Scaling limits for the peeling process on random maps

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Abstract
We study the scaling limit of the volume and perimeter of the discovered regions in the Markovian
explorations known as peeling processes for infinite random planar maps such as the uniform infinite
planar triangulation (UIPT) or quadrangulation (UIPQ). In particular, our results apply to the metric
exploration or peeling by layers algorithm, where the discovered regions are (almost) completed balls,
or hulls, centered at the root vertex. The scaling limits of the perimeter and volume of hulls can be
expressed in terms of the hull process of the Brownian plane studied in our previous work. Other
applications include first-passage percolation with exponential edge weights on the dual of random
maps, also known as the Eden model or uniform peeling.

1 Introduction
The spatial Markov property of random planar maps is one of the most important properties of these
random lattices. Roughly speaking, this property says that, after a region of the map has been explored,
the law of the remaining part only depends on the perimeter of the discovered region. The spatial Markov
property was first used in the physics literature, without a precise justification: Watabiki [26] introduced
the so-called “peeling process”, which is a growth process discovering the random lattice step by step. A
rigorous version of the peeling process and its Markovian properties was given by Angel [3] in the case
of the Uniform Infinite Planar Triangulation (UIPT), which is the local limit of uniformly distributed
plane triangulations with a fixed size [6]. The peeling process has been used since to derive information
about the metric properties of the UIPT [3], about percolation [3, 4, 22] and simple random walk [7] on
the UIPT and its generalizations, and more recently about the conformal structure [12] of random planar
maps. It also plays a crucial role in the construction of “hyperbolic” random triangulations [5, 13].

In the present paper, we derive scaling limits for the perimeter and the volume of the discovered region
in a peeling process of the UIPT. Our methods also apply to the Uniform Infinite Planar Quadrangulation
(UIPQ). By considering the special case of the peeling by layers, we get scaling limits for the volume and
the boundary length of the hull of radius $r$ centered at the root of the UIPT, or of the UIPQ (the hull
of radius $r$ is obtained by “filling in the finite holes” in the ball of radius $r$). The limiting processes that
arise in these scaling limits coincide with those that appeared in our previous work [14] dealing with the
hull process of the Brownian plane. This is not surprising since the Brownian plane is conjectured to
be the universal scaling limit of many random infinite lattices such as the UIPT, and it is known that
this conjecture holds in the special case of the UIPQ [15]. We also apply our results to the study of
first-passage percolation with exponential edge weights on the dual graph of the UIPT (this is also known
as the Eden model). In particular, we show that the volume and perimeter of the hulls with respect to
the first-passage percolation distance have the same scaling limits as those corresponding to the graph
distance, up to an explicit deterministic multiplicative factor.
For the sake of clarity, the following results are stated and proved in the case of the UIPT corresponding to type II triangulations, in the terminology of Angel and Schramm [6]. In type II triangulations, loops are not allowed but there may be multiple edges. Section 6 explains the changes that are needed for the extension of our results to other random lattices such as the UIPT for type I triangulations or the UIPQ. In these extensions, scaling limits remain the same, but different constants are involved. In the case of type II triangulations, the three basic constants that arise in our results are

\[ p_{\Delta^2} = \left( \frac{2}{3} \right)^{2/3}, \quad v_{\Delta^2} = \left( \frac{2}{3} \right)^{7/3} \quad \text{and} \quad h_{\Delta^2} = 12^{-1/3}. \]

Here the subscript \( \Delta^2 \) emphasizes the fact that these constants are relevant to the case of type II triangulations.

So, except in Section 6, all triangulations in this article are type II triangulations. The corresponding UIPT is denoted by \( T_{\infty} \). This is an infinite random triangulation of the plane given with a distinguished oriented edge whose tail vertex is called the origin (or root vertex) of the map. If \( t \) is a rooted finite triangulation with a simple boundary \( \partial t \), we denote the number of inner vertices of \( t \) by \( |t| \) and the boundary length of \( t \) by \( |\partial t| \). Furthermore, we say that \( t \) is a subtriangulation of \( T_{\infty} \) and write \( t \subset T_{\infty} \), if \( T_{\infty} \) is obtained from \( t \) by gluing an infinite triangulation with a simple boundary along the boundary of \( t \) (of course we also require that the root of \( T_{\infty} \) coincides with the root of \( t \) after this gluing operation). If \( t \subset T_{\infty} \) and \( e \) is an edge of \( \partial t \), the triangulation obtained by the peeling of \( e \) is the triangulation \( t \) to which we add the face incident to \( e \) that was not already in \( t \), as well as the finite region that the union of \( t \) and this added face may enclose (recall that the UIPT has only one end [6]). An exploration process \( (T_t)_{t \geq 0} \) is a sequence of subtriangulations of the UIPT with a simple boundary such that \( T_0 \) consists only of the root edge (viewed as a trivial triangulation) and for every \( i \geq 0 \) the map \( T_{t+1} \) is obtained from \( T_t \) by peeling one edge of its boundary. If the choice of this edge is independent of \( T_{\infty} \setminus T_t \), the exploration is said to be Markovian and we call it a peeling process. Different peeling processes correspond to different ways of choosing the edge to be peeled at each step. See Section 3.1 for a more rigorous presentation.

Our first theorem complements results due to Angel [3] by describing the scaling limit of the perimeter and volume of the discovered region in a peeling process. We let \( (S_t)_{t \geq 0} \) denote the stable Lévy process with index 3/2 and only negative jumps, which starts from 0 and is normalized so that its Lévy measure is \( 3/(4\sqrt{\pi})|x|^{-5/2}1_{\{x<0\}} \) or equivalently \( \mathbb{E}[\exp(\lambda S_t)] = \exp(\lambda^{3/2}) \) for any \( \lambda, t \geq 0 \). The process \( (S_t)_{t \geq 0} \) conditioned to stay nonnegative is then denoted by \( (S^+_t)_{t \geq 0} \) (see [8, Chapter VII] for a rigorous definition of \( (S^+_t)_{t \geq 0} \)). We also let \( \xi_1, \xi_2, \ldots \) be a sequence of independent real random variables with density

\[ \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}x^2}1_{\{x>0\}}. \]

We assume that this sequence is independent of the process \( (S^+_t)_{t \geq 0} \) and, for every \( t \geq 0 \), we set \( Z_t = \sum_{t_1 \leq t} \xi_i \cdot (\Delta S^+_i)^2 \) where \( t_1, t_2, \ldots \) is a measurable enumeration of the jumps of \( S^+ \).

**Theorem 1** (Scaling limit for general peelings). For any peeling process \( (T_n)_{n \geq 0} \) of the UIPT, we have the following convergence in distribution in the sense of Skorokhod

\[ \left( \frac{|\partial T_{[nt]}|}{p_{\Delta^2} \cdot n^{2/3}}, \frac{|T_{[nt]}|}{v_{\Delta^2} \cdot n^{5/3}} \right)_{t \geq 0} \xrightarrow{n \to \infty} (S^+_t, Z_t)_{t \geq 0}. \]

The proof of Theorem 1 relies on the explicit expression of the transition probabilities of the peeling process. It follows from this explicit expression that the process of perimeters \( (|\partial T_t|)_{t \geq 0} \) is an \( h \)-transform of a random walk with independent increments in the domain of attraction of a spectrally negative stable
distribution with index 3/2 (Proposition 5). This $h$-transform is interpreted as conditioning the random walk to stay above level 2, and in the scaling limit this leads to the process $(S_t^3)_{t \geq 0}$. The common distribution of the variables $\xi_i$ is the scaling limit of the volume of a Boltzmann triangulation (see Section 2.1) conditioned to have a large boundary size. The appearance of this distribution is explained by the fact that the “holes” created by the peeling process are filled by finite triangulations distributed according to Boltzmann weights (this is called the free distribution in [6, Definition 2.3]). As a corollary of Theorem 1, we prove that any peeling process of the UIPT will eventually discover the whole triangulation, i.e., $\bigcup T_n = T_\infty$ no matter what peeling algorithm is used (of course as long as the exploration is Markovian), see Corollary 6. We note that Theorem 1 can be applied to various peeling processes that have been considered in earlier works: peeling along percolation interfaces [3, 4], peeling along simple random walk [7], peeling along a Brownian or an SLE$_6$ exploration of the Riemann surface associated with the UIPT [12], etc. In the present work, we apply Theorem 1 to two specific peeling algorithms, each of which is related to a “metric” exploration of the UIPT. The first one is the peeling by layers, which essentially grows balls for the graph distance on the UIPT, and the second one is the uniform peeling, which is related to first-passage percolation on the dual graph of the UIPT.

Scaling limits for the hulls. For every integer $r \geq 1$, the ball $B_r(T_\infty)$ is defined as the union of all faces of $T_\infty$ whose boundary contains at least one vertex at graph distance smaller than or equal to $r - 1$ from the origin (when $r = 0$ we agree that $B_0(T_\infty)$ is the trivial triangulation consisting only of the root edge). The hull $B^*_r(T_\infty)$ is then obtained by adding to the ball $B_r(T_\infty)$ the bounded components of the complement of this ball (see Fig. 1). Note that $B^*_r(T_\infty)$ is a finite triangulation with a simple boundary. One can define a particular peeling process $(T_i)_{i \geq 0}$ (called the peeling by layers) that, for every $n \geq 0$, there exists a random integer $H_n$ such that $B^*_{H_n}(T_\infty) \subset T_n \subset B^*_{H_n+1}(T_\infty)$. Scaling limits for the volume and the boundary length of the hulls can then be derived by applying Theorem 1 to this particular peeling algorithm. A crucial step in this derivation is to get information about the asymptotic behavior of $H_n$ when $n \to \infty$ (Proposition 9). Before stating our limit theorem for the hulls, we need to introduce some notation.

![Figure 1](image.png)

**Figure 1:** From left to right, the “cactus” representation of the UIPT, the ball $B_r(T_\infty)$, whose boundary may have several components, and the hull $B^*_r(T_\infty)$, whose boundary is a simple cycle.

For every real $u \geq 0$, set $\psi(u) = u^{3/2}$. The continuous-state branching process with branching
mechanism $\psi$ is the Feller Markov process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}_+$, whose semigroup is characterized as follows: for every $x, t \geq 0$ and every $\lambda > 0$,

$$E[e^{-\lambda X_t} \mid X_0 = x] = \exp \left( -x \left( \lambda^{-1/2} + t/2 \right)^{-2} \right).$$

Note that $X$ gets absorbed at 0 in finite time. It is easy to construct a process $(L_t)_{t \geq 0}$ with càdlàg paths such that the time-reversed process $(L_{-t^{-}})_{t \leq 0}$ (indexed by negative times) is distributed as $X$ “started from $+\infty$ at time $-\infty$” and conditioned to hit zero at time 0 (see [14, Section 2.1] for a detailed presentation). We consider the sequence $(\xi_i)_{i \geq 1}$ introduced before Theorem 1, and we assume that this sequence is independent of $L$. We then set, for every $t \geq 0$

$$\mathcal{M}_t = \sum_{s_1, s_2, \ldots} \xi_i \cdot (\Delta \mathcal{L}_{s_i})^2,$$

where $s_1, s_2, \ldots$ is a measurable enumeration of the jumps of $L$.

**Theorem 2** (Scaling limit of the hull process).

We have the following convergence in distribution in the sense of Skorokhod,

$$\left( n^{-2} |\partial B^*_{|\nu|}(T_\infty)|, n^{-4} |B^*_{|\nu|}(T_\infty)| \right)_{t \geq 0} \xrightarrow{n \to \infty} \left( p_{\Delta^2} \cdot L_t/\partial B_{\Delta^2}, \psi_{\Delta^2} \cdot \mathcal{M}_t/\partial B_{\Delta^2} \right)_{t \geq 0}.$$

A scaling argument shows that the limiting process has the same distribution as

$$\left( \frac{p_{\Delta^2}}{(\partial B_{\Delta^2})^2} L_t, \frac{\psi_{\Delta^2}}{(\partial B_{\Delta^2})^4} \mathcal{M}_t \right)_{t \geq 0},$$

but the form given in Theorem 2 helps to understand the connection with Theorem 1.

We note that the convergence in distribution of the variables $r^{-2} |\partial B^*_{|\nu|}(T_\infty)|$ as $r \to \infty$ had already been obtained by Krikun [19, Theorem 1.4] via a different approach. The limiting process in Theorem 2 appeared in the companion paper [14] as the process describing the evolution of the boundary length and the volume of hulls in the Brownian plane (in the setting of the Brownian plane, the length of the boundary has to be defined in a generalized sense). The paper [14] contains detailed information about distributional properties of this limiting process (see Proposition 1.2 and Theorem 1.4 in [14]). In particular, for every fixed $s > 0$, the joint distribution of the pair $(L_s, \mathcal{M}_s)$ is known explicitly. Here we mention only the Laplace transform of the marginal laws:

$$E[e^{-\lambda L_t}] = \left( 1 + \frac{\lambda s^2}{4} \right)^{-3/2},$$

$$E[e^{-\lambda \mathcal{M}_t}] = 3^{3/2} \cosh \left( \frac{(2\lambda)^{1/4}}{\sqrt{8/3}} \right) \left( \cosh^2 \left( \frac{(2\lambda)^{1/4}}{\sqrt{8/3}} \right) + 2 \right)^{-3/2}.$$

Note in particular that $L_t$ follows a Gamma distribution with parameter $3/2$.

**First-passage percolation.** Consider now the dual graph of the UIPT, whose vertices are the faces of the UIPT, and where two vertices are connected by an edge if the corresponding faces of the UIPT share a common edge. We assign independently to each edge of the dual graph an exponential weight with parameter 1. For every $t \geq 0$, we write $F_t$ for the union of all faces that may be reached from the root face (by definition the root face is incident to the right side of the root edge) by a path whose total weight is at most $t$. As usual, $F^*_t$ stands for the hull of $F_t$, which is obtained by filling in the finite holes of $F_t$ inside $T_\infty$. Then $F^*_t$ is a triangulation with a simple boundary. If $0 = \tau_0 < \tau_1 < \ldots < \tau_n \ldots$ are the
jump times of the process $t \mapsto F_t^*$, it is not hard to verify that the sequence $(F_{t_n}^*)_{n \geq 0}$ is a uniform peeling process, meaning that at each step the edge to be peeled off is chosen uniformly at random among all edges of the boundary. See Proposition 13 for a precise statement. Then Theorem 1 leads to the following result.

Figure 2: Illustration of the exploration along first-passage percolation on the dual of the UIPT.

We represented $F_t^*$ for some value of $t > 0$. By standard properties of exponential variables, the next dual edge to be explored is uniformly distributed on the boundary.

