The Brownian disk viewed from a boundary point*

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Abstract

We provide a new construction of Brownian disks in terms of forests of continuous random trees equipped with nonnegative labels corresponding to distances from a distinguished point uniformly distributed on the boundary of the disk. This construction shows in particular that distances from the distinguished point evolve along the boundary as a five-dimensional Bessel bridge. As an important ingredient of our proofs, we show that the uniform measure on the boundary, as defined in the earlier work of Bettinelli and Miermont, is the limit of the suitably normalized volume measure on a small tubular neighborhood of the boundary. Our construction also yields a simple proof of the equivalence between the two definitions of the Brownian half-plane.

Contents

1 Introduction

Brownian disks are random compact metric spaces that serve as models of random geometry and arise as scaling limits of large random planar maps with a boundary [3, 4, 6, 7, 15]. They appear as special subsets of the Brownian map, and in particular as connected components of the complement of balls in the Brownian map [19]. Brownian disks are also closely related to the Liouville quantum gravity surfaces called quantum disks, see [22, Corollary 1.5], as well as the survey [21] and the references therein. The initial construction of Brownian disks was given by Bettinelli [6] in terms of a forest of continuous random trees equipped with Brownian labels. In this construction, labels correspond to distances from a distinguished point belonging to the interior of the Brownian disk. A different construction still based on a labeled continuous random tree appeared in [19], with labels now corresponding to distances from the boundary of the disk. The main goal of the present work is to present a new construction of Brownian disks where labels represent distances from a point chosen uniformly at random on the boundary. This construction has several interesting consequences. In particular, it shows that, if

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one starts from a distinguished point chosen at random on the boundary and then moves along the boundary, distances from the distinguished point evolve exactly like a five-dimensional Bessel bridge. In contrast with preceding constructions [6, 7, 19], we do not rely on discrete approximations to establish the validity of our method, but rather we pass to the limit in the construction of [6] by letting the distinguished point tend to the boundary.

Let us give an informal description of our construction (see Section 6 for a more precise presentation). We start from the circle, which we view as the interval $[0,1]$ with the two points 0 and 1 identified, and we assign a "label" Λ_t to each point t of $[0,1]$, in such a way that the process $(\Lambda_t)_{t\in[0,1]}$ is a five-dimensional Bessel bridge from 0 to 0 scaled by the factor $\sqrt{3}$. We then consider a Poisson forest of continuous random trees (scaled versions of the celebrated Aldous Brownian CRT) that are rooted randomly on the circle. For every tree $\mathcal{T}$ in this forest, and for every point $u \in \mathcal{T}$, we assign a label Λ_u to u: The collection $(\Lambda_u)_{u\in\mathcal{T}}$ is distributed as Brownian motion indexed by $\mathcal{T}$, started from the label of the root (recall that the root of $\mathcal{T}$ belongs to the circle). We let $\mathfrak{H}$ be the geodesic metric space consisting of the union of the circle and the collection of those trees that have only nonnegative labels (we just remove those trees where negative labels occur). In this way, every point u of $\mathfrak{H}$ has been assigned a nonnegative label Λ_u. For $u, v \in \mathfrak{H}$, we set

$$\Delta^\circ(u,v) = \Lambda_u + \Lambda_v - 2\,\max\Big(\min\{\Lambda_w : w \in [|u,v|]\}, \min\{\Lambda_w : w \in [|v,u|]\}\Big),$$

where $[|u,v|]$ is the "interval" of $\mathfrak{H}$ consisting of points visited when going from u to v in "clockwise direction" along $\mathfrak{H}$ (see Section 4 for more precise definitions). Finally, we define $\Delta(u,v)$ as the largest pseudo-metric on $\mathfrak{H}$ that is bounded above by $\Delta^\circ(u,v)$. Then we consider the quotient space $\mathbb{D} := \mathfrak{H}/\simeq$ for the equivalence relation defined by setting $x \simeq y$ if and only $\Delta(x,y) = 0$. Theorem 15 below states that $\mathbb{D}$ equipped with the distance induced by Δ is a free Brownian disk of perimeter 1 pointed at a uniform boundary point.

The preceding definitions of Δ° and Δ are of course very similar to the construction of the Brownian map (see e.g. [17]), of the Brownian disk [6], or of the Brownian plane [11]. Indeed, we derive Theorem 15 by a suitable passage to the limit from the construction of the free pointed Brownian disk that is given in [6] — note that [6] considers the slightly different model of the Brownian disk with prescribed volume and perimeter, but the same method applies to the free Brownian disk with minor changes. In this construction, labels correspond to distances from a distinguished point distributed according to the volume measure of the Brownian disk (see formula (42) below for a more precise statement describing the distribution of the distinguished point). The idea is then to condition the distinguished point to lie within distance at most ε from the boundary $\partial\mathbb{D}$ and to pass to the limit $\varepsilon \to 0$. For this passage to the limit, it is crucial to have information about the probability measure μ_ε obtained by normalizing the restriction of the volume measure of $\mathbb{D}$ to the tubular neighborhood of radius ε of the boundary. More precisely, one needs the fact that μ_ε converges when $\varepsilon \to 0$ to the uniform probability measure μ on the boundary, as defined in the construction of [6, 7]. The convergence of μ_ε as $\varepsilon \to 0$ towards a probability measure ν supported on $\partial\mathbb{D}$ had already been obtained in [19], but the equality $\mu = \nu$ was still open. Theorem 9 below shows that this equality holds, so that the two natural ways of defining a uniform measure on the boundary are indeed equivalent.

Another important ingredient consists in studying the behavior of labels on the boundary, under the condition that the distinguished point lies within distance at most ε from $\partial\mathbb{D}$. In the construction of [6, 7], labels evolve along the boundary like a Brownian bridge scaled by the constant $\sqrt{3}$, and one may replace the Brownian bridge by a normalized Brownian excursion $\mathbf{e} = (\mathbf{e}_t)_{0\le t\le 1}$ thanks to Vervaat's transformation [26]. Under the preceding conditioning, labels along the boundary evolve like $\sqrt{3}\,\mathbf{e}^\varepsilon$, where the distribution of $\mathbf{e}^\varepsilon$ is specified by

$$\mathbb{E}[F(\mathbf{e}^\varepsilon)] := C_\varepsilon^{-1}\,\mathbb{E}\Big[F(\mathbf{e})\,\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big], \tag{1}$$

where C_ε is the appropriate normalizing constant. Proposition 4 below states that $\mathbf{e}^\varepsilon$ converges in distribution as $\varepsilon \to 0$ to a five-dimensional Bessel bridge $\mathbf{b} = (\mathbf{b}_t)_{0\le t\le 1}$. For our applications, we need in fact a more precise result showing that, for every $\delta \in (0, 1/2)$, it is possible to couple $\mathbf{e}^\varepsilon$ and $\mathbf{b}$ so that the equality $\varepsilon + \mathbf{e}^\varepsilon_t = \mathbf{b}_t$ holds for every $t \in [\delta, 1-\delta]$, with high probability when $\varepsilon \to 0$.

Our construction of the Brownian bridge is closely related to the definition of the Brownian half-plane proposed by Caraceni and Curien [10], which involves a two-sided five-dimensional Bessel process. Another definition of the Brownian half-plane, which is close to the Bettinelli construction of Brownian disks, has been given independently by Gwynne and Miller [14] and by Baur, Miermont and Ray [4], but initially it was not clear that this definition yields the same random object as the Caraceni-Curien definition (see the comments in [4, Remark 2.7] and in [14, Section 1.6]). Recently, Budzinski and Riera [8] have been able to prove the equivalence of the two definitions via discrete approximations. In Section 7, we provide a short simple proof of this equivalence based on our new construction of the Brownian disk.

The paper is organized as follows. Section 2 contains a few preliminaries. In particular, we recall the formalism of snake trajectories, which provides a convenient framework to deal with continuous random trees equipped with labels, and we define the spaces of compact or non-compact measure metric spaces that are relevant to the present work. The technical Section 3 investigates the limiting behavior of the "excursions" $\mathbf{e}^\varepsilon$ distributed as in (1). We start by recalling several properties of Bessel processes, and especially of first-passage Bessel bridges, as these properties play an important role in the proof of the key technical Proposition 4. Section 4 is mainly devoted to recalling the constructions of the (free pointed) Brownian disk and of the Brownian half-plane found in [4, 6, 7, 14]. Our presentation is slightly different from the latter papers and adapted to our purposes. In Section 5, we prove Theorem 9 concerning the approximation of the uniform measure on the boundary by the volume measure on a tubular neighborhood of small radius. Here the Brownian half-plane is used as a tool: We derive a Brownian half-plane version of the desired approximation via an application of the ergodic theorem, and we then use a suitable coupling of the Brownian disk and the Brownian half-plane near a boundary point. In Section 6, we prove Theorem 15 giving our construction of the free Brownian disk pointed at a uniform boundary point. Here, the method consists in coupling the Brownian disk pointed at a point lying within distance ε from the boundary, and the candidate space for the Brownian disk pointed at a uniform boundary point, in such a way that one can get a suitable bound on the Gromov-Hausdorff distance between these two spaces. For this coupling, the precise statement of Proposition 4 is crucial. Finally, Section 7 proves the equivalence of the two definitions of the Brownian half-plane.

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2 Preliminaries

2.1 Snake trajectories

We use the formalism of snake trajectories to deal with continuous random trees whose vertices are assigned real labels. In this section, we briefly recall the notation and definitions that are relevant to the present work. We refer to [2] for more details.

We denote the space of all finite (real) paths by $\mathcal{W}$. Here a finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})} \geq 0$ is called the lifetime of w. The endpoint or tip of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. The space $\mathcal{W}$ is equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

We set $\mathcal{W}_0 := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = 0\}$. The trivial path of $\mathcal{W}_0$ with zero lifetime is identified to the point 0 of $\mathbb{R}$.

Definition 1. *A snake trajectory ω (with initial point 0) is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_0$ which satisfies the following two properties:*

(i) *We have $\omega_0 = 0$ and the number $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq 0\}$, called the duration of the snake trajectory ω, is finite (by convention $\sup \varnothing = 0$).*

(ii) (Snake property) *For every $0 \leq s \leq s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s\leq r\leq s'} \zeta_{(\omega_r)}]$.*

We denote the set of all snake trajectories by $\mathcal{S}$. If $\omega \in \mathcal{S}$, we often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geq 0$. The set $\mathcal{S}$ is equipped with the distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

It is not hard to verify that a snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$ (see [2, Proposition 8]).

Let $\omega \in \mathcal{S}$ be a snake trajectory. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact $\mathbb{R}$-tree, which will be denoted by $\mathcal{T}_{(\omega)}$. This $\mathbb{R}$-tree is the quotient space $\mathcal{T}_{(\omega)} := [0, \sigma(\omega)]/\sim$ of the interval $[0, \sigma(\omega)]$ for the equivalence relation

$$s \sim s' \text{ if and only if } \ \zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega),$$

and $\mathcal{T}_{(\omega)}$ is equipped with the distance induced by

$$d_{(\omega)}(s, s') = \zeta_s(\omega) + \zeta_{s'}(\omega) - 2 \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega).$$

(see e.g. [18, Section 3] for more information about the coding of $\mathbb{R}$-trees by continuous functions). We write $p_{(\omega)} : [0, \sigma(\omega)] \longrightarrow \mathcal{T}_{(\omega)}$ for the canonical projection. By convention, $\mathcal{T}_{(\omega)}$ is rooted at the point $\rho_{(\omega)} := p_{(\omega)}(0)$, and the volume measure on $\mathcal{T}_{(\omega)}$ is defined as the pushforward of Lebesgue measure on $[0, \sigma(\omega)]$ under $p_{(\omega)}$.

By property (ii) in the definition of a snake trajectory, the condition $p_{(\omega)}(s) = p_{(\omega)}(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ can be viewed as defined on the quotient space $\mathcal{T}_{(\omega)}$. For $u \in \mathcal{T}_{(\omega)}$, we set $\ell_u(\omega) := \widehat{W}_s(\omega)$, for any $s \in [0, \sigma(\omega)]$ such that $u = p_{(\omega)}(s)$. We interpret $\ell_u(\omega)$ as a "label" assigned to the "vertex" u of $\mathcal{T}_{(\omega)}$. Notice that the mapping $u \mapsto \ell_u(\omega)$ is continuous on $\mathcal{T}_{(\omega)}$. We set $W_*(\omega) := \min\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$ and $W^*(\omega) := \max\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$, for $\omega \in \mathcal{S}$.

We now introduce a σ-finite measure on $\mathcal{S}$ that plays an important role in the present work.

Definition 2. *The Brownian snake excursion measure $\mathbb{N}_0$ is the σ-finite measure on $\mathcal{S}$ that is characterized by the following two properties:*

(i) *The distribution of the lifetime function $(\zeta_s)_{s \geq 0}$ under $\mathbb{N}_0$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon > 0$,*

$$\mathbb{N}_0\Big(\sup_{s \geq 0} \zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

(ii) *Under $\mathbb{N}_0$ and conditionally on $(\zeta_s)_{s \geq 0}$, the tip function $(\widehat{W}_s)_{s \geq 0}$ is a centered Gaussian process with covariance function*

$$K(s, s') := \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r.$$

Informally, property (ii) says that, under $\mathbb{N}_0$ and conditionally on $(\zeta_s)_{s \geq 0}$, the labels $(\ell_u)_{u \in \mathcal{T}_{(\omega)}}$ are distributed as Brownian motion indexed by $\mathcal{T}_{(\omega)}$. The measure $\mathbb{N}_0$ can be interpreted as the excursion measure away from 0 for the Markov process in $\mathcal{W}$ called the Brownian snake (we refer to [16] for a detailed study of the Brownian snake and its excursion measures). For our purposes, it will be important to know the distribution of the minimum W_* under $\mathbb{N}_0$: For every $y < 0$, we have

$$\mathbb{N}_0(W_* \leq y) = \frac{3}{2y^2}. \tag{2}$$

See e.g. [16, Section VI.1] for a proof.

The following scaling property is often useful. For $\lambda > 0$, for every $\omega \in \mathcal{S}$, we define $\Theta_\lambda(\omega) \in \mathcal{S}$ by $\Theta_\lambda(\omega) := \omega'$, with

$$\omega'_s(t) := \sqrt{\lambda}\, \omega_{s/\lambda^2}(t/\lambda)\ , \quad \text{for } s \geq 0 \text{ and } 0 \leq t \leq \zeta'_s := \lambda \zeta_{s/\lambda^2}.$$

Then it is a simple exercise to verify that $\Theta_\lambda(\mathbb{N}_0) = \lambda\,\mathbb{N}_0$.

Let us introduce some additional notation. If I is an interval of $\mathbb{R}$ and E is a metric space, we write $C(I,E)$ for the space of all continuous functions from I into E (in particular, $C([0,t],\mathbb{R})$ is a subset of $\mathcal{W}$, for every $t \geq 0$). When I is a compact interval and $E = \mathbb{R}$ or $\mathbb{R}_+$, the space $C(I,E)$ will be equipped with the topology of uniform convergence, and then convergence of probability measures on $C(I,E)$ will be in the usual sense of weak convergence of probability measures on a Polish space.

We also write $M_p(I \times \mathcal{S})$ for the set of all point measures (countable sums of Dirac masses) on $I \times \mathcal{S}$. As usual, $M_p(I \times \mathcal{S})$ is equipped with the σ-field generated by the mappings $\gamma \mapsto \gamma(A)$, when A varies among the Borel subsets of $I \times \mathcal{S}$.

2.2 Spaces of compact and locally compact metric spaces

Recall that a compact measure metric space is a compact metric space (X,d) equipped with a Borel finite measure μ on X, which is sometimes called the volume measure on X. If there is a distinguished point $x \in X$, we say that (X,d,μ,x) is a pointed compact measure metric space.

We write $\mathbb{M}$, resp. $\mathbb{M}^\bullet$, for the set of all compact measure metric spaces, resp. of all pointed compact measure metric spaces, where two such spaces (X,d,μ) and (X',d',μ'), resp. (X,d,μ,x) and (X',d',μ',x'), are identified if there exists an isometry ϕ from X onto X' such that μ' is the pushforward of μ under ϕ (and $\phi(x) = x'$ in the pointed case). Both $\mathbb{M}$ and $\mathbb{M}^\bullet$ are Polish spaces when equipped with the Gromov-Hausdorff-Prokhorov distance (see e.g. [19, Section 2.1] for a definition).

We will also consider the case of non-compact spaces. We restrict our attention to length spaces (a metric space (E,d) is called a length space if, for every $x,y \in E$, the distance $d(x,y)$ is the infimum of lengths of continuous paths from x to y). Recall also that a metric space (E,d) is said to be boundedly compact if the closed balls of E are compact. A length space is boundedly compact if and only if it is locally compact and complete [9, Proposition 2.5.22]. We let $\mathbb{M}^\bullet_{bcl}$ denote the space of all (isometry classes of) boundedly compact length spaces (X,d) given with a distinguished point x and a measure μ which is finite on compact subsets of X. The set $\mathbb{M}^\bullet_{bcl}$ can be equipped with the "local" Gromov-Hausdorff-Prokhorov distance as defined in [1] and is then also a Polish space.

3 Convergence to the five-dimensional Bessel bridge

The main goal of this section is to prove Proposition 4 concerning the asymptotic behavior of the "excursions" $\mathbf{e}^\varepsilon$ defined in (1). Before stating and proving Proposition 4, we need to gather a few facts about Bessel processes (more information can be found in [25, Chapter XI] and especially in [23]). It will be convenient to introduce a random process $R = (R_t)_{t\geq 0}$ and probability measures $\mathbb{P}^{(5)}_x$ and $\mathbb{P}^{(-1)}_x$, for every $x \geq 0$, such that R is a five-dimensional Bessel process that starts at x under $\mathbb{P}^{(5)}_x$, and similarly R is a Bessel process of dimension -1 that starts at x under $\mathbb{P}^{(-1)}_x$. Recall that the Bessel process of dimension -1 is absorbed at 0, and that the five-dimensional Bessel can be viewed as the Bessel process of dimension -1 conditioned to escape to infinity, in the sense of h-transforms. More precisely, for every $x > 0$ and $t > 0$, for every nonnegative measurable function F on $C([0,t],\mathbb{R})$, we have

$$\mathbb{E}^{(5)}_x[F((R_s)_{s\leq t})] = \mathbb{E}^{(-1)}_x\Big[\Big(\frac{R_t}{x}\Big)^3 F((R_s)_{s\leq t})\Big]. \tag{3}$$

For every $x \geq 0$, we set $T^{(R)}_x := \inf\{t \geq 0 : R_t = x\}$ and $L^{(R)}_x := \sup\{t \geq 0 : R_t = x\}$ with the usual conventions $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. It follows from (3) that, if $0 < \varepsilon < x$, $\mathbb{P}^{(5)}_x(T^{(R)}_\varepsilon < \infty) = (\varepsilon/x)^3$ and

$$(R_t)_{0\leq t\leq T^{(R)}_\varepsilon} \text{ under } \mathbb{P}^{(5)}_x(\cdot \mid T^{(R)}_\varepsilon < \infty) \overset{\text{(d)}}{=} (R_t)_{0\leq t\leq T^{(R)}_\varepsilon} \text{ under } \mathbb{P}^{(-1)}_x. \tag{4}$$

Furthermore, as a consequence of Nagasawa's time-reversal theorem [25, Theorem VII.4.5], we have for every $x > 0$,

$$(R_t)_{0\leq t\leq L^{(R)}_x} \text{ under } \mathbb{P}^{(5)}_0 \overset{\text{(d)}}{=} \Big(R_{T^{(R)}_0 - t}\Big)_{0\leq t\leq T^{(R)}_0} \text{ under } \mathbb{P}^{(-1)}_x. \tag{5}$$

This implies that the process $(L_x^{(R)})_{x\geq 0}$ has independent increments under $\mathbb{P}_0^{(5)}$. This property (for more general Bessel processes) was first observed by Getoor [13].

