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Cartan equivalence problem and, The geometry of marked contact Engel structures.

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Abstract

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- This memory begins to understand the main part of the theory of the equivalence problem : the answer to the equivalence problem, which is **the Cartan's algorithm**.

- Then we will apply this theory in talking about the famous equivalence problem of **second-order ordinary differential equations** : this problem will be discussed in different ways ; using the **Cartan's algorithm** and the **Cartan connections**.

Doubrov-Komrakov : The geometry of second-order ordinary differential equations, arXiv : 1602.00913v1. 57 pp.

- Finally, we will talk about another application of the Cartan's method : the equivalence problem of **the marked contact Engel structures**.

Manno-Nurowski-Sagerschnig : The geometry of marked contact Engel structures, arXiv : 1809.06455v1. 43 pp

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Geometric foundations part 1

1. Manifolds

Before working about the equivalence problem, we will talk about the geometric tools useful for the problem.

Definition 1.1. An m -dimensional manifold M is a separable and Hausdorff topological space which is covered by a collection of open subset $W_\alpha \subset M$, called coordinate charts, and one-to-one maps $\chi_\alpha : W_\alpha \rightarrow V_\alpha$ onto connected open subset $V_\alpha \subset \mathbb{R}^m$, which serve to define local coordinates on M . The manifold is smooth (respectively analytic) if the composite "overlap maps" $\chi_{\beta,\alpha} = \chi_\beta \circ \chi_\alpha$ are smooth (respectively analytic) from $\chi_\alpha(W_\alpha \cap W_\beta)$ to $\chi_\beta(W_\alpha \cap W_\beta)$.

We shall also have occasion to use complex (analytic) manifolds, which are defined in the same fashion, with $\mathbb{C}^m$ replacing $\mathbb{R}^m$; the overlap maps are assumed to be holomorphic. In order to keep the exposition straightforward, we will usually state our results for real manifolds.

Remark 1.2. The topology of a manifold is induced by that of $\mathbb{R}^m$. Thus, a subset $V \subset M$ is open if and only if its intersection $V \cap W_\alpha$ with every coordinate chart maps it to an open subset $\chi_\alpha(V \cap W_\alpha) \subset \mathbb{R}^m$.

Remark 1.3. More generally, every manifold is the union of a countable number of disjoint connected components.

Example 1.4. 1) $\mathbb{R}^m$

2) $S^{m-1} = \{x \in \mathbb{R}^m \mid |x| = 1\}$

3) $\mathbb{RP}^{m-1}$ the projective plane defined as the space of lines through the origin in $\mathbb{R}^m$. Coordinate charts on $\mathbb{RP}^{m-1}$ are constructed by considering all lines with a given component, say x^i non zero; the coordinates are then provided by ratios $p^k = x^k/x^i, k \neq i$.

2. Functions

Let M and N be two smooth manifolds of dimension m and n .

Definition 2.1. A map $F : M \rightarrow N$ between two smooth manifolds is called smooth if it is smooth in local coordinates.

Definition 2.2. The rank of a map $F : M \rightarrow N$ at a point $x \in M$ is defined to be the rank of the $n \times m$ Jacobian matrix $(\frac{\partial F_i}{\partial x_j}(x))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ of any local coordinate expression for F at the point p . The map F is called regular if its rank is constant.

The first of the equivalence problems which we encounter is to determine whether two different maps $y = F(x)$ and $\bar{y} = \bar{F}(\bar{x})$ between manifolds of the same dimension are locally the same, meaning that they can be transformed into each other by appropriate changes of coordinates $\bar{x} = \chi(x)$, $\bar{y} = \psi(y)$.

In the regular case, the Implicit Function Theorem solves the local equivalence problem, and provides a canonical form for regular form.

Theorem 2.3 (of regular maps). *Let $F : M \rightarrow N$ be regular map of rank r . Then there exist local coordinates $x = (x_1, \dots, x_m)$ on M and $y = (y_1, \dots, y_n)$ on N such that F takes the canonical form*

$$y = F(x) = (x_1, \dots, x_r, 0, \dots, 0)$$

Then, all maps of constant rank are locally equivalent and can be linearized by the introduction of appropriate local coordinates. The places where the rank of a map decreases are singularities. the canonical forms at singularities are much more complicated - this is the subject of singularity theory.

We will just talk about the regular cases.

Definition 2.4. The differential of a smooth function $f : M \rightarrow \mathbb{R}$ is given by the expression

$$df = \sum_{i=1}^m \frac{\partial f}{\partial x_i} dx_i$$

in any local coordinate.

We can denote also the differential f_* .

Definition 2.5. Let $\mathcal{F}$ be a family of smooth real-valued functions $f : M \rightarrow \mathbb{R}$. The rank of $\mathcal{F}$ at a point $x \in M$ is the number

$$r(x) = \dim(\text{vect}(\{df_x\}_{f \in \mathcal{F}}))$$

The family is regular if r is constant on M .

Definition 2.6. A set $\{f_1, \dots, f_k\}$ of smooth real-valued functions on a manifold M having a common domain of definition is called functionally dependent if, for each $x_0 \in M$, there is a neighborhood U and a smooth function $H : \mathbb{R}^k \rightarrow \mathbb{R}$, not identically zero on any

subset of $\mathbb{R}^k$, such that for all $x \in U$, $H(f_1(x), \dots, f_k(x)) = 0$.

The functions are called functionally independent if they are not functionally dependent when restricted to any open subset to M .

Example 2.7. $f_1(x, y) = x/y$ and $f_2(x, y) = xy/(x^2 + y^2)$ are functionally dependent on the upper half plan $\{y > 0\}$ since $f_2 = f_1/(1 + f_1^2)$.

For a regular family of functions, the rank tells us how many functionally independent functions it contains.

Theorem 2.8. *If a family of function $\mathcal{F}$ is regular of rank r , then in a neighborhood of any point, there exist r functionally independent functions $f_1, \dots, f_r$ with the property that any other function $f \in \mathcal{F}$ can be expressed as a function thereof $f = H(f_1, \dots, f_r)$.*

Proof. Given $x \in M$, choose $f_1, \dots, f_r \in \mathcal{F}$ such that their differential $df_1, \dots, df_r$ are linearly independent at x , and hence, by continuity, in a neighborhood of x . According to the theorem of regular maps, we can locally choose coordinates (y, z) near x such that for $1 \leq i \leq r$, $f_i(y, z) = y^i$. If $f \in \mathcal{F}$ is any other function, then, since the rank is r , its differential must be a linear combination of the differentials $\{df_i\}_{1 \leq i \leq r}$, so that in the new coordinates $df = \sum_{i=1}^r h_i(y, z)dy_i$.

To conclude, we invoke the following lemma.

Lemma 2.9. *Let $U \subset \mathbb{R}^m$ be a convex open set. A function $f : U \rightarrow \mathbb{R}$ has differential $df = \sum_{i=1}^r h_i(x)dx_i$ given as a linear combination of the first r coordinate differentials if and only if $f = f(x_1, \dots, x_r)$ is a function of the first r coordinates. In particular, $df = 0$ if and only if f is constant.*

Thus, by shrinking the neighborhood of p if necessary so that it is convex in the (y, z) -coordinates. The lemma implies that $f(y_1, \dots, y_r)$ is a function of y alone. Reverting to the original x coordinates completes the proof. $\square$

Remark 2.10. Consequently, if $f_1, \dots, f_r$ is a set of functions whose $m \times r$ Jacobian matrix $(\frac{\partial f_i}{\partial x_k})_{\substack{1 \leq i \leq r \\ 1 \leq k \leq m}}$ has maximal rank r at x , then by continuity, they also have rank r in a neighborhood of x , and hence are functionally independent near x . Note also that, the theorem implies that, locally, there are at most m functionally independent functions on any m -dimensional manifold M .

3. Submanifolds

Definition 3.1. A smooth (analytic) n -dimensional immersed submanifold of a manifold M is a subset $N \subset M$ parametrized by a smooth (analytic), one-to-one map $F : \tilde{N} \rightarrow N \subset M$, whose domain $\tilde{N}$ the parameter space, is a smooth (analytic) n -dimensional manifold, and such that F is everywhere regular, of maximal rank n .

Definition 3.2. A submanifold is regular if, in addition to the regularity of the parameter map, at each point $p \in N$ there exist arbitrarily small neighborhood U of x in M such that $F^{-1}(U \cap N)$ is a connected open subset of $\tilde{N}$.

The Implicit Function Theorem gives a canonical form for the regular manifolds ;

Theorem 3.3. An n -dimensional submanifold $N \subset M$ of an m -dimensional manifold M is regular if and only if for each $x \in N$ there exist local coordinates $x = (x_1, \dots, x_m)$ defined on a neighborhood U of x such that $U \cap N = \{x \mid x_1 = \dots = x_{m-n} = 0\}$

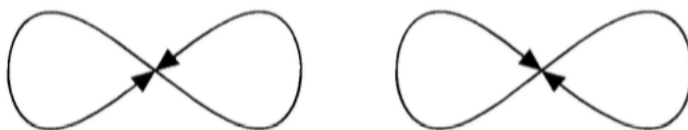
Remark 3.4. Every regular n -dimensional submanifold of an m -dimensional manifold locally looks like an n -dimensional subspace of $\mathbb{R}^{m-n}$.

Thus, all regular n -dimensional submanifolds are locally equivalent.

Definition 3.5. A curve in a manifold M is defined by a smooth map $\phi : t \rightarrow M$ where $I \subset \mathbb{R}$ is an open subinterval.

Remark 3.6. Irregular submanifolds are trickier, since the same subset can be parametrized as a submanifold in several inequivalent ways.

Example 3.7. Let the curve $\phi_0(t) = (\sin(t), 2\sin(2t))$ describes below in the plane.



The map $\phi_1(t) = (\sin(2\arctan(t)), 2\sin(4\arctan(t)))$ parametrizes the same figure, but now is one-to-one, and so describes an immersed submanifold, which is not regular since $\lim_{t \rightarrow 0} \phi(t) = 0 = \phi(0)$.

Note that the alternative one-to-one map $\phi_2(t) = (-\sin(2\arctan(t)), 2\sin(4\arctan(t)))$ defines an inequivalent means of parametrizing the figure eight.

Thus, in the irregular case, a given subset can be parametrized as a submanifold in fundamentally different ways.

Definition 3.8. Two n -dimensional manifolds N and $\bar{N}$ of a manifold M are said to overlap if their intersection $N \cap \bar{N}$ is a non empty n -dimensional submanifold of M .

They are said to strictly overlap if their union $N \cup \bar{N}$ is also a connected n -dimensional submanifold of M , containing N and $\bar{N}$ as open submanifolds.

Definition 3.9. Let $\mathcal{F}$ be a family of real-valued functions. We define the variety $S_{\mathcal{F}}$ by

$$S_{\mathcal{F}} = \{x \in M \mid \forall f \in \mathcal{F}, f(x) = 0\}$$

We will call the variety regular if $S_{\mathcal{F}}$ is not empty, and the rank if $\mathcal{F}$ is constant in a neighborhood of $S_{\mathcal{F}}$.

By the Implicit Function Theorem and the theorem of regular submanifold, we obtain the following theorem.

Theorem 3.10. *Let $\mathcal{F}$ be a family of functions defined on an m -dimensional manifold M . If $S_{\mathcal{F}}$ is regular, then it defined a regular submanifold of dimension $m - r$.*

Example 3.11. * Let $F : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$ has rank one everywhere except at the origin, and its variety - the unit sphere - is a regular 2-dimensional submanifold of $\mathbb{R}^3$.
* Let $F : (x, y, z) \mapsto xyz$ is not regular and its variety - the three coordinate planes - is not a submanifold.

4. Vector fields and 1-parameter group of transformations

Let M a manifold of dimension m .

A tangent vector to a manifold M at a point $x \in M$ is geometrically defined by the tangent vector to a smooth curve passing through x . In local coordinates, the tangent vector $X|_x$ to the parametrized curve c is determined by the derivative $X|_x = c'(t)$. The collection of all such tangent vectors forms the tangent space to M at x . Each tangent space $T_x M$ is a vector space of the same dimension of M . $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle of M which is a $2m$ -dimensional manifold, and forms a vector bundle over M .

Definition 4.1. A vector field X is smoothly (or analytically) varying assignment of tangent vector $X|_x \in T_x M$, i.e. a section of the tangent bundle TM .

We write the local coordinate formula for a vector field in the form

$$X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$$

where the coefficients ξ_i are smooth (analytic) functions.

We will now talk about 1-parameter group of transformations.

Definition 4.2. Let X be a vector field on a manifold M . A curve $c : I \rightarrow M$ in M is called an integral curve of X if, for every parameter t_0 , the vector $X_{c(t_0)}$ is tangent to the curve c at $c(t_0)$.

Remark 4.3. Let X be a vector field on M . For any point $x \in M$, there is a unique integral curve c of X , defined for $|t| < \epsilon$, for some $\epsilon > 0$, such that $x = c(0)$.

In fact, let $x_1, \dots, x_n$ be a local coordinate system in a neighborhood U of x and let $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$ on U .

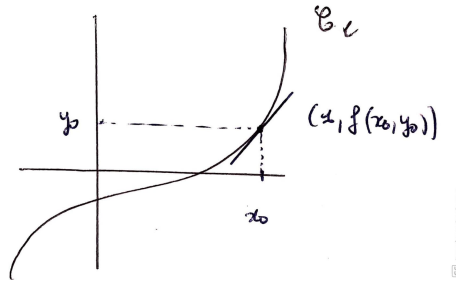
Then an integral curve c of X is a solution of

$$\frac{dc_i}{dt} = \xi_i(c_1(t), \dots, c_m(t)) \quad (1 \leq i \leq m)$$

i.e. the equation $\frac{dc}{dt} = \xi(t, c)$.

Our assertion follows from the fundamental theorem for systems of ordinary differential equations.

Example 4.4. Let $y' = f(x, y)$ be a first order of ordinary differential equation.



c is an integral curve of direction $\partial_x + f\partial_y$.

Definition 4.5. A differential transformation of M is a homeomorphism $\varphi : M \rightarrow M$ such that φ and φ^{-1} are differentiable.

Definition 4.6. A 1-parameter group of differentiable transformation of M is a map

$$\varphi : \begin{cases} \mathbb{R} \times M \longrightarrow M \\ (t, x) \longmapsto \varphi_t(x) \end{cases}$$

which satisfies the following conditions :

- 1) For each $t \in \mathbb{R}$, $\varphi_t : p \mapsto \varphi_t(x)$ is a transformation of M
- 2) For all $s, t \in \mathbb{R}$, and $x \in M$, $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$

Remark 4.7. Each 1-parameter group of transformation φ_t induces a vector field X as follow ;

For every point $x \in M$, $X|_x$ is the vector tangent to the curve $c = \varphi_{(\cdot)}(x)$ called the orbit of x at $x = \varphi_0(x)$.

By the equation solved by φ , we will be able to take, further, the following notation :

$$\varphi_t(x) = \exp(tX|_x) \cdot x \quad (t \in \mathbb{R}, x \in M)$$

To say that φ induces X means by the new notation

$$X|_x = \frac{d}{dt}(\exp(tX|_x) \cdot x)|_{t=0} \quad (x \in M)$$

Remark 4.8. For $x \in M$, The orbit $\varphi_{(\cdot)}(x)$ is an integral curve of X starting at x .

Remark 4.9. A local 1-parameter group of local transformations can be defined in the same way, except that φ is defined only for t in a neighborhood of 0 and x in an open set of M . More precisely, let $I_\epsilon = (-\epsilon, \epsilon)$ and U an open set of M . A local 1-parameter group of local transformations defined is a map

$$\varphi : \begin{cases} I_\epsilon \times U \longrightarrow M \\ (t, x) \longmapsto \varphi_t(x) \end{cases}$$

which satisfies the following conditions :

- 1) For each $t \in I_\epsilon$, $\varphi_t : x \mapsto \varphi_t(x)$ is a transformation of U onto $\varphi_t(U)$
- 2) For all $s, t \in I_\epsilon$, and $x, \varphi_s(x) \in U$, $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$.

As the case of 1-parameter group of transformations, φ induces a vector field X defined on U , an open set of M .

We have the converse :

Proposition 4.10. *Let X be a vector field on a manifold M . For each point $x \in M$, there exist a neighborhood U of x , a $\epsilon > 0$, and a local 1-parameter group of local transformations ; for $t \in I_\epsilon$, $\varphi_t : U \rightarrow M$, such that φ induces the given X .*

Definition 4.11. In the case of the proposition, we will say that X generates a local 1-parameter group of local transformations in a neighborhood of each point $x \in M$, denoted φ . Often, we say that φ is the flow generated by X .

Definition 4.12. Let X be a vector field on M .

X is said complete if X generates a global 1-parameter group of transformations of M .

Remark 4.13. If φ is defined on $I_\epsilon \times M$ for a $\epsilon > 0$, then the vector field is complete. Indeed, the solution of $\frac{dc}{dt} = \xi(t, c)$ can be prolonged by local lipschitzianity of ξ .

Proof of the Proposition. Let $p \in M$ and $(x_1, \dots, x_m)$ a local coordinate chart in a neighborhood W of x such that $x_1(x) = \dots = x_m(x) = 0$.

Let $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$ in W .

consider the following system of ordinary linear differential equations

$$\frac{df_i}{dt} = \xi_i(f_1, \dots, f_m) \quad (1 \leq i \leq m)$$

By the fundamental theorem for system of ordinary differential equations, there exists a unique set of functions

$$f_i : \begin{cases} I_\epsilon \times B(0, \delta) \longrightarrow M \\ (t, x) \longmapsto f_i(t, x) \end{cases} \quad (1 \leq i \leq m)$$

which form a solution of the differential equation for each fixed $x = (x_1, \dots, x_m)$ and satisfy the initial conditions

$$f_i(0, x) = x_i \quad (1 \leq i \leq m)$$

Set $U = B(0, \delta)$ and

$$\varphi : \begin{cases} I_\epsilon \times U \longrightarrow M \\ (t, x) \longmapsto (f_1(t, x), \dots, f_m(t, x)) \end{cases}$$

If t, s and $t + s \in I_\epsilon$, $x, \varphi_s(x) \in U$, then the functions defined by

$$g_i : \begin{cases} I_\epsilon \longrightarrow M \\ t \longmapsto f_i(t + s, x) \end{cases} \quad (1 \leq i \leq m)$$

are easily seen to be a solution of the differential equation for the initial conditions $(f_1(s, x), \dots, f_m(s, x))$.

By uniqueness of the solution, we have for all $1 \leq i \leq m$, $g_i = f(\cdot, \varphi_s(x))$.

This proves that

$$\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x) \quad (s, t \in I_\epsilon)$$

Since, φ_0 is the identity transformation of U , there exist $\delta > 0$, and $\epsilon > 0$, such that, for $U = \{x \mid \forall 1 \leq i \leq m, |x_i| < \delta\}$, $t \in I_\epsilon$, $\varphi_t(U) \subset U$.

Hence, for every $x \in U$ and $t \in I_\epsilon$, $\varphi_{-t}(\varphi_t(x)) = \varphi_t(\varphi_{-t}(x)) = \varphi_0(x) = x$.

This proves that for $t \in I_\epsilon$, φ_t is a diffeomorphism of U . Thus, φ is a local 1-parameter group of local transformations defined on $I_\epsilon \times U$ and by construction, φ induces X on U . $\square$

For $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$ on M with a local coordinate chart $(x_1, \dots, x_m)$.

There is a local 1-parameter group of local transformation φ defined on $I_\epsilon \times U$, which induces X . By the proof,

$$(x'_1, \dots, x'_m) := (f_1(t, x), \dots, f_m(t, x)) = \varphi_t(x) \quad (t \in I_\epsilon, x \in U)$$

Then we restate the proposition as follow :

Proposition 4.14. *Let X be a vector field on M .*

Let $x_1, \dots, x_m$ be a local coordinate chart.

There exists a local 1-parameter group of local transformation

$$x'_i = f_i(t; x_1, \dots, x_m) \quad (1 \leq i \leq m)$$

which induces the given X and satisfying the differential equations of the form :

$$\frac{df_i}{dt} = \xi_i(f_1, \dots, f_m) \quad (1 \leq i \leq m)$$

Remark 4.15. By definition, the flow satisfies

$$f_i(t_2; f_i(t_1; x)) = f_i(t_2 + t_1; x) \quad (1 \leq i \leq m)$$

Proposition 4.16. *On a compact manifold M , every vector field X is complete.*

Proof. For each point $x \in M$, and let $U(x)$ be a neighborhood of x and $\epsilon(x) > 0$ such that the vector field X generate the flow φ on $I_{\epsilon(x)} \times U(x)$.

Since M is compact, the open covering $\{U(x) \mid x \in M\}$ has a finite sub covering $\{U(x_i) \mid 1 \leq i \leq k\}$. Let $\epsilon := \min_{1 \leq i \leq k} \epsilon(x_i) > 0$, it is clear that φ is defined on $I_\epsilon \times M$ then on $\mathbb{R} \times M$. □

Proposition 4.17. *Let ψ a transformation of M . If a vector field X generates a flow, then the vector field ψ_*X generates*

$$\psi \circ \varphi \circ \psi : (t, x) \mapsto \psi \circ \varphi_t \circ \psi(x)$$

Proof. It is clear that $\psi \circ \varphi \circ \psi$ is a local 1-parameter group of local transformations.

Let $x \in M$ and $y = \psi^{-1}(x)$.

Since φ induces X , the vector $X|_y \in T_y M$ is tangent to the curve $c = \varphi.(y)$ at $y = c(0)$. It follows that the vector $(\psi_*X)|_x = \psi_*(X_y) \in T_x M$ is tangent to the curve $c_1 = \psi \circ \varphi.(y) = \psi \circ \varphi \circ \psi^{-1}(x)$. □

Now, we will see another equivalence problem. We need a definition.

Definition 4.18. Let X be a vector field on M .

A point $x \in M$ such that $X|_x = 0$ is said a singularity or equilibrium point.

Remark 4.19. In this case, the flow satisfies for all $t \in \mathbb{R}$, $\varphi_t(x) = \exp(tX|_x) \cdot x = x$.

Theorem 4.20 (Linearization of vector fields,1). *Let X be a vector field defined on M . If x is not a singularity, so $X|_p \neq 0$, then there exist a local coordinate $y = (y_1, \dots, y_m)$ near p such that $X = \frac{\partial}{\partial y_1}$ generates the translational flow $\exp(tX)y = (y_1 + t, y_2, \dots, y_m)$*

By the local reformulation of the flow, we can restate the theorem as follow

Theorem 4.21 (Linearization of vector fields,2). *Every local 1-parameter group of local transformations*

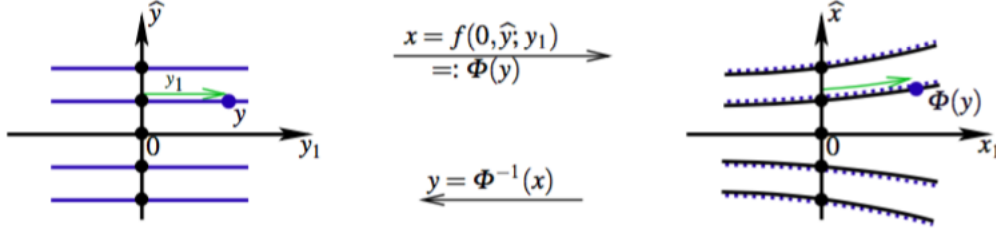
$$x'_i = f_i(t; x_1, \dots, x_m) \quad (1 \leq i \leq m)$$

which induces a vector field $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$ satisfying the differential equations of the form :

$$\frac{df_i}{dt} = \xi_i(f_1, \dots, f_m) \quad (1 \leq i \leq m)$$

is locally equivalent, through a suitable change of variables $y_i = y_i(x)$, to a group of translations :

$$y'_1 = y_1 + t, y'_2 = y_2 \dots y'_n = y_n$$



Proof. We may suppose from the beginning that the coordinate $x_1, \dots, x_m$ had been chosen so that $\xi_1(0) = 1$ and $\xi_2(0) = \dots = \xi_m(0) = 0$.

In an auxiliary space of coordinates $y_1, \dots, y_m$ drawn on the left figure, we consider all points $(0, \hat{y}) := (0, y_2, \dots, y_m)$ near the origin lying on the coordinate hyperplane which is complementary to the y_1 -axis, and we introduce the diffeomorphism

$$y \longmapsto x = x(y) := f(y_1; 0, \hat{y}) =: \Phi(y)$$

defined by the following flow up to time y_1 from $(0, \hat{y})$: this is indeed a diffeomorphism fixing the origin thanks to $\frac{\partial \Phi_1}{\partial y_1}(0) = \frac{\partial f_1}{\partial t}(0) = \xi_1(0) = 1$, to $\frac{\partial \Phi_k}{\partial y_1}(0) = \frac{\partial f_k}{\partial t}(0) = \xi_k(0) = 0$ and to $\Phi_k(0, \hat{y}) = \hat{y}_k$ for $k = 2, \dots, m$.

Consequently, we get that the flow represented on the right figure side has been straightened on the left side to be just a uniform translation directed by the y_1 -axis, because by substituting :

$$x' = f(y'_1; 0, \hat{y}') = f(y_1 + t; 0, \hat{y}) = f(t; f(y_1; 0, \hat{y})) = f(t, x)$$

We recover the uniquely defined flow $x' = f(t, x)$ when assuming that $\hat{y}' = \hat{y}$ and $y'_1 = y_1 + t$. □

Remark 4.22. By the restatement of the theorem, we see that it gives a solution to the basic equivalence problem for systems of autonomous first order systems of ordinary differential equations. However, the equivalence problem of vector fields at singularities is much more delicate.

Example 4.23. For $M = \mathbb{R}$, we have :

$$(1) \exp(t\partial_x)x = x + t$$

$$(2) \exp(tx\partial_x)x = e^t x$$

Remark 4.24. Let X be a vector field on M .

Since, $X|_x = \frac{d}{dt}(\exp(tX) \cdot p)|_{t=0}$, we have in local coordinate chart

$$\exp(tX|_x) \cdot x = x + tX|_x + O_{t \sim 0}(t^2)$$

5. Lie bracket

The most important operation on vector field is the Lie bracket or commutator.

Let M a manifold of dimension m .

Definition 5.1. Let X and Y be two vector field on M . We define the Lie bracket as the vector field noted $[X, Y]$ which satisfies for all $f : M \rightarrow \mathbb{R}$ smooth,

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

In a local coordinate chart, for $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^m \nu_i \frac{\partial}{\partial x_i}$, we have

$$[X, Y] = \sum_{i=1}^m (X(\nu_i) - Y(\xi_i)) \frac{\partial}{\partial x_i} = \sum_{i=1}^m \sum_{j=1}^m (\xi_j \frac{\partial \nu_i}{\partial x_j} - \nu_j \frac{\partial \xi_i}{\partial x_j}) \frac{\partial}{\partial x_i}$$

Proposition 5.2. Let X, Y, Z three vector fields on M . The Lie bracket is anti-symmetric ; $[X, Y] = -[Y, X]$ and satisfies the Jacobi identity ;

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Remark 5.3. The Lie bracket of X and Y two vector fields on M satisfy

$$\exp(-tY|_x) \exp(-tX|_x) \exp(tY|_x) \exp(tX|_x) x = x + t^2[X, Y]|_x + O_{t \sim 0}(t^3)$$

Example 5.4. On $\mathbb{R}$ we have $[\partial_x, x\partial_x] = \partial_x$.