**Theorem 3** (Scaling limits for first passage percolation). We have the following convergence in distribution for the Skorokhod topology

\[
(n^{-2}|\partial F_{nt}^*|, n^{-4}|F_{nt}^*|)_{t \geq 0} \xrightarrow{(d) \quad n \to \infty} \left( p_{\Delta^2} \cdot L_{p_{\Delta^2}} t, v_{\Delta^2} \cdot M_{p_{\Delta^2}} t \right)_{t \geq 0}.
\]

Set $a_{\Delta^2} = p_{\Delta^2} h_{\Delta^2} = 1/3$. If we compare Theorem 2 with Theorem 3, we see that the scaling limits of the volume and the perimeter are the same for $B_{r}(T_{\infty})$ and for $F_{r/a_{\Delta^2}}^*$. This is consistent with the conjecture saying that balls for the first-passage percolation distance grow like deterministic balls, up to a constant multiplicative factor (this property is not expected to hold for deterministic lattices such as $\mathbb{Z}^2$, but in some sense the UIPT is more isotropic). Informally, writing $d_{gr}$ for the graph distance (on the UIPT) and $d_{fpp}$ for the first-passage percolation distance, our results suggest that in large scales,

\[
d_{fpp}(. , .) \approx \frac{1}{a_{\Delta^2}} \cdot d_{gr}(. , .).
\]

Note that $d_{gr}$ is a metric on the UIPT, whereas $d_{fpp}$ is a metric on the dual graph. Still it is easy to restate the previous display in the form of a precise conjecture (see the end of Section 5). This conjecture is consistent with the recent calculations of Ambjørn and Budd [1] for two and three-point functions in first-passage percolation on random triangulations.

We finally note that our uniform peeling process can be viewed as a variant of the classical Eden model on the (dual graph of the) UIPT. The same variant has been considered by Miller and Sheffield [23] and served as a motivation for the construction of Quantum Loewner Evolutions. In fact the process $QLE(\frac{\kappa}{4}, 0)$ that is constructed in [23] is a continuum analog of the Eden model on the UIPT. See Section 2.2 in [23] for more details.
The organization of the paper follows the preceding presentation. In Section 2, we recall some enumeration results for triangulations that play an important role in the paper, and we also give a result connecting the UIPT with Boltzmann triangulations, which is of independent interest (Theorem 4). This result shows that the distributions of the ball of radius $r$ in the UIPT and in a Boltzmann triangulation are linked by an absolute continuity relation involving a martingale, which has an explicit expression in terms of the sizes of the cycles bounding the connected components of the complement of the ball.

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2 Preliminaries

Throughout this work, we consider only rooted planar maps, and we often omit the word rooted. We view planar maps as graphs drawn on the sphere, with the usual identification modulo orientation-preserving homeomorphisms. Recall that, except in Section 6 below, we restrict our attention to type II triangulations, meaning that there are no loops, but multiple edges are allowed. We define a triangulation with a boundary as a rooted planar map without loops, with a distinguished face (the external face) such that all faces except possibly the distinguished one are triangles. If $\tau$ is a triangulation with a boundary as a rooted planar map without loops, with a distinguished face (the external face) bounded by a simple cycle (called the boundary), such that all faces except possibly the distinguished one are triangles. If $\tau$ is a triangulation with a boundary, we denote its boundary by $\partial \tau$. Vertices of $\tau$ not on the boundary are called inner vertices. The size $|\tau|$ of $\tau$ is defined as the number of inner vertices of $\tau$. The length $|\partial \tau|$ of $\partial \tau$ (or perimeter of $\tau$) is the number of edges, or equivalently the number of vertices, in $\partial \tau$. Note that $|\partial \tau| \geq 2$ since loops are not allowed.
2.1 Enumeration

We gather here several results about the asymptotic enumeration of planar triangulations, see [3, 6] and the references therein. For every \( n \geq 0 \) and \( p \geq 2 \), we let \( T_{n,p} \) denote the set of all (type II) triangulations of size \( n \) with a simple boundary of length \( p \), that are rooted at an edge of the boundary oriented so that the external face lies on the right of the root edge. We have

\[
\# T_{n,p} = \frac{2^{n+1}(2p-3)![(2p+3n-4)!]}{(p-2)!n!(2p+2n-2)!} \sim n \rightarrow \infty C(p) \left( \frac{27}{2} \right)^n n^{-5/2}
\]

where

\[
C(p) = \frac{4}{3^{3/2} \sqrt{\pi}} \left( \frac{2p-3}{2} \right)! \left( \frac{9}{4} \right)^p \sim p \rightarrow \infty \frac{1}{54 \pi \sqrt{3}} 2^p 9^p \sqrt{p}.
\]

The exact formula for \( \# T_{n,p} \) in (1) gives \( \# T_{n,p} = 1 \) for \( n = 0 \) and \( p = 2 \). This formula is valid provided we make the special convention that the rooted planar map consisting of a single (oriented) edge between two vertices is viewed as a triangulation with a simple boundary of length 2: This will be called the trivial triangulation. It will be used in the sequel as the starting point of the peeling process, and also sometimes to “fill in” holes of size two arising in this process.

The exponent \( 5/2 \) in (1) is typical of the enumeration of planar maps and shows that

\[
Z(p) := \sum_{n=0}^{\infty} \left( \frac{2}{27} \right)^n \# T_{n,p} < \infty.
\]

The numbers \( Z(p) \) can be computed exactly (see [3, Proposition 1.7]): for every \( p \geq 2 \),

\[
Z(p) = \left( \frac{2p-4}{(p-2)!p!} \right) \left( \frac{9}{4} \right)^{p-1}.
\]

Triangulations in \( T_{n,p} \), for some \( n \geq 0 \), are also called triangulations of the \( p \)-gon. By definition, the (critical) Boltzmann distribution on triangulations of the \( p \)-gon is the probability measure on \( \bigcup_{n \geq 0} T_{n,p} \) that assigns mass \( (2/27)^n Z(p)^{-1} \) to each triangulation of \( T_{n,p} \). This is also called the free distribution in [6]. It follows from (3) that for every \( x \in [0, 1/9] \),

\[
\sum_{p=1}^{\infty} Z(p+1) x^p = \frac{1}{2} \left( \frac{1 - 9x}{27x} \right)^{3/2} - 1.
\]

From (3) and the last display, we get that

\[
Z(p+1) \sim p \rightarrow \infty t_{\Delta^2} \cdot 9^p p^{-5/2}, \quad \text{where} \quad t_{\Delta^2} = \frac{1}{4 \sqrt{\pi}}.
\]

\[
\sum_{p=1}^{\infty} Z(p+1) 9^{-p} = \frac{1}{6},
\]

\[
\sum_{p=1}^{\infty} p Z(p+1) 9^{-p} = \frac{1}{3}.
\]

Finally, we note that there is a bijection between rooted triangulations of the 2-gon having \( n \) inner vertices and rooted plane triangulations having \( n + 2 \) vertices: Just glue together the two boundary edges of a triangulation of the 2-gon to get a triangulation of the sphere. The Boltzmann distribution on rooted triangulations of the 2-gon thus induces a probability measure on the space of all triangulations of the sphere (including the trivial one). A random triangulation distributed according to this probability measure is called a Boltzmann triangulation of the sphere. Equivalently, the law of a Boltzmann triangulation of the sphere assigns a mass \( (2/27)^{n-2} Z(2)^{-1} \) to every triangulation of the sphere with \( n \) vertices (including the trivial triangulation for which \( n = 2 \)).
2.2 Boltzmann triangulations and the UIPT

In this section, we describe a relation between Boltzmann triangulations of the sphere and the UIPT. This relation is not really needed in what follows but it helps to understand the importance of Boltzmann triangulations in the subsequent developments.

Let $T_{\text{Bol}}$ be a Boltzmann triangulation of the sphere. As in the introduction above, for every integer $r \geq 1$, let $B_r(T_{\text{Bol}})$ denotes the ball of radius $r$ in $T_{\text{Bol}}$. So $B_r(T_{\text{Bol}})$ is the rooted planar map obtained by keeping only those faces of $T_{\text{Bol}}$ that are incident to at least one vertex at distance at most $r − 1$ from the root vertex. We view $B_r(T_{\text{Bol}})$ as a random variable with values in the space of all (type II) triangulations with holes. Here, a triangulation with holes is a planar map without loops, with a finite number of distinguished faces called the holes, such that all faces except possibly the holes are triangles, the boundary of every hole is a simple cycle, whose length is called the size of the hole, and two distinct holes cannot share a common edge (the triangulations with a simple boundary that we considered above are just triangulations with a single hole). In the case of $B_r(T_{\text{Bol}})$, holes obviously correspond to the connected components of the complement of the ball, in a way analogous to the middle part of Fig. 1.

We write $\ell_1(r), \ell_2(r), \ldots, \ell_n(r)$ for the sizes of the holes of $B_r(T_{\text{Bol}})$ enumerated in nonincreasing order. We also write $F_r$ for the $\sigma$-field generated by $B_r(T_{\text{Bol}})$ and we let $F_0$ be the trivial $\sigma$-field. Recall our notation $B_r(T_\infty)$ for the ball of radius $r$ in the UIPT, which is also viewed as a random triangulation with holes.

**Theorem 4.** Let $f(n) = \frac{n^2}{2} \cdot (n − 1) \cdot (2n − 3)$ for every integer $n \geq 3$ and $f(2) = 9$. The random process $(M_r)_{r \geq 0}$ defined by

$$M_r := \sum_{i=1}^{\ell_r} f(\ell_i(r)), \quad \text{for } r \geq 1,$$

and $M_0 = 1$, is a martingale with respect to the filtration $(F_r)_{r \geq 0}$. Moreover, if $F$ is any nonnegative measurable function on the space of triangulations with holes, we have, for every $r \geq 1$,

$$\mathbb{E}[F(B_r(T_\infty))] = \mathbb{E}[M_r F(B_r(T_{\text{Bol}}))].$$

(7)

The second part of the theorem shows that the law of a ball in the UIPT can be obtained by biasing the law of the corresponding ball in a Boltzmann triangulation using the martingale $M_r$. This is an analog of a classical result for Galton–Watson trees: In order to get the first $k$ generations of a Galton–Watson tree conditioned on non-extinction, one biases the law of the first $k$ generations of an unconditioned Galton–Watson tree using a martingale which is simply the size of generation $k$ of the tree (see e.g. [21, Chapter 12]). In a sense, the UIPT can thus be viewed as a Boltzmann triangulation conditioned to be infinite. This is related to the discussion in Section 6 of [6], which associates with a Boltzmann triangulation a multitype Galton–Watson tree describing the structure of balls, in such a way that the tree associated with the UIPT is just the same Galton–Watson tree conditioned on non-extinction.

**Proof.** It suffices to prove the second part of the theorem. Indeed, if (7) holds, we immediately get, for every $1 \leq k \leq \ell$, and every function $F$,

$$\mathbb{E}[M_r F(B_k(T_{\text{Bol}}))] = \mathbb{E}[M_k F(B_k(T_{\text{Bol}}))],$$

and it follows that $\mathbb{E}[M_r | F_k] = M_k$.

In order to verify the second assertion of the theorem, we will provide explicit formulas for the probability that the ball of radius $r$ in $T_{\text{Bol}}$, resp. in $T_\infty$, is equal to a given triangulation with holes. Let
\( t \) be a fixed triangulation with holes. Note that \( \mathbb{P}(B_r(T_{\text{Bol}}) = t) > 0 \) if and only if all vertices belonging to the boundaries of holes of \( t \) are at distance \( r \) from the root vertex, and all faces of \( t \) other than the holes are incident to (at least) one vertex at distance at most \( r - 1 \) from the root vertex. Furthermore, the preceding conditions are also necessary for \( \mathbb{P}(B_r(T_{\infty}) = t) \) to be strictly positive.

Write \( n \) for the total number of vertices of \( t \), \( m \geq 0 \) for the number of holes of \( t \) and \( p_1, \ldots, p_m \) for the respective sizes of the holes of \( t \) – the holes are enumerated in some deterministic manner given \( t \). Then, for every integer \( q \geq n \), the number of triangulations with \( q \) vertices whose ball of radius \( r \) coincides with \( t \) is equal to

\[
\sum_{n_1 + \cdots + n_m = q - n} \left( \prod_{j=1}^{m} \# T_{n_j, p_j} \right),
\]

where the sum is over all choices of the nonnegative integers \( n_1, \ldots, n_m \) such that \( n_1 + \cdots + n_m = q - n \), with the additional constraint that \( n_i > 0 \) if \( p_i = 2 \). The reason for this last constraint if the fact that a hole of size 2 cannot be filled by the trivial triangulation, because this would mean that we glue the two edges of the boundary. Note that when there is no hole \((m = 0)\) the quantity in the last display should be interpreted as equal to 1 if \( q = n \) and to 0 otherwise. The total Boltzmann weight of those triangulations whose ball of radius \( r \) coincides with \( t \) is then

\[
\sum_{q=n}^{\infty} \left( \frac{2}{27} \right)^{q-2} Z(2)^{-1} \sum_{n_1 + \cdots + n_m = q - n} \left( \prod_{j=1}^{m} \# T_{n_j, p_j} \right),
\]

where we impose the same constraint as before on the integers \( n_1, \ldots, n_m \) in the sum. We set \( Z'(p) = Z(p) \) if \( p > 2 \) and \( Z'(2) = Z(2) - 1 \). The quantity in the last display equals

\[
\left( \frac{2}{27} \right)^{n-2} Z(2)^{-1} \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \prod_{j=1}^{m} \left( \frac{2}{27} \right)^{n_j} \# T_{n_j, p_j} = \left( \frac{2}{27} \right)^{n-2} Z(2)^{-1} \prod_{j=1}^{m} Z'(p_j),
\]

and so we have proved that

\[
\mathbb{P}(B_r(T_{\text{Bol}}) = t) = \left( \frac{2}{27} \right)^{n-2} Z(2)^{-1} \prod_{j=1}^{m} Z'(p_j). \tag{8}
\]

Next consider the UIPT \( T_{\infty} \). We can similarly compute \( \mathbb{P}(B_r(T_{\infty}) = t) \), using the fact that \( T_{\infty} \) is the local limit of triangulations with a large size. If, for every integer \( q \geq 3 \), \( T_{(q)} \) denotes a uniformly distributed plane triangulation with \( q \) vertices, we have

\[
\mathbb{P}(B_r(T_{\infty}) = t) = \lim_{q \to \infty} \mathbb{P}(B_r(T_{(q)}) = t).
\]

Recalling that the number of rooted plane triangulations with \( q \) vertices is \( \# T_{q-2, 2} \) the same counting argument as above gives for \( q \geq n \),

\[
\mathbb{P}(B_r(T_{(q)}) = t) = \left( \# T_{q-2, 2} \right)^{-1} \sum_{n_1 + \cdots + n_m = q - n} \left( \prod_{j=1}^{m} \# T_{n_j, p_j} \right),
\]

where the sum is again over nonnegative integers \( n_1, \ldots, n_m \) such that \( n_1 + \cdots + n_m = q - n \), with the same additional constraint that \( n_i > 0 \) if \( p_i = 2 \). From the asymptotics in (1), it is an easy matter to verify that, for any \( \varepsilon > 0 \), we can choose \( K \) sufficiently large so that the asymptotic contribution of terms corresponding to choices of \( n_1, \ldots, n_m \) where \( n_i \geq K \) for two distinct values of \( i \in \{1, \ldots, m\} \) is bounded above by \( \varepsilon \). Thanks to this observation, we get from the asymptotics (1) that

\[
\mathbb{P}(B_r(T_{\infty}) = t) = \left( \frac{2}{27} \right)^{n-2} C(2)^{-1} \sum_{j=1}^{m} C(p_j) \sum_{n_1, \ldots, n_j-1, n_j+1, \ldots, n_m} \left( \prod_{i=1, i \neq j}^{m} \left( \frac{2}{27} \right)^{n_i} \# T_{n_i, p_i} \right),
\]

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where the second sum is over all choices of \( n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_m \geq 0 \) such that \( n_i > 0 \) if \( p_i = 2 \). It follows that
\[
P(B_r(T_\infty) = t) = \left(\frac{2}{27}\right)^{n-2} C(2)^{-1} \sum_{j=1}^{m} C(p_j) \left( \prod_{i=1, i \neq j}^{m} Z'(p_i) \right).
\]
Comparing (9) with (8), we get
\[
P(B_r(T_\infty) = t) = \left( \frac{Z(2)}{C(2)} \sum_{j=1}^{m} C(p_j) \right) P(B_r(T_{\text{Bol}}) = t).
\]
Note that, for every integer \( p \geq 2 \),
\[
\frac{Z(2)}{C(2)} \frac{C(p)}{Z'(p)} = f(p),
\]
and so we have obtained \( P(B_r(T_\infty) = t) = g(t) P(B_r(T_{\text{Bol}}) = t) \), where \( g(t) := \sum_{j=1}^{m} f(p_j) \). Formula (7) now follows since \( M_r = g(B_r(T_{\text{Bol}})) \) by definition.

