Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere

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Abstract

We prove that scaling limits of random planar maps which are uniformly distributed over the set of all rooted $2k$-angulations are a.s. homeomorphic to the two-dimensional sphere. Our methods rely on the study of certain random geodesic laminations of the disk.

1   Introduction

This paper continues the study of scaling limits of large random planar maps in the sense of the Hausdorff-Gromov topology. In the particular case of uniformly distributed $2k$-angulations, scaling limits were shown in [L2] to be homeomorphic to a (random) compact metric space which may be naturally defined as a quotient of the Continuum Random Tree (CRT), which was introduced by Aldous in [A1, A2]. The main goal of the present paper is to prove that this limiting metric space is almost surely homeomorphic to the 2-sphere $\mathbb{S}^2$.

Let us first recall some basic definitions. More details can be found in [L2]. A *planar map* is a topological embedding (without edge crossing) of a finite connected graph in the sphere $\mathbb{S}^2$. Its *faces* are the connected components of the complement of its image in $\mathbb{S}^2$. Let $k \geq 2$ be a fixed integer. A $2k$-angulation is a planar map such that each face is adjacent to $2k$ edges (one should in fact count edge sides, so that if an edge lies entirely inside a face, it should be counted twice). A planar map is called *rooted* if it has a distinguished oriented edge, which is called the *root edge*. Two rooted planar maps are said to be *equivalent* if the second one is the image of the first one under an orientation-preserving homeomorphism of the sphere, which also preserves the root edge. We systematically identify equivalent rooted planar maps. Thanks to this identification, the set of all rooted $2k$-angulations with a given number of faces is finite.

For every integer $n \geq 2$, let $\mathcal{M}_n^k$ be the set of all rooted $2k$-angulations with n faces, and let M_n be a random planar map that is uniformly distributed over $\mathcal{M}_n^k$. Denote by $\mathbf{m}_n$ the set of vertices of M_n, and write d_n for the graph distance on $\mathbf{m}_n$. We view $(\mathbf{m}_n, d_n)$

as a random compact metric space, and study its convergence in distribution as $n \longrightarrow \infty$, after a suitable rescaling.

We denote by $\mathbb{K}$ the set of all isometry classes of compact metric spaces, and equip $\mathbb{K}$ with the Hausdorff-Gromov distance d_{GH} (see [Gro], [P] or [BBI]). Then $(\mathbb{K}, d_{GH})$ is a Polish space, which makes it appropriate to study the convergence in distribution of $\mathbb{K}$-valued random variables.

We can now state our main result.

Theorem 1.1 *The sequence of the laws of the metric spaces $(\mathbf{m}_n, n^{-1/4}d_n)$ is tight (i.e. relatively compact) in the space of all probability measures on $\mathbb{K}$. If $(\mathbf{m}_\infty, d_\infty)$ is the weak limit of a subsequence of $(\mathbf{m}_n, n^{-1/4}d_n)$, then the metric space $(\mathbf{m}_\infty, d_\infty)$ is almost surely homeomorphic to the sphere $\mathbb{S}^2$.*

Remark. It is natural to conjecture that the sequence $(\mathbf{m}_n, n^{-1/4}d_n)$ does converge in distribution, or equivalently that the law of any weak limit $(\mathbf{m}_\infty, d_\infty)$ is uniquely determined, and that this law is independent of k up to multiplicative constants. This is still an open problem, even though detailed information on $(\mathbf{m}_\infty, d_\infty)$ is already available. In particular, it is known that the Hausdorff dimension of $(\mathbf{m}_\infty, d_\infty)$ is almost surely equal to 4 (see [L2, Theorem 6.1]).

The first assertion of Theorem 1.1 is already stated in Proposition 3.2 of [L2]. The new part of the theorem is the second assertion, which is proved in Section 3 below. We rely on the main theorem of [L2], which asserts in particular that any weak limit $(\mathbf{m}_\infty, d_\infty)$ is almost surely homeomorphic to a quotient of the CRT corresponding to a certain pseudo-metric D^* (see Section 3 for details). As a preparation for the proof of our main result, Section 2 investigates, in a deterministic setting, quotient spaces of compact $\mathbb{R}$-trees coded by continuous functions on the circle, and their relations with geodesic laminations of the disk. As a matter of fact, a key idea is to observe that the CRT, which is the random $\mathbb{R}$-tree coded by a normalized Brownian excursion (in the sense of Theorem 2.1 of [DL]), can also be interpreted as the quotient space induced by a certain random geodesic lamination of the hyperbolic disk. This observation is related to the work of Aldous [A3, A4] about random triangulations of the circle: The random geodesic lamination that we consider corresponds to the random triangulation in Section 5 of [A3] (or Section 2.3 in [A4]), provided we replace the Poincaré disk model of Lobatchevsky's hyperbolic plane with the Klein disk model.

Any random metric space that arises as a weak limit of rescaled planar maps is then homeomorphic to a topological space that can be obtained by taking one more quotient with respect to a second random geodesic lamination, which is not independent of the first one. To handle this setting, we introduce on the sphere $\mathbb{S}^2$ the equivalence relation for which two distinct points of the upper hemisphere, resp. of the lower hemisphere, are equivalent if they belong to the same geodesic line of the first random lamination, resp. of the second one, or to the closure of an ideal hyperbolic triangle which is a connected component of the complement of the same lamination. To get the second assertion of Theorem 1.1, we then use a theorem of Moore [Moo] giving sufficient conditions for a quotient space of the sphere $\mathbb{S}^2$ to be homeomorphic to the sphere.

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Theorem 1.1 yields information about the large scale geometry of random planar maps. Let us state a typical result in this direction. Recall that a path of length p in a planar map is a sequence $x_0, e_1, x_1, e_2, \ldots, x_{p-1}, e_p, x_p$, where $x_0, x_1, \ldots, x_p$ are vertices, $e_1, \ldots, e_p$ are edges and the endpoints of e_i are the points x_{i-1} and x_i, for every $i \in \{1, \ldots, p\}$. The path is called a cycle if $x_0 = x_p$. We say that it is an injective cycle if in addition $x_1, \ldots, x_p$ are distinct (when $p = 2$, we also require that $e_1 \neq e_2$). If C is an injective cycle, then the union of its edges $\mathcal{R}(C)$ separates the sphere in two connected components, by Jordan's theorem.

Corollary 1.2 *Let $\delta > 0$ and let $\theta : \mathbb{N} \longrightarrow \mathbb{R}_+$ be a function such that $\theta(n) = o(n^{1/4})$ as $n \to \infty$. Then, with a probability tending to 1 as $n \to \infty$, there exists no injective cycle C of the map M_n with length $\ell(C) \leq \theta(n)$ such that the set of vertices that lie in either connected component of $\mathbb{S}^2 \setminus \mathcal{R}(C)$ has diameter at least $\delta n^{1/4}$.*

Notice that the diameter of the map M_n is of order $n^{1/4}$ by Theorem 1.1 (see also Theorem 3 in [MaMi] or Theorem 2.5 in [We]). So Corollary 1.2 says that with a probability close to one when $n \to \infty$, we cannot find small "bottlenecks" in the map M_n such that both sides of the bottleneck have a diameter which is also of order $n^{1/4}$.

We refer to the introduction of [L2] for a detailed discussion of the recent work about asymptotics for random planar maps. The idea of studying the scaling limit of random quadrangulations appeared in Chassaing and Schaeffer [CS]. This paper made an extensive use of bijections between quadrangulations and trees, which have been extended to very general planar maps by Bouttier, Di Francesco and Guitter [BDG]. Marckert and Mokkadem [MaMo] proved that in a weak sense the scaling limit of random quadrangulations is given by the so-called Brownian map, which is essentially the same object as the quotient of the CRT that was mentioned above (see also [MaMi] for related work on more general planar maps). Planar maps play an important role in theoretical physics. See the pioneering paper [BIPZ] for the relation between enumeration problems for maps and the evaluation of matrix integrals. Bouttier's thesis [B] gives an overview of the connections between planar maps and statistical physics.

As a final remark, it is very likely that Theorem 1.1 can be extended to more general random planar maps, in particular to the Boltzmann distributions on bipartite planar maps which are discussed in [MaMi]. The recent work of Miermont [Mi] also suggests that similar results should hold for random triangulations.

The paper is organized as follows. Section 2 introduces the $\mathbb{R}$-tree $\mathcal{T}_g$ coded by a continuous function g on the circle, and associates with this tree a geodesic lamination L_g of the disk. Moore's theorem is used in the proof of Proposition 2.4 to verify that certain quotients of $\mathcal{T}_g$ are homeomorphic to the sphere $\mathbb{S}^2$. In addition, Section 2 gives a few properties of the lamination L_g, and in particular computes its Hausdorff dimension under suitable assumptions on the function g (Proposition 2.3). In the particular case when g is the normalized Brownian excursion, one recovers the value $3/2$ which was given in [A3] (see Proposition 3.4 below). Section 3 contains the proof of our main results. The key step is to verify that any weak limit in Theorem 1.1 can be written in the form of a quotient space which satisfies the assumptions needed to apply Proposition 2.4. The verification of

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these assumptions requires two technical lemmas, whose proofs are postponed to Section 4. The path-valued random process called the Brownian snake plays an important role in these proofs.

2 Trees and geodesic laminations

In this section, we deal with various quotient spaces. Let E be a topological space, and let $\sim$ be an equivalence relation on E. Unless otherwise stated, the quotient space $E/\sim$ will always be equipped with the quotient topology, which is the finest topology on $E/\sim$ such that the canonical projection $E \longrightarrow E/\sim$ is continuous (see for instance [Bou]). The equivalence relation $\sim$ is said to be closed if its graph $\{(x,y) \in E \times E \ : \ x \sim y\}$ is a closed subset of $E \times E$. We use several times the following simple fact: If E is a compact metric space and $\sim$ is closed, then the quotient space $E/\sim$ is a Hausdorff space, and is therefore compact, as the image of E under the canonical projection.