Theorem 5.5. Let X, Y be two vector fields on M .

$[X, Y] = 0$ if and only if for all $t, s \in I$, for all $x \in M$, $\exp(tY|_x) \exp(tX|_x) \cdot x = \exp(tX|_x) \exp(tY|_x) \cdot x$, where I is the interval where the equation is defined.

Definition 5.6. Let X be a vector field on M and ψ a transformation of M .

X is said invariant by ψ if $\psi_* X = X$.

Proposition 5.7. Let X be a vector field on M .

X is invariant by ψ if and only if ψ commute with, φ_t , the flow of X .

We now give a geometric interpretation of the Lie bracket $[X, Y]$ of two vector fields.

Proposition 5.8. *Let X and Y be two vector fields on M . If X generates a local 1-parameter groups of local transformation φ_t , then*

$$[X, Y]_{|x} = \lim_{t \rightarrow 0} \frac{1}{t} [Y|_x - ((\varphi_t)_* Y)|_x] \quad (x \in M)$$

We need two lemma.

Lemma 5.9. *Let $\epsilon > 0$, and $I_\epsilon = (-\epsilon, \epsilon)$.*

If $f : I_\epsilon \times M \rightarrow \mathbb{R}$ is a function such that for all $x \in M$, $f(0, x) = 0$, then there is a function $g : I_\epsilon \times M \rightarrow \mathbb{R}$ such that for all $(t, x) \in I_\epsilon \times M$, $f(t, x) = tg(t, x)$.

Moreover, for $x \in M$, $g(0, x) = \frac{\partial f}{\partial t}(0, x)$.

Proof. It is sufficient to define

$$g(t, x) = \int_0^1 f'(ts, x) ds \quad (t, x) \in I_\epsilon \times M$$

□

Lemma 5.10. *Let X be a vector field which generates φ_t .*

For any function $f : M \rightarrow \mathbb{R}$, there exist a $\epsilon > 0$, and a function $g : I_\epsilon \times M \rightarrow \mathbb{R}$ such that for all $t \in I_\epsilon$, $f \circ \varphi_t = f + t \cdot g_t$ and $g_0 = Xf$ on M , where $g_t(\cdot) = g(t, \cdot)$.

Proof. Consider $f(t, x) = f(\varphi_t(x)) - f(x)$ and apply the previous lemma, then $f \circ \varphi_t = f + t \cdot g_t$ and

$$\lim_{t \rightarrow 0} \frac{1}{t} [f(\varphi_t(x)) - f(x)] = \lim_{t \rightarrow 0} \frac{1}{t} f(t, x) = \lim_{t \rightarrow 0} g_t(x) = g_0(x)$$

□

Proof of the proposition. Let $f : M \rightarrow \mathbb{R}$ a function and g_t such that $f \circ \varphi_t = f + tg_t$ and $g_0 = Xf$ on M .

Set $x(t) := \varphi_t^{-1}(x)$, then

$$((\varphi_t)_* Y)|_x f = (Y(f \circ \varphi_t))|_{x(t)} = (Yf)|_{x(t)} + t(Yg_t)|_{x(t)}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} [Y - (\varphi_t)_* Y]|_x f &= \lim_{t \rightarrow 0} \frac{1}{t} [(Yf)|_x - (Yf)_{x(t)}] - \lim_{t \rightarrow 0} (Yg_t)_{x(t)} \\ &= X|_x(Y|_x f) - Y|_x g_0(x) \\ &= [X, Y]_{|x}(f) \end{aligned}$$

□

Corollary 5.11. *With the same notation as in the previous proposition, we have more generally*

$$(\varphi_s)_*[X, Y] = \lim_{t \rightarrow 0} \frac{1}{t} [(\varphi_s)_* Y - (\varphi_{s+t})_* Y] = -\frac{d}{dt} ((\varphi_t)_* Y)|_{t=s}$$

Proof. To apply the previous proposition to $(\varphi_s)_*Y$.

□

Corollary 5.12. *Let X and Y be two vector fields which generate local 1-parameter groups φ_s and ψ_s .*

For every t, s , $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ if and only if $[X, Y] = 0$.

Proof. For every t , Y is invariant by every φ_t . By the previous proposition $[X, Y] = 0$. Conversely, if $[X, Y] = 0$, then the previous corollary gives for every t ,

$$\frac{d}{dt}((\varphi_t)_*Y) = 0$$

Therefor for every t , $(\varphi_t)_*Y$ is a constant vector field at each point of M so that Y is invariant by every φ_t . Then every ψ_s commute with every φ_t .

□

6. The differential

A smooth map $F : M \rightarrow N$ between manifolds will map smooth curves on M to smooth curves on N , and thus induce a map between their tangent vectors.

This result is a linear map called the differential of F , $dF_x : T_p M \rightarrow T_{F(x)} N$. If the parametrized curve c has tangent vector $X|_x = c'(t)$ at $x = c(t)$, then the image curve $\tilde{c} = F(c)$ will have tangent vector $Y|_{F(x)} = dF_x(X|_x) = \tilde{c}'$.

Definition 6.1. Let M, N be two manifolds, and $F : M \rightarrow N$ a smooth map.

We define $dF : TM \rightarrow TN$ the differential of F as follow ; In local coordinate chart, for $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i} \in TM$

$$dF_x(X|_x) = \sum_{j=1}^m \left(\sum_{i=1}^m \xi_i \frac{\partial F_j}{\partial x_i} \right)(x) \frac{\partial}{\partial y_j} \Big|_x$$

We can also denote the differential F_* .

Remark 6.2. If X is vector field is mapped to a well-defined vector field $dF(X)$ on N then the flows match up, meaning

$$F(\exp(tX) \cdot x) = \exp(tdF(X))F(x) \quad (x \in M)$$

The definition of the Lie bracket of two vector field give us

Proposition 6.3. *Let X and Y be two vector fields which are tangent to a submanifold $N \subset M$; $X, Y \in TN$. Then their Lie bracket $[X, Y] \in TN$.*

Example 6.4. Let $X = x\partial_y - y\partial_x$ and $Y = y\partial_z - z\partial_y$ generates the rotational flows around the z -axis and the x -axis respectively. They are both tangent to $S^2 \subset \mathbb{R}^3$, then their Lie bracket $[X, Y] = x\partial_z - z\partial_x$, which generates the rotational flows around the y -axis, is also tangent to S^2 .

7. Differential forms

In this section, we firstly talk about some recalls of differential forms $\mathbb{R}$ -valued before talking about V -valued (V a finite-dimensional vector space).

Let M a manifold of dimension m .

Definition 7.1. Let $x \in M$, a real-valued linear function $\omega : T_x M \rightarrow \mathbb{R}$ on the tangent space is said to define a one-form at x .

Definition 7.2. The space of one-forms at p is the dual vector space of $T_x M$ and is called the cotangent space, denoted $T_x^* M$.

We define also the cotangent bundle $T^* M = \bigcup_{x \in M} T_x^* M$.

Definition 7.3. A frame on an M is an ordered set of vector fields $\mathcal{V} = \{v_1, \dots, v_m\}$ such that for all $x \in M$, $\mathcal{V}|_x$ is a basis of $T_x M$.

Example 7.4. The coordinate vector field $\{\frac{\partial}{\partial x_i}\}_{1 \leq i \leq m}$ is a frame on M .

Definition 7.5. A coframe on an M is an ordered set of one-form $\mathcal{W} = \{\omega_1, \dots, \omega_m\}$ such that for all $x \in M$, $\mathcal{W}|_x$ is a basis of $T_x^* M$.

Example 7.6. The set $\{dx_i\}_{1 \leq i \leq m}$ is a coframe on M .

Definition 7.7. A differential k -form map $\Omega : TM^k \rightarrow \mathbb{R}$ such that at a point $x \in M$, $\Omega(x) : (T_x M)^k \rightarrow \mathbb{R}$ is a k -linear map which is skew-symmetric (or antisymmetric) in its arguments, meaning that

$$\Omega(x)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sign}(\sigma) \Omega(x)(v_1, \dots, v_k) \quad (v_1, \dots, v_k \in T_x M, \sigma \in \mathfrak{S}_k)$$

Proposition-Definition 7.8. Let $\omega_1, \dots, \omega_k$ be k one-form. We define the wedge product $\omega_1 \wedge \dots \wedge \omega_k$ as follow

$$(\omega_1 \wedge \dots \wedge \omega_k)(x)(v_1, \dots, v_k) = \det((\omega_i(x)(v_j))_{1 \leq i, j \leq k}) \quad (x \in M, v_1, \dots, v_k \in T_x M)$$

Remark 7.9. For a k -form ω , if k is odd, $\omega \wedge \omega = 0$.

Proposition 7.10. *Let $f_1, \dots, f_k : M \rightarrow \mathbb{R}$ smooth functions.
If $df_1 \wedge \dots \wedge df_k \neq 0$, then they are functionally independent.
If $df_1 \wedge \dots \wedge df_k = 0$, then $f_1, \dots, f_k$ are functionally dependent.*

Definition 7.11. Let $F : M \rightarrow N$ be a smooth map.

We define the pull-back of F denoted F^* , which maps a differential form on N back to a differential form on M .

Let $x \in M$, if θ is one-form of N at $y = F(x)$, then $\omega := F^*(\theta)$ is a one-form of M defined by

$$\omega(X) := F^*(\theta)(X) = \theta(F(X)) \quad (X \in T_x M)$$

Theorem 7.12. *Let $F : M \rightarrow N$ be a smooth map and Let Ω be a differential form on N .*

$$d(F^*\Omega) = F^*(d\Omega)$$

Definition 7.13. Let $F : \tilde{N} \rightarrow M$ be a parametrized map of a submanifold $N \subset M$.

We define the restriction of the submanifold N , and denote it $\Omega|_N$ as the pull-back $F^*\Omega$ of a differential form Ω on M .

Corollary 7.14. *Let $N \subset M$ a submanifold and Ω be a differential form such that $\Omega|_N = 0$, then $d\Omega|_N = 0$.*

Theorem 7.15. *Let Ω be a differential form.*

$$d(d\Omega) = 0$$

Definition 7.16. A differential form Ω is said to be closed if $d\Omega = 0$.

Let $k > 0$. A differential k -form Ω is said exact if it exists Θ a differential $(k-1)$ -form such that $\Omega = d\Theta$.

Theorem 7.17. *Let $k > 0$. Let $M \subset \mathbb{R}^m$ be a convex open subset, and Ω be a closed k -form defined on M , then Ω is exact.*

Then, we discuss about a generalization of $\mathbb{R}$ -valued differential forms.

Let V be a finite dimensional vector space such that $v := \dim(V)$.

Definition 7.18. Let $k \geq 0$. A smooth V -valued k -form on M is a smooth map

$$\omega : \bigoplus_{i=1}^k TM \rightarrow V$$

such that for all $x \in M$, $\omega(x) : \bigoplus_{i=1}^k T_x M \rightarrow V$ is multilinear and totally skew-symmetric in its vector arguments.

Definition 7.19. The k -forms make up an infinite dimensional vector space denoted by $A^k(M, V)$. Let $A(M, V) = \bigoplus_{k=1}^m A^k(M, V)$.

Remark 7.20. Let $f_1, \dots, f_v$ a basis of V .

1) we have $\omega \in A(M, V)$ if and only if $\omega \in A^k(M, V)$ for some $0 \leq k \leq m$.
Let $x \in M$ and $(U; x_1, \dots, x_m)$ a local coordinate.
For all $y \in U(x)$, a neighborhood of x

$$\begin{aligned} \omega(y) &= \sum_{i=1}^v \omega_i(y) f_i \\ &= \sum_{i=1}^v \left(\sum_{|I|=k} a_{i,I}(y) dy_I \right) f_i \end{aligned}$$

where for $0 \leq k \leq m$, $\omega_i(y) \in A^k(M, V)$, $(y_1, \dots, y_m)$ are local coordinate of y and $I = (1 \leq i_1 < i_2 < \dots < i_k \leq m)$.

2) A smooth map $f : M \rightarrow N$ induces the pullback map $f^* : A^k(N, V) \rightarrow A^k(M, V)$ defined by

$$(f^*\omega)(x)(X_1, \dots, X_k) = \omega(f(x))(f_*X_1, \dots, f_*X_k) \quad (x \in M, X_1, \dots, X_k \in T_x M)$$

Definition 7.21. The exterior product of a V_1 -valued k -form ω_1 on M with a V_2 -valued l -form ω_2 on M is a $V_1 \otimes V_2$ -valued $(k + l)$ -form noted $\omega_1 \wedge \omega_2$ defined by

$$(\omega_1 \wedge \omega_2)(x)(X_1, \dots, X_{k+l}) = \sum_{\sigma \in \mathfrak{S}_{k+l}} \epsilon(\sigma) \omega_1(x)(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \otimes \omega_2(x)(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})$$

$(x \in M, X_1, \dots, X_{k+l} \in T_x M)$

where $\epsilon(\sigma)$ is the signature of σ .

Remark 7.22. If $f : M \rightarrow N$ is a smooth map, then $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$.

As for the case of $\mathbb{R}$ -valued differential forms, we define the k^{th} exterior derivative $d^k : A^k(M, V) \rightarrow A^{k+1}(M, V)$ by the following way ;

Let $f_1, \dots, f_v$ a basis of V , $p \in M$ and $(U; x_1, \dots, x_m)$ a coordinate chart of p .

For $x \in U$, $\omega_x = \sum_{i=1}^v \sum_{|I|=k} a_{i,I}(x) dx_I f_i$

$$(d^k \omega)(x) = \sum_{i=1}^v \sum_{|I|=k} da_{i,I}(x) \wedge dx_I f_i$$

Quickly, we use the notation $d = d^k$ without specified the dimension of the form space.

This value is independent of the choice of the coordinate chart on M (known with the $\mathbb{R}$ -valued differential forms).

Moreover, if we have $(f_i)_{1 \leq i \leq v}$ and $(e_j)_{1 \leq j \leq v}$ two basis of V , then $(d\omega)(x)^{(f)} = (d\omega)(x)^{(e)}$ with the following relations

$$\left\{ \begin{array}{l} f_i = \sum_{j=1}^v P_{i,j} e_j \\ a_{i,I,(e)}(x) = \sum_{j=1}^v P_{j,i} a_{j,I,(f)}(x) \end{array} \right. \quad ((i,j,x,P) \in [1,v] \times M \times GL_v(\mathbb{R}))$$

Then the exterior derivative for V -valued k -form is well defined.

It follows by $\mathbb{R}$ -valued forms the following lemmas.

Lemma 7.23. (1) Let ω_1, ω_2 be V -valued forms,

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$$

(2) Let V_1 -valued k -form ω_1 on M with V_2 -valued l -form ω_2 on M ,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$$

(3) Let ω be a V -valued form.

$$d(d\omega) = 0$$

Lemma 7.24. If two vector-valued operators on forms satisfy the three properties of the previous lemma and agree on functions, then they are identical.

Corollary 7.25. There is a unique operator $d : A^k(M, V) \rightarrow A^{k+1}(M, V)$ defined for all $k \in \mathbb{N}$ and all V , that satisfies (1), (2), (3) and for $f : M \rightarrow V$, reduces to $df = \sum_{i=1}^m \frac{\partial f}{\partial x_i} dx_i$.

Let $X \in TM$ and $f : M \rightarrow \mathbb{R}$, we know that $d_p f(X) = X|_p(f)$. Then we have

Proposition 7.26. *For all $x \in M$, $X \in TM$, and $f : M \rightarrow \mathbb{R}$ smooth,*

$$df_x(X) = X|_x(f)$$

There are similar expressions for the derivate $d\omega$ of a k -form ω valued on $k + 1$ vector field $X_1, \dots, X_{k+1}$.

Lemma 7.27 (Cartan's formula, 1). *Let ω be a V -valued 1-form on M and X, Y be two vector fields on M .*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

Proof. By linearity of d , it enough to assume $\omega = f dx_i \otimes e$ with $f : M \rightarrow \mathbb{R}$, $e \in V$ and $(U; x_1, \dots, x_m)$ a coordinate chart.

If the lemma is true for $\omega = f dx_i$, it will be true for $\omega = f dx_i \otimes e$. Then assume $\omega = f dx_i$ or more generally $\omega = f dg$ where $g : M \rightarrow \mathbb{R}$.

We have $d\omega = df \wedge dg$ and so

$$d\omega(X, Y) = df(X)dg(Y) - df(Y)dg(X) = X(f)Y(g) - Y(f)X(g)$$

and

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) &= X(fdg(Y)) - Y(fdg(X)) - fdg([X, Y]) \\ &= X(fY(g)) - Y(fX(g)) - f[X, Y](g) \\ &= X(f)Y(g) + fX(Y(g)) - Y(f)X(g) - fY(X(g)) \\ &\quad - f[X, Y](g) \\ &= X(f)Y(g) - Y(f)X(g) \end{aligned}$$

□

Corollary 7.28 (Cartan's formula, 2). *Let ω be a V -valued k form on M .*

We have for $X_1, \dots, X_{k+1} \in TM$

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \end{aligned}$$

Proof. Let $\omega \in A^p(M, V)$.

To prove this, we use the following theorem proved in Spivak [1] page 118 (Theorem 2)

Theorem 7.29. *If $\mathcal{A} : TM^k \rightarrow \mathcal{F} = \{f \in \mathcal{C}^\infty\}$ is multilinear over $\mathcal{F}$ then there exists a unique tensor field A with $\mathcal{A} = \overline{A}$ where $\overline{A}(X_1, \dots, X_p)(x) = A(x)(X_1(x), \dots, X_p(x))$.*

Call Σ_1 et Σ_2 the two sums above. Clearly, the operator which send $(X_1, \dots, X_k, X_{k+1})$ on $\Sigma_1 + \Sigma_2$ is multilinear for the multiplication of vector field by real constants.

Moreover, we can see that this operator is, in fact, multilinear for the multiplication by $\mathcal{C}^\infty$ functions on M .

Indeed, for Σ_1 , take any index $1 \leq i_0 \leq k+1$, replace the vector field X_{i_0} by $f X_{i_0}$ where $f: M \rightarrow \mathbb{R}$ is $\mathcal{C}^\infty$. From definition of k -forms, they are linear compared to multiplication by functions.

Then for all $i \neq i_0$, we have :

$$\omega(X_1, \dots, f X_{i_0}, \dots, \widehat{X}_i, \dots, X_{k+1}) = f \omega(X_1, \dots, X_{i_0}, \dots, \widehat{X}_i, \dots, X_{k+1}),$$

where we have fixed ideas, assume in the written that $i_0 < i$.

Then the first sum where we extract the term $i = i_0$:

$$\Sigma_1 := (-1)^{i_0+1} X_{i_0} \left(\omega(X_1, \dots, \widehat{X}_{i_0}, \dots, X_{k+1}) \right) + \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} X_i \left(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right),$$

becomes by Leibniz's formula :

$$\begin{aligned} \Sigma'_1 &:= (-1)^{i_0+1} f X_{i_0} \left(\omega(X_1, \dots, \widehat{X}_{i_0}, \dots, X_{k+1}) \right) + \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} X_i \left(f \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right) \\ &= (-1)^{i_0+1} f X_{i_0} \left(\omega(X_1, \dots, \widehat{X}_{i_0}, \dots, X_{k+1}) \right) + \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} f X_i \left(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right) \\ &\quad + \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} X_i f \cdot \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &= f \Sigma_1 + \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} X_i f \cdot \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}). \end{aligned}$$

Now, for Σ_2 , take any index $1 \leq i_0 \leq k+1$, replace the vector field X_{i_0} by $f X_{i_0}$ where $f: M \rightarrow \mathbb{R}$ is $\mathcal{C}^\infty$.

To proceed, we decompose Σ'_2 plainly in three sub-sums :

$$\sum_{1 \leq i < j \leq k+1} = \sum_{1 \leq i = i_0 \leq j \leq k+1} + \sum_{1 \leq i \leq j = i_0 \leq k+1} + \sum_{i_0 \notin \{i, j\}}.$$

For the two first sums, we use

$$[f X_{i_0}, X_j] = f [X_{i_0}, X_j] - X_i(f) \cdot X_{i_0} \quad (1 \leq i_0 < j \leq k+1)$$

$$[X_i, f X_{i_0}] = f [X_i, X_{i_0}] + X_i(f) \cdot X_{i_0} \quad (1 \leq i < i_0 \leq k+1)$$

Then

$$\begin{aligned}
\Sigma'_2 &= \sum_{1 \leq i=i_0 < j \leq k+1} (-1)^{i_0+j} X_i \left(\omega([fX_{i_0}, X_j], X_0, \dots, \widehat{X}_{i_0}, \dots, \widehat{X}_j, \dots, X_{k+1}) \right) \\
&+ \sum_{1 \leq i < j=i_0 \leq k+1} (-1)^{i+i_0} X_i \left(\omega([X_i, fX_{i_0}], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_{i_0}, \dots, X_{k+1}) \right) \\
&+ f \sum_{i_0 \notin \{i, j\}} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \\
&= f\Sigma_2 \\
&+ \sum_{1 \leq i < j=i_0 \leq k+1} (-1)^{i+i_0} X_i(f) \cdot (\omega(X_{i_0}, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_{i_0}, \dots, X_{k+1})) \\
&- \sum_{1 \leq i=i_0 < j \leq k+1} (-1)^{i_0+j} X_j(f) \cdot (\omega(X_{i_0}, X_0, \dots, \widehat{X}_{i_0}, \dots, \widehat{X}_j, \dots, X_{k+1})) \\
&= f\Sigma_2 - \sum_{\substack{1 \leq i \leq k+1 \\ i \neq i_0}} (-1)^{i_0+i} X_i(f) \cdot (\omega(X_{i_0}, X_0, \dots, \widehat{X}_i, \dots, X_{k+1}))
\end{aligned}$$

Then by multilinearity under the multiplication by $\mathcal{C}^\infty$ functions, there exists a unique tensor field noted $\delta\omega$ satisfying

$$\begin{aligned}
\delta\omega &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right) \\
&+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}).
\end{aligned}$$

where $\delta : A^k(M, V) \rightarrow A^{k+1}(M, V)$.

To show that $\delta = d$, we can assume $\omega = f dx_1 \wedge \dots \wedge dx_k$ with $(U, x_1, \dots, x_m)$ a local coordinate on M .

We have

$$\delta\omega = \sum_{1 \leq i_1 < \dots < i_{k+1} \leq m} \delta\omega\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_{k+1}}}\right) dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}}.$$

We have $\delta\omega\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_{k+1}}}\right) = 0$ unless some $(i_1, \dots, \widehat{i}_k, \dots, i_{k+1})$ is a permutation of $(1, \dots, k)$.

Since the i 's are increasing, this happened only if

$$(i_1, \dots, i_k) = (1, \dots, k, j), \text{ for } j > k$$

Then $\delta\omega(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}}, \frac{\partial}{\partial x_j}) = (-1)^k \frac{\partial f}{\partial x_j}$

Therefore

$$\begin{aligned}\delta\omega &= \sum_{j>k} (-1)^k \frac{\partial f}{\partial x_j} dx_1 \wedge \dots \wedge dx_k \wedge dx_j \\ &= \sum_{j>k} \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \dots \wedge dx_k \\ &= \sum_{1 \leq j \leq m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \dots \wedge dx_k \\ &= d\omega\end{aligned}$$

□

Proposition 7.30. *If $f : M \rightarrow N$ is a smooth map, then the following diagram commutes for all $k \in \mathbb{N}$*

$$\begin{array}{ccc} A^k(N, V) & \xrightarrow{d} & A^{k+1}(N, V) \\ f^* \downarrow & & \downarrow f^* \\ A^k(M, V) & \xrightarrow{d} & A^{k+1}(M, V) \end{array}$$

In other words,

$$f^*(d\omega) = d(f^*\omega) \quad (k \in \mathbb{N}, \omega \in A^k(M, V))$$

Proof. The proof is by induction on k .

Let $X \in TM$.

* $k = 0$:

$$f^*(dg(X)) = dg(f_*(X)) = f_*(X)g = X(f^*g) = X(gf) = d(fg)(X)$$

* Let $k \in \mathbb{N}$, and assume the result for the j -forms with $j < k$.

By linearity we can assume $\omega = gdx_I$ where $|I| = k$.

We separate ω as follow :

$$\omega = gdx_J \wedge dx_K$$

with $|J|, |K| < k$ and $|J| + |K| = |I|$.

Then

$$\begin{aligned}d(f^*\omega) &= d(f^*(gdx_J \wedge dx_K)) \\ &= d(f^*(gdx_J) \wedge f^*(dx_K)) \\ &= d(f^*(gdx_J)) \wedge f^*(dx_K) \pm f^*(gdx_J) \wedge d(f^*(dx_K)) \\ &= f^*d(gdx_J) \wedge f^*(dx_K) \pm f^*(gdx_J) \wedge f^*(dx_K) \\ &= f^*(d(gdx_J \wedge dx_K)) \\ &= f^*(d\omega)\end{aligned}$$

□

We can notice that the wedge product of two V -valued differential forms gives a new differential form but which not live in the same space ; $V \otimes V$. We will talk about map allowing to solve this problem.

Let $\varphi : V_1 \rightarrow V_2$ be a linear map of vector spaces, then it induces a linear map $\varphi_* : A(M, V_1) \rightarrow A(M, V_2)$ defined by

$$\varphi_*(\omega)(X_1, \dots, X_k) = \varphi(\omega(X_1, \dots, X_k)) \quad (k; X_1, \dots, X_k) \in \mathbb{N} \times TM^k$$

We can remark that φ_* commute with d ;

$$\varphi_*(d\omega) = d(\varphi_*(\omega)) \quad (\omega \in A(M, V_1))$$

Let $V_1 = V \otimes V$ and $V_2 = V$.

We assume that $\varphi =: m$ is not necessarily commutative or associate (in fact we can think $m := [,]$ is the Lie bracket).

Then we obtain a corresponding map

$$m_* : \begin{cases} A(M, V \otimes V) \longrightarrow A(M, V) \\ \omega \longmapsto m_*(\omega) \end{cases}$$

This map turns $A(M, V)$ into an algebra with multiplication given by $m_* \circ \wedge$.

Remark 7.31. Let ω_1, ω_2 be V -valued k -forms and l -forms respectively.

$$d(m_*(\omega_1 \wedge \omega_2)) = m_*(d\omega_1 \wedge \omega_2) + (-1)^k m_*(\omega_1 \wedge d\omega_2)$$

An important particular case is $V = \mathfrak{g}$ a Lie group and $m : \begin{cases} \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \\ v \otimes w \longmapsto [v, w] \end{cases}$ the Lie bracket.