**Remark.** Formula (9) is obviously related to Proposition 4.10 in [6]. We did not use directly that result because it is apparently restricted to type III triangulations (the formula of Proposition 4.10 in [6] does not seem to take into account the possibility of holes of size 2).

### 3 Asymptotics for a general peeling process

#### 3.1 Peeling

The peeling process is an algorithmic procedure that “discovers” the UIPT step by step. We give a brief presentation of this algorithm and refer to [2, 3, 4, 7] for details.

Formally, the algorithm produces a nested sequence of rooted triangulations with a simple boundary \( T_0 \subset T_1 \subset \ldots \subset T_n \subset \ldots \subset T_\infty \), such that, for every \( i \geq 0 \), conditionally on \( T_i \), the remaining part \( T_\infty \setminus T_i \) has the same distribution as a UIPT of the \( |\partial T_i| \)-gon (see [3, Section 1.2.2] for the definition of the UIPT of the \( p \)-gon).

Assuming that we are given the UIPT \( T_\infty \), the sequence \( T_0, T_1, \ldots \) is constructed inductively as follows. First \( T_0 \) is the trivial triangulation. Then, for every \( n \geq 0 \), conditionally on \( T_n \) we pick an edge \( e_n \) on \( \partial T_n \), either deterministically (i.e. as a deterministic function of \( T_n \)) or via a randomized algorithm that may involve only random quantities independent of \( T_\infty \). The triangulation \( T_{n+1} \) is then obtained by adding to \( T_n \) the triangle incident to \( e_n \) which was not contained in \( T_n \) (this is called the revealed triangle) and the bounded region that may be enclosed in the union of \( T_n \) and the revealed triangle. We sometimes say that \( T_{n+1} \) is obtained from \( T_n \) by peeling the edge \( e_n \). Notice that, at the first step, there is only one (oriented) edge in the boundary of \( T_0 \), but we can choose to reveal the triangle on the right or on the left of this oriented edge.

The point is the fact that the distribution of the whole sequence \( T_0, T_1, \ldots \) can be described in a simple way and provides a construction of \( T_\infty \) (although this is not obvious, we shall see later that \( T_\infty \) is the limit of the finite triangulations \( T_n \)). Remarkably, the description of the law of \( T_0, T_1, \ldots \) is essentially the same independently of the (deterministic or randomized) algorithm that we use to choose the peeled edge at step \( n \).

In order to describe the conditional law of \( T_{n+1} \) given \( T_n \) and the peeled edge \( e_n \), we need to distinguish several cases. Suppose that at step \( n \geq 0 \) the triangulation \( T_n \) has a boundary of length \( p \). The revealed triangle at time \( n \) may be of several different types (see Fig. 3):

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1. **Type C:** The revealed triangle has a vertex in the “unknown region”. This occurs with probability
\[
P(C | |\partial T_n| = p) = q^{(p)}_{-1} = \frac{2}{27} \frac{C(p+1)}{C(p)}.
\] (10)

2. **Types \(L_k\) and \(R_k\):** The three vertices of the revealed triangle lie on the boundary of \(T_n\). This triangle thus “swallows” a piece of the boundary of \(\partial T_n\) of length \(k \in \{1, \ldots, p - 2\}\). These events are denoted by \(R_k\) or \(L_k\), depending on whether the edge of the revealed triangle that comes after the peeled edge in clockwise order is incident or not to the infinite part of the triangulation (see Fig. 3). These events have a probability equal to
\[
P(L_k | |\partial T_n| = p) = P(R_k | |\partial T_n| = p) := q^{(p)}_k = Z(k + 1) \frac{C(p - k)}{C(p)}.
\] (11)

![Figure 3: Illustration of cases C, L_k, and R_k.](image)

In cases \(R_k\) and \(L_k\), we also need to specify the distribution of the triangulation with a boundary of length \(k + 1\) that is enclosed in the union of \(T_k\) and the revealed triangle. If by convention we root this triangulation at the unique edge of its boundary incident to the revealed triangle, we specify its distribution by saying that it is a Boltzmann triangulation of the \((k+1)\)-gon. Note that when \(k = 1\), there is a positive probability that this Boltzmann triangulation is the trivial one, and this simply means that the enclosed region is empty, or equivalently that the revealed triangle has two edges on the boundary of \(T_n\).

The preceding considerations completely describe the distribution of the sequence \(T_0, T_1, \ldots\) – modulo of course the deterministic or randomized algorithm that is used at every step to select the peeled edge. The choices of types \(C, L_k,\) and \(R_k\), and of the Boltzmann triangulations that are used (whenever needed) to “fill in the holes” are made independently at every step with the probabilities given above.

At this point, we note that the geometry of the random triangulations \(T_n\) depends on the peeling algorithm used to choose the peeled edge at every step. On the other hand, it should be clear from the previous description that the law of the process \((|T_n|, |\partial T_n|)_{n \geq 0}\) does not depend on this algorithm. In the present section, we will be interested only in this process, and for this reason we do not need to specify the peeling algorithm. Later, in Section 3 and 4, we will consider particular choices of the peeling algorithm, which are useful to investigate various properties of the UIPT.
To simplify notation, we set, for every $n \geq 0$,

$$P_n = |\partial T_n| \quad \text{and} \quad V_n = |T_n|.$$ 

In the remaining part of this section, we will prove Theorem 1 describing the scaling limit of the process $(P_n, V_n)_{n \geq 0}$ (see [3] and [7, Theorem 5] in the quadrangular case for related statements). We will also establish a few consequences of Theorem 1, which are of independent interest.

### 3.2 The scaling limit of perimeters

The description of the previous section shows that both processes $(P_n)_{n \geq 0}$ and $(P_n, V_n)_{n \geq 0}$ are Markov chains. The Markov chain $(P_n)_{n \geq 0}$ starts from $P_0 = 2$ and takes values in $\{2, 3, \ldots\}$. Its transition probabilities are given by

$$\mathbb{E}[f(P_{n+1}) | P_n] = f(P_n + 1) \cdot q_{n-1}^{(P_n)} + 2 \sum_{k=1}^{p-2} f(P_n - k) \cdot q_k^{(P_n)}.$$ 

Using (2), we may set $q_{-1} = \lim_{p \to \infty} q_{-1}^{(p)} = \frac{2}{3}$ and similarly $q_k = \lim_{p \to \infty} q_k^{(p)} = Z(k+1)g^{-k}$ for every $k \geq 1$. From (5) and (6), it is an easy matter to verify that

$$q_{-1} + 2 \sum_{k \geq 1} q_k = 1 \quad \text{and} \quad q_{-1} - 2 \sum_{k \geq 1} k q_k = 0,$$

so that the probability measure $\nu$ on $\mathbb{Z}$ given by $\nu(1) = q_{-1}$ and $\nu(-k) = 2q_k$ for every $k \geq 1$ is centered (note that $\nu$ is supported on $\{-3, -2, -1, 1\}$). In fact, the weights $q_k$ describe the law of the one-step peeling in the half-plane version of the UIPT, see [2, 4].

We write $(W_n)_{n \geq 0}$ for a random walk with values in $\mathbb{Z}$, started from $W_0 = 2$ and with jump distribution $\nu$. Notice that the jumps of $W$ are bounded above by 1. Furthermore, using (4) we have for every $n \geq 0$,

$$\nu(-k) = 2q_k \sim 2t_{\Delta^2}k^{-5/2}.$$ 

It follows that $\nu$ is in the domain of attraction of a spectrally negative stable law of index $3/2$. This implies the convergence in distribution in the Skorokhod sense,

$$\left( \frac{W_n}{p_{\Delta^2} \cdot n^{2/3}} \right)_{t \geq 0} \overset{(d)}{\underset{n \to \infty}{\longrightarrow}} (S_t)_{t \geq 0},$$

where

$$p_{\Delta^2} = \left( \frac{8t_{\Delta^2} \sqrt{n}}{3} \right)^{2/3} = (2/3)^{2/3},$$

and $S$ is the stable Lévy process with index $3/2$ and no positive jumps, whose distribution is determined by the Laplace transform $\mathbb{E}[\exp(\lambda S_t)] = \exp(t\lambda^{2/3})$ for every $t, \lambda \geq 0$. Note that the Lévy measure of $S$ is

$$\frac{3}{4\sqrt{\pi}} |x|^{-5/2} 1_{\{x < 0\}} \, \mathrm{d}x.$$

Our first objective is to get a scaling limit analogous to (14) for $(P_n)_{n \geq 0}$. To this end, recall from [8, Section VII.3] that one can define a process $(S_t^+)_{t \geq 0}$ with càdlàg sample paths, which is distributed as $(S_t)_{t \geq 0}$ “conditioned to stay positive forever”. The scaling limit in the following result was suggested in [3] before Lemma 3.1. To simplify notation we write $[k, \infty[ = \{k, k+1, k+2, \ldots\}$ and $]-\infty, k[ = \{\ldots, k-2, k-1, k\}$ for every integer $k \in \mathbb{Z}$. 

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Proposition 5. (i) The Markov chain \((P_n)_{n \geq 0}\) is distributed as the random walk \((W_n)_{n \geq 0}\) conditioned not to hit \([-\infty, 1]\). Equivalently, \((P_n)_{n \geq 0}\) is distributed as the \(h\)-transform of the random walk \((W_n)_{n \geq 0}\) killed upon hitting \([-\infty, 1]\), where the function \(h\) defined on \(\mathbb{Z}\) by

\[
h(p) := \begin{cases} 
9^{-p} C(p) & \text{if } p \geq 2, \\
0 & \text{if } p \leq 1,
\end{cases} \tag{15}
\]

is, up to multiplication by a positive constant, the unique nontrivial nonnegative function that is \(\nu\)-harmonic on \([-2, \infty]\) and vanishes on \([-\infty, 1]\).

(ii) The following convergence in distribution holds in the Skorokhod sense,

\[
\left( \frac{P_{\lfloor n \rfloor}}{p_{\lfloor n \rfloor} \cdot n^{2/3}} \right)_{t \geq 0} \xrightarrow{n \to \infty} \left( S^+_{t} \right)_{t \geq 0}, \tag{16}
\]

where we recall that \(p_{\lfloor n \rfloor} = (2/3)^{2/3}\).

Proof. (i) Let \(h\) be defined by (15). From the explicit formulas (10) and (11), one immediately gets that, for every \(p \geq 2\) and every \(k \in \{-1, 1, 2, \ldots, p-2\},\)

\[
q_k^{(p)} = \frac{h(p-k)}{h(p)} q_k. \tag{17}
\]

It then follows from (12) and the definition of \(\nu\) that, for every \(p \geq 2\) and \(k \in \{-p+2, -p+3, \ldots, -1, 1\},\)

\[
\mathbb{P}(P_{n+1} = p + k \mid P_n = p) = \frac{h(p+k)}{h(p)} \nu(k) = \frac{h(p+k)}{h(p)} \mathbb{P}(W_{n+1} = p + k \mid W_n = p). \tag{18}
\]

By summing over \(k\), we get, for every \(p \geq 2,\)

\[
\sum_{k \in \mathbb{Z}} \frac{h(p+k)}{h(p)} \nu(k) = 1
\]

so that \(h\) is \(\nu\)-harmonic on \([2, \infty]\). Note that the uniqueness (up to a multiplicative constant) of a positive function that is \(\nu\)-harmonic on \([2, \infty]\) and vanishes on \([-\infty, 1]\) is easy, since, for every \(p \geq 2\), the value of this function at \(p+1\) is determined from its values for \(2 \leq i \leq p\). Furthermore, formula (18) precisely says that \((P_n)_{n \geq 0}\) is distributed as the \(h\)-transform of the random walk \((W_n)_{n \geq 0}\) killed upon hitting \([-\infty, 1]\). The fact that this \(h\)-transform can be interpreted as the random walk \(W\) conditioned to stay in \([2, \infty]\) is classical, see e.g. [9].

(ii) This follows from the invariance principle proved in [11]. \qed

From (2), we have

\[
h(p) \underset{p \to \infty}{\sim} \frac{1}{54\pi \sqrt{3}} \sqrt{p}. \tag{19}
\]

Still from (2), we can write, for \(p \geq 2,\)

\[
h(p) = \frac{1}{3^{p/2} 4 \sqrt{\pi}} \frac{(2p-3) \times (2p-5) \times \cdots \times 3 \times 1}{(2p-4) \times (2p-6) \times \cdots \times 4 \times 2},
\]

so that \(h(p+1)/h(p) = (2p-1)/(2p-2),\) proving that \(h\) is monotone increasing on \([2, \infty]\). Then, for every \(j \geq 1,\) and every \(p\) with \(p \geq j+2,\)

\[
q_j^{(p)} = \frac{h(p-j)}{h(p)} q_j \leq q_j \tag{20}
\]

and similarly, for every \(p \geq 2,\)

\[
q_{-1}^{(p)} = \frac{h(p+1)}{h(p)} q_{-1} \geq q_{-1}. \tag{21}
\]

These bounds will be useful later.
3.3 A few applications

Let us give a few applications of Proposition 5. First, it is easy to recover from this proposition the known fact (see [3, Claim 3.3]) that the Markov chain \((P_n)_{n \geq 0}\) is transient,

\[
P_n \xrightarrow{a.s. \ n \to \infty} +\infty. \tag{22}
\]

To see this, let \(p \geq 2\) and write \(\mathbb{P}_p\) for a probability measure under which the random walk \(W\) starts from \(p\). For every \(y \in \mathbb{Z}\), set \(T_y = \min\{n \geq 0 : W_n = y\}\). Note that \(T_y < \infty\) a.s. because the random walk \(W\) is recurrent. Similarly, suppose that \(\tilde{T}_y\) is distributed under \(\mathbb{P}_p\) as the hitting time of \(y\) for a Markov chain with the same transition kernel as \((P_n)_{n \geq 0}\) but started from \(p\). Then, standard properties of h-transforms give for every \(p, y \in [2, \infty]\),

\[
\mathbb{P}_p(\tilde{T}_y < \infty) = \frac{h(y)}{h(p)} \mathbb{P}_p(W_k \geq 2, \forall k \leq T_y).
\]

Since \(h\) is monotone increasing on \([2, \infty]\), the right-hand side is strictly smaller than 1 when \(p > y\), giving the desired transience.