Fix $0 < \varepsilon < x$. By (5), the law of $T_\varepsilon^{(R)}$ under $\mathbb{P}_x^{(-1)}$ is equal to the law of $L_x^{(R)} - L_\varepsilon^{(R)}$ under $\mathbb{P}_0^{(5)}$. Thus, for every $\lambda > 0$,

$$\mathbb{E}_x^{(-1)}[\exp(-\lambda T_\varepsilon^{(R)})] = \mathbb{E}_0^{(5)}[\exp(-\lambda(L_x^{(R)} - L_\varepsilon^{(R)}))] = \frac{\mathbb{E}_0^{(5)}[\exp(-\lambda L_x^{(R)})]}{\mathbb{E}_0^{(5)}[\exp(-\lambda L_\varepsilon^{(R)})]}.$$

From the main result of [13], the density of $L_x^{(R)}$ under $\mathbb{P}_0^{(5)}$ is the function

$$t \mapsto r_t(x,0) := \frac{x^3}{\sqrt{2\pi t^5}} \exp(-\frac{x^2}{2t}),$$

from which one easily computes the Laplace transform $\mathbb{E}_0^{(5)}[\exp(-\lambda L_x^{(R)})] = (1 + x\sqrt{2\lambda})e^{-\sqrt{2\lambda}}$. Hence,

$$\mathbb{E}_x^{(-1)}[\exp(-\lambda T_\varepsilon^{(R)})] = \frac{1 + x\sqrt{2\lambda}}{1 + \varepsilon\sqrt{2\lambda}} e^{-(x-\varepsilon)\sqrt{2\lambda}}.$$

We will need the explicit formula for the density of $T_\varepsilon^{(R)}$ under $\mathbb{P}_x^{(-1)}$, which we can obtain by inverting the Laplace transform in the preceding display. We note that

$$\frac{1 + x\sqrt{2\lambda}}{1 + \varepsilon\sqrt{2\lambda}} e^{-(x-\varepsilon)\sqrt{2\lambda}} = e^{-(x-\varepsilon)\sqrt{2\lambda}} + \frac{x-\varepsilon}{\varepsilon} \frac{\varepsilon\sqrt{2\lambda}}{1 + \varepsilon\sqrt{2\lambda}} e^{-(x-\varepsilon)\sqrt{2\lambda}}. \tag{6}$$

We have $e^{-(x-\varepsilon)\sqrt{2\lambda}} = \int_0^\infty \mathrm{d}t\, e^{-\lambda t}\, q_t(x,\varepsilon)$, where the function

$$t \mapsto q_t(x,\varepsilon) := \frac{x-\varepsilon}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(x-\varepsilon)^2}{2t}\Big) \tag{7}$$

is the density of the hitting time of ε for a linear Brownian motion started at x. On the other hand, for every $a, b > 0$ and $t \geq 0$, set

$$g_{a,b}(t) := e^{ab + a^2 t}\, \mathrm{erfc}(a\sqrt{t} + \frac{b}{2\sqrt{t}}),$$

where we recall the standard notation $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\, \mathrm{d}y$. Via an integration by parts and simple calculations, one gets that the Laplace transform of $g_{a,b}$ is

$$\int_0^\infty \mathrm{d}t\, e^{-\lambda t}\, g_{a,b}(t) = \frac{e^{-b\sqrt{\lambda}}}{\sqrt{\lambda}(a + \sqrt{\lambda})}.$$

Notice that $g_{a,b}(0) = 0$ and $g_{a,b}$ tends to 0 at infinity. It follows that

$$\int_0^\infty \mathrm{d}t\, e^{-\lambda t}\, g'_{a,b}(t) = \lambda \int_0^\infty \mathrm{d}t\, e^{-\lambda t}\, g_{a,b}(t) = \frac{\sqrt{\lambda}}{a + \sqrt{\lambda}}\, e^{-b\sqrt{\lambda}}.$$

Recalling (6), and using the last display with $b = \sqrt{2}(x-\varepsilon)$ and $a = 1/(\varepsilon\sqrt{2})$, we get that the law of $T_\varepsilon^{(R)}$ under $\mathbb{P}_x^{(-1)}$ has a density given by

$$t \mapsto r_t(x,\varepsilon) := q_t(x,\varepsilon) + \frac{x-\varepsilon}{\varepsilon}\, g'_{(\varepsilon\sqrt{2})^{-1},\sqrt{2}(x-\varepsilon)}(t).$$

From the explicit expression for $g_{a,b}$, we get

$$r_t(x,\varepsilon) = \left(\frac{1}{2\varepsilon^3} \mathrm{erfc}\Big(\frac{\sqrt{t}}{\varepsilon\sqrt{2}} + \frac{x-\varepsilon}{\sqrt{2t}}\Big) \exp\Big(\Big(\frac{\sqrt{t}}{\varepsilon\sqrt{2}} + \frac{x-\varepsilon}{\sqrt{2t}}\Big)^2\Big) - \frac{1}{\varepsilon^2\sqrt{2\pi t}} + \frac{x}{\varepsilon\sqrt{2\pi t^3}}\right) (x-\varepsilon)\, e^{-(x-\varepsilon)^2/(2t)}.$$

Using the asymptotic expansion $\mathrm{erfc}(z)\exp(z^2) = (z\sqrt{\pi})^{-1}(1 - \frac{1}{2}z^{-2} + o(z^{-2}))$ as $z \to \infty$, one easily verifies that $r_t(x, \varepsilon) \longrightarrow r_t(x, 0)$ and $r_t(x + \varepsilon, \varepsilon) \longrightarrow r_t(x, 0)$ as $\varepsilon \to 0$.

We will need to introduce the "first-passage Bessel bridge" giving the distribution of $(R_s)_{0\leq s\leq T_\varepsilon^{(R)}}$ under $\mathbb{P}_x^{(-1)}(\cdot \,|\, T_\varepsilon^{(R)} = t)$, for $0 \leq \varepsilon < x$ and $t > 0$ (beware that this first-passage bridge should not be confused with the usual Bessel bridges studied in [24]). Before giving a precise definition of this bridge, let us introduce the transition densities of the Bessel process of dimension -1 killed upon hitting ε. For every $\varepsilon \geq 0$, these transition densities are the (continuous) functions $p_t^{(\varepsilon)}(y, z)$, defined for $t > 0$ and $y, z > \varepsilon$ and such that

$$\mathbb{E}_y^{(-1)}[\varphi(R_t)\,\mathbf{1}_{\{T_\varepsilon^{(R)}>t\}}] = \int_{(\varepsilon,\infty)} \mathrm{d}z\, p_t^{(\varepsilon)}(y, z)\, \varphi(z),$$

for every nonnegative measurable function φ on $\mathbb{R}_+$. Let $p_t(y, z)$, $t, y, z > 0$, denote the transition densities of the Bessel process of dimension -1. Then, using the strong Markov property at time $T_\varepsilon^{(R)}$, one easily gets, for $\varepsilon > 0$,

$$p_t^{(\varepsilon)}(y, z) = p_t(y, z) - \int_0^t \mathrm{d}s\, r_s(y, \varepsilon)\, p_{t-s}(\varepsilon, z). \tag{8}$$

For $\varepsilon = 0$, we have just $p_t^{(0)}(y, z) = p_t(y, z)$. Furthermore, we have for every $0 < s < t$ and $x > \varepsilon > 0$,

$$r_t(x, \varepsilon) = \int_\varepsilon^\infty \mathrm{d}y\, p_s^{(\varepsilon)}(x, y)\, r_{t-s}(y, \varepsilon). \tag{9}$$

For $y, z > 0$, let $G(y, z) = \int_0^\infty \mathrm{d}t\, p_t(y, z)$ be the Green function of the Bessel process of dimension -1. Then, $G(y, z) = \frac{2}{3}z(1 \wedge \frac{y^3}{z^3})$ (a simple way to get this formula is to use (3) to observe that $G(y, z) = \frac{y^3}{z^3}G'(y, z)$, where G' is the Green function of the five-dimensional Bessel process, which is easily computed from the Green function of Brownian motion). Using (8), it follows that, for $y, z > \varepsilon$,

$$G^{(\varepsilon)}(y, z) := \int_0^\infty \mathrm{d}t\, p_t^{(\varepsilon)}(y, z) = G(y, z) - G(\varepsilon, z) = \frac{2}{3}z(1 \wedge \frac{y^3}{z^3}) - \frac{2}{3}\frac{\varepsilon^3}{z^2}.$$

where we made the convention $G(0, z) = 0$. If $y > z$, $G^{(\varepsilon)}(y, z)$ does not depend on y and is equal to

$$G^{(\varepsilon)}(\infty, z) := \frac{2}{3}z\,(1 - \frac{\varepsilon^3}{z^3}).$$

Proposition 3. *Let $x > \varepsilon \geq 0$. For every $t > 0$, we can define a probability measure $\Pi_t^{(x,\varepsilon)}(\mathrm{dw})$ on $C([0, t], \mathbb{R}_+)$ in such a way that:*

(i) *The collection $(\Pi_t^{(x,\varepsilon)})_{t>0}$ is a regular version of the conditional distributions of $(R_s)_{0\leq s\leq T_\varepsilon^{(R)}}$ knowing $T_\varepsilon^{(R)} = t$ under $\mathbb{P}_x^{(-1)}$.*

(ii) *For every $0 \leq s < t$, the distribution of $(\mathrm{w}(u))_{0\leq u\leq s}$ under $\Pi_t^{(x,\varepsilon)}(\mathrm{dw})$ is absolutely continuous with respect to the distribution of $(R_u)_{0\leq u\leq s}$ under $\mathbb{P}_x^{(-1)}$, with a Radon-Nikodym density given by*

$$(\mathrm{w}(u))_{0\leq u\leq s} \mapsto \mathbf{1}_{\{\mathrm{w}(u)>\varepsilon:\forall u\in[0,s]\}}\,\frac{r_{t-s}(\mathrm{w}(u), \varepsilon)}{r_t(x, \varepsilon)}. \tag{10}$$

Furthermore, for every $t > 0$, $\Pi_t^{(x+\varepsilon,\varepsilon)}(\mathrm{dw})$ converges weakly to $\Pi_t^{(x,0)}(\mathrm{dw})$ as $\varepsilon \to 0$.

In what follows, we will write $\mathbb{E}_x^{(-1)}[F((R_s)_{0\leq s\leq t}) \,|\, T_\varepsilon^{(R)} = t]$ instead of $\int \Pi_t^{(x,\varepsilon)}(\mathrm{dw})\, F(\mathrm{w})$ when F is a measurable function on $C([0, t], \mathbb{R}_+)$, .

Remark. The proof below applies to the more general setting where the Bessel process of dimension -1 is replaced by a Bessel process of dimension $2(1 - \nu)$ ($\nu > 0$) and the role of the Bessel process of dimension 5 is played by a Bessel process of dimension $2(1 + \nu)$. In particular, the (classical) case $\nu = 1/2$ involving linear Brownian motion and the three-dimensional Bessel process corresponds to the first-passage bridges used in [5] or [7]. We refrained from giving a more general statement because our interest lies mainly in the case considered in the proposition.

Proof. Let us fix $0 \le \varepsilon < x$. For every $t > 0$ and $s \in [0,t)$, let P_x^s be the law of $(R_u)_{0\le u\le s}$ under $\mathbb{P}_x^{(-1)}$. We define another probability measure $P_x^{\varepsilon,s,t}$ on $C([0,s],\mathbb{R}_+)$, which is absolutely continuous with respect to P_x^s, by letting the Radon-Nikodym derivative of $P_x^{\varepsilon,s,t}$ with respect to P_x^s be given by formula (10) (note that $P_x^{\varepsilon,s,t}$ is a probability measure by (9)). Then it is straightforward to verify that the collection $(P_x^{\varepsilon,s,t})_{t\in(s,\infty)}$ forms a regular version of the conditional distributions of $(R_u)_{0\le u\le s}$ knowing $T_\varepsilon^{(R)} = t$, under $\mathbb{P}_x^{(-1)}(\cdot \cap \{T_\varepsilon^{(R)} > s\})$. Furthermore, for every fixed $t > 0$, the probability measures $P_x^{\varepsilon,s,t}$ are consistent when s varies, in the sense that, if $0 \le s < s' < t$, $P_x^{\varepsilon,s,t}$ is the image of $P_x^{\varepsilon,s',t}$ under the obvious restriction mapping. It follows that we can define a process $(X_u^{(t)})_{0\le u<t}$ with continuous sample paths on the time interval $[0,t)$ and such that for every $s \in [0,t)$, the distribution of $(X_u^{(t)})_{0\le u\le s}$ is $P_x^{\varepsilon,s,t}$. From the Radon-Nikodym density (10), we can compute the finite-dimensional marginals of $X^{(t)}$,

$$\mathbb{E}[\varphi_1(X_{t_1}^{(t)})\varphi_2(X_{t_2}^{(t)})\cdots\varphi(X_{t_p}^{(t)})] \tag{11}$$
$$= \frac{1}{r_t(x,\varepsilon)}\int_{(\varepsilon,\infty)^p} \mathrm{d}y_1\ldots\mathrm{d}y_p\, p_{t_1}^{(\varepsilon)}(x,y_1)p_{t_2-t_1}^{(\varepsilon)}(y_1,y_2)\cdots p_{t_p-t_{p-1}}^{(\varepsilon)}(y_{p-1},y_p)r_{t-t_p}(y_p,\varepsilon)\,\varphi_1(y_1)\cdots\varphi_p(y_p),$$

for every $0 < t_1 < \cdots < t_p < t$ and every nonnegative measurable functions $\varphi_1,\ldots,\varphi_p$. We also set $X_t^{(t)} := \varepsilon$. Then it is not obvious that the sample paths of $X^{(t)}$ are continuous at time t. To verify that this property holds, we use a time-reversal argument. It follows from (5) and simple manipulations that, for every $0 < t_1 < \cdots < t_p$, the distributions of $(X_{t-t_1}^{(t)}, X_{t-t_2}^{(t)},\ldots,X_{t-t_p}^{(t)})$ when t varies in (t_p,∞) also form a regular version of the conditional distributions of $(R_{L_\varepsilon^{(R)}+t_1}, R_{L_\varepsilon^{(R)}+t_2},\ldots,R_{L_\varepsilon^{(R)}+t_p})$ knowing $L_x^{(R)} - L_\varepsilon^{(R)} = t$, under $\mathbb{P}_0^{(5)}(\cdot\cap\{L_x^{(R)} - L_\varepsilon^{(R)} > t_p\})$. By (11), the density of $(X_{t-t_1}^{(t)}, X_{t-t_2}^{(t)},\ldots,X_{t-t_p}^{(t)})$ is the function

$$(y_1,\ldots,y_p) \mapsto \frac{1}{r_t(x,\varepsilon)}p_{t-t_p}^{(\varepsilon)}(x,y_p)p_{t_p-t_{p-1}}^{(\varepsilon)}(y_p,y_{p-1})\cdots p_{t_2-t_1}^{(\varepsilon)}(y_2,y_1)r_{t_1}(y_1,\varepsilon).$$

If we integrate this density with respect to the measure $\mathbf{1}_{(t_p,\infty)}(t)r_t(x,\varepsilon)\mathrm{d}t$, we obtain that the density of $(R_{L_\varepsilon^{(R)}+t_1}, R_{L_\varepsilon^{(R)}+t_2},\ldots,R_{L_\varepsilon^{(R)}+t_p})$ under $\mathbb{P}_0^{(5)}(\cdot\cap\{L_x^{(R)} - L_\varepsilon^{(R)} > t_p\})$ is

$$(y_1,\ldots,y_p) \mapsto G^{(\varepsilon)}(x,y_p)\,p_{t_p-t_{p-1}}^{(\varepsilon)}(y_p,y_{p-1})\cdots p_{t_2-t_1}^{(\varepsilon)}(y_2,y_1)r_{t_1}(y_1,\varepsilon).$$

By letting $x\to\infty$, we get that the density of $(R_{L_\varepsilon^{(R)}+t_1}, R_{L_\varepsilon^{(R)}+t_2},\ldots,R_{L_\varepsilon^{(R)}+t_p})$ under $\mathbb{P}_0^{(5)}$ is given by the same formula with $G^{(\varepsilon)}(x,y_p)$ replaced by $G^{(\varepsilon)}(\infty,y_p)$.

From these finite-dimensional marginals distributions, we obtain that, for every $0 < s < t$, the distribution of $(X_{t-u}^{(t)})_{0<u\le s}$ is absolutely continuous with respect to the distribution of $(R_{L_\varepsilon^{(R)}+u})_{0<u\le s}$ under $\mathbb{P}_0^{(5)}$, with a density given by

$$\mathrm{w} \mapsto \frac{p_{t-s}^{(\varepsilon)}(x,\mathrm{w}(s))}{r_t(x,\varepsilon)G^{(\varepsilon)}(\infty,\mathrm{w}(s))}.$$

From this absolute continuity property, we get that $X_s^{(t)} \longrightarrow \varepsilon$ when $s\to t$, a.s. So we can define a probability measure $\Pi_t^{(x,\varepsilon)}(\mathrm{dw})$ on $C([0,t],\mathbb{R}_+)$ as the distribution of $(X_s^{(t)})_{0\le s\le t}$. It should be clear from our construction that the collection $(\Pi_t^{(x,\varepsilon)})_{t>0}$ is a regular version of the conditional distributions of $(R_s)_{0\le s\le T_\varepsilon^{(R)}}$ knowing $T_\varepsilon^{(R)} = t$ under $\mathbb{P}_x^{(-1)}$.

Finally, from the fact that $r_t(x+\varepsilon,\varepsilon) \longrightarrow r_t(x,0)$ as $\varepsilon\to 0$, and the analogous convergence $p_t^{(\varepsilon)}(y,z) \longrightarrow p_t(y,z)$, which is derived from (8), it is a simple matter to verify that the finite-dimensional marginals of $\Pi_t^{(x+\varepsilon,\varepsilon)}$ converge to those of $\Pi_t^{(x,0)}$. Tightness of the collection $(\Pi_t^{(x+\varepsilon,\varepsilon)})_{\varepsilon\in(0,1)}$ is also easy from the absolute continuity properties stated above. The last assertion of the proposition follows. $\square$

The probability measure $\Pi_t^{(x,0)}$ is also the law of the usual Bessel bridge of dimension 5 from x to 0 over the time interval $[0,t]$. This may be verified from the finite-dimensional marginals in (11), noting that the transition densities $p'_t(x,y)$, $t,x,y>0$, of the five-dimensional Bessel process satisfy $p'_t(x,y) = (\frac{y}{x})^3 p_t(x,y)$ by the the h-transform relation (3) (we refer to [24] for detailed information about Bessel bridges). The latter Bessel bridge can be defined in a simpler way using the fact that the five-dimensional Bessel process is the norm of a five-dimensional Brownian motion, and the additivity properties of squares of Bessel bridges (see e.g. [24]). This interpretation also makes it possible to define a five-dimensional Bessel bridge $\mathbf{b} = (\mathbf{b}_t)_{0\leq t\leq 1}$ from 0 to 0 over the time interval $[0,1]$. The process $\mathbf{b}$ may indeed be obtained as the square root of the sum of the squares of five independent standard (one-dimensional) Brownian bridges from 0 to 0 over $[0,1]$. One then easily verifies that the distribution of $\mathbf{b}_{1/2}$ has density

$$\rho(x) = \frac{64}{3\sqrt{\pi}}\, x^4\, e^{-2x^2}\ , \quad x>0.$$

Furthermore, conditionally on $\mathbf{b}_{1/2} = x$, the two processes $(\mathbf{b}_{\frac{1}{2}-t})_{0\leq t\leq 1/2}$ and $(\mathbf{b}_{\frac{1}{2}+t})_{0\leq t\leq 1/2}$ are independent and distributed according to $\Pi^{(x,0)}_{1/2}$ (that is, they are independent Bessel bridges of dimension 5 from x to 0).

We now turn to the main result of this section.

Proposition 4. *Let $\mathbf{e} = (\mathbf{e}_t)_{0\leq t\leq 1}$ be a normalized Brownian excursion. For every $\varepsilon>0$, set*

$$C_\varepsilon := \mathbb{E}\Big[\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big],$$

and write $\mathbf{e}^\varepsilon = (\mathbf{e}^\varepsilon_t)_{0\leq t\leq 1}$ for a random element of $C([0,1],\mathbb{R}_+)$ whose distribution is specified by

$$\mathbb{E}[F(\mathbf{e}^\varepsilon)] := C_\varepsilon^{-1}\,\mathbb{E}\Big[F(\mathbf{e})\,\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big],$$

for any nonnegative measurable function F on $C([0,1],\mathbb{R}_+)$. Then, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} C_\varepsilon = 3, \tag{12}$$

and

$$\mathbf{e}^\varepsilon \underset{\varepsilon\to 0}{\overset{(\mathrm{d})}{\longrightarrow}} \mathbf{b}$$

where $\mathbf{b} = (\mathbf{b}_t)_{0\leq t\leq 1}$ is a five-dimensional Bessel bridge from 0 to 0 over the time interval $[0,1]$. Finally, for every $\delta\in(0,\frac{1}{2})$, the total variation distance between the distribution of $(\mathbf{e}^\varepsilon_t+\varepsilon)_{\delta\leq t\leq 1-\delta}$ and the distribution of $(\mathbf{b}_t)_{\delta\leq t\leq 1-\delta}$ tends to 0 as $\varepsilon\to 0$.

Remark. As we will see later (cf. formula (30) below), the quantity C_ε can also be interpreted as the probability in a free pointed Brownian disk of perimeter 1 that the distance from the distinguished point to the boundary is smaller than $\varepsilon\sqrt{3}$. The asymptotics of C_ε when $\varepsilon\to 0$ could therefore be derived from the distribution of the latter distance, which is known explicitly [20].

Proof. It is well known that, conditionally on $\mathbf{e}_{1/2}$, the two processes $(\mathbf{e}_{\frac{1}{2}+t})_{0\leq t\leq 1/2}$ and $(\mathbf{e}_{\frac{1}{2}-t})_{0\leq t\leq 1/2}$ are independent and follow the distribution of a linear Brownian motion started from $\mathbf{e}_{1/2}$ and conditioned to hit 0 for the first time at time $1/2$. This conditioned process can be defined in a way similar to Proposition 3 (see Section 5.1 in [5] or Section 2.1 in [7]). Fix $x>0$ and $t>0$, and, for every $s\in(0,t)$, write $\mathcal{F}_s$ for the σ-field on $C([0,t],\mathbb{R})$ generated by $\mathrm{w}\mapsto(\mathrm{w}(r))_{0\leq r\leq s}$. Then the Radon-Nikodym derivative on $\mathcal{F}_s$ of the law of Brownian motion started at x and conditioned to hit 0 for the first time at time t, with respect to the law of Brownian motion started at x, is

$$\mathbf{1}_{\{\mathrm{w}(r)>0,\forall r\in[0,s]\}}\,\frac{q_{t-s}(\mathrm{w}(s),0)}{q_t(x,0)},$$

where the function $q_t(x,0)$ is as in (7) .