Let $\mathbb{S}^1$ be the unit circle in the complex plane $\mathbb{C}$. If $a, b \in \mathbb{S}^1$ and $a \neq b$, we denote by $[a, b]$ the closed arc in $\mathbb{S}^1$ going from a to b in the counterclockwise order. Similarly, $]a, b[$ denotes the corresponding open arc. By convention, $[a, a] = \{a\}$ and $]a, a[= \emptyset$.

Let $g : \mathbb{S}^1 \longrightarrow \mathbb{R}$ be a continuous function. For every $a, b \in \mathbb{S}^1$, we set

$$m_g(a, b) = \max \left\{ \min_{c \in [a,b]} g(c), \min_{c \in [b,a]} g(c) \right\} \ ,$$

and

$$d_g(a, b) = g(a) + g(b) - 2m_g(a, b) \ .$$

Since $m_g(a, b) \leq \min\{g(a), g(b)\}$, we have $d_g(a, b) = 0$ if and only if $g(a) = g(b) = m_g(a, b)$. We define an equivalence relation $\sim_g$ on $\mathbb{S}^1$ by setting $a \sim_g b$ if and only if $d_g(a, b) = 0$. Since g is continuous, this equivalence relation is closed.

Then d_g induces a metric, still denoted by d_g, on the quotient space $\mathcal{T}_g = \mathbb{S}^1/\sim_g$. Furthermore, $\mathcal{T}_g$ equipped with this metric is a compact $\mathbb{R}$-tree. See Theorem 2.1 in [DL], which deals with a slightly different but equivalent setting. It is also easy to verify that the topology of the metric space $(\mathcal{T}_g, d_g)$ coincides with the quotient topology. Indeed, the canonical projection $\mathbb{S}^1 \longrightarrow \mathcal{T}_g$ is continuous when $\mathcal{T}_g$ is equipped with the metric d_g, hence induces a continuous bijection from $\mathcal{T}_g$ endowed with the quotient topology, onto $\mathcal{T}_g$ endowed with the topology induced by the metric d_g. As $\mathcal{T}_g$ is compact for both topologies, the desired result follows.

From now on, we make the following additional assumption on g.

$$(H_g) \qquad \text{Local minima of } g \text{ are distinct.}$$

This means that if $]a, b[$ and $]c, d[$ are two disjoint open arcs in $\mathbb{S}^1$, and if the lower bound of the values of g over $]a, b[$, respectively over $]c, d[$, is attained at a point of $]a, b[$, respectively at a point of $]c, d[$, then

$$\min_{x \in [a,b]} g(x) \neq \min_{x \in [c,d]} g(x) .$$

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and let $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ be the closed disk. We equip $\mathbb{D}$ with the usual hyperbolic metric and for every $a, b \in \mathbb{S}^1$ with $a \neq b$, we denote by ab the (hyperbolic) geodesic line joining a to b in $\mathbb{D}$. We also denote by $\overline{ab}$ the union of ab and of the points a and b. By convention, $aa = \emptyset$ and $\overline{aa} = \{a\}$. We then let L_g be the union of the geodesic lines ab for all pairs $\{a, b\}$ of distinct points of $\mathbb{S}^1$ such that $a \sim_g b$.

Recall that a *(hyperbolic) geodesic lamination* in $\mathbb{D}$ is a closed subset of $\mathbb{D}$ which is the union of a collection of pairwise disjoint geodesic lines. A geodesic lamination is said to be *maximal* if it is maximal for the inclusion relation among geodesic laminations. As a general reference about geodesic laminations, we will use [Bon] and the references therein.

Proposition 2.1 *Under Assumption (H_g), the set L_g is a maximal geodesic lamination of the hyperbolic disk $\mathbb{D}$.*

Proof. An elementary argument shows that, under assumption (H_g), equivalence classes for $\sim_g$ can have at most three points. Then, let $\{a, b\}$ and $\{c, d\}$ be two pairs of distinct points in $\mathbb{S}^1$ such that $a \sim_g b$ and $c \sim_g d$. We claim that either the open arcs $]a, b[$ and $]c, d[$ are disjoint, or one of them is contained in the other one. Indeed, if this were not the case, then it would follow from the definition of d_g that the four points a, b, c, d are distinct and equivalent for $\sim_g$, which contradicts the first observation of the proof. We conclude that the geodesic lines ab and cd are disjoint, or coincide if $\{a, b\} = \{c, d\}$. Hence L_g is a disjoint union of geodesic lines.

As the equivalence relation $\sim_g$ is closed, its graph is compact in $\mathbb{S}^1 \times \mathbb{S}^1$. It immediately follows that L_g is a closed subset of the hyperbolic disk.

It remains to verify that L_g is maximal. To this end, we argue by contradiction. Let a and b be two distinct points in $\mathbb{S}^1$, and suppose that ab does not intersect L_g. Without loss of generality, we may assume that

$$m := \min_{x \in [a,b]} g(x) \geq \min_{x \in [b,a]} g(x) .$$

Suppose first that $g(a) > m$, and let $\varepsilon \in]0, g(a) - m[$. By the continuity of g, we can find $c \in]a, b[$ such that $g(c) = g(a) - \varepsilon$ and $g(c) = \min\{g(x) : x \in [a, c]\}$. Similarly we can find $d \in]b, a[$ such that $g(d) = g(a) - \varepsilon$ and $g(d) = \min\{g(x) : x \in [d, a]\}$. We have then $c \sim_g d$, and the geodesic line cd intersects ab, which contradicts our initial assumption that ab does not intersect L_g. We conclude that

$$g(a) = \min_{x \in [a,b]} g(x) ,$$

and similarly we have

$$g(b) = \min_{x \in [a,b]} g(x) .$$

It follows that $a \sim_g b$, which is again a contradiction. $\square$

Since L_g is a maximal geodesic lamination, we know (see for instance [Bon]) that every connected component of $\mathbb{D} \setminus L_g$ is an ideal hyperbolic triangle. Clearly, these connected components are in one-to-one correspondence with triples $\{a, b, c\}$ of distinct points in $\mathbb{S}^1$ such that $a \sim_g b \sim_g c$.

We can extend the equivalence relation $\sim_g$ to $\overline{\mathbb{D}}$ as follows. If $x, y \in \overline{\mathbb{D}}$ and $x \neq y$, we put $x \sim_g y$ if and only if x and y belong to the same arc $\overline{ab}$ with $a \sim_g b$, or if x and y belong to the closure of the same ideal geodesic triangle which is a connected component of $\mathbb{D} \setminus L_g$. In order to verify that this extension is still an equivalence relation, we observe that a given geodesic line ab cannot be contained in the boundary of two distinct components of $\mathbb{D} \setminus L_g$. This again follows from the fact that equivalence classes for $\sim_g$ contain at most three points of $\mathbb{S}^1$. For the extended equivalence relation on $\overline{\mathbb{D}}$, equivalence classes are of three possible types, either singletons $\{a\}$ for certain values of $a \in \mathbb{S}^1$, or arcs $\overline{ab}$ for $a, b \in \mathbb{S}^1$, $a \neq b$ and $a \sim_g b$, or closures of ideal hyperbolic triangles with vertices a, b, c in $\mathbb{S}^1$ such that $a \sim_g b \sim_g c$.

By the preceding remarks, the inclusion map $\mathbb{S}^1 \longrightarrow \overline{\mathbb{D}}$ induces a bijection $\mathbb{S}^1 / \sim_g \longrightarrow \overline{\mathbb{D}} / \sim_g$, and we use this bijection to identify these two sets. Note that this identification is also an homeomorphism. Indeed, the inclusion map $\mathbb{S}^1 \longrightarrow \overline{\mathbb{D}}$ is continuous and both $\mathbb{S}^1 / \sim_g$ and $\overline{\mathbb{D}} / \sim_g$ are compact (note that the equivalence relation $\sim_g$ on $\overline{\mathbb{D}}$ is also closed).

The following two propositions are not used in the proofs of our main results. Still they contain useful information about the geodesic lamination L_g. Before stating these results, we recall that geodesic laminations are fundamental objects in low-dimensional topology and geometry, and that transversally measured geodesic laminations are of particular importance as illustrated by the work of Thurston [Thu] or the recent paper [BCM]. Transversally measured geodesic laminations have a well-known close relationship with $\mathbb{R}$-trees (cf Morgan and Shalen [MS]), which indeed motivated much of the initial interest for $\mathbb{R}$-trees. The first proposition below answers the basic question of the existence of a transverse measure on the geodesic lamination L_g, and identifies the associated $\mathbb{R}$-tree. We refer for instance to [Bon, page 12] for the definition of a transverse measure on a geodesic lamination, and to [LP, page 84] for the definition of its space of leaves made Hausdorff.

Proposition 2.2 *Under Assumption (H_g), the geodesic lamination L_g carries a natural transverse measure μ, whose support is L_g, such that the space of leaves made Hausdorff of (L_g, μ) is an $\mathbb{R}$-tree whose completion is isometric to the $\mathbb{R}$-tree $(\mathcal{T}_g, d_g)$. Furthermore, if the times of local minima of g are dense in $\mathbb{S}^1$, then L_g has empty interior.*

Proof. Let $\pi : \mathbb{D} \longrightarrow \overline{\mathbb{D}} / \sim_g$ be the composition of the inclusion map $\mathbb{D} \longrightarrow \overline{\mathbb{D}}$ with the canonical projection $\overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}} / \sim_g$. Consider in $\mathbb{D}$ a non trivial (hyperbolic) geodesic segment $[u, v]$, with $u, v \in \mathbb{D}$, and assume that this segment is transverse to L_g. As a geodesic line in $\mathbb{D}$, that does not contain $[u, v]$, cuts (transversely) $[u, v]$ at one point at most, the restriction of the map π to $L_g \cap [u, v]$ is continuous and injective, except that

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the endpoints of a connected component of $[u, v] \setminus L_g$ are mapped to the same point. In particular, the image of this restriction is the geodesic segment in $\mathcal{T}_g$ between $\pi(u)$ and $\pi(v)$. Denote by λ the Lebesgue measure on this segment, which is isometrically identified with an interval of the real line. Since λ has no atom, there exists a unique finite measure $\mu_{[u,v]}$ on $[u, v]$, which is supported on $L_g \cap [u, v]$, such that the image measure of $\mu_{[u,v]}$ under π is λ. As the support of λ is $[\pi(u), \pi(v)]$, it follows that the support of $\mu_{[u,v]}$ is exactly $L_g \cap [u, v]$. By construction, it is easy to check that the transverse measure $\mu = (\mu_{[u,v]})_{[u,v]}$ is invariant by holonomy along the leaves of L_g. Hence (L_g, μ) is a transversely measured geodesic lamination of $\mathbb{D}$ (see [FLP, Bon]).

