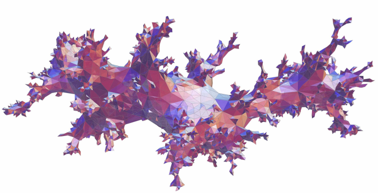
Compact and non-compact models of random geometry

Jean-François Le Gall
(partly joint with Armand Riera)

Probability and the City Seminar, November 2020

European Research Council



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Goal of the lecture: Describe how the basic models of random geometry (obtained as scaling limits of random graphs embedded in the sphere) can be constructed from random trees equipped with Brownian labels.

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Non-compact models:

- The Brownian plane
- The Brownian half-plane
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Our construction allows us to study relations between different models.

These models can also be viewed as quantum surfaces: cf. the work of Sheffield, Miller, Gwynne, Holden, Sun, etc.

1. Brownian spheres and Brownian disks

Definition

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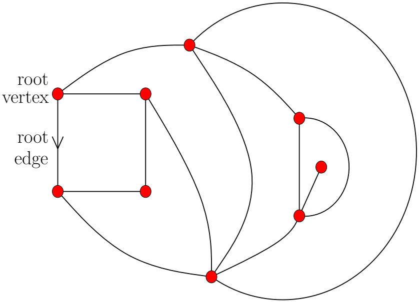

A rooted quadrangulation with 7 faces

Faces = connected components of the complement of edges

p-angulation:

- each face is incident to p half-edges

$p = 3$: triangulation

$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

The Brownian sphere (or Brownian map)

Let M_n be uniform over $\mathbb{M}_n^4 = \{$rooted quadrangulations with n faces$\}$.

- $V(M_n)$ vertex set of M_n
- $d_{\rm gr}$ graph distance on $V(M_n)$
- π_n uniform probability measure on $V(M_n)$

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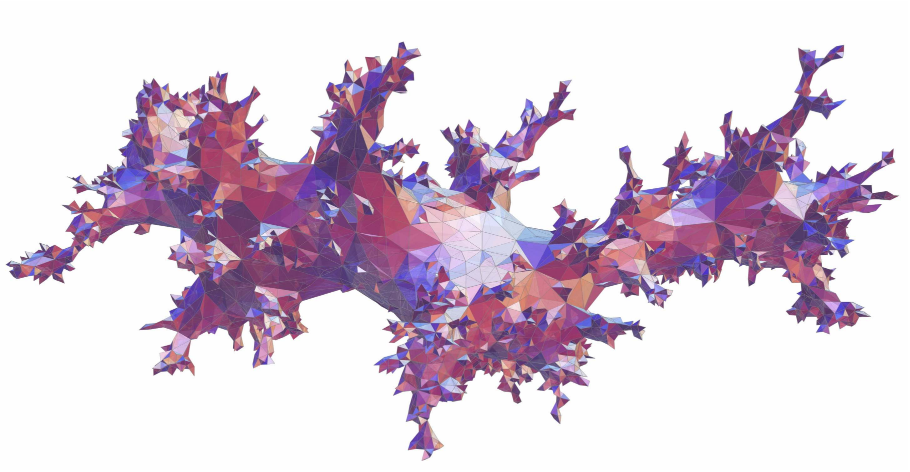
Theorem (LG 2013, Miermont 2013)

$$\Big(V(M_n), (\frac{9}{8n})^{1/4}\, d_{\mathrm{gr}}, \pi_n\Big) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\mathbf{m}_\infty, D, Vol)$$

in the Gromov-Hausdorff-Prokhorov sense. The limit $(\mathbf{m}_\infty, D, Vol)$ is a random compact metric measure space called the Brownian sphere (or Brownian map).

Remark A similar result holds for random triangulations and for much more general random planar maps, with the same limit (universality of the Brownian sphere). One can even randomize edge lengths: assigning i.i.d. lengths to edges does not change the scaling limit (Curien-LG, Ann. Sci. ENS 2019).

A large triangulation of the sphere (simulation: N.Curien)
(an approximation of the Brownian sphere)



Two properties of the Brownian sphere

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad\qquad a.s.$$

Uniform control of the volume of balls: for $\delta > 0$,

$$c_\delta(\omega)\, r^{4+\delta} \leq Vol(B(x,r)) \leq c'_\delta(\omega)\, r^{4-\delta},$$

with (random) constants $c_\delta(\omega), c'_\delta(\omega)$ independent of $x \in \mathbf{m}_\infty$ and $r \in (0,1]$.

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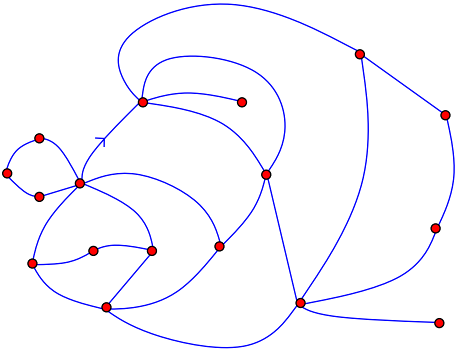
Theorem (topological type, LG-Paulin 2008)

Almost surely, $(\mathbf{m}_\infty, D)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Quadrangulations with a boundary



A quadrangulation with a boundary of size 14.

A quadrangulation with a boundary is a rooted planar map M such that

- The root face (to the left of the root edge) has an arbitrary even degree. (In the figure, the root face is the "external" face)
- All other faces have degree 4.

The degree of the root face is the boundary size of M.

Boltzmann quadrangulations with a boundary

For $p \geq 1$, let $\mathbb{M}^{4,p}$ be the set of all (rooted) quadrangulations with a boundary of size $2p$.
If $Q \in \mathbb{M}^{4,p}$, let $|Q|$ stand for the number of faces of Q

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A Boltzmann quadrangulation with boundary size $2p$ is a random quadrangulation with a boundary $\mathbf{Q}_p$ such that :

$$\mathbb{P}(\mathbf{Q}_p = Q) = c_p \, 12^{-n} \text{ for every } Q \in \mathbb{M}^{4,p} \text{ with } |Q| = n$$

here $c_p > 0$ is the appropriate normalizing constant (depending on p).

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This makes sense because

$$\#\{Q \in \mathbb{M}^{4,p} : |Q| = n\} \underset{n\to\infty}{\approx} c'_p\, n^{-5/2}\, 12^n$$

Convergence to the Brownian disk

Recall that $\mathbf{Q}_p$ is a Boltzmann quadrangulation with boundary size $2p$. Equip the vertex set $V(\mathbf{Q}_p)$ with the graph distance d_{gr} and the counting measure m_p.