The following corollary was conjectured in [7, Section 5.1].

**Corollary 6.** Any peeling \((T_n)_{n \geq 0}\) of the UIPT will eventually discover \(T_\infty\) entirely, that is

\[
\bigcup_{n \geq 0} T_n = T_\infty, \quad a.s.
\]

**Proof.** It is enough to prove that, if \(n_0 \geq 1\) is fixed, then a.s. every vertex of \(\partial T_{n_0}\) belongs to the interior of \(T_{n_1}\) for some \(n_1 > n_0\) sufficiently large. Indeed, if this property holds, an inductive argument shows that the minimal distance between a vertex outside \(T_n\) and the root tends to infinity as \(n \to \infty\), which gives the desired result.

So let us fix \(n_0\) and a vertex \(v\) of \(\partial T_{n_0}\), and argue conditionally on \(T_{n_0}\) and \(v\). We note that, for every \(n \geq n_0\), conditionally on the event that \(v\) is still on the boundary of \(T_n\), the probability that \(v\) will be “surrounded” by the revealed triangle at step \(n + 1\), and therefore will belong to the interior of \(T_{n+1}\), is at least

\[
\sum_{k=\lfloor P_n/2 \rfloor+1}^{P_n-2} q_k^{(P_n)}
\]

with the convention that the sum is 0 if \(\lfloor P_n/2 \rfloor + 1 > P_n - 2\). If \(P_n\) is large enough, the latter quantity is bounded below by

\[
\sum_{k=\lfloor P_n/2 \rfloor+1}^{\lfloor 3P_n/4 \rfloor} q_k^{(P_n)} \geq \sum_{k=\lfloor P_n/2 \rfloor+1}^{\lfloor 3P_n/4 \rfloor} \frac{h(P_n-k)}{h(P_n)} q_k \geq c P_n^{-3/2},
\]

where \(c\) is a positive constant and we used (4) and (19) in the last inequality. Recalling that \(P_n \to \infty\) a.s., we see that the proof will be complete if we can verify that the series

\[
\sum_{n=1}^{\infty} P_n^{-3/2}
\]

diverges a.s.

To this end, we argue by contradiction and assume that we can find two constants \(M < \infty\) and \(\varepsilon > 0\) such that the probability of the event

\[
\left\{ \sum_{n=1}^{\infty} P_n^{-3/2} \leq M \right\}
\]

is
is greater than \( \varepsilon \). On this event, for any \( t > 1 \) and any \( n \geq 1 \), we have
\[
\int_1^t \frac{P_{\lfloor nt \rfloor}}{n^{2/3}} \, du \leq \frac{1}{n} \sum_{i=n}^{\lfloor nt \rfloor} \left( \frac{P_i}{n^{2/3}} \right)^{-3/2} = \sum_{i=n}^{\lfloor nt \rfloor} P_i^{-3/2} \leq M.
\]
Using the convergence of Proposition 5 (ii), we obtain that, for every \( t > 1 \), the probability of the event \( \{ \int_1^t du (S_n^+)^{-3/2} \leq (\varepsilon M)^{-3/2} \} \) is greater than \( \varepsilon \). Letting \( t \to \infty \) we get that
\[
P\left( \int_1^\infty \frac{du}{(S_n^+)^{3/2}} \leq (\varepsilon M)^{-3/2} \right) \geq \varepsilon.
\]
This is a contradiction because
\[
\int_1^\infty \frac{du}{(S_n^+)^{3/2}} = \infty \quad \text{a.s.}
\]
as can be seen by an application of Jeulin’s lemma [17], noting that we have \( (S_n^+)^{-3/2} \leq (\varepsilon M)^{-3/2} \) by scaling.

The next lemma will be an important tool in the proof of Theorems 2 and 3.

**Lemma 7.** There exist two constants \( 0 < c_1 < c_2 < \infty \) such that, for all \( n \geq 1 \), we have
\[
c_1 n^{-2/3} \leq \mathbb{E} \left[ \frac{1}{P_n} \right] \leq c_2 n^{-2/3}.
\]

**Proof.** The lower bound is easy since Proposition 5 (ii) gives
\[
\mathbb{E} \left[ \frac{n^{2/3}}{P_n} \right] \geq \mathbb{E} \left[ \frac{n^{2/3}}{P_n} \wedge 1 \right] \xrightarrow{\text{n-\to\infty}} \mathbb{E} \left[ \frac{1}{P_{\Delta^2} S_n^+} \wedge 1 \right] > 0.
\]
To prove the upper bound, we first fix \( k \geq 2 \) and \( n \geq 1 \), and we evaluate \( P(P_n = k) \). By Proposition 5 (i) and properties of \( h \)-transforms, we have
\[
P(P_n = k) = \frac{h(k)}{h(2)} \cdot P(\{W_i \geq 2, \forall i \leq n\} \cap \{W_n = k\}).
\]
We set \( \tilde{W}_i = W_n - W_{n-1} \) for \( 0 \leq i \leq n \) and note that we can also define \( \tilde{W}_i \) for \( i > n \) in such a way that \( (\tilde{W}_i)_{i \geq 0} \) is a random walk with the same jump distribution as \( W \) and \( \tilde{W}_0 = 0 \). We have then
\[
P(\{W_i \geq 2, \forall 0 \leq i \leq n\} \cap \{W_n = k\}) = P(\{\tilde{W}_i = k - 2, \forall 0 \leq i \leq n\} \cap \{\tilde{W}_n = k\}) = \frac{P(\tilde{T}_{k-1} = n + 1)}{q_{k-1}},
\]
where we have set \( \tilde{T}_{k-1} = \min\{i \geq 0 : \tilde{W}_i = k - 1\} \). Note that \( \tilde{W} \) has positive jumps only of size 1. We can thus use Kemperman’s formula (see e.g. [24, p.122]) to get
\[
P(\tilde{T}_{k-1} = n + 1) = \frac{k - 1}{n + 1} P(\tilde{W}_{n+1} = k - 1).
\]
From the last three displays, we have
\[
P(P_n = k) = \frac{3 h(k)}{2 h(2)} \frac{k - 1}{n + 1} P(\tilde{W}_{n+1} = k - 1).
\]
Using the local limit theorem for random walk in the domain of attraction of a stable distribution (see e.g. [16, Theorem 4.2.1]), we can find a constant \( c'' \) such that
\[
P(\tilde{W}_n = k) \leq c'' n^{-2/3}, \quad (23)
\]
for every $n \geq 1$ and $k \in \mathbb{Z}$. Then, for every $n \geq 1$,

\[
\mathbb{E} \left[ \frac{1}{P_n} \right] = \mathbb{E} \left[ \frac{1}{P_n} \mathbf{1}_{\{P_n > n^{2/3}\}} \right] + \mathbb{E} \left[ \frac{1}{P_n} \mathbf{1}_{\{P_n \leq n^{2/3}\}} \right] \\
\leq n^{-2/3} + \sum_{k=1}^{\lfloor n^{2/3} \rfloor} \frac{3}{2} \frac{h(k)}{h(2)} \frac{k - 1}{n + 1} \mathbb{P}(W_n + 1 = k - 1) \\
\leq n^{-2/3} + \frac{3\epsilon''}{2h(2)} n^{-5/3} \sum_{k=1}^{\lfloor n^{2/3} \rfloor} h(k).
\]

The upper bound of the lemma follows using (19).

\[ \square \]

### 3.4 The scaling limit of volumes

Our goal is now to study the scaling limit of the process $(V_n)_{n \geq 0}$. We start with a result similar to [3, Proposition 6.4] about the distribution of the size of a Boltzmann triangulation with a large perimeter. For every $p \geq 2$, we let $T^{(p)}$ denote a random triangulation of the $p$-gon with Boltzmann distribution.

**Proposition 8.** Set $b_{\Delta z} = \frac{2}{3}$.

1. We have $\mathbb{E}[|T^{(p)}|] \sim b_{\Delta z} \cdot p^2$ as $p \to \infty$.

2. The following convergence in distribution holds:

\[ p^{-2} |T^{(p)}| \xrightarrow{d} b_{\Delta z} \cdot \xi, \]

where $\xi$ is a random variable with density $e^{-1/2x} x^{-5/2} / 2\pi$ on $\mathbb{R}_+$.

**Remark.** We have $\mathbb{E}[\xi] = 1$ and the size-biased version of the distribution of $\xi$ (with density $e^{-1/2x} x^{-5/2} / 2\pi$ on $\mathbb{R}_+$) is the $1/2$-stable distribution with Laplace transform $e^{-\sqrt{2x}}$. Consequently, for $\lambda > 0$, we have

\[ \mathbb{E}[e^{-\lambda \xi}] = (1 + \sqrt{2\lambda}) e^{-\sqrt{2\lambda}}. \]

**Proof.** The first assertion follows from the formula $\mathbb{E}[|T^{(p)}|] = \frac{1}{2} (p - 1)(2p - 3)$ for $p \geq 2$ which is easily derived from the exact formula for the generating function of the sequence $(\#T^{(p)}_n)_{n \geq 0}$ found in [6, Proposition 2.4]. See also [25, Proposition 6.4].

For the second assertion, we proceed as in [3, Proposition 6.4]. From the explicit expressions (1) and (3), an asymptotic expansion using Stirling’s formula shows that, for every fixed $x > 0$, we have

\[ p^2 \mathbb{P}(|T^{(p)}| = [p^2 x]) = p^2 \left( \frac{2/27}{[p^2 x]} \#T^{(p^2 x)}_n \right) \frac{Z(p)}{Z(p^2 x)} \xrightarrow{p \to \infty} 2 e^{-1/(3x)} \frac{3 x^{5/2} \sqrt{3\pi}}{3x^{5/2} \sqrt{3\pi}}, \]

and the convergence holds uniformly when $x$ varies over a compact subset of $\mathbb{R}_+$. Since the right-hand side of the last display is the density of the variable $2\xi/3$, the desired result follows.

\[ \square \]

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We will verify that

\[ \left( \frac{P_{[nt]}}{b_{\Delta z} \cdot n^{2/3}}, \frac{V_{[nt]}}{\sqrt{2} \cdot n^{1/3}} \right)_{0 \leq t \leq 1} \xrightarrow{d} \left( S_t^V, Z_t \right)_{0 \leq t \leq 1}. \]
The statement of Theorem 1 follows, noting that there is no loss of generality in restricting the time interval to $[0,1]$.

The convergence of the first component in (24) is given by Proposition 5. We will thus study the conditional distribution of the second component given the first one, and Proposition 8 will be our main tool. We first note that, for every $n \geq 1$, we can write

$$V_n = |T_n| = V^*_n + \tilde{V}_n,$$

where $V^*_n$ denotes the number of inner vertices of $T_n$ that belong to $\partial T_i$ for some $i \leq n - 1$, and $\tilde{V}_n$ is thus the total number of inner vertices in the Boltzmann triangulations that were used to fill in the holes in the case of occurrence of events $L_k$ or $R_k$ at some step $i \leq n$ of the peeling process. Since $\#(\partial T_i \setminus \partial T_{i-1}) \leq 1$ for $1 \leq i \leq n$, it is clear that $V^*_n \leq n + 2$ for every $n \geq 0$. It follows that (24) is equivalent to the same statement where $V_{[nt]}$ is replaced by $\tilde{V}_{[nt]}$.

Next we can write, for every $k \in \{1, \ldots, n\}$,

$$\tilde{V}_k = \sum_{i=1}^{k} 1_{\{P_i < P_{i-1}\}} U_i,$$

(25)

where, conditionally on $(P_0, P_1, \ldots, P_n)$, the random variables $U_i$ (for $i$ such that $P_i < P_{i-1}$) are independent, and $U_i$ is distributed as $|T^{(P_i - P_{i-1})}|$, with the notation of Proposition 8.

Fix $\varepsilon > 0$ and set, for every $k \in \{1, \ldots, n\}$,

$$\tilde{V}_k^{\leq \varepsilon} = \sum_{i=1}^{k} 1_{\{0 < P_{i-1} - P_i \leq \varepsilon n^{2/3}\}} U_i, \quad \tilde{V}_k^{> \varepsilon} = \sum_{i=1}^{k} 1_{\{P_{i-1} - P_i \leq \varepsilon n^{2/3}\}} U_i.$$

(26)

We first observe that $n^{-4/3} \tilde{V}_n^{\leq \varepsilon}$ is small uniformly in $n$ when $\varepsilon$ is small. Indeed, it follows from Proposition 8 that there is a constant $C'$ such that $\mathbb{E}[|T^{(p)}|] \leq C' p^2$ for every $p \geq 2$, which gives

$$\mathbb{E}[\tilde{V}_n^{\leq \varepsilon}] \leq C \sum_{i=1}^{n} \mathbb{E}[(P_{i-1} - P_i + 1)2^i 1_{\{0 < P_{i-1} - P_i \leq \varepsilon n^{2/3}\}}].$$

On the other hand, from the bound (20) and (4), it is straightforward to verify that, for every $i \geq 1$ and every $p \geq 2$,

$$\mathbb{E}[(P_{i-1} - P_i + 1)2^i 1_{\{0 < P_{i-1} - P_i \leq \varepsilon n^{2/3}\}} | P_{i-1} = p] \leq C' \sum_{j=1}^{\lfloor \varepsilon n^{2/3}\rfloor} (j+1)^2 j^{-5/2} \leq C'' \sqrt{\varepsilon} n^{1/3},$$

with some constants $C'$ and $C''$ independent of $n$ and $\varepsilon$. By combining the last two displays, we obtain, for every $n \geq 1$,

$$n^{-4/3} \mathbb{E}[\tilde{V}_n^{\leq \varepsilon}] \leq C C'' \sqrt{\varepsilon}.$$

(27)

Let us turn to $\tilde{V}_n^{> \varepsilon}$. We write $s_1, s_2, \ldots$ for the jump times of $S^+$ before time 1 listed in decreasing order of their absolute values. For every $n \geq 1$, let $\ell_1^{(n)}(n), \ldots, \ell_k(n)$ be all integers $i \in \{1, \ldots, n\}$ such that $P_{i-1} - P_i > 0$, listed in decreasing order of the quantities $P_{i-1} - P_i$ (and in the usual order of $N$ for indices such that $P_{i-1} - P_i$ is equal to a given value). For definiteness, we also set $\ell_1^{(n)} = 1$ if $i > k_n$. It follows from (16) that, for every integer $K \geq 1$,

$$\left(n^{-1/2} \ell_1^{(n)}, \ldots, n^{-1/2} \ell_K^{(n)}, n^{-2/3} (P_{\ell_1^{(n)}} - P_{\ell_1^{(n)}-1}), \ldots, n^{-2/3} (P_{\ell_K^{(n)}} - P_{\ell_K^{(n)}-1}) \right) \xrightarrow{(d)} (s_1, \ldots, s_K, p_{\Delta s} \Delta S^+_{s_1}, \ldots, p_{\Delta s} \Delta S^+_{s_K}),$$

(28)
and this convergence in distribution holds jointly with (16). Furthermore, using the conditional distribution of the variables \( U_i \) given \((P_0, \ldots, P_n)\) and Proposition 8, we also get, for every integer \( K \geq 1 \),

\[
\left( \frac{U_{(n)}^{i_1}}{(P_{i_1}^{(n)} - P_{i_1}^{(n-1)})^2}, \ldots, \frac{U_{(n)}^{i_K}}{(P_{i_K}^{(n)} - P_{i_K}^{(n-1)})^2} \right) \frac{(d)}{n \to \infty} \left( b_{\Delta z} \xi_1, \ldots, b_{\Delta z} \xi_K \right),
\]

where \( \xi_1, \xi_2, \ldots \) are independent copies of the variable \( \xi \) of Proposition 8. This convergence holds jointly with (16) and (28), provided that we assume that the sequence \( \xi \) completes the proof.

