mathbb{R}^p)$ for the space of all bounded Lipschitz continuous functions on $\mathbb{R}^p$ that vanish on a neighborhood of 0. We fix $g \in C_*(\mathbb{R}^p)$ such that $|g| \leq 1$, and we choose $\alpha > 0$ such that $g(z_1, \ldots, z_p) = 0$ if $|z_i| \leq \alpha$ for every $i = 1, \ldots, p$. Following the notation in [12], we set

$$(g * \mu_{(a_n)})_t = \int_{[0,t]\times\mathbb{R}^p} \mu_{(a_n)}(dt, dz_1 \ldots dz_p)\, g(z_1, \ldots, z_p).$$

From formula (11) we have

$$\begin{aligned} B_t^{n,i} &= \int_0^t ds\, \beta^{n,i}(Y_s^{n,1}, \ldots, Y_s^{n,p}) \\ \tilde{C}_t^{n,i,j} &= \int_0^t ds\, \gamma^{n,i,j}(Y_s^{n,1}, \ldots, Y_s^{n,p}) \\ (g * \mu_{(a_n)})_t &= \int_0^t ds\, \varphi^n(Y_s^{n,1}, \ldots, Y_s^{n,p}), \end{aligned} \tag{13}$$

where the functions $\beta^{n,i}, \gamma^{n,i,j}, \varphi^n$ are defined by

$$\begin{aligned} \beta^{n,i}(y_1, \ldots, y_p) &= -\int_{\mathbb{R}^p} R^{(a_n)}(y_1, \ldots, y_p; dz_1, \ldots, dz_p)\,(z_i - h(z_i)) \\ \gamma^{n,i,j}(y_1, \ldots, y_p) &= \int_{\mathbb{R}^p} R^{(a_n)}(y_1, \ldots, y_p; dz_1, \ldots, dz_p)\, h(z_i) h(z_j) \\ \varphi^n(y_1, \ldots, y_p) &= \int_{\mathbb{R}^p} R^{(a_n)}(y_1, \ldots, y_p; dz_1, \ldots, dz_p)\, g(z_1, \ldots, z_p). \end{aligned}$$

Similarly, the (modified) characteristics of the semimartingale Z are

$$(B, \tilde{C}, \theta)$$

where θ is as in Proposition 1, and

$$\begin{aligned} B_t^i &= \int_0^t ds\, \beta^i(Z_s^1, \ldots, Z_s^p) \\ \tilde{C}_t^{i,j} &= \int_0^t ds\, \gamma^{i,j}(Z_s^1, \ldots, Z_s^p) \\ (g * \theta)_t &= \int_0^t ds\, \varphi(Z_s^1, \ldots, Z_s^p), \end{aligned} \tag{14}$$

where the functions $\beta^i, \gamma^{i,j}, \varphi$ are respectively defined by

$$\begin{aligned}
\beta^i(y_1,\dots,y_p) &= -\int_{\mathbb{R}^p} U(y_1,\dots,y_p;dz_1,\dots,dz_p)\,(z_i - h(z_i)) \\
\gamma^{i,j}(y_1,\dots,y_p) &= \int_{\mathbb{R}^p} U(y_1,\dots,y_p;dz_1,\dots,dz_p)\,h(z_i)h(z_j) \\
\varphi(y_1,\dots,y_p) &= \int_{\mathbb{R}^p} U(y_1,\dots,y_p;dz_1,\dots,dz_p)\,g(z_1,\dots,z_p).
\end{aligned}$$

Lemma 1 *For every $(y_1,\dots,y_p) \in \mathcal{D}_p$,*

$$\begin{aligned}
|\beta^{n,i}(y_1,\dots,y_p)| &\le 2y_i \int \nu_{(a_n)}(dr)\, r\, \mathbf{1}_{\{r>\delta\}} \\
|\gamma^{n,i,j}(y_1,\dots,y_p)| &\le (y_i + y_j) \int \nu_{(a_n)}(dr)\,(r \wedge r^2) \\
|\varphi^n(y_1,\dots,y_p) &\le \frac{2}{\alpha}\, y_p \int \nu_{(a_n)}(dr)\, r\, \mathbf{1}_{\{r>\alpha\}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \beta^{n,i}(y_1,\dots,y_p) &= \beta^i(y_1,\dots,y_p) \\
\lim_{n\to\infty} \gamma^{n,i,j}(y_1,\dots,y_p) &= \gamma^{i,j}(y_1,\dots,y_p) \\
\lim_{n\to\infty} \varphi^n(y_1,\dots,y_p) &= \varphi(y_1,\dots,y_p),
\end{aligned}$$

uniformly on bounded subsets of $\mathcal{D}_p$.

Let us postpone the proof of the lemma and complete that of the theorem. The first step is to check the sequence of the laws of the processes Y^n is tight in the space of probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^p)$. This will follow from Theorem VI.4.18 in [12] provided we can check that:

(i) We have for every $N > 0$ and $\varepsilon > 0$,

$$\lim_{b\uparrow\infty} \Big(\limsup_{n\to\infty} P[\mu_{(a_n)}([0,N] \times \{z \in \mathbb{R}^p : |z| > b\}) > \varepsilon] \Big) = 0.$$

(ii) The laws of the processes $B^{n,i}, \tilde{C}^{n,i,j}, g * \mu_{(a_n)}$ are tight in the space of probability measures on $C(\mathbb{R}_+, \mathbb{R})$.

To prove (i), set

$$T_A^n = \inf\{t \ge 0 : Y_t^{p,n} > A\}$$

for every $A > x_p$. Since $Y^{n,p}$ is a (bounded) nonnegative martingale, a classical result states that

$$\mathbb{P}[\sup\{Y_t^{n,p}, t \ge 0\} > A] = \mathbb{P}[T_A^n < \infty] \le \frac{x_p}{A}. \tag{15}$$

From formulas (11) and (12), we have on the event $\{\sup\{Y_t^{p,n}, t \ge 0\} \le A\}$

$$\mu_{(a_n)}([0,N] \times \{z \in \mathbb{R}^p : |z| > b\}) \le N\Big(A\, \nu_{(a_n)}(]\frac{b}{p}, \infty[) + a_n\, \nu_{(a_n)}(]\frac{b a_n}{pA}, \infty[) \Big)$$

Under Assumption (H), we have

$$\lim_{n\to\infty} a_n\, \nu_{(a_n)}(]\frac{ba_n}{pA},\infty[) = 0$$

and so, on the event $\{\sup\{Y_t^{p,n}, t\geq 0\} \leq A\}$,

$$\limsup_{n\to\infty} \mu_{(a_n)}([0,N]\times\{z\in\mathbb{R}^p : |z|>b\}) \leq NA\,\pi([\frac{b}{p},\infty[).$$

If we first choose A so that x_p/A is small, and then b large enough so that $NA\,\pi([\frac{b}{p},\infty[) < \varepsilon$, we see that the statement in (i) follows from (15). Part (ii) is a straightforward consequence of formulas (13), the bounds of the first part of Lemma 1 and (15) again. This completes the proof of the tightness of the sequence of the laws of the processes Y^n.

Then, we can assume that, at least along a suitable subsequence, Y^n converges in distribution towards a limiting process $Y^\infty = (Y^{\infty,1},\dots,Y^{\infty,p})$. We claim that Y^∞ is a semimartingale whose triplet of (modified) characteristics $(B^\infty, \tilde{C}^\infty, \mu_\infty)$ is such that

$$\begin{aligned} B_t^{\infty,i} &= \int_0^t ds\, \beta^i(Y_s^{\infty,1},\dots,Y_s^{\infty,p}) \\ \tilde{C}_t^{\infty,i,j} &= \int_0^t ds\, \gamma^{i,j}(Y_s^{\infty,1},\dots,Y_s^{\infty,p}) \\ (g*\mu_\infty)_t &= \int_0^t ds\, \varphi(Y_s^{\infty,1},\dots,Y_s^{\infty,p}), \end{aligned} \tag{16}$$

with $\beta^i, \gamma^{i,j}, \varphi$ as above. To see this, it is enough to verify that the 4-tuples $(Y^n, B^n, \tilde{C}^n, g*\mu_{(a_n)})$ converge in distribution to $(Y^\infty, B^\infty, \tilde{C}^\infty, g*\mu_\infty)$ (see Theorem IX.2.4 in [12]). The latter convergence readily follows from the convergence of Y^n towards Y^∞, formulas (13) and the second part of Lemma 1.