Write $(B_t)_{t\geq 0}$ for a linear Brownian motion that starts at x under the probability measure $\mathbb{P}_x$, and $T_y = \inf\{t \geq 0 : B_t = y\}$ for every $y \in \mathbb{R}$. Let F_1 and F_2 be bounded Lipschitz continuous functions on $C([0, \frac{1}{2}], \mathbb{R})$. We can summarize the first observation of the proof by the equality

$$\begin{aligned}
&\mathbb{E}\Big[F_1\Big((\mathbf{e}_{\frac{1}{2}-t})_{0\leq t\leq\frac{1}{2}}\Big)F_2\Big((\mathbf{e}_{\frac{1}{2}+t})_{0\leq t\leq\frac{1}{2}}\Big)\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big] \qquad (13)\\
&= \int_0^\infty \mathrm{d}x\,\pi(x)\,\mathbb{E}_x\Big[F_1\Big((B_t)_{0\leq t\leq\frac{1}{2}}\Big)\,e^{-\int_0^{1/2}\frac{\mathrm{d}t}{(\varepsilon+B_t)^2}}\,\Big|\,T_0=\frac{1}{2}\Big]\mathbb{E}_x\Big[F_2\Big((B_t)_{0\leq t\leq\frac{1}{2}}\Big)\,e^{-\int_0^{1/2}\frac{\mathrm{d}t}{(\varepsilon+B_t)^2}}\,\Big|\,T_0=\frac{1}{2}\Big],
\end{aligned}$$

where $\pi(x) = \frac{16}{\sqrt{\pi}}\,x^2\,e^{-2x^2}$ is the density of $\mathbf{e}_{1/2}$. We note that

$$\mathbb{E}_x\Big[F_1\Big((B_t)_{0\leq t\leq\frac{1}{2}}\Big)\,e^{-\int_0^{1/2}\frac{\mathrm{d}t}{(\varepsilon+B_t)^2}}\,\Big|\,T_0=\frac{1}{2}\Big] = \mathbb{E}_{x+\varepsilon}\Big[F_1\Big((B_t-\varepsilon)_{0\leq t\leq\frac{1}{2}}\Big)e^{-\int_0^{1/2}\frac{\mathrm{d}t}{(B_t)^2}}\,\Big|\,T_\varepsilon=\frac{1}{2}\Big]. \qquad (14)$$

To study the right-hand side, we rely on the next lemma, where we use the notation introduced at the beginning of the section.

Lemma 5. *Let $x > \varepsilon > 0$. Then, for every $t > 0$ and every nonnegative measurable function G on $C([0,t],\mathbb{R})$,*

$$\mathbb{E}_x\Big[G((B_s)_{0\leq s\leq t})e^{-\int_0^t\frac{\mathrm{d}s}{(B_s)^2}}\,\Big|\,T_\varepsilon=t\Big] = \frac{\varepsilon}{x}\frac{r_t(x,\varepsilon)}{q_t(x,\varepsilon)}\,\mathbb{E}_x^{(-1)}\Big[G((R_s)_{0\leq s\leq t})\,\Big|\,T_\varepsilon^{(R)}=t\Big]. \qquad (15)$$

Proof. We first prove that

$$\mathbb{E}_x\Big[G((B_s)_{0\leq s\leq T_\varepsilon})\,\exp\Big(-\int_0^{T_\varepsilon}\frac{\mathrm{d}s}{(B_t)^2}\Big)\Big] = \varepsilon^{-2}x^2\,\mathbb{E}_x^{(5)}\Big[\mathbf{1}_{\{T_\varepsilon^{(R)}<\infty\}}\,G\Big((R_s)_{0\leq s\leq T_\varepsilon^{(R)}}\Big)\Big], \qquad (16)$$

for every nonnegative measurable function G on $\mathcal{W}$. This is basically a consequence of the absolute continuity relations between Bessel processes (see e.g. [25, Exercise XI.1.22]). These relations give the equality

$$\mathbb{E}_x\Big[\mathbf{1}_{\{T_0>u\}}\,G((B_s)_{0\leq s\leq u})\,\exp\Big(-\int_0^u\frac{\mathrm{d}s}{(B_s)^2}\Big)\Big] = x^2\,\mathbb{E}_x^{(5)}\Big[(R_u)^{-2}\,G((R_s)_{0\leq s\leq u})\Big],$$

for every $u \geq 0$. So to get (16), we just need to justify the replacement of the constant time u by the hitting time of ε in the last display. This can be done by standard approximation techniques. For every $y > 0$ and $n \geq 1$, write $[y]_n$ for the unique real of the form $k2^{-n}$, $k \in \mathbb{N}$, such that $(k-1)2^{-n} < y \leq k2^{-n}$. Then, assuming that G is bounded and continuous,

$$\begin{aligned}
\mathbb{E}_x\Big[G((B_s)_{0\leq s\leq T_\varepsilon})\,e^{-\int_0^{T_\varepsilon}\frac{\mathrm{d}s}{(B_s)^2}}\Big] &= \lim_{n\to\infty}\mathbb{E}_x\Big[1_{\{[T_\varepsilon]_n<T_0\}}\,G((B_s)_{0\leq s\leq[T_\varepsilon]_n})\,e^{-\int_0^{[T_\varepsilon]_n}\frac{\mathrm{d}s}{(B_s)^2}}\Big]\\
&= \lim_{n\to\infty}\sum_{k=1}^\infty \mathbb{E}_x\Big[\mathbf{1}_{\{(k-1)2^{-n}<T_\varepsilon\leq k2^{-n}<T_0\}}\,G((B_s)_{0\leq s\leq k2^{-n}})\,e^{-\int_0^{k2^{-n}}\frac{\mathrm{d}s}{(B_s)^2}}\Big]\\
&= \lim_{n\to\infty}\sum_{k=1}^\infty x^2\,\mathbb{E}_x^{(5)}\Big[\mathbf{1}_{\{(k-1)2^{-n}<T_\varepsilon^{(R)}\leq k2^{-n}\}}\,(R_{k2^{-n}})^{-2}\,G((R_s)_{0\leq s\leq k2^{-n}})\Big]\\
&= \lim_{n\to\infty} x^2\,\mathbb{E}_x^{(5)}\Big[(R_{[T_\varepsilon^{(R)}]_n})^{-2}\,G\Big((R_s)_{0\leq s\leq[T_\varepsilon^{(R)}]_n}\Big)\,\mathbf{1}_{\{T_\varepsilon^{(R)}<\infty\}}\Big]\\
&= \varepsilon^{-2}\,x^2\,\mathbb{E}_x^{(5)}\Big[G\Big((R_s)_{0\leq s\leq T_\varepsilon^{(R)}}\Big)\,\mathbf{1}_{\{T_\varepsilon^{(R)}<\infty\}}\Big],
\end{aligned}$$

where the use of dominated convergence in the last equality is justified by the fact that the variable $(\inf_{s\geq 0} R_s)^{-2}$ is integrable under $\mathbb{P}_x^{(5)}$. This completes the proof of (16).

Recalling that $\mathbb{P}_x^{(5)}(T_\varepsilon^{(R)} < \infty) = (\varepsilon/x)^3$ and using (4), we get that the right-hand side of (16) can be written in the form

$$\frac{\varepsilon}{x}\,\mathbb{E}_x^{(-1)}\Big[G\Big((R_s)_{0\leq s\leq T_\varepsilon^{(R)}}\Big)\Big].$$

For $x > \varepsilon \geq 0$, the density of T_ε under $\mathbb{P}_x$ is the function $t \mapsto q_t(x,\varepsilon)$ defined in (7). Also recall that the density of $T_\varepsilon^{(R)}$ under $\mathbb{P}_x^{(-1)}$ is the function $t \mapsto r_t(x,\varepsilon)$. It follows from (16) that, for $x > \varepsilon > 0$,

$$\int_0^\infty \mathrm{d}t\, q_t(x,\varepsilon)\, \mathbb{E}_x\Big[G((B_s)_{0\le s\le t}) e^{-\int_0^t \frac{\mathrm{d}s}{(B_s)^2}} \,\Big|\, T_\varepsilon = t\Big] = \frac{\varepsilon}{x}\int_0^\infty \mathrm{d}t\, r_t(x,\varepsilon)\, \mathbb{E}_x^{(-1)}\Big[G((R_s)_{0\le s\le t}) \,\Big|\, T_\varepsilon^{(R)} = t\Big].$$

Replacing $G((\mathrm{w}(s))_{0\le s\le t})$ by $g(t)G((\mathrm{w}(s))_{0\le s\le t})$, with an arbitrary nonnegative measurable function g on $\mathbb{R}_+$, we get that (15) holds for Lebesgue almost every $t > 0$.

To verify that (15) indeed holds for *every* $t > 0$, it is enough to consider the special case where $G((\mathrm{w}(s))_{s\le t}) = \mathbf{1}_{\{t>t_p\}} g_1(\mathrm{w}(t_1)) \dots g_p(\mathrm{w}(t_p))$ where $0 < t_1 < \dots < t_p$ and $g_1,\dots,g_p$ are bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}_+$. Then formula (11) shows that the right-hand side of (15) is a continuous function of t on (t_p,∞). On the other hand, for $\delta > 0$ and $t > t_p + \delta$, the left-hand side of (15) is bounded above by $I_\delta(t)$ and bounded below by $e^{-\delta/\varepsilon^2} I_\delta(t)$, where

$$\begin{aligned} I_\delta(t) &= \mathbb{E}_x\Big[g_1(B_{t_1})\dots g_p(B_{t_p})\, e^{-\int_0^{t-\delta} \frac{\mathrm{d}s}{(B_s)^2}} \,\Big|\, T_\varepsilon = t\Big] \\ &= \mathbb{E}_x\Big[g_1(B_{t_1})\dots g_p(B_{t_p})\, e^{-\int_0^{t-\delta} \frac{\mathrm{d}s}{(B_s)^2}}\, \mathbf{1}_{\{T_\varepsilon > t-\delta\}}\, \frac{q_\delta(B_{t-\delta},\varepsilon)}{q_t(x,\varepsilon)}\Big]. \end{aligned}$$

Thanks to dominated convergence, the last formula implies that $I_\delta(t)$ is a continuous function of $t \in (t_p+\delta,\infty)$. Letting $\delta \to 0$, it follows that the left-hand side of (15) is also a continuous function of t on (t_p,∞). We conclude that (15) holds for every $t > t_p$. This completes the proof of Lemma 5. $\square$

We return to the proof of Proposition 4. From (13), (14) and (15) (with $t = 1/2$ and x replaced by $x+\varepsilon$), we get,

$$\begin{aligned} &\varepsilon^{-2}\mathbb{E}\Big[F_1\Big((\mathbf{e}_{\frac12 - t})_{0\le t\le 1/2}\Big) F_2\Big((\mathbf{e}_{\frac12 + t})_{0\le t\le 1/2}\Big) \exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big] \qquad (17) \\ &= \int_0^\infty \mathrm{d}x\, \frac{\pi(x)}{(x+\varepsilon)^2}\, \frac{r_{\frac12}(x+\varepsilon,\varepsilon)^2}{q_{\frac12}(x+\varepsilon,\varepsilon)^2} \mathbb{E}_{x+\varepsilon}^{(-1)}\Big[F_1((R_s-\varepsilon)_{0\le s\le\frac12})\Big| T_\varepsilon^{(R)} = \frac12\Big]\, \mathbb{E}_{x+\varepsilon}^{(-1)}\Big[F_2((R_s-\varepsilon)_{0\le s\le\frac12})\Big| T_\varepsilon^{(R)} = \frac12\Big]. \end{aligned}$$

We have the explicit expression

$$\frac{r_{\frac12}(x+\varepsilon,\varepsilon)}{q_{\frac12}(x+\varepsilon,\varepsilon)} = \frac{\sqrt{\pi}}{4\varepsilon^3}\,\mathrm{erfc}(\frac{1}{2\varepsilon}+x)\, \exp\Big((\frac{1}{2\varepsilon}+x)^2\Big) - \frac{1}{2\varepsilon^2} + \frac{x}{\varepsilon} + 1,$$

from which it is a simple matter to get that

$$\lim_{\varepsilon\to 0} \frac{r_{\frac12}(x+\varepsilon,\varepsilon)}{q_{\frac12}(x+\varepsilon,\varepsilon)} = \frac{r_{\frac12}(x,0)}{q_{\frac12}(x,0)} = 2x^2, \qquad (18)$$

and

$$\frac{r_{\frac12}(x+\varepsilon,\varepsilon)}{q_{\frac12}(x+\varepsilon,\varepsilon)} \le K_1 x^2 + K_2 \qquad (19)$$

with constants K_1 and K_2 that do not depend on $x > 0$ and $\varepsilon \in (0,1]$. Moreover, using the fact that F_1 and F_2 are Lipschitz continuous, we have for $i = 1,2$, and for every $x > 0$,

$$\begin{aligned} \lim_{\varepsilon\to0} \mathbb{E}_{x+\varepsilon}^{(-1)}\Big[F_i((R_s-\varepsilon)_{0\le s\le\frac12}) \,\Big|\, T_\varepsilon^{(R)} = \frac12\Big] &= \lim_{\varepsilon\to0} \mathbb{E}_{x+\varepsilon}^{(-1)}\Big[F_i((R_s)_{0\le s\le\frac12}) \,\Big|\, T_\varepsilon^{(R)} = \frac12\Big] \\ &= \mathbb{E}_x^{(-1)}\Big[F_i((R_s)_{0\le s\le\frac12}) \,\Big|\, T_0^{(R)} = \frac12\Big], \qquad (20) \end{aligned}$$

by the last assertion of Proposition 3.

Thanks to (18) and (20), we can now pass to the limit $\varepsilon \to 0$ in the right-hand side of (17), using (19) to justify dominated convergence. It follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathbb{E}\Big[F_1\Big((\mathbf{e}_{\frac{1}{2}-t})_{0\leq t\leq \frac{1}{2}}\Big)F_2\Big((\mathbf{e}_{\frac{1}{2}+t})_{0\leq t\leq \frac{1}{2}}\Big)\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big]\\ &\quad = 4\int_0^\infty \mathrm{d}x\, \pi(x)\, x^2\, \mathbb{E}_x^{(-1)}\Big[F_1((R_s)_{0\leq s\leq\frac{1}{2}})\,\Big|\, T_0^{(R)}=\frac{1}{2}\Big]\,\mathbb{E}_x^{(-1)}\Big[F_2((R_s)_{0\leq s\leq\frac{1}{2}})\,\Big|\, T_0^{(R)}=\frac{1}{2}\Big].\end{aligned} \tag{21}$$

The particular case $F_1 = F_2 = 1$ of (21) gives

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}C_\varepsilon = 4\int_0^\infty \mathrm{d}x\, x^2\,\pi(x) = 3.$$

Furthermore, the function $x\mapsto \frac{4}{3}x^2\pi(x)$ is the density of $\mathbf{b}_{1/2}$, and it follows from (21) that we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}(C_\varepsilon)^{-1}\mathbb{E}\Big[F_1\Big((\mathbf{e}_{\frac{1}{2}-t})_{0\leq t\leq \frac{1}{2}}\Big)F_2\Big((\mathbf{e}_{\frac{1}{2}+t})_{0\leq t\leq \frac{1}{2}}\Big)\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big]\\ &\quad = \mathbb{E}\Big[F_1\Big((\mathbf{b}_{\frac{1}{2}-t})_{0\leq t\leq \frac{1}{2}}\Big)F_2\Big((\mathbf{b}_{\frac{1}{2}+t})_{0\leq t\leq \frac{1}{2}}\Big)\Big].\end{aligned}$$

This gives the convergence in distribution of $\mathbf{e}^\varepsilon$ toward $\mathbf{b}$.

It remains to prove the last assertion of the proposition. To this end, fix $\varepsilon > 0$, and let A_1 and A_2 be measurable subsets of $C([0,\frac{1}{2}-\delta],\mathbb{R}_+)$ such that $\min\{\mathrm{w}(t) : 0\leq t\leq \frac{1}{2}-\delta\} > \varepsilon$ for every $\mathrm{w}\in A_1\cup A_2$. Then,

$$\begin{aligned}&\mathbb{P}\Big(\Big((\varepsilon+\mathbf{e}^\varepsilon_{\frac{1}{2}-t})_{0\leq t\leq\frac{1}{2}-\delta},(\varepsilon+\mathbf{e}^\varepsilon_{\frac{1}{2}+t})_{0\leq t\leq\frac{1}{2}-\delta}\Big)\in A_1\times A_2\Big)\\ &\quad = C_\varepsilon^{-1}\mathbb{E}\Big[\mathbf{1}_{A_1}((\varepsilon+\mathbf{e}_{\frac{1}{2}-t})_{0\leq t\leq\frac{1}{2}-\delta})\,\mathbf{1}_{A_2}((\varepsilon+\mathbf{e}_{\frac{1}{2}+t})_{0\leq t\leq\frac{1}{2}-\delta})\exp\Big(-\int_0^1\frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\Big]\\ &\quad = C_\varepsilon^{-1}\int_0^\infty \mathrm{d}x\,\pi(x)\,\mathbb{E}_x\Big[\mathbf{1}_{A_1}((\varepsilon+B_t)_{0\leq t\leq\frac{1}{2}-\delta})\exp\Big(-\int_0^{1/2}\frac{\mathrm{d}t}{(\varepsilon+B_t)^2}\Big)\,\Big|\, T_0=\frac{1}{2}\Big]\\ &\qquad\qquad \times\mathbb{E}_x\Big[\mathbf{1}_{A_2}((\varepsilon+B_t)_{0\leq t\leq\frac{1}{2}-\delta})\exp\Big(-\int_0^{1/2}\frac{\mathrm{d}t}{(\varepsilon+B_t)^2}\Big)\,\Big|\, T_0=\frac{1}{2}\Big]\\ &\quad = C_\varepsilon^{-1}\int_\varepsilon^\infty \mathrm{d}x\,\pi(x-\varepsilon)\,\mathbb{E}_x\Big[\mathbf{1}_{A_1}((B_t)_{0\leq t\leq\frac{1}{2}-\delta})\exp\Big(-\int_0^{1/2}\frac{\mathrm{d}t}{(B_t)^2}\Big)\,\Big|\, T_\varepsilon=\frac{1}{2}\Big]\\ &\qquad\qquad \times\mathbb{E}_x\Big[\mathbf{1}_{A_2}((B_t)_{0\leq t\leq\frac{1}{2}-\delta})\exp\Big(-\int_0^{1/2}\frac{\mathrm{d}t}{(B_t)^2}\Big)\,\Big|\, T_\varepsilon=\frac{1}{2}\Big].\end{aligned}$$

By (15), the quantities in the last display are equal to

$$\begin{aligned}&C_\varepsilon^{-1}\varepsilon^2\int_\varepsilon^\infty \mathrm{d}x\,\pi(x-\varepsilon)\Big(\frac{r_{1/2}(x,\varepsilon)}{x\,q_{1/2}(x,\varepsilon)}\Big)^2\,\mathbb{P}_x^{(-1)}\Big((R_t)_{0\leq t\leq\frac{1}{2}-\delta}\in A_1\,\Big|\, T_\varepsilon^{(R)}=\frac{1}{2}\Big)\\ &\qquad\qquad \times\mathbb{P}_x^{(-1)}\Big((R_t)_{0\leq t\leq\frac{1}{2}-\delta}\in A_2\,\Big|\, T_\varepsilon^{(R)}=\frac{1}{2}\Big).\end{aligned}$$

Recalling the Radon-Nikodym derivative (10), this is also equal to

$$C_\varepsilon^{-1}\varepsilon^2\int_\varepsilon^\infty \mathrm{d}x\,\frac{\pi(x-\varepsilon)}{x^2 q_{1/2}(x,\varepsilon)^2}\,\mathbb{E}_x^{(-1)}\Big[\mathbf{1}_{A_1}((R_t)_{0\leq t\leq\frac{1}{2}-\delta})r_\delta(R_{\frac{1}{2}-\delta},\varepsilon)\Big]\,\mathbb{E}_x^{(-1)}\Big[\mathbf{1}_{A_2}((R_t)_{0\leq t\leq\frac{1}{2}-\delta})r_\delta(R_{\frac{1}{2}-\delta},\varepsilon)\Big].$$

It is convenient to consider that, under each probability measure $\mathbb{P}_x^{(-1)}$, we have an independent copy $(R'_t)_{t\geq 0}$ of the Bessel process $(R_t)_{t\geq 0}$. The last display can then be written as

$$C_\varepsilon^{-1}\varepsilon^2\int_\varepsilon^\infty \mathrm{d}x\,\frac{\pi(x-\varepsilon)}{x^2 q_{1/2}(x,\varepsilon)^2}\,\mathbb{E}_x^{(-1)}\Big[\mathbf{1}_{A_1}((R_t)_{0\leq t\leq\frac{1}{2}-\delta})\mathbf{1}_{A_2}((R'_t)_{0\leq t\leq\frac{1}{2}-\delta})r_\delta(R_{\frac{1}{2}-\delta},\varepsilon)r_\delta(R'_{\frac{1}{2}-\delta},\varepsilon)\Big].$$

We have thus obtained that

$$\begin{aligned}&\mathbb{P}\Big(\big((\varepsilon+\mathbf{e}^{\varepsilon}_{\frac12-t})_{0\le t\le\frac12-\delta},(\varepsilon+\mathbf{e}^{\varepsilon}_{\frac12+t})_{0\le t\le\frac12-\delta}\big)\in A\Big)\\&\quad=C_\varepsilon^{-1}\varepsilon^2\int_\varepsilon^\infty \mathrm{d}x\,\frac{\pi(x-\varepsilon)}{x^2q_{1/2}(x,\varepsilon)^2}\,\mathbb{E}^{(-1)}_x\Big[\mathbf{1}_A\big((R_t)_{0\le t\le\frac12-\delta},(R'_t)_{0\le t\le\frac12-\delta}\big)\,r_\delta(R_{\frac12-\delta},\varepsilon)\,r_\delta(R'_{\frac12-\delta},\varepsilon)\Big],\end{aligned}\tag{22}$$

for any measurable subset A of $C([0,\frac12-\delta],(\varepsilon,\infty))^2$.