Now consider the pseudo-distance $\widetilde{d}$ on $\mathbb{D}$, where $\widetilde{d}(u, v)$ is defined as the lower bound over all piecewise transverse paths γ from u to v of the total mass placed on γ by the transverse measure μ. Then the leaf space made Hausdorff $T_{L_g, \mu}$ of (L_g, μ) is by definition the quotient metric space of $(\mathbb{D}, \widetilde{d})$ (obtained by identifying u and v if and only if $\widetilde{d}(u, v) = 0$), which is an $\mathbb{R}$-tree (see [MO, GS]).

Note that for every u and v in $\mathbb{D}$, if the geodesic segment $[u, v]$ is transverse to L_g, then $\widetilde{d}(u, v) = d_g(\pi(u), \pi(v))$, as any piecewise transverse path from u to v has transverse measure at least the transverse measure of $[u, v]$, by standard arguments. Hence the map π induces an isometric embedding from $T_{L_g, \mu}$ into $(\mathcal{T}_g, d_g)$. As $\mathbb{D}$ is dense in $\overline{\mathbb{D}}$, the image of this embedding is dense. As $(\mathcal{T}_g, d_g)$ is compact, it is hence (isometric to) the completion of $T_{L_g, \mu}$.

Suppose that the times of local minima of g are dense. Let $a, b \in \mathbb{S}^1$ be two distinct points such that $a \sim_g b$. Without loss of generality, assume that $\min_{x \in [a,b]} g(x) \geq \min_{x \in [b,a]} g(x)$. We argue by contradiction and assume that there is a point of the geodesic line ab that lies in the interior of L_g. Then we can find two points c and c' of $]a, b[$ such that the arcs $[a, c]$ and $[c', b]$ are disjoint, and every point $x \in [a, c]$ is an endpoint of a geodesic line of L_g whose other endpoint $\phi(x)$ belongs to $[c', b]$ (we use the fact that a point of $\mathbb{S}^1$ can be the endpoint of at most two geodesics of L_g). Let d be a point of local minimum of g that belongs to $]a, c[$. Since $g(d) \geq g(a)$, there exists $e \in [a, d] \setminus \{d\}$ such that $d \sim_g e$. But then the geodesic line de intersects all the geodesic lines $x\phi(x)$ for $x \in]e, d[$, which is a contradiction. $\qquad \square$

The Hausdorff dimension of geodesic laminations on compact connected hyperbolic surfaces has been much studied: See in particular Birman and Series [BS]. It is natural to compare fractal properties of the lifts to the disk (by a universal covering map) of geodesic laminations on these surfaces with those of random geodesic laminations on the disk. The next proposition, which gives the Hausdorff dimension of L_g under suitable assumptions on the function g, will allow us to derive the dimension of the random geodesic laminations that are considered in the next section.

If A is a subset of the closed disk $\overline{\mathbb{D}}$ equipped with the usual Euclidean distance, we denote by $\dim(A)$ the Hausdorff dimension of A, and by $\underline{\dim}_M(A)$ the lower Minkowski dimension of A (also called the lower box-counting dimension, see for instance [Mat, page 77]). Recall that $\dim(A) \leq \underline{\dim}_M(A)$.

Let A_g denote the set of all $x \in \mathbb{S}^1$ such that the equivalence class of x under $\sim_g$ is not

a singleton. We also let $\mathcal{J}$ be the countable set of all (ordered) pairs (I, J) where I and J are two disjoint closed subarcs of $\mathbb{S}^1$ with nonempty interior and rational endpoints. If $(I, J) \in \mathcal{J}$, we denote by $A_g^{(I,J)}$ the set of all $x \in I$ such that $x \sim_g y$ for some $y \in J$. Plainly,

$$A_g = \bigcup_{(I,J) \in \mathcal{J}} A_g^{(I,J)} . \tag{1}$$

Proposition 2.3 (i) *We have*

$$\dim(L_g) \geq 1 + \dim(A_g) .$$

(ii) *Assume that* $\underline{\dim}_M(A_g^{(I,J)} \cup A_g^{(J,I)}) \leq \dim(A_g)$ *for every* $(I, J) \in \mathcal{J}$. *Then,*

$$\dim(L_g) = 1 + \dim(A_g) .$$

Proof. (i) We assume that $\dim(A_g) > 0$, because otherwise the result is easy. Let $\alpha \in]0, \dim(A_g)[$. By (1), we can find a pair $(I, J) \in \mathcal{J}$ such that $\dim(A_g^{(I,J)}) > \alpha$. Since $A_g^{(I,J)}$ is a compact subset of $\mathbb{S}^1$, Frostman's lemma [Mat, page 112] yields the existence of a nontrivial finite Borel measure ν supported on $A_g^{(I,J)}$ such that

$$\nu(B(x,r)) \leq r^\alpha \tag{2}$$

for every $r > 0$ and $x \in \mathbb{S}^1$. Here $B(x, r)$ denotes the (Euclidean) disk of radius r centered at x. Let $\widetilde{A}_g^{(I,J)}$ be the subset of $A_g^{(I,J)}$ consisting of points x such that the equivalence class of x under $\sim_g$ contains exactly two points, and for every $x \in \widetilde{A}_g^{(I,J)}$, let $s_g(x)$ be the unique element of J such that $x \sim_g s_g(x)$. Notice that $A_g^{(I,J)} \backslash \widetilde{A}_g^{(I,J)}$ is countable, and so ν is supported on $\widetilde{A}_g^{(I,J)}$. For every $x \in \widetilde{A}_g^{(I,J)}$, let λ_x denote the one-dimensional Hausdorff measure on the arc $x s_g(x)$ (equipped with the Euclidean distance). Define a finite Borel measure Λ by setting for every Borel subset B of the plane

$$\Lambda(B) = \int \nu(dx) \int \lambda_x(dz)\, \mathbf{1}_B(z).$$

By construction, Λ is supported on L_g. Then fix $R \in]0, 1[$ such that $\Lambda(B(0, R)) > 0$. Let $z_0 \in L_g$ be such that $|z_0| \leq R$, and choose $x_0, y_0 \in A_g$ such that $z_0 \in x_0 y_0$. Let $\varepsilon \in]0, 1]$. A simple geometric argument shows that the conditions $x \in \widetilde{A}_g^{(I,J)}$ and $x s_g(x) \cap B(z_0, \varepsilon) \neq \emptyset$ imply $|x - x_0| \leq C\varepsilon$, where the constant C only depends on R. Hence, using (2),

$$\Lambda(B(z_0, \varepsilon)) = \int_{\{|x-x_0| \leq C\varepsilon\}} \nu(dx)\, \lambda_x(B(z_0, \varepsilon)) \leq C' \varepsilon^{1+\alpha}$$

where the constant C' does not depend on ε nor on z_0. Frostman's lemma now gives $\dim(L_g) \geq 1 + \alpha$ as desired.

(ii) We now prove that $\dim(L_g) \leq 1 + \dim(A_g)$ under the assumption in (ii). For $(I, J) \in \mathcal{J}$, let $F_g^{(I,J)}$ be the union of all geodesic lines xy for $x \in I$, $y \in J$ and $x \sim_g y$. It is enough to prove that $\dim(F_g^{(I,J)}) \leq 1 + \dim(A_g)$ for a fixed choice of $(I, J) \in \mathcal{J}$.

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Let $\beta > \dim(A_g)$. By the assumption in (ii), we can find a sequence $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ decreasing to 0, such that the following holds for every ε belonging to this sequence. There exist a positive integer $M(\varepsilon) \leq \varepsilon^{-\beta}$ and $M(\varepsilon)$ disjoint subarcs $I_1, I_2, \ldots, I_{M(\varepsilon)}$ of $\mathbb{S}^1$, with length less than ε, such that $A_g^{(I,J)}$ is contained in $I_1 \cup \cdots \cup I_{M(\varepsilon)}$. Similarly, we can find a positive integer $N(\varepsilon) \leq \varepsilon^{-\beta}$ and $N(\varepsilon)$ disjoint subarcs $J_1, J_2, \ldots, J_{N(\varepsilon)}$, with length less than ε, such that $A_g^{(J,I)}$ is contained in $J_1 \cup \cdots \cup J_{N(\varepsilon)}$. Then, let H be the set of all pairs $(i,j) \in \{1, \ldots, M(\varepsilon)\} \times \{1, \ldots, N(\varepsilon)\}$ such that there exists a geodesic line $xy \subset L_g$ with $x \in I_i$ and $y \in J_j$. Because geodesic lines in L_g are not allowed to cross, a simple argument shows that $\#(H) \leq M(\varepsilon) + N(\varepsilon) \leq 2\varepsilon^{-\beta}$. It easily follows that the two-dimensional Lebesgue measure of the Euclidean tubular neighborhood of $F_g^{(I,J)}$ with radius ε is bounded above by $C\varepsilon^{1-\beta}$, where the constant C does not depend on ε in our sequence. This implies [Mat, page 79] that $\underline{\dim}_M(F_g^{(I,J)}) \leq 1 + \beta$, and a fortiori $\dim(F_g^{(I,J)}) \leq 1 + \beta$. $\hfill\square$

The next proposition will be an essential ingredient in the proof of our main result Theorem 1.1. We let $h : \mathbb{S}^1 \longrightarrow \mathbb{R}$ be another continuous function. We again assume that local minima of h are distinct, i.e. that (H_h) holds. Furthermore, we assume that the following condition holds.

