Lewy's curves in para-CR geometry

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Overview

I will talk about a construction in para-CR geometry in dimension 3. The construction builds upon a similar concept in the CR-case from early '80, and recent works related to the so-called 'dancing construction' (Bor, Lamoneda, Nurowski (2018), Dunajski (2022)) .
Plan:

1. Overview of the CR-case
2. Para-CR counterpart
3. Dancing curves and generalizations
4. Result
5. Relations to chains

Based on a joint work with O. Makhmali (arXiv:2303.08807)

Lewy's curves in 3-dimensional CR-gometry

The curves appeared in 1981 in the work of J. Faran (Transactions AMS), who attributed the construction to H. Lewy.

Consider a CR-manifold

$$N = \{r(z, \bar{z}) = 0\} \subset \mathbb{C}^2,$$

where r is a real-analytic function, $z \in \mathbb{C}^2$. Then r extends to a complex function $\tilde{r}$ on $\mathbb{C}^2 \times \mathbb{C}^2$

$$\tilde{r}(z, \xi) = r(z, \xi).$$

The Segre curves in $\mathbb{C}^2$ are defined as

$$Q_\xi = \{z \in \mathbb{C}^2 \mid \tilde{r}(z, \xi) = 0\}.$$

This is a family parameterized by $\xi \in \mathbb{C}^2$ and can be also considered as a 3-dimensional complex manifold

$$\mathcal{M} \subset \mathbb{C}^2 \times \mathbb{C}^2.$$

Lewy's curves in 3-dimensional CR-gometry

Lewy's curves on N are defined as intersections

$$\gamma_\xi = Q_\xi \cap N.$$

These are real curves and they define a path geometry on N (in a generalized sense: they emerge from each point in directions transverse to the contact distribution). They are well defined by the CR-structure, independently on the choice of the original function r.

- In the flat case, Lewy's curves coincide with chains.
- Open problem: do they coincide in other cases?

3-dimensional para-CR geometry

A para-CR structure on N is a contact distribution

$$\mathcal{C} \subset TN$$

equipped with a para-complex structure $I\colon \mathcal{C} \to \mathcal{C}$, $I^2 = Id$, which induces a splitting

$$\mathcal{C} = \mathcal{D}_1 \oplus \mathcal{D}_2$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are 1-dimensional eigen-spaces corresponding to ± 1.

This descents to the following double fibration

$$M \longleftarrow N \longrightarrow T$$

where $M = N/\mathcal{D}_2$ and $T = N/\mathcal{D}_1$. Moreover M and T posses canonical path geometries defined by projections of integral curves of $\mathcal{D}_1$ and $\mathcal{D}_2$. In this setting T is called the twistor space of the path geometry on M, and N is called the correspondence space.

3-dimensional para-CR geometry

A path geometry on 2-manifold M is locally described by a second order ODE

$$x'' = F(t, x, x')$$

determined uniquely up to a point transformation. Here (t, x) are local coordinates on M.

Local coordinates (a, b) on T parameterize solution curves on M. A general solution function $\Phi\colon M \times T \to \mathbb{R}$ defines the path geometry on M by formula

$$\varphi_{ab} = \{(t, x) \in M \mid \Phi(t, x, a, b) = 0\}$$

and the dual path geometry on T by formula

$$\varphi_{tx} = \{(a, b) \in T \mid \Phi(t, x, a, b) = 0\}.$$

Moreover, we get an identification

$$N = \{\Phi = 0\} \subset M \times T.$$

Lewy's curves for para-CR geometries

The CR case translates to the para-CR case in the following way

- $M \times T (\simeq \mathbb{R}^2 \times \mathbb{R}^2)$ is a 4-dimensional manifold with a para-complex structure (split structure).
- N is defined as a hyper-surface in $M \times T$ by function Φ.
- Φ gives rise to a function on $(M \times T)^2$ defined as $\tilde{\Phi}(p, q, \hat{p}, \hat{q}) = (\Phi(p, \hat{q}), \Phi(\hat{p}, q))$.

In this setting, Lewy's curves are introduced as

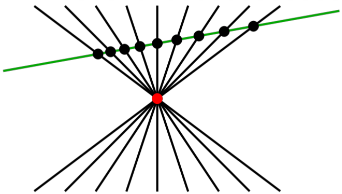
$$\gamma_{\hat{p}\hat{q}} = \{(p, q) \in N \mid \tilde{\Phi}(p, q, \hat{p}, \hat{q}) = 0\},$$

where we assume that $(\hat{p}, \hat{q}) \notin N$.

As in the CR-case the curves define a path geometry on N.

Interpretation as dancing

The curves in the para-CR setting appeared in the context of the dancing construction. The construction on the correspondence space (in B-L-N and Dunajski the primary interest is on null curves on 4-dim. space $M \times T \setminus N$) in the flat case is presented on the picture



It defines a path (black punctured lines) in the correspondence space

$$N = \{(p, q) \in M \times T \mid p \in q\} \subset M \times T$$

The path is defined by $\hat{p} \in M$ (red point) and $\hat{q} \in T$ (green line) such that $\hat{p} \notin \hat{q}$.

Explicit formulas

In terms of the general solution function Φ, Lewy's curves (dancing curves) are given by

$$\gamma_{\hat{p}\hat{q}} = \{\Phi(t, x, a, b) = 0,\ \Phi(\hat{t}, \hat{x}, a, b) = 0,\ \Phi(t, x, \hat{a}, \hat{b}) = 0\}$$

where $\hat{p} = (\hat{t}, \hat{x})$, $\hat{q} = (\hat{a}, \hat{b})$ are constant.