The convergence (24), with \( \ell \) following reason. One easily proves by induction that, for every \( n \geq 1 \), one and only one of the two

4.1 Peeling by layers

In this section, we focus on a particular peeling algorithm, which we call the peeling by layers. As previously, we start from the trivial triangulation that consists only of the root edge. At the first step, we discover the triangle on the left side of the root edge to get \( T_1 \). To get \( T_2 \), we then discover the triangle on the right side of the root edge. Then we continue by induction in the following way. We note that the triangle revealed at step \( n \) has either one or two edges in the boundary of \( T_n \). If it has one edge in the boundary, we discover at step \( n+1 \) the triangle incident to this edge which is not already in \( T_n \). If it has two edges in the boundary, we do the same for the right-most among these two edges (this makes sense because in that case the boundary of \( T_n \) must contain at least 3 edges). See Fig. 5 for an example.

This algorithm is particularly well suited to the study of distances from the root vertex, for the following reason. One easily proves by induction that, for every \( n \geq 1 \), one and only one of the two

The convergence (24), with \( V \) replaced by \( \tilde{V} \), follows from (30) and the preceding considerations. This completes the proof. \( \square \)

4 Distances in the peeling process

4.1 Peeling by layers
following possibilities occurs. Either all vertices of $\partial T_n$ are at the same distance $h$ from the root vertex. Or there is an integer $h \geq 0$ such that $\partial T_n$ contains both vertices at distance $h$ and at distance $h+1$ from the root vertex. In the latter case, vertices at distance $h$ form a connected subset of $\partial T_n$, and the edge that will be “peeled off” at step $n+1$ is the only edge of the boundary whose left end is at distance $h+1$ and whose right end is at distance $h$. In both cases we write $H_n = h$, so that the boundary $\partial T_n$ does contain vertices at distance $H_n$ and may also contain vertices at distance $H_n + 1$. We also set $H_0 = 0$ by convention.

Figure 4: The peeling by layers algorithm in a random triangulation drawn in the plane via Tutte’s barycentric embedding. The successive layers are represented with different colors. Courtesy of Timothy Budd.

Since the peeling algorithm discovers the whole triangulation $T_\infty$ (Corollary 6), it is clear that $H_n$ tends to $\infty$ as $n \to \infty$. Also obviously $0 \leq H_{n+1} - H_n \leq 1$ for every $n \geq 1$, hence we may set $\sigma_r := \min\{n \geq 0 : H_n = r\}$ for every integer $r \geq 1$. A simple argument shows that for $n = \sigma_r$, all vertices of $\partial T_n$ are at distance $r$ from the root vertex (this however does not characterize $\sigma_r$ since there may exist other times $n > \sigma_r$ with the same property). Furthermore, any vertex lying outside $T_{\sigma_r}$ must be at distance at least $r+1$ from the root vertex, and any triangle of $T_{\sigma_r}$ that is incident to an edge of the boundary contains a vertex at distance $r-1$ from the root vertex (indeed this triangle has been discovered by the peeling algorithm at a time where the boundary still contained vertices at distance $r-1$, and the corresponding peeled edge had to connect a vertex at distance $r$ to a vertex at distance $r-1$). It follows from the previous considerations that we have $T_{\sigma_r} = B^*_r(T_\infty)$ for every $r \geq 1$. Furthermore, for every
\[ n \geq 1 \text{ such that } H_n > 0, \text{ we have } \sigma_{H_n} \leq n < \sigma_{H_{n+1}} \text{ and therefore } \]
\[ B_{H_n}^\bullet(T_\infty) \subset T_n \subset B_{H_{n+1}}^\bullet(T_\infty). \tag{31} \]

This also holds for \( n \) such that \( H_n = 0 \), provided we define \( B_0^\bullet(T_\infty) \) as the trivial triangulation consisting only of the root edge.

An important consequence is the following fact, which needs not be true for a general peeling algorithm.

If \( F_n \) stands for the \( \sigma \)-field generated by \( T_0, T_1, \ldots, T_n \), then the graph distances of vertices of \( T_n \) from the root vertex are measurable with respect to \( F_n \). This is clear since (31) shows that a geodesic from any vertex of \( T_n \) to the root visits only vertices of \( T_n \).

At an intuitive level, the peeling algorithm “turns” around the boundary of the hull of balls of the UIPT in clockwise order and discovers \( T_\infty \) layer after layer. When turning around \( \partial B_r^\bullet(T_\infty) \), the peeling process creates new vertices at distance \( r+1 \) from the root vertex in a way similar to a front propagation. See Fig. 5.

\[ r+1 \]
\[ r \]

**Figure 5:** Illustration of the peeling by layers. When \( B_r^\bullet(T_\infty) \) has been discovered, we turn around the boundary \( \partial B_r^\bullet(T_\infty) \) from left to right in order to reveal the next layer and obtain \( B_{r+1}^\bullet(T_\infty) \).

To simplify notation, we write \( B_r^\bullet \) and \( \partial B_r^\bullet \) instead of \( B_r^\bullet(T_\infty) \) and \( \partial B_r^\bullet(T_\infty) \) in this section. As (31) suggests, the proof of Theorem 2 will rely on the convergence in distribution of a rescaled version of the process \( H_n \). Let us sketch some ideas of the proof of the latter convergence. Between times \( \sigma_r \) and \( \sigma_{r+1} \), the peeling process needs to turn around \( \partial B_r^\bullet \), which roughly takes a time linear in \( |\partial B_r^\bullet| \) (see Proposition 10 below for a precise statement). We thus expect that for some positive constant \( a \),

\[ \sigma_{r+1} - \sigma_r \approx \frac{1}{a} |\partial B_r^\bullet| = \frac{1}{a} P_{\sigma_r}. \tag{32} \]

and therefore

\[ \sigma_r \approx \frac{1}{a} \sum_{i=1}^{r-1} P_{\sigma_i}. \]

A formal inversion now gives for \( k \) large,

\[ H_k = \sup \{ r \geq 0 : \sigma_r \leq k \} \approx a \sum_{i=1}^{k} \frac{1}{P_i}, \]

and the limit behavior of the right-hand side can be derived from the fact that \( (n^{-2/3} P_{[nt]})_{t \geq 0} \) converges in distribution to \( (\Delta^2 S^*_t)^{1/3})_{t \geq 0} \) (Proposition 5).

The following proposition shows that the previous heuristic considerations are indeed correct.

**Proposition 9** (Distances in the peeling by layers). We have the following convergence in distribution for the Skorokhod topology

\[ \left( \frac{P_{[nt]}}{\Delta^2 \cdot n^{2/3}}, \frac{V_{[nt]}}{\Delta^2 \cdot n^{1/3}}, \frac{H_{[nt]}}{\Delta^2 \cdot n^{1/3}} \right)_{t \geq 0} \xrightarrow{(d)} \left( S^*_t, Z_t, \int_0^t \frac{du}{S^*_u} \right)_{t \geq 0}, \]

where \( \Delta^2 = 12^{-1/3} \).
Noting that $|B_r^*| = V_r$, and $|\partial B_r^*| = P_r$, we will derive Theorem 2 from the last proposition via a time change argument in Section 4.4. This derivation involves time-changing the limiting processes $S^+_t$ and $Z_t$ by the inverse of the increasing process $\int_0^t \frac{du}{u}$, which is clearly related to the Lamperti transformation connecting continuous-state branching processes to spectrally positive Lévy processes. In the next section, we state and prove Proposition 10, which is the key ingredient of the proof of Proposition 9. The latter proof will be given in Section 4.3.

4.2 Turning around layers

We write $L$ for the set of all vertices of $T_\infty$ that belong to $\partial B_r^*$ for some integer $r \geq 1$. Note that not all these vertices belong to $\partial T_n$ for some $n \geq 1$, but there may also be vertices of $\partial T_n$ that do not belong to $L$ (a vertex at distance $r$ from the root may appear on $\partial T_n$ for some $n < r$, and then be “swallowed” by the peeling algorithm before time $\sigma_r$). For every $n \geq 0$, we write $A_n$ for the number of inner vertices of $T_n$ that belong to $L$.

Clearly, $(A_n)_{n \geq 0}$ is an increasing process. Also, recalling our notation $F_n$ for the $\sigma$-field generated by $T_0, T_1, \ldots, T_n$, the random variable $A_n$ is measurable with respect to $F_n$. The point is that, on one hand, the hulls $B^*_1, \ldots, B^*_n$ are measurable functions of $T_n$, and on the other hand vertices of $T_n \setminus \partial T_n$ are at distance at most $H_n$ from the root (here it is important that we considered only inner vertices of $T_n$ in the definition of $A_n$, since the $\sigma$-field $F_n$ does not give enough information to decide whether a vertex of $\partial T_n$ which is at distance $H_n + 1$ from the root belongs to $L$ or not).

**Proposition 10.** We have

$$\frac{A_n}{n} \xrightarrow{p} \frac{1}{3} =: a_{\Delta^2}.$$

**Proof.** We use the notation $\Delta A_n = A_{n+1} - A_n$ for every $n \geq 0$. We note that inner vertices of the Boltzmann triangulations that are used to fill in the holes created by the peeling algorithm cannot be in $L$, and it follows that we have $0 \leq \Delta A_n \leq (\Delta P_n)_-$ for every $n \geq 0$ (plainly, $\Delta A_n = 0$ if $\Delta P_n = 1$). In particular $\mathbb{E}[\Delta A_n] < \infty$ and $\mathbb{E}[A_n] < \infty$. We then set, for every $i \geq 0$,

$$\eta_i = \mathbb{E}[\Delta A_i \mid F_i],$$

so that $M_n := A_n - \sum_{i=0}^{n-1} \eta_i$ is a martingale with respect to the filtration $(F_n)$.

We first prove that $M_n/n \to 0$ in probability. To this end, we use bounds on the second moment of $\Delta M_n$. Recall our bound $\Delta A_n \leq (\Delta P_n)_-$, and note that, for every $k \geq 1$ and every $p \geq 2$, (13) and (20) give

$$\mathbb{P}(\Delta P_n = -k \mid P_n = p) = \frac{h(p-k)}{h(p)} \mathbb{P}(\Delta W_n = -k) \leq C k^{-5/2},$$

for some constant $C > 0$ independent of $p$ and $k$. It follows that

$$\mathbb{E}[(\Delta A_n)^2 \mid P_n = p] \leq C \sum_{k=1}^{p-2} k^{-1/2} = O(\sqrt{p}).$$

Since $P_n \leq n + 2$, we deduce from the last display that

$$\mathbb{E}[(\Delta M_n)^2] = \mathbb{E}[(\Delta A_n - \eta_n)^2] \leq 2 \left( \mathbb{E}[(\Delta A_n)^2] + \mathbb{E}[\Delta A_n \mid F_n]^2 \right) \leq 4 \mathbb{E}[(\Delta A_n)^2] = O(\sqrt{n}).$$

Since the martingale $M$ has orthogonal increments, we get $\mathbb{E}[M_n^2] = O(n^{3/2})$ and it follows that $M_n/n \to 0$ in $L^2$.
Lemma 11. For every integer \( L \geq 1 \), we have

\[
\frac{1}{n} \sum_{i=0}^{n-1} \eta_i \xrightarrow{P} \frac{1}{3} \quad \text{as} \ n \to \infty.
\]

(33)

The idea of the proof is as follows. For most times \( n \), the boundary \( \partial \mathcal{T}_n \) has both a “large” number of vertices at distance \( H_n \) and a “large” number of vertices at distance \( H_n + 1 \) from the root. Then, except on a set of small probability, the only events leading to a nonzero value of \( \Delta A_n \) are events of type \( R_k \) for which \( \Delta A_n = -\Delta P_n = k \). The conditional expectation of \( \Delta A_n \) is thus computed using the probabilities of the events \( R_k \).

To make the preceding argument rigorous, we introduce some notation. For every integer \( n \geq 0 \), write \( U_n \) for the number of vertices in \( \partial \mathcal{T}_n \) that are at distance \( H_n \) from the root vertex. Note that the function \( n \mapsto U_n \) is nonincreasing on every interval \([\sigma_r, \sigma_{r+1}]\) where \( H_n \) is equal to \( r \). We also set \( G_n = P_n - U_n \), which represents the number of vertices in \( \partial \mathcal{T}_n \) that are at distance \( H_n + 1 \) from the root vertex.

Lemma 11. For every integer \( L \geq 1 \), we have

\[
\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{\{U_i \leq L \text{ or } G_i \leq L\}} \xrightarrow{P} \frac{1}{3} \quad \text{as} \ n \to \infty.
\]

(33)

Let us postpone the proof of this lemma. To complete the proof of (33), we first use the bound (20) to deduce from the bound \( \Delta A_n \leq |\Delta P_n| \) that, for every \( n \geq 0 \),

\[
\eta_n = \mathbb{E}[\Delta A_n \mid F_n] \leq \mathbb{E}[|\Delta P_n| \mid F_n] \leq C_1,
\]

(34)

for some finite constant \( C_1 \). Furthermore, using (20) again, we have also, for every integer \( L \geq 1 \),

\[
\mathbb{E}[\Delta A_n \mathbf{1}_{\{|\Delta P_n| \geq L\}} \mid F_n] \leq \mathbb{E}[|\Delta P_n| \mathbf{1}_{\{|\Delta P_n| \geq L\}} \mid F_n] \leq c(L)
\]

(35)

where the constants \( c(L) \) are such that \( c(L) \to 0 \) as \( L \to \infty \). Then, on the event \( \{U_n \geq L, G_n \geq L\} \), the condition \( |\Delta P_n| < L \) ensures that the only transitions of the peeling algorithm at step \( n + 1 \) leading to a positive value of \( \Delta A_n \) are of type \( R_k \) for some \( k \), and in that case \( \Delta A_n = -\Delta P_n = k \). It follows that, still on the event \( \{U_n \geq L, G_n \geq L\} \),

\[
\mathbb{E}[\Delta A_n \mathbf{1}_{\{|\Delta P_n| < L\}} \mid F_n] = \sum_{k=1}^{L-1} k q_k^{(p)} \leq \sum_{k=1}^{\infty} k q_k = \frac{1}{3}.
\]

Note that we have \( P_n \geq 2L \) on the event \( \{U_n \geq L, G_n \geq L\} \). Since \( q_k^{(p)} \) converges to \( q_k \) as \( p \to \infty \), the preceding considerations and (35) entail that, for every \( \varepsilon > 0 \), we can fix \( L_0 > 0 \) so that, for every \( L \geq L_0 \) and every \( n \), we have, on the event \( \{U_n \geq L, G_n \geq L\} \),

\[
\frac{1}{3} - \varepsilon \leq \mathbb{E}[\Delta A_n \mid F_n] \leq \frac{1}{3} + \varepsilon.
\]

(36)

Finally, we have, using (34),

\[
\left| \frac{1}{n} \sum_{i=0}^{n-1} \eta_i - \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{\{U_i \leq L, G_i \leq L\}} \mathbb{E}[\Delta A_i \mid F_i] \right| \leq \frac{C_1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{\{U_i \leq L \text{ or } G_i \leq L\}},
\]

and we can now combine (36) and Lemma 11 to get our claim (33). This completes the proof of Proposition 10, but we still have to prove Lemma 11.