Theorem (Bettinelli and Miermont)

$$\Big(V(\mathbf{Q}_p), (\frac{3}{2p})^{1/2} d_{\mathrm{gr}}, \frac{2}{p^2}\, m_p\Big) \xrightarrow[p\to\infty]{(\mathrm{d})} (\mathbb{D}, \Delta, \mathrm{Vol})$$

in the Gromov-Hausdorff-Prokhorov sense. The limit $(\mathbb{D}, \Delta, \mathrm{Vol})$ *is a random compact metric measure space called the free Brownian disk with perimeter* 1*.*

(A similar result for the simple boundary case has been obtained by Gwynne and Miller, see also Albenque-Holden-Sun for triangulations)

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By scaling one can define the free Brownian disk with perimeter r. By conditioning on $\mathrm{Vol}(\mathbb{D}) = v$, one defines the Brownian disk with perimeter r and volume v.

Properties of the Brownian disk

Fact (Bettinelli): The free Brownian disk $\mathbb{D}$ (with perimeter $r > 0$) is homeomorphic to the closed unit disk.

Hence one can make sense of the boundary $\partial\mathbb{D}$.

Remark. Similarly as Brownian spheres, Brownian disks can be viewed as Liouville quantum gravity surfaces: recent work of Miller, Sheffield, Gwynne, Holden, ...

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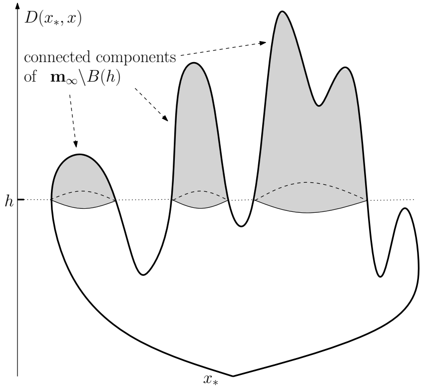
Special subsets of the Brownian sphere $(\mathbf{m}_\infty, D)$ can be identified as Brownian disks.

Brownian disks in the Brownian sphere



For $h > 0$, let $B(h)$ be the ball of radius h centered at the distinguished point x_* in the Brownian sphere $(\mathbf{m}_\infty, D)$

Let $\mathcal{D}_j$, $j \in J$ be the connected components of $\mathbf{m}_\infty \backslash B(h)$. We can equip each $\mathcal{D}_j$ with its intrinsic metric $D^{(j)}$

Vol : volume measure on $\mathbf{m}_\infty$

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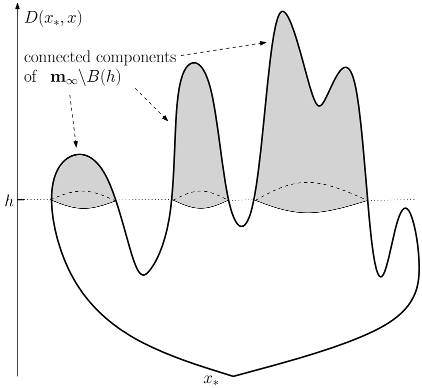

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Theorem

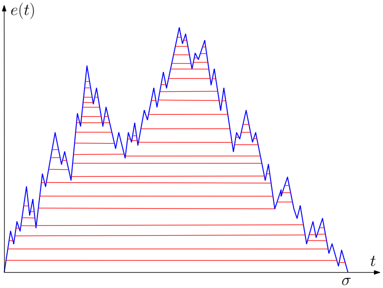
For every j, the limit

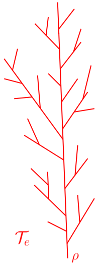
$$|\partial \mathcal{D}_j| := \lim_{\varepsilon \to 0} \varepsilon^{-2} \mathrm{Vol}\{x \in \mathcal{D}_j : D(x, \partial \mathcal{D}_j) < \varepsilon\}$$

exists, and, conditionally on $(|\partial \mathcal{D}_j|, \mathrm{Vol}(\mathcal{D}_j))_{j \in J}$*, the metric spaces* $(\bar{\mathcal{D}}_j, D^{(j)})$ *are independent Brownian disks with the prescribed volumes and perimeters.*

2. The construction of the Brownian sphere

A key ingredient: The Brownian tree (Aldous' CRT), or tree coded by a Brownian excursion under $\mathbf{n}_+(\mathrm{d}e)$ (the Itô excursion measure).





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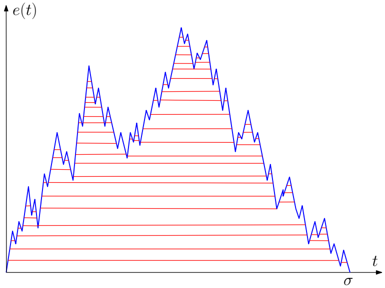

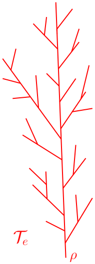

Informally, glue $s, t \in [0, \sigma]$ if they correspond to the ends of a horizontal chord drawn below the graph of e.

Formally, say that $s \sim t$ iff $e(s) = e(t) = \min_{u \in [s \wedge t, s \vee t]} e(u)$.
The Brownian tree is $\mathcal{T}_e := [0, \sigma] / \sim$, with the metric induced by

$$d_e(s, t) = e(s) + e(t) - 2 \min_{u \in [s \wedge t, s \vee t]} e(u).$$

The Brownian tree

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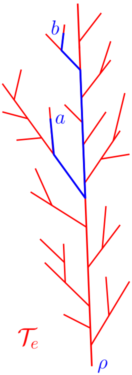
Then $(\mathcal{T}_e, d_e)$ is a compact $\mathbb{R}$-tree

(means that two points of $\mathcal{T}_e$ are connected by a unique arc $[[a,b]]$, which is isometric to a line segment — $d(a,b)$ is the length of the blue path connecting a to b)



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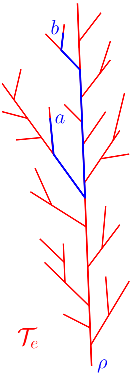

Let $p_e : [0,\sigma] \to \mathcal{T}_e = [0,\sigma]/\sim$ be the canonical projection:

- $\mathcal{T}_e$ is rooted at $\rho := p_e(0) = p_e(\sigma)$
- the volume measure is the push forward of Lebesgue measure under p_e.
- the Brownian tree $\mathcal{T}_e$ also inherits a cyclic ordering from the projection p_e (it makes sense to explore the tree "clockwise" from one point to another)

Brownian motion indexed by the Brownian tree

Conditionally on $\mathcal{T}_e$, $Z = (Z_a)_{a\in\mathcal{T}_e}$ is the centered Gaussian process characterized by:

- $Z_\rho = 0$
- $\mathbb{E}[(Z_a - Z_b)^2] = d_e(a,b)$ for every $a, b \in \mathcal{T}_e$

(Technical difficulty: Z is a random process indexed by a random set. Since $\mathcal{T}_e = [0,\sigma]/\sim$, one can as well define Z as indexed by $[0,\sigma]$ — this is the Brownian snake construction)

Fact: Z has continuous sample paths.