On the other hand, by the last observation before the statement of the proposition, and using again the Radon-Nidodym derivative (10), we have, for any measurable subset A of $C([0,\frac12-\delta],(0,\infty))^2$,

$$\begin{aligned}&\mathbb{P}\Big(\big((\mathbf{b}_{\frac12-t})_{0\le t\le\frac12-\delta},(\mathbf{b}_{\frac12+t})_{0\le t\le\frac12-\delta}\big)\in A\Big)\\&\quad=\frac43\int_0^\infty \mathrm{d}x\,\frac{x^2\pi(x)}{r_{1/2}(x,0)^2}\,\mathbb{E}^{(-1)}_x\Big[\mathbf{1}_A\big((R_t)_{0\le t\le\frac12-\delta},(R'_t)_{0\le t\le\frac12-\delta}\big)\,r_\delta(R_{\frac12-\delta},0)\,r_\delta(R'_{\frac12-\delta},0)\Big].\end{aligned}\tag{23}$$

By comparing the right-hand sides of (22) and (23), we get that the total variation distance between the distribution of $(\mathbf{e}^\varepsilon_t+\varepsilon)_{\delta\le t\le1-\delta}$ and the distribution of $(\mathbf{b}_t)_{\delta\le t\le 1-\delta}$ is bounded above by the sum of the quantities $\mathbb{P}(\min_{\delta\le t\le1-\delta}\mathbf{b}_t\le\varepsilon)$ and

$$\int_\varepsilon^\infty \mathrm{d}x\,\mathbb{E}^{(-1)}_x\left[\mathbf{1}_{\{m_{R,R'}>\varepsilon\}}\left|\frac{\varepsilon^2\pi(x-\varepsilon)}{C_\varepsilon\, x^2q_{1/2}(x,\varepsilon)^2}r_\delta(R_{\frac12-\delta},\varepsilon)\,r_\delta(R'_{\frac12-\delta},\varepsilon)-\frac{4x^2\pi(x)}{3r_{1/2}(x,0)^2}r_\delta(R_{\frac12-\delta},0)\,r_\delta(R'_{\frac12-\delta},0)\right|\right]$$

where we have written $m_{R,R'}:=\min\{R_t\wedge R'_t:0\le t\le\frac12-\delta\}$. Clearly, $\mathbb{P}(\min_{\delta\le t\le1-\delta}\mathbf{b}_t\le\varepsilon)$ tends to 0 as $\varepsilon\to0$, so we need only check that the quantity in the last display also tends to 0 as $\varepsilon\to0$.

We first note that

$$\frac{\varepsilon^2\pi(x-\varepsilon)}{C_\varepsilon\,x^2q_{1/2}(x,\varepsilon)^2}=\frac{\varepsilon^2}{C_\varepsilon}\,\frac{4\sqrt{\pi}}{x^2}\quad\text{and}\quad\frac{4x^2\pi(x)}{3r_{1/2}(x,0)^2}=\frac{4\sqrt{\pi}}{3x^2}.$$

Recalling (12), we see that the desired result will follow if we can prove that

$$\lim_{\varepsilon\to0}\int_\varepsilon^\infty\frac{\mathrm{d}x}{x^2}\,\mathbb{E}^{(-1)}_x\Big[\mathbf{1}_{\{m_{R,R'}>\varepsilon\}}\,\Big|r_\delta(R_{\frac12-\delta},\varepsilon)\,r_\delta(R'_{\frac12-\delta},\varepsilon)-r_\delta(R_{\frac12-\delta},0)\,r_\delta(R'_{\frac12-\delta},0)\Big|\Big]=0,\tag{24}$$

and

$$\int_0^\infty\frac{\mathrm{d}x}{x^2}\,\mathbb{E}^{(-1)}_x\Big[\mathbf{1}_{\{m_{R,R'}>0\}}r_\delta(R_{\frac12-\delta},0)\,r_\delta(R'_{\frac12-\delta},0)\Big]<\infty.\tag{25}$$

The proof of (25) is immediate since the integral in (25) is equal, up to a multiplicative constant, to the right-hand side of (23) with $A=C([0,\frac12-\delta],(0,\infty))^2$. Then, we observe that the integral in (24) is equal to

$$\int_\varepsilon^\infty\frac{\mathrm{d}x}{x^2}\int_\varepsilon^\infty\mathrm{d}y\int_\varepsilon^\infty\mathrm{d}z\,p^{(\varepsilon)}_{\frac12-\delta}(x,y)p^{(\varepsilon)}_{\frac12-\delta}(x,z)\,\Big|r_\delta(y,\varepsilon)\,r_\delta(z,\varepsilon)-r_\delta(y,0)\,r_\delta(z,0)\Big|.$$

and we can bound $p^{(\varepsilon)}_{\frac12-\delta}(x,y)p^{(\varepsilon)}_{\frac12-\delta}(x,z)$ by $p_{\frac12-\delta}(x,y)p_{\frac12-\delta}(x,z)$. Furthermore, we know that, for every fixed $y,z>0$, the quantities $|r_\delta(y,\varepsilon)\,r_\delta(z,\varepsilon)-r_\delta(y,0)\,r_\delta(z,0)|$ tend to 0 as $\varepsilon\to0$, and these quantities (for $y>\varepsilon$ and $z>\varepsilon$) are uniformly bounded by a constant depending only on δ (this follows from our explicit formula for $r_t(x,\varepsilon)$). In order to justify dominated convergence and to get (24), it suffices to verify that

$$\int_0^\infty\frac{\mathrm{d}x}{x^2}\int_0^\infty\mathrm{d}y\int_0^\infty\mathrm{d}z\,p_{\frac12-\delta}(x,y)p_{\frac12-\delta}(x,z)<\infty.\tag{26}$$

However, using (4) and writing $p'_t(x,y)$ for the transition densities of the five-dimensional Bessel process, we have for every $x>0$,

$$\int_0^\infty\mathrm{d}y\,p_{\frac12-\delta}(x,y)=\int_0^\infty\mathrm{d}y\,\frac{x^3}{y^3}\,p'_{\frac12-\delta}(x,y)=x^3\,\mathbb{E}^{(5)}_x[(R_{\frac12-\delta})^{-3}]\le x^3\,(x^{-3}\wedge K),$$

with a constant K depending only on δ. It follows that the integral in (26) is bounded above by $\int_0^\infty\mathrm{d}x\,x^{-2}\,(1\wedge Kx^3)^2<\infty$. This completes the proof of (24) and of the proposition. $\square$

4 The free pointed Brownian disk and the Brownian half-plane

In this section we recall the definitions of the (free pointed) Brownian disk and of the Brownian half-plane along the lines of [4, 6, 7, 14]. Our presentation is a little different from the latter papers and better suited to our applications.

The free pointed Brownian disk. We consider a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{(t_i,\omega_i)}$ on $[0,1]\times\mathcal{S}$ with intensity

$$2\,\mathbf{1}_{[0,1]}(t)\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega).$$

We then introduce the compact metric space $\mathfrak{H}$, which is obtained from the disjoint union

$$[0,1]\cup\Big(\bigcup_{i\in I}\mathcal{T}_{(\omega_i)}\Big) \tag{27}$$

by identifying 0 with 1 and, for every $i\in I$, the root $\rho_{(\omega_i)}$ of $\mathcal{T}_{(\omega_i)}$ with the point t_i of $[0,1]$. The metric $\mathrm{d}_{\mathfrak{H}}$ on $\mathfrak{H}$ is defined as follows. First, the restriction of $\mathrm{d}_{\mathfrak{H}}$ to each tree $\mathcal{T}_{(\omega_i)}$ is the metric $d_{(\omega_i)}$. Then, if $u,v\in[0,1]$, we take $\mathrm{d}_{\mathfrak{H}}(u,v)=\min\{u\vee v-u\wedge v, 1-u\vee v+u\wedge v)$. If $u\in[0,1]$, and $v\in\mathcal{T}_{(\omega_i)}$ for some $i\in I$, $\mathrm{d}_{\mathfrak{H}}(u,v)=\mathrm{d}_{\mathfrak{H}}(u,\rho_{(\omega_i)})+d_{(\omega_i)}(\rho_{(\omega_i)},v)$. Finally if $u\in\mathcal{T}_{(\omega_i)}$ and $v\in\mathcal{T}_{(\omega_j)}$, with $j\neq i$,

$$\mathrm{d}_{\mathfrak{H}}(u,v)=d_{(\omega_i)}(u,\rho_{(\omega_i)})+d_{\mathfrak{H}}(\rho_{(\omega_i)},\rho_{(\omega_j)})+d_{(\omega_j)}(\rho_{(\omega_j)},v).$$

The volume measure on $\mathfrak{H}$ is just the sum of the volume measures on the trees $\mathcal{T}_{(\omega_i)}$, $i\in I$.

If $\Sigma:=\sum_{i\in I}\sigma(\omega_i)$ is the total mass of the volume measure, we can define a cyclic clockwise exploration $(\mathcal{E}_t)_{0\leq t\leq\Sigma}$ of $\mathfrak{H}$, informally by concatenating the mappings $p_{(\omega_i)}:[0,\sigma(\omega_i)]\longrightarrow\mathcal{T}_{(\omega_i)}$ in the order prescribed by the t_i's. To give a more precise definition, set

$$\beta_s:=\sum_{i\in I}\mathbf{1}_{\{t_i\leq s\}}\,\sigma(\omega_i)\,,\ \ \beta_{s-}:=\sum_{i\in I}\mathbf{1}_{\{t_i<s\}}\,\sigma(\omega_i)\,,$$

for every $s\in[0,1]$. Then, for every $t\in[0,\Sigma]$, we define $\mathcal{E}_t\in\mathfrak{H}$ as follows. We observe that there is a unique $s\in[0,1]$ such that $\beta_{s-}\leq t\leq\beta_s$, and:

- Either there is a (unique) $i\in I$ such that $s=t_i$, and we set $\mathcal{E}_t:=p_{(\omega_i)}(t-\beta_{t_i-})$.
- Or there is no such i and we set $\mathcal{E}_t:=s$.

Note that $\mathcal{E}_\Sigma=\mathcal{E}_0$ (because 1 is identified to 0 in $\mathfrak{H}$).

The clockwise exploration allows us to define "intervals" in $\mathfrak{H}$. Let us make the convention that, if $s,t\in[0,\Sigma]$ and $s>t$, the (real) interval $[s,t]$ is defined by $[s,t]:=[s,\Sigma]\cup[0,t]$ (of course, if $s\leq t$, $[s,t]$ is the usual interval). Then, for every $u,v\in\mathfrak{H}$, such that $u\neq v$, there is a smallest interval $[s,t]$, with $s,t\in[0,\Sigma]$, such that $\mathcal{E}_s=u$ and $\mathcal{E}_t=v$, and we define

$$[|u,v|]:=\{\mathcal{E}_r:r\in[s,t]\}.$$

We have typically $[|u,v|]\neq[|v,u|]$. Of course, we take $[|u,u|]=\{u\}$. Note that we use the notation $[|u,v|]$ rather than $[u,v]$ to avoid confusion with intervals of the real line.

We next assign real labels to the points of $\mathfrak{H}$. To this end, we let $(\mathbf{e}_t)_{0\leq t\leq 1}$ be a normalized Brownian excursion, which is independent of $\mathcal{N}$. For $t\in[0,1]$, we set $\Lambda_t:=\sqrt{3}\,\mathbf{e}_t$, and for $u\in\mathcal{T}_{(\omega_i)}$, $i\in I$,

$$\Lambda_u:=\sqrt{3}\,\mathbf{e}_{t_i}+\ell_u(\omega_i),$$

where we recall that $\ell_u(\omega_i)$ is the label of u in $\mathcal{T}_{(\omega_i)}$. By [6, Lemma 11], $\min\{\Lambda_u:u\in\mathfrak{H}\}$ is attained at a unique point v_* of $\mathfrak{H}$, and we set $\Lambda_*:=\Lambda_{v_*}$ to simplify notation.

Labels allow us to define the pseudo-metric D on $\mathfrak{H}$ as follows. For every $u,v\in\mathfrak{H}$, we first set

$$D^\circ(u,v):=\Lambda_u+\Lambda_v-2\max\Big(\inf_{w\in[|u,v|]}\Lambda_w,\inf_{w\in[|v,u|]}\Lambda_w\Big), \tag{28}$$

and then

$$D(u,v) := \inf_{u_0=u,u_1,\dots,u_p=v} \sum_{i=1}^{p} D^\circ(u_{i-1},u_i), \tag{29}$$

where the infimum is over all choices of the integer $p \geq 1$ and of the finite sequence $u_0, u_1, \dots, u_p$ in $\mathfrak{H}$ such that $u_0 = u$ and $u_p = v$. One immediately verifies that $D(u,v) \geq |\Lambda_u - \Lambda_v|$ for every $u, v \in \mathfrak{H}$. It easily follows that, for every $u \in \mathfrak{H}$, $D(u,v_*) = D^\circ(u,v_*) = \Lambda_u - \Lambda_*$. We also notice that the mapping $(u,v) \mapsto D(u,v)$ is continuous on $\mathfrak{H} \times \mathfrak{H}$ (note that $D^\circ(u_n,u) \longrightarrow 0$ if $u_n \to u$ in $\mathfrak{H}$, and use the triangle inequality).

We abuse notation by writing $\mathfrak{H}/\{D=0\}$ for the quotient space of $\mathfrak{H}$ with respect to the equivalence relation defined by setting $u \sim v$ if and only if $D(u,v) = 0$.

Definition 6. *The free pointed Brownian disk with perimeter* 1 *is the quotient space* $\mathbb{D}^\bullet := \mathfrak{H}/\{D=0\}$, *which is equipped with the distance induced by* D *and with a distinguished point which is the equivalence class of* v_*. *The volume measure on* $\mathbb{D}^\bullet$ *is the pushforward of the volume measure on* $\mathfrak{H}$ *under the canonical projection.*

We may and will view $\mathbb{D}^\bullet$ as a (random) pointed compact measure metric space, that is, as an element of the space $\mathbb{M}^\bullet$ of Section 2.2. The reader will easily check that this presentation of the free pointed Brownian disk is consistent with the one in [7]. Note that the role of the Brownian excursion $\mathbf{e}$ is played in [6, 7] by a standard Brownian bridge. The celebrated Vervaat transformation [26] connecting the Brownian bridge with the Brownian excursion shows that this makes no difference (note that adding a random constant to all labels does not change the definition of D).

We will use the notation Π for the canonical projection from $\mathfrak{H}$ onto $\mathbb{D}^\bullet$. We note that $\mathbb{D}^\bullet$ is a length space. This can be verified by observing that, for every $u, v \in \mathfrak{H}$, $D^\circ(u,v)$ is the length of a continuous curve from $\Pi(u)$ to $\Pi(v)$ in $\mathbb{D}$, namely the curve obtained by concatenating the respective simple geodesics from $\Pi(u)$ and $\Pi(v)$ to $\Pi(v_*)$ until the point where they merge (see e.g. [17, Section 2.6] for a definition of simple geodesics in the Brownian map, which is immediately adapted to the present setting).

Furthermore, the space $\mathbb{D}^\bullet$ is homeomorphic to the closed unit disk of the plane [6], and, in this homeomorphism, the unit circle corresponds to $\partial\mathbb{D}^\bullet := \Pi([0,1))$. We define the uniform measure μ on $\partial\mathbb{D}^\bullet$ as the pushforward of Lebesgue measure on $[0,1)$ under Π. From [6, Lemma 14], one knows that a.s. for every $u \in [0,1) \subset \mathfrak{H}$, the equivalence class of u in the quotient $\mathfrak{H}/\{D=0\}$ is a singleton (no point of $[0,1)$ is identified with another point of $\mathfrak{H}$). Notice that [6] deals with the slightly different model where the total volume of $\mathfrak{H}$ is fixed (corresponding to the Brownian disk with fixed volume and perimeter), but the result also applies to our setting. In particular the mapping $[0,1) \ni u \mapsto \Pi(u)$ is injective.

We will keep the notation D for the metric of $\mathbb{D}^\bullet$ and, without risk of confusion, we identify v_* with $\Pi(v_*)$. For every $x \in \mathbb{D}^\bullet$, we set $\Lambda_x := \Lambda_u$, where u is a point of $\mathfrak{H}$ such that $\Pi(u) = x$ (the bound $|\Lambda_u - \Lambda_v| \leq D(u,v)$ shows that this does not depend on the choice of u). Then, we have $D(v_*, x) = \Lambda_x - \Lambda_*$ for every $x \in \mathbb{D}^\bullet$.

The fact that the Brownian bridge used in the presentation of [6, 7] is replaced here by a Brownian excursion has an important consequence. In [6, 7], the equivalence class of 0 is a typical point of the boundary, in a sense that can be made precise, whereas here $\Pi(0)$ is the point of $\partial\mathbb{D}^\bullet$ that is closest to v_* (and is therefore a very special point). The distance from the distinguished point v_* to $\partial\mathbb{D}^\bullet$ is

$$D(v_*, \partial\mathbb{D}^\bullet) = D(v_*, \Pi(0)) = -\Lambda_*.$$

The explicit distribution of $D(v_*, \partial\mathbb{D}^\bullet)$ is given in [20], but for us it will be sufficient to know the asymptotics of $\mathbb{P}(D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon)$ when $\varepsilon > 0$. To this end, note that the event $\{D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon\} = \{\Lambda_* \geq -\varepsilon\}$ occurs if and only if we have $\sqrt{3}\mathbf{e}_{t_i} + W_*(\omega_i) \geq -\varepsilon$ for every atom (t_i, ω_i) of $\mathcal{N}$. Using (2), we obtain that

$$\mathbb{P}(D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon) = \mathbb{E}\Big[\exp\Big(-3\int_0^1 \frac{\mathrm{d}t}{(\sqrt{3}\,\mathbf{e}_t + \varepsilon)^2}\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\mathbf{e}_t + \varepsilon/\sqrt{3})^2}\Big)\Big] \tag{30}$$

and (12) then yields

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathbb{P}(D(v_*,\partial\mathbb{D}^\bullet)\leq\varepsilon)=1. \tag{31}$$

The Brownian half-plane.

We now present the construction of the Brownian half-plane along the lines of [4] or [14]. This construction is very similar to that of the free pointed Brownian disk presented above, and we will therefore omit a few details.

We consider a Poisson point measure $\mathcal{N}_\infty=\sum_{j\in J}\delta_{(t_j^\infty,\omega_j^\infty)}$ on $\mathbb{R}\times\mathcal{S}$ with intensity $2\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega)$. We introduce the locally compact metric space $\mathfrak{H}_\infty$, which is obtained from the disjoint union

$$\mathbb{R}\cup\Big(\bigcup_{j\in J}\mathcal{T}_{(\omega_j^\infty)}\Big) \tag{32}$$

by identifying, for every $j\in J$, the root $\rho_{(\omega_j^\infty)}$ of $\mathcal{T}_{(\omega_j)}$ with the point t_j^∞ of $\mathbb{R}$. The metric $\mathrm{d}_{\mathfrak{H}_\infty}$ on $\mathfrak{H}_\infty$ is defined in the same way as the metric $\mathrm{d}_{\mathfrak{H}}$ on $\mathfrak{H}$ was defined above (the restriction of $\mathrm{d}_{\mathfrak{H}_\infty}$ to $\mathbb{R}$ is the usual Euclidean metric). We note that $\mathfrak{H}_\infty$ is a (non-compact) $\mathbb{R}$-tree. The volume measure on $\mathfrak{H}_\infty$ is the σ-finite measure that puts no mass on $\mathbb{R}$ and whose restriction to each tree $\mathcal{T}_{(\omega_j)}$, $j\in J$, is the volume measure on this tree.