\( \square \)
Proof of Lemma 11. We start with some preliminary observations. From the definition of the peeling by layers, one easily checks that the triple \((P_n, G_n, H_n)_{n \geq 0}\) is a Markov chain with respect to the filtration \((\mathcal{F}_n)\), taking values in \{(p, \ell, h) \in \mathbb{Z}^3 : p \geq 2, 0 \leq \ell \leq p - 1, h \geq 0\}, and whose transition kernel \(Q\) is specified as follows:

\[
Q((p, \ell, h), (p + 1, \ell + 1, h)) = q^{(p)}_{-1}
Q((p, \ell, h), (p - k, \ell - k, h)) = q^{(p)}_{k}
\text{for } 1 \leq k \leq \ell - 1
Q((p, \ell, h), (p - k, \ell, h)) = q^{(p)}_{k}
\text{for } 1 \leq k \leq p - \ell - 1
Q((p, \ell, h), (p - k, 0, h + 1)) = q^{(p)}_{k}
\text{for } \ell \leq k \leq p - 2
Q((p, \ell, h), (p - k, 0, h)) = q^{(p)}_{k}
\text{for } p - \ell \leq k \leq p - 2.
\]

The Markov chain \((P_n, G_n, H_n)_{n \geq 0}\) starts from the initial value \((2, 1, 0)\).

Obviously, the triple \((P_n, U_n, H_n)_{n \geq 0}\) is also a Markov chain, now with values in \{(p, \ell, h) \in \mathbb{Z}^3 : p \geq 2, 1 \leq \ell \leq p, h \geq 0\}, and its transition kernel \(Q'\) is expressed by the formula analogous to (37), where only the first and the last two lines are different and replaced by

\[
Q'((p, \ell, h), (p + 1, \ell, h)) = q^{(p)}_{-1}
Q'((p, \ell, h), (p - k, \ell, h)) = q^{(p)}_{k}
\text{for } \ell \leq k \leq p - 2
Q'((p, \ell, h), (p - k, h + 1)) = q^{(p)}_{k}
\text{for } p - \ell \leq k \leq p - 2.
\]

We now fix \(k \in \{0, 1, \ldots, L\}\). We will prove that

\[
\frac{1}{n} \sum_{i=0}^{n} \mathbb{P}(G_i = k) \longrightarrow 0.
\]

Let us explain why the lemma follows from (39). If \(k' \in \{1, \ldots, L\}\), a simple argument using the Markov chain \((P_n, U_n, H_n)\) shows that, for every \(i \geq 1\),

\[
\mathbb{P}(G_{i+k+1} = k | \mathcal{F}_i) \geq \mathbb{P}(U_i = k') \mathbb{P}(p_i \geq k' + 2),
\]

where \(\gamma > 0\) is chosen so that \(q^{(p)}_{-1} \geq \gamma\) for every \(p \geq 2\). It follows that there exists a constant \(\beta > 0\) (depending on \(k\) and \(k'\)) such that

\[
\mathbb{P}(G_{i+k+1} = k) \geq \beta \mathbb{P}(U_i = k', P_i \geq k' + 2).
\]

If we assume that (39) holds, the latter bound (together with the transience of the Markov chain \((P_n)\)) implies that

\[
\frac{1}{n} \sum_{i=0}^{n} \mathbb{P}(U_i = k') \longrightarrow 0.
\]

Clearly the lemma follows from (39) and (40).

Let us prove (39). Let \(N \geq 1\), and write \(T^N_1, T^N_2, \ldots\) for the successive passage times of the Markov chain \((P_n, G_n, H_n)\) in the set \{(p, \ell, h) : p \geq N, \ell = k\}. We claim that there exist two positive constants \(c\) and \(\alpha\) (which depend on \(k\) but not on \(N\)) such that, for every sufficiently large \(N\) and for every integer \(i \geq 1\),

\[
\mathbb{P}(T^N_{i+1} - T^N_i \geq \alpha N | \mathcal{F}_{T^N_i}) \geq c.
\]

If the claim holds, simple arguments show that we have a.s.

\[
\liminf_{j \to \infty} \frac{T^N_j}{j} \geq \alpha c N
\]

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and it follows that, a.s.,
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} 1_{\{P_i \geq N, G_i = k\}} \leq \frac{1}{\alpha c N}.
\]
We can remove \( P_i \geq N \) in the indicator function since the Markov chain \((P_n)_{n \geq 0}\) is transient. This gives (39) since \( N \) can be taken arbitrarily large.

Let us verify the claim. Applying the strong Markov property at time \( t_i \) leads to a Markov chain \((\tilde{P}_n, \tilde{G}_n, \tilde{H}_n)\) with transition kernel \( Q \) but now started from some triple \((p_0, \ell_0, h_0)\) such that \( p_0 \geq N \) and \( \ell_0 = k \). We also set \( \tilde{U}_n = \tilde{P}_n - \tilde{G}_n \). The bound (41) reduces to finding two positive constants \( \alpha \) and \( c \) such that, for every sufficiently large \( N \),
\[
P(\tau_k \geq \alpha N) \geq c,
\]
where \( \tau_k = \min\{j \geq 1 : \tilde{G}_j = k\} \). We set \( \tilde{T} := \inf\{n \geq 0 : \tilde{P}_n = \tilde{U}_n\} \), and observe that we have either \( \tilde{H}_{\tilde{T}} = h_0 + 1 \) or \( \tilde{H}_{\tilde{T}} = h_0 \).

By looking at the transition kernel \( Q \) and using the bounds (20) and (21), we see that we can couple the Markov chain \((\tilde{P}_n, \tilde{G}_n, \tilde{H}_n)\) with a random walk \((Y_n)\) started from \( \ell_0 = k \), whose jump distribution \( \mu \) is given by \( \mu(1) = q_{-1}, \mu(-j) = q_j \) for every \( j \geq 1 \), and \( \mu(0) = 1 - \mu(1) - \sum_{j \geq 1} \mu(-j) \), in such a way that
\[
\tilde{G}_n \geq Y_n, \quad \text{for every } 0 \leq n < \tilde{T},
\]
and on the event where \( Y_1 = k + 1 \) and \( \min_{j \geq 1} Y_j = k + 1 \) we have \( \tilde{H}_{\tilde{T}} = h_0 + 1 \) (the point is that on the latter event, the transition corresponding to the last line of (38) will not occur, at any time \( n \) such that \( 0 \leq n < \tilde{T} \)). Since the random walk \( Y \) has a positive drift to \( \infty \), the latter event occurs with probability \( c_0 > 0 \). We have thus obtained that
\[
P(\{\tilde{G}_n \geq k + 1, \text{ for every } 1 \leq n < \tilde{T}\} \cap \{\tilde{H}_{\tilde{T}} = h_0 + 1\}) \geq c_0.
\]

Next we observe that there is a positive constant \( c_1 \) such that, for every \( \varepsilon > 0 \), we have, for all sufficiently large \( N \),
\[
P(\{\tilde{T} \leq c_1(N - k)\} \cap \{H_{\tilde{T}} = h_0 + 1\}) < \varepsilon.
\]
To get this bound, we now consider the transition kernel \( Q' \): We use (20) to observe that we can couple \((\tilde{P}_n, \tilde{U}_n, \tilde{H}_n)\) with a random walk \( Y' \) started from \( N - k \), with only nonpositive jumps distributed according to \( \mu'(k) = q_k \) for every \( k \geq 1 \) (and of course \( \mu'(0) = 1 - \sum_{k \geq 1} \mu'(k) \)), in such a way that
\[
\tilde{U}_n \geq Y'_n, \quad \text{for every } 0 \leq n < \tilde{T},
\]
and \( Y'_n \leq 0 \) on the event \( \{H_{\tilde{T}} = h_0 + 1\} \). In particular on the event \( \{H_{\tilde{T}} = h_0 + 1\} \) the hitting time of the negative half-line by \( Y' \) must be smaller than or equal to \( \tilde{T} \). Since \( \mu' \) has a finite first moment, the law of large numbers gives a constant \( c_1 \) such that (44) holds.

By combining (43) and (44), and recalling the definition of \( \tau_k \), we get
\[
P(\tau_k \geq c_1(N - k))
\geq P(\{\tilde{G}_n \geq k + 1, \text{ for every } 1 \leq n < \tilde{T}\} \cap \{H_{\tilde{T}} = h_0 + 1\}) - P(\{\tilde{T} \leq c_1(N - k)\} \cap \{H_{\tilde{T}} = h_0 + 1\})
\geq c_0 - \varepsilon,
\]
Our claim (42) now follows since we can choose \( \varepsilon < c_0 \).
4.3 Distances in the peeling by layers

We need another lemma before we proceed to the proof of Proposition 9.

**Lemma 12.** There exists a constant $C$ such that $\mathbb{E}[H_n] \leq Cn^{1/3}$, for every $n \geq 1$.

**Proof.** It will be convenient to introduce a process $H'_n$ which coincides with $H_n$ at times of the form $\sigma_r$, $r \geq 1$, but which “interpolates” $H_n$ on every interval $[\sigma_r, \sigma_{r+1}]$. To be specific, we recall the notation introduced in the proof of Lemma 11, and we set for every $n \geq 0$,

$$H'_n = H_n + \frac{G_n}{F_n}.$$

From the form of the transition kernel of the Markov chain $(P_n, G_n, H_n)$ (see the proof of Lemma 11), we get, for every triple $(p, \ell, h)$ such that $P(P_n = p, G_n = \ell, H_n = h) > 0$,

$$\mathbb{E}[\Delta H'_n | P_n = p, G_n = \ell, H_n = h] = q^{-1}_n \left( \frac{\ell + 1}{p + 1} - \frac{\ell}{p} \right) + \sum_{k=1}^{p-\ell-1} q_k \left( \frac{\ell}{p - k} - \frac{\ell}{p} \right).$$

Then it is not hard to verify that each term in the right-hand side is bounded above by $c/p$, with some constant $c$ independent of $(p, \ell, h)$. Indeed, writing $c$ for a constant that may vary from line to line, and using (20), we have

$$q^{-1}_n \left( \frac{\ell + 1}{p + 1} - \frac{\ell}{p} \right) \leq \frac{1}{p + 1},$$

and similarly,

$$\sum_{k=1}^{\ell} q_k \left( \frac{\ell - k}{p - k} - \frac{\ell}{p} \right) = q(p) \sum_{k=1}^{\ell} k(p-k) \leq \frac{1}{p} \sum_{k=1}^{\infty} k q_k = \frac{c}{p},$$

$$\sum_{k=1}^{p-\ell-1} q_k \left( \frac{\ell}{p - k} - \frac{\ell}{p} \right) = \sum_{k=1}^{p-\ell-1} q_k \left( \frac{\ell}{p - k} \right) \leq \frac{p-\ell-1}{p} \sum_{k=1}^{p-\ell} q_k \frac{k}{p} \leq \frac{c}{p},$$

$$\sum_{k=p-\ell}^{p-2} q_k \left( 1 - \frac{\ell}{p} \right) \leq \left( 1 - \frac{\ell}{p} \right) \sum_{k=p-\ell}^{\infty} q_k \leq \left( 1 - \frac{\ell}{p} \right) \sum_{k=p-\ell}^{\infty} q_k \leq (1 - \frac{\ell}{p}) \times c (p-\ell)^{-3/2} \leq \frac{c}{p} (p-\ell)^{-1/2}.$$
and every \( n \) with \( \sigma_r \leq n < \sigma_{r+1} \), we have

\[
A_{\sigma_{r+1}} - A_n = U_n \leq P_n,
\]

\[
A_n - A_{\sigma_r} = P_{\sigma_r} - U_n \leq P_{\sigma_r}.
\]

It easily follows that, for every \( 0 \leq n_1 \leq n_2 \), we have

\[
\frac{A_{n_2} - A_{n_1}}{\max_{n_1 \leq i \leq n_2} P_i} \leq H_{n_2} - H_{n_1} + 1, \tag{46}
\]

and

\[
H_{n_2} - H_{n_1} \leq \frac{A_{n_2} - A_{n_1}}{\min_{n_1 \leq i \leq n_2} P_i} + 1. \tag{47}
\]

Fix \( 0 < s < t \). By (45),

\[
n^{-2/3} \min_{|n_1 - k| \leq |nt|} P_k \xrightarrow{d \to \infty} p_{\Delta^2} \inf_{s \leq u \leq t} S_u^+,
\]

and the limit is a (strictly) positive random variable. Using also Proposition 10, we then deduce from the bound (47) that the sequence \( n^{-1/3}(H_{[nt]} - H_{[ns]}) \) is tight. Hence we can assume that along a suitable subsequence, for every integer \( k \geq 0 \), for every \( 1 \leq i \leq 2^k \), we have the convergence in distribution

\[
n^{-1/3}(H_{[n(s+i2^{-k}(t-s))]} - H_{[n(s+(i-1)2^{-k}(t-s))])}) \xrightarrow{d \to \infty} \Lambda_{k,i}(s,t) \tag{48}
\]

where \( \Lambda_{k,i}(s,t) \) is a nonnegative random variable. Moreover, we can assume that the convergences (48) hold jointly, and jointly with (45). It then follows from the bounds (46) and (47) that, for every \( k \) and \( i \),

\[
\frac{a_{\Delta^2}}{p_{\Delta^2}} \sup_{s+(i-1)2^{-k}(t-s) \leq u \leq s+i2^{-k}(t-s)} S_u^+ \leq \Lambda_{k,i}(s,t) \leq \frac{a_{\Delta^2}}{p_{\Delta^2}} \inf_{s+(i-1)2^{-k}(t-s) \leq u \leq s+i2^{-k}(t-s)} S_u^+.
\]

Note that \( a_{\Delta^2}/p_{\Delta^2} = 12^{-1/3} =: h_{\Delta^2} \). By summing over \( i \), we get

\[
h_{\Delta^2} \sum_{i=1}^{2^k} \sup_{s+(i-1)2^{-k}(t-s) \leq u \leq s+i2^{-k}(t-s)} S_u^+ \leq \Lambda_{0,1}(s,t) \leq h_{\Delta^2} \sum_{i=1}^{2^k} \inf_{s+(i-1)2^{-k}(t-s) \leq u \leq s+i2^{-k}(t-s)} S_u^+.
\]

When \( k \to \infty \), both the right-hand side and the left-hand-side of the previous display converge a.s. to

\[
h_{\Delta^2} \int_s^t \frac{du}{S_u^+}.
\]

This argument (and the fact that the limit does not depend on the chosen subsequence) thus gives

\[
n^{-1/3}(H_{[nt]} - H_{[ns]}) \xrightarrow{n \to \infty} h_{\Delta^2} \int_s^t \frac{du}{S_u^+}, \tag{49}
\]

and this convergence holds jointly with (45).