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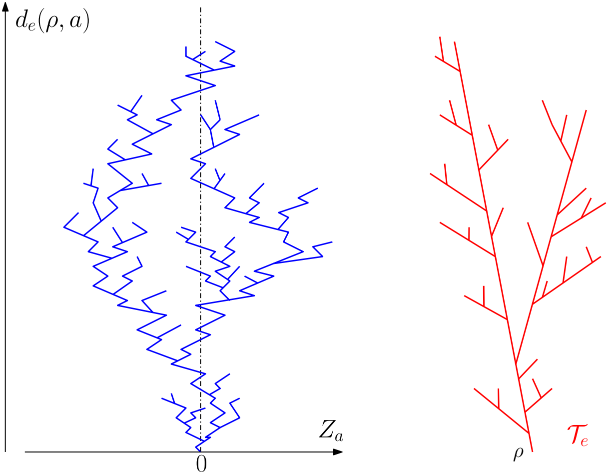
One views Z_a as a Brownian label assigned to $a \in \mathcal{T}_e$. When moving along a line segment of $\mathcal{T}_e$, labels evolve like linear Brownian motion.

Motivations for studying $\mathcal{T}_e$ and $(Z_a)_{a \in \mathcal{T}_e}$: These objects arise in a number of asymptotics for discrete models, in combinatorics, interacting particle systems, statistical physics, etc.

Brownian motion indexed by the Brownian tree 2



The collection $(Z_a)_{a \in \mathcal{T}_e}$ forms a "tree of Brownian paths" whose genealogy is prescribed by $\mathcal{T}_e$.
Z_a is also interpreted as a "label" assigned to vertex $a \in \mathcal{T}_e$.

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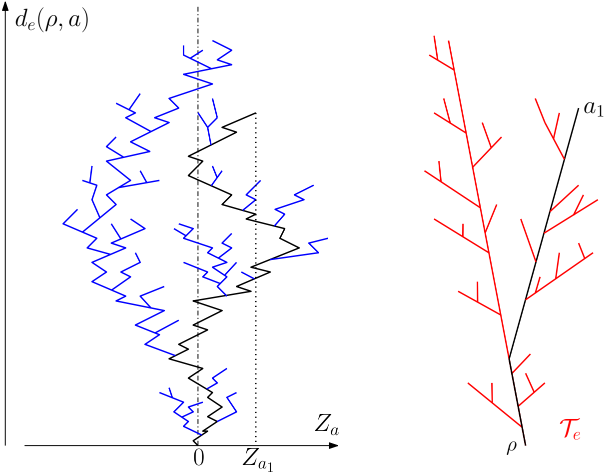

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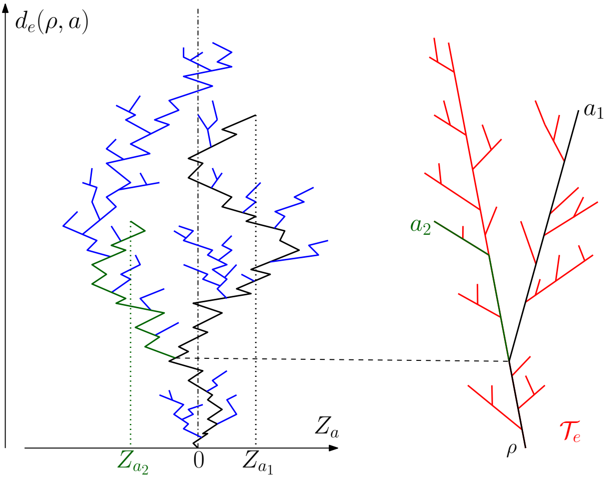

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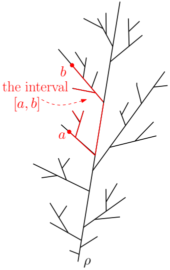
The construction of the Brownian sphere

$\mathcal{T}_e$ is the Brownian tree, $(Z_a)_{a\in\mathcal{T}_e}$ Brownian motion indexed by $\mathcal{T}_e$ (Two levels of randomness!).

Set, for every $a, b \in \mathcal{T}_e$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the "interval" from a to b corresponding to the cyclic ordering on $\mathcal{T}_e$ (vertices visited when going from a to b in clockwise order around the tree).



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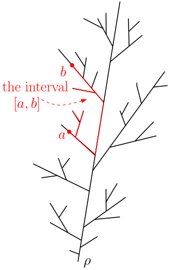

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Then let D be the maximal symmetric function on $\mathcal{T}_e \times \mathcal{T}_e$ that is bounded above by D^0 and satisfies the triangle inequality. Also set

$$a \approx b \text{ if and only if } D(a,b) = 0 \quad \text{(equivalent to } D^0(a,b) = 0\text{)}.$$

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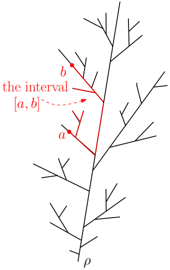

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Definition

The free Brownian sphere $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_e / \approx$, which is equipped with the distance induced by D.

To get the "standard" Brownian sphere, condition on $\sigma (= \mathrm{Vol}(\mathcal{T}_e)) = 1$.

Summary and interpretation

Starting from the Brownian tree $\mathcal{T}_e$, with Brownian labels $Z_a, a \in \mathcal{T}_e$,
$\rightarrow$ Identify two vertices $a, b \in \mathcal{T}_e$ if $D^\circ(a, b) = 0$, meaning that:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree (in clockwise or in counterclockwise order) visiting only vertices with label greater than or equal to $Z_a = Z_b$.