Similarly as above, the clockwise exploration $(\mathcal{E}_s^\infty)_{s\in\mathbb{R}}$ of $\mathfrak{H}_\infty$ is defined by concatenating the mappings $p_{(\omega_j^\infty)}:[0,\sigma(\omega_j^\infty)]\longrightarrow\mathcal{T}_{(\omega_j^\infty)}$ in the order prescribed by the t_j^∞'s, in such a way that $\mathcal{E}_0^\infty=0$ (so the points $(\mathcal{E}_s^\infty,s\geq 0)$ correspond exactly to the union of $\mathbb{R}_+$ and of the trees $\mathcal{T}_{(\omega_j^\infty)}$ for indices j such that $t_j^\infty\geq 0$). We omit the precise description of $(\mathcal{E}_s^\infty)_{s\in\mathbb{R}}$, which should be obvious from the analogous definition of $(\mathcal{E}_s)_{s\in[0,\Sigma]}$ given above.

The clockwise exploration allows us to define intervals in the space $\mathfrak{H}_\infty$. We now make the convention that, if $s,t\in\mathbb{R}$ and $s>t$, $[s,t]=[s,\infty)\cup(-\infty,t]$. Then, for every $u,v\in\mathfrak{H}_\infty$, such that $u\neq v$, we set $[|u,v|]:=\{\mathcal{E}_r^\infty:r\in[s,t]\}$, where $[s,t]$ is the smallest "interval" such that $\mathcal{E}_s^\infty=u$ and $\mathcal{E}_t^\infty=v$. Note that at least one of the two intervals $[|u,v|]$ and $[|v,u|]$ is compact.

In order to assign real labels to the points of $\mathfrak{H}_\infty$, we consider a two-sided Brownian motion $B=(B_t)_{t\in\mathbb{R}}$ (in other words, $(B_t)_{t\geq 0}$ and $(B_{-t})_{t\geq 0}$ are two independent linear Brownian motions started from 0), and we assume that B is independent of $\mathcal{N}_\infty$. We set $\Lambda_u^\infty:=\sqrt{3}\,B_u$ if $u\in\mathbb{R}$, and for $u\in\mathcal{T}_{(\omega_j^\infty)}$, $j\in J$,

$$\Lambda_u^\infty:=\sqrt{3}\,B_{t_j^\infty}+\ell_u(\omega_j^\infty).$$

Then, for every $u,v\in\mathfrak{H}_\infty$, we set

$$D^{\infty,\circ}(u,v):=\Lambda_u^\infty+\Lambda_v^\infty-2\max\Big(\inf_{w\in[|u,v|]}\Lambda_w^\infty,\inf_{w\in[|v,u|]}\Lambda_w^\infty\Big), \tag{33}$$

and

$$D^\infty(u,v):=\inf_{u_0=u,u_1,\dots,u_p=v}\sum_{i=1}^p D^{\infty,\circ}(u_{i-1},u_i) \tag{34}$$

where the infimum is over all choices of the integer $p\geq 1$ and of the finite sequence $u_0,u_1,\dots,u_p$ in $\mathfrak{H}_\infty$ such that $u_0=u$ and $u_p=v$. It is immediate that $D^\infty(u,v)\geq|\Lambda_u^\infty-\Lambda_v^\infty|$ for every $u,v\in\mathfrak{H}_\infty$, and we also note that the mapping $(u,v)\mapsto D^\infty(u,v)$ is continuous on $\mathfrak{H}_\infty\times\mathfrak{H}_\infty$. Furthermore, the so-called cactus bound states that, for every $u,v\in\mathfrak{H}_\infty$,

$$D^\infty(u,v)\geq\Lambda_u^\infty+\Lambda_v^\infty-2\min_{w\in[\![u,v]\!]}\Lambda_w^\infty, \tag{35}$$

where $[\![u,v]\!]$ denotes the geodesic segment between u and v in the $\mathbb{R}$-tree $\mathfrak{H}_\infty$ (not to be confused with the interval $[|u,v|]$). See formula (4) in [11] for a short proof in the case of the Brownian map, which is immediately extended to the present setting.

We finally notice that, if $u,v\in\mathfrak{H}_\infty$ are such that $[|u,v|]=\{\mathcal{E}_r^\infty:r\in[s,t]\}$ with $s>t$, then trivially $\inf_{w\in[|u,v|]}\Lambda_w^\infty=-\infty$ and thus the maximum in (33) is equal to $\inf_{w\in[|v,u|]}\Lambda_w^\infty$ (this occurs in particular if $u\in\mathcal{T}_{(\omega_i)}$ and $v\in\mathcal{T}_{(\omega_j)}$ with $t_j<t_i$).

Definition 7. *The Brownian half-plane is the quotient space $\mathbb{H} := \mathfrak{H}_\infty / \{D^\infty = 0\}$, which is equipped with the distance induced by D^∞ and with a distinguished point which is the equivalence class of* 0*. The volume measure on $\mathbb{H}$ is the pushforward of the volume measure on $\mathfrak{H}_\infty$ under the canonical projection.*

We use the notation Π_∞ for the canonical projection from $\mathfrak{H}_\infty$ onto $\mathbb{H}$, and keep the notation D^∞ for the metric on $\mathbb{H}$. By the same argument as for the Brownian disk, $\mathbb{H}$ is a length space, and (for instance by using Lemma 8 below) it is easy to verify that closed balls in $\mathbb{H}$ are compact. Thus we may and will view $\mathbb{H}$ as a random element of the space $\mathbb{M}^\bullet_{bcl}$ of Section 2.2.

The space $\mathbb{H}$ is homeomorphic to the usual upper half-plane and in this homeomorphism the real line corresponds to $\partial\mathbb{H} := \Pi_\infty(\mathbb{R})$ (see [14, Section 1.5]). The uniform measure μ_∞ on $\partial\mathbb{H}$ is defined as the pushforward of Lebesgue measure on $\mathbb{R}$ under Π_∞. For our purposes, it is important to note that the equivalence class of any $u \in \mathbb{R}$ in the quotient $\mathbb{H} = \mathfrak{H}_\infty / \{D^\infty = 0\}$ is a singleton (no point of $\mathbb{R}$ is identified to another point of $\mathfrak{H}_\infty$). Indeed, comparing the constructions of the Brownian half-plane and of the free pointed Brownian disk (and also using Lemma 8 below), one may observe that the existence of a pair $\{u, v\}$ of distinct points of $\mathfrak{H}_\infty$ such that $u \in \mathbb{R}$ and $D^\infty(u, v) = 0$ would imply the existence of a pair with similar properties in the Brownian disk, and we know from [6] that this does not occur.

For reals $a < b$, we set

$$\mathfrak{H}_\infty^{[a,b]} := [a, b] \cup \Big(\bigcup_{j \in J, a \le t_j \le b} \mathcal{T}_{(\omega_j)} \Big)$$

of course with the same identifications as in the definition of $\mathfrak{H}_\infty$, so that $\mathfrak{H}_\infty^{[a,b]}$ is a subset of $\mathfrak{H}_\infty$.

Lemma 8. *We have a.s.*

$$\lim_{a \to \infty} \Big(\inf_{u \in \mathfrak{H}_\infty \setminus \mathfrak{H}_\infty^{[-a,a]}} D^\infty(0, u) \Big) = \infty.$$

Proof. Let $u \in \mathfrak{H}_\infty \setminus \mathfrak{H}_\infty^{[-a,a]}$, and assume for definiteness that u belongs to a tree $\mathcal{T}_{(\omega_j)}$ with $t_j > a$, or that $u \in (a, \infty)$. The cactus bound (35) ensures that

$$D^\infty(0, u) \ge \Lambda^\infty_u - 2 \min_{v \in [\![0,u]\!]} \Lambda^\infty_v \ge - \min_{v \in [\![0,u]\!]} \Lambda^\infty_v .$$

Since the geodesic segment $[\![0, u]\!]$ contains the interval $[0, a]$, the right-hand side of the preceding display is bounded below by $-\min_{0 \le t \le a} B_t$. Finally, the infimum in the lemma is bounded below by

$$\Big(- \min_{0 \le t \le a} B_t \Big) \wedge \Big(- \min_{-a \le t \le 0} B_t \Big)$$

which tends to ∞ as $a \to \infty$. □

Notation. Without risk of confusion, we will use the same notation $\mathrm{Vol}(\cdot)$ for the volume measure on any of the spaces $\mathfrak{H}, \mathfrak{H}_\infty, \mathbb{D}^\bullet$ and $\mathbb{H}$, as well as on the space $\mathbb{D}$ introduced below.

5 Approximating the uniform measure on the boundary

We consider the free pointed Brownian disk $(\mathbb{D}^\bullet, D)$ as defined in the previous section. Recall the definition of the uniform probability measure μ on $\partial\mathbb{D}^\bullet$ as the pushforward of Lebesgue measure on $[0, 1)$ under the canonical projection Π. The goal of this section is to prove the following useful approximation result.

Theorem 9. *For every $\varepsilon > 0$, let μ_ε be the finite measure on $\mathbb{D}^\bullet$ defined by*

$$\langle \mu_\varepsilon, \varphi \rangle = \varepsilon^{-2} \int_{\mathbb{D}^\bullet} \mathrm{Vol}(\mathrm{d}x)\, \mathbf{1}_{\{D(x, \partial\mathbb{D}^\bullet) \le \varepsilon\}}\, \varphi(x).$$

Then a.s. μ_ε converges weakly to μ as $\varepsilon \to 0$.

It is proved in [19, Proposition 2] that the measures μ_ε converge weakly to a probability measure ν, which is also called the uniform probability measure on the boundary in [19]. So Theorem 9 is equivalent to the statement $\mu = \nu$. Unfortunately, this equality is not easy to prove, because the construction of the free pointed Brownian disk in [19] is very different from the one presented in Section 4 (see the comments in the introduction of [19]). So below, we will essentially prove the convergence of μ_ε to μ independently of the results of [19] — we still need these results to get the value of the constant κ that appears in Lemma 10 below.

Before we proceed to the proof of Theorem 9, we need a few preliminary lemmas, which are mainly concerned with the case of the Brownian half-plane $(\mathbb{H}, D^\infty)$ constructed in the previous section as a quotient space of $\mathfrak{H}_\infty$. As previously, we view $\mathbb{R}$ as a subset of $\mathfrak{H}_\infty$. Recall the notation $\mathfrak{H}_\infty^{[a,b]}$ introduced before Lemma 8.

Lemma 10. *There exists a constant $\kappa \in (0,\infty]$ such that, for any reals a, b with $a < b$, we have*

$$\varepsilon^{-2}\,\mathrm{Vol}(\{u \in \mathfrak{H}_\infty^{[a,b]} : D^\infty(u,\mathbb{R}) \le \varepsilon\}) \underset{\varepsilon\to 0}{\longrightarrow} \kappa(b-a),$$

in probability.

Remark. We will see later that $\kappa = 1$, but at the present stage, we do not exclude the possibility that $\kappa = \infty$.

Proof. Consider first the case $a = 0, b = 1$. Simple arguments relying on the invariance of B and $\mathbb{N}_0$ under scaling transformations show that

$$\mathrm{Vol}(\{u \in \mathfrak{H}_\infty^{[0,1]} : D^\infty(u,\mathbb{R}) \le \varepsilon\}) \overset{\text{(d)}}{=} \varepsilon^4\,\mathrm{Vol}(\{u \in \mathfrak{H}_\infty^{[0,1/\varepsilon^2]} : D^\infty(u,\mathbb{R}) \le 1\}).$$

So we will get the desired convergence for $a = 0, b = 1$ if we can verify that

$$\frac{1}{n}\,\mathrm{Vol}(\{u \in \mathfrak{H}_\infty^{[0,n]} : D^\infty(u,\mathbb{R}) \le 1\}) \underset{\varepsilon\to 0}{\longrightarrow} \kappa, \tag{36}$$

in probability, with some constant $\kappa \in (0,\infty]$. To this end, we may assume that the pair $(B, \mathcal{N}_\infty)$ is defined on the canonical space $\Omega_\circ := C(\mathbb{R},\mathbb{R}) \times M_p(\mathbb{R}\times\mathcal{S})$, in such a way that $B_t(\mathrm{w},\gamma) = \mathrm{w}(t)$ and $\mathcal{N}_\infty(\mathrm{w},\gamma) = \gamma$ for $(\mathrm{w},\gamma) \in \Omega_\circ$. The space $\Omega_\circ$ is equipped with the unique probability measure $\mathbb{P}$ under which B and $\mathcal{N}_\infty$ have the required properties. The shift θ on $\Omega_\circ$ is then defined by

$$\theta\Big(\mathrm{w}, \sum_{k\in I} \delta_{(t_k,\omega_k)}\Big) = \Big(\mathrm{w}(1+\cdot) - \mathrm{w}(1), \sum_{k\in I} \delta_{(t_k-1,\omega_k)}\Big),$$

and $\mathbb{P}$ is invariant under θ.

For every integers $i < j$, set

$$V_{i,j} := \mathrm{Vol}(\{u \in \mathfrak{H}_\infty^{[i,j]} : D^\infty(u,\mathbb{R}) \le 1\}).$$

Then,

$$V_{0,n} = \sum_{i=0}^{n-1} V_{i,i+1} = \sum_{i=0}^{n-1} V_{0,1}\circ\theta^i.$$

The ergodic theorem then implies that $n^{-1}V_{0,n}$ converges a.s. as $n \to \infty$. The limit must be constant since it is a shift-invariant function of the i.i.d. sequence $(\xi_n)_{n\in\mathbb{Z}}$ defined by

$$\xi_n := \Big((B_{n+t} - B_n)_{0\le t\le 1}, \mathfrak{R}_{[0,1]}(\mathcal{N}_\infty\circ\theta^n)\Big),$$

where $\mathfrak{R}_{[0,1]}(\gamma)$ stands for the restriction of γ to $[0,1]\times\mathcal{S}$. Our claim (36) follows. Finally, for arbitrary $a < b$, the convergence in the lemma follows from the special case $a = 0, b = 1$ using scaling and translation invariance properties of the model. $\square$

Our goal is to prove that a result similar to Lemma 10 holds for the free pointed Brownian disk. We need a couple of preliminary lemmas.

Lemma 11. *Let $\eta > 0$ and $\delta > 0$. Then a.s. there exists a (random) real $\varepsilon_0 > 0$ such that the following holds for every $0 < \varepsilon < \varepsilon_0$: for every $u \in \mathfrak{H}_\infty^{[-\eta,\eta]}$, the property $D^\infty(u, \mathbb{R}) < \varepsilon$ implies that there exists $v \in [-\eta - \delta, \eta + \delta]$ such that $D^\infty(u, v) < \varepsilon$, and morever*

$$D^\infty(u, v) = \inf_{\substack{u_0=u,u_1,\dots,u_{p-1},u_p=v \\ u_1,\dots,u_{p-1}\in\mathfrak{H}_\infty^{[-\eta-2\delta,\eta+2\delta]}}} \sum_{i=1}^{p} D^{\infty,\circ}(u_{i-1}, u_i). \tag{37}$$

Proof. We start by observing that we have a.s.

$$\inf\Big\{D^\infty\Big(u, (-\infty, -\eta - \delta] \cup [\eta + \delta, \infty)\Big) : u \in \mathfrak{H}_\infty^{[-\eta,\eta]}\Big\} > 0 \tag{38}$$

and

$$\inf\Big\{D^\infty\Big(u, [-\eta - \delta, \eta + \delta]\Big) : u \in \mathfrak{H}_\infty \backslash \mathfrak{H}_\infty^{[-\eta+2\delta,\eta+2\delta]}\Big\} > 0. \tag{39}$$

Let us prove (38). We argue by contradiction. If the infimum in (38) is zero, this means that we can find a sequence $(u_n)_{n\geq1}$ in $\mathfrak{H}_\infty^{[-\eta,\eta]}$ such that $D^\infty(u_n, (-\infty, -\eta - \delta] \cup [\eta + \delta, \infty))$ tends to 0 as $n \to \infty$. By compactness, we may assume that $u_n \longrightarrow u_\infty \in \mathfrak{H}_\infty^{[-\eta,\eta]}$ as $n \to \infty$ (in the sense of the topology of $\mathfrak{H}_\infty$). Then, using the continuity of the mapping $(u, v) \mapsto D^\infty(u, v)$, we have necessarily $D^\infty(u_\infty, (-\infty, -\eta - \delta] \cup [\eta + \delta, \infty)) = 0$ and (using Lemma 8) this is only possible if there exists $v \in (-\infty, -\eta - \delta] \cup [\eta + \delta, \infty)$ such that $D^\infty(u_\infty, v) = 0$. This is a contradiction since we know that the equivalence class of any point of $\mathbb{R}$ in the quotient $\mathfrak{H}_\infty/\{D^\infty = 0\}$ is a singleton.

The proof of (39) is similar. If the infimum in (39) is zero, we can find a sequence $(v_n)_{n\geq1}$ in $\mathfrak{H}_\infty \backslash \mathfrak{H}_\infty^{[-\eta+2\delta,\eta+2\delta]}$ such that $D^\infty(v_n, [-\eta - \delta, \eta + \delta])$ tends to 0, and, thanks to Lemma 8, we can extract a subsequence converging to v_∞. The fact that $D^\infty(v_\infty, [-\eta - \delta, \eta + \delta]) = 0$ gives a contradiction.

We let ε_1 and ε_2 be the infima appearing in formulas (38) and (39) respectively, and take $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2$. The first assertion of the lemma follows from the definition of ε_1. To get the second one, let $u \in \mathfrak{H}_\infty^{[-\eta,\eta]}$ and $v \in [-\eta - \delta, \eta + \delta]$ such that $D^\infty(u, v) < \varepsilon$. Then, for every ε' such that $D^\infty(u, v) < \varepsilon' < \varepsilon$, we can find an integer $p \geq 1$ and $u_1, \dots, u_{p-1} \in \mathfrak{H}_\infty$ such that

$$\sum_{i=1}^{p} D^{\infty,\circ}(u_{i-1}, u_i) < \varepsilon'$$

where $u_0 = u$ and $u_p = v$. We claim that we must have $u_1, \dots, u_{p-1} \in \mathfrak{H}_\infty^{[-\eta-2\delta,\eta+2\delta]}$. Indeed, if there exists $j \in \{1, \dots, p-1\}$ such that $u_j \in \mathfrak{H}_\infty \backslash \mathfrak{H}_\infty^{[-\eta-2\delta,\eta+2\delta]}$, then the bound

$$\sum_{i=j+1}^{p} D^{\infty,\circ}(u_{i-1}, u_i) < \varepsilon'$$

implies $D^\infty(u_j, v) < \varepsilon'$, which contradicts the definition of ε_2. □

Let us now turn to the free pointed Brownian disk $\mathbb{D}^\bullet$. We recall the construction of $(\mathbb{D}^\bullet, D)$ in Section 4 as a quotient of the space $\mathfrak{H}$ defined from a Poisson measure $\mathcal{N}$ on $[0, 1] \times \mathcal{S}$. Without loss of generality, we may and will assume that $\mathcal{N}$ is the restriction of $\mathcal{N}_\infty$ to $[0, 1] \times \mathcal{S}$. Then, for any $0 \leq a < b < 1$, the subset of $\mathfrak{H}$ defined by

$$\mathfrak{H}^{[a,b]} := [a, b] \cup \Big(\bigcup_{j\in J, a\leq t_j\leq b} \mathcal{T}_{(\omega_j)} \Big)$$

is identified with the subset $\mathfrak{H}_\infty^{[a,b]}$ of $\mathfrak{H}_\infty$. For $u, v \in \mathfrak{H}^{[a,b]}$, it will be useful to introduce the quantity $D^{\circ,[a,b]}(u, v)$: This quantity is defined by the same formula (28) as $D^\circ(u, v)$, except that, if one of the two intervals $[|u, v|]$ and $[|v, u|]$ of $\mathfrak{H}$ is not contained in $\mathfrak{H}^{[a,b]}$ (this holds for at most one of the two intervals), we replace the infimum of labels on this interval by $-\infty$, or, equivalently, we disregard the infimum over this interval. Obviously, $D^{\circ,[a,b]}(u, v) \geq D^\circ(a, b)$.

The following lemma is then an analog of Lemma 11.

Lemma 12. *Let $\eta \in (0,1/8)$ and $\delta \in (0,1/8)$. Then a.s. there exists $\varepsilon_0' > 0$ such that the following holds for every $0 < \varepsilon < \varepsilon_0'$: for every $u \in \mathfrak{H}^{[\frac{1}{2}-\eta,\frac{1}{2}+\eta]}$, the property $D(u,[0,1]) < \varepsilon$ implies that there exists $v \in [\frac{1}{2}-\eta-\delta, \frac{1}{2}+\eta+\delta]$ such that $D(u,v) < \varepsilon$, and morever*

$$D(u,v) = \inf_{\substack{u_0=u,u_1,\ldots,u_{p-1},u_p=v \\ u_1,\ldots,u_{p-1}\in\mathfrak{H}^{[\frac{1}{2}-\eta-2\delta,\frac{1}{2}+\eta+2\delta]}}} \sum_{i=1}^{p} D^{\circ,[\frac{1}{2}-\eta-2\delta,\frac{1}{2}+\eta+2\delta]}(u_{i-1},u_i). \tag{40}$$

Proof. The beginning of the proof is exactly similar to that of Lemma 11, using the obvious analogs of (38) and (39), which hold thanks to the fact that no point of $[0,1)$ is identified to another point of $\mathfrak{H}$ in the quotient $\mathfrak{H}/\{D=0\}$ (we leave the details to the reader). This leads to the variant of formula (40) where the quantities $D^{\circ,[\frac{1}{2}-\eta-2\delta,\frac{1}{2}+\eta+2\delta]}(u_{i-1},u_i)$ are replaced by $D^\circ(u_{i-1},u_i)$. So to get the statement of Lemma 12, it suffices to prove that the following claim holds for every $0<a<b<1$: a.s. for ε small enough, if $u',v' \in \mathfrak{H}^{[a,b]}$ are such that $D(u',[0,1]) < \varepsilon$, $D(v',[0,1])<\varepsilon$, and $D^\circ(u',v')<\varepsilon$, then we have automatically $D^\circ(u',v') = D^{\circ,[a,b]}(u',v')$.