At this point, we use Lemma 12, which tells us that \( \mathbb{E}[n^{-1/3}H_{[ns]}] \) can be made arbitrarily small, uniformly in \( n \), by choosing \( s \) small. Also Lemma 12, (49) and Fatou’s lemma imply that

\[
\mathbb{E}\left[ \int_s^t \frac{du}{S_u^+} \right]
\]

is bounded above independently of \( s \in (0,t] \), and therefore \( \int_0^t \frac{du}{S_u^+} < \infty \) a.s. (we could have obtained this more directly). Letting \( s \to 0 \), we deduce from the previous considerations that

\[
n^{-1/3}(H_{[nt]} \xrightarrow{n \to \infty} h_{\Delta^2} \int_0^t \frac{du}{S_u^+}. \tag{50}
\]

jointly with (45). The statement of Proposition 9 now follows from monotonicity arguments using the fact that the limit in (50) is continuous in \( t \).
4.4 From Proposition 9 to Theorem 2

In this section, we deduce Theorem 2 from Proposition 9 via a time change argument. We start with some preliminary observations.

We fix $x > 0$ and write $(\Gamma_t^x)_{t \geq 0}$ for the stable Lévy process with index $3/2$ and no negative jumps started from $x$, whose distribution is characterized by the formula

$$E[\exp(-\lambda(\Gamma_t^x - x))] = \exp(\lambda^{3/2}), \quad \lambda, t \geq 0.$$  

Equivalently, $\Gamma_t^x = x - S_t$ where $S_t$ is as in the introduction. Set $\gamma_x := \inf\{t \geq 0 : \Gamma_t^x = 0\}$. Then $\gamma_x < \infty$ a.s., and a classical time-reversal theorem (see e.g. [8, Theorem VII.18]) states that the law of $(\Gamma_t^x - t)_{0 \leq t \leq \gamma_x}$ (with $\Gamma_{\gamma_x}^x = x$) coincides with the law of $(S_t^+)_{0 \leq t \leq \rho_x}$, where $\rho_x := \sup\{t \geq 0 : S_t^+ = x\}$.

On the other hand, consider the process $\mathcal{L}$ of Section 1. If $\lambda_x := \sup\{t \geq 0 : \mathcal{L}_t \leq x\}$, then $\lambda_x < \infty$ a.s. and setting $X_t^x = \mathcal{L}_{(\lambda_x - t)^-}$ for $0 \leq t \leq \lambda_x$ (with $\mathcal{L}_{\lambda_x} = 0$), the process $(X_t^x)_{0 \leq t \leq \lambda_x}$ is distributed as the continuous-state branching process with branching mechanism $\psi(u) = u^{3/2}$ started from $x$ and stopped when it hits $0$. See [14, Section 2.1] for more details.

The classical Lamperti transformation asserts that, if we set

$$\tau_t^x := \inf\{s \geq 0 : \int_0^s \frac{du}{\Gamma_u} \geq t\}$$

for $0 \leq t \leq R_x := \int_0^{\gamma_x} \frac{du}{\Gamma_u}$, the time-changed process $(\Gamma_t^x)_{0 \leq t \leq R_x}$ has the same distribution as $(X_t^x)_{0 \leq t \leq \lambda_x}$.

We can then combine the Lamperti transformation with the preceding observations to obtain that, if

$$\eta_t := \inf\{s \geq 0 : \int_0^s \frac{du}{S_u^+} \geq t\},$$

for every $t \geq 0$, the process

$$\left(S_{\eta_t}^+, Z_{\eta_t}\right)_{t \geq 0} \xrightarrow{(d)} \left(\mathcal{L}_t, \mathcal{M}_t\right)_{t \geq 0},$$

with the notation of Section 1.

Let us turn to the proof of Theorem 2. We recall that, for every integer $r \geq 1$, we have $|\partial B_r^x| = P_{\sigma_r}$ and $|B_r^x| = V_{\sigma_r}$, with $\sigma_r = \min\{n : H_n \geq r\}$. We use the convergence in distribution of Proposition 9 and the Skorokhod representation theorem to find, for every $n \geq 1$, a triple $(P^{(n)}, V^{(n)}, H^{(n)})$ having the same distribution as $(P, V, H)$, in such a way that we now have the almost sure convergence

$$\left(\frac{P^{(n)}_{[nt]}}{P_{\Delta^2 \cdot n^{2/3}} \cdot \nu_{\Delta^2 \cdot n^{1/3}}}, \frac{V^{(n)}_{[nt]}}{h_{\Delta^2 \cdot n^{1/3}}}, \frac{H^{(n)}_{[nt]}}{h_{\Delta^2 \cdot n^{1/3}}}\right)_{t \geq 0} \xrightarrow{n \to \infty} \left(\mathcal{L}_t, Z_t, \int_0^t \frac{du}{S_u^+}\right)_{t \geq 0},$$

for the Skorokhod topology. For every $n \geq 1$, and every $r \geq 1$, set

$$\sigma^{(n)}_r = \min\{k : H^{(n)}_k \geq r\}.$$  

Then it easily follows from (52) that

$$\left(\frac{1}{n^{1/3} \cdot \sigma^{(n)}_r}\right)_{t \geq 0} \xrightarrow{n \to \infty} \left(\gamma_{t/\Delta^2}\right)_{t \geq 0},$$
uniformly on every compact time set. By combining the latter convergence with (52) we arrive at the a.s. convergence in the Skorokhod sense,
\[
\left( n^{-2/3} P(n), n^{-4/3} V(n) \right)_{t \geq 0} \xrightarrow{n \to \infty} \left( P_{\Delta^2 \eta / n^{1/3}}^+, V_{\Delta^2 Z / n^{1/3}} \right)_{t \geq 0}.
\]
Recalling the identity in distribution (51), we get the convergence in distribution of Theorem 2 since
\[
(P(n), V(n))_{t \geq 0} \overset{d}{=} (P_{\sigma}, V_{\sigma})_{t \geq 0} = (|\partial B_{\tau} |, |B_{\tau} |)_{t \geq 0}.
\]
This completes the proof.

5 First-passage percolation

We consider the dual graph of the UIPT, whose vertices are the faces of the UIPT, and each edge \( e \) of the UIPT corresponds to an edge of the dual graph between the two faces incident to \( e \). By convention, the root of the dual graph or root face is the face incident to the right-hand side of the root edge of the UIPT. We assign independent weights exponentially distributed with parameter 1 to the edges of the UIPT. We assign independent weights exponentially distributed with parameter 1 to the edges of the UIPT which are connected to the root face by a dual path whose weight is less than or equal to \( t \). We then let \( F^* \) be the hull of \( F \). We set \( \tau_0 = 0 \) and we let \( 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots \) be the successive jump times of the process \( F^* \) (a simple argument shows that \( \tau_n \to \infty \) as \( n \to \infty \), which will also follow from the next proposition). Note that, at each time \( \tau_n \) with \( n \geq 1 \), a new triangle incident to the boundary of \( F^*_{\tau_n-1} \) is added to \( F^*_{\tau_n-1} \), together with the triangles in the “hole” that this addition may create.

By convention we let \( F_0^* \) be the trivial triangulation, and we set, for every \( n \geq 1 \)
\[
F_n^* = F^*_{\tau_n-1}.
\]
The following proposition shows that the process \( (F_n^*)_{n \geq 0} \) is a particular instance of a peeling process, which is called the uniform peeling process or Eden model on the UIPT. See also [1, Section 6].

**Proposition 13.** The sequence \( (F_n^*)_{n \geq 0} \) has the same law as the sequence \( (T_n)_{n \geq 0} \) corresponding to a peeling process where at step 1 we reveal the triangle incident to the right-hand side of the root edge, and for every \( n \geq 2 \), conditionally on \( T_0, \ldots, T_{n-1} \), the peeled edge at step \( n \) is chosen uniformly at random among the edges of \( \partial T_{n-1} \). Furthermore, conditionally on the sequence \( (F_n^*)_{n \geq 1} \), the increments \( \tau_1 - \tau_0, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots \) are independent, and, for every \( k \geq 1 \), \( \tau_k - \tau_{k-1} \) is exponentially distributed with parameter \( |\partial F_k^*| \).

**Remark.** Since \( |\partial F_k^*| \leq k + 2 \), the last assertion shows that \( \tau_k \uparrow \infty \) a.s. as \( k \to \infty \).

**Proof.** Let \( n \geq 1 \). Consider an edge \( e \) of \( \partial F_n^* \). Then, \( e \) is incident to a unique face \( f_e \) of \( F_{\tau_{n-1}}^* \), and we write \( d_{\text{FPP}}(f_e) \) for the first-passage percolation distance between \( f_e \) and the root face (in other words, this is the minimal weight of a dual path connecting the root face and \( f_e \)). We also write \( w_e \) for the weight of \( e \), or rather of its dual edge. Since \( f_e \) is contained in \( F_{\tau_{n-1}}^* \), we have \( d_{\text{FPP}}(f_e) \leq \tau_{n-1} \), with equality only if \( f_e \) is the triangle that was added at time \( \tau_{n-1} \). Also it is clear that
\[
w_e > \tau_{n-1} - d_{\text{FPP}}(f_e)
\]
because otherwise this would contradict the fact that the other face incident to \( e \) is not in \( F_{\tau_{n-1}}^* \).
Next the lack of memory of the exponential distribution ensures that, conditionally on the variables \( \tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_n, \tau_1, \ldots, \tau_{n-1} \), the random variables
\[
w_e - (\tau_{n-1} - d_{\text{FPP}}(f_e)),
\]
where \( e \) varies over the edges of \( \partial \tilde{F}_n \), are independent and exponentially distributed with parameter 1. Now observe that the next jump will occur at time
\[
\tau_n = \tau_{n-1} + \min \{w_e - (\tau_{n-1} - d_{\text{FPP}}(f_e)) : e \text{ edge of } \partial \tilde{F}_n \}.
\]
It follows that, conditionally on \( \tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_n, \tau_1, \ldots, \tau_{n-1} \), the variable \( \tau_n - \tau_{n-1} \) is exponential with parameter \( |\partial \tilde{F}_n| \), and furthermore, the new triangle added to \( \tilde{F}_n \) corresponds to the edge attaining the preceding minimum, which is therefore uniformly distributed over edges of \( \partial \tilde{F}_n \). This completes the proof.

**Proof of Theorem 3.** As in the previous sections, we use the notation \( V_n \) and \( P_n \) for the volume and the perimeter of \( \tilde{F}_n \). We will establish the following convergence in distribution for the Skorokhod topology
\[
\left( \frac{P_{[nt]}}{p_{\Delta}^2 \cdot n^{2/3}}, \frac{V_{[nt]}}{v_{\Delta}^2 \cdot n^{4/3}}, \frac{\tau_{[nt]}}{1/p_{\Delta}^2 \cdot n^{1/3}} \right)_{t \geq 0} \xrightarrow{n \to \infty} \left( S^+_t, Z_t, \int_0^t \frac{du}{S_u^+} \right)_{t \geq 0}.
\]
(53)

Theorem 3 then follows from (53) by the very same arguments we used to deduce Theorem 2 from Proposition 9 in Section 4.4.

The joint convergence of the first two components in (53) is given by Theorem 1. So we need to prove the convergence of the third component and to check that it holds jointly with the first two. As in the proof of Proposition 9, we fix \( 0 < s < t \) and we first consider \( \tau_{[nt]} - \tau_{[ns]} \). Writing
\[
\tau_{[nt]} - \tau_{[ns]} = \sum_{i=[ns]+1}^{[nt]} (\tau_i - \tau_{i-1})
\]
and using Proposition 13, we see that conditionally on \( (P_k)_{k \geq 0} \), the variable \( \tau_{[nt]} - \tau_{[ns]} \) is distributed as
\[
\sum_{i=[ns]+1}^{[nt]} \frac{e_i}{P_i},
\]
where the random variables \( e_1, e_2, \ldots \) are independent and exponentially distributed with parameter 1, and are also independent of \( (P_k)_{k \geq 0} \). By the convergence of the first component in (53), we have
\[
n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i} = \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{du}{n^{-2/3} P_{[ns]}^4} \xrightarrow{(d)} \frac{1}{p_{\Delta}^2} \int_s^t \frac{du}{S_u^+},
\]
and, on the other hand,
\[
E \left[ \left( n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{e_i}{P_i} - n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i} \right)^2 \mid (P_k)_{k \geq 0} \right] = n^{-2/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{(P_i)^2} = \frac{1}{n} \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{du}{(n^{-2/3} P_{[ns]}^4)^2}
\]

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converges to 0 in probability as \( n \to \infty \). It easily follows that
\[
 n^{-1/3} (\tau_{[nt]} - \tau_{[ns]}) \xrightarrow{(d) \ n \to \infty} \frac{1}{p_{\Delta^2}} \int_{\Delta^2} \frac{\, \mathrm{d}u}{S_u^g},
\] (54)
and the previous argument also shows that this convergence holds jointly with that of the first two components in (53). We can complete the proof by arguing in a way similar to the end of the proof of Proposition 9. It suffices to verify that
\[
\sup_{n \geq 1} \mathbb{E}[n^{-1/3} \tau_{[ns]}] \xrightarrow{s \to 0} 0.
\]
This is however very easy, since
\[
\mathbb{E}[\tau_{[ns]}] = \mathbb{E}\left[ \sum_{j=1}^{[ns]} \frac{1}{P_j} \right]
\]
and we can use Lemma 7 to obtain that \( \mathbb{E}[\tau_{[ns]}] \leq C (ns)^{1/3} \), for some constant \( C \).

**Remark.** One conjectures that balls for the first-passage percolation distance grow asymptotically like “deterministic” balls for the graph distance. More precisely, one expects that there exists a constant \( c > 0 \) such that, for every \( \varepsilon > 0 \), one has
\[
B_{(c-\varepsilon)r} \subset F_r \subset B_{(c+\varepsilon)r}.
\] (55)
with high probability when \( r \) is large. The reason for this belief is the fact that the UIPT is “isotropic” in contrast with deterministic lattices like \( \mathbb{Z}^d \). We hope to address this problem in a future work.

Our results support the previous conjecture since the scaling limits for the perimeter and volume of hulls are the same for \( B_r \) and for \( F_r \) up to multiplicative constants. Note that, if (55) holds, we must have also
\[
B_{(c-\varepsilon)r}^* \subset F_r^* \subset B_{(c+\varepsilon)r}^*.
\]
By comparing the limits in distribution of \( |B_r^*| \) (Theorem 2) and of \( |F_r^*| \) (Theorem 3), we see that if the previous conjecture is true, the constant \( c \) must be equal to
\[
c = a_{\Delta^2} = \frac{1}{3}.
\]

See [1, Remark 5] for related calculations about two-point and three-point functions for first-passage percolation on type I triangulations (in that case, the analog of the constant \( a_{\Delta^2} \) is \( a_{\Delta^2} = 1/(2\sqrt{3}) \), as we shall see below).

## 6 Other models

In this section we briefly explain how our results can be extended to other classes of infinite random planar maps. We consider type I triangulations and quadrangulations, and leave aside the case of type III triangulations, which seems to be less amenable to a study via the peeling process.

### 6.1 Type I triangulations

Let us consider the case of type I triangulations, where both loops and multiple edges are allowed. This case is not considered by Angel and Schramm [6], but the techniques of [6] can easily be extended using the corresponding enumeration results (see below). We denote the UIPT for type I triangulations by
Let us list the enumeration results corresponding to those of Section 2.1. These results may be found in Krikun [20] (Krikun uses the number of edges as the size parameter and in order to apply his formulas we note that a triangulation of the $p$-gon with $n$ inner vertices has $3n + 2p - 3$ edges).