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$\rightarrow$ conn.comp. of complement of a ball = excursions of Z above a level
$\rightarrow$ Brownian disks correspond to excursions of the process Z !!

A different approach to the Brownian sphere

Miller, Sheffield (2015-2016) have developed a program aiming to relate the Brownian sphere with Liouville quantum gravity:

- new construction of the Brownian sphere using the Gaussian free field and the random growth process called Quantum Loewner Evolution (an analog of the celebrated SLE processes studied by Lawler, Schramm and Werner)
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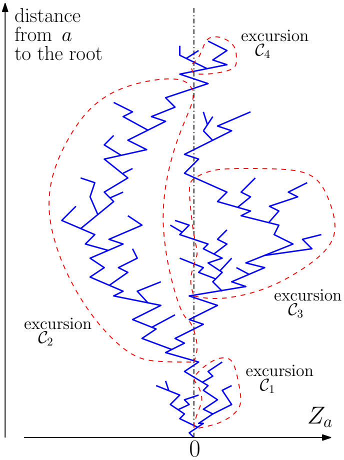
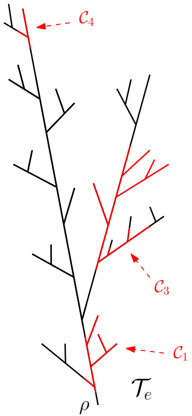
More recently: the Miller-Sheffield construction has been simplified by a direct construction of the Liouville quantum gravity metric from the Gaussian free field (Gwynne-Miller 2019 after the work of several other authors).

3. Excursions of Brownian motion indexed by the Brownian tree





Recall:

$\mathcal{T}_e$ Brownian tree

$(Z_a)_{a\in\mathcal{T}_e}$ Brownian motion indexed by $\mathcal{T}_e$

Let $(\mathcal{C}_i)_{i\in I}$ be the connected components of $\{a \in \mathcal{T}_e : Z_a \neq 0\}$.

The excursions of Z are $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$, $i \in I$, viewed as $\mathbb{R}$-trees equipped with continuous labels (here $\bar{\mathcal{C}}_i$ is the closure of $\mathcal{C}_i$)

The law of excursions

For each "excursion" $\big(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i}\big)$, one can define its boundary size

$$|\partial\mathcal{C}_i| = \lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathrm{Vol}\big(\{a\in\mathcal{C}_i : |Z_a| < \varepsilon\}\big)$$

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Theorem (Abraham-LG, JEMS 2018)

There exists a σ-finite measure $\mathbb{M}$ (with appropriate scaling properties) on the space of compact $\mathbb{R}$-trees $\mathcal{T}$ equipped with a volume measure $\mathrm{Vol}(\cdot)$ and with labels $(z(a))_{a\in\mathcal{T}}$, such that, conditionally on $(|\partial\mathcal{C}_i|)_{i\in I}$,

- *the "excursions" $(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i})$, $i\in I$ are independent*
- *for every $i\in I$, the distribution of $(\bar{\mathcal{C}}_i, (Z_a)_{a\in\bar{\mathcal{C}}_i})$ knowing $|\partial\mathcal{C}_i| = r$ is*

$$\mathbb{M}^{(r)} := \mathbb{M}\big(\cdot \mid \Sigma = r\big)$$

where $\Sigma = \lim_{\varepsilon\to 0}\varepsilon^{-2}\,\mathrm{Vol}\big(\{a\in\mathcal{T} : |z(a)| < \varepsilon\}\big)$
(the limit exists $\mathbb{M}$ a.e.)

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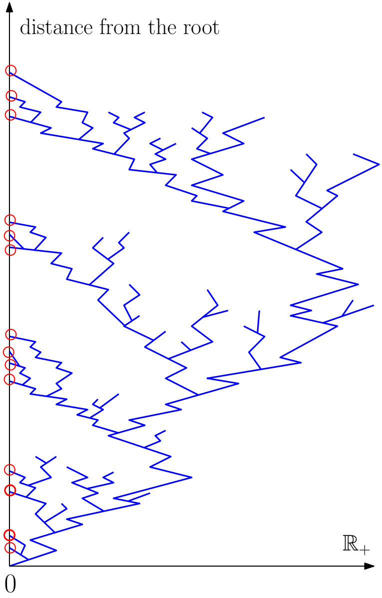
(the limit exists $\mathbb{M}$ a.e.)

We can write $\mathbb{M} = \mathbb{M}_+ + \mathbb{M}_-$ and interpret $\mathbb{M}_+$ as a measure on "trees of Brownian paths in $[0,\infty)$". One similarly defines $\mathbb{M}_+^{(r)}$.

The tree of paths under $\mathbb{M}_+$



Under $\mathbb{M}_+$, we now have a tree of nonnegative "Brownian paths" all starting from 0, which stay positive during some interval $(0, \varepsilon]$ and are stopped at the time when they return to 0, if they do return to 0.

Informally, the boundary size Σ counts the number of paths that return to 0 (circled points on the figure).

Explicit formulas under $\mathbb{M}_+$

Joint distribution of boundary size and volume: The distribution of the pair $(\Sigma, \mathrm{Vol}(\mathcal{T}))$ under $\mathbb{M}_+$ has density

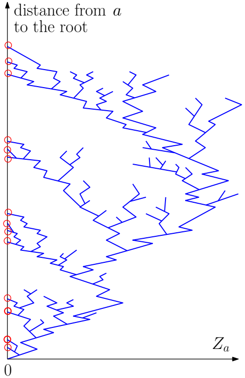
$$f(s,v) = \frac{\sqrt{3}}{2\pi}\,\sqrt{s}\,v^{-5/2}\,\exp\Big(-\frac{s^2}{2v}\Big)$$

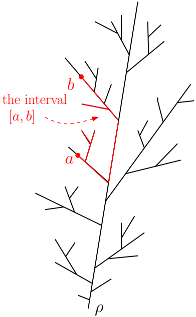
As a consequence, for every $s > 0$, the density of $\mathrm{Vol}(\mathcal{T})$ under $\mathbb{M}_+^{(s)} := \mathbb{M}_+(\cdot \mid \Sigma = s\Big)$ is

$$g_s(v) = \frac{1}{\sqrt{2\pi}}\,s^3\,v^{-5/2}\,\exp\Big(-\frac{s^2}{2v}\Big)$$

(this is the asymptotic distribution of the volume of a Boltzmann quadrangulation with a boundary of size n when $n \to \infty$ and the volume is rescaled by n^{-2})

4. The construction of Brownian disks under $\mathbb{M}_+$





Under $\mathbb{M}_+^{(r)} = \mathbb{M}_+(\cdot \mid \Sigma = r)$,

- we have an $\mathbb{R}$-tree $\mathcal{T}$
- and nonnegative labels $z(a)$, $a \in \mathcal{T}$

Also cyclic order structure on $\mathcal{T}$ that allows one to define intervals $[a, b]$ (informally, points visited when going from a to b around the tree).