Definition 7.32. Let ω^1, ω^2 be V -valued differential forms. We define the Lie bracket of two forms noted $[\omega^1, \omega^2]$ by

$$\omega^1 \wedge_{\mathfrak{g}} \omega^2 := m_*(\omega^1 \wedge \omega^2)$$

Lemma 7.33. Let ω_1, ω_2 be V -valued differential forms and $X, Y \in TM$. We have

$$(\omega^1 \wedge_{\mathfrak{g}} \omega^2)(X, Y) = [\omega^1(X), \omega^2(Y)] + [\omega^2(X), \omega^1(Y)]$$

In particular for ω a V -valued form,

$$[\omega(X), \omega(Y)] = \frac{1}{2}(\omega \wedge_{\mathfrak{g}} \omega)(X, Y);$$

$$\omega^1 \wedge_{\mathfrak{g}} \omega^2 = \omega^2 \wedge_{\mathfrak{g}} \omega^1.$$

Proof.

$$\begin{aligned}
(\omega^1 \wedge_{\mathfrak{g}} \omega^2)(X, Y) &= m_*(\omega^1 \wedge \omega^2)(X, Y) \\
&= m(\omega^1 \wedge \omega^2(X, Y)) \\
&= m(\omega^1(X) \otimes \omega^2(Y) - \omega^1(Y) \otimes \omega^2(X)) \\
&= m(\omega^1(X) \otimes \omega^2(Y)) - m(\omega^1(Y) \otimes \omega^2(X)) \\
&= [\omega^1(X), \omega^2(Y)] - [\omega^1(Y), \omega^2(X)] \\
&= [\omega^1(X), \omega^2(Y)] + [\omega^2(X), \omega^1(Y)]
\end{aligned}$$

□

For $\{e_1, \dots, e_r\}$ a basis of $\mathfrak{g}$, a 1-form $\omega \in A(M, \mathfrak{g})$

$$\omega = \sum_{1 \leq k \leq r} \omega_k e_k$$

where $\omega_k \in A(M, \mathbb{R})$ and for ω^1 and $\omega^2 \in A(M, \mathfrak{g})$, we have

$$\omega^1 \wedge_{\mathfrak{g}} \omega^2 = \sum_{1 \leq k_1, k_2 \leq r} \omega_{k_1}^1 \wedge \omega_{k_2}^2 [e_{k_1}, e_{k_2}]_{\mathfrak{g}}$$

Remark 7.34. Let $\omega^p \in A^p(M, \mathfrak{g})$, $\omega^q \in A^q(M, \mathfrak{g})$ and $\omega^r \in A^r(M, \mathfrak{g})$. We have :

- (1) $d(\omega^p \wedge_{\mathfrak{g}} \omega^q) = d\omega^p \wedge_{\mathfrak{g}} \omega^q + (-1)^p(\omega^p \wedge_{\mathfrak{g}} d\omega^q)$
- (2) $\omega^q \wedge_{\mathfrak{g}} \omega^p = (-1)^{pq+1}(\omega^p \wedge_{\mathfrak{g}} \omega^q)$
- (3) $(-1)^{rq}((\omega^p \wedge_{\mathfrak{g}} \omega^q) \wedge_{\mathfrak{g}} \omega^r) + (-1)^{pq}((\omega^q \wedge_{\mathfrak{g}} \omega^r) \wedge_{\mathfrak{g}} \omega^p) + (-1)^q((\omega^r \wedge_{\mathfrak{g}} \omega^p \wedge_{\mathfrak{g}} \omega^q)) = 0.$

Remark 7.35. Let $f : M \rightarrow N$ be a smooth map and let $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism. Then $\varphi_* f^* : A(N, \mathfrak{h}) \rightarrow A(M, \mathfrak{g})$ is a homomorphism of Lie algebras.

8. Cartan's lemmas

Now, we see important lemmas.

Lemma 8.1 (of Cartan, 1). *On $\mathbb{R}^{N \geq 1}$, equipped with a number $1 \leq M \leq N$ of local independent differential one-forms $\{\tau_1, \dots, \tau_M\}$, if given differential one-form $\{\lambda_1, \dots, \lambda_M\}$ satisfy*

$$0 = \sum_{1 \leq j \leq M} \tau_j \wedge \lambda_j$$

Then

$$\lambda_i = 0 \text{ mod } (\tau_1, \dots, \tau_M) \quad (1 \leq i \leq M)$$

Precisely, there exist functions $(A_{i,j})_{1 \leq i,j \leq M}$ such that

$$\lambda_i = \sum_{1 \leq j \leq M} A_{i,j} \tau_j \quad (1 \leq i \leq M)$$

$$A_{i,j} = A_{j,i} \quad (1 \leq i < j \leq M)$$

Proof. Complete $\{\tau_1, \dots, \tau_M\}$ as a coframe $\{\tau_1, \dots, \tau_M, \chi_1, \dots, \chi_{N-M}\}$ and decompose

$$\lambda_i = \sum_{1 \leq j \leq M} A_{i,j} \tau_j + \sum_{1 \leq k \leq N-M} B_{i,k} \chi_k \quad (1 \leq i \leq M)$$

By

$$0 = \sum_{1 \leq j \leq M} \tau_j \wedge \lambda_j = \sum_{1 \leq i \leq M} \sum_{1 \leq j \leq M} A_{i,j} \tau_i \wedge \tau_j + \sum_{1 \leq i \leq M} \sum_{1 \leq k \leq N-M} B_{i,k} \tau_i \wedge \chi_k$$

Since $\{\tau, \chi\} =: \{\lambda_i \mid 1 \leq i \leq N\}$ is a coframe, then $\{\lambda_{i_1} \wedge \lambda_{i_2} \mid 1 \leq i_1 < i_2 \leq N\}$ form a basis of 2-form.

Then

$$B_{i,k} = 0 \quad (1 \leq i \leq M, 1 \leq k \leq N-M)$$

Then

$$\begin{aligned} 0 &= \sum_{1 \leq i \leq M} \sum_{1 \leq j \leq M} A_{i,j} \tau_i \wedge \tau_j \\ \lambda_i &= \sum_{1 \leq j \leq M} A_{i,j} \tau_j \end{aligned} \quad (1 \leq i \leq M)$$

We have

$$\begin{aligned} 0 &= \sum_{1 \leq i \leq M} \sum_{1 \leq j \leq M} A_{i,j} \tau_i \wedge \tau_j \\ &= \sum_{1 \leq i \leq M} \left(\sum_{1 \leq i < j \leq M} A_{i,j} \tau_i \wedge \tau_j + \sum_{1 \leq j < i \leq M} A_{i,j} \tau_i \wedge \tau_j + \sum_{i=j} 0 \right) \\ &= \sum_{1 \leq i \leq M} \left(\sum_{1 \leq i < j \leq M} A_{i,j} \tau_i \wedge \tau_j + \sum_{1 \leq i < j \leq M} A_{j,i} \tau_j \wedge \tau_i \right) \\ &= \sum_{1 \leq i \leq M} \sum_{1 \leq i < j \leq M} (A_{i,j} - A_{j,i}) \tau_i \wedge \tau_j \end{aligned}$$

Then

$$A_{i,j} = A_{j,i} \quad (1 \leq i \leq M)$$

□

Lemma 8.2 (of Cartan,2). *On $\mathbb{R}^{N \geq 1}$ equipped with a number $1 \leq M \leq N$ of local independent differential 1-forms $\{\tau_1, \dots, \tau_M\}$, if given differential 2-forms $\{\Omega_1, \dots, \Omega_M\}$ satisfy*

$$0 = \sum_{1 \leq i \leq M} \tau_i \wedge \Omega_i$$

Then

$$\Omega_i = 0 \text{ mod } (\tau_1, \dots, \tau_M) \quad (1 \leq i \leq M)$$

Precisely, completing $\{\tau_1, \dots, \tau_M\}$ as a coframe $\{\tau_1, \dots, \tau_M, \chi_1, \dots, \chi_{N-M}\}$, there exist functions A_{i,j_1,j_2} and $B_{i,j,k}$ such that

$$\Omega_i = \sum_{1 \leq j_1 < j_2 \leq M} A_{i,j_1,j_2} \tau_{j_1} \wedge \tau_{j_2} + \sum_{1 \leq j \leq M} \sum_{1 \leq k \leq N-M} B_{i,j,k} \tau_j \wedge \chi_k \quad (1 \leq i \leq M)$$

where

$$\begin{aligned} A_{i,j_1,j_2} - A_{j_1,i,j_2} + A_{j_2,i,j_1} &= 0 \quad (1 \leq i < j_1 < j_2 \leq M) \\ B_{i,j,k} &= B_{j,i,k} \quad (1 \leq i < j \leq M, 1 \leq k \leq N-M) \end{aligned}$$

Proof. For $1 \leq i \leq M$, decompose Ω_i in

$$\{\tau_{j_1} \wedge \tau_{j_2}, \tau_j \wedge \chi_k, \chi_{k_1} \wedge \chi_{k_2}\}_{\substack{1 \leq k \leq N-M, 1 \leq j \leq M, 1 \leq j_1 < j_2 \leq M \\ 1 \leq k_1 < k_2 \leq N-M}}$$

Then

$$\Omega_i = \sum_{1 \leq j_1 < j_2 \leq M} A_{i,j_1,j_2} \tau_{j_1} \wedge \tau_{j_2} + \sum_{1 \leq j \leq M} \sum_{1 \leq k \leq N-M} B_{i,j,k} \tau_j \wedge \chi_k + \sum_{1 \leq k_1 < k_2 \leq N-M} C_{i,k_1,k_2} \chi_{k_1} \wedge \chi_{k_2}$$

By

$$\begin{aligned} 0 &= \sum_{1 \leq i \leq M} \tau_i \wedge \Omega_i \\ &= \sum_{\substack{1 \leq i \leq M \\ 1 \leq j_1 < j_2 \leq M}} A_{i,j_1,j_2} \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} + \sum_{\substack{1 \leq j \leq M \\ 1 \leq k \leq N-M}} B_{i,j,k} \tau_i \wedge \tau_j \wedge \tau_k \\ &\quad + \sum_{\substack{1 \leq i \leq M \\ 1 \leq k_1 < k_2 \leq N-M}} C_{i,k_1,k_2} \tau_i \wedge \chi_{k_1} \wedge \tau_{k_2} \end{aligned}$$

The coframe $\{\tau, \chi\} =: \{\lambda_i \mid 1 \leq i \leq N\}$ then $\{\lambda_{i_1} \wedge \lambda_{i_2} \wedge \lambda_{i_3}\}_{1 \leq i_1 < i_2 < i_3 \leq N}$ form a basis of 3-form.

Then

$$C_{i,k_1,k_2} = 0 \quad (1 \leq i \leq M, 1 \leq k_1 < k_2 \leq N-M)$$

Then

$$\Omega_i = \sum_{1 \leq j_1 < j_2 \leq M} A_{i,j_1,j_2} \tau_{j_1} \wedge \tau_{j_2} + \sum_{1 \leq j \leq M} \sum_{1 \leq k \leq N-M} B_{i,j,k} \tau_j \wedge \chi_k \quad (1 \leq i \leq M)$$

and

$$\begin{aligned} 0 &= \sum_{1 \leq j_1 < j_2 \leq M} \left(\sum_{1 \leq i < j_1 < j_2 \leq M} A_{i,j_1,j_2} \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} + \sum_{1 \leq j_1 < i < j_2 \leq M} A_{i,j_1,j_2} \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} \right. \\ &\quad \left. + \sum_{1 \leq j_1 < j_2 < i \leq M} A_{i,j_1,j_2} \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} \right) + \sum_{1 \leq k \leq N-M} \left(\sum_{1 \leq i < j \leq M} B_{i,j,k} \tau_i \wedge \tau_j \wedge \chi_k \right. \\ &\quad \left. + \sum_{1 \leq j < i \leq M} B_{i,j,k} \tau_i \wedge \tau_j \wedge \chi_k \right) \\ 0 &= \sum_{1 \leq j_1 < j_2 \leq M} \left(\sum_{1 \leq i < j_1 < j_2 \leq M} [A_{i,j_1,j_2} \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} + A_{j_1,i,j_2} \tau_{j_1} \wedge \tau_i \wedge \tau_{j_2} \right. \\ &\quad \left. + A_{j_2,i,j_1} \tau_{j_2} \wedge \tau_i \wedge \tau_{j_1}] \right) + \sum_{1 \leq k \leq N-M} ([B_{i,j,k} \tau_i \wedge \tau_j \wedge \chi_k + B_{j,i,k} \tau_j \wedge \tau_i \wedge \chi_k]) \\ 0 &= \sum_{1 \leq j_1 < j_2 \leq M} (A_{i,j_1,j_2} - A_{j_1,i,j_2} + A_{j_2,i,j_1}) \tau_i \wedge \tau_{j_1} \wedge \tau_{j_2} + \sum_{\substack{1 \leq i < j \leq M \\ 1 \leq k \leq N-M}} (B_{i,j,k} - B_{j,i,k}) \tau_i \wedge \tau_j \wedge \chi_k \end{aligned}$$

Then

$$A_{i,j_1,j_2} - A_{j_1,i,j_2} + A_{j_2,i,j_1} = 0 \quad (1 \leq i < j_1 < j_2 \leq M)$$

$$B_{i,j,k} = B_{j,i,k} \quad (1 \leq i < j \leq M, 1 \leq k \leq N-M)$$

□

9. Equivalence of differential forms

Now, we talk about the equivalence problem of differential forms. We begin by noting that any one-form ω on a m -dimensional manifold M defines, using the differential and wedge product, a sequence of auxiliary differential forms :

$$\begin{aligned}\omega_1 &= \omega \\ \omega_2 &= d\omega \\ \omega_4 &= d\omega_3 = d\omega \wedge d\omega \\ \omega_5 &= \omega \wedge d\omega \wedge d\omega, \text{ etc...}\end{aligned}$$

Note that each ω_k is k -form, so $\omega_{m+1} = 0$ automatically.

Definition 9.1. The rank of a one-form ω at a point $x \in M$ is the integer $0 \leq r = r(x) \leq m$, such that for $0 \leq r \leq k$, $\omega_k(x) \neq 0$, whereas for $k > r$, $\omega_k(x) = 0$.

Remark 9.2. The rank of a one-form is not always well defined, since if k is odd, it is possible $\omega_k(x) = 0$ while $\omega_{k+1}(x) \neq 0$.

Definition 9.3. A one-form is regular if its rank is constant.

Theorem 9.4. Let ω is a one-form of constant rank r on an m -dimensional manifold M . There exist local coordinates $x = (x_1, \dots, x_m)$ such that ω has the canonical form.

$$\omega = \begin{cases} x_1 dx_2 + \dots + x_{2s-1} dx_{2s}, & r = 2s \\ x_1 dx_2 + \dots + x_{2s-1} dx_{2s} + dx_{2s+1}, & r = 2s + 1 \end{cases}$$

depending on whether the rank is even or odd.

Corollary 9.5. Every regular one-form of the same rank is locally equivalent.

Definition 9.6. We define the rank of Ω a two-form at x to be the smallest integer $0 \leq r \leq \frac{m}{2}$ such that

$$\begin{aligned}(\Omega)^r(x) &\neq 0 \\ (\Omega)^{r+1}(x) &= 0\end{aligned}$$

We have also the equivalence problem of closed two-forms :

Theorem 9.7. Let Ω a closed two-form such that the rank of Ω is constant equal to r . There exist local coordinates $x = (x_1, \dots, x_m)$ such that

$$\Omega = dx_1 \wedge dx_2 + \dots + dx_{2r-1} \wedge dx_{2r}$$

10. Tensors

Now, we talk about tensors.

Definition 10.1. Let V be an n -dimensional $\mathbb{R}$ -vector space with its dual space V^* . The elements in the tensor product

$$V_s^r = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*,$$

where r is the number of V and s of V^* , are called the (r, s) -type tensors.

We call r the contravariant order and s the covariant order.

In particular, the element in V_0^r are called contravariant tensors of order r and those in V_s^0 are called covariant tensors of order s .

Remark 10.2. $V_0^0 = \mathbb{R}$; $V_0^1 = V$; $V_1^0 = V$.

The elements of V are called contravectors; the vectors.

The elements of V^* are called covectors; the linear form on V .

For $s, r \in \mathbb{N}$,

$$\dim(V_s^r) = n^{r+s}$$

$$V_s^r = \mathcal{L}(V^*; \dots; V^*; V; \dots; V),$$

with r terms of V^* and s terms of V .

Then $V_0^r = \mathcal{L}(V^*; \dots; V^*, \mathbb{R})$ and $V_s^0 = \mathcal{L}(V; \dots; \mathbb{R})$.

Suppose $\{e_i\}_{1 \leq i \leq n}$ and $\{e^{*i}\}_{1 \leq i \leq n}$ are dual basis in V and V^* respectively, then

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_s} \quad (1 \leq i_1, \dots, i_r \leq n; 1 \leq k_1, \dots, k_s \leq n)$$

form a basis of V_s^r .

Therefore a (r, s) -type tensor T can be uniquely expressed as

$$T = \sum_{\substack{1 \leq i_1, \dots, i_r \leq n \\ 1 \leq k_1, \dots, k_s \leq n}} T_{k_1 \dots k_s}^{i_1 \dots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_s},$$

where $T_{k_1 \dots k_s}^{i_1 \dots i_r} = T(e^{*i_1}, \dots, e^{*i_r}, e_{k_1}, \dots, e_{k_s}) \in \mathbb{R}$.

Definition 10.3. Let T_1 be a (r_1, s_1) -type tensor and T_2 be a (r_2, s_2) -type tensor on V . We define the tensor product $T_1 \otimes T_2$ a $(r_1 + r_2, s_1 + s_2)$ -type tensor given by

$$T_1 \otimes T_2 : (v^{*1}, \dots, v^{*r_1+r_2}, v_1, \dots, v_{s_1+s_2}) \mapsto T_1(v^{*1}, \dots, v^{*r_1}, v_1, \dots, v_{s_1}) T_2(v^{*r_1+1}, \dots, v^{*r_1+r_2}, v_{s_1+1}, \dots, v_{s_1+s_2})$$

Remark 10.4.

$$T_1 \otimes T_2 = \sum_{\substack{1 \leq k_1, \dots, k_{s_1+s_2} \leq n \\ 1 \leq i_1, \dots, i_{r_1+r_2} \leq n}} T_1^{i_1 \dots i_{r_1}} T_2^{i_{r_1+1} \dots i_{r_1+r_2}} e_{i_1} \otimes \cdots \otimes e_{i_{r_1+r_2}} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_{s_1+s_2}}$$

Definition 10.5. $V_0^* := \sum_{r \geq 0} V_0^r$ is an infinite-dimensional vector space.

Any element $T \in V_0^*$ can be expressed as the formal form

$$T = \sum_{r \geq 0} T^r,$$

where $T^r \in V_0^r$.

Let $\mathfrak{S}_r$ the permutation group of $\{1, \dots, r\}$.

Any element of $\sigma \in \mathfrak{S}_r$ determines an automorphism of $T^r(V)$.

Let $T \in V_0^r$, we define $\sigma^{tensor} : V_0^r \rightarrow V_0^r$, denote again σ , by

$$\sigma(T)(v^{*1}, \dots, v^{*r}) = T(v^{*\sigma(1)}, \dots, v^{*\sigma(r)}) \quad (v^{*1}, \dots, v^{*r} \in V^*)$$

Then, for $T = v_1 \otimes \dots \otimes v_r$,

$$\sigma(T) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(r)}$$

Definition 10.6. Let $T \in V_0^r$.

We say that T is a symmetric contravariant tensor of order r if $\sigma(T) = T$.

We let $\mathcal{S}^r(V)$ the set of such tensors.

We say that T is an alternating contravariant tensor of order r if $\sigma(T) = \epsilon(\sigma)T$, where $\epsilon(\sigma)$ is the signature of σ .

We let $\Lambda^r(V)$ the set of such tensors.

Definition 10.7. Let $T \in V_0^r$.

We define automorphisms of V_0^r by

$$S_r : \begin{cases} V_0^r \longrightarrow V_0^r \\ T \longmapsto \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \sigma(T) \end{cases}$$

the symmetric map on contravariant tensors of order r and

$$A_r : \begin{cases} V_0^r \longrightarrow V_0^r \\ T \longmapsto \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) \sigma(T) \end{cases}$$

We have the theorem.

Theorem 10.8. For $s, r \in \mathbb{N}$,

$$\begin{aligned} \mathcal{S}^r(V) &= S_r(V_0^r) \\ \Lambda^r(V) &= A_r(V_0^r). \end{aligned}$$

We can define a bigger space.

Definition 10.9. We define the space of symmetric contravariant tensors by

$$\mathcal{S}(V) = \sum_{r \geq 0} \mathcal{S}^r(V)$$

and the space of alternating contravariant tensors by

$$\Lambda(V) = \sum_{r \geq 0} \Lambda^r(V)$$

Moreover, we define the symmetric and alternating map, noted $S : V_0^* \rightarrow \mathcal{S}(V)$ and $A : V_0^* \rightarrow \Lambda(V)$, on contravariant tensors by

$$S(T) = S_r(T) \quad (T \in V_0^r)$$

$$A(T) = A_r(T) \quad (T \in V_0^r)$$

Remark 10.10. Similarly, we can do it for V^* .

In particular, for $V^* = TM^*$, with M a n -dimensional manifold, we have $\Lambda(V^*)$ the space of differential $\mathbb{R}$ -valued forms.

As for differential forms in $\Lambda(TM^*)$, we can define a product on $\mathcal{S}(V)$.

Definition 10.11. We define a product on $\mathcal{S}(V)$ by

$$T_1 \odot T_2 = S(T_1 \otimes T_2)$$

called the symmetrized tensor product.

Then, we can use the notations

$$\odot V = \mathcal{S}(V)$$

and

$$\odot^r V = \mathcal{S}^r(V) \quad (r \geq 0)$$

Example 10.12. Let $\{e_i\}_{1 \leq i \leq n}$ a basis of V . We have

(1) $\{e_i \odot e_j\}_{1 \leq i \leq j \leq n}$ a basis of $\odot^2 V$.

(2) $\{e_i \wedge e_j\}_{1 \leq i < j \leq n}$ a basis of $\Lambda^2 V$.

Proposition 10.13.

$$\dim \odot^r V = \binom{n+r-1}{r}.$$

Proof. Let $T \in \mathcal{S}^r(V)$ and write

$$T = \sum_{1 \leq i_1, \dots, i_r \leq n} T^{i_1 \dots i_r} e_{i_1} \otimes \dots \otimes e_{i_r}$$

for some basis $\{e_1, \dots, e_{i_r}\}$ of V .

We need to determine the number of k combinations with possible repetitions, of integers $\{1, \dots, n\}$.

This number is $\binom{n+r-1}{r}$.

□

Example 10.14. $\odot^3 \mathbb{R}^2$ is a 4-dimensional vector space.

Then we can have the identification

$$\odot^3 \mathbb{R}^2 = \mathbb{R}^4.$$

We end with a definition.

Definition 10.15. Let $l : \odot^r V \rightarrow \odot^r V$ a map.

l is an endomorphism of $\odot^r V$ if there is $(m_1, \dots, m_r) \in (\odot^r V)^r$ such that

$$l = m_1 \odot \dots \odot m_r$$

i.e. for $(v_i)_{1 \leq i \leq r} \in V^r$,

$$l(v_1 \odot \dots \odot v_r) = m_1(v_1) \odot \dots \odot m_r(v_r)$$

Lie groups

1. Introduction

Definition 1.1. A Lie group G is a smooth manifold which is also a group such that the group multiplication $(g, h) \mapsto g \cdot h$ and inversion $g \mapsto g^{-1}$ define smooth maps.

Remark 1.2. A local Lie group G is a set such that there exists a neighborhood of e satisfying that $G|_U = U$ is a Lie group.

Remark 1.3. Often, an r -dimensional Lie group is referred to as an r parameter group, the "group parameters" referring to a choice of local coordinates on the group manifold.

Example 1.4. The simplest example of an r parameter Lie group is $\mathbb{R}^r$.

Example 1.5. The prototypical example of a real Lie group is the general group $GL(n, \mathbb{R})$.

Definition 1.6. The subgroup H is called a Lie subgroup if it is also an immersed submanifold parametrized by a smooth group homomorphism $F : \tilde{H} \rightarrow H \subset G$.

Proposition 1.7. *If $H \subset G$ is a subgroup of a Lie group G , which is also a topological closed subset, then H is a Lie subgroup of G .*

Example 1.8. We have :

- * If G is a Lie group, then the connected component of G containing the identity element, denoted G^+ is itself a Lie group.
- * $GL(n, \mathbb{R})^{*+} = \{A \in GL(n, \mathbb{R}) \mid \det(A) > 0\}$ is a Lie group.
- * $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) = 1\}$ is a Lie group.
- * $O(n) = \{A \in GL(n, \mathbb{R}) \mid {}^t AA = I_n\}$ is a Lie group.
- * $Sp(2r) = \{A \in GL(n, \mathbb{R}) \mid {}^t A J A = J\}$ is a Lie group with $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.
- * $E(n) = O(n) \rtimes \mathbb{R}^n$ The Euclidean group generated by the groups of orthogonal transformations and translations.
- * $SE(n) = SO(n) \rtimes \mathbb{R}^n$.

Definition 1.9. A discrete subgroup $\Gamma \subset G$ of a Lie group is a subgroup whose intersection with some neighborhood $\{e\} \subset U \subset G$ of the identity consists only for the identity : $\Gamma \cap U = \{e\}$.

Example 1.10. The integer lattices $\mathbb{Z}^r$ and $SL(n, \mathbb{Z})$ are discrete subgroups.

Example 1.11. The quotient group G/Γ by a discrete normal subgroup, is a Lie group which is locally isomorphic to G itself.

2. Transformation groups

Let M be a smooth manifold of dimension m and G be a Lie group.

Definition 2.1. A transformation group acting on M is determined by a Lie group G and a smooth map $\Phi : \begin{cases} G \times M \longrightarrow M \\ (g, x) \longmapsto g \cdot x \end{cases}$ which satisfies :

- (1) For all $x \in M$, $e \cdot x = x$
- (2) For all $x \in M$, $g, h \in G$, $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.

The definition assume that the action is global, but an action may only be defined "locally", meaning that, for a given $x \in M$, transformation $g \cdot x$ is only defined for a group element g sufficiently near the identity. Thus

Definition 2.2. A local transformation group acting on M is determined by a Lie group G , and $\mathcal{V}$ an open subset such that $\{e\} \times M \subset \mathcal{V} \subset G \times M$ and a smooth map $\Phi : \begin{cases} \mathcal{V} \longrightarrow M \\ (g, x) \longmapsto g \cdot x \end{cases}$ which satisfies :

- (1) For all $(e, x) \in \mathcal{V}$, $e \cdot x = x$
- (2) For all $(g, h, x) \in G^2 \times M$ such that $(h, x) \in \mathcal{V}$, $(g, h \cdot x) \in \mathcal{V}$, we have $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.

Example 2.3. We give some examples of transformations :

- * The linear action of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$: $\begin{cases} GL(2, \mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ ((\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x, y))) \longmapsto (ax + by, cx + dy) \end{cases}$
- * The Möbius transformation : $\begin{cases} GL(2, \mathbb{R}) \times \mathbb{RP}^1 \longrightarrow \mathbb{RP}^1 \\ ((\begin{pmatrix} a & b \\ c & d \end{pmatrix}, p) \longmapsto \frac{ap + b}{cp + d} \end{cases}$

Definition 2.4. The isotropy subgroup of a point $x \in M$ is $G_x = \{g \in G \mid g \cdot x = x\}$.
Let $x \in M$.

x is said fixed by the group action if $G_x = G$.

Remark 2.5. For all $x \in M$, G_x is a Lie group.