Finally, knowing the triplet of characteristics of Y^∞, Theorem II.2.34 in [12] shows that Y^∞ is a purely discontinuous martingale, and the compensator of its jump measure is

$$dt\; U(Y_t^{\infty,1},\dots,Y_t^{\infty,p}; dz_1,\dots,dz_p).$$

By Proposition 1, this implies that Y^∞ has the same distribution as Z, and this completes the proof of Theorem 1. □

Proof of Lemma 1: By definition, for $(y_1,\dots,y_p)\in\mathcal{D}_p^{a_n}$,

$$\begin{aligned} \beta^{n,i}(y_1,\dots,y_p) &= -\int \nu^{(a_n)}(dr)\int_0^{a_n} du\Big(r(\mathbf{1}_{\{u\leq y_i\}} - a_n^{-1}y_i) - h(r(\mathbf{1}_{\{u\leq y_i\}} - a_n^{-1}y_i))\Big) \\ &= -\int \nu^{(a_n)}(dr)\, y_i(r(1-a_n^{-1}y_i) - h(r(1-a_n^{-1}y_i))) \\ &\quad + \int \nu^{(a_n)}(dr)\,(a_n - y_i)(a_n^{-1}ry_i + h(-a_n^{-1}ry_i)). \end{aligned}$$

Recalling that $h(x)=x$ if $|x|\leq\delta$, we immediately get the bound

$$\beta^{n,i}(y_1,\dots,y_p) \leq 2y_i \int \nu^{(a_n)}(dr)\, r\mathbf{1}_{\{r>\delta\}}.$$

Furthermore, using the fact that h is Lipschitz with Lipschitz constant 1, we have

$$\begin{aligned}&|\beta^{n,i}(y_1,\ldots,y_p)+y_i\int\nu^{(a_n)}(dr)\,(r-h(r))|\\&\leq 2a_n^{-1}y_i^2\int\nu^{(a_n)}(dr)\,r\mathbf{1}_{\{r>\delta\}}+y_i\int\nu^{(a_n)}(dr)\,r\mathbf{1}_{\{a_n^{-1}ry_i>\delta\}}\end{aligned}$$

and it is easy to verify from Assumption (H) that the right-hand side tends to 0 as $n\to\infty$, uniformly when y_i varies over a bounded subset in $\mathbb{R}_+$. Since Assumption (H) also implies that

$$\lim_{n\to\infty}\int\nu^{(a_n)}(dr)\,(r-h(r))=\int\pi(dr)\,(r-h(r)),$$

we get the first limit of the lemma.

Consider now, for $(y_1,\ldots,y_p)\in\mathcal{D}_p^{a_n}$, and $1\leq i\leq j\leq p$,

$$\begin{aligned}\gamma^{n,i,j}(y_1,\ldots,y_p)&=\int\nu^{(a_n)}(dr)\int_0^{a_n}du\,h(r(\mathbf{1}_{\{u\leq y_i\}}-a_n^{-1}y_i))h(r(\mathbf{1}_{\{u\leq y_j\}}-a_n^{-1}y_j))\Big)\\&=\int\nu^{(a_n)}(dr)\,y_i\,h(r(1-a_n^{-1}y_i))h(r(1-a_n^{-1}y_j))\\&\quad+\int\nu^{(a_n)}(dr)\,(y_j-y_i)\,h(-a_n^{-1}ry_i)\,h(r(1-a_n^{-1}y_j))\\&\quad+\int\nu^{(a_n)}(dr)\,(a_n-y_j)\,h(-a_n^{-1}ry_i)\,h(-a_n^{-1}ry_j).\end{aligned}\tag{17}$$

Using the bounds $|h|\leq 1$ and $|h(x)|\leq|x|$, we get

$$|\gamma^{n,i,j}(y_1,\ldots,y_p)|\leq y_j\int\nu^{(a_n)}(dr)(r^2\wedge 1)+y_i\int\nu^{(a_n)}(dr)\,r(r\wedge 1),$$

which gives the second bound of the lemma. Then, using the Lipschitz property of h,

$$\begin{aligned}&\Big|\int\nu^{(a_n)}(dr)\,h(r(1-a_n^{-1}y_i))h(r(1-a_n^{-1}y_j))-\int\nu^{(a_n)}(dr)\,h(r)^2\Big|\\&\leq 2a_n^{-1}y_j\int\nu^{(a_n)}(dr)\,rh(r)\longrightarrow 0\end{aligned}$$

as $n\to\infty$. Notice that

$$\lim_{n\to\infty}y_i\int\nu^{(a_n)}(dr)\,h(r)^2=y_i\int\pi(dr)\,h(r)^2=\gamma^{i,j}(y_1,\ldots,y_p),$$

uniformly when $(y_1,\ldots,y_p)$ varies over a bounded set. To complete the verification of the second limit in the lemma, we need to check that the last two terms in the right-hand side of (17) tend to 0 as $n\to\infty$. We have first

$$\int\nu^{(a_n)}(dr)\,h(-a_n^{-1}ry_i)h(r(1-a_n^{-1}y_j))\leq\int\nu^{(a_n)}(dr)\,ra_n^{-1}y_i\,h(r)\longrightarrow 0$$

as $n\to\infty$. It remains to bound

$$\begin{aligned}&\Big|a_n\int\nu^{(a_n)}(dr)\,h(-a_n^{-1}ry_i)\,h(-a_n^{-1}ry_j)\Big|\\&\leq\int\nu^{(a_n)}(dr)\,ry_i((a_n^{-1}ry_j)\wedge 1)\\&\leq y_i\int\nu^{(a_n)}(dr)\,r\mathbf{1}_{\{r>A\}}+y_iy_ja_n^{-1}\int\nu^{(a_n)}(dr)\,r^2\mathbf{1}_{\{r\leq A\}}\end{aligned}$$

where $A > 0$ is arbitrary. If $\eta > 0$ is given, we can first choose A sufficiently large so that

$$\limsup_{n\to\infty} \int \nu^{(a_n)}(dr)\, r\mathbf{1}_{\{r>A\}} \leq \int \pi(dr)\, r\mathbf{1}_{\{r\geq A\}} < \eta.$$

On the other hand, we have also

$$\lim_{n\to\infty} a_n^{-1} \int \nu^{(a_n)}(dr)\, r^2\mathbf{1}_{\{r\leq A\}} = 0$$

and together with the preceding estimates, this gives the second limit of the lemma.

Finally, we have

$$\varphi^n(y_1,\ldots,y_p) = \int \nu^{(a_n)}(dr) \int_0^{a_n} du\, g(r(\mathbf{1}_{\{u\leq y_1\}} - a_n^{-1}y_1),\ldots,r(\mathbf{1}_{\{u\leq y_p\}} - a_n^{-1}y_p)).$$

Since $|g| \leq 1$ and $g(z_1,\ldots,z_p) = 0$ if $\sup|z_i| \leq \alpha$, we easily get the bound

$$\begin{aligned} |\varphi^n(y_1,\ldots,y_p)| &\leq y_p \int \nu^{(a_n)}(dr)\, \mathbf{1}_{\{r>\alpha\}} + a_n \int \nu^{(a_n)}(dr)\, \mathbf{1}_{\{a_n^{-1}ry_p>\alpha\}} \\ &\leq y_p \int \nu^{(a_n)}(dr)\, \mathbf{1}_{\{r>\alpha\}} + \frac{y_p}{\alpha} \int \nu^{(a_n)}(dr)\, r\mathbf{1}_{\{r>\alpha\}} \end{aligned}$$

which gives the third bound of the lemma. Then, if M denotes a Lipschitz constant for g,

$$\begin{aligned} &\Big|\varphi^n(y_1,\ldots,y_p) - \int \nu^{(a_n)}(dr) \int_0^{a_n} du\, g(r\mathbf{1}_{\{u\leq y_1\}},\ldots,r\mathbf{1}_{\{u\leq y_p\}})\Big| \\ &\leq Mp \int \nu^{(a_n)}(dr) \int_0^{a_n} du(\mathbf{1}_{\{u\leq y_p\}} a_n^{-1} r y_p\, \mathbf{1}_{\{r>\alpha\}} + \mathbf{1}_{\{u>y_p\}} a_n^{-1} r y_p\, \mathbf{1}_{\{a_n^{-1}ry_p>\alpha\}}) \\ &\leq Mp\Big(\int \nu^{(a_n)}(dr)\, r\mathbf{1}_{\{r>\alpha\}}\Big) a_n^{-1} y_p^2 + Mpy_p \int \nu^{(a_n)}(dr)\, r\mathbf{1}_{\{a_n^{-1}ry_p>\alpha\}} \end{aligned}$$

which tends to 0 as n tends to ∞, uniformly when y_p varies over a compact subset of $\mathbb{R}_+$. The last convergence of the lemma now follows from Assumption (H). This completes the proof. □

4 Hydrodynamic limits for exchangeable coalescents

The motivation for this section stems from hydrodynamic limit theorems leading from stochastic coalescents to Smoluchowski's coagulation equation, which we now summarize.