$(H'_{g,h})$ Let a, b, c be three points in $\mathbb{S}^1$ such that $a \sim_g b$ and $a \sim_h c$.
Then $a = b$ or $a = c$.

In other words, if the equivalence class of $a \in \mathbb{S}^1$ with respect to $\sim_g$ is not a singleton, then its equivalence class with respect to $\sim_h$ must be a singleton.

We can define an equivalence relation, which we still denote by $\sim_h$, on the quotient $\mathcal{T}_g = \mathbb{S}^1/\sim_g$ by declaring for $\alpha, \beta \in \mathcal{T}_g$ that $\alpha \sim_h \beta$ if and only if there exists a representative a of α in $\mathbb{S}^1$, respectively a representative b of β in $\mathbb{S}^1$, such that $a \sim_h b$. Note that our assumption $(H'_{g,h})$ is used to verify that this prescription defines an equivalence relation on $\mathcal{T}_g$.

Proposition 2.4 *Under Assumptions (H_g), (H_h) and $(H'_{g,h})$, the quotient space $\mathcal{T}_g/\sim_h$ is homeomorphic to the sphere $\mathbb{S}^2$.*

Proof. We embed the complex plane into $\mathbb{R}^3$ by identifying it with the horizontal plane $\{x_3 = 0\}$. We write $H_+ = \{x = (x_1, x_2, x_3) \in \mathbb{S}^2 \ : \ x_3 \geq 0\}$ for the (closed) upper hemisphere, and similarly $H_- = \{x = (x_1, x_2, x_3) \in \mathbb{S}^2 \ : \ x_3 \leq 0\}$ for the lower hemisphere. We can use the stereographic projection from the South pole to identify (topologically) H_+ with the closed unit disk $\overline{\mathbb{D}}$. Thanks to this identification, we may define the equivalence relation $\sim_g$ on H_+, and by previous observations, the quotient space $H_+/\sim_g$ is homeomorphic to $\mathcal{T}_g$. Similarly, we can use the stereographic projection from the North pole to identify H_- with $\overline{\mathbb{D}}$, and then define the equivalence relation $\sim_h$ on H_-.

Let $\sim$ be the equivalence relation on $\mathbb{S}^2$ whose graph is the union of the graphs of $\sim_g$ and $\sim_h$ viewed as equivalence relations on H_+ and H_- respectively. Note that $(H'_{g,h})$ is

used to verify that $\sim$ is an equivalence relation on $\mathbb{S}^2$. Any equivalence class for $\sim$ is an equivalence class for $\sim_g$, or an equivalence class for $\sim_h$. It may be both if and only if it is a singleton. As a consequence, any equivalence class of $\sim$ is a compact path-connected subset of $\mathbb{S}^2$ whose complement is also connected. Furthermore, as $\sim_g$ and $\sim_h$ are both closed, it follows that the equivalence relation $\sim$ is closed.

At this point, we use the following theorem of Moore [Moo, page 416] (see also [Thu] for a previous application of this theorem, and in particular Figure 10, page 376, of [Thu]).

Theorem 2.5 (Moore) *Let $\sim$ be a closed equivalence relation on $\mathbb{S}^2$. Assume that every equivalence class of $\sim$ is a compact path-connected subset of the sphere whose complement is connected. Then the quotient space $\mathbb{S}^2/\sim$ is homeomorphic to $\mathbb{S}^2$.*

Clearly Moore's theorem applies to our setting, and we get that the quotient $\mathbb{S}^2/\sim$ is homeomorphic to $\mathbb{S}^2$.

To complete the proof of Proposition 2.4, it remains to verify that $\mathcal{T}_g/\sim_h$ is homeomorphic to $\mathbb{S}^2/\sim$. We first observe that $\mathcal{T}_g/\sim_h$ is compact. Indeed, $\sim_h$ viewed as an equivalence relation on $\mathcal{T}_g$ is closed, as $\sim_h$ is closed on $\mathbb{S}^1$ and the canonical projection $\mathbb{S}^1 \longrightarrow \mathcal{T}_g$ is a closed map. Then, by composing the inclusion map $\mathbb{S}^1 \longrightarrow \mathbb{S}^2$ with the projection $\mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$, we get a continuous mapping, which factorizes through the equivalence relation $\sim_g$ and thus yields a continuous mapping from $\mathcal{T}_g$ onto $\mathbb{S}^2/\sim$. Again, this mapping factorizes through the equivalence relation $\sim_h$ and we obtain that the canonical bijection from $\mathcal{T}_g/\sim_h$ onto $\mathbb{S}^2/\sim$ is continuous. Since both $\mathcal{T}_g/\sim_h$ and $\mathbb{S}^2/\sim$ are compact, this bijection is a homeomorphism. $\qquad\square$

Remark. Assumption $(H'_{g,h})$ in Proposition 2.4 can be weakened. The application of Moore's theorem is possible under less stringent assumptions.

3   Proof of the main result

In this section, we prove Theorem 1.1 and Corollary 1.2. On a given probability space, we consider a normalized Brownian excursion $(\mathbf{e}_t)_{0\le t\le 1}$ and a process $(Z_t)_{0\le t\le 1}$ which is distributed as the head of the one-dimensional Brownian snake driven by $\mathbf{e}$. This means that the process Z has continuous sample paths and that, conditionally given $\mathbf{e}$, it is a centered real-valued Gaussian process with (conditional) covariance function

$$E[Z_s Z_t \mid \mathbf{e}] = \min_{s\wedge t\le u\le s\vee t} \mathbf{e}_u \tag{2}$$

for every $s, t \in [0, 1]$. See Section 4 below for more information about Z and the Brownian snake. Notice that $\mathbf{e}_0 = \mathbf{e}_1 = 0$ and $Z_0 = Z_1 = 0$, a.s.

We also need to introduce the pair $(\mathbf{e}, Z)$ "re-rooted at the minimal spatial position". Set

$$\underline{Z} = \min_{0\le s\le 1} Z_s$$

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and let s_* be the almost surely unique time such that $Z_{s_*} = \underline{Z}$ (the uniqueness of s_* follows from Proposition 2.5 in [LW], and is also a consequence of Lemma 3.1 below). For every $s, t \in [0, 1]$, set $s \oplus t = s + t$ if $s + t \leq 1$ and $s \oplus t = s + t - 1$ if $s + t > 1$. Then, for every $t \in [0, 1]$, define

- $\overline{\mathbf{e}}_t = \mathbf{e}_{s_*} + \mathbf{e}_{s_* \oplus t} - 2 \min_{s_* \wedge (s_* \oplus t) \leq r \leq s_* \vee (s_* \oplus t)} \mathbf{e}_r$;

- $\overline{Z}_t = Z_{s_* \oplus t} - Z_{s_*}$.

Note again that $\overline{\mathbf{e}}_0 = \overline{\mathbf{e}}_1 = 0$ and $\overline{Z}_0 = \overline{Z}_1 = 0$ a.s. The pair $(\overline{\mathbf{e}}, \overline{Z})$ can be interpreted as the pair $(\mathbf{e}, Z)$ conditioned on the event $\{\underline{Z} = 0\}$ (see [LW]).

In view of applying the results of Section 2, it will be convenient to view the random functions $\mathbf{e}, Z, \overline{\mathbf{e}}$ and $\overline{Z}$ as parametrized by the circle $\mathbb{S}^1$ rather than by the interval $[0, 1]$. This is of course easily achieved by setting, for instance,

$$\mathbf{e}(e^{2i\pi r}) = \mathbf{e}_r , \quad r \in [0, 1].$$

Then Assumption $(H_\mathbf{e})$ holds a.s. This follows from the well-known analogous result for linear Brownian motion, which is a very easy application of the Markov property.

Lemma 3.1 *Assumption (H_Z) holds almost surely. In other words, local minima of Z are distinct, with probability one.*

Lemma 3.2 *Asumption $(H'_{\mathbf{e}, Z})$ holds almost surely. In other words, it is almost surely true that, for every $a, b, c \in \mathbb{S}^1$, the conditions $a \sim_\mathbf{e} b$ and $a \sim_Z c$ imply that $a = b$ or $a = c$.*

We postpone the proof of these two lemmas to Section 4. Thanks to these lemmas, we can apply the results of Section 2 to the pair $(\mathbf{e}, Z)$. In particular, we can consider the quotient space $\mathcal{T}_\mathbf{e} / \sim_Z$ and we know from Proposition 2.4 that this quotient space is almost surely homeomorphic to the sphere $\mathbb{S}^2$.

In order to complete the proof of Theorem 1.1, it will be sufficient to verify that the (random) metric space that appears in [L2] as the weak limit of rescaled random maps is a.s. homeomorphic to $\mathcal{T}_\mathbf{e} / \sim_Z$. We first need to recall the topological description of this limiting random metric space that is given in [L2].

We start by observing that outside a set of probability zero, the value of Z_a for $a \in \mathbb{S}^1$, respectively the value of $\overline{Z}_a$, only depends on the equivalence class of a in $\mathcal{T}_\mathbf{e}$, respectively in $\mathcal{T}_{\overline{\mathbf{e}}}$: This essentially follows from the form of the covariance function in (2), see Section 2.4 in [L2]. Thanks to this observation, we may and will sometimes view Z, respectively $\overline{Z}$, as parametrized by $\mathcal{T}_\mathbf{e}$, respectively $\mathcal{T}_{\overline{\mathbf{e}}}$.