Proposition 2.6. Let $x, y \in M$.

If $y = g \cdot x$ then $G_y = g \cdot G_x \cdot g^{-1}$.

Definition 2.7. A transformation group acts freely if all isotropy subgroup are trivial.

A transformation group acts locally freely if it exists U a neighborhood of e such that for all $x \in M$, $G_x \cap U = \{e\}$.

A transformation group acts effectively if for all $g, h \in G$, $x \in M$, we have $g \cdot x = h \cdot x$ if and only if $g = h$.

Definition 2.8. We call the global isotropy subgroup $G_M = \bigcap_{x \in M} G_x = \{g \in G \mid g \cdot x = x, \text{ for all } x \in M\}$.

Remark 2.9. A transformation group acts effectively if $G_M = \{e\}$.

Definition 2.10. A transformation group acts locally effectively if it exists U a neighborhood of e such that for all $x \in M$, $G_M \cap U = \{e\}$.

In other words, the restriction of the group action to $U \times M$ acts effectively.

We have :

Proposition 2.11. Let G be a Lie group acting on M .

The global isotropy subgroup G_M is a normal Lie subgroup of G .

Moreover, there is a well-defined effective action of the quotient group G/G_M on M , which "coincides" with that of G in the sense that two group elements g and $\tilde{g}$ have the same action on M , so for all $x \in M$, $g \cdot x = \tilde{g} \cdot x$, if and only if it exists $h \in G_M$ such that $\tilde{g} = g \cdot h$.

Remark 2.12. Thus, if a transformation group G does not act effectively, we can without any sufficient loss of information or generality, replace it by the quotient group G/G_M .

Remark 2.13. A group action acts effectively freely if and only if G/G_M acts freely. It is equivalent to that for every $x \in M$, $G_x = G_M$.

Example 2.14. $GL(n, \mathbb{R})$ acts effectively on $\mathbb{R}^n$.

3. Invariant subsets and equations

Let M be a smooth manifold of dimension m and G be a Lie group.

Definition 3.1. A subset $S \subset M$ is called G -invariant if for every $g \in G$, $x \in S$, we have $g \cdot x \in S$.

Definition 3.2. Let $F_1, \dots, F_k : M \rightarrow \mathbb{R}$.

A group G is called a symmetry group of a system of equation

$$F_1(x) = \dots = F_k(x) = 0$$

if the variety $S_F = \{x \in M \mid F_1(x) = \dots = F_k(x) = 0\}$ is a G -invariant subset of M .

Definition 3.3. An orbit of a transformation group is a minimal nonempty invariant subset.

Example 3.4. For a global action, the orbit through a point $x \in M$ is $\mathcal{O}_x = \{g \cdot x \mid g \in G\}$.

Example 3.5. For a local action, the orbit through a point $x \in M$ is $\mathcal{O}_x = \{g_1 \cdot g_2 \cdots g_n \cdot x \mid g \in G, n \in \mathbb{N}\}$.

Remark 3.6. If G is connected, then the orbits are connected.

Proposition 3.7. A subset $S \subset M$ is G -invariant if and only if it is the union of orbits.

Definition 3.8. The group action is called transitive if there is only one orbit.

Example 3.9. A fixed point is a zero-dimensional orbit.

For a connected group, the converse holds.

Example 3.10. * The linear action $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ has two orbits : the origin $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$.

* The same hold for $SL(n)$ on $\mathbb{R}^n$.

* For the orthogonal group $O(n)$ it is different : the orbits are spheres $\{|x| = c\}$ for $c \in \mathbb{R}$. Any other invariant for this action is a union of spheres.

* The induces projective actions of each of these three groups on $\mathbb{RP}^{n-1}$ are all transitive.

We have an important theorem.

Theorem 3.11. A Lie group G acts globally and transitively on M if and only if $M \simeq G/H$ where G/H is called the homogeneous space of G by the isotropy subgroup $H = G_x$ of any designated point $x \in M$.

Remark 3.12. Mostow showed that every local transitive group action on M of dimension $m \geq 4$ is locally isomorphic to a homogeneous space G/H where $\dim(G) - \dim(H) = m$. However, it is not true that M itself is globally isomorphic to an open subset of a homogeneous space.

Moreover, Mostow showed that even the local result is not true if $m \geq 5$.

Example 3.13. The affine group $A(n)$ is defined as the group of affine transformation $x \mapsto Ax + a$ in $\mathbb{R}^n$ with $A \in GL(n, \mathbb{R})$ and $a \in \mathbb{R}^n$. The affine group can be realized as a subgroup of $GL(n+1, \mathbb{R})$ by identifying the group element (A, a) with the matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$.

With the identification we have $\mathbb{R}^n \setminus \{0\} \simeq GL(n)/A(n-1)$ another possible identification.

Definition 3.14. A group action is called semi-regular if its orbits have the same dimension. A group action is called regular if its orbits have the same dimension and each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset.

Proposition 3.15. An r -dimensional Lie group G acts locally freely on M if and only if its orbits have the same dimension r as G itself.

The group acts effectively freely if and only if its orbits have dimension $s = \dim(G) - \dim(G_M)$.

Theorem 3.16. Let G be a Lie group acting regularly on M with s -dimensional orbits.

Near every point of M , there exists rectifying local coordinates $(y_1, \dots, y_s, z_1, \dots, z_{m-s})$ having the property that any orbit intersects the coordinate chart in at most one slice $N_c = \{z_1 = c_1, \dots, z_{m-s} = c_{m-s}\}$ for constants $c = (c_1, \dots, c_{m-s})$.

If the action is semi-regular, the same statement holds except that an orbit may intersect the chart in more than one such slice.

4. Canonical forms

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

Definition 4.1. A canonical form of an element $x \in M$ is representative x_0 of the orbit containing x .

Example 4.2. The Jordan canonical form of a matrix correspond to the conjugation action $X \mapsto AXA^{-1}$ of $GL(n, \mathbb{C})$ on $M(n, \mathbb{C})$.

Definition 4.3. Two objects are equivalent if they have the same canonical form.

Example 4.4. The canonical form of $x \in \mathbb{R}^n$ under the action of the rotation group $SO(n)$ can be chosen to be a positive multiple of the first unit basis vector; re_1 with $r = |x| \geq 0$.

5. Invariant functions

The determination of a complete set of invariant of a given group action is a problem of supreme importance for the study of equivalence and canonical forms.

Definition 5.1. An invariant of G is a real-valued function $I : M \rightarrow \mathbb{R}$ such that for all $g \in G$, $I(g \cdot x) = I(x)$.

Proposition 5.2. Let $I : M \rightarrow \mathbb{R}$. The following conditions are equivalent :

- (1) I is a G -invariant function.
- (2) I is constant on the orbits of G .
- (3) The level sets $\{x \in M \mid I(x) = c\}$ of I are G -invariant.

Example 5.3. In the case of $O(n)$ acting on $\mathbb{R}^n$, the orbits are sphere of constant rays r , hence any orthogonal invariant is a function of the radius : $I = F(r)$.

Remark 5.4. If G acts transitively on M , then there is no nonconstant invariant. If G acts transitively on a dense $M_0 \subset M$, then the only continuous invariants are the constants.

Example 5.5. The only continuous invariants of the standard action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ are the constant functions.

Remark 5.6. Note that the canonical form x_0 of any element $x \in M$, must have the same invariants : $I(x) = I(x_0)$.

Proposition 5.7. Let $x, y \in M$ and a regular group actions. x and y are in the same orbit if and only if for every invariant I of the group action, $I(x) = I(y)$.

Remark 5.8. Note that if $I_1, \dots, I_k : M \rightarrow \mathbb{R}$ are invariants and $H : \mathbb{R}^r \rightarrow \mathbb{R}$ is any function then $I = H(I_1, \dots, I_k)$ is also invariant. Therefore, we need only find a complete set of functionally independent invariants having the property that any other invariant can be written as a function of these fundamental invariants.

Definition 5.9. Let G be a Lie group acting on M .

A function $I : M \rightarrow \mathbb{R}$ defined on an open subset $U \subset M$ is called a local invariant of G if there is U a neighborhood in M , and for all $x \in U$, there is V_x a neighborhood in G such that for all $x \in M, g \in V_x, I(g \cdot x) = I(x)$.

I is called a global invariant If for all $x \in U$, and $g \in G$, such that $g \cdot x \in U$, we have $I(g \cdot x) = I(x)$.

Remark 5.10. The number of independent local invariants of a regular transformation group is completely determined by the orbit dimension.

Theorem 5.11. *Let G be a Lie group acting semi-regularly on M of dimension m with s -dimensional orbits.*

At each $x \in M$, there exist $m - s$ functionally independent local invariant $I_1, \dots, I_{m-s}$ defined on a neighborhood U of x , with the property that any other local invariant I defined on U can be written as a function of the fundamental invariants : $I = H(I_1, \dots, I_{m-s})$.

If G acts regularly, then we can choose the I_k 's to be global invariants on U .

Moreover, in the regular case, two point $y, z \in U$ lie in the same orbit of G if and only if for all $1 \leq k \leq m - s$, $I_k(y) = I_k(z)$.

Remark 5.12. The theorem provides a complete answer to the question of local invariant of group actions.

Global considerations are more delicate.

Definition 5.13. If G acts simultaneously on $M_1, \dots, M_n$ then be a joint invariant we mean an ordinary invariant $I : M_1 \times \dots \times M_n \rightarrow \mathbb{R}$.

The invariants of regular group action can be used to completely characterize the invariant submanifolds ;

Remark 5.14. If G acts on M and $I_1, \dots, I_k$ are any invariants, then $\{x \in M \mid I_1(x) = c_1, \dots, I_k(x) = c_k\}$ is an invariant subset of M .

Conversely, if G acts regularly any invariant submanifold can be expressed locally as the vanishing set for some collection of invariant functions.

Theorem 5.15. *Let G act regularly on M .*

A regular n -dimensional submanifold $N \subset M$ is G -invariant if and only if at each point $x \in N$, there exists a neighborhood U and invariant $I_1, \dots, I_{m-n}$ such that $N \cap U = \{x \in U \mid I_1(x) = \dots = I_{m-n}(x) = 0\}$.

Proof. Choose rectifying local coordinate $(y_1, \dots, y_n, z_1, \dots, z_{m-n})$. Since any invariant submanifold must be a collection of orbits which, in the given coordinate chart, are just the slices where the invariant coordinates z are constant., we can characterize its intersection with the coordinate chart by the vanishing of one or more functions depending on $z_1, \dots, z_{m-n}$. But any function of the z 's is a local invariant of the group action, so the result follows. □

6. Lie algebra

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r acting on M by $\Phi : (g, x) \mapsto g \cdot x$.

Definition 6.1. A vector field v on M is called G -invariant if for all $g \in G$, $x \in M$ such that $g \cdot x$ is defined, we have $d(\Phi_g)_e(v|_x) = v|_{g \cdot x}$. Sometime we write $dg(v|_x) = d(\Phi_g)_e(v|_x)$.

Remark 6.2. The most import example is provided by the action of Lie group G on itself be left and right multiplication ;

Definition 6.3. Let $g \in G$.

We define $L_g : h \mapsto g \cdot h$ and $R_g : h \mapsto h \cdot g$ respectively the left and right multiplication maps.

A vector field v on G is called left-invariant if for all $g \in G$, $d(L_g)_e(v|_e) = v|_g$.

A vector field v on G is called right-invariant if for all $g \in G$, $d(R_g)_e(v|_e) = v|_g$.

Definition 6.4. The left (respectively right) Lie algebra of a Lie group G is the space of all left-invariant (respectively right-invariant) vector field on G .

Remark 6.5. There are two different Lie algebras associated with a Lie group denoted by $\mathfrak{g}_L$ and $\mathfrak{g}_R$.

But when we will talk about the Lie algebra associate with a Lie group, we will mean $\mathfrak{g} := \mathfrak{g}_R$.

Remark 6.6. Every $v \in \mathfrak{g}$ is uniquely determined by its value at the identity by definition. Then we can identify $\mathfrak{g} = \mathfrak{g}_R = T_e G$.

So $\mathfrak{g}$ is a finite-dimensional vector space of dimension r .

Similarly $\mathfrak{g}_L \simeq T_e G$.

Thus a given tangent vector $v|_e \in T_e G$ determines both a left- and a right-invariant vector field on the Lie group G , which are different.

Moreover, we have $\mathfrak{g}_R \simeq \mathfrak{g}_L$.

Remark 6.7. By the previous remark we use the identification $v = v|_e$.

Remark 6.8. Each Lie algebra associated with a Lie group comes equipped with a natural bracket, induces by the Lie bracket of vector field.

Definition 6.9. A Lie algebra $\mathfrak{g}$ is a vector space equipped with a bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is bilinear, skew-symmetric ; for all $X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$ and satisfies the Jacobi identity.

Example 6.10. The Lie algebra $\mathfrak{gl}(n)$ of $GL(n)$ can be identified with the space $M(n, \mathbb{R})$. In terms of the local coordinates provided by the matrix $X = (x_{i,j}) \in GL(n)$, the left-invariant vector field associated with a matrix $A = (a_{i,j})$ has the explicit formula

$$\hat{v}_A = \sum_{1 \leq i, j, k \leq n} x_{i,k} a_{k,j} \frac{\partial}{\partial x_{i,j}}$$

and the right-invariant vector field associated with a matrix $A = (a_{i,j})$ has the explicit formula

$$v_A = \sum_{1 \leq i,j,k \leq n} a_{i,k} x_{k,j} \frac{\partial}{\partial x_{i,j}}$$

7. Structure constants

Definition 7.1. Let $v_1, \dots, v_r$ be a basis of a Lie algebra $\mathfrak{g}$.

We define the associated structure constants $c_{i,j}^k$ by the braces relations

$$[v_i, v_j] = \sum_{1 \leq k \leq r} c_{i,j}^k v_k \quad (1 \leq i, j \leq r)$$

By the skew-symmetry of the Lie bracket and the Jacobi identity we have

$$c_{i,j}^k = -c_{j,i}^k \quad (1 \leq i, j, k \leq r)$$

and

$$\sum_{1 \leq l \leq r} (c_{i,j}^l c_{l,k}^m + c_{k,i}^l c_{l,j}^m + c_{j,k}^l c_{l,i}^m) = 0 \quad (1 \leq i, j, k, m \leq r)$$

Conversely, given $\frac{1}{2}r^2(r-1)$ constants satisfying the previous identities, we can reconstruct the Lie algebra $\mathfrak{g}$ by introducing a basis $v_1, \dots, v_r$ and then imposing the bracket relations.

Remark 7.2. The theory of Lie groups can be essentially reduced to the study of its structure constants.

8. The exponential map

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

Definition 8.1. Let $\mathfrak{g}$ the right-invariant Lie algebra of a Lie group G .

Let $v \in \mathfrak{g}$.

We recall the flow at t is $\exp(tv) : G \rightarrow G$

Remark 8.2. In fact by continuous argument, for $v \in \mathfrak{g}$, the flow is defined on all $t \in \mathbb{R}$.

Definition 8.3. The vector field v of the subgroup $\{\exp(tv) \cdot e \mid t \in \mathbb{R}\}$ of G is called the infinitesimal generator.

Proposition 8.4. *The right-invariant vector fields form the infinitesimal generators of the action of G on itself by left multiplication.*

Vice versa, the infinitesimal generators of the action of G on itself by right multiplication is the Lie algebra $\mathfrak{g}_L$.

Proof. For $g \in G$, $t \in \mathbb{R}$, $\varphi_t(g) = \exp(tv) \cdot g = R_g(\exp(tv))$ and

$$\frac{d}{dt}\bigg|_{t=0} \varphi_t(g) = \frac{d}{dt}\bigg|_{t=0} (R_g(\exp(tv))) = d(R_g)_e(v)$$

□

Example 8.5. Consider the group $GL(n)$.

The flow corresponding to the right-invariant vector field v_A already given is given by the usual matrix exponential $\exp(tv_A)X = e^{tA}X$.

The flow corresponding to the left-invariant vector field $\tilde{v}_A$ already given is given by the usual matrix exponential $\exp(tv_A)X = Xe^{tA}$.

Evaluation of the flow $\exp(tv)$ at $t = 1$ for each $v \in \mathfrak{g}$ serves to define the exponential map.

Definition 8.6. We call $\exp : \begin{cases} \mathfrak{g} \longrightarrow G \\ v \longmapsto \exp(v) \end{cases}$ the exponential map.

Remark 8.7. Since $\exp(0) = e$, and $d\exp = \mathbf{1}$, then exponential map defines a local diffeomorphism in a neighborhood of $0 \in \mathfrak{g}$.

Consequently, all Lie groups having the same Lie algebra look locally the same in a neighborhood of the identity.

Proposition 8.8. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Every element $g \in G$ can be written as a product of exponentials :

$$g = \exp(v_1) \cdot \exp(v_2) \cdots \exp(v_k)$$

for some $v_1, \dots, v_k \in \mathfrak{g}$.

Remark 8.9. This result forms the basis of the infinitesimal or Lie algebraic approach to symmetry groups and invariants.

Proof. The set $\mathcal{S} = \{g \in G \mid \exists k \geq 1, \exists v_1, \dots, v_k \in \mathfrak{g}, g = \exp(v_1) \cdot \exp(v_2) \cdots \exp(v_k)\}$ is nonempty, an open and closed subgroup of G .

By connectivity, it follows.

□

9. Subgroups and subalgebra

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

Definition 9.1. A sub algebra of a Lie algebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is invariant under the Lie bracket.

Example 9.2. Every one-dimensional subspace forms a sub algebra, and generates an associated one-parameter subgroup of the associated Lie group via exponentiation.

Theorem 9.3. *Let G be a Lie group with its Lie algebra $\mathfrak{g}$. There is a one-to-one correspondence between connected s -dimensional subgroup $H \subset G$ and s -dimensional Lie sub algebra $\mathfrak{h} \subset \mathfrak{g}$.*

Remark 9.4. By the previous section, to determine the subalgebra $\mathfrak{h}$, we need only find $T_e H$.

Example 9.5. * $\mathfrak{sl}(n) = \{A \in M(n, \mathbb{R}) \mid \text{Tr}(A) = 0\}$.
 * $\mathfrak{o}(n) = \mathfrak{so}(n)$ is the set of all skew-symmetric matrix.

Remark 9.6. From "Lie Algebras", Interscience Publ. Inc., New York, 1962 of Jacobson, we can see that the Ado's theorem. It says that any finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a sub algebra of $\mathfrak{gl}(n)$ for some n . Consequently, $\mathfrak{g}$ is realized as the Lie algebra of the associated matrix Lie group $G \subset GL(n)$.

The full connection between Lie groups and Lie algebras is stated in the following theorem.

Theorem 9.7. *Each finite-dimensional Lie algebra $\mathfrak{g}$ corresponds to a unique connected, simply connected Lie group $\tilde{G}$. Moreover, any other connected Lie group G having the same Lie algebra $\mathfrak{g}$ is isomorphic to the quotient $\tilde{G}/\Gamma$, where Γ is a normal discrete subgroup.*

We can restate the theorem as follow :

Theorem 9.8. *Let $\mathfrak{g}$ an arbitrary Lie algebra. There exists a local connected Lie group G such that $\mathfrak{g}_G$ is isomorphic to $\mathfrak{g}$. Therefore, Let G, G' be two connected local Lie groups, and $\mathfrak{g}_G, \mathfrak{g}_{G'}$ their infinitesimal groups (Lie algebras). If there exists an isomorphism mapping $\mathfrak{g}_G$ to $\mathfrak{g}_{G'}$, then there exists a local isomorphism mapping G to G' .*

Proof. To see the theorem 73 of "Topological Groups", Princeton Univ. Press, Princeton, 1946 of Pontrjagin, L.

□

The projection $\pi : \tilde{G} \rightarrow G$ from the theorem defines a covering map, so that $\tilde{G}$ is as a covering group for any other connected G having the same Lie algebra.

Corollary 9.9. *Any two Lie group having the same Lie algebra $\mathfrak{g}$ are not only locally isomorphic but are in fact both covered by a common Lie group.*

Remark 9.10. By the theorem, the study of G is equivalent to the study of $\mathfrak{g}$.

Definition 9.11. A Lie algebra of dimension greater than 1 is called simple if it contains no proper nonzero ideal.

Example 9.12. For $n \geq 2$, $\mathfrak{sl}(n)$ is simple.

For $n \geq 3$ and $n \neq 4$, $\mathfrak{so}(n)$ is simple.

For $r \geq 2$, $\mathfrak{sp}(2r)$ is simple.

10. Infinitesimal group actions

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

As a one-parameter group of transformations is generated as the flow of vector field, so a general Lie group of transformations G acting on M will be generated by a set of vector fields on M , known as the infinitesimal generators of the group action.

By the group action G on M , firstly determine the Lie algebra $\mathfrak{g}$ and a basis of itself. For $v \in \mathfrak{g}$, we want a vector field $\tilde{v}$ on M .

Recall, $v = v|_e$.

We use the flow generate by v :

$$\varphi_t(x) = \exp(tv)x \quad (\text{where } t \text{ is defined, } x \in M)$$

Note that the interval I where t is defined contain always 0 and $I = \{t \in \mathbb{R} \mid e^{tv} \in G\}$.

Then we define the infinitesimal generator of the group action G on M by

$$v_x^\dagger = \frac{d}{dt}(\exp(tv)x)|_{t=0} \quad (x \in M)$$

Then we identify v with $\hat{v}$.

Then for a basis $\{v_1, \dots, v_r\}$ of $\mathfrak{g}$ we obtain r infinitesimal generator of the group action : $\{v_1^\dagger, \dots, v_r^\dagger\}$.

Example 10.1. Consider the action group of $SL(2)$ on $\mathbb{RP}^1$ by the Möbius transformation. Its Lie algebra $\mathfrak{sl}(2) = \{A \in M(2) \mid \text{Tr}(A) = 0\}$ is generated by

$$J^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, J^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

By the identification $J = v_J$, we have

$$\Phi_t^{v_{J^-}}(p) = e^{tJ^-} \cdot p = (I_2 + tJ^-) \cdot p = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot p = p + t$$

Then $v_{J^-|_p}^\dagger = \frac{d}{dt}(\Phi_t^{v_{J^-}}(p))|_{t=0} = 1 = \frac{\partial}{\partial p}$ by the identification.

If we do that for the other matrix, we find finally the basis actions :

(1) Translation action : $p \mapsto p + t$ with $v_{J^-|p}^\dagger = \frac{\partial}{\partial p}$.

(2) Scaling action : $p \mapsto e^{2t}p$ with $v_{J^0|p}^\dagger = 2p \frac{\partial}{\partial p}$.

(3) Inversion action : $p \mapsto \frac{p}{tp+1}$ with $v_{J^+|p}^\dagger = -p^2 \frac{\partial}{\partial p}$.

The group action is decomposed in three fundamental actions.

By the section "Structure constants", since we have

$$[v_{J^-|p}^\dagger, v_{J^0|p}^\dagger] = 2v_{J^-|p}^\dagger, [v_{J^+|p}^\dagger, v_{J^0|p}^\dagger] = -2v_{J^+|p}^\dagger, [v_{J^-|p}^\dagger, v_{J^+|p}^\dagger] = -v_{J^0|p}^\dagger$$

$$[J^-, J^0] = -2J^-, [J^+, J^0] = 2J^+, [J^-, J^+] = J^0$$

then the Lie algebra of vector fields generated by the $v^\dagger$'s coincide with $\mathfrak{sl}(2)$.

Remark 10.2. This previous example, show us the method to determine the Lie algebra of vector fields of a group action ;

Let G a Lie group acting on M .

1) Determine $\mathfrak{g}$ of G view in a some $GL(n)$.

2) Determine a basis $B_{\mathfrak{g}}$.

3) For every $v \in B_{\mathfrak{g}}$, compute the flow associated to v ; $\Phi_t : x \mapsto e^{tv} \cdot x$ where $\cdot$ is the group action.

4) For every $v \in B_{\mathfrak{g}}$, compute $v^\dagger$ by

$$v_x^\dagger = \frac{d}{dt}(\exp(tv) \cdot x)|_{t=0}$$

Remark 10.3. For $B_{\mathfrak{g}^\dagger} = \{v_1^\dagger, \dots, v_r^\dagger\}$ a basis of the Lie algebra of vector field on M , the flow at $v \in B_{\mathfrak{g}^\dagger}$, Φ_t^v , determine a basis action of the group action. We can understand the group action as a composition of several basis actions.

Why can we make this identification ?

Since we have the following theorem.

Theorem 10.4. Let G be a transformation group actin on M .

The linear map $\rho : \begin{cases} \mathfrak{g} \longrightarrow TM \\ v \longmapsto v^\dagger \end{cases}$ defines a Lie algebra homomorphism satisfying for

$$v, w \in \mathfrak{g}, \rho([v, w]) = [\rho(v), \rho(w)].$$

Moreover the image $\mathfrak{g}^\dagger = \rho(\mathfrak{g})$ forms a finite-dimensional Lie algebra of the effectively action quotient group G/G_M .

In particular, G acts locally effectively on M if and only if ρ is injective.

Moreover, if G acts freely on M , then for each $v \in \mathfrak{g}$ non zero, $v^\dagger$ never vanishes on M .

Remark 10.5. If $v_1, \dots, v_r$ is a basis of $\mathfrak{g}$, the condition for local effectiveness is that the corresponding vector fields $v_1^\dagger, \dots, v_r^\dagger$ be linearly independent over $\mathbb{R}$.

Remark 10.6. By this theorem, and the fact we can assume without lost generality the local effectiveness of G , then we can use the identification $v = v^\dagger$ and $\mathfrak{g} = \mathfrak{g}^\dagger$.

Given a finite-dimensional Lie algebra of vector fields on M , we can reconstruct a local action of the corresponding Lie group via the exponentiation process.

Theorem 10.7. *Let $\mathfrak{g}$ be finite-dimensional Lie algebra of vector fields on M . Let G denote a Lie group having Lie algebra $\mathfrak{g}$. There is a local action of G whose infinitesimal generators coincide with the given Lie algebra.*

Proof. To see the theorem 88 of "Topological Groups", Princeton Univ. Press, Princeton, 1946 of Pontrjagin, L. □

Proposition 10.8. *Let G be a Lie group acting on M with a Lie algebra $\mathfrak{g}$ of G . For each $x \in M$, $\mathfrak{g}|_x = \{\hat{v}|_x \mid v \in \mathfrak{v}\} \subset T_x M$. In particular, for each $x \in M$, $\dim(\mathcal{O}_x) = \dim(\mathfrak{g}|_x)$.*

Corollary 10.9. *A Lie group acts transitively on a connected manifold M if and only if for all $x \in M$, $\mathfrak{g}|_x = T_x M$.*

There is an important connection between the dimension of the isotropy subgroup at a point and the dimension of the orbit through that point.

Proposition 10.10. *If G is an r -dimensional Lie group acting on M , then the isotropy subgroup G_x of any point $x \in M$ has dimension $r - s$, where $s = \dim(\mathcal{O}_x)$. In particular, G acts semi-regularly if and only if all its isotropy subgroups have the same dimension.*

11. Classification of group actions

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

The problem of classifying all possible Lie group actions on M is, by the previous theorem, essentially the same as the problem of classifying finite-dimensional Lie algebras of vector fields on M .