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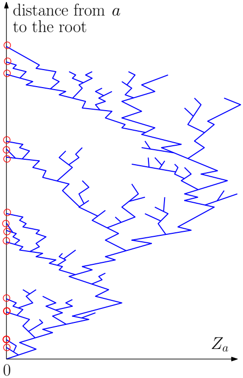

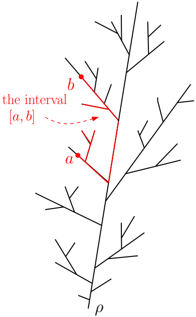

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Also cyclic order structure on $\mathcal{T}$ that allows one to define intervals $[a, b]$ (informally, points visited when going from a to b around the tree).

For $a, b \in \mathcal{T}$, set

$$D^\circ(a,b) = z(a) + z(b) - 2\max\Big\{\min_{c\in[a,b]} z(c), \min_{c\in[b,a]} z(c)\Big\}.$$

Imitating the construction of the Brownian sphere would require identifying a and b if $D^\circ(a,b) = 0$. But here this would mean identifying all boundary points (all c such that $z(c) = 0$)!

Constructing free Brownian disks

Recall $D^\circ(a,b) = z(a) + z(b) - 2\max\Big\{\min_{c\in[a,b]} z(c), \min_{c\in[b,a]} z(c)\Big\}$.

Set $\partial\mathcal{T} = \{c \in \mathcal{T} : z(c) = 0\}$, $\mathcal{T}^\circ = \mathcal{T}\backslash\partial\mathcal{T}$ and, for $a, b \in \mathcal{T}^\circ$,

$$\Delta^\circ(a,b) = \begin{cases} D^\circ(a,b) & \text{if } \max\Big\{\min_{[a,b]} z(c), \min_{[b,a]} z(c)\Big\} > 0, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\Delta(a,b) = \inf_{\substack{a=a_0,a_1,\ldots,a_k=b \\ a_i\in\mathcal{T}^\circ}} \sum_{i=1}^{k} \Delta^\circ(a_{i-1}, a_i).$$

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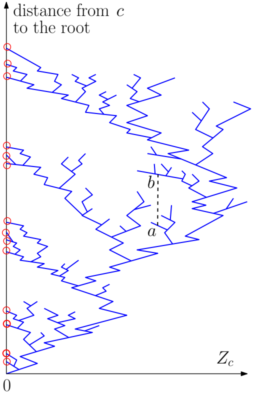
Theorem (LG, Ann.IHP 2019)

Under $\mathbb{M}_+^{(r)}$, $(\Delta(a,b), a, b \in \mathcal{T}^\circ)$ has a continuous extension to $\mathcal{T}\times\mathcal{T}$, which is a pseudo-metric on $\mathcal{T}$. The associated quotient space $\mathbb{D}$ equipped with the distance induced by Δ is a free Brownian disk with perimeter r.

Remark: $\partial\mathbb{D}$ corresponds to $\partial\mathcal{T}$ in the quotient space.

Uniform measure on the boundary



Interpretation: We glue $a, b \in \mathcal{T}^\circ$ if

- they have the same label $z(a) = z(b) > 0$
- going from a to b "around" the tree $\mathcal{T}$ one encounters only vertices with greater label.

The Bettinelli-Miermont construction also relied on using a labeled forest, but here we have the additional remarkable interpretation of labels:

$z(c) = \Delta(c, \partial\mathbb{D})$ coincides with the distance from (the equivalence class of) c to $\partial\mathbb{D}$.

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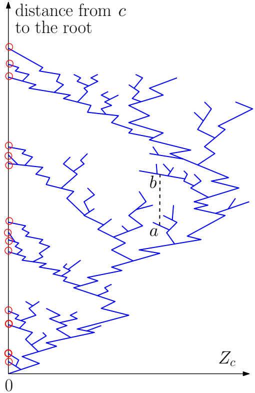

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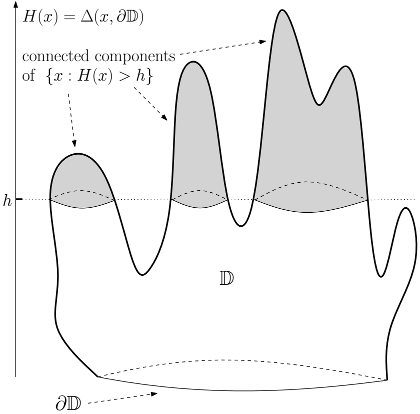
One can use this to construct the uniform measure on the boundary.

Proposition

The formula $\langle \mu, \varphi \rangle = \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, \varphi(x)\, \mathbf{1}_{\{\Delta(x,\partial\mathbb{D})<\varepsilon\}}$

defines a finite measure on the boundary with total mass r.

5. Cutting Brownian disks at heights



$(\mathbb{D}, \Delta)$ is the free Brownian disk with perimeter r

For $x \in \mathbb{D}$, $H(x) = \Delta(x, \partial\mathbb{D})$ is called the height of x.

Fix $h > 0$. For each connected component $\mathcal{C}$ of $\{x : H(x) > h\}$, can define its boundary size (perimeter)

$$|\partial\mathcal{C}| = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \mathrm{Vol}(\{x \in \mathcal{C} : H(x) < h + \varepsilon\})$$

Theorem (LG-Riera AOP 2020)

Conditionally on their boundary sizes, the connected components of $\{x \in \mathbb{D} : H(x) > h\}$, equipped with their intrinsic metrics, are independent free Brownian disks with the prescribed perimeters.

Question. How does the collection of perimeters of connected components of $\{x \in \mathbb{D} : H(x) > h\}$ evolve as h varies ?