4.1 Stochastic coalescents and Smoluchowski's coagulation equation

Consider a symmetric measurable function $K :]0,\infty[\times]0,\infty[\to \mathbb{R}_+$ which will be referred to as a coagulation kernel. A stochastic coalescent with coagulation kernel K can be viewed as a Markov chain in continuous time $C = (C_t, t \geq 0)$ with values in the space of finite integer-valued measures on $]0,\infty[$ with the following dynamics. Suppose that the process starts from some state $\sum_{i=1}^k \delta_{x_i}$, where $k \geq 2$ and $x_i \in]0,\infty[$ for $i = 1,\ldots,k$. For $1 \leq i < j \leq k$, let $\mathbf{e}_{i,j}$ be an exponential variable with parameter $K(x_i, x_j)$, such that to different pairs correspond independent variables. The first jump of the process C occurs at time $\min_{1\le i<j\le k}\mathbf{e}_{i,j}$, and if this minimum is reached for the indices $1\le \ell < m\le k$ (i.e. ℓ and k are the indices such that $\min_{1\le i<j\le k}\mathbf{e}_{i,j}=\mathbf{e}_{\ell,m}$), then the state after the jump is

$$\delta_{x_\ell+x_m}+\sum_{i\neq \ell,m}\delta_{x_i}\,.$$

In other words, a stochastic coalescent with coagulation kernel K is a finite particle system in $]0,\infty[$ such that each pair of particles (x_i,x_j) in the system merges at rate $K(x_i,x_j)$, independently of the other pairs.

Now consider a sequence $(\tilde C^{(n)}_t, t\ge 0)_{n\in\mathbb{N}}$ of stochastic coalescents with coagulation kernel K and set $C^{(n)}_t=n^{-1}\tilde C^{(n)}_{t/n}$ for $t\ge 0$. Suppose that the sequence of initial states $C^{(n)}_0$ converges in probability in $\mathcal{M}_{\mathrm{R}}$ to a Radon measure μ_0. Then under some technical assumptions on the coagulation kernel K (see e.g. Norris [15]), the sequence $(C^{(n)}_t, t\ge 0)$ converges in probability on the space of càdlàg trajectories with values in $\mathcal{M}_{\mathrm{R}}$ towards a deterministic limit $(\mu_t, t\ge 0)$. Moreover this limit is characterized as the solution to Smoluchowski's coagulation equation

$$\frac{d\langle \mu_t, f\rangle}{dt} = \frac{1}{2}\int_{]0,\infty[^2}\left(f(x+y)-f(x)-f(y)\right)K(x,y)\mu_t(dx)\mu_t(dy)\,, \tag{18}$$

where $f:]0,\infty[\to\mathbb{R}$ denotes a generic continuous function with compact support.

4.2 Hydrodynamic limits

Let ν denote a σ-finite measure on $]0,1]$ such that $\int_{]0,1]} r^2\nu(dr)<\infty$, and write $\Lambda(dr)=r^2\nu(dr)$, which is thus a finite measure on $]0,1]$. The so-called Λ-coalescent (or coalescent with multiple collisions, see [16]) is a Markov process $(\Pi_t, t\ge 0)$ taking values in the set of all partitions of $\mathbb{N}$. Unless otherwise specified, we assume that Π_0 is the partition of $\mathbb{N}$ into singletons. For every $t\ge 0$, write D_t for the sequence of asymptotic frequencies of the blocks of Π_t, ranked in nonincreasing order (if the number k of blocks is finite, then the terms of index greater than k in the sequence are all equal to 0). Then ([16], section 2.2) the process $(D_t, t\ge 0)$ is a time-homogeneous Markov process with values in the space $\mathcal{S}_1^{\downarrow}$ of nonincreasing numerical sequences $\mathbf{s}=(s_1,\dots)$ with $\sum_{i=1}^{\infty}s_i\le 1$.

The following connection with generalized Fleming-Viot processes can be found in [4]. Let $(F_t, t\ge 0)$ be the generalized Fleming-Viot process associated with ν, and for every $t\ge 0$, let J_t be the sequence of sizes of jumps of the mapping $x\to F_t(x)$, ranked again in nonincreasing order, and with the same convention if there are finitely many jumps. Then, for each fixed $t\ge 0$, J_t and D_t have the same distribution (Theorem 1 in [4] indeed gives a deeper connection, which has been briefly described in Section 1).

For each $a>0$, let $\tilde\nu^{(a)}$, $\nu^{(a)}$, $\tilde F^{(a)}$ and $F^{(a)}$ be as in Section 3. Denote by $\tilde\mu^{(a)}_t$ the point measure whose atoms are given by the jump sizes of the increasing process $x\to \tilde F^{(a)}_t(x)$:

$$\tilde\mu^{(a)}_t=\sum_{\{x\in]0,1]:\tilde F^{(a)}_t(x)-\tilde F^{(a)}_t(x-)>0\}}\delta_{\tilde F^{(a)}_t(x)-\tilde F^{(a)}_t(x-)}.$$

Fom the preceding observations, the atoms of $\tilde{\mu}_t^{(a)}$ also correspond to the sizes of the blocks in a Λ-coalescent at time t, for $\Lambda(dr) = r^2\tilde{\nu}_t^{(a)}(dr)$. We then consider the rescaled version $\mu_t^{(a)}$, given as the image of $a^{-1}\tilde{\mu}_{at}^{(a)}$ under the dilation $r \to ar$. Equivalently, $\mu^{(a)}$ is a^{-1} times the sum of the Dirac point masses at the jump sizes of the mapping $x \to F_t^{(a)}(x)$.

Theorem 2 *Suppose that* (H) *holds and let* $(Z(t,x); t, x \geq 0)$ *be the flow of continuous-state branching processes associated with* π. *Then for every* $t \geq 0$, $\mu_t^{(a)}$ *converges to the Lévy measure* λ_t *of the subordinator* $Z(t,\cdot)$ *as* $a \to \infty$ *in probability in* $\mathcal{M}_{\mathrm{R}}$.

Theorem 2 is an immediate consequence of Theorem 1 and the following lemma.

Lemma 2 *Let* $\sigma = (\sigma_t, t \geq 0)$ *be a subordinator with Lévy measure* λ. *For each* $a > 0$, *let* $X^{(a)} = (X_t^{(a)}, 0 \leq t \leq a)$ *be an increasing càdlàg process with exchangeable increments, with* $X_0^{(a)} = 0$ *and* $X_a^{(a)} = a$ *a.s. Suppose that* $X^{(a)}$ *converges to* σ *as* $a \to \infty$ *in the sense of finite-dimensional distributions. Then the random point measure*

$$a^{-1} \sum_{0<t<a} \delta_{\Delta X_t^{(a)}}$$

converges to λ *in probability in* $\mathcal{M}_{\mathrm{R}}$ *as* $a \to \infty$.

Proof: Pick some nonnegative continuous function $f :]0,\infty[\to \mathbb{R}$ with compact support and write

$$c := \int_{]0,\infty[} f(x)\lambda(dx)\,.$$

By the Lévy-Itô decomposition for subordinators, the random point measure $\sum_{\Delta\sigma_t>0} \delta_{(t,\Delta\sigma_t)}$ on $\mathbb{R}_+ \times]0,\infty[$ is Poisson with intensity $dt \otimes \lambda(dx)$. Let $\rho > 0$. The law of large numbers ensures the existence of a real number $a_\rho > 0$ such that

$$\mathbb{E}\Big[\Big|a_\rho^{-1} \sum_{0<t<a_\rho} f(\Delta\sigma_t) - c\Big|\Big] < \rho\,. \tag{19}$$

Then consider for $a > a_\rho$ the bridges with exchangeable increments, bounded variation and no negative jumps on the time interval $[0, a_\rho]$, defined by

$$B_t^{(a)} := X_t^{(a)} - t a_\rho^{-1} X_{a_\rho}^{(a)} \quad , \quad B_t = \sigma_t - t a_\rho^{-1} \sigma_{a_\rho}\,, \qquad t \in [0, a_\rho]\,.$$