Let us denote by $p_\mathbf{e}$, respectively $p_{\overline{\mathbf{e}}}$, the canonical projection from $\mathbb{S}^1$ onto $\mathcal{T}_\mathbf{e}$, respectively onto $\mathcal{T}_{\overline{\mathbf{e}}}$. If $\alpha, \beta \in \mathcal{T}_\mathbf{e}$, we denote by $[\alpha, \beta]$ the image under $p_\mathbf{e}$ of the smallest arc

$[a, b]$ in $\mathbb{S}^1$ such that $p_{\mathbf{e}}(\alpha) = a$ and $p_{\mathbf{e}}(\beta) = b$. We similarly define $[\alpha, \beta]$ when $a, \beta \in \mathcal{T}_{\overline{\mathbf{e}}}$. Then, for every $\alpha, \beta \in \mathcal{T}_{\overline{\mathbf{e}}}$, we set

$$D^{\circ}(\alpha, \beta) = \overline{Z}_{\alpha} + \overline{Z}_{\beta} - 2 \max \left(\min_{\gamma \in [\alpha, \beta]} \overline{Z}_{\gamma}, \min_{\gamma \in [\beta, \alpha]} \overline{Z}_{\gamma} \right)$$

and

$$D^{*}(\alpha, \beta) = \inf \left\{ \sum_{i=1}^{p} D^{\circ}(\alpha_{i-1}, \alpha_i) \right\}$$

where the last lower bound is over all choices of the integer $p \geq 1$ and of the finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_p$ in $\mathcal{T}_{\overline{\mathbf{e}}}$, such that $\alpha_0 = \alpha$ and $\alpha_p = \beta$. We set $\alpha \approx \beta$ if and only if $D^{*}(\alpha, \beta) = 0$. According to Theorem 3.4 in [L2], it is almost surely true that, for every $\alpha, \beta \in \mathcal{T}_{\overline{\mathbf{e}}}$, the condition $D^{*}(\alpha, \beta) = 0$ holds if and only if $D^{\circ}(\alpha, \beta) = 0$.

Recall the notation introduced in Section 1. According to the same theorem of [L2] and Remark (a) following it, any weak limit of the sequence $(\mathbf{m}_n, n^{-1/4} d_n)$ is a.s. homeomorphic to the quotient space $\mathcal{T}_{\overline{\mathbf{e}}} / \approx$ equipped with the metric induced by D^{*}, which is still denoted by D^{*}. Thus Theorem 1.1 follows from the next proposition.

Proposition 3.3 *The metric space $(\mathcal{T}_{\overline{\mathbf{e}}} / \approx, D^{*})$ is almost surely homeomorphic to the quotient space $\mathcal{T}_{\mathbf{e}} / \sim_Z$.*

Proof. We first construct a (canonical) bijection between $\mathcal{T}_{\overline{\mathbf{e}}} / \approx$ and $\mathcal{T}_{\mathbf{e}} / \sim_Z$ and then verify that this bijection is a homeomorphism. Let $\rho : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the rotation with angle $2\pi s_{*}$. According to the re-rooting lemma (Lemma 2.2 in [DL]), ρ induces an isometry R from $(\mathcal{T}_{\overline{\mathbf{e}}}, d_{\overline{\mathbf{e}}})$ onto $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$. Furthermore, for every $\alpha \in \mathcal{T}_{\mathbf{e}}$,

$$\overline{Z}_{\alpha} = Z_{R(\alpha)} - \underline{Z} \,. \tag{3}$$

Now recall that almost surely, for every $\alpha, \beta \in \mathcal{T}_{\overline{\mathbf{e}}}$, the relation $\alpha \approx \beta$ holds if and only if $D^{\circ}(\alpha, \beta) = 0$, or equivalently

$$\overline{Z}_{\alpha} = \overline{Z}_{\beta} = \max \left(\min_{\gamma \in [\alpha, \beta]} \overline{Z}_{\gamma}, \min_{\gamma \in [\beta, \alpha]} \overline{Z}_{\gamma} \right) \,.$$

From our definitions and the identity (3), this is immediately seen to be equivalent to

$$Z_{R(\alpha)} = Z_{R(\beta)} = \max \left(\min_{\gamma \in [R(\alpha), R(\beta)]} Z_{\gamma}, \min_{\gamma \in [R(\beta), R(\alpha)]} Z_{\gamma} \right) \,,$$

that is to $R(\alpha) \sim_Z R(\beta)$.

Thus R induces a bijection, which we denote by $\widetilde{R}$, from $\mathcal{T}_{\overline{\mathbf{e}}} / \approx$ onto $\mathcal{T}_{\mathbf{e}} / \sim_Z$. To prove that $\widetilde{R}$ is a homeomorphism, it is enough to verify that $\widetilde{R}^{-1}$ is continuous, since both $\mathcal{T}_{\overline{\mathbf{e}}} / \approx$ (equipped with the metric D^{*}) and $\mathcal{T}_{\mathbf{e}} / \sim_Z$ are compact. The canonical projection from $\mathcal{T}_{\overline{\mathbf{e}}}$ onto $(\mathcal{T}_{\overline{\mathbf{e}}} / \approx, D^{*})$ is continuous: Using the continuity of the mapping $\mathbb{S}^1 \ni a \to \overline{Z}_a$, a direct inspection of the definition of D° shows that if α_n tends to α in $\mathcal{T}_{\overline{\mathbf{e}}}$ then $D^{\circ}(\alpha_n, \alpha)$

tends to 0 and a fortiori $D^*(\alpha_n, \alpha)$ tends to 0 as $n \to \infty$. By composing the isometry R^{-1} from $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ onto $(\mathcal{T}_{\overline{\mathbf{e}}}, d_{\overline{\mathbf{e}}})$ with the previous projection, we get a continuous mapping from $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ onto $(\mathcal{T}_{\overline{\mathbf{e}}}/\approx, D^*)$, which in turn induces a continuous mapping from the space $\mathcal{T}_\mathbf{e}/\sim_Z$, equipped with the quotient topology, onto $(\mathcal{T}_{\overline{\mathbf{e}}}/\approx, D^*)$. The latter mapping is just $\widetilde{R}^{-1}$, and so we have obtained that $\widetilde{R}^{-1}$ is continuous, which completes the proof. $\square$

Proof of Corollary 1.2. For every integer $n \geq 2$, denote by A_n the event consisting of all ω's in our underlying probability space such that there exists an injective cycle of the map $M_n(\omega)$ satisfying the properties stated in the corollary, with a fixed choice of $\delta > 0$ and the function $\theta(n)$. We argue by contradiction, assuming that $P(A_n)$ does not converge to 0. Then we can find $\eta > 0$ and a sequence (n_k) converging to $+\infty$ such that $P(A_{n_k}) \geq \eta$ for every k. From now on, we restrict our attention to values of n belonging to this sequence. By extracting another subsequence if necessary, we can also assume that $(\mathbf{m}_n, n^{-1/4} d_n)$ converges in distribution along this sequence. The convergence in distribution can be replaced by an almost sure convergence thanks to the Skorokhod representation theorem. Thus we have almost surely

$$(\mathbf{m}_n, n^{-1/4} d_n) \longrightarrow (\mathbf{m}_\infty, d_\infty) \tag{4}$$

as $n \to \infty$, in the sense of the Hausdorff-Gromov distance. By Theorem 1.1, $(\mathbf{m}_\infty, d_\infty)$ is almost surely homeomorphic to the sphere $\mathbb{S}^2$.

From now on, we argue with a fixed value of ω in our probability space, such that $\omega \in \limsup A_n$, the convergence (4) holds and $(\mathbf{m}_\infty, d_\infty)$ is homeomorphic to $\mathbb{S}^2$ (note that the event $\limsup A_n$ has probability greater than η by the above). Let us show that this leads to a contradiction. By the definition of the events A_n, we can find a subsequence (depending on ω) such that for every n belonging to this subsequence, there exists an injective cycle C_n of the map M_n, with length $\ell(C_n) \leq \theta(n)$ and two vertices $a_n, b_n \in \mathbf{m}_n$, which are separated by the cycle C_n (in the sense that every path from a_n to b_n has to cross C_n) and such that $\min\{d_n(a_n, C_n), d_n(b_n, C_n)\} > \delta n^{1/4}/3$. Here $d_n(a_n, C_n)$ denotes as usual the minimal distance between a_n and a vertex of C_n.

Say that a map φ from a metric space (E, d) into another metric space (E', d') is an ε-isometry if $|d'(\varphi(x), \varphi(y)) - d(x, y)| \leq \varepsilon$ for every $x, y \in E$. From the convergence (4) and the definition of the Hausdorff-Gromov topology (see e.g. [BBI]), we can find a sequence $\varepsilon_n \to 0$, and ε_n-isometries $f_n : (\mathbf{m}_n, n^{-1/4} d_n) \to (\mathbf{m}_\infty, d_\infty)$ and $g_n : (\mathbf{m}_\infty, d_\infty) \to (\mathbf{m}_n, n^{-1/4} d_n)$ such that $n^{-1/4} d_n(g_n \circ f_n(x), x) \leq \epsilon_n$ for every x in $\mathbf{m}_n$. Let $a'_n = f_n(a_n), b'_n = f_n(b_n)$ and let C'_n be the image under f_n of the vertex set of C_n. Note that the diameter of C'_n tends to 0 by our assumption $\ell(C_n) \leq \theta(n)$. Using the compactness of $\mathbf{m}_\infty$ and again extracting a subsequence if necessary, we can assume that the points a'_n, b'_n converge respectively to a_∞, b_∞ in $\mathbf{m}_\infty$ and the (finite) sets C'_n converge (for the Hausdorff distance) to a singleton $\{c_\infty\}$, such that $\min\{d_\infty(a_\infty, c_\infty), d_\infty(b_\infty, c_\infty)\} \geq \delta/3$. Since the complement of a single point in the sphere $\mathbb{S}^2$ is path connected, there exists a (continuous) path $\gamma : [0, 1] \to \mathbf{m}_\infty$ from a_∞ to b_∞ avoiding c_∞. Let $\epsilon' = d_\infty(c_\infty, \gamma) > 0$, and let $N \in \mathbb{N}$ be large enough so that $d_\infty(\gamma(\frac{k}{N}), \gamma(\frac{k+1}{N})) \leq \frac{\epsilon'}{3}$ for every integer k with $0 \leq k \leq N-1$. For $0 \leq k \leq N-1$, define $x_{n,k} = g_n(\gamma(\frac{k}{N}))$, and $x_{n,-1} = a_n$, $x_{n,N+1} = b_n$. Then if n is large enough, $(x_{n,k})_{-1 \leq k \leq N+1}$ is a sequence of points in $\mathbf{m}_n$

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such that $d_n(x_{n,k}, C_n) \geq \frac{\epsilon'}{2} n^{1/4}$ and $d_n(x_{n,k}, x_{n,k+1}) < \frac{\epsilon'}{2} n^{1/4}$. Connecting $x_{n,k}$ and $x_{n,k+1}$ by a geodesic path in the graph M_n, we get a path from a_n to b_n in the map M_n that avoids C_n, which is a contradiction. $\qquad\square$

We conclude this section with an application of Proposition 2.3 to the Hausdorff dimension of the random lamination $L_{\mathbf{e}}$. The result is already stated in [A3], but the proof there is only sketched. Proposition 3.4 below should be compared with [BS], which considers geodesic laminations of the disk obtained as lifts of geodesic laminations on a closed connected hyperbolic surface, and proves that the Hausdorff dimension of such geodesic laminations of the disk is equal to 1.