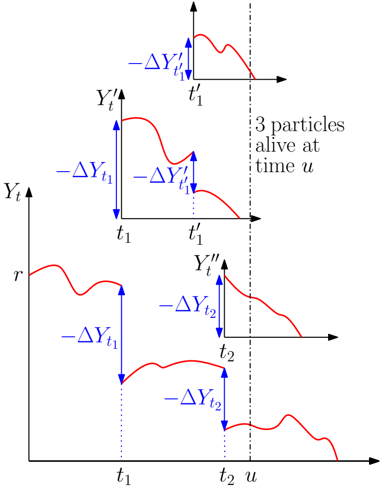
Write $\mathcal{C}^{1,h}, \mathcal{C}^{2,h}, \ldots$ for the connected components of $\{x \in \mathbb{D} : H(x) > h\}$ ranked in decreasing order of their boundary sizes, and

$$\mathbf{X}(h) = (|\partial \mathcal{C}^{1,h}|, |\partial \mathcal{C}^{2,h}|, \ldots)$$

The preceding theorem suggests that $(\mathbf{X}(h))_{h \geq 0}$ satisfies a kind of branching property analogous to that of growth-fragmentation processes.

Growth-fragmentation processes

Basic ingredient: Y self-similar Markov process with values in $\mathbb{R}_+$ and only negative jumps, absorbed at 0.



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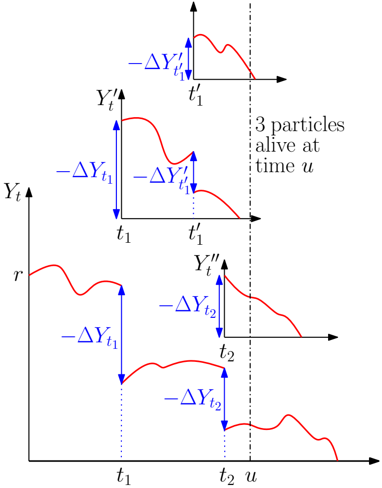

The process starts with an initial particle (Eve particle) whose mass evolves in time according to the law of Y started at r.
When the mass of the initial particle has a (negative) jump of size $-\delta$, a new particle (child of the Eve particle) is created, whose mass then evolves according to the law of Y started at δ.
In turn, each child of the Eve particle has children at jump times of its mass process, and so on.

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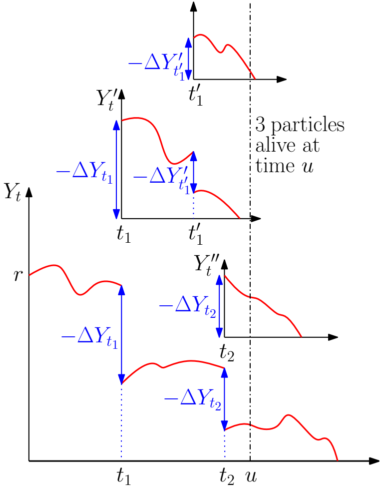

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In turn, each child of the Eve particle has children at jump times of its mass process, and so on.

The associated growth-fragmentation process is:

$\mathbf{Y}(t) =$ ranked sequence of masses of particles alive at time t.

Growth-fragmentation process in the Brownian disk

Recall that $\mathbb{D}$ is the free Brownian disk with perimeter r, and

$$\mathbf{X}(h) = (|\partial\mathcal{C}^{1,h}|, |\partial\mathcal{C}^{2,h}|, \ldots)$$

Here $\mathcal{C}^{1,h}, \mathcal{C}^{2,h}, \ldots$ are the connected components of $\{x \in \mathbb{D} : H(x) > h\}$.

Theorem (LG-Riera, AOP 2020)

$(\mathbf{X}(h))_{h\geq 0}$ *is a growth-fragmentation process whose Eve particle mass process X (starting from 1) can be obtained as follows:*

$$X_t = \exp(\xi_{\tau(t)}),$$

where

$$\tau(t) = \inf\left\{u \geq 0 : \int_0^u e^{\xi_s/2}\,\mathrm{d}s > t\right\}$$

and ξ is the spectrally negative Lévy process with Laplace exponent

$$\psi(q) = \sqrt{\frac{3}{2\pi}}\left(-\frac{8}{3}q + \int_{1/2}^{1}(x^q - 1 + q(1-x))\,(x(1-x))^{-5/2}\,\mathrm{d}x\right).$$

Remarks

- The formula
$$X_t = \exp(\xi_{\tau(t)})$$
is the Lamperti representation of a self-similar Markov process in terms of a Lévy process.
- The theorem is closely related to the work of Bertoin, Curien, Kortchemski who studied asymptotics for a discrete analog of the process $\mathbf{X}(h)$ (for triangulations with a boundary).
- The measure
$$(x(1-x))^{-5/2}\,\mathrm{d}x$$
that appears in the formula for ψ should be compared with the dislocation measure $(x(1-x))^{-3/2}\,\mathrm{d}x$ corresponding to the (pure) fragmentation process obtained by cutting the Brownian tree at heights (Bertoin).

6. Non-compact models of random geometry

The Brownian plane

Let M_n be uniform over {rooted quadrangulations with n faces}.

$V(M_n)$ vertex set of M_n, $V(M_n)$ is pointed at the root vertex

$d_{\rm gr}$ graph distance on $V(M_n)$

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$d_{\rm gr}$ graph distance on $V(M_n)$

Theorem (Curien-LG)

Let a_n be a sequence of positive reals converging to 0 such that $n^{1/4} a_n \longrightarrow \infty$. Then

$$\Big(V(M_n), a_n\, d_{\rm gr}\Big) \xrightarrow[n\to\infty]{\rm (d)} (\mathcal{P}, D)$$

in the local Gromov-Hausdorff sense. The limit $(\mathcal{P}, D)$ is a random pointed (non-compact) metric space called the Brownian plane.

The Brownian plane is also the scaling limit of the Uniform Infinite Planar Quadrangulation (or of the UIPT, cf. Budzinski).