Definition 11.1. Two Lie algebras of vector fields, defined on open subsets of $\mathbb{R}^m$, are considered to be equivalent if there is a local diffeomorphism on $\mathbb{R}^m$ taking one to the other.

which is equivalent to say that

Definition 11.2. Two action groups are considered to be equivalent if the two Lie algebras of vector fields associated are equivalent.

Remark 11.3. Often, to simplify we said that two groups G and G' are equivalent if their group action are equivalent.

Example 11.4. A one-dimensional Lie group is generated by a vector field, and hence, away from singularities, all such group action / Lie algebras are isomorphic to the group of translations in a single coordinate direction.

Lie investigates the general classification problem. He succeeded in completely classifying all non singular finite-dimensional Lie group actions in one and two dimensions, determining a complete list of local canonical forms for Lie algebras of vector fields on one-dimensional and two-dimensional manifold.

Theorem 11.5. *Let G be a connected finite-dimensional Lie group action locally effectively on a one-dimensional real or complex manifold without singular point.*

Then, locally the action group of G is equivalent to one of the following group actions :

- (1) *Translation action : $x \mapsto x + b$, Group : $\mathbb{R}$, and generators : $\frac{\partial}{\partial x}$.*
- (2) *Affine action : $x \mapsto ax + b$, Group : $A(1)$, and generators : $\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}$.*
- (3) *Projective action : $x \mapsto ax + b$, Group : $SL(2)$, and generators : $\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x}$.*

.

Proof. We outline the proof in the complex case, the real case being similar but slightly more complicated. The assumption that G has no singularities means that the vector fields in the Lie algebra $\mathfrak{g}$ do not all simultaneously vanish at a point.

Fix a point x_0 and choose $v_1 \in \mathfrak{g}$ such that $v_1|_{x_0} \neq 0$. According to the linearization of vector field, we can introduce a local coordinate x near x_0 which straightens out $v_1 = \frac{\partial}{\partial x}$. Every other infinitesimal generator in $\mathfrak{g}$ has the form $v_f = f(x) \frac{\partial}{\partial x}$ for some function $f(x)$; let $\mathcal{F} = \{f\}$ denote the finite-dimensional vector space spanned by these functions. Since $[v_1, v_f] = f'(x) \frac{\partial}{\partial x}$ must also be an infinitesimal generator, we deduce that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$.

Smoothness implies that $\mathcal{F}$ is a finite-dimensional translationally invariant space of functions : if $f \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f(\cdot + c) \in \mathcal{F}$.

The following useful lemma now comes into play :

Lemma 11.6. *A finite-dimensional space $\mathcal{F}$ of smooth functions of a single variable x is translationally invariant if and only if it is the space of solutions to a linear, homogenous, constant coefficient ordinary differential equation.*

Proof. Let $f_1, \dots, f_n$ a basis of $\mathcal{F}$. Each derivative f'_i for $1 \leq i \leq n$, belongs to $\mathcal{F}$ and is therefore a constant coefficient linear combination of the basis elements : $f'_i = \sum_{1 \leq k \leq n} a_i^k f_k$.

Therefore, the column vector $f = \begin{pmatrix} f_1 \\ \dots \\ f_n \end{pmatrix}$ solves the first order linear system $f' = Af$

where $A = (a_{i,j})$ is a constant matrix.

Let $p(\lambda) = \det(\lambda I_n - A)$ denote the characteristic polynomial of the matrix A . The Cayley-Hamilton theorem demonstrates that each entry f_i satisfies the n^{th} order linear homogeneous constant coefficient ordinary differential equation $p(D)y = 0$ obtained by replacing λ by the deviated operator $D = \frac{d}{dx}$. Moreover, the solution space to the n^{th} order differential equation has dimension n , and coincide with $\mathcal{F}$. □

Consequently, over $\mathbb{C}$, our Lie algebra has, as a basis, a collection of vector fields of the form $x^i e^{\lambda x} \frac{\partial}{\partial x}$ where $\lambda \in S$, a finite set of complex number containing 0, and $0 \leq i \leq n_\lambda$. (The real case is similar but also involves sines and cosines.)

We compute the commutator of two such vector fields

$$[x^i e^{\lambda x} \frac{\partial}{\partial x}, x^j e^{\mu x} \frac{\partial}{\partial x}] = [(\mu - \lambda)x^{i+j} + (j - i)x^{i+j-1}]e^{(\lambda+\mu)x} \frac{\partial}{\partial x} \quad (\lambda, \mu \in S, 0 \leq i \leq n_\lambda)$$

The simplest case occurs when $S = \{0\}$, so that $\mathfrak{g}$ is spanned by $x^i \frac{\partial}{\partial x}$ for $0 \leq i \leq n_0$.

Taking $i = n_0$, $j = n_0 - 1$, we have $x^{2n_0-2} \frac{\partial}{\partial x} \in \mathfrak{g}$, which implies $2n_0 - 2 \leq n_0$ then $n_0 \leq 2$.

The cases $n_0 = 0, 1, 2$ are precisely the three Lie algebras stated in the theorem.

If $S = \{0, \lambda\}$ contains just one nonzero complex number, then by the expression it is easy to see that $n_0 = n_\lambda = 0$; in this case $\mathfrak{g}$ is spanned by $\frac{\partial}{\partial x}$, $e^{\lambda x} \frac{\partial}{\partial x}$, and is mapped to the affine algebra by the change of variables $x \mapsto e^{\lambda x}$.

The only other possible case is $S = \{0, \lambda, -\lambda\}$, with $n_0 = n_\lambda = n_{-\lambda} = 0$, which is mapped to the projective algebra by the same change of variables. □

Definition 11.7. A group action has no fixed point if it has no zero-dimensional orbits.

Remark 11.8. A group action has no fixed point if and only if not all vector fields in the associated Lie algebra vanish at point.

Definition 11.9. A manifold has a foliation if it is decomposed in submanifold with same dimension and locally pile up like $\mathbb{R}^n \times \mathbb{R}^{m-n}$ for a some $0 \leq n \leq m$.

A group action is called imprimitive if there is an invariant foliation of the manifold.

Example 11.10. The group generated by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}\}$ admits two invariants foliations of $\mathbb{R}^2$ - by horizontal lines and by vertical lines.

Remark 11.11. Invariance is the fact that each group action maps every horizontal (vertical) line to another horizontal (vertical) line.

Then we expose, next pages, tables of classifications of

- (1) Transitive, imprimitive Lie algebras of vector fields in $\mathbb{C}^2$.
- (2) Primitive Lie algebras of vector fields in $\mathbb{C}^2$.
- (3) Intransitive Lie algebras of vector fields in $\mathbb{C}^2$.

Transitive, Imprimitive Lie algebras of vector fields in $\mathbb{C}^2$ (Table 1) :

Generators	Dim.	Structure
1.1	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} - 2xu \frac{\partial}{\partial u}$	3 $\mathfrak{sl}(2)$
1.2	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} - (2xu + 1) \frac{\partial}{\partial u}$	3 $\mathfrak{sl}(2)$
1.3	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}$	4 $\mathfrak{gl}(2)$
1.4	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x}, \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}, u^2 \frac{\partial}{\partial u}$	6 $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
1.5	$\frac{\partial}{\partial x}, \nu_1(x) \frac{\partial}{\partial u}, \dots, \nu_k(x) \frac{\partial}{\partial u}$	$k + 1$ $\mathbb{C} \rtimes \mathbb{C}^k$
1.6	$\frac{\partial}{\partial x}, u \frac{\partial}{\partial u}, \nu_1(x) \frac{\partial}{\partial u}, \dots, \nu_k(x) \frac{\partial}{\partial u}$	$k + 2$ $\mathbb{C}^2 \rtimes \mathbb{C}^k$
1.7	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial u}, \dots, x^{k-1} \frac{\partial}{\partial u}$	$k + 2$ $\mathfrak{a}(1) \rtimes \mathbb{C}^k$
1.8	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} + (ku + x^k) \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial u}, \dots, x^{k-1} \frac{\partial}{\partial u}$	$k + 2$ $\mathfrak{a}(1) \rtimes \mathbb{C}^k$
1.9	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial u}, \dots, x^{k-1} \frac{\partial}{\partial u}$	$k + 3$ $\mathfrak{a}(1) \oplus \mathbb{C} \rtimes \mathbb{C}^k$
1.10	$\frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} + (k-1)u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} + (k-1)xu \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial u}, \dots, x^{k-1} \frac{\partial}{\partial u}$	$k + 3$ $\mathfrak{sl}(2) \rtimes \mathbb{C}^k$
1.11	$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} + (k-1)u \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial u}, \dots, x^{k-1} \frac{\partial}{\partial u}$	$k + 4$ $\mathfrak{gl}(2) \rtimes \mathbb{C}^k$

In cases 1.5, 1.6, the functions $\nu_1(x), \dots, \nu_k(x)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.

In cases 1.5 – 1.11, we require $k \geq 1$. Note though, that if we set $k = 0$ in case 1.10, and replace u by u^2 , we obtain the case 1.1.

Similarly, if we set $k = 0$ in case 1.11, we obtain was 1.3. Case 1.7 and 1.8 for $k = 0$ are equivalent to the Lie algebra $\{\frac{\partial}{\partial x}, e^x \frac{\partial}{\partial u}\}$ of type 1.5. Case 1.9 for $k = 0$ is equivalent to the Lie algebra $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}\}$ of type 1.6.

Primitive Lie algebras of vector fields in $\mathbb{C}^2$ (Table 2) :

Generators	Dim.	Structure
2.1	$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}$	5 $\mathfrak{sa}(2)$
2.2	$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}$	6 $\mathfrak{a}(2)$
2.3	$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} + xu \frac{\partial}{\partial u}, xu \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u}$	8 $\mathfrak{sl}(3)$

Intransitive Lie algebras of vector fields in $\mathbb{C}^2$ (Table 3) :

Generators	Dim.	Structure
3.1	$\nu_1(x) \frac{\partial}{\partial x}, \dots, \nu_k(x) \frac{\partial}{\partial u}$	k $\mathbb{C}^k$
3.2	$\nu_1(x) \frac{\partial}{\partial u}, \dots, \nu_k(x) \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}$	$k + 1$ $\mathbb{C} \rtimes \mathbb{C}^k$
3.3	$\frac{\partial}{\partial u}, u \frac{\partial}{\partial u}, u^2 \frac{\partial}{\partial u}$	3 $\mathfrak{sl}(2)$

Primitive lie algebras of vector fields in $\mathbb{R}^2$:

Generators	Dim.	Structure	ℂ Type
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \alpha(x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}) + u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}$	3	$\mathbb{R} \rtimes \mathbb{R}^2$	1.7
$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, (x^2 - u^2) \frac{\partial}{\partial x} + 2xu \frac{\partial}{\partial u}$	3	$\mathfrak{sl}(2)$	1.1
$u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}, (1 + x^2 - u^2) \frac{\partial}{\partial x} + 2xu \frac{\partial}{\partial u}$	3	$\mathfrak{so}(3)$	1.1
$2xu \frac{\partial}{\partial x} + (1 - x^2 + u^2) \frac{\partial}{\partial u}$			
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}$	4	$\mathbb{R}^2 \rtimes \mathbb{R}^2$	1.9
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}$	5	$\mathfrak{sa}(2)$	2.1
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}$	6	$\mathfrak{a}(2)$	2.2
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u},$ $(x^2 - u^2) \frac{\partial}{\partial x} + 2xu \frac{\partial}{\partial u},$	6	$\mathfrak{so}(3, 1)$	1.4
$2xu \frac{\partial}{\partial x} + (u^2 - x^2) \frac{\partial}{\partial u}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial x}, u \frac{\partial}{\partial u},$ $x^2 \frac{\partial}{\partial x} + xu \frac{\partial}{\partial u}, xu \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u}$	8	$\mathfrak{sl}(3)$	2.3

12. Infinitesimal invariance

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

The fundamental feature of connected Lie groups ability of work infinitesimally, thereby effectively linearizing complicated invariance criteria.

We begin with the infinitesimal criterion for the invariance of a real-valued-function.

Theorem 12.1. *Let G be a connected group of transformations acting on M . A function $I : M \rightarrow \mathbb{R}$ is G -invariant if and only if*

$$v|_x[I](x) = 0 \quad (x \in M, v \in \mathfrak{g})$$

Proof. Let $v \in \mathfrak{g}$ be fixed.

We differentiate the invariance condition $I(\exp(tv)x) = I(x)$ ($x \in M$) and set $t = 0$ to deduce the infinitesimal condition of the theorem.

Conversely, if we have the infinitesimal condition, then for all t , $\frac{d}{dt}(I(\exp(tv)x)) = 0$, then $I(\exp(tv)x)$ is constant under the one-parameter subgroup generated by v .

The invariance of I under arbitrary group elements then follows from that fact that every group element $g = \exp(v_1) \cdots \exp(v_k)$ with $v_1, \dots, v_k \in \mathfrak{g}$.

□

According the theorem, the invariant I of a one-parameter group with infinitesimal generator $v = \sum_{1 \leq i \leq m} \xi_i(x) \frac{\partial}{\partial x_i}$ satisfy the first order, linear, homogeneous partial differential equation

$$\sum_{1 \leq i \leq m} \xi_i(x) \frac{\partial I}{\partial x_i} = 0$$

The solutions are effectively found by the method of characteristics.

We replace the partial differential equation by the characteristic system of ordinary differential equations

$$\frac{dx_i}{\xi_1(x)} = \frac{dx_2}{\xi_2(x)} = \dots = \frac{dx_m}{\xi_m(x)}$$

The general solution can be locally written in the form $I_1(x) = c_1, \dots, I_{m-1}(x) = c_{m-1}$, where c_i are constants of integration.

Then $I_1, \dots, I_{m-1}$ form a complete set of functionally independent invariants of the one-parameter group generated by v .

Example 12.2. Consider the local one-parameter group generated by

$$v = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + (1 + z^2) \frac{\partial}{\partial z}$$

Solving the system

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \\ z'(t) = 1 + z^2(t) \end{cases}$$

allow us to find the group transformation which is

$$\begin{cases} \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (t, x, y, z) \longmapsto (x \cos(t) - y \sin(t), x \sin(t) + y \cos(t), \frac{\sin(t) + z \cos(t)}{\cos(t) - z \sin(t)}) \end{cases}$$

The characteristic system is

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1 + z^2}$$

The first equation reduces to a simple separable ordinary differential equation $\frac{dy}{dx} = -\frac{x}{y}$ with general solution $x^2 + y^2 = c_1$, with $c_1 \in \mathbb{R}$. Therefore, $r = \sqrt{x^2 + y^2}$ provides one invariant.

To solve the second characteristic equation, we replace x by $\sqrt{r^2 - y^2}$, and treat r as a constant. The solution is $\tan^{-1}(z) - \sin^{-1}(y/r) = \tan^{-1}(z) - \tan^{-1}(y/x) = c_2$, hence $\tan^{-1}(z) - \tan^{-1}(y/x)$ is a second invariant.

A more convenient choice is provided by the tangent of this invariant. We deduce that for $yz + x \neq 0$, the function $r = \sqrt{x^2 + y^2}$ and $w = \frac{xz - y}{yz + x}$ form a complete system of functionally independent invariants, whose common level sets describe the integral curves.

Remark 12.3. For multi-parameter groups, the invariants are simultaneous solutions to an overdetermined system of linear, homogeneous, first order partial differential equations. One solution methods is to look for invariants of each generator in turn and try to re-express subsequent generators in terms of the invariants. However, this method can become complicated.

Infinitesimal techniques are also effective for the determination of invariant submanifolds and invariant systems of equations.

Theorem 12.4. A closed submanifold $N \subset M$ is G -invariant if and only if for every $x \in N$, $\mathfrak{g}|_x \subset T_x N$.

Proof. It follows from the uniqueness of the flown, which implies that if $v \in TN$, then $\exp(tv)N \subset N$. □

Corollary 12.5. *If the submanifold N is not closed, then N is merely locally G -invariant ; meaning that for every $x \in N$, and every $g \in G_x$ in a neighborhood of the identity, possibly depending on x , we have $g \cdot x \in N$.*

Example 12.6. If G acts on $\mathbb{R}^2$ by $(t, x, y) \mapsto x + t, y$ then any horizontal line segment, e.g. $\{(x, 0) \mid 0 < x < 1\}$ is locally, but not globally, translationally invariant.

An important consequence of the theorem is the characterization of symmetry groups of regular systems of equations.

Theorem 12.7. *(Characterization of symmetry group of regular systems of equations) A connected Lie group G is a symmetry group of the regular system of equations $F_1(x) = \dots = F_k(x) = 0$ if and only if*

$$v|_x[F_\nu](x) = 0 \quad (x \text{ such that } F_1(x) = \dots = F_k(x) = 0, 1 \leq \nu \leq k, v \in \mathfrak{g})$$

Proof. By the previous theorem and the fact that for $S_F = \{F_1 = \dots = F_k\}$ a regular variety in M , we have

$$T_x S_F = \{v \in T_x M \mid d(F_\nu)_x = 0, 1 \leq \nu \leq k\}$$

□

Example 12.8. The equation $x^2 + y^2 = 1$ defines a circle, which is rotationally invariant ; We use the group action of $O(2)$ on $\mathbb{R}^2$. We know that $\mathfrak{o}(2)$ is the set of skew-symmetric matrix generated by $v = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The flow associated to v is $\Phi_t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto e^{tv} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$.

with $J^{2k} = (-1)^k I_2$ and $J^{2k+1} = (-1)^k J$ allows us to obtain

$$\Phi_t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1 + e^{-t^2})x + \sin(t)y \\ (1 + e^{-t^2})y - x\sin(t) \end{pmatrix}$$

Then

$$\hat{v} = \begin{pmatrix} -y \\ x \end{pmatrix} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

Let $F : (x, y) \mapsto x^2 + y^2$ and $v = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$.

We compute $v(F)$ and find 0.

13. Invariant vector fields

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

Definition 13.1. A vector field on M is G -invariant if for all $g, x \in G, M$, $d(\Phi_g)_e(w|_x) = w|_{g \cdot x}$.

Theorem 13.2. *Let G a connected transformation group acting on M . A vector field w on M is G -invariant if and only if for all $v \in \mathfrak{g}$, $[v, w] = 0$.*

Proof. Let $v \in \mathfrak{g}$.

We need to compare the value of w at $\exp(tv)x$ with its value at x . Since these two vectors lie in different spaces, to effect the comparison we must map $w|_{\exp(tv)x}$ back to x using the differential $d\exp(-tv)$. We expand the resulting expression in powers of t to obtain

$$d\exp(-tv)[w|_{\exp(tv)x}] = w|_x + t[v, w]|_x + \dots$$

As a consequence, the Lie bracket $[v, w]$ represents the infinitesimal change of the vector field w under the action of the flow induced by the vector field v .

In particular, if w is invariant under the flow induced by v then the Lie bracket vanishes.

The converse follows from the uniqueness theorem for ordinary differential equations.

Then we have the infinitesimal invariance condition for v : $d\exp(-tv)[w|_{\exp(tv)x}] = w|_x$.

Finally, if the infinitesimal invariance condition holds for every $v \in \mathfrak{g}$, then by the fact for $g \in G$ is decomposed in product of $\exp(tv)$ implies the result. $\square$

Proposition 13.3. *If w is a G -invariant vector field, and I is any invariant function, then Iw is also a G -invariant vector field.*

More generally, if $w_1, \dots, w_k$ are pointwise linearly independent invariant vector fields, then $w = \sum_{1 \leq i \leq k} I_i w_i$ is G -invariant if and only if the I_j 's are G -invariant functions.

Remark 13.4. In contrast to the situation for invariant functions and invariant vector field on Lie group, the existence of invariant vector field of general transformation groups is not so straightforward.

Theorem 13.5. *Let G be an r -dimensional Lie group acting effectively freely on M . Then locally, there exists m point wise linearly independent G -invariant vector field forming a G -invariant frame on M .*

Example 13.6. Let $SE(2)$ the Euclidean group acting on $\mathbb{R}^2$, which is generated by

$$v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + (1 + z^2) \frac{\partial}{\partial z}$$

Let $w = \xi \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}$.

w is $SE(2)$ -invariant if and only if $w \in \text{vect}\left(\frac{\frac{\partial}{\partial x} + z \frac{\partial}{\partial y}}{\sqrt{1+z^2}}, \frac{z \frac{\partial}{\partial x} - \frac{\partial}{\partial y}}{\sqrt{1+z^2}}, (1 + z^2) \frac{\partial}{\partial z}\right)$.

14. Lie-derivatives-and-invariant-differential-forms

Let M be a smooth manifold of dimension m and G be a Lie group of dimension r .

Definition 14.1. A differential form Ω on M is called G -invariant for all $(g, x) \in G \times M$, $(\Phi_g)^*(\Omega)(x) = \Omega(x)$.

Proposition 14.2. *If I is an G -invariant function, its differential dI is G -invariant one-form.*

Definition 14.3. Let v be a vector field on M with flow $\exp(tv)$.
The Lie derivative $v(\Omega)$ of a differential form Ω with respect to v is defined as

$$v(\Omega)|_x = \frac{d}{dt}(\exp(tv)^*(\Omega)(x))|_{t=0} \quad (x \in M)$$

Proposition 14.4. *The Lie derivative satisfies for Ω, Θ differential forms and v a vector field :*

$$\begin{aligned} v(\Omega + \Theta) &= v(\Omega) + v(\Theta) \\ v(\Omega \wedge \Theta) &= v(\Omega) \wedge \Theta + \Omega \wedge v(\Theta) \\ v(d\Omega) &= dv(\Omega) \end{aligned}$$

Definition 14.5. Given a k -form Ω , we define the interior product with a vector field v as the $(k-1)$ -form $v \lrcorner \Omega$ by

$$(v \lrcorner \Omega)(x)(w_1, \dots, w_{k-1}) = \Omega(x)(v, w_1, \dots, w_{k-1}) \quad (x \in M, w_1, \dots, w_{k-1} \in T_x M)$$

The interior product satisfies

$$v(\Omega) = v \lrcorner (d\Omega) + d(v \lrcorner \Omega)$$

Proposition 14.6. *Let G be a connected Lie group.*

A differential form Ω is G -invariant if and only if its Lie derivative with respect to every $v \in \mathfrak{g}$, gives $v(\Omega) = 0$.

Example 14.7. Let $dx = dx_1 \wedge \dots \wedge dx_m$ the form volume on $\mathbb{R}^m$.

For $v = \sum_{1 \leq i \leq m} \xi_i(x) \frac{\partial}{\partial x_i}$ is given by $v(dx) = (div(\xi))dx$ where $div(\xi) = \sum_{1 \leq i \leq m} \frac{\partial \xi_i}{\partial x_i}$.
Then a group action on $\mathbb{R}^m$ preserve the volume if and only if each of its infinitesimal generators satisfy $div(\xi) = 0$.

Theorem 14.8. *Let G be an r -dimensional Lie group acting effectively freely on M . Locally, there exist m pointwise linearly independent G -invariant one-forms $\omega^1, \dots, \omega^m$.*

Remark 14.9. The vector fields $v_1, \dots, v_m$ form a G -invariant frame on M if and only if the duals one-forms $\omega_1, \dots, \omega_m$ is a G -invariant coframe on M .

Example 14.10. Consider the action of $SE(2)$ on $\mathbb{R}^2$. The invariant one-forms are the dual to the invariant vector field, so for ω a one-form of $SE(2)$:
 ω is $SE(2)$ -invariant if and only if $\omega \in \text{vect}\left(\frac{dx+zy}{\sqrt{1+z^2}}, \frac{zdx-dy}{\sqrt{1+z^2}}, (1+z^2)dz\right)$.

15. The Maurer-Cartan forms and Maurer-Cartan form

Definition 15.1. A differential form Ω on a Lie group G is right-invariant if for all $g \in G$, $(R_g)_*\Omega = \Omega$.

Definition 15.2. The canonical one-form α on G called the Maurer-Cartan one-form is the right-invariant $\mathfrak{g}$ -valued one-form uniquely determined by

$$\alpha(g)(v) = v \quad (g \in G, v \in \mathfrak{g})$$

or by

$$\alpha(g)(v) = (R_{g^{-1}})_*(v) \quad (g \in G, v \in T_g G)$$

Let $\{v_1, \dots, v_r\}$ be a basis of $\mathfrak{g}$.
 For every $g \in G$, $v \in T_g G$, $\alpha(g)(v) \in \mathfrak{g}$, hence there exists $(\beta_1^g(v), \dots, \beta_r^g(v))$ such that

$$\alpha(g)(v) = \sum_{1 \leq k \leq r} \beta_k^g(v) v_k$$

and set also for every $1 \leq i \leq r$, $\alpha_i : g \mapsto \beta_i^g$. Therefore,

$$\alpha = \sum_{1 \leq i \leq r} \alpha_i v_i$$

Note that by definition of α , for every $1 \leq k \leq r$, $\alpha(v_k) = v_k$ then

$$\alpha_i(v_j) = \delta_{i,j} \quad (g \in G, v \in T_g G)$$

Then $\{\alpha_1, \dots, \alpha_r\}$ is the dual basis of the $\{v_1, \dots, v_r\}$, precisely $\{\alpha_1, \dots, \alpha_r\}$ is a basis of the right-invariant one-forms called the basis of the right-invariant Maurer-Cartan one-forms associated to $\{v_1, \dots, v_r\}$.

The v 's satisfy

$$[v_i, v_j] = \sum_{1 \leq k \leq r} c_{i,j}^k v_k \quad (1 \leq i, j \leq r)$$

where the $c_{i,j}^k$'s are the structure constants associated to the basis $\{v_1, \dots, v_r\}$.

Then the α 's satisfy the fundamental structure equation

$$d\alpha_k = -\frac{1}{2} \sum_{1 \leq i, j \leq r} c_{i,j}^k \alpha_i \wedge \alpha_j \quad (1 \leq k \leq r)$$

Remark 15.3. If $G \subset GL(n)$ for some $n \geq 1$, then $\alpha(g) = dg \cdot g^{-1}$.

Indeed :

If $g : G \rightarrow G$ is the identity map then $dg : TG \rightarrow TG$ is the identity map.

Then $\alpha(g) = (R_{g^{-1}})_* \cdot dg = R_{g^{-1}}(dg) = dg \cdot g^{-1}$.

Then a basis of Maurer-Cartan forms is given among the entries of the matrix $dg \cdot g^{-1}$.

Example 15.4. We know that $A(1)$ can be identified with the group 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

A basis of Maurer-Cartan forms is given among the entries of the matrix

$$dg \cdot g^{-1} = \begin{pmatrix} da & db \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1}da & a^{-1}(adb - bda) \\ 0 & 0 \end{pmatrix} \quad (a \in \mathbb{R}^*, b \in \mathbb{R})$$

Then the two Maurer-Cartan forms are

$$\alpha_1 : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \frac{da}{a}$$

and

$$\alpha_2 : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \frac{adb - bda}{a}$$

The Maurer-Cartan structure equation for the group $A(1)$ are then

$$d\alpha_1 = 0, \quad d\alpha_2 = \frac{da \wedge db}{a} = \alpha_1 \wedge \alpha_2$$

Geometric foundations part 2

1. Distributions

Now, we wish to generalize the part of study of integral curves to include leaves of larger dimension. We shall formalize the idea of "prescribing the tangent planes" in the notion of a distribution.