Proposition 3.4 *We have* $\dim(L_{\mathbf{e}}) = 3/2$ *almost surely.*

Proof. Denote by $\overline{\dim}_M(A)$ the upper Minkowski dimension of a set A, and recall that $\overline{\dim}_M(A \cup B) = \max(\overline{\dim}_M(A), \overline{\dim}_M(B))$. Also recall the notation introduced before Proposition 2.3. It is then enough to prove that $\dim(A_{\mathbf{e}}) = 1/2$ and $\overline{\dim}_M(A_{\mathbf{e}}^{(I,J)}) \leq 1/2$, for every $(I, J) \in \mathcal{J}$, a.s. In this proof, it is more convenient to view $\mathbf{e}$ as parametrized by the time interval $[0, 1]$. Recall that $\mathbf{e}_0 = \mathbf{e}_1 = 0$ and $\mathbf{e}_t > 0$ for every $t \in \,]0, 1[$, a.s., and set

$$ H_a := \{ t \in \,]a, 1] : \mathbf{e}_t = \min_{a \leq r \leq t} \mathbf{e}_r \} \,. $$

for every $a \in \,]0, 1[$. Then $H_a \subset A_{\mathbf{e}}$ for every $a \in \,]0, 1[$, and $A_{\mathbf{e}}^{(I,J)} \subset H_a$ whenever $(I, J) \in \mathcal{J}$ and $a \in\,]0, 1[$ are such that $I = [u, v]$, $J = [u', v']$ and $0 \leq u' < v' < a < u < v \leq 1$. Using the invariance of the Brownian excursion under time reversal, we hence see that the required properties follow from the identities

$$ \dim(H_a) = \overline{\dim}_M(H_a) = \frac{1}{2} \,, \qquad \text{for almost all } a \in \,]0, 1[, \quad \text{a.s.} $$

(almost all refers to Lebesgue measure on $]0, 1[$). By a scaling argument, it is enough to verify that a similar property holds under the Itô measure of Brownian excursions. Using the Markov property at time $a > 0$ under the Itô measure, it then suffices to prove that the following holds. If $(\beta_t)_{t \geq 0}$ is a standard linear Brownian motion, and

$$ K_u := \{ t \in [0, u] : \beta_t = \min_{0 \leq r \leq t} \beta_r \} \,, $$

we have

$$ \dim(K_u) = \overline{\dim}_M(K_u) = \frac{1}{2} $$

for every $u > 0$, a.s. By a classical theorem of Lévy, the random set K_u has the same distribution as $\{ t \in [0, u] : \beta_t = 0 \}$. The preceding claim now follows from standard results about the zero set of linear Brownian motion. $\qquad\square$

4 Proof of the technical results

In this section, we prove Lemma 3.1 and Lemma 3.2. In both these proofs, it is more convenient to view the processes $\mathbf{e}$ and Z as parametrized by the interval $[0, 1]$ rather than by the unit circle (see the beginning of Section 3). We will make extensive use of properties of the Brownian snake. We start with a brief discussion of this path-valued Markov process, referring to [L1] for a more thorough presentation (see also Section 4 in [L2]).

A finite path in $\mathbb{R}$ is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})}$ is a nonnegative real number called the lifetime of w. The set $\mathcal{W}$ of all finite paths is a Polish space when equipped with the distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \max_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})| \; .$$

Let $x \in \mathbb{R}$. The one-dimensional Brownian snake with initial point x is a continuous strong Markov process taking values in the set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. Thus for every $s \geq 0$, $W_s = (W_s(t), 0 \leq t \leq \zeta_s)$ is a random continuous path in $\mathbb{R}$, with a (random) lifetime $\zeta_s = \zeta_{(W_s)}$ and such that $W_s(0) = x$. The behavior of the Brownian snake can be described informally as follows. The lifetime ζ_s evolves like reflecting Brownian motion in $\mathbb{R}_+$, and when ζ_s decreases the path W_s is erased from its tip, whereas when ζ_s increases the path W_s is extended by adding little pieces of Brownian paths at its tip. We denote by $\widehat{W}_s = W_s(\zeta_s)$ the terminal point (head of the snake) of the path W_s.

Let us fix $\mathrm{w} \in \mathcal{W}_x$. We denote by $\mathbb{P}_{\mathrm{w}}$ the law of the Brownian snake started from w. We also let $\mathbb{N}_x$ be the excursion measure of the Brownian snake away from the trivial path with lifetime 0 in $\mathcal{W}_x$. Both measures $\mathbb{P}_{\mathrm{w}}$ and $\mathbb{N}_x$ may be defined on the space $C(\mathbb{R}_+, \mathcal{W})$ of all continuous functions from $\mathbb{R}_+$ into $\mathcal{W}$. Under $\mathbb{P}_{\mathrm{w}}$, the lifetime process $(\zeta_s, s \geq 0)$ evolves like reflecting Brownian motion in $\mathbb{R}_+$, whereas under $\mathbb{N}_x$ it is distributed according to the Itô measure of positive excursions of linear Brownian motion. In particular the quantity $\sigma := \sup\{s \geq 0 : \zeta_s > 0\}$ is finite $\mathbb{N}_x$ a.e., and is called the duration of the excursion. We denote by $\mathbb{N}_x^{(1)}$ the Itô measure conditioned on the event $\{\sigma = 1\}$. Then, the pair $(\mathbf{e}_s, Z_s)_{0 \leq s \leq 1}$ of Section 3 has the same distribution as $(\zeta_s, \widehat{W}_s)_{0 \leq s \leq 1}$ under $\mathbb{N}_0^{(1)}$, which explains why Lemma 3.1 and Lemma 3.2 will be reduced to statements about the Brownian snake.

For every $s, s' \geq 0$, set

$$m(s, s') = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r \; .$$

We use several times the so-called *snake property*: $\mathbb{P}_{\mathrm{w}}$ a.s., or $\mathbb{N}_x$ a.e. (or $\mathbb{N}_x^{(1)}$ a.s.) for every $s, s' \geq 0$, we have

$$W_s(t) = W_{s'}(t) \; , \; \text{for every } 0 \leq t \leq m(s, s') \; .$$

We now state a lemma which plays an important role in our proofs. Recall that we have fixed an element w of $\mathcal{W}_x$. Under the probability measure $\mathbb{P}_{\mathrm{w}}$, we set

$$T = \inf\{s \geq 0 : \zeta_s = 0\}$$

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and we denote by $]\alpha_i, \beta_i[$, $i \in I$, the connected components of the open set

$$\{s \in [0, T] : \zeta_s > m(0, s)\} .$$

Then, for every $i \in I$, we define a random element W^i of $C(\mathbb{R}_+, \mathcal{W})$ by setting for every $s \geq 0$,

$$W_s^i(t) = W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + t) , \qquad 0 \leq t \leq \zeta_s^i := \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i} .$$

From the snake property, W_s^i belongs to $\mathcal{W}_{\mathrm{w}(\zeta_{\alpha_i})}$. The following result is Lemma V.5 in [L1].

Lemma 4.1 *The point measure*

$$\sum_{i \in I} \delta_{(\zeta_{\alpha_i}, W^i)}(dt\, d\omega)$$

is under $\mathbb{P}_\mathrm{w}$ *a Poisson point measure on* $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ *with intensity*

$$2\, \mathbf{1}_{[0,\zeta_{(\mathrm{w})}]}(t)dt\, \mathbb{N}_{\mathrm{w}(t)}(d\omega) .$$

We will also use the explicit form of the law of the minimal value of the Brownian snake under $\mathbb{N}_x$: For every $x, y \in \mathbb{R}$ with $y < x$,

$$\mathbb{N}_x\left(\min_{s\geq 0} \widehat{W}_s < y \right) = \frac{3}{2}\, (x-y)^{-2} . \tag{5}$$

For a short proof, simply note that the left-hand side, viewed as a function of x, solves the differential equation $u'' = 4u^2$ on $]y, +\infty[$, with boundary condition $u(y+) = +\infty$: see e.g. Lemma 2.1 in [LW] for a detailed argument. We use (5) in connection with Lemma 4.1 in the following calculation.

Let $\mathrm{w} \in \mathcal{W}$ and $t_0 \in [0, \zeta_{(\mathrm{w})}[$, $x \in \mathbb{R}$. Suppose that $\mathrm{w}(t) \geq x$ for every $t \in [t_0, \zeta_{(\mathrm{w})}]$. Then, recalling the notation introduced before Lemma 4.1, we have

$$\mathbb{P}_\mathrm{w}(\widehat{W}_r \geq x, \forall r \in [0, \inf\{s \geq 0 : \zeta_s = t_0\}]) = \mathbb{P}_\mathrm{w}(\widehat{W}_r^j \geq x , \forall r \geq 0, \forall j \in \{i \in I : \zeta_{\alpha_i} > t_0\})$$

$$= \exp\left(-2 \int_{t_0}^{\zeta_{(\mathrm{w})}} dt\, \mathbb{N}_{\mathrm{w}(t)}\left(\min_{s\geq 0} \widehat{W}_s < x \right) \right)$$

$$= \exp\left(-3 \int_{t_0}^{\zeta_{(\mathrm{w})}} dt\, (\mathrm{w}(t) - x)^{-2} \right). \tag{6}$$

The first equality follows from the fact that $\widehat{W}_r = \widehat{W}_{r-\alpha_j}^j$ if $r \in [\alpha_j, \beta_j]$ (we also use the snake property to observe that $\widehat{W}_r = \mathrm{w}(\zeta_r)$ if $m(0, r) = \zeta_r$). The second equality is an application of Lemma 4.1, and the last one comes from (5).