In order to prove our claim, we argue by contradiction. If the claim fails, then we can find a sequence $(\varepsilon_n)_{n\geq 1}$ decreasing to 0, and, for every $n\geq 1$, two points $u^{(n)}$ and $v^{(n)}$ in $\mathfrak{H}^{[a,b]}$, such that:

(a) $D(u^{(n)},[0,1]) < \varepsilon_n$ and $D(v^{(n)},[0,1]) < \varepsilon_n$;

(b) the interval $[|u^{(n)},v^{(n)}|]$ is not contained in $\mathfrak{H}^{[a,b]}$;

(c) $\Lambda_{u^{(n)}} + \Lambda_{v^{(n)}} - 2\inf_{w\in[|u^{(n)},v^{(n)}|]} \Lambda_w < \varepsilon_n$.

Recall the cyclic exploration $(\mathcal{E}_s)_{s\in[0,\Sigma]}$ in Section 4. Let s_n be as large as possible such that $\mathcal{E}_{s_n} = u^{(n)}$ and similarly let t_n be as small as possible such that $\mathcal{E}_{t_n} = v^{(n)}$. Because of property (b) we must have $t_n < s_n$ and $[|u^{(n)},v^{(n)}|] = \{\mathcal{E}_r : r \in [s_n,\Sigma]\cup[0,t_n]\}$. Up to extracting a subsequence, we can assume that $s_n \longrightarrow s_\infty$ and $t_n \longrightarrow t_\infty$ as $n\to\infty$. Set $u^{(\infty)} = \mathcal{E}_{s_\infty}$ and $v^{(\infty)} = \mathcal{E}_{t_\infty}$. By property (a) and the fact that the equivalence class of any point of $[0,1)$ for the equivalence relation $\{D=0\}$ is a singleton, $u^{(\infty)}$ and $v^{(\infty)}$ must belong to $[0,1]$. On the other hand, property (c) gives

$$\Lambda_{u^{(\infty)}} + \Lambda_{v^{(\infty)}} - 2\inf_{r\in[s_\infty,\Sigma]\cup[0,t_\infty]} \Lambda_{\mathcal{E}_r} = 0,$$

which implies in particular that $D(u^{(\infty)},v^{(\infty)}) = 0$. This means that $u^{(\infty)} = v^{(\infty)}$. Then two cases may occur. Either $t_\infty < s_\infty$, which implies that $u^{(\infty)}$ is the root of one of the trees $\mathcal{T}_{(\omega_i)}$, but then, using the fact that $\{\mathcal{E}_r : r\in[s_\infty,t_\infty]\}$ contains $[0,1]$, the last display would imply that the minimal value of β over $[0,1]$ is attained at the root of one of the trees $\mathcal{T}_{(\omega_i)}$, which does not hold a.s. Or $t_\infty = s_\infty$, but then the last display shows that the minimal label on $\mathfrak{H}$ is attained at a point of $[0,1]$, which means that $v_* \in \partial\mathbb{D}$, contradicting (31). This contradiction completes the proof. $\square$

Proposition 13. *For every $0\leq a<b\leq 1$, we have*

$$\varepsilon^{-2}\,\mathrm{Vol}\Big(\{u\in\mathfrak{H}^{[a,b]} : D(u,[0,1])<\varepsilon\}\Big) \underset{\varepsilon\to 0}{\longrightarrow} b-a,$$

in probability.

Proof. We first show that the convergence in the proposition holds with $b-a$ replaced by $\kappa(b-a)$, and at the end of the proof we explain why $\kappa=1$. Thanks to the symmetries of the model (and also using the fact that $\mathfrak{H}^{[a,c]} = \mathfrak{H}^{[a,b]}\cup\mathfrak{H}^{[b,c]}$ if $0\leq a<b<c\leq 1$), it is enough to consider the case $a=\frac{1}{2}-\eta$, $b=\frac{1}{2}+\eta$, where $\eta\in(0,1/8)$. We also fix $\delta\in(0,1/8)$. The idea is to combine the convergence of Lemma 10 with an absolute continuity argument. Let us introduce some notation. We set

$$V_\varepsilon := \mathrm{Vol}\Big(\{u\in\mathfrak{H}^{[\frac{1}{2}-\eta,\frac{1}{2}+\eta]} : D(u,[0,1])<\varepsilon\}\Big)\ ,\ V_\varepsilon^\infty := \mathrm{Vol}\Big(\{u\in\mathfrak{H}_\infty^{[\frac{1}{2}-\eta,\frac{1}{2}+\eta]} : D^\infty(u,\mathbb{R})<\varepsilon\}\Big).$$

We also let $\bar V_\varepsilon$ be the volume of the subset of $\mathfrak{H}$ consisting of all $u \in \mathfrak{H}^{[\frac12-\eta,\frac12+\eta]}$ such that there exist $v \in [\frac12-\eta-\delta, \frac12+\eta+\delta]$ and $u_1,\ldots,u_{p-1} \in \mathfrak{H}^{[\frac12-\eta-2\delta,\frac12+\eta+2\delta]}$ with the property

$$\sum_{i=1}^{p} D^{\circ,[\frac12-\eta-2\delta,\frac12+\eta+2\delta]}(u_{i-1},u_i) < \varepsilon,$$

where $u_0 = u$ and $u_p = v$. Similarly, we let $\bar V^\infty_\varepsilon$ be the volume of the subset of $\mathfrak{H}_\infty$ consisting of all $u \in \mathfrak{H}_\infty^{[\frac12-\eta,\frac12+\eta]}$ such that there exist $v \in [\frac12-\eta-\delta, \frac12+\eta+\delta]$ and $u_1,\ldots,u_{p-1} \in \mathfrak{H}_\infty^{[\frac12-\eta-2\delta,\frac12+\eta+2\delta]}$ with the property

$$\sum_{i=1}^{p} D^{\infty,\circ}(u_{i-1},u_i) < \varepsilon,$$

with the same convention for u_0 and u_p.

By Lemma 10, we know that $\varepsilon^{-2}V^\infty_\varepsilon$ converges in probability to $2\kappa\eta$. Lemma 11 shows that $\mathbb{P}(\bar V^\infty_\varepsilon = V^\infty_\varepsilon)$ tends to 1 as $\varepsilon \to 0$, so that we also get that $\varepsilon^{-2}\bar V^\infty_\varepsilon$ converges in probability to $2\kappa\eta$. Now the point is that $\bar V^\infty_\varepsilon$ is a measurable function of $\mathcal{N}$ (recall that we have assumed that $\mathcal{N}$ is the restriction of $\mathcal{N}_\infty$ to $[0,1]\times\mathcal{S}$) and of the random path $(B_t)_{\frac12-\eta-2\delta\le t\le\frac12+\eta+2\delta}$, whereas $\bar V_\varepsilon$ is the *same* measurable function of $\mathcal{N}$ and of the random path $(\beta_t)_{\frac12-\eta-2\delta\le t\le\frac12+\eta+2\delta}$ — here it is crucial that we use $D^{\circ,[\frac12-\eta-2\delta,\frac12+\eta+2\delta]}$ instead of D° in the definition of $\bar V_\varepsilon$. The distribution of $(\beta_t)_{\frac12-\eta-2\delta\le t\le\frac12+\eta+2\delta}$ is absolutely continuous with respect to that of $(B_t)_{\frac12-\eta-2\delta\le t\le\frac12+\eta+2\delta}$, with a Radon-Nikodym derivative bounded above by a constant K. It follows that the distribution of $\bar V_\varepsilon$ is also absolutely continuous with respect to that of $\bar V^\infty_\varepsilon$, with a Radon-Nikodym derivative bounded by the same constant K independent of ε. Hence $\varepsilon^{-2}\bar V_\varepsilon$ also converges in probability to $2\kappa\eta$. Then, Lemma 12 shows that $\mathbb{P}(\bar V_\varepsilon = V_\varepsilon)$ tends to 1 as $\varepsilon\to 0$, and we get that $\varepsilon^{-2}V_\varepsilon$ converges in probability to $2\kappa\eta$.

We have thus obtained the statement of the proposition, except for the value $\kappa = 1$. However, taking $a = 0$ and $b = 1$, we have $\varepsilon^{-2}\mathrm{Vol}(\{x \in \mathbb{D}^\bullet : D(x,\partial\mathbb{D}^\bullet) < \varepsilon\}) \longrightarrow \kappa$ as $\varepsilon \to \infty$, and, comparing with [19, Proposition 2], we get that $\kappa = 1$. $\square$

Proof of Theorem 9. Thanks to Proposition 13, we may choose a sequence $(\varepsilon_n)_{n\ge1}$ decreasing to 0 such that, a.s. for every integer $N \ge 3$ and every $k \in \{0,1,\ldots,2^N-1\}$, we have

$$\varepsilon_n^{-2}\int_{\mathfrak{H}^{[k2^{-N},(k+1)2^{-N}]}} \mathrm{Vol}(\mathrm{d}u)\,\mathbf{1}_{\{D(u,[0,1])<\varepsilon_n\}} \underset{n\to\infty}{\longrightarrow} 2^{-N}. \tag{41}$$

By Lemma 12, we also know that a.s. for every $N \ge 3$ and every $k \in \{0,1,\ldots,2^N-1\}$, if n is large enough (depending on the choice of N), the conditions $u \in \mathfrak{H}^{[k2^{-N},(k+1)2^{-N}]}$ and $D(u,[0,1]) < \varepsilon_n$ imply that there exists $v \in [(k-1)2^{-N},(k+2)2^{-N}]$ such that $D(u,v) < \varepsilon_n$ (in the case $k = 0$, the notation $[-2^{-N}, 2\cdot 2^{-N}]$ of course refers to $[0,2\cdot2^{-N}]\cup[1-2^{-N},1]$, and similarly if $k = 2^N-1$). From now on, we fix an element of the underlying probability space such that the preceding property and (41) both hold.

Let $\delta > 0$ and consider a bounded continuous function $\varphi : \mathbb{D}^\bullet \longrightarrow \mathbb{R}_+$. We fix N such that $|\varphi(\Pi(u)) - \varphi(\Pi(v))| \le \delta$ whenever u, v both belong to $[k2^{-N},(k+1)2^{-N}]$ for some $k \in \{0,1,\ldots,2^N-1\}$. Then, for n large enough (such that $D(x,y) < \varepsilon_n$ implies $|\varphi(x)-\varphi(y)| < \delta$ and also such that the property stated after (41) holds), we have, for every $k \in \{0,1,\ldots,2^N-1\}$,

$$\begin{aligned}&\int_{\mathfrak{H}^{[k2^{-N},(k+1)2^{-N}]}} \mathrm{Vol}(\mathrm{d}u)\,\mathbf{1}_{\{D(u,[0,1])<\varepsilon_n\}}\,\varphi(\Pi(u))\\ &\qquad\le \Big(\max_{v\in[(k-1)2^{-N},(k+2)2^{-N}]}\varphi(\Pi(v)) + \delta\Big)\int_{\mathfrak{H}^{[k2^{-N},(k+1)2^{-N}]}} \mathrm{Vol}(\mathrm{d}u)\,\mathbf{1}_{\{D(u,[0,1])<\varepsilon_n\}}.\end{aligned}$$

Using (41), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\varepsilon_n^{-2}\int_{\mathfrak{H}^{[k2^{-N},(k+1)2^{-N}]}} \mathrm{Vol}(\mathrm{d}u)\,\mathbf{1}_{\{D(u,[0,1])<\varepsilon_n\}}\,\varphi(\Pi(u)) &\le 2^{-N}\Big(\max_{v\in[k2^{-N},(k+1)2^{-N}]}\varphi(\Pi(v)) + 2\delta\Big)\\ &\le \int_{k2^{-N}}^{(k+1)2^{-N}}\varphi(\Pi(v))\,\mathrm{d}v + 3\delta\,2^{-N}.\end{aligned}$$

By summing over k, we get

$$\limsup_{n\to\infty} \varepsilon_n^{-2} \int_{\mathfrak{H}} \mathrm{Vol}(\mathrm{d}u)\, \mathbf{1}_{\{D(u,[0,1])<\varepsilon_n\}}\, \varphi(\Pi(u)) \leq \int_0^1 \varphi(\Pi(v))\,\mathrm{d}v + 3\delta,$$

and similar arguments give the corresponding result for the liminf behavior. Since δ was arbitrary, this shows that $\langle \mu_{\varepsilon_n}, \varphi\rangle \longrightarrow \langle \mu, \varphi\rangle$, with the notation of Theorem 9. So we have proved that μ_{ε_n} converges weakly to μ, a.s. This is the desired result, except that we have restricted ourselves to a particular sequence $(\varepsilon_n)_{n\geq 1}$ decreasing to 0. However, we may use [19, Proposition 2], which already gives the a.s. weak convergence of μ_ε as $\varepsilon \to 0$ to a limiting probability measure ν on $\partial\mathbb{D}^\bullet$. Then necessarily $\mu = \nu$ and this completes the proof. □

6 Conditioning the distinguished point to belong to the boundary

Our goal in this section is to give a description of the free Brownian disk pointed at a point chosen uniformly on the boundary. To this end, we will condition the free pointed Brownian disk $\mathbb{D}^\bullet$ on the event $\{D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon\}$ and pass to the limit $\varepsilon \to 0$.

We first need to introduce the (non-pointed) free Brownian disk $\mathbb{D}$. To this end, write $\mathbb{D}^\circ$ for the (non-pointed) space obtained by forgetting the distinguished point of $\mathbb{D}^\bullet$. Then the distribution of the free Brownian disk $\mathbb{D}$ is given by the identity

$$\mathbb{E}[F(\mathbb{D})] = \mathbb{E}\Big[\frac{1}{\mathrm{vol}(\mathbb{D}^\circ)}\, F(\mathbb{D}^\circ)\Big]$$

for any nonnegative measurable function F on the space $\mathbb{M}$ of Section 2.2 (see [7, Section 1.5]). Conversely, we can recover the distribution of $\mathbb{D}^\bullet$ from that of $\mathbb{D}$ via the formula

$$\mathbb{E}[F(\mathbb{D}^\bullet)] = \mathbb{E}\Big[\int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}, x])\Big], \tag{42}$$

where we use the notation $[\mathbb{D}, x]$ for the space $\mathbb{D}$ pointed at x, and F is now defined on $\mathbb{M}^\bullet$. Let us give a brief justification of (42). For every $r > 0$, write $\mathbb{D}_{(r)}$, resp. $\mathbb{D}^\bullet_{(r)}$, for the Brownian disk, resp. the pointed Brownian disk, of perimeter 1 and volume r. Note that $\mathbb{D}^\bullet_{(r)}$ is constructed by the very same method as in Section 4 under the probability measure $\mathbb{P}(\cdot \mid \Sigma = r)$, and that $\mathbb{D}_{(r)}$ is derived from $\mathbb{D}^\bullet_{(r)}$ by forgetting the distinguished point. Hence, the law of $\mathbb{D}^\bullet$ is obtained by integrating the law of $\mathbb{D}^\bullet_{(r)}$ with respect to the density $(2\pi r^3)^{-1/2} \exp(-1/(2r))$ of Σ, and similarly the law of $\mathbb{D}$ is obtained by integrating the law of $\mathbb{D}_{(r)}$ with respect to the density $(2\pi r^5)^{-1/2} \exp(-1/(2r))$ (see [7, Section 1.5]). From these considerations, we see that (42) is a consequence of the identity

$$\mathbb{E}[F(\mathbb{D}^\bullet_{(r)})] = \mathbb{E}\Big[\frac{1}{r}\int_{\mathbb{D}_{(r)}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}_{(r)}, x])\Big]. \tag{43}$$

This identity follows from Lemma 18 in [7], which is itself derived from the (trivial) discrete analog of (43) for quadrangulations. To be precise, [7] considers Brownian disks as random metric spaces, without including the volume measures, but the argument of [7] immediately extends to our setting.

We next observe that the uniform measure μ on the boundary also makes sense for $\mathbb{D}$ (it may be defined by the almost sure approximation in Theorem 9). We then define the free Brownian disk with perimeter 1 pointed at a uniform boundary point as the pointed compact measure metric space $\bar{\mathbb{D}}^\bullet$, whose distribution is given by

$$\mathbb{E}[F(\bar{\mathbb{D}}^\bullet)] = \mathbb{E}\Big[\int_{\partial\mathbb{D}} \mu(\mathrm{d}x)\, F([\mathbb{D}, x])\Big].$$

Proposition 14. *The conditional distribution of the pointed metric measure space $\mathbb{D}^\bullet$ given that $D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon$ converges as $\varepsilon \to 0$ to the distribution of the free Brownian disk with perimeter* 1 *pointed at a uniform boundary point.*

Proof. Let F be a bounded continuous function on $\mathbb{M}^\bullet$, and assume that $0 \leq F \leq 1$. Then,

$$\mathbb{E}[F(\mathbb{D}^\bullet) \,|\, D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon] = \frac{\mathbb{E}[F(\mathbb{D}^\bullet)\, \mathbf{1}_{\{D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon\}}]}{\mathbb{P}(D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon)}, \tag{44}$$

and we know from (31) that

$$\lim_{\varepsilon \to 0} \varepsilon^{-2}\, \mathbb{P}(D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon) = 1.$$

On the other hand, (42) gives

$$\mathbb{E}\Big[F(\mathbb{D}^\bullet)\, \mathbf{1}_{\{D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon\}}\Big] = \mathbb{E}\Big[\int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}, x])\, \mathbf{1}_{\{D(x, \partial\mathbb{D}) \leq \varepsilon\}}\Big]. \tag{45}$$

From Theorem 9 (and the continuity of the mapping $x \mapsto [\mathbb{D}, x]$), we have

$$\varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}, x])\, \mathbf{1}_{\{D(x, \partial\mathbb{D}) \leq \varepsilon\}} \xrightarrow[\varepsilon \to 0]{a.s.} \int \mu(\mathrm{d}y)\, F([\mathbb{D}, y]).$$

The desired convergence of $\mathbb{E}[F(\mathbb{D}^\bullet) | D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon]$ to $\mathbb{E}[\int \mu(\mathrm{d}y) F([\mathbb{D}, y])]$ will follow from (44) and (45) if we can prove that the convergence in the last display also holds for expected values. Arguing along a sequence of values of ε tending to 0, Fatou's lemma gives

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2} \mathbb{E}\Big[\int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}, x])\, \mathbf{1}_{\{D(x, \partial\mathbb{D}^\bullet) \leq \varepsilon\}}\Big] \geq \mathbb{E}\Big[\int \mu(\mathrm{d}y)\, F([\mathbb{D}, y])\Big]. \tag{46}$$

By the case $F = 1$ of (45), we have

$$\varepsilon^{-2} \mathbb{E}\Big[\int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, \mathbf{1}_{\{D(x, \partial\mathbb{D}) \leq \varepsilon\}}\Big] = \varepsilon^{-2} \mathbb{P}(D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 1.$$

Replacing F by $1 - F$ in (46), we get the corresponding upper bound for the limsup, and we conclude that we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \mathbb{E}\Big[\int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, F([\mathbb{D}, x])\, \mathbf{1}_{\{D(x, \partial\mathbb{D}^\bullet) \leq \varepsilon\}}\Big] = \mathbb{E}\Big[\int \mu(\mathrm{d}y)\, F([\mathbb{D}, y])\Big].$$

This completes the proof. $\square$

We will now combine Proposition 4 and Proposition 14 to get our construction of the free Brownian disk with perimeter 1 pointed at a uniform boundary point. We start from a pair $(\mathbf{b}, \mathcal{N}')$, where $\mathbf{b} = (\mathbf{b}_t)_{0 \leq t \leq 1}$ is a five-dimensional Bessel bridge from 0 to 0 over the time interval $[0,1]$ and, conditionally on $\mathbf{b}$, $\mathcal{N}'(\mathrm{d}t\mathrm{d}\omega)$ is a Poisson measure on $[0,1] \times \mathcal{S}$ with intensity

$$2\, \mathbf{1}_{\{W_*(\omega) \geq -\sqrt{3}\, \mathbf{b}_t\}}\, \mathrm{d}t\, \mathbb{N}_0(\mathrm{d}\omega).$$

We write

$$\mathcal{N}' = \sum_{j \in J} \delta_{(t'_j, \omega'_j)},$$

and $\Sigma' := \sum_{j \in J} \sigma(\omega'_j)$. From $\mathcal{N}'$, we can define a compact measure metric space $\mathfrak{H}'$, and the associated cyclic exploration $(\mathcal{E}'_s)_{s \in [0, \Sigma']}$, in exactly the same way as $\mathfrak{H}$ and $(\mathcal{E}_s)_{s \in [0, \Sigma]}$ were defined from $\mathcal{N}$ at the beginning of Section 4. Intervals in $\mathfrak{H}'$ are defined as previously from the exploration $(\mathcal{E}'_s)_{s \in [0, \Sigma']}$, and we now specify labels $(\Lambda'_u)_{u \in \mathfrak{H}'}$ by setting $\Lambda'_t := \sqrt{3}\, \mathbf{b}_t$ for $t \in [0,1]$, and for $u \in \mathcal{T}_{(\omega'_j)}$, $j \in J$,

$$\Lambda'_u := \sqrt{3}\, \mathbf{b}_{t'_j} + \ell_u(\omega'_j).$$

A fundamental difference is now that $\Lambda'_u \geq 0$ for every $u \in \mathfrak{H}'$ (because by construction $W_*(\omega'_j) \geq -\sqrt{3}\, \mathbf{b}_{t'_j}$ for every $j \in J$). Furthermore 0 is the unique element of $\mathfrak{H}'$ with zero label.