Our assumptions entail that $B^{(a)}$ converges in the sense of finite dimensional distributions to B, so according to Kallenberg [13], the random measure

$$\sum_{0<t<a_\rho} \delta_{\Delta B_t^{(a)}} = \sum_{0<t<a_\rho} \delta_{\Delta X_t^{(a)}}$$

converges in law on $\mathcal{M}_{\mathrm{R}}$ towards

$$\sum_{0<t<a_\rho} \delta_{\Delta B_t} = \sum_{0<t<a_\rho} \delta_{\Delta \sigma_t}\,,$$

and in particular, when $a \to \infty$,

$$a_\rho^{-1} \sum_{0<t<a_\rho} f(\Delta X_t^{(a)}) \xrightarrow{\text{(d)}} a_\rho^{-1} \sum_{0<t<a_\rho} f(\Delta \sigma_t)\,. \tag{20}$$

Let us check that the variables

$$\Big| \sum_{0<t<a_\rho} f(\Delta X_t^{(a)}) \Big|\,, \qquad a \in [a_\rho, \infty[$$

are uniformly integrable. Let $[u,v]$ be a compact subinterval of $]0,\infty[$ such that the support of f is contained in $[u,v]$. Denote by $N^a_{[u,v]}$ the number of jumps of the process $X^{(a)}$ with size in $[u,v]$. By classical results about processes with exchangeable increments, conditionally on $N^a_{[u,v]} = n$, the number

$$N^{(a,a_\rho)}_{[u,v]} := \sum_{0<t<a_\rho} \mathbf{1}_{[u,v]}(\Delta X_t^{(a)})$$

has a binomial $\mathcal{B}(n, \frac{a_\rho}{a})$ distribution. Notice that $N^a_{[u,v]} \le \frac{a}{u}$ since $X_a^{(a)} = a$. We see that $N^{(a,a_\rho)}_{[u,v]}$ is bounded above in distribution by a binomial $\mathcal{B}([\frac{a}{u}], \frac{a_\rho}{a})$ distribution, and the desired uniform integrability readily follows.

It then follows from (19) and (20) that

$$\lim_{a\to\infty} \mathbb{E}\Big[\Big|a_\rho^{-1} \sum_{0<t<a_\rho} f(\Delta X_t^{(a)}) - c\Big|\Big] = \mathbb{E}\Big[\Big|a_\rho^{-1} \sum_{0<t<a_\rho} f(\Delta \sigma_t) - c\Big|\Big] \le \rho\,.$$

Moreover, an easy exchangeability argument shows that we have also

$$\limsup_{a\to\infty} \mathbb{E}\Big[\Big|a^{-1} \sum_{0<t<a} f(\Delta X_t^{(a)}) - c\Big|\Big] \le \rho\,.$$

Since ρ may be taken arbitrarily small, we have thus shown that

$$\lim_{a\to\infty} a^{-1} \sum_{t\le a} f(\Delta X_t^{(a)}) = \int_{]0,\infty[} f(x)\lambda(dx)\,,$$

in L^1 for every continuous function f with compact support. The conclusion now follows by a standard argument. $\square$

We will now show that the family $(\lambda_t, t>0)$ of Lévy measures, which appears in Theorem 2, solves a certain coagulation equation with multiple collisions. To this end, we introduce the following additional assumption, which also plays a key role in the study of the genealogical structure of continuous-state branching processes (see e.g. [9]).

Assumption (E) *The Ψ-CSBP becomes extinct almost surely.*

Equivalently, this assumption holds iff $\mathbb{P}[Z(t,x) = 0] > 0$ for every $t > 0$ and $x \ge 0$. By solving (3), it is easy to verify that Assumption (E) is equivalent to

$$\int_1^\infty \frac{du}{\Psi(u)} < \infty. \tag{21}$$

In particular, Assumption (E) holds in the so-called stable case $\Psi(u) = u^\gamma$, $\gamma \in]1,2[$, that will be considered in Section 5 below.

From (4), we see that under Assumption (E) we have $d_t = 0$, and the total mass $\lambda_t(]0,\infty[) = -\log \mathbb{P}[Z(t,1) = 0]$ is finite for every $t > 0$. Moreover the function $t \to \lambda_t(]0,\infty[)$ is nonincreasing.

We denote by $C_\bullet(\mathbb{R}_+)$ the space of all bounded continuous functions f on $\mathbb{R}_+$ such that $f(0) = 0$ and $f(x)$ has a limit as $x \to +\infty$. The space $C_\bullet(\mathbb{R}_+)$ is equipped with the uniform norm, which is denoted by $\|f\|$. For every integer $k \geq 2$ and $q > 0$, we denote by $\Psi^{(k)}(q)$ the k-th derivative of Ψ at q. It is immediately checked that

$$\Psi^{(k)}(q) = (-1)^k \int \pi(dr)\, r^k \mathrm{e}^{-qr}. \tag{22}$$

Obviously, $(-1)^k \Psi^{(k)}(q) \geq 0$ for every $k \geq 2$ and $q > 0$.

Proposition 3 *Under Assumption* (E)*, for every $f \in C_\bullet(\mathbb{R}_+)$, the function $t \to \langle \lambda_t, f\rangle$ solves the equation*

$$\frac{d\langle \lambda_t, f\rangle}{dt} = \sum_{k=2}^{\infty} \frac{(-1)^k \Psi^{(k)}(\langle \lambda_t, 1\rangle)}{k!} \int_{]0,\infty[^k} (f(x_1+\cdots+x_k) - (f(x_1)+\cdots+f(x_k)))\, \lambda_t(dx_1)\ldots\lambda_t(dx_k) \tag{23}$$

where the series in the right-hand side converges absolutely.

It is interesting to observe that (23) also holds when $\Psi(q) = cq^2$ for some constant $c > 0$. Take $\Psi(q) = \frac{1}{2}u^2$ for definiteness (then the Ψ-CSBP is the classical Feller diffusion) in such a way that (23) exactly reduces to (18) with $K \equiv 1$. Then $u_t(q) = 2q\,(2+qt)^{-1}$, and it follows that

$$\lambda_t(dx) = \frac{4}{t^2} \exp(-\frac{2x}{t})\, dx \tag{24}$$

so that the density of λ_t is the classical solution, arising from infinitesimally small initial clusters, of the Smoluchovski equation (18) in the case $K \equiv 1$ (cf Section 2.2 of [1]).

We can rewrite equation (23) in a somewhat more synthetic way by introducing the following notation. If μ is a measure on $]0,\infty[$ such that $\int_{]0,\infty[}(1 \wedge x)\mu(dx) < \infty$, we write $\mu^\oplus$ for the distribution on $[0,\infty[$ of the sum of the atoms of a Poisson random measure on $]0,\infty[$ with intensity μ. Note that $\mu^\oplus$ is a probability measure and that, by Campbell's formula,

$$\int_{[0,\infty[} \mathrm{e}^{-qx} \mu^\oplus(dx) \;=\; \exp\Big\{ -\int_{]0,\infty[} (1 - \mathrm{e}^{-qx})\mu(dx) \Big\}, \qquad q \geq 0. \tag{25}$$

As we will see in the proof below, (23) follows from the equation

$$\frac{d\langle \lambda_t, f\rangle}{dt} \;=\; \int_{]0,\infty[} \pi(da) \Big(\langle (a\lambda_t)^\oplus, f\rangle - \langle a\lambda_t, f\rangle \Big). \tag{26}$$

Informally, we may think of $\lambda_t(dx)$ as the density at time t of particles with size x in some infinite system of particles. The right-hand side in (26) can be interpreted by saying at rate $\pi(da)$, a 'quantity' a of particles coagulates at time t. More precisely, this 'quantity' is sampled in a Poissonian way, viewing at $a\lambda_t$ as an intensity measure for the sampling (so, loosely speaking, the particles involved into the coagulation are sampled uniformly at random amongst the particles present at time t).

As the proof below will show, (26) still holds without Assumption (E) at least for functions f of the type $f(x) = 1 - \exp(-qx)$, provided that $d_t = 0$ for every $t > 0$ (recall from Silverstein [19] that the latter holds whenever $\int_{]0,1[} r\pi(dr) = \infty$). In that case however, the measures λ_t may be infinite, and then coagulations involve infinitely many components, so that one cannot write an equation of the form (23).