Differentiating w.r.t t one can eliminate the constants and get a system of ODEs for the curves. E.g. in the flat case

$$\Phi(x, y, a, b) = y - bx - a$$

and the system takes the form

$$x'' = 0, \qquad b'' = -\frac{2(b')^2}{x' - b}.$$

Properties and questions

- 3-dimensional path geometry defined by Lewy's curves is never flat.
- Can be viewed as an extension of a scalar ODE to a system. In fact it can be interpreted as a geometric pairing of a second order ODE and its dual.
- The system is torsion-free (Wuenschman invariant=0) if and only if the original equation is flat. Consequently, it defines a conformal structure on its 4-dimensional solution space ($M \times T \setminus N$) only in the flat case (Dunajski (2022))

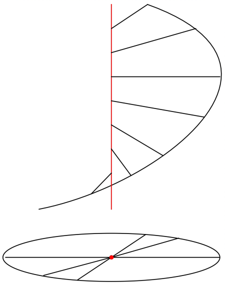
Goals:

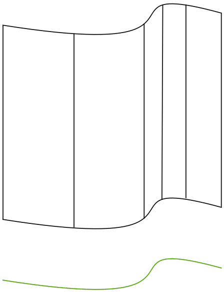
- Characterize path geometries defined by Lewy's curves in the general class of 3-dimensional path geometries.
- Compare with chains (coincide in the flat case).

Intrinsic viewpoint

To eliminate Φ from the picture, we introduce two families of surfaces, parameterized by points in $M = N/\mathcal{D}_2$ and $T = N/\mathcal{D}_1$:

$$\Sigma^1_p = \textit{union of all integral curves of } \mathcal{D}_1 \textit{ emerging}$$
$$\textit{from an integral curve of } \mathcal{D}_2 \textit{ corresponding to } p \in M$$
$$\Sigma^2_q = \textit{union of all integral curves of } \mathcal{D}_2 \textit{ emerging}$$
$$\textit{from an integral curve of } \mathcal{D}_1 \textit{ corresponding to } q \in T$$





Intrinsic viewpoint

Lemma

$\gamma_{pq} = \Sigma_p^1 \cap \Sigma_q^2$

Corollary

Each submanifold Σ_p^1 or Σ_q^2 is totally geodesic. Moreover, the corresponding path geometries restricted to the submanifolds are equivalent to the original one on S or its dual on T, respectively.

Invariants of 3-dimensional path geometries

3-dimensional path geometries give rise to (quasi)-contact geometry $\mathcal{C} = \mathcal{V} \oplus \mathcal{X}$ on a 5-manifold. In general, it is a subspace of Q=PTN. Locally, if coordinates are chosen, it is identified with the jet space $J^1(\mathbb{R}, \mathbb{R}^2)$. Invariants are of two types

1. Torsion (the Wuenschmann invariant) $T = (T^i_j)_{i,j=1,2}$: trace free 2×2 matrix (endomorphism of vertical distribution $\mathcal{V}$, tangent to fibers of $PTN \to N$).
2. Curvature $W = (W_i)_{i=0,1,2,3,4}$: if $T = 0$ then it is the Weyl curvature of the corresponding metric on the solution space; in general it can be considered as a binary quartic.

Characterization of Lewy's curves

Theorem

A 3-dimensional path geometry is defined by Lewy's curves iff

(a) *T has two eigenvectors V_1 and V_2 (the eigenvectors of T define splitting of the vertical distribution $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$),*

(b) *distributions $B_i = span(\mathcal{X}, \mathcal{V}_i, [\mathcal{X}, \mathcal{V}_i])$, for $i = 1, 2$, are integrable and each B_i has a corank-1 integrable subdistribution K_i such that $K_i + \mathcal{V}$ is also integrable (it follows that projections of K_i to M are well defined line bundles)*

(c) *projections of K_i's to N span a contact distribution (and consequently define a 2D path geometry)*

(d) *naturally defined path geometries on the leaf spaces of B_i are all equivalent to the 2D path geometry (or its dual, respectively).*

Characterization of Lewy's curves

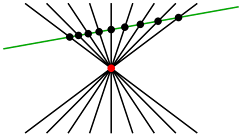
The integrability conditions on B_i and K_i imply the following

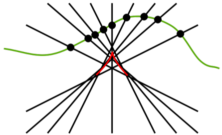
Corollary

If $T \neq 0$ then the curvature W has at least two distinct real roots and at most one real root of multiplicity 2.

A generalization of Lewy's curves

If the last condition in the Theorem is dropped then it describes the following generalization of Lewy's curves:





It is defined by a triple of para-CR structures on the same contact manifold: $(\mathcal{D}_1, \mathcal{D}_2)$, $(\tilde{\mathcal{D}}_1, \mathcal{D}_2)$, $(\mathcal{D}_1, \tilde{\mathcal{D}}_2)$.

Similarly to the regular case one also gets nice formulas in terms of general solution functions. We proved a generalization of Dunajski (2022)

Proposition

The generalized construction defines 4-dimensional metric only in the case of flat path geometry on M.

Characterization of chains

Theorem (K.-Makhmali, Annali SNS Pisa (accepted))

A 3-dim. path geometry arises from chains of a 2-dim. path geometry iff

(a) *the binary quartic W has two distinct real roots of multiplicity 2,*

(b) *certain invariantly defined 2-form is closed,*

(c) *the entries of T have no dependency on fibers (i.e. are functions on $N = Q/\mathcal{V}$).*

Theorem (K.-Makhmali)

A 3-dim. path geometry arises from chains of a 3-dim. CR-geometry iff

(a) *the binary quartic W has a non-real complex root of multiplicity 2,*

(b) *certain invariantly defined 2-form is closed,*

(c) *the entries of T have no dependency on fibers (i.e. are functions on $N = Q/\mathcal{V}$).*

Consequently, chains and Lewy's curves coincide only for $T = 0$.

Thank you for your attention!