Definition 1.1. An r -dimensional distribution on M is a collection $\mathcal{D} = \{\mathcal{D}_x \mid x \in M\}$ of r -dimensional subspaces $\mathcal{D}_x \subset T_x M$, one for each $x \in M$, that are smooth in the sense that they may be described on any sufficiently small open set $U \subset M$ as the span of r smooth linearly independent vector fields $\{X_1, \dots, X_r\}$ around a point $x \in M$.

These vector fields themselves are called a local basis for $\mathcal{D}$ on U .

A distribution on M is said to be integrable or involutive if for any point $x \in M$ there is a local coordinate $(U; x_1, \dots, x_m)$ around x such that $\{x_*^{-1}(\frac{\partial}{\partial x_1}|_x), \dots, x_*^{-1}(\frac{\partial}{\partial x_r}|_x)\}$ is a basis of $\mathcal{D}$ on U .

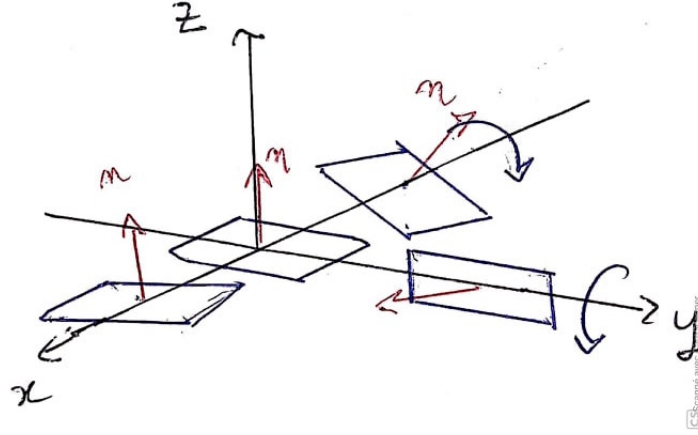
Definition 1.2. A connected r -dimensional submanifold N of M is called an integral submanifold for the r -dimensional distribution $\mathcal{D}$ if for each $x \in N$, $T_x N = \mathcal{D}_x$.

Remark 1.3. For an integrable distribution $\mathcal{D}$ on a M , there is locally a r -dimensional integrable submanifold N ; for all $x \in M$, there is a U neighborhood of x such that $\mathcal{D}_x = \text{vect}(\{x_*^{-1}(\frac{\partial}{\partial x_i}|_x)\}_{1 \leq i \leq r}) = T_x N$.

Indeed, let $x \in M$ and $(U; x_1, \dots, x_m)$ a local coordinate around x . Set $(a_1, \dots, a_m) := (x_1(x), \dots, x_m(x))$ are the local coordinate of x , and $N = x^{-1}(\{y \in x(U) \mid \forall r+1 \leq j \leq m, x_j(y) = a_j\})$.

In contrast to the case of a single vector field with integral curve, not every distribution has an integral submanifold through every point.

Example 1.4. Let the 2-dimensional distribution on $\mathbb{R}^3$ consisting of planes normal to the vector field $n(x, y, z) := (y, -x, 1)$.



It it exists an integral surface, around $0 \in \mathbb{R}^3$; If we draw a loop at a neighborhood of 0 in the counterclockwise, necessarily the surface has singularity. This is absurd since we always use smooth manifolds.

Suppose that N is a integral r -dimensional submanifold for a distribution on M . Let $x \in N$, and assume that a local basis for the distribution is given, on a neighborhood U , by $X_1, \dots, X_r$.

For all $1 \leq i \leq r$, and $y \in U$, $X_{i|y} \in T_y N$.

Then for all $1 \leq i, j \leq r$, and $y \in U$, $[X_i, X_j]_{|y} = [X_{i|y}, X_{j|y}] \in T_y N = \text{vect}(X_1|_y, \dots, X_r|_y) = \mathcal{D}_y$.

Definition 1.5. The system of vector fields $\{X_1, \dots, X_r\}$ on $U \subset M$ is called Frobenius integrable (or involutive) if for each $1 \leq i, j \leq r$, $[X_i, X_j] \in \text{vect}(X_1, \dots, X_r)$ on U .

Remark 1.6. We have just seen that Frobenius integrable is a necessary condition that a system of $r < m$ vector fields $\{X_1, \dots, X_r\}$ on a m -dimensional manifold have an integrable r -submanifold through each point.

Remark 1.7. The existence of an integral manifold through each point is in turn a necessary condition for the distribution to be Frobenius integrable. But for the previous example, a local basis for the distribution is given by

$$X_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}; X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$$

Indeed, let $X = (\alpha, \beta, \gamma) \in \mathbb{R}^3$

$$\begin{aligned}
 X \in \mathcal{D}_{(x,y,z)} &\iff \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot \begin{pmatrix} y \\ -x \\ 1 \end{pmatrix} = 0 \\
 &\iff \alpha y - \beta x + \gamma = 0 \\
 &\iff \begin{cases} \alpha = \alpha \\ \beta = \beta \\ \gamma = -y\alpha + x\beta \end{cases} \\
 &\iff X = \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ -y \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \\
 &\iff X = \alpha X_1 + \beta X_2
 \end{aligned}$$

And

$$\begin{aligned}
 [X_1, X_2] &= (X_1(0) - X_2(0)) \frac{\partial}{\partial x} + (X_1(1) - X_2(0)) \frac{\partial}{\partial y} + (X_1(x) - X_2(-y)) \frac{\partial}{\partial z} \\
 [X_1, X_2] &= 2 \frac{\partial}{\partial z} \notin \text{vect}(X_1, X_2)
 \end{aligned}$$

Then it is an other proof that the distribution has no integral surface.

Proposition 1.8. *Let $\{X_1, \dots, X_r\}$ and $\{Y_1, \dots, Y_r\}$ be two local basis on an open set $U \subset M$ for the distribution $\mathcal{D}$. One local basis is Frobenius integrable if and only if the other is.*

Proof. For all $x \in U$, $\mathcal{D}_x = \text{vect}(X_{1|x}, \dots, X_{r|x}) = \text{vect}(Y_{1|x}, \dots, Y_{r|x})$.

Assume that $\{X_1, \dots, X_r\}$ is Frobenius integrable,

For each $1 \leq i, j \leq r$, $[X_i, X_j] \in \text{vect}(X_1, \dots, X_r)$ on U .

Let $1 \leq i, j \leq r$, and $x \in U$, $Y_{i|x} = \sum_{k=1}^r \alpha_i^{k_i}(x) X_{k|x}$.

$$[Y_{i|x}, Y_{j|x}] = \sum_{1 \leq k_i, j_j \leq r} \alpha_i^{k_i}(x) \alpha_j^{k_j}(x) [X_{k_i|x}, X_{k_j|x}]$$

But for all $1 \leq i, j \leq r$, $[X_{k_i|x}, X_{k_j|x}] \in \text{vect}(X_{1|x}, \dots, X_{r|x}) = \text{vect}(Y_{1|x}, \dots, Y_{r|x})$, then

$$[Y_{i|x}, Y_{j|x}] \in \text{vect}(Y_{1|x}, \dots, Y_{r|x})$$

Then $\{Y_1, \dots, Y_r\}$ is Frobenius integrable.

Conversely, be the same way, $\{Y_1, \dots, Y_r\}$ is Frobenius integrable implies $\{X_1, \dots, X_r\}$ is Frobenius integrable. □

Then by this result, the Frobenius integrability is a property of the distribution itself ;

Definition 1.9. A r -dimensional distribution $\mathcal{D}$ on M is Frobenius integrable if for any local basis $\mathcal{B}$ for $\mathcal{D}$, $\mathcal{B}$ is Frobenius integrable.

Remark 1.10. If a distribution $\mathcal{D}$ is integrable, then $\mathcal{D}$ is Frobenius integrable.

2. Frobenius theorem

By the previous for a distribution, a necessary condition to be involutive is to be Frobenius involutive. Moreover, we can see by the following that this condition is also sufficient.

Theorem 2.1 (Frobenius). *Let $\mathcal{D}$ be an r -dimensional distribution on M . The following condition are equivalents :*

- (1) (1) $\mathcal{D}$ is Frobenius integrable.
- (2) (2) $\mathcal{D}$ is integrable.

Remark 2.2. This theorem allow us to dispense henceforth with the adjective "Frobenius" in the "Frobenius integrable".

Proof. (1) $\implies$ (2) is clear with what we have seen above.

(2) $\implies$ (1)

* For $r = 1$, the theorem is implied by the theorem of linearization of vector fields.

* For $r \geq 2$, we process by induction on r .

The method is to use the inductive hypothesis and the theorem of linearization of vector fields to successively improve a local basis of $\mathcal{D}$.

The theorem is local, so we can may assume $M = U$ an open set of $\mathbb{R}^m$, $p = 0$, and $\mathcal{D}$ is given on U by a local basis $\{X_1, \dots, X_r\}$.

At each step the vector fields and the open set will change, but we will keep the same notation for them.

Our goal is to produce a local coordinate $(U, x = (x_1, \dots, x_m))$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1 = \frac{\partial}{\partial x_1}, \dots, X_r = \frac{\partial}{\partial x_r}\}$.

Step 1 : There is a local coordinate system $x = (x_1, \dots, x_m)$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1, X_2, \dots, X_r = \frac{\partial}{\partial x_r}\}$.

Indeed, it is clear by the theorem of linearization of vector fields.

Step 2 : There is a local coordinate system $x = (x_1, \dots, x_m)$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1, X_2, \dots, X_r = \frac{\partial}{\partial x_r}\}$ such that $\{X_1, \dots, X_{r-1}\}$ is Frobenius integrable and for all $1 \leq j \leq r-1$, $X_j(x_r) = 0$.

Indeed, Set

$$\begin{aligned} X'_j &:= X_j - X_j(x_r)X_r \\ X'_r &:= X_r \end{aligned} \quad (1 \leq j \leq r-1)$$

Note that

$$\begin{aligned} X'_j(x_r) &= 0 \\ X'_r(x_r) &= 1 \end{aligned} \quad (1 \leq j \leq r-1)$$

Writing for $1 \leq i, j \leq r-1$

$$[X'_i, X'_j] = [X_i - X_i(x_r)X_r, X_j - X_j(x_r)X_r]$$

$$[X'_i, X'_j] = [X_i, X_j] - X_j(x_r)[X_i, X_r] - X_i(x_r)[X_r, X_j] + X_i(x_r)X_j(x_r)[X_r, X_r]$$

$$[X'_i, X'_j] \in \text{vect}(X_1, \dots, X_r)$$

But for all $1 \leq k \leq r-1$, $X_k = X'_k + X_k(x_r)X_r$, then

$$[X'_i, X'_j] \in \text{vect}(X'_1, \dots, X'_{r-1}, X'_r = X_r)$$

Then there exist $a_{i,j} : U \rightarrow \mathbb{R}$, such that

$$[X'_i, X'_j] - a_{i,j}X_r \in \text{vect}(X'_1, \dots, X'_{r-1})$$

For $p \in U$,

$$[X'_{i|p}, X'_{j|p}] = a_{i,j}(p)X_{r|p} + \text{vect}(X'_{1|p}, \dots, X'_{r-1|p})$$

By the function $x \mapsto x_r$ we find $a_{i,j} = 0$.

Then, for all $1 \leq i, j \leq r-1$,

$$[X'_i, X'_j] \in \text{vect}(X'_1, X'_{r-1})$$

Then $\{X'_1, \dots, X'_{r-1}\}$ is Frobenius integrable.

Step 3 : There is a local coordinate system $x = (x_1, \dots, x_m)$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1 = \frac{\partial}{\partial x_1}, X_2, \dots, X_{r-1} = \frac{\partial}{\partial x_{r-1}}, X_r = \sum_{1 \leq A \leq m} \zeta_A \frac{\partial}{\partial x_A}\}$ where the final coefficients ζ_A for $r \leq A \leq m$, are functions of $x_r, x_{r+1}, \dots, x_m$.

Indeed, By inductive hypothesis applied to the result of the step 2, we can find a local coordinate system $y = (y_1, \dots, y_m)$ such that

$$\text{vect}(X_1, \dots, X_{r-1}) = \text{vect}\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_r}\right)$$

It follows that $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{r-1}}\}$ is a local basis for the distribution, and moreover for $1 \leq j \leq r-1$, $\frac{\partial x_r}{\partial y_j} = 0$.

Like the step 2, we write for $x \in M$,

$$\left[\frac{\partial}{\partial y_j|_x}, \frac{\partial}{\partial x_r|_x}\right] = b_j(x) \frac{\partial}{\partial x_r|_x} \text{ mod } \left(\frac{\partial}{\partial y_1|_x}, \dots, \frac{\partial}{\partial y_{r-1}|_x}\right) \quad (1 \leq j \leq r-1)$$

where $A = B \text{ mod } (C)$ means $A - B \in \text{vect}(C)$ and $b_j : U \rightarrow \mathbb{R}$ By evaluating by $x \mapsto x_r$, we see that for all $1 \leq j \leq r-1$, $b_j = 0$.

Thus,

$$\left[\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_r}\right] = \sum_{1 \leq k \leq r-1} c_{j,k} \frac{\partial}{\partial y_k} \quad (1 \leq j \leq r-1)$$

Now, expressing $\frac{\partial}{\partial x_r}$ in terms of the y 's we may write be the local basis,

$$\frac{\partial}{\partial x_r} = \sum_{1 \leq k \leq r-1} \zeta_A \frac{\partial}{\partial y_A}$$

Putting this in the previous formula yields

$$\sum_{1 \leq k \leq r-1} \zeta_A \frac{\partial}{\partial y_A} = \sum_{1 \leq k \leq r-1} c_{j,k} \frac{\partial}{\partial y_k} \quad (1 \leq j \leq r-1)$$

Then for $r \leq A \leq m$, $1 \leq j \leq r-1$, $\frac{\partial \zeta_A}{\partial y_j} = 0$.

It follows that $\zeta_r, \dots, \zeta_m$ are functions of $x_r, x_{r+1}, \dots, x_m$.

Step 4 : There is a local coordinate system $x = (x_1, \dots, x_m)$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1 = \frac{\partial}{\partial x_1}, X_2, \dots, X_{r-1} = \frac{\partial}{\partial x_{r-1}}, X_r = \sum_{r \leq A \leq m} \zeta_A(x_r, \dots, x_m) \frac{\partial}{\partial x_A}\}$.
Indeed, set

$$X'_j := X_j \quad (1 \leq j \leq r-1)$$

$$X'_r := X_r - \sum_{1 \leq A \leq r-1} \zeta_A X_A$$

Then $\{X'_1, \dots, X'_r\}$ is still a local basis for $\mathcal{D}$ and X'_r has the form

$$X'_r := \sum_{r \leq A \leq m} \zeta(x_r, \dots, x_m) \frac{\partial}{\partial x_A}$$

Step 5 : There is a local coordinate system $x = (x_1, \dots, x_m)$ and a local basis for $\mathcal{D}$ on U of the form $\{X_1 = \frac{\partial}{\partial x_1}, \dots, X_r = \frac{\partial}{\partial x_r}\}$.

Indeed, By linearization of X_r , we can find change of variable $x_1, \dots, x_m$ that make $X'_r = \frac{\partial}{\partial x_r}$ as a coordinate field without affecting the other variables. □

Remark 2.3. The proof of Frobenius theorem not only tells us that the leaves through any point exist, but also tell us something about their mutual disposition ; it says that locally, an integrable r -dimensional distribution on a m -dimensional manifold looks like the affix r -planes in $\mathbb{R}^r \times \mathbb{R}^{m-r}$ with "second coordinate" constants.

More formally, we have

Corollary 2.4. *Let $\mathcal{D}$ be an integrable r -dimensional distribution on M . For each point $x \in M$, there is a local coordinate $(U; x_1, \dots, x_m)$ about x such that the integral manifold through p meets U in a set containing*

$$V(x) := \{y \in U \mid \forall m-r \leq j \leq m \mid x_j(y) = x_j(x)\}$$

Moreover, if $(U; x_1, \dots, x_m)$ and $(W; y_1, \dots, y_m)$ are two such local coordinate, then the coordinate changes $\Phi := xy^{-1}$ have the form

$$(\Phi_1(x_1, \dots, x_m), \dots, \Phi_{m-r}(x_1, \dots, x_m), \Phi_{m-r+1}(x_{m-r+1}, \dots, x_m), \dots, \Phi_m(x_{m-r+1}, \dots, x_m))$$

Proof. This is a consequence of step 5 in the proof of Frobenius theorem. □

Remark 2.5. * For $r = 1$, it exists always an integral curve for any vector field X given ; indeed $[X, X] = 0$ and it follows by Frobenius theorem.

* For $r \geq 2$, it's not always true, by the previous example.

For us, the most useful formulation of the Frobenius theorem is in terms of differential forms. This is the context in which a distribution is given, not in local terms as a the span of linearly independent smooth vector field, but in global terms as the kernel of a V -valued 1-form.

Proposition 2.6. *Let ω be a smooth V -valued 1-form on M .*

If $\begin{cases} M \longrightarrow \mathbb{N} \\ p \longmapsto \dim(\ker(\omega(p))) \end{cases}$ is constant equal to r , then for all $p \in M$, $\ker(\omega(p))$ is a r -dimensional distribution.

Proof. Let $p \in M$, we have to show that, locally, $\ker(\omega(p))$ is spanned by r linearly independent smooth vector fields.

Let $\{e_1, \dots, e_v\}$ a basis of V and write $\omega(p) = \sum_{1 \leq i \leq v} \omega_i(p) e_i$ where for all $1 \leq i \leq v$, $\omega_i \in A^1(M, \mathbb{R})$.

Let (U, x) a local coordinate around p , and

$$\omega_i(x) = \sum_{1 \leq j \leq m} a_{i,j}(x) dx_j \quad (1 \leq i \leq v)$$

so that

$$\omega(x) = \sum_{\substack{1 \leq j \leq m \\ 1 \leq i \leq v}} a_{i,j}(x) dx_j e_i$$

Set $A = (a_{i,j})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq v}}$ a $v \times m$ matrix.

Note that for $X \in T_x M$,

$$\omega(x)(X) = 0 \iff \sum_{1 \leq j \leq m} a_{i,j}(x) X_j = 0 \quad (1 \leq i \leq v)$$

So

$$\dim(\ker(\omega(x))) = r \iff \text{rank}(a_{i,j}(x)) = m - r$$

Thus we may assume, possibly after changing the basis of V and reordering the x_j that the first $(m - r) \times (m - r)$ block of $A(p)$ is invertible. Then, the same is true on some neighborhood of p we again call it U , and because the rank, condition the last $(v - (m - r))$ rows are linearly dependent on the first $(m - r)$ rows.

Thus, on U we clearly have $\ker(\omega) = \bigcap_{1 \leq i \leq m-r} \ker(\omega_i)$.

Now set

$$\nu_i := \begin{cases} \omega_i, & 1 \leq i \leq m - r \\ dx_i, & m - r < i \leq m \end{cases} \quad (1 \leq i \leq v)$$

The ν_i form a basis of the 1-form on U . We write for all $1 \leq i \leq m$, $x \in U$, $\nu_i(x) = \sum_{1 \leq j \leq m} b_{i,j}(x) dx_j$.

For all $1 \leq i \leq m$, the vector field $X_j = \sum_{1 \leq k \leq m} c_{j,k}(x) \frac{\partial}{\partial x_k}$ dual to the ν_i are smooth since the coefficients are given, on U , by $(b_{i,j})_{1 \leq i, j \leq m} (c_{j,k})_{1 \leq j, k \leq m} = I_m$ and $(b_{i,j})_{1 \leq i, j \leq m}$ invertible.

Moreover, the smooth vector fields X_j for $m - r < j \leq m$, form a local basis for the distribution $\ker(\omega(x))$.

□

Now, we are in the position to ask what form the integrability conditions for a distribution assumed when the distribution is given as the kernel of a V -valued 1-form ω . We have the next result which is the differential form version of the Frobenius' theorem.

Theorem 2.7 (Frobenius). *Let ω be a V -valued 1-form on M .*

Assume that $\begin{cases} M \longrightarrow \mathbb{N} \\ x \longmapsto \dim(\ker(\omega(x))) \end{cases}$ is constant equal to r , and let $\mathcal{D} = \{\ker(\omega(x)) \mid x \in M\}$ be the associated distribution. The following conditions are equivalents :

- (1) $\mathcal{D}$ is integrable on M .
- (2) For $X, Y \in TM$, such that $\omega(X) = \omega(Y) = 0$, we have $d\omega(X, Y) = 0$.

.

Proof. Let us choose a local basis $X_1, \dots, X_r$ for $\mathcal{D}$ in a neighborhood U of a point M . Then,

$\mathcal{D}$ is integrable on U if and only if for all $1 \leq j, k \leq n$, $[X_j, X_k] \in \text{vect}(X_1, \dots, X_r)$

And for all $1 \leq j \leq r$, $X_i \in \ker(\omega)$, then

$\mathcal{D}$ is integrable on U if and only if for all $1 \leq j, k \leq n$, $\omega([X_j, X_k]) = 0$. By the Cartan's formula ;

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \quad (X, Y \in TM)$$

We have

$$d\omega(X_j, X_k) = -\omega([X_j, X_k]) \quad (1 \leq j, k \leq r)$$

Then,

$\mathcal{D}$ is integrable if and only if for all $1 \leq j, k \leq n$, $d\omega(X_j, X_k) = 0$.

□

Corollary 2.8. *Let ω be a V -valued 1-form on M and $W \subset V$ be a subspace.*

Assume that $\begin{cases} M \longrightarrow \mathbb{N} \\ x \longmapsto \dim(\ker(\omega(x))) \end{cases}$ is constant equal to r , and let $\mathcal{D} = \{\omega(x)^{-1}(W) \mid x \in M\}$ be a associated distribution.

- (1) $\mathcal{D}$ is integrable on M .
- (2) For $X, Y \in TM$, such that $\omega(X), \omega(Y) \in W$, we have $d\omega(X, Y) \in W$.

.

Proof. The distribution $\mathcal{D}$ is the kernel of the $\omega \bmod(W)$, then by the theorem, the corollary follows.

□

Proposition 2.9. *Let $\omega = \begin{pmatrix} \omega_1 \\ \dots \\ \omega_v \end{pmatrix}$ be a $\mathbb{R}^v$ -valued 1-form on M .*

Assume that $\begin{cases} M \longrightarrow \mathbb{N} \\ x \longmapsto \dim(\ker(\omega(x))) \end{cases}$ is constant equal to r , and let $\mathcal{D} = \{\ker(\omega(x)) \mid x \in M\}$ be the associated distribution. The following conditions are equivalents :

- (1) $\mathcal{D}$ is integrable on M .
- (2) For all $1 \leq i \leq v$, $d\omega_i \in I(\omega_1, \dots, \omega_v)$

where $I(\omega_1, \dots, \omega_v)$ is the ideal in $A(M)$ generated by the ω_i 's.

Proof. We have for $\omega = \begin{pmatrix} \omega_1 \\ \dots \\ \omega_v \end{pmatrix}$, and $1 \leq i \leq v$,

$$d\omega_i \in I(\omega_1, \dots, \omega_v) \iff d\omega_i = \sum_{1 \leq k \leq v} A_{i,k} \omega_k \wedge \eta_k$$

where for $1 \leq i, k \leq v$, $A_{i,k}$ a smooth function on M , $\eta_k \in A^1(M)$. By $A = (A_{i,j})_{1 \leq i,j \leq m}$, we have

$$d\omega_i \in I(\omega_1, \dots, \omega_v) \iff d\omega = A\omega \wedge \eta$$

The we know that, $\mathcal{D}$ is integrable if and only of for all $X, Y \in TM$, such that $\omega(X) = \omega(Y) = 0$, we have $d\omega(X, Y) = 0$.

Then (2) implies (1).

Conversely, we assume $\mathcal{D}$ integrable.

We can, without lost generality, assume $\dim(V) = 1$.

Let $\mathbb{R}^{n+1} \ni (x_1, \dots, x_n, x_{n+1}) = x$, and let :

$$\omega = a_1(x) dx_1 + \dots + a_n(x) dx_n + a_{n+1}(x) dx_{n+1},$$

a differential 1-form $\mathcal{C}^\infty$ defined at a neighborhood of $0 \in \mathbb{R}^{n+1}$. By linear transformation, we can assume that at the origin $x = 0$, the hyperplane $\{\omega = 0\}$ is horizontal, given by $\{dx_{n+1} = 0\}$, it means that :

$$0 = a_1(0) = \dots = a_n(0) \quad \text{whereas} \quad 0 \neq a_{n+1}(0).$$

With other functions $\mathcal{C}^\infty$, we can solve :

$$dx_{n+1} = b_1(x) dx_1 + \dots + b_n(x) dx_n + b_{n+1}(x) \omega.$$

It is not useful to express the formulas.

Every differential 2-form can be written :

$$\begin{aligned} \Omega &= \sum_{1 \leq i, j \leq n+1} T_{i,j}(x) dx_i \wedge dx_j \\ &= \sum_{1 \leq i < j \leq n} (T_{i,j}(x) - T_{j,i}(x)) dx_i \wedge dx_j + \sum_{1 \leq i \leq n} T_{i,n+1}(x) dx_i \wedge dx_{n+1} \\ &= \sum_{1 \leq i < j \leq n} U_{i,j}(x) dx_i \wedge dx_j + \sum_{1 \leq i \leq n} V_i(x) dx_i \wedge \omega, \end{aligned}$$

with other function $\mathcal{C}^\infty$ — not useful to give the formulas.

Then, by hypothesis of integrability of (1), we assume that we have n vector fields $\mathcal{C}^\infty$ defined at $0 \in \mathbb{R}^{n+1}$, which are linearly independent, forming a local basis of $\mathcal{D}$.

Then we have for

$$X_1|_0 = \frac{\partial}{\partial x_1}|_0, \dots, X_n|_0 = \frac{\partial}{\partial x_n}|_0.$$

$$0 = \omega(X_1) = \dots = \omega(X_n).$$

For $\Omega = d\omega$, we see that for all $1 \leq i < j \leq n+1$, $U_{i,j}(0) = 0$. Then for x near 0 and for $k = i < j = l$,

$$\begin{aligned} d\omega(x)(X_j X_k) &= \sum_{1 \leq i < j \leq n} U_{i,j}(x) (dx_i \wedge dx_j)(X_j, X_k) + \sum_{1 \leq i \leq n} V_i(x) (dx_i \wedge \omega)(X_j, X_k) \\ &= U_{k,l}(x) + \sum_{1 \leq i \leq n} V_i(x) (\delta_{i,k} \omega(X_k) - \delta_{i,l} \omega(X_k)) \\ &= U_{k,l} + 0 \end{aligned}$$

But by the Cartan's formula

$$\begin{aligned} d\omega(x)(X_k, X_j) &= X_k(\omega(X_l)) - X_l(\omega(X_k)) - \omega([X_k, X_j]) \\ &= -\omega([X_k, X_j]) \end{aligned}$$

By integrability of $\mathcal{D}$, $\omega([X_k, X_j]) = 0$ then for all $k < l$, $U_{k,l}(x) = 0$.
Then

$$d\omega = \left(\sum_{1 \leq i \leq n} V_i(x) dx_i \right) \wedge \omega.$$

The result is proved near 0 and by translation it is true for all $x \in M$. □

Example 2.10. Take again the same example of this section, we will show in another way why there is no integral surface for the distribution. We use the Frobenius theorem viewed by 1-forms.