For every $p \geq 1$ and $n \geq 0$, let $\mathcal{T}_{n,p}^{(1)}$ stand for the set of all type I triangulations with $n$ inner vertices and a simple boundary of length $p$, which are rooted on an edge of the boundary in the way explained in Section 2.1. We have:

$$\# \mathcal{T}_{n,p}^{(1)} = 4^{n-1}p (2p)! (2p + 3n - 5)! / (p!)^2 n! (2p + n - 1)! \sim C^{(1)}(p) (12\sqrt{3})^n n^{-5/2},$$

where

$$C^{(1)}(p) = \frac{3^{p-2} p (2p)!}{4\sqrt{2\pi} (p!)^2} \sim \frac{1}{36\pi \sqrt{2}} \frac{1}{\sqrt{p}} 12^p.$$

We then set $Z^{(1)}(p) = \sum_{n \geq 0} \# \mathcal{T}_{n,p}^{(1)} (12\sqrt{3})^{-n}$ and we have the formula (see [4, Section 2.2])

$$Z^{(1)}(p) = \frac{6^p (2p - 5)!}{8\sqrt{3} p!} \quad \text{if} \quad p \geq 2, \quad Z^{(1)}(1) = \frac{2 - \sqrt{3}}{4}.$$

The generating series of $Z^{(1)}(p)$ can also be computed explicitly from [20, formula (4)] and an appropriate change of variables (we omit the details):

$$\sum_{p \geq 0} Z^{(1)}(p+1) z^p = \frac{1}{2} + \frac{(1 - 12z)^{3/2} - 1}{24\sqrt{3} z}.$$

In particular, the analog of (4) is

$$Z^{(1)}(p+1) \sim \frac{\sqrt{3}}{8\sqrt{\pi}} 12^p p^{-5/2},$$

and similarly as in (4), we set

$$t^{\triangle 1} = \frac{\sqrt{3}}{8\sqrt{\pi}}.$$

The peeling algorithm discovering $T^{(1)}_{\infty}$ is then described in a very similar way as in Section 3.1. The only difference is that we now need to consider the possibility of loops. With the notation of Section 3.1, and supposing that the revealed region has a boundary of size $p \geq 1$, events of type $L_0$ or $R_0$, or of type $L_{p-1}$ or $R_{p-1}$, may occur (the definition of these events should be obvious from Fig. 3). The respective probabilities of events $C, L_k$ or $R_k$ are given by formulas analogous to (10) and (11), where $2/27$ is replaced by $1/(12\sqrt{3})$, the functions $C$ and $Z$ are replaced respectively by $C^{(1)}$ and $Z^{(1)}$, and finally $k$ is allowed to vary in $\{1, \ldots, p - 1\}$.

An analog of Proposition 5 holds, and the constant $p^{\triangle 2}$ has to be replaced by

$$p^{\triangle 1} = \left( \frac{8t^{\triangle 1} \sqrt{\pi}}{3} \right)^{2/3} = 3^{-1/3}.$$

Similarly, there is a version of Proposition 8 in the type I case, and the constant $b^{\triangle 2}$ is replaced by

$$b^{\triangle 1} = \frac{4}{3},$$

whereas the limiting distribution remains the same. Finally, the analog of Proposition 10 involves the new constant

$$a^{\triangle 1} = \frac{1}{2\sqrt{3}}.$$
The proofs of Theorems 1, 2 and 3 can then be adapted easily to the UIPT $T_{\infty}^{(1)}$. In these statements, $p_{\Delta^2}$ is replaced by $p_{\Delta^1}$ and the other constants $v_{\Delta^2}$ and $h_{\Delta^2}$ are replaced respectively by

$$v_{\Delta^1} = (p_{\Delta^1})^2 b_{\Delta^1} = 4 \cdot 3^{-5/3} \quad \text{and} \quad h_{\Delta^1} = \frac{a_{\Delta^1}}{p_{\Delta^1}} = \frac{1}{2} 3^{-1/6}.$$ 

We note that $p_{\Delta^1}/(h_{\Delta^1})^2 = p_{\Delta^2}/(h_{\Delta^2})^2$, which, by Theorem 2 and its type I analog, means that the scaling limit of the perimeter of hulls is exactly the same for type I and for type II triangulations. This fact can be explained by a direct relation between the UIPTs of type I and of type II, but we omit the details.

### 6.2 Quadrangulations

Let us now consider the Uniform Infinite Planar Quadrangulation (UIPQ), which is denoted here by $Q_\infty$. This case requires more changes in the arguments. We first note that a quadrangulation with a simple boundary necessarily has an even perimeter. For every $p \geq 1$, let $Q_{n,p}$ stand for the set of all quadrangulations with a simple boundary of perimeter $2p$ and $n$ inner vertices, which are rooted at an oriented edge of the boundary in such a way that the external face lies on the right of the root edge. For $n \geq 0$ and $p \geq 1$, we read from [10, Eq. (2.11)] that

$$\# Q_{n,p} = 3^{n-1} \frac{(3p)! (3p - 3 + 2n)!}{n! p! (2p - 1)! (n + 3p - 1)!} \sim C^\square(p) 12^n n^{-5/2},$$

where

$$C^\square(p) = \frac{8^{p-1} (3p)!}{3 \sqrt{\pi p! (2p - 1)!}} \sim \frac{1}{8 \sqrt{3 \pi}} 54^p \sqrt{p}.$$ 

We have also, for every $p \geq 2$,

$$Z^\square(p) = \sum_{n \geq 0} \# Q_{n,p} 12^{-n} = \frac{8^p (3p - 4)!}{(p - 2)! (2p)!},$$

and $Z^\square(1) = 4/3$. Furthermore,

$$Z^\square(p + 1) \sim \frac{1}{\sqrt{3 \pi}} 54^p p^{-5/2}, \quad \sum_{k \geq 0} Z^\square(k + 1) 54^{-k} = 3/2, \quad \sum_{k \geq 0} k Z^\square(k + 1) 54^{-k} = 1/2. \quad (56)$$

The transitions in the peeling process of the UIPQ are more complicated than previously because of additional cases. If at step $n \geq 0$ the perimeter of the discovered quadrangulation $Q_n$ is equal to $2m$, then the revealed quadrangle at the next step may have three different shapes (see Fig. 3):

1. **Shape C**: The revealed quadrangle has two vertices in the unknown region, an event of probability

   $$\mathbb{P}(C \mid \partial Q_n = 2m) = q^{(m)}_{C} = 12^2 C^\square(m + 1) C^\square(m).$$

2. **Shapes L_k and R_k**, for $k \in \{0, 1, \ldots, 2m - 1\}$: The revealed quadrangle has three vertices on the boundary of $Q_n$. This quadrangle then “swallows” a part of the boundary of $\partial Q_n$ of length $k$. This event is denoted by $L_k$ or $R_k$ according to whether the part of the boundary that is swallowed is on the right or on the left of the peeled edge. Note that the revealed face encloses a finite quadrangulation of perimeter $k + 1$ if $k$ is odd and $k + 2$ if $k$ is even. These events have probability

   $$\mathbb{P}(L_k \mid \partial Q_n = 2m) = \mathbb{P}(L_{k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m) = \mathbb{P}(R_{2k+1} \mid \partial Q_n = 2m).$$

   $$= a^{(m)}_{2k+1} = q^{(m)}_{2k} = \frac{Z^\square(k + 1) C^\square(m - k)}{12} C^\square(m).$$

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3. Shapes $L_{k_1,k_2}$, $R_{k_1,k_2}$ and $C_{k_1,k_2}$ for $k_1, k_2 \geq 1$ odd and such that $k_1 + k_2 < 2m$: This last case occurs when the revealed quadrangle has its four vertices on $\partial Q_n$. It then encloses two finite quadrangulations of respective perimeters $k_1 + 1$ and $k_2 + 1$ either both on the left side of the peeled edge in case $L_{k_1,k_2}$, or one on each side of the peeled edge in case $C_{k_1,k_2}$, or both on the right side of the peeled edge in case $R_{k_1,k_2}$. These three events have the same probability: writing $k_1 = 2j_1 + 1$ and $k_2 = 2j_2 + 1$, with $j_1 + j_2 < m - 1$,

$$\mathbb{P}(L_{k_1,k_2} \mid |\partial Q_n| = 2m) = \mathbb{P}(R_{k_1,k_2} \mid |\partial Q_n| = 2m) = \mathbb{P}(C_{k_1,k_2} \mid |\partial Q_n| = 2m)$$

$$= q_{k_1,k_2}^{(m)} = Z^{\square}(j_1 + 1)Z^{\square}(j_2 + 1) \frac{\sqrt{2}^m (m - j_1 - j_2 - 1)}{\sqrt{2}^m (m)}.$$

**Figure 6**: A few peeling transitions in the quadrangular case.

Furthermore, conditionally on each of the above cases, the finite quadrangulations enclosed by the revealed face are independent Boltzmann quadrangulations with the prescribed perimeters. Let $P_n^{\square}$ stand for the half-perimeter at step $n$ in the peeling process. Then, similarly as in the triangular case, the Markov chain $(P_n^{\square})$ is obtained by conditioning a random walk $X$ on $\mathbb{Z}$ to stay (strictly) positive, and the increments of $X$ are now distributed as follows:

$$\mathbb{E}[f(X_{n+1}) \mid X_n] = f(X_n + 1) \cdot q_{-2} + \sum_{k=0}^{\infty} f(X_n - k) \cdot \left(2q_{2k} + q_{2k+1} + 3 \sum_{\substack{k_1+k_2=2k \\text{k_1,k_2 \geq 1 odd}}} q_{k_1,k_2}\right),$$

where $q_j = \lim_{m \to \infty} q_j^{(m)}$ and $q_{k_1,k_2} = \lim_{m \to \infty} q_{k_1,k_2}^{(m)}$ as in the triangular case. From the enumeration results, we get, for every $k \geq 0$,

$$\mathbb{P}(\Delta X = -k) = 2(q_{2k} + q_{2k+1} + 3 \sum_{\substack{k_1+k_2=2k \\text{k_1,k_2 \geq 1 odd}}} q_{k_1,k_2}) \sim \frac{1}{2\sqrt{3\pi} k^{5/2}}.$$

(57)

The results of Sections 3.2 and 3.4 can then be extended to the UIPQ $Q_\infty$. Comparing (57) with (13), we see that the role of the constant $t_{\Delta^2}$ is now played by $t_\square = 1/(4\sqrt{3\pi})$. Then the convergence in distribution of Proposition 5 holds for $P_n^{\square}$, with the constant $p_{\Delta^2}$ replaced by

$$p_\square = \left(\frac{8\sqrt{2}}{3}\right)^{2/3} = \frac{2^{2/3}}{3}.$$

An analog of Proposition 8, where we now consider a Boltzmann quadrangulation $Q^{(p)}$ of the $2p$-gon, also holds in the form

$$p^{-2} \mathbb{E}[|Q^{(p)}|] \to_{p \to \infty} \frac{9}{2} =: b_\square.$$
The peeling by layers requires certain modifications in the case of quadrangulations. As previously, the ball \( B_r(Q_\infty) \) is the planar map obtained by keeping only those faces of \( Q_\infty \) that are incident to at least one vertex whose graph distance from the root vertex is smaller than or equal to \( r - 1 \), and the hull \( B^*_r(Q_\infty) \) is obtained by filling in the finite holes of \( B_r(Q_\infty) \). The boundary \( \partial B^*_r(Q_\infty) \) is now a simple cycle that visits alternatively vertices at distance \( r \) and \( r + 1 \) from the root vertex. If we move around the boundary of this cycle in clockwise order, we encounter two types of (oriented) edges, edges \( r + 1 \rightarrow r \) connecting a vertex at distance \( r + 1 \) to a vertex at distance \( r \), and edges \( r \rightarrow r + 1 \) connecting a vertex at distance \( r \) to a vertex at distance \( r + 1 \).

To describe the peeling by layers algorithm, suppose that, at a certain step of the peeling process, the revealed region is the hull \( B^*_r(Q_\infty) \). Then we choose (deterministically or using some independent randomization) an edge of the boundary of type \( r + 1 \rightarrow r \). We reveal the face incident to this edge that is not already in \( B^*_r(Q_\infty) \) and as usual we fill in the holes that may have been created. At the next step, either the (new) boundary has an edge of type \( r + 1 \rightarrow r \) that is incident to the quadrangle revealed in the previous step, and we peel this edge, or we peel the first edge of type \( r + 1 \rightarrow r \) coming after the revealed quadrangle in clockwise order. We continue inductively, “moving around the boundary in clockwise order”. See Fig. 7 for an example. After a finite number of steps, the boundary does not contain any vertex at distance \( r \), and it is easy to verify that the revealed region is then the hull \( B^*_{r+1}(Q_\infty) \), so that we can continue the construction by induction.

**Figure 7:** Illustration of the peeling by layers in the quadrangular case: we choose an edge to start discovering the new layer and then peel from left to right all the edges that contain a vertex at distance \( r \) from the root vertex.

Proposition 10 is adapted as follows. For every \( r \geq 1 \), let \( \mathcal{L}^r_\Box \) be the set of all vertices of \( \partial B^*_r(Q_\infty) \) that are at distance exactly \( r \) from the root vertex. Clearly the perimeter \( |\partial B^*_r(Q_\infty)| \) is equal to \( 2\# \mathcal{L}^r_\Box \). We also denote the union of all \( \mathcal{L}^r_\Box \) for \( r \geq 1 \) by \( \mathcal{L}_\Box \). Finally, for \( n \geq 1 \), we let \( A^\Box_n \) be the number of vertices of \( \mathcal{L}_\Box \) that are in the interior of the discovered region at step \( n \). Then the analog of Proposition 10 reads

\[
\frac{A_n}{n} \xrightarrow{n \to \infty} \frac{1}{3} =: a_\Box.
\]

The idea of the proof is the same but technicalities become somewhat more complicated (we omit the details).

Versions of Theorems 1, 2 and 3 then hold for the UIPQ \( Q_\infty \). In these statements we now interpret the size of the boundary as half its perimeter, the constant \( p_\Delta^2 \) is replaced by \( p_\Box \) and the other constants \( v_\Delta^2 \) and \( h_\Delta^2 \) are replaced respectively by

\[
v_\Box = (p_\Box)^2 b_\Box = 2^{1/3} \quad \text{and} \quad h_\Box = \frac{a_\Box}{p_\Box} = 2^{-2/3}.
\]

Furthermore the limiting process in the analog of Theorem 3 should be \( (p_\Box \cdot \mathcal{L}_{2p_\Box t}, v_\Box \cdot \mathcal{M}_{2p_\Box t})_{t \geq 0} \), the extra multiplicative factor 2 in the time parameter coming from the fact that we are dealing with
half-perimeters. We finally observe that the convergence of volumes in the analog of Theorem 2 for the UIPQ was already obtained in [14] as a consequence of the invariance principles relating the UIPQ and the Brownian plane (see Theorems 5.1 and 1.3 in [14]). It would be significantly harder to derive the convergence of boundary lengths from the same invariance principles. On the other hand, Krikun [18] has a version of the scaling limit for boundary lengths in the case of quadrangulations, but with a different definition of hull boundaries leading to different constants.

References