We use the analogs of (28) and (29), with Λ_u replaced by Λ'_u, to define $D'^{\circ}(u,v)$ and $D'(u,v)$ for $u, v \in \mathfrak{H}'$. Then $D'(u,v)$ is a pseudo-metric on $\mathfrak{H}'$. Furthermore, it is immediate that $D'(0,u) = \Lambda'_u$ for every $u \in \mathfrak{H}'$, and that the bound $|\Lambda'_u - \Lambda'_v| \leq D'(u,v)$ holds for every u, v.

Theorem 15. *The quotient space $\mathbb{D}' := \mathfrak{H}'/\{D'=0\}$ equipped with the metric induced by D', with the volume measure which is the pushforward of the volume measure on $\mathfrak{H}'$, and with the distinguished point which is the equivalence class of* 0, *is a free Brownian disk with perimeter* 1 *pointed at a uniform boundary point.*

We write Π' for the canonical projection from $\mathfrak{H}'$ onto $\mathbb{D}'$. As previously, we can define the label Λ'_x of $x \in \mathbb{D}'$ by setting $\Lambda'_x := \Lambda'_u$, for any $u \in \mathfrak{H}'$ such that $\Pi'(u) = x$. In a way similar to the formula $D(v_*, u) = \Lambda_u - \Lambda_*$ for the free pointed Brownian disk, labels Λ'_x exactly correspond to distances from the distinguished point 0 lying on the boundary.

Proof. Thanks to Proposition 14, it is enough to verify that the (pointed measure metric) space $\mathbb{D}^\bullet$ conditioned on the event $\{D(v_*, \partial\mathbb{D}^\bullet) \leq \varepsilon\}$ converges in distribution to $\mathbb{D}'$ as $\varepsilon \to 0$, in the sense of the pointed Gromov-Hausdorff-Prokhorov topology. For the sake of simplicity, we will content ourselves with proving the pointed Gromov-Hausdorff convergence (a few additional technicalities, using Lemma 4 in [19], yield the stronger Gromov-Hausdorff-Prokhorov convergence).

Let $\mathcal{N}$ be as previously a Poisson point measure on $[0,1] \times \mathcal{S}$ with intensity $2\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega)$. Let us fix $\eta > 0$ and $\alpha > 0$. We first choose $\delta \in (0, 1/2)$ small enough so that

$$\mathbb{P}\Big(\sup_{t\in[0,\delta]\cup[1-\delta,1]} \sqrt{3}\,\mathbf{b}_t < \alpha\Big) > 1-\eta, \tag{47}$$

and

$$\mathbb{P}\Big(W^*(\omega) \leq \alpha \text{ for every atom } (t,\omega) \text{ of } \mathcal{N} \text{ such that } t \in [0,\delta] \cup [1-\delta,1]\Big) \geq 1-\eta. \tag{48}$$

For every $\varepsilon > 0$, let $\mathbf{e}^\varepsilon$ be distributed as in Proposition 4. Thanks to (47) and to the convergence in distribution in the latter proposition, we can find $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we have also

$$\mathbb{P}\Big(\sup_{t\in[0,\delta]\cup[1-\delta,1]} \sqrt{3}\,\mathbf{e}^\varepsilon_t < \alpha\Big) > 1-\eta. \tag{49}$$

Furthermore, we can also fix $\varepsilon_1 \in (0, \varepsilon_0)$ such that, for every $\varepsilon \in (0, \varepsilon_1]$, the total variation distance between the distribution of $(\mathbf{e}^\varepsilon_t + \varepsilon)_{\delta\leq t\leq 1-\delta}$ and the distribution of $(\mathbf{b}_t)_{\delta\leq t\leq 1-\delta}$ is smaller than η.

Let us fix $\varepsilon \in (0, \varepsilon_1 \wedge (\alpha/\sqrt{3})]$. On a suitable probability space, we can construct both $\mathbf{b}$ and $\mathbf{e}^\varepsilon$ so that

$$\mathbb{P}\Big(\varepsilon + \mathbf{e}^\varepsilon_t = \mathbf{b}_t \text{ for every } t \in [\delta, 1-\delta]\Big) > 1-\eta. \tag{50}$$

We may also assume that the Poisson point measure $\mathcal{N}$ is defined on the same probability space, and is independent of the pair $(\mathbf{b}, \mathbf{e}^\varepsilon)$. We then define two other random point measures $\mathcal{N}'$ and $\mathcal{N}_\varepsilon$ by

$$\mathcal{N}'(\mathrm{d}t\mathrm{d}\omega) := \mathbf{1}_{\{W_*(\omega)\geq -\sqrt{3}\,\mathbf{b}_t\}}\,\mathcal{N}(\mathrm{d}t\mathrm{d}\omega) \quad \text{and} \quad \mathcal{N}_\varepsilon(\mathrm{d}t\mathrm{d}\omega) := \mathbf{1}_{\{W_*(\omega)\geq -\sqrt{3}(\varepsilon+\mathbf{e}^\varepsilon_t)\}}\,\mathcal{N}(\mathrm{d}t\mathrm{d}\omega).$$

Note that the pair $(\mathbf{b}, \mathcal{N}')$ has the same distribution as described before the statement of the theorem, and so we may assume that the pointed metric space $\mathbb{D}'$ is constructed from this pair as explained above.

Similarly, we claim that the pair $(\mathbf{e}^\varepsilon, \mathcal{N}_\varepsilon)$ has the conditional distribution of the pair $(\mathbf{e}, \mathcal{N})$ introduced in Section 4 to construct the free pointed Brownian disk, given the event $\{\Lambda_* \geq -\varepsilon\sqrt{3}\}$. Let us briefly explain this. Noting that $\{\Lambda_* \geq -\varepsilon\sqrt{3}\} = \{W_*(\omega) \geq -\sqrt{3}(\mathbf{e}_t+\varepsilon) \text{ for every atom } (t,\omega) \text{ of } \mathcal{N}\}$, we get, using (2),

$$\mathbb{P}(\Lambda_* \geq -\varepsilon\sqrt{3} \,|\, \mathbf{e}) = \exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big).$$

Then, conditionally on $\mathbf{e}$, the law of $\mathcal{N}$ given the event $\{\Lambda_* \geq \varepsilon\sqrt{3}\}$ is the law $\mathbf{P}^{(\varepsilon,\mathbf{e})}$ of a Poisson point measure with intensity $2\,\mathbf{1}_{\{W_*(\omega)\geq-\sqrt{3}(\mathbf{e}_t+\varepsilon)\}}\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega)$. Hence, for any nonnegative measurable function G on $M_p([0,1]\times\mathcal{S})$,

$$\frac{\mathbb{E}[G(\mathcal{N})\,\mathbf{1}_{\{\Lambda_*\geq-\varepsilon\sqrt{3}\}} \mid \mathbf{e}]}{\mathbb{P}(\Lambda_* \geq -\varepsilon\sqrt{3}\,|\,\mathbf{e})} = \int \mathbf{P}^{(\varepsilon,\mathbf{e})}(\mathrm{d}\gamma)\,G(\gamma).$$

If F is a nonnegative measurable function on $C([0,1],\mathbb{R}_+)$, it follows that

$$\mathbb{E}[F(\mathbf{e})G(\mathcal{N})\,\mathbf{1}_{\{\Lambda_*\geq -\varepsilon\sqrt{3}\}}] = \mathbb{E}\Big[F(\mathbf{e})\exp\Big(-\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+\mathbf{e}_t)^2}\Big)\int \mathbf{P}^{(\varepsilon,\mathbf{e})}(\mathrm{d}\gamma)\,G(\gamma)\Big] = C_\varepsilon\,\mathbb{E}[F(\mathbf{e}^\varepsilon)\,G(\mathcal{N}_\varepsilon)],$$

which gives our claim.

Recall the construction of the free pointed Brownian disk from the pair $(\mathbf{e},\mathcal{N})$ at the beginning of Section 4. If in this construction we replace the pair $(\mathbf{e},\mathcal{N})$ with $(\mathbf{e}^\varepsilon,\mathcal{N}_\varepsilon)$, we can define a space $\mathfrak{H}_\varepsilon$ analogous to $\mathfrak{H}$ and assign labels $(\Lambda^\varepsilon_u)_{u\in\mathfrak{H}_\varepsilon}$ to the points of $\mathfrak{H}_\varepsilon$ (with $\Lambda^\varepsilon_u = \sqrt{3}\,\mathbf{e}^\varepsilon_t + \ell_u(\omega)$ if $u\in\mathcal{T}_{(\omega)}$, for any atom (t,ω) of $\mathcal{N}_\varepsilon$). We then consider the quotient space $\mathbb{D}^{(\varepsilon)} = \mathfrak{H}_\varepsilon/\{D_\varepsilon = 0\}$, where D°_ε and then the pseudo-metric D_ε are defined by the analogs of formulas (28) and (29) in terms of the labels Λ^ε_u. We write v^ε_* for the (unique) point with minimal label in $\mathfrak{H}_\varepsilon$, and Π_ε for the canonical projection from $\mathfrak{H}_\varepsilon$ onto $\mathbb{D}^{(\varepsilon)}$. The preceding claim shows that $\mathbb{D}^{(\varepsilon)}$ (viewed as a compact metric space pointed at v^ε_*) has the conditional distribution of $\mathbb{D}^\bullet$ given the event $\{D(v_*,\partial\mathbb{D}^\bullet)\leq\sqrt{3}\,\varepsilon\}$. So, to complete the proof, we now need to a get an upper bound on the Gromov-Hausdorff distance between $\mathbb{D}^{(\varepsilon)}$ and $\mathbb{D}'$.

Let $\mathfrak{R}_\delta(\mathcal{N}')$ and $\mathfrak{R}_\delta(\mathcal{N}_\varepsilon)$ denote the respective restrictions of $\mathcal{N}'$ and $\mathcal{N}_\varepsilon$ to $[\delta,1-\delta]\times\mathcal{S}$. Similarly, let $\widehat{\mathfrak{R}}_\delta(\mathcal{N}')$ and $\widehat{\mathfrak{R}}_\delta(\mathcal{N}_\varepsilon)$ denote the respective restrictions of $\mathcal{N}'$ and $\mathcal{N}_\varepsilon$ to $([0,\delta)\cup(1-\delta,1])\times\mathcal{S}$. We consider the event E_ε where the following properties hold:

(i) $\displaystyle\sup_{t\in[0,\delta]\cup[1-\delta,1]}\sqrt{3}(\mathbf{b}_t\vee\mathbf{e}^\varepsilon_t) < \alpha$;

(ii) $\varepsilon+\mathbf{e}^\varepsilon_t = \mathbf{b}_t$ for every $t\in[\delta,1-\delta]$;

(iii) for every atom (t,ω) of $\widehat{\mathfrak{R}}_\delta(\mathcal{N}')$ or of $\widehat{\mathfrak{R}}_\delta(\mathcal{N}_\varepsilon)$, we have $W^*(\omega)\leq\alpha$.

Thanks to (47), (48), (49) and (50), we have $\mathbb{P}(E_\varepsilon)\geq 1-4\eta$. From now on, we argue on the event E_ε. It is convenient to write

$$\mathcal{N}_\varepsilon = \sum_{i\in I_\varepsilon}\delta_{(t^\varepsilon_i,\omega^\varepsilon_i)}$$

and to introduce the subset $\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon$ of $\mathfrak{H}_\varepsilon$ defined by

$$\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon := [\delta,1-\delta]\cup\Bigg(\bigcup_{\{i\in I_\varepsilon:\delta\leq t^\varepsilon_i\leq 1-\delta\}}\mathcal{T}_{(\omega^\varepsilon_i)}\Bigg)$$

where we recall that the root of $\mathcal{T}_{(\omega^\varepsilon_i)}$ is identified with t^ε_i. In a similar way, we define the subset $\mathfrak{H}'^{[\delta,1-\delta]}$ of $\mathfrak{H}'$. From property (ii) and the way $\mathcal{N}_\varepsilon$ and $\mathcal{N}'$ have been constructed, we have $\mathfrak{R}_\delta(\mathcal{N}')=\mathfrak{R}_\delta(\mathcal{N}_\varepsilon)$, and so $\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon$ is identified with $\mathfrak{H}'^{[\delta,1-\delta]}$.

We define a correspondence $\mathcal{R}_\varepsilon$ between the spaces $\mathbb{D}^{(\varepsilon)}$ and $\mathbb{D}'$ by saying that $(x,x')\in\mathbb{D}^{(\varepsilon)}\times\mathbb{D}'$ belongs to $\mathcal{R}_\varepsilon$ if and only if (at least) one of the following properties hold:

(a) $x=\Pi_\varepsilon(u)$ and $x'=\Pi'(u)$, for some $u\in\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon=\mathfrak{H}'^{[\delta,1-\delta]}$;

(b) $x=\Pi_\varepsilon(u)$ and $x'=\Pi'(0)$, for some $u\in\mathfrak{H}_\varepsilon\backslash\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon$;

(c) $x=\Pi_\varepsilon(v^\varepsilon_*)$ and $x'=\Pi'(u)$, for some $u\in\mathfrak{H}'\backslash\mathfrak{H}'^{[\delta,1-\delta]}$;

We note that $\mathcal{R}_\varepsilon$ contains the pair $(\Pi_\varepsilon(v^\varepsilon_*),\Pi'(0))$ consisting of the respective distinguished points of $\mathbb{D}^{(\varepsilon)}$ and $\mathbb{D}'$.

We now need to bound the distortion of $\mathcal{R}_\varepsilon$. To this end, we first observe that $0\leq\Lambda'_u\leq 2\alpha$ if $u\in\mathfrak{H}'\backslash\mathfrak{H}'^{[\delta,1-\delta]}$ (by properties (i) and (iii)), and similarly $-\alpha\leq\Lambda^\varepsilon_u\leq 2\alpha$ if $u\in\mathfrak{H}_\varepsilon\backslash\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon$. Here we use the fact that $\varepsilon\leq\alpha/\sqrt{3}$ to obtain that $\sqrt{3}\,\mathbf{e}^\varepsilon_t+W_*(\omega)\geq-\alpha$ for every atom (t,ω) of $\mathcal{N}_\varepsilon$, and we note that the lower bound $\Lambda^\varepsilon_u\geq-\alpha$ thus holds for every $u\in\mathfrak{H}_\varepsilon$, and in particular for $u=v^\varepsilon_*$. Furthermore, we have

$$\varepsilon\sqrt{3}+\Lambda^\varepsilon_u=\Lambda'_u\quad\text{ for every } u\in\mathfrak{H}^{[\delta,1-\delta]}_\varepsilon=\mathfrak{H}'^{[\delta,1-\delta]},\tag{51}$$

by property (ii) and our construction of the Poisson measures $\mathcal{N}^\varepsilon$ and $\mathcal{N}'$.

For $u,v \in \mathfrak{H}_\varepsilon$, resp. for $u,v\in\mathfrak{H}'$, let us introduce the notation $[|u,v|]^\varepsilon$, resp. $[|u,v|]'$, for the interval from u to v in $\mathfrak{H}_\varepsilon$, resp. in $\mathfrak{H}'$. For $u,v\in\mathfrak{H}_\varepsilon$, we use the bounds

$$D_\varepsilon(u,v) \leq D^\circ_\varepsilon(u,v) = \Lambda^\varepsilon_u + \Lambda^\varepsilon_v - 2\max\Big(\min_{w\in[|u,v|]^\varepsilon}\Lambda^\varepsilon_w,\ \min_{w\in[|v,u|]^\varepsilon}\Lambda^\varepsilon_w\Big) \leq \Lambda^\varepsilon_u + \Lambda^\varepsilon_v + 2\alpha \tag{52}$$

and $D_\varepsilon(u,v) \geq |\Lambda^\varepsilon_u - \Lambda^\varepsilon_v|$. Similarly, we have for $u,v\in\mathfrak{H}'$,

$$D'(u,v) \leq D'^\circ(u,v) = \Lambda'_u + \Lambda'_v - 2\max\Big(\min_{w\in[|u,v|]'}\Lambda'_w,\ \min_{w\in[|v,u|]'}\Lambda'_w\Big) \leq \Lambda'_u + \Lambda'_v \tag{53}$$

and $D'(u,v)\geq|\Lambda'_u-\Lambda'_v|$. We recall that $D'(0,u)=\Lambda'_u$ for every $u\in\mathfrak{H}'$, and similarly we note that $D_\varepsilon(v^\varepsilon_*,u)=\Lambda^\varepsilon_u-\Lambda^\varepsilon_{v^\varepsilon_*}$ for every $u\in\mathfrak{H}_\varepsilon$.

Let (x,x') and (y,y') be two pairs in $\mathcal{R}_\varepsilon$. In order to bound $|D_\varepsilon(x,y)-D'(x',y')|$, we need to distinguish several cases. Suppose first that (x,x') and (y,y') both satisfy property (b) above, so that $x'=y'=\Pi'(0)$ and there exist $u,v\in\mathfrak{H}_\varepsilon\backslash\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$ such that $x=\Pi_\varepsilon(u)$ and $y=\Pi_\varepsilon(v)$. Then of course $D'(x',y')=0$, whereas the preceding bounds on labels give $D_\varepsilon(x,y)\leq\Lambda^\varepsilon_u+\Lambda^\varepsilon_v+2\alpha\leq6\alpha$, and thus $|D_\varepsilon(x,y)-D'(x',y')|\leq6\alpha$. The same arguments show that $|D_\varepsilon(x,y)-D'(x',y')|\leq4\alpha$ if (x,x') and (y,y') both satisfy (c).

Then suppose that (x,x') satisfies (b) and (y,y') satisfies (c), and pick $u\in\mathfrak{H}_\varepsilon\backslash\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$ and $v\in\mathfrak{H}'\backslash\mathfrak{H}'^{[\delta,1-\delta]}$ such that $x=\Pi_\varepsilon(u)$ and $y'=\Pi'(v)$. We have $D_\varepsilon(x,y)=D_\varepsilon(u,v^\varepsilon_*)=\Lambda^\varepsilon_u-\Lambda^\varepsilon_{v^\varepsilon_*}\leq3\alpha$, whereas $D'(x',y')=D'(0,v)=\Lambda'_v\leq2\alpha$. So, in that case, we get $|D_\varepsilon(x,y)-D'(x',y')|\leq3\alpha$.

Consider next the case where (x,x') satisfies (a) and (y,y') satisfies (b). Pick $u\in\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}=\mathfrak{H}'^{[\delta,1-\delta]}$ such that $x=\Pi_\varepsilon(u)$ and $x'=\Pi'(u)$, and $v\in\mathfrak{H}_\varepsilon\backslash\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$ such that $y=\Pi_\varepsilon(v)$. Note that $\Lambda'_u=\Lambda^\varepsilon_u+\varepsilon\sqrt{3}$, and $\varepsilon\sqrt{3}\leq\alpha$. Since $y'=\Pi'(0)$, we have $D'(x',y')=D'(u,0)=\Lambda'_u$. On the other hand, $D_\varepsilon(x,y)=D_\varepsilon(u,v)\geq|\Lambda^\varepsilon_u-\Lambda^\varepsilon_v|\geq\Lambda^\varepsilon_u-2\alpha$, and $D_\varepsilon(x,y)\leq\Lambda^\varepsilon_u+\Lambda^\varepsilon_v+2\alpha\leq\Lambda^\varepsilon_u+4\alpha$. Using (51), it follows that $|D_\varepsilon(x,y)-D'(x',y')|\leq4\alpha$. Similar arguments show that $|D_\varepsilon(x,y)-D'(x',y')|\leq3\alpha$ if (x,x') satisfies (a) and (y,y') satisfies (c).

It remains to consider the more delicate case when (x,x') and (y,y') both satisfy (a). In this case, we use the following lemma.

Lemma 16. *On the event E_ε, we have, for every $u,v\in\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}=\mathfrak{H}'^{[\delta,1-\delta]}$,*

$$|D_\varepsilon(u,v)-D'(u,v)|\leq7\alpha.$$

Proof. Let us fix $u,v\in\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}=\mathfrak{H}'^{[\delta,1-\delta]}$. We observe that at least one of the following two properties holds:

- $[|u,v|]_\varepsilon=[|u,v|]'$ and this interval is contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}=\mathfrak{H}'^{[\delta,1-\delta]}$;
- $[|v,u|]_\varepsilon=[|v,u|]'$ and this interval is contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}=\mathfrak{H}'^{[\delta,1-\delta]}$.