Proof: We first prove (26). For $q > 0$, let $f_{(q)} \in C_\bullet(\mathbb{R}_+)$ be defined by $f_{(q)}(x) = 1 - \mathrm{e}^{-qx}$. By (25) and (4),

$$\langle (a\lambda_t)^\oplus, f_{(q)} \rangle = 1 - \exp\Big(- a \int \lambda_t(dr)(1 - \mathrm{e}^{-qr}) \Big) = 1 - \exp(-a u_t(q)).$$

On the other hand, by (4) again,

$$\langle \lambda_t, f_{(q)} \rangle = u_t(q).$$

Thus when $f = f_{(q)}$ the right-hand side of (26) makes sense and is equal to

$$\int_{]0,\infty[} \pi(da) \Big(1 - \exp(-a u_t(q)) - a u_t(q) \Big) = -\Psi(u_t(q)).$$

Therefore (26) reduces to (3) in that case. Note that we have not used Assumption (E) at this stage (except for the fact that $d_t = 0$ for every $t > 0$).

Denote by $\mathcal{H}$ the subspace of $C_\bullet(\mathbb{R}_+)$ that consists of linear combinations of the functions $f_{(q)}$. Then $\mathcal{H}$ is dense in $C_\bullet(\mathbb{R}_+)$. Obviously, for every $f \in \mathcal{H}$, (26) holds, and the right-hand side of (26) is a continuous function of $t \in]0, \infty[$. Fix $f \in C_\bullet(\mathbb{R}_+)$ and a sequence $(f_n)_{n\geq 1}$ in $\mathcal{H}$ that converges to f. If we also fix $0 < \varepsilon < t$, we have for every $n \geq 1$,

$$\langle \lambda_t, f_n \rangle = \langle \lambda_\varepsilon, f_n \rangle + \int_\varepsilon^t ds \int \pi(da) \Big(\langle (a\lambda_s)^\oplus, f_n \rangle - \langle a\lambda_s, f_n \rangle \Big). \tag{27}$$

Plainly, for every $s > 0$,

$$\langle \lambda_s, f_n \rangle \underset{n\to\infty}{\longrightarrow} \langle \lambda_s, f \rangle \quad \text{and} \quad \langle (a\lambda_s)^\oplus, f_n \rangle \underset{n\to\infty}{\longrightarrow} \langle (a\lambda_s)^\oplus, f \rangle.$$

We claim that there exists a constant C_ε such that, for every $s \geq \varepsilon$ and $n \geq 1$, and every $h \in C_\bullet(\mathbb{R}_+)$,

$$|\langle (a\lambda_s)^\oplus, h \rangle - \langle a\lambda_s, h \rangle| \leq C_\varepsilon (a^2 \wedge a)\, \|h\|. \tag{28}$$

As the quantities $\langle \lambda_s, 1 \rangle$, $s \in [\varepsilon, \infty[$ are bounded above, it is clearly enough to consider $a \leq 1$. Since $h(0) = 0$, the definition of $(a\lambda_s)^\oplus$ immediately gives

$$\langle (a\lambda_s)^\oplus, h \rangle = \mathrm{e}^{-a\langle \lambda_s, 1\rangle}\, a\langle \lambda_s, h \rangle + O(a^2 \|h\|)$$

where the remainder $O(a^2\|h\|)$, which corresponds to the event that a Poisson measure with intensity $a\lambda_s$ has at least two atoms, is uniform in $h \in C_\bullet(\mathbb{R}_+)$ and $s \geq \varepsilon$. The estimate (28) follows.

Using (28) and dominated convergence, we get

$$\lim_{n\to\infty}\int \pi(da)\Big(\langle (a\lambda_s)^{\oplus}, f_n\rangle - \langle a\lambda_s, f_n\rangle\Big) = \int \pi(da)\Big(\langle (a\lambda_s)^{\oplus}, f\rangle - \langle a\lambda_s, f\rangle\Big) \tag{29}$$

uniformly in $s \in [\varepsilon, \infty[$, and the right-hand side of (29) is a continuous function of s. Equation (26) in the general case follows by passing to the limit $n \to \infty$ in (27).

Then, to derive (26) from (23), we write

$$\begin{aligned}
&\int_{]0,\infty[} \pi(da)\Big(\langle (a\lambda_t)^{\oplus}, f\rangle - \langle a\lambda_t, f\rangle\Big) \\
&= \int \pi(da)\Big(\Big(\sum_{k=1}^{\infty} \frac{a^k}{k!}\, \mathrm{e}^{-a\langle \lambda_t, 1\rangle} \int f(x_1+\cdots+x_k)\, \lambda_t(dx_1)\ldots\lambda_t(dx_k)\Big) - a\langle \lambda_t, f\rangle\Big) \\
&= \int \pi(da) \sum_{k=1}^{\infty} \frac{a^k}{k!}\, \mathrm{e}^{-a\langle \lambda_t, 1\rangle} \int (f(x_1+\cdots+x_k) - (f(x_1)+\cdots+f(x_k)))\, \lambda_t(dx_1)\ldots\lambda_t(dx_k).
\end{aligned}$$

Notice that the term $k = 1$ in the last series vanishes. Moreover, bounding the other terms by their absolute value gives a convergent series, whose sum is integrable with respect to $\pi(da)$. Hence we may interchange the sum and the integral with respect to $\pi(da)$, and we get the statement of the proposition from (26). □

Remark. To conclude this section, let us observe that Assumption (E) is closely related to the property for a Λ-coalescent to come down from infinity (cf Pitman [16] and Schweinsberg [18]). Let ν denote a σ-finite measure on $]0, 1]$ such that $\int_{]0,1]} r^2 \nu(dr) < \infty$, and let $\Lambda(dx) = x^2\nu(dx)$. Let Ψ be given by (1) with $\pi = \nu$ (and $\beta = 0$). Then the Λ-coalescent comes down from infinity if and only if the Ψ-CSBP becomes extinct almost surely. To see this, recall from Schweinsberg [18] that a necessary and sufficient condition for the Λ-coalescent to come down from infinity is

$$\sum_{b=2}^{\infty}\Big(\sum_{k=2}^{b}(k-1)\begin{pmatrix} b \\ k \end{pmatrix}\int r^k(1-r)^{b-k}\nu(dr)\Big)^{-1} < \infty. \tag{30}$$

Using the binomial formula, we can rewrite this condition as

$$\sum_{b=2}^{\infty}\Big(\int (br - 1 + (1-r)^b)\, \nu(dr)\Big)^{-1} < \infty,$$

or equivalently, if we put $\Phi(q) = \int (qr - 1 + (1-r)^q)\, \nu(dr)$ for every real $q \geq 1$,

$$\int_2^{\infty} \frac{dq}{\Phi(q)} < \infty. \tag{31}$$

(note that the function Φ is nondecreasing on $[1, \infty[$). Simple estimates give the existence of a constant $c \in]0, 1[$ such that, for every $q \geq 2$,

$$c\Psi(q) \leq \Phi(q) \leq \Psi(q).$$

It follows that (30) and (21) are equivalent. In the spirit of the present work, it would be interesting to give a direct probabilistic proof of the equivalence between the property for a Λ-coalescent to come down from infinity and Assumption (E) for the associated branching process.

5 Small time behavior of flows and coalescents

In this section, we fix a measure ν on $]0,1]$ such that $\int r^2\nu(dr) < \infty$ and we consider the associated generalized Fleming-Viot process $(F_t, t \geq 0)$.

From now on until the end of the section, we make the following assumption on ν.

Assumption (A). *The function $\nu([\varepsilon,1])$ is regularly varying with index $-\gamma$ as $\varepsilon \to 0$, for some $\gamma \in]1,2[$.*

As a consequence, there exists a function $L(\varepsilon)$, $\varepsilon \in]0,1]$ that is slowly varying as $\varepsilon \to 0$, such that, for every $\varepsilon \in]0,1]$,

$$\nu([\varepsilon,1]) = \varepsilon^{-\gamma} L(\varepsilon).$$

Fix $\varepsilon_0 > 0$ such that $\nu([\varepsilon_0,1]) > 0$. For $\varepsilon \in]0,\varepsilon_0]$ we have $L(\varepsilon) > 0$ and so we can set

$$F_t^\varepsilon(x) = \frac{1}{\varepsilon}\, F_{L(\varepsilon)^{-1}\varepsilon^{\gamma-1}t}(\varepsilon x)$$

for $0 \leq x \leq \varepsilon^{-1}$ and $t \geq 0$. We also let ν_ε be the measure on $[0,\varepsilon^{-1}]$ defined by

$$\int \nu_\varepsilon(dr)\,\varphi(r) = L(\varepsilon)^{-1}\varepsilon^\gamma \int \nu(dr)\,\varphi(\frac{r}{\varepsilon}).$$