$(\mathcal{P}, D)$ is homeomorphic to the usual plane

The infinite Brownian disk

For $p, n \geq 1$, let $\mathbb{M}^{4,p,n}$ be the set of all (rooted) quadrangulations with a boundary of size $2p$ and n internal faces.
Let $\mathbf{Q}_{p,n}$ be uniformly distributed over $\mathbb{M}^{4,p,n}$.
The vertex set $V(\mathbf{Q}_{p,n})$ of $\mathbf{Q}_{p,n}$ is a pointed metric space for $d_{\rm gr}$

Theorem (Baur-Miermont-Ray)

Let $(n_p)_{p\in\mathbb{N}}$ be a sequence of positive integers with $n_p/p^2 \longrightarrow \infty$ as $p \to \infty$. Then

$$\Big(V(\mathbf{Q}_{p,n_p}), \big(\frac{3}{2p}\big)^{1/2} d_{\rm gr}\Big) \xrightarrow[p\to\infty]{\rm (d)} (\mathbb{D}_1^\infty, \Delta^\infty)$$

in the local Gromov-Hausdorff sense. The limit $(\mathbb{D}_1^\infty, \Delta^\infty)$ is a random pointed non-compact metric space called the infinite Brownian disk with perimeter 1.

The infinite Brownian disk is homeomorphic to the complement of the (open) unit disk of the plane.

The Brownian half-plane

Keep the notation of the preceding slide : $\mathbf{Q}_{p,n}$ is uniformly distributed over the set of all quadrangulations with a boundary of size $2p$ and n internal faces.

Theorem (Baur-Miermont-Ray)

Let $(n_p)_{p\in\mathbb{N}}$ be a sequence of positive integers with $n_p/p^2 \longrightarrow \infty$ as $p \to \infty$, and let $(a_p)_{p\in\mathbb{N}}$ be a sequence of positive reals tending to 0, with $\sqrt{p}\, a_p \longrightarrow \infty$. Then

$$\Big(V(\mathbf{Q}_{p,n_p}), a_p\, d_{\rm gr}\Big) \xrightarrow[p\to\infty]{\rm (d)} (\mathbb{H}, D^\infty)$$

in the local Gromov-Hausdorff sense. The limit $(\mathbb{H}, D^\infty)$ is a random pointed non-compact metric space called the Brownian half-plane.

The Brownian half-plane is homeomorphic to the usual half-plane.

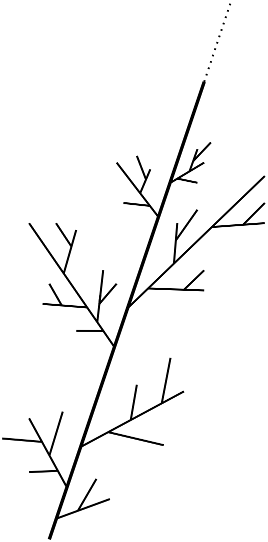
Baur-Miermont-Ray (AOP 2019) classify all possible scaling limits of quadrangulations with a boundary: the Brownian plane, the Brownian half-plane and the infinite Brownian disk are the basic non-compact models that can appear in the limit.

Constructing the non-compact models

Basic ingredient: the infinite Brownian tree (Aldous 1990)

This tree consists of

- An infinite spine isometric to $[0, \infty)$
- Brownian subtrees branching off the left side and the right side of the spine



To get the subtrees branching off the left side, recall $\mathbf{n}_+$ is the Brownian excursion measure, consider a Poisson point measure

$$\sum_{i \in I} \delta_{(h_i, e_i)}$$

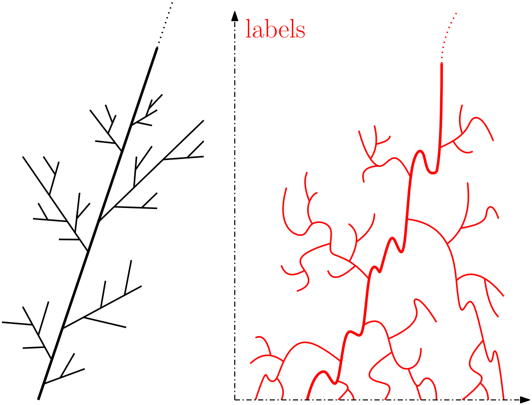
with intensity $2\,\mathrm{d}h\,\mathbf{n}_+(\mathrm{d}e)$, and for every $i \in I$ graft the tree $\mathcal{T}_{e_i}$ coded by e_i at height h_i of the spine.

Proceed similarly for the right side.

$\longrightarrow$ Get a non-compact $\mathbb{R}$-tree

Labels and truncation of the infinite Brownian tree

Assign a label to each point of the infinite Brownian tree.



Labels along the spine evolve like a 3-dimensional Bessel process (started from 0)

For each subtree $\mathcal{T}_e$, labels on this subtree are given by Brownian motion indexed by $\mathcal{T}_e$ (started from the label of the point of the spine where $\mathcal{T}_e$ is grafted)

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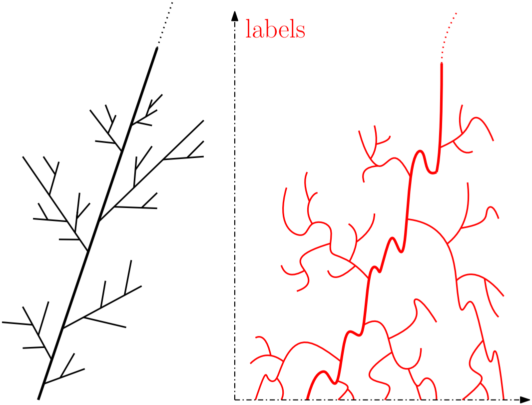

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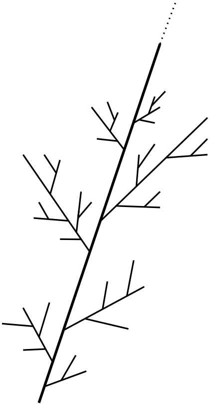
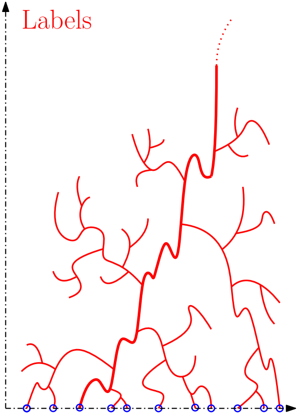
KEY POINT: Subtrees are truncated at points where the label vanishes (so that all labels remain nonnegative) — of course after this truncation, the subtrees are no longer Brownian trees.

The construction of non-compact models (LG-Riera)

The preceding model of a labeled tree allows us to construct the three non-compact models of Brownian geometry.





Can define a quantity Z measuring the number of points with zero label.