Let the distribution given by $n(x, y, z) = (y, -x, 1)$ in $\mathbb{R}^3$.

The plan is generated by

$$X_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}; \quad X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$$

Then we need to find ω such that

$$\langle X_1, X_2 \rangle = \ker(\omega)$$

Let $(x, y, z) \in \mathbb{R}^3$, $\omega(x, y, z) = A dx + B dy + C dz$, we have

$$0 = \omega(x, y, z)(X_1) = A - yC$$

and

$$0 = \omega(x, y, z)(X_2) = B + xC$$

Then $xA + yB = 0$.

For $A = y$ and $B = -x$ we obtain

$$\omega(x, y, z) = y dx - x dy + dz$$

Now, assume $\mathcal{D} = \{\ker(\omega(x, y, z)) \mid (x, y, z) \in \mathbb{R}^3\}$ is integrable.

Then $d\omega \in I(\omega)$, thus $d\omega = \alpha \omega \wedge \eta$ for $\eta \in A^1(\mathbb{R}^3)$ and $\alpha \in C^\infty(\mathbb{R}^3)$.

We note that $\omega \wedge d\omega = 0$ but $d\omega = -2dx dy$.

Then $0 = \omega \wedge d\omega = -2dx dy dz$ and it is absurd.

3. Fibre bundle

Now, we will talk about fibre bundles. Then, let M a manifold and G a Lie group.

Definition 3.1. A differentiable principal fibre bundle over M with a Lie group G consist of a manifold P and an action G on P satisfying the following conditions :

- (1) (1) G acts freely on P on the right by, for $g \in G$,

$$L_g : \begin{cases} P \longrightarrow P \\ p \longmapsto g \cdot p \end{cases}$$

- (2) (2) M is the quotient space of P by the equivalence relation induced by G , $M = P/G$, and the canonical projection $\pi : P \rightarrow M$ is differentiable.

- (3) (3) P is locally trivial ; every point $x \in M$, has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism

$$\psi : \begin{cases} \pi^{-1}(U) \longrightarrow U \times G \\ p \longmapsto (\pi(p), \varphi(p)) \end{cases}$$

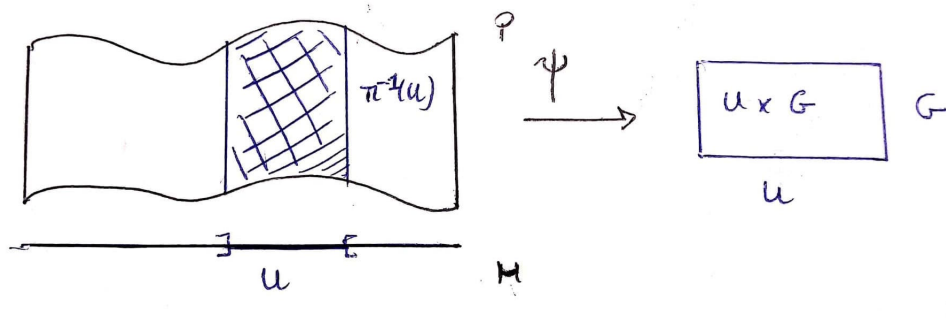
where $\varphi : \pi^{-1}(U) \longrightarrow G$ such that for all $p \in \pi^{-1}(U)$ and $g \in G$, $\varphi(L_g(p)) = L_g(\varphi(p))$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or P . We call P the total space, M the base space, G the structure group and π the projection.

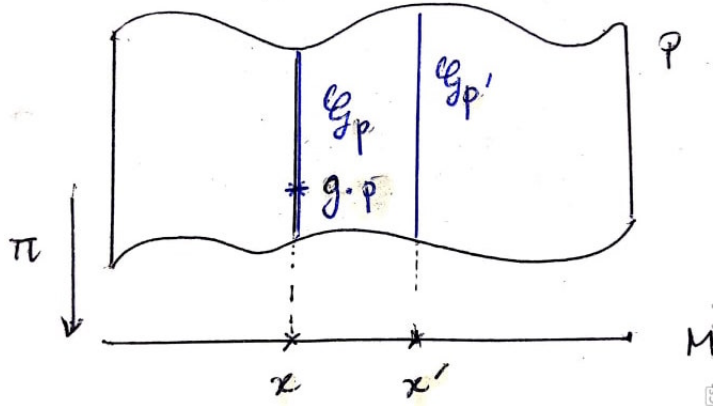
For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P , called the fibre over p .

Let $p \in \pi^{-1}(x)$, $\pi^{-1}(x) = \{g \cdot p \mid g \in G\} =: \mathcal{G}_p$ called the fibre through p .

Remark 3.2. We take this convention of fibre bundle and not with R_g since we want to use the flow $\varphi_t(p) = \exp(tv) \cdot p$.



Remark 3.3. Every fibre $\pi^{-1}(x) = \mathcal{G}_p = \{g \cdot p \mid g \in G\}$ is diffeomorphic to G . Then P is foliated by copies of G and $\mathcal{G} = \bigcup_{p \in P} \mathcal{G}_p$ is the foliation of P .



Example 3.4. $P = M \times G$ is called the trivial bundle.

Example 3.5. If W is a submanifold of M , then $\pi^{-1}(W)(W, G)$ is a principal fibre bundle. We call it the restriction of P to W and denote it $P|_W$.

Let P be a principal fibre bundle $P(M, G)$, the action of G on P induces a homomorphism σ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{X}(P)$ of the vector fields on P ; For each $v \in \mathfrak{g}$, $v^\dagger := \sigma(v)$ is called the fundamental vector corresponding to v .

Since the action sends each fibre into itself, for all $p \in P$, $v^\dagger|_p := \frac{d}{dt}|_0 (\exp(tv) \cdot p) \in T_p \mathcal{G}_p$.

As G acts freely on P , if $v \neq 0$, $v^\dagger$ never vanishes on P .

The dimension of each fibre is equal to $\dim(\mathfrak{g})$; $\begin{cases} \mathfrak{g} \longrightarrow T_p \mathcal{G}_p \\ v \longmapsto v^\dagger|_p \end{cases}$ is a linear isomorphism.

Proposition 3.6. Let $v^\dagger$ be the fundamental vector field corresponding to $v \in \mathfrak{g}$. For each $g \in G$, $(L_g)_* v^\dagger$ is the fundamental vector field corresponding to $(\text{Ad}(g^{-1}))v \in \mathfrak{g}$, where $\text{Ad}(g) : h \in \mathfrak{g} \mapsto ghg^{-1}$.

Proof. Since $v^\dagger$ is induced by the local 1-parameter group of transformation L_{g_t} where $g_t = \exp(tv)$, the vector field $(L_g)_* v^\dagger$ is induced by $L_g L_{g_t} L_{g^{-1}} = L_{g^{-1}g_t g}$. The proposition follows from the fact that $g^{-1}g_t g$ is generated by $(\text{Ad}(g^{-1}))v$. □

The concept of fundamental vector fields will prove to be useful in the theory of connections.

In order to relate our intrinsic definition of a principal fibre bundle to the definition and the construction by means of an open covering, we need the concept of transition functions. By (3) for a principal fibre bundle $P(M, G)$, it is possible to choose an open covering $\{U_\alpha\}$

of M , each $\pi^{-1}(U_\alpha)$ provided with a diffeomorphism $\psi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G \\ p \longmapsto (\pi(p), \varphi_\alpha(p)) \end{cases}$ where $\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow G$ such that for all $p \in \pi^{-1}(U_\alpha)$ and $g \in G$, $\varphi(L_g(u)) =$

$L_g(\varphi(p))$.

If $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, then $\varphi_\beta(L_g(p))(\varphi_\alpha(L_g(p)))^{-1} = \varphi_\beta(p)(\varphi_\alpha(p))^{-1}$, which shows the $\varphi_\beta(\varphi_\alpha)^{-1}$ depends only on $\pi(p)$ not on p . Then, we can define a mapping $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$ by; for all $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, $\psi_{\alpha,\beta}(\pi(p)) = \varphi_\beta(p)(\varphi_\alpha(p))^{-1}$.

The family $\{\psi_{\alpha,\beta}\}_{\alpha,\beta}$ is called transition functions of the bundle $P(M, G)$ corresponding to the open covering $\{U_\alpha\}$ of M .

We have the condition $(*)$:

$$\psi_{\gamma,\alpha}(x) = \psi_{\gamma,\beta}(x) \cdot \psi_{\beta,\alpha}(x) \quad (x \in U_\alpha \cap U_\beta \cap U_\gamma)$$

Conversely, we have :

Proposition 3.7. *Let $\{U_\alpha\}$ an open covering of M and for all α, β , such that $U_\alpha \cap U_\beta \neq \emptyset$, let a mapping $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$ in such way that the relations $(*)$ are satisfied. Then, we can construct a differentiable principal fibre bundle $P(M, G)$ with the transition functions $\psi_{\alpha,\beta}$.*

Proof. We first observe that the relation $(*)$ imply for all $x \in U_\alpha$ $\psi_{\alpha,\alpha} = e$ and for all $x \in U_\alpha \cap U_\beta$, $\psi_{\alpha,\beta}(x)\psi_{\beta,\alpha}(x) = e$.

For each α , let $X_\alpha = U_\alpha \times G$ and $X = \bigcup_\alpha X_\alpha$ be the topological sum of X_α ;

$$w \in X \iff w = (\alpha, x, g)$$

for a some $\alpha, x \in U_\alpha$ and $g \in G$.

Since each X_α is a differentiable manifold, X is a differentiable manifold in a natural way.

We introduce an equivalence relation ρ in X as follows ;

We say that $(\alpha, x, g) \in \{\alpha\} \times X_\alpha$ is equivalent to $(\beta, y, h) \in \{\beta\} \times X_\beta$ if and only if $x = y \in U_\alpha \cap U_\beta$ and $h = \psi_{\beta,\alpha}(p)(g)$.

We remark that (α, x, g) and (α, y, h) are equivalent if and only if $x = y$ and $g = h$. Let P be the quotient space of X by this equivalence relation ρ .

We first show that G acts freely on P on the right and that $P/G = M$.

By definition, each $h \in G$ maps the ρ -equivalence class of (α, x, g) into the ρ -equivalence class of (α, x, gh) . It is easy to see that this definition is independent of the choice of representative (α, x, g) and that G acts freely on P on the right.

The projection $\pi : P \rightarrow M$ maps, by definition, the ρ -equivalence class of (α, x, g) into x ; the definition of π is independent of the choice of representative (α, x, g) .

For $p, q \in P$, $\pi(p) = \pi(q)$ if and only if $v = L_h(u)$ for some $h \in G$. In fact, let (α, x, g) and (β, y, h) be representatives for u and v respectively. If $v = L_k(u)$ for some $k \in G$, then $x = y$ and hence $\pi(p) = \pi(q)$. Conversely, if $\pi(p) = x = y = \pi(q) \in U_\alpha \cap U_\beta$, then $v = L_k(u)$ where $k = g^{-1}\psi_{\beta,\alpha}(x)^{-1}h \in G$.

In order to make P into a differentiable manifold, we first note that, by the natural mapping $X \rightarrow P = X/\rho$, each $X_\alpha = U_\alpha \times G$ is mapped one-to-one onto $\pi^{-1}(U_\alpha)$.

We introduce a differentiable structure in P by requiring that $\pi^{-1}(U_\alpha)$ is an open submanifold of P and that the mapping $X \rightarrow P$, induces a diffeomorphism of $X_\alpha = U_\alpha \times G$ onto $\pi^{-1}(U_\alpha)$. This is possible since every point of P is contained in $\pi^{-1}(U_\alpha)$ for some α and the identification of (α, p, g) with $(\beta, p, \psi_{\beta,\alpha}(p)g)$ is made by means of differentiable mappings. It is easy to check that the action of G on P is differentiable and $P(M, G, \pi)$ is differentiable principal fibre bundle.

Finally, the transition functions on P corresponding to the covering $\{U_\alpha\}$ are precisely the

given $\psi_{\beta,\alpha}$ if we define $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ by $\psi_\alpha(p) = (x, g)$ where $u \in \pi^{-1}(U)$ is the ρ -equivalence class of (α, x, g) .

□

Definition 3.8. A homeomorphism f of a principal fibre bundle P' into another P consists of a mapping $f' : P' \rightarrow P$ and $f'' : G' \rightarrow G$ such that for all $(u, g) \in P' \times G'$, $f'(L_g(u)) = L_{f''(g)}f'(u)$.

For the sake of simplicity, we shall denote f' and f'' by the same letter f .

Remark 3.9. Every homomorphism $f : P' \rightarrow P$ maps each fibre of P' into a fibre of P and hence, induces a mapping of M' into M , which denoted by f .

Definition 3.10. A homomorphism $f : P' \rightarrow P$ is called an imbedding or injection if f is an imbedding and if $f : G' \rightarrow G$ is a monomorphism.

Remark 3.11. If $f : P' \rightarrow P$ is an imbedding, the the induced mapping $f : M' \rightarrow M$ is also an imbedding.

By identifying P' with $f(P)$, and G' with $f(G')$ and M' with $f(M')$, we say that $P'(M', G')$ is a subbundle of $P(M, G)$.

Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the lefts :

$$\begin{cases} G \times F \longrightarrow F \\ (g, \xi) \longmapsto L_g(\xi) \end{cases}$$

We will construct a fibre bundle $E(M, F, G, P)$ associated with P with the standard fibre F .

On the manifold $P \times F$, we let G act on the right as follows : for all $g \in G$,

$$\begin{cases} P \times F \longrightarrow P \times F \\ (p, \xi) \longmapsto (L_g(p), R_{g^{-1}}(\xi)) \end{cases}$$

The quotient space of $P \times F$ by this action group is denoted $E = P \times_G F$.

At this moment E is just a set, we will introduce a differentiable structure.

From $\begin{cases} P \times F \longrightarrow M \\ (p, \xi) \longmapsto \pi(p) \end{cases}$ we define $\pi_E : \begin{cases} E \longrightarrow P \times M \\ [(p, \xi)] \longmapsto \pi(p) \end{cases}$.

For each $x \in M$, $\pi_E^{-1}(x)$ is called the fibre of E over x .

Every point $x \in M$, has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Then, we can set $\pi^{-1}(U) = U \times G$, and we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$\begin{cases} \pi^{-1}(U) \times F \longrightarrow \pi^{-1}(U) \times F \\ (p, a, \xi) \longmapsto (p, L_g(p), R_{g^{-1}}(\xi)) \end{cases}$$

It follow that the isomorphism $\pi^{-1}(U) \simeq U \times G$ induces an isomorphism $\pi_E^{-1}(U) \simeq E \times F$. Then, we can introduce a differentiable structure in E by the requirement that $\pi_E^{-1}(U)$ is an open submanifold of E , which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E^{-1}(U) \simeq E \times F$.

The projection π_E is then a differentiable mapping of E onto M .

Definition 3.12. We call $E = E(M, F, G, P) = P \times_G F$ the fibre bundle over the base M , with fibre F and the structure group G , which is associated with the principal fibre bundle P .

We have easily,

Proposition 3.13. Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left. Let $E(M, F, G, P)$ be the fibre bundle associated with P . For each, $p \in P$ and $\xi \in F$, denote by $p\xi := [(p, \xi)]$. Then each $p \in P$, is a mapping of F onto $F_x = \pi_E^{-1}(x)$ where $x = \pi(p)$ and

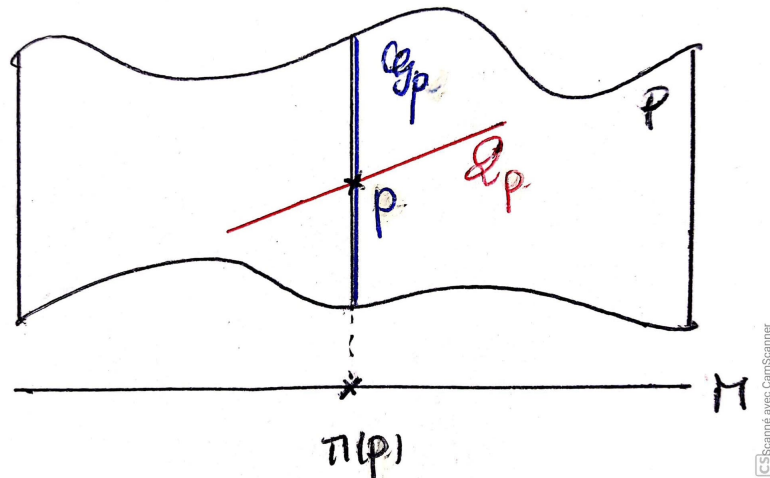
$$(L_g(p))\xi = p(L_g(\xi)) \quad (p \in P, g \in G, \xi \in F)$$

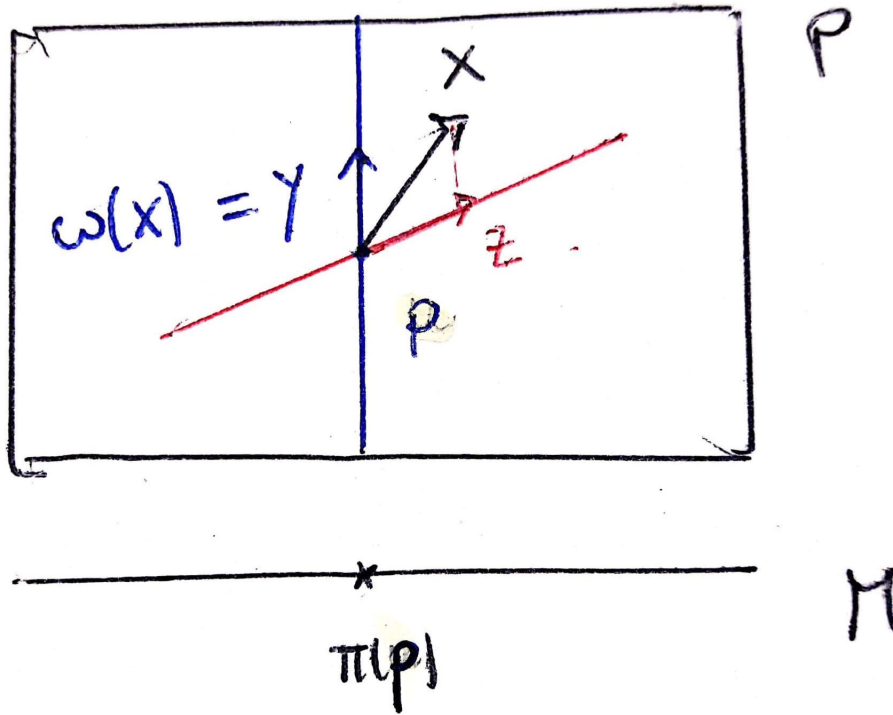
4. Connections in a principal fiber bundle

Let $P(M, G)$ be a principal fibre bundle over a manifold M with a Lie group G . For each $p \in P$, let $T_p P$ be the tangent space of P at p and $\mathcal{G}_p$ the subspace of $T_p P$ consisting of vectors tangent to the fibre through p .

Definition 4.1. A connection Γ in P is a collection of a subspace $T\mathcal{Q}_p$ of $T_p P$ at each $p \in P$ such that

- (1) $T_p P = T\mathcal{G}_p \oplus \mathcal{Q}_p$.
- (2) For $p \in P, g \in G, \mathcal{Q}_{L_g(p)} = (L_g)_* \mathcal{Q}_p$.
- (3) $\mathcal{Q}_p$ depends differentially on p .





Let Γ a connection in P , we define a 1-form ω on P with values in the Lie algebra $\mathfrak{g}$ of G as follows; We know that every $v \in \mathfrak{g}$ induces a vector field $v^\dagger$ on P , called the fundamental vector field corresponding to v , and that $\begin{cases} \mathfrak{g} \longrightarrow T_p \mathcal{G}_p \\ v \longmapsto v^\dagger_p \end{cases}$ is a linear isomorphism for each $p \in P$.

Definition 4.5. Let Γ a connection in P .

For each $p \in P$, and $X \in T_p P$, we define $\omega(X)$ to be the unique $v \in \mathfrak{g}$ such that $v^\dagger_p = X^{\text{vertical}}$.

This form ω is called the connection form of the given connection Γ .

Remark 4.6. It is clear that

$$\omega(X) = 0 \iff X \in \mathcal{Q}_p \quad (X \in T_p P)$$

Proposition 4.7. The connection form ω of a connection Γ satisfies the following conditions :

(1') For every $v \in \mathfrak{g}$, $\omega(v^\dagger) = v$.

(2') For every $g \in G$, $X \in TP$, $\omega((L_g)_* X) = \text{Ad}(g^{-1})(\omega(X))$.

Conversely, given a $\mathfrak{g}$ -valued 1-form ω on P satisfying (1') and (2'), there is a unique connection Γ in P whose connection form is ω .

Remark 4.8. (2') means that each

$$\omega(p) : T_p \mathcal{G}_p \rightarrow \mathfrak{g}$$

is the identity isomorphism, if we interpret $T_p \mathcal{G}_p$ as the tangent space to the Lie group copy $\mathcal{G}_p \simeq G$.

Proof. Let ω be the connection form of a connection. The condition (1') follows immediately from the definition of ω . Since every vector field of P can be decomposed into a horizontal and vertical vector fields, it is sufficient to verify (2') in the following two special cases : (a) X horizontal and (b) X vertical.

If X is horizontal, so is $(L_g)_*X$ for every $g \in G$, by the condition (2) for a connection. Thus both $\omega((L_g)_*X)$ and $Ad(g^{-1})(\omega(X))$ vanish.

In the case when X is vertical, we may further assume that X is a fundamental vector field $v^\dagger$. Then $(L_g)_*X$ is the fundamental vector field corresponding to $Ad(g^{-1})v$ (to see previous section).

Thus we have

$$(L_g^*\omega)(p)(X) = \omega(L_g(p))((L_g)_*X) = Ad(g^{-1})v = Ad(g^{-1})(\omega(p)(X))$$

Conversely, given a form ω satisfying (1') and (2'), we define

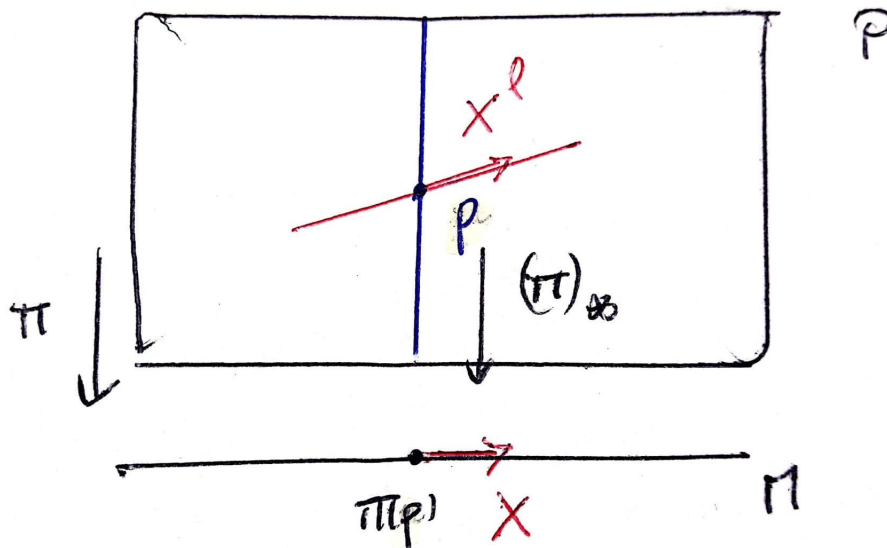
$$\mathcal{Q}_p = \{X \in T_p P \mid \omega(X) = 0\} \quad (p \in P)$$

$\{\mathcal{Q}_p \mid p \in P\}$ is also a connection whose connection form is ω . □

The projection $\pi : P \rightarrow M$ induces, for every $p \in P$, a linear mapping $\pi_* : T_p P \rightarrow T_x M$, where $x = \pi(p)$.

When a connection is given, for every $p \in P$, π_* maps $\mathcal{Q}_p$ isomorphically onto $T_p M$.

Definition 4.9. The horizontal lift of a vector field X on M is a unique vector field X^l on P which is horizontal and which projects onto X , that is, for every $p \in P$, $\pi_*(X^l_p) = X_{\pi(p)}$.



Proposition 4.10. *Let Γ be a connection in P and X be a vector field on M . There is a unique horizontal lift X^l of X . The lift X^l is invariant by the right action of G . Conversely, every horizontal vector field Y on P invariant by the right action of G is the lift of a vector field X on M ; $Y = X^l$*

Proof. The existence and uniqueness of X^l is clear from the isomorphism $\pi_* : \mathcal{Q}_p \rightarrow T_{\pi(p)}M$ for every $p \in P$.

To prove that if X is differentiable then X^l is differentiable, we take $x \in M$ and a neighborhood $U(x)$ such that $\pi^{-1}(U) \simeq U \times G$.

Using this isomorphism, we first obtain a differentiable vector field Y on $\pi^{-1}(U)$ such that $\pi(Y) = X$. Then X^l is the horizontal component of Y and hence is differentiable.

The invariance of X^l by G is clear from the invariance of the horizontal subspace by G . Finally, let Y be a horizontal vector field on P invariant by G . For every $x \in M$, take a point $p \in P$ such that $\pi(p) = x$ and define $X|_x := \pi(Y|_p)$. The vector $X|_x$ is independent of the choice of p such that $\pi(p) = x$, since if for $g \in G$, $p' = L_g(p)$, then $\pi(Y|_{p'}) = \pi(L_g(Y|_p)) = \pi(Y|_p)$. It is obvious that X^l is then the lift of the vector field X . $\square$

Proposition 4.11. *Let X^l and Y^l be the horizontal lifts of X and Y respectively. Then*

- (1) $X^l + Y^l$ is the horizontal lift of $X + Y$.
- (2) For every function f on M , $f^l \cdot X^l$ is the horizontal lift of fX where $f^l := f \circ \pi$.
- (3) The horizontal component of $[X^l, Y^l]$ is the horizontal lift of $[X, Y]$.

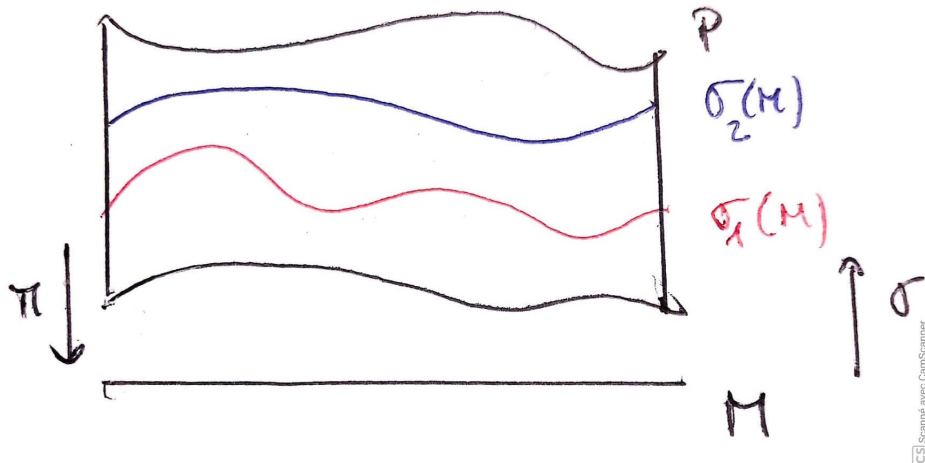
Proof. The first two assertions are trivial. As for the third, we have

$$\pi([X^l, Y^l]^{horizontal}) = \pi([X^l, Y^l]) = [X, Y]$$

$\square$

Definition 4.12. A section $\sigma : M \rightarrow P$ is a smooth map satisfying $\pi \circ \sigma = id_M$.