A simple scaling transformation shows that for every $(x_1,\ldots,x_p) \in \mathcal{D}_p^{1/\varepsilon}$, $(F_t^\varepsilon(x_1),\ldots,F_t^\varepsilon(x_p))$ is a purely discontinuous martingale, with values in $\mathcal{D}_p^{1/\varepsilon}$, and the compensator of its jump measure is

$$dt\, R_\varepsilon(F_t^\varepsilon(x_1),\ldots,F_t^\varepsilon(x_p); dz_1,\ldots,dz_p)$$

where

$$\begin{aligned}
&\int R_\varepsilon(y_1,\ldots,y_p; dz_1,\ldots,dz_p)\,\varphi(z_1,\ldots,z_p)\\
&= \int \nu_\varepsilon(dr) \int_0^{1/\varepsilon} du\, \varphi(r(\mathbf{1}_{\{u\leq y_1\}} - \varepsilon y_1),\ldots,r(\mathbf{1}_{\{u\leq y_p\}} - \varepsilon y_p)).
\end{aligned} \tag{32}$$

Let π_γ be the measure on $]0,\infty[$ such that $\pi_\gamma(]a,\infty[) = a^{-\gamma}$ for every $a > 0$, and let

$$\Psi_\gamma(q) = \int \pi_\gamma(dr)\,(\mathrm{e}^{-qr} - 1 + qr) = \frac{\Gamma(2-\gamma)}{\gamma-1}\, q^\gamma.$$

We let $(Z(t,x), t \geq 0, x \geq 0)$ be the flow of continuous-state branching processes constructed in Section 2, with $\Psi = \Psi_\gamma$.

Theorem 3 *Under Assumption* (A), *for every $(x_1,\ldots,x_p) \in \mathcal{D}_p$,*

$$((F_t^\varepsilon(x_1),\ldots,F_t^\varepsilon(x_p)); t \geq 0) \xrightarrow[\varepsilon\to 0]{\text{(d)}} ((Z(t,x_1),\ldots,Z(t,x_p)); t \geq 0)$$

in the Skorokhod space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^p)$.

Proof: This is a simple consequence of Theorem 1, or rather of its proof. Indeed, we immediately see that the kernel $R_\varepsilon(y_1,\ldots,y_p;dz_1,\ldots,dz_p)$ coincides with $R^{(1/\varepsilon)}(y_1,\ldots,y_p;dz_1,\ldots,dz_p)$ defined in (12), provided we take $\nu^{(1/\varepsilon)}=\nu_\varepsilon$. From the observation at the beginning of the proof of Theorem 1, we see that Theorem 3 will follow if we can check that Assumption (H) holds in the present setting, that is if

$$\lim_{\varepsilon\to 0}(r\wedge r^2)\,\nu_\varepsilon(dr)=(r\wedge r^2)\,\pi_\gamma(dr) \tag{33}$$

in the sense of weak convergence in $\mathcal{M}_{\mathrm{F}}$.

In order to prove (33), first note that when $\varepsilon\to 0+$,

$$\int_{[0,\varepsilon]}x^2\nu(dx)=2\int_0^\varepsilon y\nu([y,1])dy-\varepsilon^2\nu([\varepsilon,1])\sim\frac{\gamma}{2-\gamma}\varepsilon^{2-\gamma}L(\varepsilon),$$

where the equivalence follows from Assumption (A) and a classical property of integrals of regularly varying functions. We immediately deduce that

$$\lim_{\varepsilon\to 0}r^2\,\nu_\varepsilon(dr)=r^2\,\pi_\gamma(dr) \tag{34}$$

in the sense of *vague* convergence in the space of Radon measures on $[0,\infty[$.

Next, note that

$$\int\nu_\varepsilon(dr)\,(r-r\wedge a)=\int_a^\infty dr\,\nu_\varepsilon([r,\infty[)=\varepsilon^\gamma L(\varepsilon)^{-1}\int_a^\infty dr\,\nu([r\varepsilon,1])\underset{a\to\infty}{\longrightarrow}0 \tag{35}$$

uniformly in $\varepsilon\in]0,\varepsilon_0]$. From (34) and (35) the family $((r\wedge r^2)\,\nu_\varepsilon(dr),0<\varepsilon<1)$ is tight for the weak topology in $\mathcal{M}_{\mathrm{F}}$. Together, with (34), this establishes the weak convergence (33). $\square$

Remark. Suppose that $(F_t,t\geq 0)$ is the flow of bridges associated with the Kingman coalescent, corresponding to $\Lambda=\delta_0$ in our notation (cf Section 4 in [5]). If we fix $(y_1,\ldots,y_p)\in\mathcal{D}_p^1$, the process $(F_t(y_1),\ldots,F_t(y_p))$ is a diffusion process in $\mathcal{D}_p^1$ with generator

$$\mathcal{A}g(x)=\frac{1}{2}\sum_{i,j=1}^p x_{i\wedge j}(1-x_{i\vee j})\frac{\partial^2 g}{\partial x_i\partial x_j}(x)$$

(see Theorem 3 in [5]). Putting $F_t^\varepsilon(x)=\frac{1}{\varepsilon}F_{\varepsilon t}(\varepsilon x)$, it is a simple matter to verify that our Theorem 3 still holds in that setting, provided we let $(Z(t,x),t\geq 0,x\geq 0)$ be the flow associated with the Feller diffusion ($\Psi(q)=\frac{1}{2}q^2$). Indeed, if we specialize to the case $p=1$ and if we let $(B_t,t\geq 0)$ be a standard linear Brownian motion, this is just saying that, for the Fisher-Wright diffusion $(X_t(x),t\geq 0)$ solving

$$dX_t=\sqrt{X_t(1-X_t)}\,dB_t\quad,\quad X_0=x,$$

the rescaled processes $X_t^\varepsilon:=\frac{1}{\varepsilon}X_{\varepsilon t}(\varepsilon x)$ converge in distribution as $\varepsilon\to 0$ towards the Feller diffusion $Y_t(x)$ solving

$$dY_t=\sqrt{Y_t}\,dB_t\quad,\quad Y_0=x.$$

We will now use Theorem 3 to derive precise information on the sizes of blocks in a Λ-coalescent (for $\Lambda(dr) = r^2\nu(dr)$) in small time. As previously, we denote by $\lambda_1(dr)$ the Lévy measure of the subordinator $(Z(1,x), x \geq 0)$. We have for every $q \geq 0$

$$\exp -x \int (1 - \mathrm{e}^{-qr})\lambda_1(dr) = \mathbb{E}(\exp -qZ(1,x)) = \exp -xu_1(q)$$

and the function $u_1(q)$ can be calculated from equation (3), with $\Psi = \Psi_\gamma$. It follows that

$$\int (1 - \mathrm{e}^{-qr})\lambda_1(dr) = (\Gamma(2-\gamma) + q^{1-\gamma})^{1/(1-\gamma)}$$

and in particular, the total mass of λ_1 is

$$(\Gamma(2-\gamma))^{1/(1-\gamma)}.$$

We will need the fact that λ_1 has no atoms. An easy way to derive this property is to argue by contradiction as follows. Suppose that $a > 0$ is an atom of λ_1. From the Lévy-Khintchin decomposition of $Z(t,x)$ (see the discussion after (4)), it follows that a is also an atom of the distribution of $Z(1,x)$, for every $x > 0$. By a simple scaling argument, for every $s > 0$, the image of $\lambda_1(dr)$ under the mapping $r \to s^{1/(\gamma-1)}r$ is $s^{1/(\gamma-1)}\lambda_s(dr)$. Therefore, for every $s \in]0,1[$, $s^{1/(\gamma-1)}a$ is also an atom of λ_s, hence of the distribution of $Z(s,x)$ for every $x > 0$. However, applying the Markov property to the process $(Z(t,1))_{t\geq 0}$ at time $1-s$, this would imply that for every $s \in]0,1[$, $s^{1/(\gamma-1)}a$ is an atom of the distribution of $Z(1,1)$, which is absurd.

We set $g(\varepsilon) = L(\varepsilon)^{-1}\varepsilon^{\gamma-1}$ for every $\varepsilon \in]0,\varepsilon_0]$.

Theorem 4 *Assume that* (A) *holds and let $\Lambda(dr) = r^2\nu(dr)$. For every $t \geq 0$ and $r \in [0,\infty]$, denote by $N_t(]0,r[)$ the number of blocks at time t with frequencies less than r in a Λ-coalescent started from the partition of $\mathbb{N}$ in singletons. Then,*

$$\sup_{x\in]0,\infty[} \left|\varepsilon N_{g(\varepsilon)}(]0,\varepsilon x[) - \lambda_1(]0,x[)\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in probability.