Proof of Lemma 3.1. Using the fact that the distribution of $(\mathbf{e}_s, Z_s)_{0\leq s \leq 1}$ is the same as that of $(\zeta_s, \widehat{W}_s)_{0\leq s \leq 1}$ under $\mathbb{N}_0^{(1)}$, together with a simple scaling argument, it is enough to prove that local minima of $\widehat{W}_s$ over the time interval $[0, \sigma]$ are distinct $\mathbb{N}_0$ a.e. This is less

easy than the analogous result for linear Brownian motion, because the process $(\widehat{W}_s)_{s\geq 0}$ is not Markovian. We will prove that, for every fixed choice of the rationals u, v, u', v' such that $0 < u < v < u' < v'$ we have

$$\min_{r\in[u,v]} \widehat{W}_r \neq \min_{r\in[u',v']} \widehat{W}_r, \tag{7}$$

$\mathbb{N}_0$ a.e. on the event $\{\sigma > v'\}$. By considering the countable union, over all possible choices of the rationals u, v, u', v', of the negligible sets on which (7) does not hold, the statement of Lemma 3.1 will then immediately follow.

In order to prove (7), we apply the Markov property at time u' under $\mathbb{N}_0$. Notice that the law of $W_{u'}$ under $\mathbb{N}_0(\cdot \mid \sigma > u')$ is that of a Brownian path started from 0 and stopped at an independent random time whose law is explicitly known but unimportant for what follows. So let us fix $\ell > 0$ and denote by $Q_\ell(dw)$ the probability measure on $\mathcal{W}_0$ which is the law of a linear Brownian motion started from 0 and stopped at time ℓ. The proof of (7) reduces to verifying that

$$\mathbb{P}_w\Big(\min_{r\in[0,v'-u']} \widehat{W}_r = a, \ T > v' - u' \Big) = 0 \ , \ \forall a \in \mathbb{R} \ , \ Q_\ell(dw) \text{ a.s.} \tag{8}$$

To simplify notation, set $h = v' - u'$. From the snake property, we have for any $w \in \mathcal{W}_0$

$$\min_{0\leq r\leq h} \widehat{W}_r \leq \Big(\min_{m(0,h)\leq t\leq \ell} w(t) \Big) \wedge \Big(\min_{m(0,h)\leq t\leq \zeta_h} W_h(t) \Big) \tag{9}$$

$\mathbb{P}_w$ a.s. on $\{T > h\}$. Hence, our claim (8) will follow if we can prove that $Q_\ell(dw)$ a.s.,

$$\mathbb{P}_w\Big(\min_{0\leq r\leq h} \widehat{W}_r = a, \ \min_{m(0,h)\leq t\leq \ell} w(t) > a, \ T > h \Big) = 0 \ , \ \forall a \in \mathbb{R} \ , \tag{10}$$

and

$$\mathbb{P}_w\Big(\min_{0\leq r\leq h} \widehat{W}_r = a = \min_{m(0,h)\leq t\leq \ell} w(t), \ T > h \Big) = 0 \ , \ \forall a \in \mathbb{R} \ . \tag{11}$$

We first prove (10), which in fact holds for every choice of w with $\zeta_{(w)} = \ell$, and not only $Q_\ell(dw)$ a.s. We fix $a \in \mathbb{R}$. By properties of the Brownian snake, we know that under $\mathbb{P}_w(\cdot \mid T > h)$ and conditionally on the pair $(m(0,h), \zeta_h)$, the random path $(W_h(m(0,h)+t), 0 \leq t \leq \zeta_h - m(0,h))$ is distributed as a linear Brownian motion started from $w(m(0,h))$ and stopped at time $\zeta_h - m(0,h)$. In particular, on the event

$$\Big\{ \min_{0\leq r\leq h} \widehat{W}_r = a, \ \min_{m(0,h)\leq t\leq \ell} w(t) > a, \ T > h \Big\}$$

we have $w(m(0,h)) > a$ and thus

$$\min_{m(0,h)\leq t\leq \zeta_h} W_h(t) > a \ , \ \mathbb{P}_w \text{ a.s.} \tag{12}$$

because the law of the minimum of a Brownian path over a nontrivial interval has a density, and we already know from (9) that the minimum in (12) is greater than or equal to a on the event in consideration. We then argue by contradiction, assuming that

$$\mathbb{P}_w\Big(\min_{0\leq r\leq h} \widehat{W}_r = a, \ \min_{m(0,h)\leq t\leq \ell} w(t) > a, \ T > h \Big) > 0 \ .$$

Using the Markov property under $\mathbb{P}_w$ at time h, together with Lemma 4.1, the property (12) and the fact that $\mathbb{N}_x(\min_{s\geq 0} \widehat{W}_s < y) < \infty$ for every $y < x$, it follows that

$$\mathbb{P}_w\Big(\min_{0\leq r\leq S_h} \widehat{W}_r = a, \ \min_{m(0,h)\leq t\leq \ell} w(t) > a, \ T > h \Big) > 0 \,,$$

where

$$S_h := \inf\{s \geq h : \zeta_s = m(0,h)\} \,.$$

From the definition of the "excursions" W^i before Lemma 4.1, we then see that with positive probability under $\mathbb{P}_w$, there exists some $i \in I$ such that

$$\min_{s\geq 0} \widehat{W}^i_s = a \,, \ w(\zeta_{\alpha_i}) > a \,.$$

However, by Lemma 4.1 and properties of Poisson measures, the probability of the latter event is

$$1 - \exp\Big(-2\int_0^\ell dt\, \mathbf{1}_{\{w(t)>a\}}\, \mathbb{N}_{w(t)}\Big(\min_{s\geq 0} \widehat{W}_s = a \Big) \Big) = 0$$

because the law of $\min_{s\geq 0} \widehat{W}_s$ under $\mathbb{N}_x$ has no atoms by (5). This contradiction completes the proof of (10).

It remains to prove (11). We again fix $a \in \mathbb{R}$. In contrast with the previous argument, it will be important to disregard certain sets of values of w which have zero Q_ℓ-measure. We first note that $Q_\ell(dw)$ a.s.,

$$\mathbb{P}_w\Big(w(m(0,h)) = \min_{m(0,h)\leq t\leq \ell} w(t) \Big) = 0$$

so that the minimum of w over $[m(0,h),\ell]$ is attained $\mathbb{P}_w$ a.s. at a point of $]m(0,h),\ell]$. It follows that $\mathbb{P}_w$ a.s. on the event

$$\Big\{ \min_{0\leq r\leq h} \widehat{W}_r = a = \min_{m(0,h)\leq t\leq \ell} w(t), \ T > h \Big\}$$

we can find a rational $q \in]0,\ell[$ such that, if $T_q = \inf\{s \geq 0 : \zeta_s = q\}$,

$$\min_{0\leq r\leq T_q} \widehat{W}_r = a = \min_{q\leq t\leq \ell} w(t) \,.$$

Thus we need only check that the latter event has probability zero for every rational $q \in]0,\ell[$. Using Lemma 4.1 once again, we have

$$\begin{aligned}
\mathbb{P}_w\Big(\min_{0\leq r\leq T_q} \widehat{W}_r = \min_{q\leq t\leq \ell} w(t) \Big) &= \exp\Big(-2\int_q^\ell dr\, \mathbb{N}_{w(r)}\Big(\min_{s\geq 0} \widehat{W}_s \geq \min_{q\leq t\leq \ell} w(t) \Big) \Big) \\
&= \exp\Big(-3\int_q^\ell dr\, \Big(w(r) - \min_{q\leq t\leq \ell} w(t) \Big)^{-2} \Big)
\end{aligned}$$

by (5). However, an application of Lévy's modulus of continuity for Brownian motion shows that

$$\int_q^\ell dr\, \Big(w(r) - \min_{q\leq t\leq \ell} w(t) \Big)^{-2} = \infty \,, \ Q_\ell(dw) \text{ a.s.}$$

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This completes the proof of (11) and of Lemma 3.1. $\qquad\square$

Proof of Lemma 3.2. We start by recalling that the law of the pair $(\mathbf{e}_s, Z_s)_{0 \le s \le 1}$ is invariant under time-reversal. More precisely, $(\mathbf{e}_{1-s}, Z_{1-s})_{0 \le s \le 1}$ has the same distribution as $(\mathbf{e}_s, Z_s)_{0 \le s \le 1}$ (this is indeed a straightforward consequence of the well-known invariance of the law of the normalized Brownian excursion under time-reversal). Then the proof of Lemma 3.2 reduces to checking that, almost surely for every $s \in \,]0,1[$ such that

$$\mathbf{e}_s = \min_{r \in [s-\varepsilon,s]} \mathbf{e}_r \quad , \quad \text{for some } \varepsilon \in \,]0,s[\, , \tag{13}$$

we have

$$Z_s > \min_{r \in [s-\delta,s]} Z_r \quad , \quad \text{for every } \delta \in \,]0,s[\tag{14}$$

and

$$Z_s > \min_{r \in [s,s+\delta]} Z_r \quad , \quad \text{for every } \delta \in \,]0,1-s[\, . \tag{15}$$

Indeed, if the equivalence class of $s \in \,]0,1[$ for $\sim_{\mathbf{e}}$ is not a singleton, then either (13) holds and (14) and (15) imply that the equivalence class of s for $\sim_Z$ is a singleton, or the analogue of (13) holds with the interval $[s-\varepsilon, s]$ replaced by $[s, s+\varepsilon]$, and a time-reversal argument yields the same result.