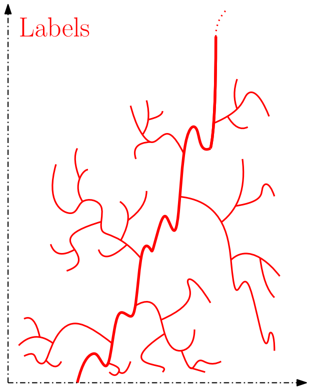
By imitating the construction of the Brownian disk, one gets

- The Brownian plane under the conditioning $Z = 0$
- The infinite Brownian disk with perimeter a under the conditioning $Z = a$
- The Brownian half-plane under $Z = \infty$

In fact in the third case, no conditioning is needed ($Z = \infty$ a.s. in the unconditioned model!)

The case of the Brownian plane

The conditioning $Z = 0$ means that labels do not vanish on any subtree branching off the spine.



Under this conditioning:

- Labels on the spine evolve like a 9-dimensional Bessel process
- The subtrees are Brownian trees equipped with Brownian labels conditioned to have only positive labels

In the Brownian plane, labels correspond to distances from the distinguished point x_* which is the bottom of the spine.

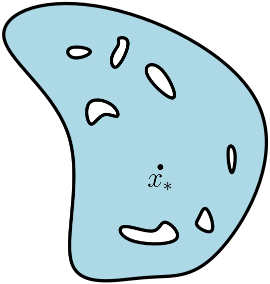
(Similarly, in the construction of the infinite Brownian disk or of the Brownian half-plane, labels correspond to distances from the boundary)

Hulls in the Brownian plane

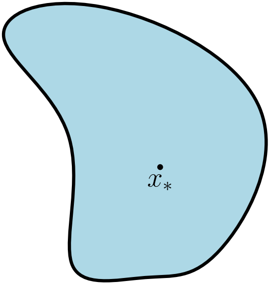
$B(r)$ ball of radius r centered at the distinguished point x_* in the Brownian plane $\mathcal{P}$

Then $\mathcal{P}\backslash B(r)$ is not connected, but has a unique unbounded component: the hull $B^\bullet(r)$ is the complement of this unbounded component

(informally, $B^\bullet(r)$ is obtained by filling in the bounded holes in $B(r)$)



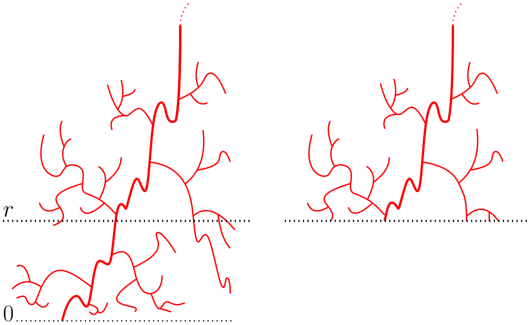
Ball $B(r)$



Hull $B^\bullet(r)$

Complement of hulls in the Brownian plane

Recall: $B^\bullet(r)$ is the hull of radius r in the Brownian plane $\mathcal{P}$



In the infinite tree picture, the complement of $B^\bullet(r)$ corresponds to:

- Keeping the part of the spine above the last point with label r on the spine
- Keeping the subtrees branching off this part of the spine, but truncating them at level r

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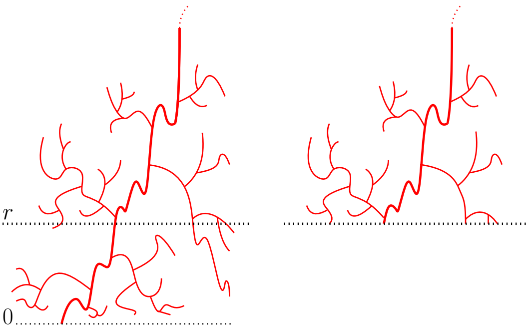

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This resembles the spine representation of the infinite Brownian disk.

Spatial Markov property in the Brownian plane

Recall:

- $\mathcal{P}$ is the Brownian plane
- $B^\bullet(r)$ is the hull of radius r ($\mathcal{P}\backslash B^\bullet(r)$ is the unbounded connected component of $\mathcal{P}\backslash B(r)$)

Equip both $B^\bullet(r)$ and $\mathcal{P}\backslash B^\bullet(r)$ with their intrinsic metrics.

One can define a boundary size $|\partial B^\bullet(r)|$ as the limit of ε^{-2} times the volume of the ε-tubular neighborhood of $\partial B^\bullet(r)$.

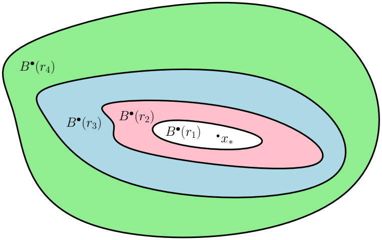
Theorem (LG-Riera)

Conditionally on $|\partial B^\bullet(r)|$, the hull $B^\bullet(r)$ and its complement $\mathcal{P}\backslash B^\bullet(r)$ are independent, and moreover (the closure of) $\mathcal{P}\backslash B^\bullet(r)$ is an infinite Brownian disk with perimeter $|\partial B^\bullet(r)|$.

Isoperimetric inequalities in the Brownian plane

Let $0 < r_1 < r_2 < r_3 < \cdots$.
Then the annuli $B^\bullet(r_2) - B^\bullet(r_1)$, $B^\bullet(r_3) - B^\bullet(r_2)$, $B^\bullet(r_3) - B^\bullet(r_2)$,...
are independent conditionally on their boundary sizes.

$\longrightarrow$Key ingredient for next theorem.



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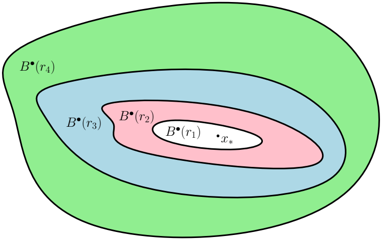

Let $\mathcal{K}$ be the collection of all (closures of) Jordan domains containing x_* in the Brownian plane. (A Jordan domain D is the interior of a simple loop, $|\partial D|$ is the length of this loop and and $|D|$ is the volume of D)

Theorem (Riera)

For any nondecreasing funtion $f : [0,\infty) \longrightarrow (0,\infty)$, we have

$$\inf_{D\in\mathcal{K}} \frac{|\partial D|}{|D|^{1/4}} f(|\log|D||) > 0 \text{ or } = 0 \text{ a.s.}$$

according as $\sum_{n\in\mathbb{N}} \frac{1}{f(n)^2} < \infty$ *or* $= \infty$.

Thank you for your attention