Again Theorem 4 is a generalization of classical results for the Kingman coalescent. In that case, one has

$$\sup_{x\in]0,\infty[} \left|\varepsilon N_\varepsilon(]0,\varepsilon x[) - 2(1 - 2\mathrm{e}^{-2x})\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$$

almost surely (cf Section 4.2 of [1]). This is consistent with Theorem 4 since in the case $\Psi(q) = \frac{1}{2}q^2$, (24) shows that

$$2(1 - 2\mathrm{e}^{-2x}) = \int_0^x 4\mathrm{e}^{-2x}\, dx = \lambda_1(]0,x[).$$

Proof: By the results of [4] recalled at the beginning of subsection 4.2, we know that, for each $t \geq 0$ the collection $(N_t(]0,r[), r \geq 0)$ has the same distribution as

$$\Big(\sum_{0<u<1} \mathbf{1}_{\{0<F_t(u)-F_t(u-)\leq r\}}\,,\ r \geq 0\Big),$$

where $(F_t, t \geq 0)$ is the generalized Fleming-Viot process associated with ν. It then follows from our definitions that

$$\Big(\varepsilon\, N_{g(\varepsilon)}(]0, x\varepsilon[), x \geq 0\Big) \overset{\text{(d)}}{=} \Big(\varepsilon \sum_{0<u<1/\varepsilon} \mathbf{1}_{\{0<F_1^\varepsilon(u)-F_1^\varepsilon(u-)\leq x\}}\,,\ x \geq 0\Big).$$

By combining Theorem 3 and Lemma 2, we get that

$$\varepsilon \sum_{0<u<1/\varepsilon} \delta_{F_t^\varepsilon(u)-F_t^\varepsilon(u-)} \underset{\varepsilon\to 0}{\longrightarrow} \lambda_1 \tag{36}$$

in probability in $\mathcal{M}_{\mathrm{R}}$. This is indeed the same result as Theorem 2 in our present setting. The preceding convergence is not quite sufficient to conclude: Recalling that λ_1 has no atoms and using Dini's theorem, we see that the statement of the theorem will follow if we can prove that the convergence (36) holds in the sense of weak convergence in the space $\mathcal{M}_{\mathrm{F}}$. To get this strengthening of (36), it suffices to prove the convergence of the total masses. Therefore the proof of Theorem 4 will be complete once we have established the following lemma.

Lemma 3 *We have*

$$\lim_{\varepsilon\to 0} \varepsilon\, N_{g(\varepsilon)}(]0,\infty[) = \lambda_1(]0,\infty[) = (\Gamma(2-\gamma))^{1/(1-\gamma)},$$

in probability.

Remark. The recent paper [2] gives closely related results that were obtained independently of the present work.

Proof: Write $N_t = N_t(]0,\infty[)$ to simplify notation. Then, for every $t \geq 0$ and $x \in]0,1]$, we have

$$\mathbb{E}[x^{N_t}] = \mathbb{P}[F_t(x) = 1]$$

(cf formula (8) in [5]). By exchangeability,

$$\mathbb{P}[F_t(x) = 1] = \mathbb{P}[F_t(x) = F_t(1)] = \mathbb{P}[F_t(1-x) = 0].$$

Hence, for $x \in]0,1[$,

$$\mathbb{P}[F_t(x) = 0] = \mathbb{E}[(1-x)^{N_t}],$$

and it follows that

$$\mathbb{P}[F_1^\varepsilon(x) = 0] = \mathbb{E}[(1-\varepsilon x)^{N_{g(\varepsilon)}}].$$

From the convergence in distribution in Theorem 3, we have for every $x > 0$,

$$\limsup_{\varepsilon\to 0} \mathbb{P}[F_1^\varepsilon(x) = 0] \leq \mathbb{P}[Z(1,x) = 0] = \exp -x\lambda_1(]0,\infty[).$$

We have thus obtained that, for every $x > 0$,

$$\limsup_{\varepsilon\to 0} \mathbb{E}[(1-\varepsilon x)^{N_{g(\varepsilon)}}] \leq \exp -x\lambda_1(]0,\infty[).$$

By standard arguments, this implies that for every $\eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}[\varepsilon N_{g(\varepsilon)} < \lambda_1(]0,\infty[) - \eta] = 0. \tag{37}$$

To complete the proof, we need to verify that we have also, for every $\eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}[\varepsilon N_{g(\varepsilon)} > \lambda_1(]0,\infty[) + \eta] = 0. \tag{38}$$

Fome now on, we fixe $\eta > 0$ and we prove (38). We will use a different method based on the knowledge of the law of the process of the number of blocks in a Λ-coalescent. For every integer $n \geq 1$, write N_t^n for the number of blocks at time t in a Λ-coalescent started initially with n blocks. Then according to Pitman [16] (Section 3.6), the process $(N_t^n, t \geq 0)$ is a time-homogeneous Markov chain with values in $\{1, 2, \ldots, n\}$, with only downward jumps, such that for $2 \leq k \leq b \leq n$, the rate of jumps from b to $b - k + 1$ is

$$\alpha_{b,k} = \begin{pmatrix} b \\ k \end{pmatrix} \int_{]0,1]} x^k (1-x)^{b-k} \nu(dx).$$

The total rate of jumps from b is thus

$$\alpha_b = \sum_{k=2}^{b} \alpha_{b,k} = \int_{]0,1]} (1 - (1-x)^b - b(1-x)^{b-1}) \nu(dx).$$

Lemma 4 *Under Assumption* (A), *we have*

$$\lim_{b\to+\infty} (b^\gamma L(1/b))^{-1}\, \alpha_b = \Gamma(2-\gamma)$$

and, for every integer $k \geq 2$,

$$\lim_{b\to+\infty} (b^\gamma L(1/b))^{-1}\, \alpha_{b,k} = \frac{\gamma\Gamma(k-\gamma)}{k!}.$$

We leave the easy proof to the reader. Note that

$$\sum_{k=2}^{\infty} \frac{\gamma\Gamma(k-\gamma)}{k!} = \Gamma(2-\gamma). \tag{39}$$

This is easily proved by using the definition of the function Γ and then an integration by parts. Similarly, we have

$$\sum_{k=2}^{\infty} \frac{\gamma\Gamma(k-\gamma)}{k!\Gamma(2-\gamma)} (k-1) = \frac{1}{\gamma - 1}. \tag{40}$$

Let us fix $\rho \in]0, 1/8[$ sufficiently small so that

$$(\Gamma(2-\gamma)^{1/(1-\gamma)} + \eta)^{1-\gamma} < (1 - 6\rho)\Gamma(2-\gamma).$$

Thanks to (39) and (40), we may choose an integer $K \geq 2 \vee \varepsilon_0^{-1}$ sufficiently large so that

$$\frac{1}{\Gamma(2-\gamma)} \sum_{k=2}^{K} \frac{\gamma\Gamma(k-\gamma)}{k!} \geq 1 - \rho$$

and

$$\sum_{k=2}^{K} \frac{\gamma\Gamma(k-\gamma)}{k!\Gamma(2-\gamma)}(k-1) \geq \frac{1}{\gamma-1} - \rho. \tag{41}$$

Then, for every $k \in \{2, 3, \ldots, K\}$, we may choose $\rho_k \in]0, \gamma\Gamma(k-\gamma)/k![$ sufficiently small so that

$$\frac{1}{\Gamma(2-\gamma)} \sum_{k=2}^{K} (k-1)\rho_k < \rho. \tag{42}$$

Now set

$$\beta_{b,k} = \Big(\frac{\gamma\Gamma(k-\gamma)}{k!} - \rho_k\Big) b^\gamma L(\frac{1}{b})$$

for $b \geq K$ and $k \in \{2, 3, \ldots, K\}$. We also put

$$\beta_b = \sum_{k=2}^{K} \beta_{b,k}.$$

Notice that

$$\beta_p = \sum_{k=2}^{K} \Big(\frac{\gamma\Gamma(k-\gamma)}{k!} - \rho_k\Big) b^\gamma L(\frac{1}{b}) \geq (1-2\rho)\Gamma(2-\gamma)\, b^\gamma L(\frac{1}{b}). \tag{43}$$