The fact that (13) implies (14) is exactly Lemma 2.2 in [L2] (combined with time-reversal invariance). We thus concentrate on the proof of (15). The argument is similar to the proof of Lemma 2.2 in [L2]. We rely on some ideas of Abraham and Werner [AW], which were already exploited in Section 4 of [LW].

Once again, we can reformulate the desired result in terms of the Brownian snake. It is enough to verify that, $\mathbb{N}_0$ a.e. for every $s \in \,]0,\sigma[$ such that

$$\zeta_s = m(s-\varepsilon, s) \quad , \quad \text{for some } \varepsilon \in \,]0,s[\, ,$$

we have

$$\widehat{W}_s > \min_{r \in [s,s+\delta]} \widehat{W}_r \quad , \quad \text{for every } \delta \in \,]0,\sigma-s[\, .$$

We first get rid of the case when s corresponds to a local minimum of the lifetime process $(\zeta_s)_{s \ge 0}$. Let u and v be two rational numbers such that $0 < u < v$, and argue under the probability measure $\mathbb{N}_0(\cdot \mid \sigma > v)$. We know that with probability one there exists a unique time $s \in \,]u,v[$ such that $\zeta_s = m(u,v)$. Furthermore, $\widehat{W}_s = W_v(m(u,v))$ (by the snake property) and conditionally on the pair $(m(u,v), \zeta_v)$, $(W_v(m(u,v) + t) - W_v(m(u,v)), 0 \le t \le \zeta_v - m(u,v))$ is a linear Brownian path started from 0. From the snake property once again, we know that, for every $\delta > 0$, the set $\{\widehat{W}_r, r \in [s, s+\delta]\}$ contains $\{W_v(t), m(u,v) \le t \le m(u,v) + \eta\}$ for some random $\eta > 0$ depending on δ. By the preceding observations and standard properties of linear Brownian paths, it follows that $\{\widehat{W}_r, r \in [s, s+\delta]\}$ contains values strictly less than $W_v(m(u,v)) = \widehat{W}_s$, as desired.

We now turn to the case when s is not a time of local minimum of the lifetime process. Let us fix $u > 0$, $\delta > 0$ and an integer $A \ge 1$. It is enough to prove that, $\mathbb{N}_0$ a.e. on the event $\{u < \sigma\} \cap \{\zeta_u \le A\}$, there exists no time $s \in \,]u,\sigma[$ such that

$$\zeta_s = m(u,s) > 2\delta \tag{16}$$

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and
$$\widehat{W}_r \geq \widehat{W}_s \quad, \quad \text{for every } r \text{ such that } s \leq r \leq \inf\{t > s : \zeta_t = \zeta_s - 2\delta\}\,. \tag{17}$$

To simplify notation, denote by $\mathbb{N}_0^u$ the conditional probability measure $\mathbb{N}_0(\cdot \mid \sigma > u, \zeta_u \leq A)$. Also set $s_u(t) = \inf\{s \geq u : \zeta_s = t\}$ for every $t \leq \zeta_u$. For every integer $n \geq 1$ and every $i \in \{0, 1, \ldots, A2^n\}$, set

$$T_i^n = \inf\{r \geq u : \zeta_r = \zeta_u - i2^{-n}\} = s_u(\zeta_u - i2^{-n})$$

and

$$S_i^n = \inf\{r > T_i^n : \zeta_r = \zeta_{T_i^n} - \delta\} = s_u(\zeta_u - i2^{-n} - \delta)$$

with the usual convention $\inf \emptyset = \infty$. Let $\alpha \in \,]0, 1]$. We can use the strong Markov property at time T_i^n, to evaluate the probability

$$\mathbb{N}_0^u(T_i^n < S_i^n < \infty; \widehat{W}_r \geq \widehat{W}_{T_i^n} - \alpha, \forall r \in [T_i^n, S_i^n]) =$$
$$= \mathbb{N}_0^u\Big(\mathbf{1}\{T_i^n < \infty, \zeta_{T_i^n} \geq \delta\}\,\mathbf{1}\{W_{T_i^n}(t) \geq \widehat{W}_{T_i^n} - \alpha, \forall t \in [\zeta_{T_i^n} - \delta, \zeta_{T_i^n}]\}$$
$$\times \mathbb{P}_{W_{T_i^n}}\Big[\widehat{W}_r \geq x - \alpha, \forall r \in [0, \inf\{s \geq 0 : \zeta_s = \zeta_0 - \delta\}]\Big]_{x=\widehat{W}_{T_i^n}}\Big)$$

(the snake property implies that $\widehat{W}_{s_u(t)} = W_u(t) = W_{T_i^n}(t)$ for every $t \in [0, \zeta_{T_i^n}]$, so that the condition $\widehat{W}_r \geq \widehat{W}_{T_i^n} - \alpha$ for every $r \in [T_i^n, S_i^n]$ forces $W_{T_i^n}(t) \geq \widehat{W}_{T_i^n} - \alpha$ for every $t \in [\zeta_{T_i^n} - \delta, \zeta_{T_i^n}]$). The probability in the last line of the previous display is calculated thanks to (6), and we obtain that

$$\mathbb{N}_0^u(T_i^n < S_i^n < \infty; \widehat{W}_r \geq \widehat{W}_{T_i^n} - \alpha, \forall r \in [T_i^n, S_i^n]) = \tag{18}$$
$$= \mathbb{N}_0^u\Big(\mathbf{1}\{T_i^n < \infty, \zeta_{T_i^n} \geq \delta\}\,\mathbf{1}\{W_{T_i^n}(t) \geq \widehat{W}_{T_i^n} - \alpha, \forall t \in [\zeta_{T_i^n} - \delta, \zeta_{T_i^n}]\}$$
$$\times \exp\Big(-3\int_{\zeta_{T_i^n} - \delta}^{\zeta_{T_i^n}} dt\,(W_{T_i^n}(t) - \widehat{W}_{T_i^n} + \alpha)^{-2}\Big)\Big)\,.$$

By the snake property, on the event $\{T_i^n < \infty\} = \{\zeta_u \geq i2^{-n}\}$, $W_{T_i^n}$ is the restriction of the path W_u to the time interval $[0, \zeta_u - i2^{-n}]$. Therefore under the probability measure $\mathbb{N}_0^u$ and conditionally on ζ_u, the path $W_{T_i^n}$ is distributed as a linear Brownian path started from 0 with lifetime $\zeta_u - i2^{-n}$. It now follows that the quantity (18) is equal to

$$\mathbb{N}_0^u(T_i^n < \infty, \zeta_{T_i^n} \geq \delta)\,E\Big[\mathbf{1}_{\{B_t \geq -\alpha, \forall t \in [0, \delta]\}}\,\exp\Big(-3\int_0^\delta dt\,(B_t + \alpha)^{-2}\Big)\Big]$$
$$\leq E\Big[\mathbf{1}_{\{B_t \geq -\alpha, \forall t \in [0, \delta]\}}\,\exp\Big(-3\int_0^\delta dt\,(B_t + \alpha)^{-2}\Big)\Big] \tag{19}$$

where $(B_t, t \geq 0)$ is a linear Brownian motion starting from 0 under the probability measure P. As an immediate application of the absolute continuity relations between Bessel processes (see e.g. Proposition 2.6 in [LW]), the right-hand side of (19) is equal to

$$\alpha^3\,E[(R_\delta^{(\alpha)})^{-3}]$$

where $(R_t^{(\alpha)}, t \geq 0)$ is a seven-dimensional Bessel process started from α. The last quantity is bounded above by $C_\delta \alpha^3$, where the constant C_δ only depends on δ (compare with the estimate of Lemma 5.2 in [L2]). Therefore we have obtained the bound

$$\mathbb{N}_0^u(T_i^n < S_i^n < \infty; \widehat{W}_r \geq \widehat{W}_{T_i^n} - \alpha, \forall r \in [T_i^n, S_i^n]) \leq C_\delta \, \alpha^3 \ .$$

We apply this estimate with $\alpha = 2^{-2n/5}$. By summing over possible values of i, and using the Borel-Cantelli lemma, we get that, $\mathbb{N}_0^u$ a.e., for every n sufficiently large, for every $i \in \{0, 1, \dots, A2^n\}$ such that $T_i^n < S_i^n < \infty$, the condition

$$\widehat{W}_r \geq \widehat{W}_{T_i^n} - 2^{-2n/5} \text{ for every } r \in [T_i^n, S_i^n] \tag{20}$$

does not hold.

To complete the proof, we argue by contradiction. Suppose that there exists $s \in \,]u, \sigma[$ such that both (16) and (17) hold, and moreover $\zeta_u \leq A$. For every integer n such that $2^{-n} \leq \delta$, choose $i \in \{1, \dots, A2^n\}$ such that $T_{i-1}^n \leq s < T_i^n$. By (16), we have $\zeta_{T_{i-1}^n} \geq \zeta_s$ and so $\zeta_{T_i^n} = \zeta_{T_{i-1}^n} - 2^{-n} \geq \zeta_s - \delta$. Hence,

$$S_i^n \leq \inf\{r > s : \zeta_r = \zeta_s - 2\delta\} \ ,$$

and by (17) we see that $\widehat{W}_r \geq \widehat{W}_s$ for every $r \in [T_i^n, S_i^n]$. On the other hand, the snake property and (16) ensure that $\widehat{W}_s = W_u(\zeta_s)$, and we already noticed that $\widehat{W}_{T_i^n} = W_u(\zeta_{T_i^n})$. Since $0 \leq \zeta_s - \zeta_{T_i^n} \leq 2^{-n}$, by applying to W_u the classical Hölder continuity properties of Brownian paths, we get that $\widehat{W}_s \geq \widehat{W}_{T_i^n} - 2^{-2n/5}$, for every n sufficiently large. So we see that for every n sufficiently large, for i chosen so that $T_{i-1}^n \leq s < T_i^n$, the condition (20) holds. This contradiction completes the proof of Lemma 3.2. $\qquad\square$

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