Without loss of generality, we may and will assume that the first property holds (otherwise, interchange u and v). Note that we have then

$$\min_{w\in[|u,v|]'}\Lambda'_w=\varepsilon\sqrt{3}+\min_{w\in[|u,v|]^\varepsilon}\Lambda^\varepsilon_w$$

by (51). Furthermore, the interval $[|v,u|]^\varepsilon$ is not contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$ if and only if $[|v,u|]'$ is not contained in $\mathfrak{H}'^{[\delta,1-\delta]}$, and then both these intervals contain 0. In that case $\min_{w\in[|v,u|]'}\Lambda'_w=0$, so that the maximum appearing in the formula for $D'^\circ(u,v)$ in (53) must be equal to $\min_{w\in[|u,v|]'}\Lambda'_w$.

If $[|v,u|]^\varepsilon$ is contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$, the same holds for $[|v,u|]'$, and the formulas for $D^\circ_\varepsilon(u,v)$ and $D'^\circ(u,v)$ in (52) and (53) show that $D^\circ_\varepsilon(u,v)=D'^\circ(u,v)$. If $[|v,u|]^\varepsilon$ is not contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$, then either the maximum in the formula for $D^\circ_\varepsilon(u,v)$ in (52) is $\min_{w\in[|u,v|]^\varepsilon}\Lambda^\varepsilon_w$, and this implies again that $D^\circ_\varepsilon(u,v)=D'^\circ(u,v)$, or this maximum is $\min_{w\in[|v,u|]^\varepsilon}\Lambda^\varepsilon_w$, which belongs to $[-\alpha,0]$, and we have

$$D^\circ_\varepsilon(u,v)\geq\Lambda^\varepsilon_u+\Lambda^\varepsilon_v. \tag{54}$$

Next, from the definition of $D_\varepsilon(u,v)$ as an infimum, we can find an integer $p \geq 1$ and $u_0, u_1, \dots, u_p \in \mathfrak{H}_\varepsilon$ such that $u_0 = u$, $u_p = v$, and

$$\sum_{i=1}^{p} D_\varepsilon^\circ(u_{i-1}, u_i) \leq D_\varepsilon(u,v) + \alpha.$$

We then distinguish several cases:

- All u_i, $i \in \{1, \dots, p-1\}$, belong to $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]} = \mathfrak{H}'^{[\delta,1-\delta]}$, and, for every $j \in \{1, \dots, p\}$, the maximum in the formula for $D_\varepsilon^\circ(u_{j-1}, u_j)$ is attained for one of the two intervals $[|u_{j-1}, u_j|]^\varepsilon$ and $[|u_j, u_{j-1}|]^\varepsilon$ that is contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$. In that case, the preceding considerations show that we have $D_\varepsilon^\circ(u_{j-1}, u_j) = D'^\circ(u_{j-1}, u_j)$, for every $j \in \{1, \dots, p\}$, and then

$$D'(u,v) \leq \sum_{i=1}^{p} D'^\circ(u_{i-1}, u_i) = \sum_{i=1}^{p} D_\varepsilon^\circ(u_{i-1}, u_i) \leq D_\varepsilon(u,v) + \alpha.$$

- All u_i, $i \in \{1, \dots, p-1\}$, belong to $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]} = \mathfrak{H}'^{[\delta,1-\delta]}$, but, for some $j \in \{1, \dots, p\}$, the maximum in the formula for $D_\varepsilon^\circ(u_{j-1}, u_j)$ is attained for one of the two intervals $[|u_{j-1}, u_j|]^\varepsilon$ and $[|u_j, u_{j-1}|]^\varepsilon$ that is not contained in $\mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$. In that case, using (54) together with the lower bound $D_\varepsilon^\circ(u_{i-1}, u_i) \geq |\Lambda^\varepsilon_{u_i} - \Lambda^\varepsilon_{u_{i-1}}|$, and then (51), we have

$$\begin{aligned} D_\varepsilon(u,v) \geq -\alpha + \sum_{i=1}^{p} D_\varepsilon^\circ(u_{i-1}, u_i) &\geq -\alpha + \sum_{i=1}^{j-1} |\Lambda^\varepsilon_{u_i} - \Lambda^\varepsilon_{u_{i-1}}| + \Lambda^\varepsilon_{u_{j-1}} + \Lambda^\varepsilon_{u_j} + \sum_{i=j+1}^{p} |\Lambda^\varepsilon_{u_i} - \Lambda^\varepsilon_{u_{i-1}}| \\ &\geq -\alpha + \Lambda^\varepsilon_u + \Lambda^\varepsilon_v \\ &= -\alpha + \Lambda'_u + \Lambda'_v - 2\varepsilon\sqrt{3} \\ &\geq D'(u,v) - 3\alpha. \end{aligned}$$

- For some $j \in \{1, \dots, p-1\}$, $u_j \notin \mathfrak{H}_\varepsilon^{[\delta,1-\delta]}$, which implies $\Lambda^\varepsilon_{u_j} \leq 2\alpha$. In that case,

$$\begin{aligned} D_\varepsilon(u,v) \geq -\alpha + \sum_{i=1}^{p} D_\varepsilon^\circ(u_{i-1}, u_i) \geq -\alpha + |\Lambda^\varepsilon_u - \Lambda^\varepsilon_{u_j}| + |\Lambda^\varepsilon_v - \Lambda^\varepsilon_{u_j}| &\geq -\alpha + \Lambda^\varepsilon_u + \Lambda^\varepsilon_v - 4\alpha \\ &= \Lambda'_u + \Lambda'_v - 2\varepsilon\sqrt{3} - 5\alpha \\ &\geq D'(u,v) - 7\alpha. \end{aligned}$$

So in all cases we have obtained $D_\varepsilon(u,v) \geq D'(u,v) - 7\alpha$. A symmetric argument (in fact simpler since we do not have to consider the second case) shows similarly that $D'(u,v) \geq D_\varepsilon(u,v) - 7\alpha$. This completes the proof of Lemma 16. □

Let us complete the proof of Theorem 15. If the pairs (x, x') and (y, y') in $\mathcal{R}_\varepsilon$ both satisfy property (a) above, we can find $u, v \in \mathfrak{H}_\varepsilon^{[\delta,1-\delta]} = \mathfrak{H}'^{[\delta,1-\delta]}$ such that $x = \Pi_\varepsilon(u), x' = \Pi'(u)$ and $y = \Pi_\varepsilon(v), y' = \Pi'(v)$. Then Lemma 16 gives

$$|D_\varepsilon(x,y) - D'(x',y')| = |D_\varepsilon(u,v) - D'(u,v)| \leq 7\alpha.$$

Recalling the bounds obtained before the statement of Lemma 16, we conclude that the distortion of $\mathcal{R}_\varepsilon$ is bounded above by 7α.

From the relation between the Gromov-Hausdorff distance and the infimum of distortions of correspondences ([18, Proposition 3.6], see [9, Theorem 7.3.25] for a proof in the non-pointed case, which is easily adapted), we get that, on the event E_ε, the (pointed) Gromov-Hausdorff distance between $\mathbb{D}^{(\varepsilon)}$ and $\mathbb{D}'$ is bounded above by $7\alpha/2$. Since E_ε has probability at least $1 - 4\eta$, and both α and η were arbitrary, this shows that $\mathbb{D}^{(\varepsilon)}$ converges in probability to $\mathbb{D}'$, in the sense of the pointed Gromov-Hausdorff topology. This completes the proof. □

7 The Caraceni-Curien construction of the Brownian half-plane

In this section, we show that the construction of the Brownian half-plane from [4, 14], which is presented in Section 4, is equivalent to the construction proposed by Caraceni and Curien [10, Section 5.3]. This is a direct application of our construction of the free Brownian disk pointed at a uniform boundary point (Theorem 15).

We start by recalling the construction of [10]. We consider a random process $(X_t)_{t\in\mathbb{R}}$ such that $(X_t)_{t\geq 0}$ and $(X_{-t})_{t\geq 0}$ are two independent five-dimensional Bessel processes started from 0. We then consider a random point measure $\mathcal{N}^\star(\mathrm{d}t\mathrm{d}\omega)$ on $\mathbb{R}\times\mathcal{S}$, such that, conditionally on X, $\mathcal{N}^\star$ is Poisson with intensity

$$2\,\mathbf{1}_{\{W_*(\omega)\geq -\sqrt{3}\,X_t\}}\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega).$$

We write

$$\mathcal{N}^\star=\sum_{i\in I^*}\delta_{(t_i^\star,\omega_i^\star)}$$

and, in a way similar to the previous sections, we let $\mathfrak{H}^\star$ be obtained from the disjoint union

$$\mathbb{R}\cup\Bigg(\bigcup_{i\in I^\star}\mathcal{T}_{(\omega_i^\star)}\Bigg),$$

by identifying the root of $\mathcal{T}_{(\omega_i^\star)}$ with $t_i^\star$, for every $i\in I^\star$. As in the preceding section, for any reals $a<b$, we introduce the subset of $\mathfrak{H}^\star$ defined by

$$\mathfrak{H}^{\star,[a,b]}:=[a,b]\cup\Bigg(\bigcup_{i\in I^\star,t_i\in[a,b]}\mathcal{T}_{(\omega_i^\star)}\Bigg).$$

We assign nonnegative labels to the points of $\mathfrak{H}^\star$ by setting $\Lambda_u^\star:=\sqrt{3}\,X_u$ if $u\in\mathbb{R}$, and $\Lambda_u^\star:=\sqrt{3}\,X_{t_i^\star}+\ell_u(\omega_i^\star)$ if $u\in\mathcal{T}_{(\omega_i^\star)}$, $i\in I^\star$. The exploration process $(\mathcal{E}_t^*)_{t\in\mathbb{R}}$ of $\mathfrak{H}^\star$ is then defined as in Section 4, and allows us to consider intervals $[|u,v|]$, for $u,v\in\mathfrak{H}^\star$. The functions $D^{\star,\circ}(u,v)$ and $D^\star(u,v)$, for $u,v\in\mathfrak{H}^\star$, are defined by the exact analogs of formulas (33) and (34), just replacing Λ^∞ with $\Lambda^\star$. Then, it is straightforward to verify that $D^\star(0,u)=D^{\star,\circ}(0,u)=\Lambda_u^\star$, for every $u\in\mathfrak{H}^\star$.

We set $\mathbb{H}^\star:=\mathfrak{H}^\star/\{D^\star=0\}$, and equip $\mathbb{H}^\star$ with the distance induced by $D^\star$, and with the distinguished point that is the equivalence class of 0. We view $(\mathbb{H}^\star,D^\star)$ as a random pointed boundedly compact length space (we could also consider the volume measure on $\mathbb{H}^\star$, but we refrain from doing so for the sake of simplicity). The compactness of closed balls in $\mathbb{H}^\star$ follows from the property

$$\lim_{a\to\infty}\inf\{\Lambda_u^\star: u\in\mathfrak{H}^\star\backslash\mathfrak{H}^{\star,[-a,a]}\}=+\infty, \tag{55}$$

whose easy proof is left to the reader (see [12, Lemma 3.3] for a very similar argument).

We note that $(\mathbb{H}^\star,D^\star)$ is scale invariant, meaning that, for every $\lambda>0$, the space obtained when multiplying the metric $D^\star$ by the factor λ has the same distribution as the original space. This scale invariance property is a straightforward consequence of our construction.

Theorem 17. *The pointed boundedly compact length spaces* $(\mathbb{H},D^\infty)$ *and* $(\mathbb{H}^\star,D^\star)$ *have the same distribution.*

Proof. Recall the space $(\mathbb{D}',D')$ in Theorem 15, which is a free Brownian disk with perimeter 1 pointed at a uniform boundary point. For every $r>0$, we write $B_r(\mathbb{D}')$ for the closed ball of radius r centered at the distinguished point of $(\mathbb{D}',D')$, and we use the similar notation $B_r(\mathbb{H})$ or $B_r(\mathbb{H}^\star)$. We view these balls as (random) pointed compact metric spaces. For $\lambda>0$, we also use the notation $\lambda\cdot B_r(\mathbb{D}')$ for the "same" space with the metric multiplied by the factor λ.

It follows from Corollary 3.9 in [4] that $(\mathbb{H},D^\infty)$ is the tangent cone in distribution of $(\mathbb{D}',D')$ at its distinguished point, meaning that, for every $r>0$, $\lambda\cdot B_{r/\lambda}(\mathbb{D}')$ converges in distribution to $B_r(\mathbb{H})$ when $\lambda\to\infty$, in the sense of the pointed Gromov-Hausdorff topology. So, to get the theorem, it will be enough to prove that $\lambda\cdot B_{r/\lambda}(\mathbb{D}')$ converges in distribution to $B_r(\mathbb{H}^\star)$ as $\lambda\to\infty$. Recalling the scale invariance property of $(\mathbb{H}^\star,D^\star)$, this follows immediately from the next lemma, which is analogous to [4, Corollary 3.9].

Lemma 18. *For every $\delta > 0$, we can find $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$, we can couple the spaces $(\mathbb{D}', D')$ and $(\mathbb{H}^\star, D^\star)$ in such a way that the balls $B_\varepsilon(\mathbb{D}')$ and $B_\varepsilon(\mathbb{H}^\star)$ are equal with probability at least $1 - \delta$.*

Proof. We fix $\delta > 0$. Recall the notation introduced before Theorem 15, and the construction of $(\mathbb{D}', D')$ as a quotient space of the (labeled) space $\mathfrak{H}'$, which is defined from the pair $(\mathbf{b}, \mathcal{N}')$. For every $\eta \in (0, 1/2)$, let $\mathfrak{H}'^{(\eta)}$ be the subset of $\mathfrak{H}'$ defined by

$$\mathfrak{H}'^{(\eta)} := [0, \eta] \cup [1 - \eta, 1] \cup \left(\bigcup_{j \in J, t'_j \in [0,\eta] \cup [1-\eta,1]} \mathcal{T}_{(\omega'_j)} \right),$$

with the same identifications as previously (0 is identified to 1, and the root of $\mathcal{T}_{(\omega'_j)}$ is identified to t'_j).

By choosing $\eta \in (0, 1/2)$ sufficiently small, we may couple the pairs $(\mathbf{b}, \mathcal{N}')$ and $(X, \mathcal{N}^*)$ in such a way that the following two properties hold with probability at least $1 - \delta/2$:

(i) $\mathbf{b}_t = X_t$ and $\mathbf{b}_{1-t} = X_{-t}$ for every $t \in [0, \eta]$;

(ii) the restriction of $\mathcal{N}'(\mathrm{d}t\mathrm{d}\omega)$ to $[0, \eta] \times \mathcal{S}$ coincides with the restriction of $\mathcal{N}^\star(\mathrm{d}t\mathrm{d}\omega)$ to $[0, \eta] \times \mathcal{S}$, and the restriction of $\mathcal{N}'(\mathrm{d}t\mathrm{d}\omega)$ to $[1 - \eta, 1] \times \mathcal{S}$ coincides with the pushforward of the restriction of $\mathcal{N}^\star(\mathrm{d}t\mathrm{d}\omega)$ to $[-\eta, 0] \times \mathcal{S}$ under the translation $(t, \omega) \mapsto (1 + t, \omega)$.

In what follows, we assume that $(\mathbf{b}, \mathcal{N}')$ and $(X, \mathcal{N}^*)$ have been coupled in this way. Then, we can choose $\varepsilon > 0$ small enough so that the two properties

(iii) $\Lambda'_u > 3\varepsilon$ for every $u \in \mathfrak{H}' \backslash \mathfrak{H}'^{(\eta)}$,

(iv) $\Lambda^\star_u > 3\varepsilon$ for every $u \in \mathfrak{H}^\star \backslash \mathfrak{H}^{\star,[-\eta,\eta]}$,

hold except on a set of probability at most $\delta/2$ (we use (55) for (iv)).

Let $E_{\eta,\varepsilon}$ be the event where properties (i)—(iv) hold, so that $\mathbb{P}(E_{\eta,\varepsilon}) \geq 1 - \delta$. We will verify that the property $B_\varepsilon(\mathbb{D}') = B_\varepsilon(\mathbb{H}^\star)$ holds on $E_{\eta,\varepsilon}$. From now on we argue on the event $E_{\eta,\varepsilon}$.

We first observe that there is an obvious one-to-one correspondence between the sets $\mathfrak{H}'^{(\eta)}$ and $\mathfrak{H}^{\star,[-\eta,\eta]}$. In particular, the subset $[1 - \eta, 1]$ of $\mathfrak{H}'^{(\eta)}$ corresponds to the subset $[-\eta, 0]$ of $\mathfrak{H}^{\star,[-\eta,\eta]}$ via the translation $u \mapsto u - 1$ and (thanks to (ii)) the trees $\mathcal{T}_{(\omega'_j)}$ for indices j such that $1 - \eta < t'_j < 1$ correspond to the trees $\mathcal{T}_{(\omega^\star_i)}$ for indices i such that $-\eta < t^\star_i < 0$. If $u \in \mathfrak{H}'^{(\eta)}$, we will write $u^\star$ for the corresponding point in $\mathfrak{H}^{\star,[-\eta,\eta]}$. Using (i) and (ii), we have $\Lambda'_u = \Lambda^\star_{u^\star}$ for every $u \in \mathfrak{H}'^{(\eta)}$. Moreover, if $u, v \in \mathfrak{H}'^{(\eta)}$, the set $[|u^\star, v^\star|] \cap \mathfrak{H}^{\star,[-\eta,\eta]}$ coincides with the set of all points $w^\star$ for $w \in [|u, v|] \cap \mathfrak{H}'^{(\eta)}$.

Recall that $\Lambda'_u = D'(0, u)$ for every $u \in \mathfrak{H}'$, and $\Lambda^\star_v = D^\star(0, v)$ for every $v \in \mathfrak{H}^\star$. It follows from (iii) and (iv) that the sets $\{u \in \mathfrak{H}' : D'(0, u) \leq 3\varepsilon\}$ and $\{v \in \mathfrak{H}^\star : D^\star(0, v) \leq 3\varepsilon\}$ are contained in $\mathfrak{H}'^{(\eta)}$ and in $\mathfrak{H}^{\star,[-\eta,\eta]}$ respectively, and these two sets are equal modulo the preceding identification of $\mathfrak{H}'^{(\eta)}$ with $\mathfrak{H}^{\star,[-\eta,\eta]}$. To complete the proof, it remains to verify that, for any $u, v \in \mathfrak{H}'$ such that $D'(0, u) \leq \varepsilon$ and $D'(0, v) \leq \varepsilon$, we have $D'(u, v) = D^\star(u^*, v^*)$.

To get this, first observe that, if $\widetilde{u}, \widetilde{v} \in \mathfrak{H}'$ are such that $D'(0, \widetilde{u}) \leq 3\varepsilon$ and $D'(0, \widetilde{v}) \leq 3\varepsilon$ (so that in particular $\widetilde{u}, \widetilde{v} \in \mathfrak{H}'^{(\eta)}$), we have

$$\inf_{w \in [|\widetilde{u}, \widetilde{v}|]} \Lambda'_w = \inf_{w \in [|\widetilde{u}, \widetilde{v}|] \cap \mathfrak{H}'^{(\eta)}} \Lambda'_w = \inf_{w \in [|\widetilde{u}^\star, \widetilde{v}^\star|] \cap \mathfrak{H}^{\star,[-\eta,\eta]}} \Lambda^\star_w = \inf_{w \in [|\widetilde{u}^\star, \widetilde{v}^\star|]} \Lambda^\star_w,$$

and thus $D'^\circ(\widetilde{u}, \widetilde{v}) = D^{\star,\circ}(\widetilde{u}^\star, \widetilde{v}^\star)$.

Then, let $u, v \in \mathfrak{H}'$ such that $D'(0, u) \leq \varepsilon$ and $D'(0, v) \leq \varepsilon$. We observe that, in the definition of $D'(u, v)$ as an infimum over choices of $u_0 = u, u_1, \ldots, u_{p-1}, u_p = v$, we may disregard the case when $D'(0, u_j) = \Lambda'_{u_j} > 3\varepsilon$ for some $j \in \{1, \ldots, p - 1\}$, because, if this happens, the lower bound $D'^\circ(u_{i-1}, u_i) \geq |\Lambda'_{u_i} - \Lambda'_{u_{i-1}}|$ implies that

$$\sum_{i=1}^{p} D'^\circ(u_{i-1}, u_i) \geq |\Lambda'_{u_j} - \Lambda'_u| + |\Lambda'_{u_j} - \Lambda'_v| \geq 4\varepsilon,$$

whereas we have $D'(u,v) \leq D'(0,u) + D'(0,v) \leq 2\varepsilon$. A similar remark applies to the definition of $D^\star(u^\star, v^\star)$, and we conclude from the equality $D'^\circ(\widetilde{u}, \widetilde{v}) = D^{\star,\circ}(\widetilde{u}^\star, \widetilde{v}^\star)$ when $D'(0,\widetilde{u}) \leq 3\varepsilon$ and $D'(0,\widetilde{v}) \leq 3\varepsilon$ that the infima in the respective definitions of $D'(u,v)$ and $D^\star(u^*,v^*)$ are equal, so that $D'(u,v) = D^\star(u^*,v^*)$ as desired. This completes the proof of Lemma 18 and Theorem 17. □

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