By Lemma 4, we can choose an integer $B \geq 2K$ sufficiently large so that, for every $b \geq B - K$, $b' \in \{b, b+1, \ldots, b+K\}$ and $k \in \{2, \ldots, K\}$, one has

$$\beta_{b',k} \leq \alpha_{b,k}. \tag{44}$$

Denote by $(U_t^n)_{t\geq 0}$ the continuous-time Markov chain with values in $\mathbb{N}$, with initial value $U_0^n = n$, which is absorbed in the set $\{1, \ldots, B-1\}$ and has jump rate $\beta_{b,k}$ from b to $b-k+1$ when $b \geq B$ and $k \in \{2, 3, \ldots, K\}$. Fix $n \geq B$. Then thanks to inequality (44), we can couple the Markov chains $(U_t^n)_{t\geq 0}$ and $(N_t^n)_{t\geq 0}$ in such a way that

$$U_t^n \geq N_t^n \quad , \text{ for every } t \leq T_B^n := \inf\{s : U_s^n < B\}.$$

Now it is easy to describe the behavior of the Markov chain (U_t^n). Note that for $k \in \{2, \ldots, K\}$ and $b \geq K$ the ratio $\beta_{b,k}/\beta_b$ does not depend on b. Then denote by $S_i = \xi_1 + \cdots + \xi_i$ $(i = 0, 1, 2, \ldots)$ a discrete random walk on the nonnegative integers started from the origin and with jump distribution

$$\mathbb{P}[\xi_i = k-1] = \frac{\beta_{b,k}}{\beta_b} = \frac{(\gamma\Gamma(k-\gamma)/k!) - \rho_k}{\sum_{\ell=2}^{K}((\gamma\Gamma(\ell-\gamma)/\ell!) - \rho_\ell)}\ , \qquad 2 \leq k \leq K.$$

From (41) and (42) we have

$$\mathbb{E}[\xi_i] \geq \frac{1}{\gamma-1} - 2\rho. \tag{45}$$

Let $\mathbf{e}_0, \mathbf{e}_1, \ldots$ be a sequence of independent exponential variables with mean 1, which are also independent of the random walk $(S_i)_{i\geq 0}$. We can construct the Markov chain (U_t^n) by setting:

$$\begin{aligned} U_t^n &= n && \text{if } 0 \leq t < \frac{\mathbf{e}_0}{\beta_n} \\ U_t^n &= n - S_1 && \text{if } \frac{\mathbf{e}_0}{\beta_n} \leq t < \frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} \end{aligned}$$

and more generally,

$$U_t^n = n - S_p \qquad \text{if } \frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_{p-1}}{\beta_{n-S_{p-1}}} \le t < \frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_p}{\beta_{n-S_p}}$$

provided $p \le p_B^n := \inf\{i : n - S_i < B\}$.

Recall that our goal is to prove (38). To this end, note that for $a > B$,

$$\mathbb{P}[N_{g(\varepsilon)}^n > a] \le \mathbb{P}[U_{g(\varepsilon)}^n > a] \le \mathbb{P}\Big[g(\varepsilon) \le \frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_{p_a^n}}{\beta_{n-S_{p_a^n}}}\Big] \tag{46}$$

where $p_a^n := \inf\{i : n - S_i < a\}$.

Lemma 5 *For $\varepsilon > 0$ set $a(\varepsilon) = (\lambda_1(]0,\infty[) + \eta)/\varepsilon$. Then,*

$$\lim_{\varepsilon\to 0}\Big(\sup_{n\ge a(\varepsilon)} \mathbb{P}\Big[g(\varepsilon) \le \frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_{p_{a(\varepsilon)}^n}}{\beta_{n-S_{p_{a(\varepsilon)}^n}}}\Big]\Big) = 0.$$

The desired bound (38) immediately follows from Lemma 5. Indeed standard properties of Λ-coalescent give

$$\mathbb{P}[\varepsilon N_{g(\varepsilon)} > \lambda_1(]0,\infty[) + \eta] = \lim_{n\uparrow\infty} \uparrow \mathbb{P}[\varepsilon N_{g(\varepsilon)}^n > \lambda_1(]0,\infty[) + \eta] = \lim_{n\uparrow\infty} \uparrow \mathbb{P}[N_{g(\varepsilon)}^n > a(\varepsilon)]$$

and by combining (46) and Lemma 5, we see that the latter quantity tends to 0 as $\varepsilon \to 0$.

Proof of Lemma 5: By (43), we have for $a > B$,

$$\frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_{p_a^n}}{\beta_{n-S_{p_a^n}}} \le ((1-2\rho)\Gamma(2-\gamma))^{-1} \sum_{i=0}^{p_a^n} \frac{\mathbf{e}_i}{(n-S_i)^\gamma L(\frac{1}{n-S_i})}. \tag{47}$$

Note that

$$\mathbb{E}\Big[\sum_{i=0}^{p_a^n} \frac{\mathbf{e}_i}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} \;\Big|\; S_i, i \ge 0\Big] = \sum_{i=0}^{p_a^n} \frac{1}{(n-S_i)^\gamma L(\frac{1}{n-S_i})}.$$

Let $m \ge 2$ be an integer. For $a > B$ and $n > ma$, a trivial bound shows that

$$a^{\gamma-1} L(\frac{1}{a}) \sum_{i=0}^{p_{ma}^n} \frac{1}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} \le a^{\gamma-1} L(\frac{1}{a}) \sum_{j=[ma]-K}^{\infty} \frac{1}{j^\gamma L(\frac{1}{j})}$$

and the right-hand side tends to 0 as $m \to \infty$, uniformly in $a > B$. On the other hand, an easy argument using the law of large numbers for the sequence $(S_i)_{i\ge 0}$ shows that, for each fixed $m \ge 2$,

$$\lim_{a\to\infty}\Big(\sup_{n>ma} \mathbb{E}\Big[\Big|a^{\gamma-1} L(\frac{1}{a}) \sum_{i=p_{ma}^n}^{p_a^n} \frac{1}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} - \frac{1}{\mathbb{E}[\xi_1]} \int_1^m \frac{dx}{x^\gamma}\Big|\Big]\Big) = 0.$$

Now recall the bound (45) for $\mathbb{E}[\xi_1]$. It follows from the preceding considerations that

$$\lim_{a\to\infty}\Big(\sup_{n>a} \mathbb{P}\Big[a^{\gamma-1} L(\frac{1}{a}) \sum_{i=0}^{p_a^n} \frac{1}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} > \frac{1}{1-3\rho}\Big]\Big) = 0. \tag{48}$$

Now we can also get an estimate for the conditional variance

$$\begin{aligned} \mathrm{var}\Big(\sum_{i=0}^{p_a^n} \frac{\mathbf{e}_i}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} \,\Big|\, S_i, i \geq 0\Big) &= \sum_{i=0}^{p_a^n} \frac{1}{(n-S_i)^{2\gamma} L(\frac{1}{n-S_i})^2} \\ &\leq \sum_{j=[a-K]}^{n} \frac{1}{j^{2\gamma} L(\frac{1}{j})^2} \\ &\leq C a^{1-2\gamma} L(\frac{1}{a})^{-2} \end{aligned}$$

for some constant C independent of a and n. From this estimate, (48) and an application of the Bienaymé-Cebycev inequality, we get

$$\lim_{a\to\infty} \Big(\sup_{n>a} \mathbb{P}\Big[a^{\gamma-1} L(\frac{1}{a}) \sum_{i=0}^{p_a^n} \frac{\mathbf{e}_i}{(n-S_i)^\gamma L(\frac{1}{n-S_i})} > \frac{1}{1-4\rho}\Big]\Big) = 0. \qquad (49)$$

Recalling (47), we arrive at

$$\lim_{\varepsilon\to 0} \Big(\inf_{n\geq a(\varepsilon)} \mathbb{P}\Big[\frac{\mathbf{e}_0}{\beta_n} + \frac{\mathbf{e}_1}{\beta_{n-S_1}} + \cdots + \frac{\mathbf{e}_{p^n_{a(\varepsilon)}}}{\beta_{n-S_{p^n_{a(\varepsilon)}}}} \leq \frac{a(\varepsilon)^{1-\gamma} L(\frac{1}{a(\varepsilon)})^{-1}}{(1-2\rho)(1-4\rho)\Gamma(2-\gamma)}\Big]\Big) = 1.$$

However, from our choice of ρ, we have for ε sufficiently small

$$g(\varepsilon) > \frac{a(\varepsilon)^{1-\gamma} L(\frac{1}{a(\varepsilon)})^{-1}}{(1-2\rho)(1-4\rho)\Gamma(2-\gamma)},$$

and this completes the proof. □

Remark. It is rather unfortunate that the simple argument we used to derive (37) does not apply to (38). On the other hand, it is interesting to observe that the techniques involved in our proof of (38) would become more complicated if we were trying to use them to get (37).

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