Remark 4.13. We see by the drawing that the section map means we section the fibre bundle P .



Let $x = (x_1, \dots, x_m)$ be local coordinate in a coordinate neighborhood U in M . For every $1 \leq i \leq m$, let X_i^l be the horizontal lift in $\pi^{-1}(U)$ of the vector field $X_i = \frac{\partial}{\partial x_i}$ in U . Then $\{X_1^l, \dots, X_m^l\}$ form a local basis for the distribution $\{\mathcal{Q}_p \mid p \in \pi^{-1}(U)\}$.

We will now express a connection form ω on P by a family of forms each defined in an open subset of the base manifold.

Definition 4.14. Let $\{U_\alpha\}$ be an open covering of M with a family of transition function $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$.

For each α , let $\sigma_\alpha : U_\alpha \rightarrow P$ be the cross section on U_α defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$, $x \in U_\alpha$ and e the identity of G .

Let θ be the left invariant $\mathfrak{g}$ -valued canonical Maurer-Cartan 1-form.

For each non-empty $U_\alpha \cap U_\beta$ define a $\mathfrak{g}$ -valued 1-form $\theta_{\alpha,\beta}$ on $U_\alpha \cap U_\beta$ by

$$\theta_{\alpha,\beta} = \psi_{\alpha,\beta}^* \theta$$

For each α , define a $\mathfrak{g}$ -valued 1-form ω_α on U_α by

$$\omega_\alpha = \sigma_\alpha^* \omega$$

Proposition 4.15. *The forms $\{\theta_{\alpha,\beta}\}_{\alpha,\beta}$ and $\{\omega_\alpha\}_\alpha$ are subject to the conditions :*

$$\omega_\beta = Ad(\psi_{\alpha,\beta}^{-1})\omega_\alpha + \theta_{\alpha,\beta} \quad (\text{on } U_\alpha \cap U_\beta)$$

Conversely, for every family of $\mathfrak{g}$ -valued 1-forms $\{\omega_\alpha\}_\alpha$ each defined on U_α and satisfying the preceding conditions, there is a unique connection form ω on P which gives rise to $\{\omega_\alpha\}_\alpha$ in the described manner.

Proof. For all $x \in U_\alpha \cap U_\beta$, if $U_\alpha \cap U_\beta \neq \emptyset$, then $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha,\beta}(x)$.

For every $x \in U_\alpha \cap U_\beta$, and vector $X \in T_x(U_\alpha \cap U_\beta)$ the vector $(\sigma_\alpha)_*(X) \in T_u(P)$ where $u = \sigma_\beta(x)$, is the image of $((\sigma_\alpha)_*(X), (\psi_{\alpha,\beta})_*(X)) \in T_{p'}P \oplus T_gG$ where $p' = \sigma_\alpha(x)$ and $g = \psi_{\alpha,\beta}(x)$ under the mapping $P \times G \rightarrow P$.

By Leibniz's formula, we have

$$(\sigma_\beta)_*(X) = (\sigma_\alpha)_*(X)\psi_{\alpha,\beta}(x) + \sigma_\alpha(x)(\psi_{\alpha,\beta})_*(X)$$

where $(\sigma_\alpha)_*(X)\psi_{\alpha,\beta}(p) = R_{\psi_{\alpha,\beta}(p)}((\sigma_\alpha)_*(X))$ and $\sigma_\alpha(p)(\psi_{\alpha,\beta})_*(X)$ is the image of $(\psi_{\alpha,\beta})_*(X)$ by the differential of $\sigma_\alpha(x)$, $\sigma_\alpha(x)$ being considered as a mapping of G into P which maps $h \in G$ into $\sigma_\alpha h$.

Taking the values of ω on both sides of the equality, we obtain

$$\omega_\beta(X) = Ad(\psi_{\alpha,\beta}^{-1})\omega_\alpha(X) + \theta_{\alpha,\beta}(X)$$

Indeed, if $v \in \mathfrak{g}$ is the left invariant vector field on G which is equal to $(\psi_{\alpha,\beta})_*(X)$ at $g = \psi_{\alpha,\beta}(x)$ so that $\theta((\psi_{\alpha,\beta})_*(X)) = v$, then $\sigma_\alpha(x)(\psi_{\alpha,\beta})_*(X)$ is the value of the fundamental vector field v^* at $p = \sigma_\alpha(x)\psi_{\alpha,\beta}(x)$ and hence $\omega(\sigma_\alpha(p)(\psi_{\alpha,\beta})_*(X)) = v$.

The converse can be verified by following back the process of obtaining $\{\omega_\alpha\}$ from ω .

□

Let $\{U_\alpha\}$ be an open covering of M with a family of transition functions $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$.

By transition functions, look locally, and so take a α and assume $P = \pi^{-1}(U_\alpha) = U_\alpha \times G$. Then we can assume $P = U \times G$ where U is an open set of M .

Let $\sigma, \tilde{\sigma} : M \rightarrow P$ two sections.

We have, by the identification, $\sigma_0 : p \mapsto (p, e)$ the zero (cross) section.

We want a relation between σ and $\tilde{\sigma}$ and we see that for all $x \in M$, $\sigma(x) = (x, g) = g \cdot (x, e) = g \cdot \sigma_0(x)$.

Then

$$\tilde{\sigma} = \varphi \circ \sigma$$

where $\varphi : M \rightarrow G$ a smooth function. Moreover, by the previous proposition, they satisfy

$$\tilde{\sigma}^* \omega = (Ad(\varphi^{-1}))\sigma^*(\omega) + \varphi^* \theta$$

5. Curvature form and structure equation

Let $P(M, G)$ be a principal fibre bundle and ρ a representation of G on a finite dimensional space V ; for all $g \in G$, $\rho(g)$ is a linear transformation and for all $g, h \in G$, $\rho(gh) = \rho(g)\rho(h)$.

Definition 5.1. A pseudotensorial form of degree r on P of type (ρ, V) is a V -valued r -form φ on P such that

$$L_g^* \varphi = \rho(g^{-1}) \cdot \varphi \quad (g \in G)$$

A such form φ is called a tensorial form if it is horizontal in the sense that for all $p \in P$, and $X_1, \dots, X_r \in T_p P$, if at least one of the tangent vector is tangent to a fibre, then $\varphi(p)(X_1, \dots, X_r) = 0$.

Example 5.2. If ρ_0 is the trivial representation of G on V , that is, for each $g \in G$, $\rho_0(g) = id_V$, then a tensorial form of degree r of type (ρ_0, V) is a form φ on P on which can be expressed as $\varphi = \pi^* \varphi_M$ where φ_M is a V -valued r -form on the base M .

Example 5.3. Let ρ be a representation of G on V and E the bundle associated with P with standard fibre V on which G acts through ρ .

A tensorial form φ of degree r of type (ρ, V) is defined by the following relation

$$\tilde{\varphi}(x)(X_1, \dots, X_r) = p(\varphi)(p)(X_1^l, \dots, X_r^l) \quad (X_1, \dots, X_r \in T_x M)$$

where $p \in P$, with $\pi(p) = p$, $(\pi)_*(X_i^l) = X_i$ and

$$\left\{ \begin{array}{l} M \longrightarrow \mathcal{L}((T_x M)^r, \pi_E^{-1}(x)) \\ x \longmapsto \left\{ \begin{array}{l} (T_x M)^r \longrightarrow \pi_E^{-1}(x) \\ (X_1, \dots, X_r) \longmapsto \tilde{\varphi}(x)(X_1, \dots, X_r) \end{array} \right. \end{array} \right.$$

and for $x \in M$, $\tilde{\varphi}(x)$ is a multilinear skew-symmetric mapping.

Let Γ be a connection in $P(M, G)$.

For $p \in P$, let G_u and Q_u be the vertical and the horizontal subspace of $T_p P$ respectively.

Let $h : \begin{cases} T_p P \longrightarrow Q_p \\ X \longmapsto X^{\text{horizontal}} \end{cases}$ be the projection.

Proposition 5.4. *Let φ be a pseudotensorial r -form on P of type (ρ, V) . We have*

- (1) *The form φh defined by $\begin{cases} P \longrightarrow \mathcal{L}(T_p P, \pi_E^{-1}(x)) \\ p \longmapsto \begin{cases} (T_p P)^r \longrightarrow \pi_E^{-1}(x) \\ (X_1, \dots, X_r) \longmapsto \varphi(x)(h(X_1), \dots, h(X_r)) \end{cases} \end{cases}$ is a tensorial form of type (ρ, V) .*
- (2) *$d\varphi = \varphi_*$ is a pseudotensorial $(r+1)$ -form of type (ρ, V) .*
- (3) *The $(r+1)$ -form $D\varphi$ defined by $D\varphi = (d\varphi)h$ is a tensorial form of type (ρ, V) .*

Proof. From $L_g \circ h = h \circ L_g$, $g \in G$, it follows that φh is a pseudotensorial form of type (ρ, V) . It is evident that if one of X_i 's is vertical, then $(\varphi h)(p)(X_1, \dots, X_r) = 0$.

(2) follows from $L_g^* \circ d = d \circ L_g^*$, $g \in G$.

(3) follows from (1) and (2). □

Definition 5.5. The form $D\varphi := (d\varphi)h$ is called the exterior covariant derivative of φ and D is called exterior covariant differentiation.

If ρ is the adjoint representation of G in the Lie algebra $\mathfrak{g} = V$, a (pseudo) tensorial form of type $(\rho, \mathfrak{g})$ is said to be of type $Ad(G)$. The connection form ω is a pseudotensorial 1-form of type $Ad(G)$. Then by the previous proposition, we have

Definition 5.6. Let Γ be a connection in P and ω the connection form of Γ .

$D\omega$ is tensorial 2-form of type $Ad(G)$ called the curvature form of ω .

We denote it $\Omega := D\omega$.

Theorem 5.7 (Structure equation). *Let ω be a connection form and Ω its curvature form. We have*

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)] \quad (X, Y \in TP)$$

and more formally,

$$\Omega(p)(X, Y) = d\omega(p)(X, Y) + [\omega(p)(X), \omega(p)(Y)] \quad (X, Y \in T_p P, p \in P)$$

Proof. Every vector of P is a sum of vertical vector field and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric in X and Y , it is sufficient to verify the equality in the following three cases :

(1) X and Y are horizontal ;

We have $\omega(X) = \omega(Y) = 0$ and the equality reduces to the definition of Ω .

(2) X and Y are vertical ;

Let $X = v^\dagger$ and $Y = w^\dagger$ at u , where $v, w \in \mathfrak{g}$.

We have by Cartan's formula

$$d\omega(v^\dagger, w^\dagger) = v^\dagger(\omega(w^\dagger)) - w^\dagger(\omega(v^\dagger)) - \omega([v^\dagger, w^\dagger]) = -[v, w] = -[\omega(v^\dagger), \omega(w^\dagger)]$$

since $\omega(v^\dagger) = v$ and $\omega(w^\dagger) = w$ and $[v^\dagger, w^\dagger] = [v, w]^\dagger$. On the other hand, $\Omega(v^\dagger, w^\dagger) = 0$.
 (3) X is horizontal and Y is vertical ;

We extend X to a horizontal vector field on P , which will be also denoted by X . Let $Y = v^\dagger$ at u , where $v \in \mathfrak{g}$. Since the right hand side of the equality vanishes, it is sufficient to show that $d\omega(X, v^\dagger) = 0$.

We have by Cartan's formula

$$d\omega(X, v^\dagger) = X(\omega(v^\dagger)) - v^\dagger(\omega(X)) - \omega([X, v^\dagger]) = -\omega([X, v^\dagger])$$

Now it is sufficient to prove the following lemma. □

Lemma 5.8. *If $v^\dagger$ is the fundamental vector field corresponding to an element $v \in \mathfrak{g}$ and X is a horizontal vector field, then $[X, v^\dagger]$ is horizontal.*

Proof. The fundamental vector field $v^\dagger$ is induced by L_{g_t} where g_t is the 1-parameter subgroup of G generated by $v \in \mathfrak{g}$. We have

$$[X, v^\dagger] = \lim_{t \rightarrow 0} \frac{1}{t} [L_{g_t}(X) - X]$$

If X is horizontal, so is $L_{g_t}(X)$. Thus $[X, v^\dagger]$ is horizontal. □

Then we have

Corollary 5.9. *Let X, Y be two horizontal vector fields on P . We have*

$$\omega([X, Y]) = -\Omega(X, Y)$$

The structure equation (often called "the structure equation of E. Cartan") is sometimes written, for the sake of simplicity, as follows :

$$\Omega = d\omega + [\omega, \omega]$$

In fact, we set $[\omega, \omega](X, Y) := [\omega(X), \omega(Y)]$.

Or we can use the definition of $\wedge_{\mathfrak{g}}$;

$$\Omega = d\omega + \frac{1}{2} \omega \wedge_{\mathfrak{g}} \omega$$

Let $\{e_1, \dots, e_r\}$ a basis of the Lie algebra $\mathfrak{g}$ and $c_{j,k}^i$ for $1 \leq i, j, k \leq r$ the structure constants of $\mathfrak{g}$ with respect $e_1, \dots, e_r$ that is,

$$[e_j, e_k] = \sum_{1 \leq i \leq r} c_{j,k}^i e_i \quad (1 \leq j, k \leq r)$$

Let $\omega = \sum_{1 \leq i \leq r} \omega_i e_i$ and $\Omega = \sum_{1 \leq i \leq r} \Omega_i e_i$.

Then the structure equation can be expressed as follows :

$$\Omega_i = d\omega_i + \frac{1}{2} \sum_{1 \leq j, k \leq r} c_{j,k}^i \omega_j \wedge \omega_k \quad (1 \leq i \leq r)$$

Theorem 5.10 (Bianchi's identity). *We have*

$$D\Omega = 0$$

Proof. By the definition of D , it suffices to prove that $d\Omega(X, Y, Z) = 0$ whenever X, Y, Z are all horizontal vectors. We apply the exterior differentiation d to the structure equation. Then

$$0 = dd\omega_i = -\frac{1}{2} \sum_{1 \leq j, k \leq r} c_{j,k}^i d\omega_j \wedge \omega_k + \frac{1}{2} \sum_{1 \leq j, k \leq r} c_{j,k}^i \omega_j \wedge d\omega_k + d\Omega_i$$

For $1 \leq i \leq r$, since, if X is horizontal then $\omega_i(X) = 0$, we have for X, Y, Z horizontal, $d\Omega_i(X, Y, Z) = 0$. □

Proposition 5.11. *The curvature form Ω of a connection form ω satisfies*

(1) *For all $g \in G$, $L_g^* \Omega = (Ad(g^{-1})) \Omega$.*

(2) *If at least one of the tangent vectors X, Y is vertical then $\Omega(X, Y) = 0$*

(3)

$$d\Omega = \frac{1}{2} \Omega \wedge_{\mathfrak{g}} \omega = \sum_{1 \leq k \leq r} \left(\sum_{1 \leq i_1, i_2 \leq r} c_{i_1, i_2}^k \Omega_{i_1} \wedge \omega_{i_2} \right) e_k$$

Proof. For (1), it follows from definition of ω .

For (2), we have seen in the proof of structure equation.

For (3), we use $\Omega = d\omega + \frac{1}{2} \omega \wedge_{\mathfrak{g}} \omega$;

$$\begin{aligned} d\Omega &= dd\omega + \frac{1}{2} d\omega \wedge_{\mathfrak{g}} \omega - \frac{1}{2} \omega \wedge_{\mathfrak{g}} d\omega \\ &= \frac{1}{2} (\Omega - \frac{1}{2} \omega \wedge_{\mathfrak{g}} \omega) \wedge_{\mathfrak{g}} \omega - \frac{1}{2} \omega \wedge_{\mathfrak{g}} (\Omega - \frac{1}{2} \omega \wedge_{\mathfrak{g}} \omega). \end{aligned}$$

(3) follows from the following lemma. □

Lemma 5.12. *For any $\mathfrak{g}$ -valued 1-form $\varpi = \sum_{1 \leq k \leq r} \varpi_k e_k$, we have*

$$0 = (\varpi \wedge_{\mathfrak{g}} \varpi) \wedge_{\mathfrak{g}} \varpi$$

and

$$0 = \varpi \wedge_{\mathfrak{g}} (\varpi \wedge_{\mathfrak{g}} \varpi)$$

Proof. We check the first identity, the second one being similar.

We expand

$$\begin{aligned} (\varpi \wedge_{\mathfrak{g}} \varpi) \wedge_{\mathfrak{g}} \varpi &= \left(\sum_{1 \leq k_1 \leq r} \varpi_{k_1} e_{k_1} \wedge_{\mathfrak{g}} \sum_{1 \leq k_2 \leq r} \varpi_{k_2} e_{k_2} \right) \wedge_{\mathfrak{g}} \sum_{1 \leq k_3 \leq r} \varpi_{k_3} e_{k_3} \\ &= \left(\sum_{1 \leq k_1, k_2 \leq r} \varpi_{k_1} \wedge \varpi_{k_2} [e_{k_1}, e_{k_2}] \right) \wedge_{\mathfrak{g}} \sum_{1 \leq k_3 \leq r} \varpi_{k_3} e_{k_3} \\ &= \sum_{1 \leq k_1, k_2, k_3 \leq r} \varpi_{k_1} \wedge \varpi_{k_2} \wedge \varpi_{k_3} [[e_{k_1}, e_{k_2}], e_{k_3}] \end{aligned}$$

We decompose the sum $\sum_{k_1, k_2, k_3}$ plainly in six sub-sums :

$$\begin{aligned} \sum_{k_1, k_2, k_3} &= \sum_{k_1 < k_2 < k_3} + \sum_{k_3 < k_1 < k_2} + \sum_{k_2 < k_3 < k_1} \\ &+ \sum_{k_2 < k_1 < k_3} + \sum_{k_3 < k_2 < k_1} + \sum_{k_1 < k_3 < k_2} \end{aligned}$$

We see, thanks to the Jacobi identity, that this decomposition is equal to 0. □

6. Cartan connections

Now, we talk about Cartan connections.

Let M a manifold.

Let G be a Lie group of dimension r and H be a Lie subgroup of G of dimension s . Denote by $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras, with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

Definition 6.1. A Cartan geometry on M modeled on $(\mathfrak{g}, \mathfrak{h})$ consists of the following data :

* A principal fibre bundle $P := P(M, H)$.

* A $\mathfrak{g}$ -valued 1-form ω on $P(M, H)$ which satisfies the following three properties :

(1) For every $p \in P$, $\omega(p) : T_p P \rightarrow \mathfrak{g}$ is an isomorphism.

(2) For every $y \in \mathfrak{h}$, we have $\omega(y^\dagger) = y$.

(3)

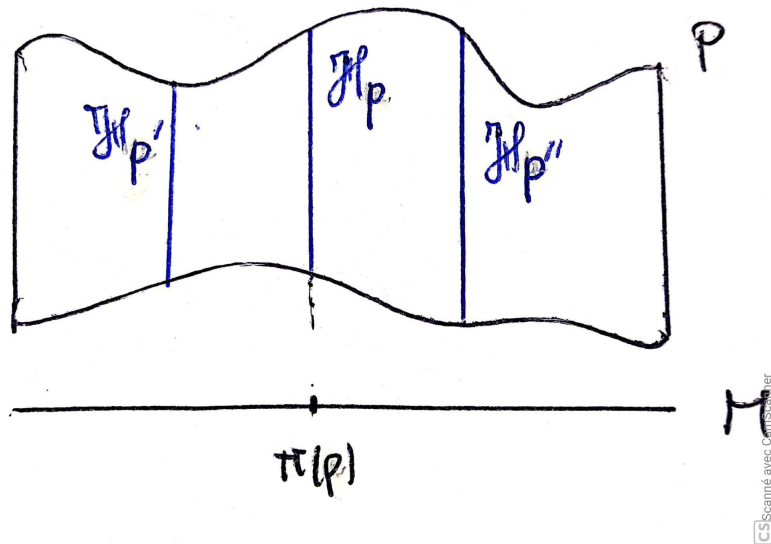
$$\omega(hp)((L_h)_*(v_p)) = \text{Ad}(h^{-1})(\omega(p)(v_p)) \quad (p \in P, v_p \in T_p P)$$

where $L_h : p \mapsto hp$ on P

We call Cartan connection the 1-form ω .

The difference between Cartan connection and connection on a principal fibre bundle is that for Cartan connection we add H a Lie subgroup of G ; the fibre of P is given by the foliation $\mathcal{H} = \bigcup_{p \in P} \mathcal{H}_p$, with $\mathcal{H}_p = \{p \cdot h \mid h \in H\}$.

Moreover, $T_p \mathcal{H}_p$ is the set of vertical vector fields.



Remark 6.2. * For ω a Cartan connection,

$$\dim(P) = \dim(T_p P) = \dim(\mathfrak{g}) > \dim(\mathfrak{h}) \quad (p \in P)$$

* For a general connection,

$$\dim(P) \geq \dim(\mathfrak{g})$$

Moreover, for a general connection, it is possible that $\dim(P) > \dim(\mathfrak{g})$; take $P = G$, $M = G/H$ and ω the Maurer-Cartan 1-form.

By (3) we have an isomorphism and so :

Definition 6.3. Let $x \in \mathfrak{g}$. We define a vector field on P by

$$\widehat{x}|_p := \omega(p)^{-1}(x) \quad (p \in P)$$

called the constant vector field associated to x .

Proposition 6.4. For any choice of a basis $(e_k)_{1 \leq k \leq r}$ of $\mathfrak{g}$, we have a collection of r vector fields

$$\widehat{e}_k := \omega^{-1}(e_k) \quad (1 \leq k \leq r)$$

such that for every $p \in P$, $\{(\widehat{e}_1)|_p, \dots, (\widehat{e}_r)|_p\}$ is a basis of $T_p P$.

Furthermore, by (2), we have for every $y \in \mathfrak{h}$

$$y|_p^\dagger = \omega^{-1}(p)(y) = \widehat{y}|_p \quad (p \in P)$$

Remark 6.5. So that only for the element $y \in \mathfrak{h}$, we have

$$y^\dagger = \widehat{y}$$

Remark 6.6. The Cartan curvature given by a Cartan connection satisfies the same relations that an ordinary curvature connection ;

(1)

$$\Omega = d\omega + \frac{1}{2}\omega \wedge_{\mathfrak{g}} \omega$$

(2) For all $g \in G$, $L_g^* \Omega = (Ad(g^{-1}))\Omega$.

(3) If at least one of the tangent vectors X, Y is vertical then $\Omega(X, Y) = 0$

(4)

$$d\Omega = \frac{1}{2}\Omega \wedge_{\mathfrak{g}} \omega = \sum_{1 \leq k \leq r} \left(\sum_{1 \leq i_1, i_2 \leq r} c_{i_1, i_2}^k \Omega_{i_1} \wedge \omega_{i_2} \right) e_r$$

Now, at each point $p \in P$, for every $x \in \mathfrak{g}$, the inverse image through $\omega(p)$ of the element $x + \mathfrak{h}$ of the quotient $\mathfrak{g}/\mathfrak{h}$:

$$\widehat{x + \mathfrak{h}}|_p = \omega(p)^{-1}(x + \mathfrak{h}) = \omega(p)^{-1}(x) + \omega(p)^{-1}(\mathfrak{h}) = \omega(p)^{-1}(x) + T_p \mathcal{H}_p$$

By (3), we have

Corollary 6.7.

$$\Omega(p)(\widehat{x + \mathfrak{h}}|_p, \widehat{x' + \mathfrak{h}}|_p) = \Omega(p)(\widehat{x}|_p, \widehat{x'}|_p) \quad (p \in P, x, x' \in T_p P)$$

In other words, the curvature acts in fact on the quotient $TP/T\mathcal{H}$ by the vertical foliation $\mathcal{H}$.

Since $\Omega : TP/T\mathcal{H} \times TP/T\mathcal{H} \rightarrow \mathfrak{g}$ is a skew-symmetric and linear map, we can define the curvature function of a Cartan connection ω .

Definition 6.8. The curvature function of a Cartan connection $\omega : TP \rightarrow \mathfrak{g}$ is the map $c \in \mathcal{C}^\infty(P, \mathcal{L}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}))$ such that

$$c(p) : \begin{cases} \mathfrak{g}/\mathfrak{h} \wedge \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g} \\ (\mathbf{x}' + \mathfrak{h}, \mathbf{x}'' + \mathfrak{h}) \longmapsto \Omega(p)(\widehat{\mathbf{x}'}|_p, \widehat{\mathbf{x}''}|_p) \end{cases}$$

Applying (3) of the definition of Cartan connection for $\widehat{\mathbf{x}}$, where $\mathbf{x} \in \mathfrak{g}$, we have

$$\omega(gp)(d(L_g)_p(\widehat{\mathbf{x}})) = Ad(g^{-1})(\mathbf{x})$$

Then by $\omega(gp)^{-1}$

$$d(L_g)_p(\widehat{\mathbf{x}}) = Ad(\widehat{g^{-1}})(\mathbf{x})|_{gp} \quad (p \in P, g \in G)$$

Then for $\mathbf{x} \leftarrow Ad(g)(\mathbf{x})$

$$\widehat{\mathbf{x}}|_{gp} = d(L_g)_p(Ad(g)(\mathbf{x})|_u) \quad (p \in P, g \in G)$$

By this relation, we can have

Lemma 6.9. For all $p \in P, g \in G, c(gp) = (g^{-1}) \cdot c(p)$.

Proof. For $\mathbf{x}, \mathbf{x}' \in \mathfrak{g}$,

$$\begin{aligned} c(gp)(\mathbf{x} + \mathfrak{g}, \mathbf{x}' + \mathfrak{g}) &= \Omega(gp)(\widehat{\mathbf{x}}|_{gu}, \widehat{\mathbf{x}'}|_{gp}) \\ &= \Omega(gp)(d(L_g)_p(\widehat{Ad(g)(\mathbf{x})}|_p), d(L_g)_p(\widehat{Ad(g)(\mathbf{x}')}|_p)) \\ &= (L_g^* \Omega)(p)(\widehat{Ad(g)(\mathbf{x})}|_p, \widehat{Ad(g)(\mathbf{x}')}|_p) \\ &= Ad(g^{-1})(\Omega(p)(\widehat{Ad(g)(\mathbf{x})}|_p, \widehat{Ad(g)(\mathbf{x}')}|_p)) \\ &= (g^{-1} \cdot c(p))(\mathbf{x} + \mathfrak{h}, \mathbf{x}' + \mathfrak{h}) \end{aligned}$$

□

We fix a basis $\{e_1, \dots, e_r\}$ of $\mathfrak{g}$ such that $\{e_{r-s+1}, \dots, e_r\}$ is a basis of $\mathfrak{h}$. For $p \in P, v \in T_p P, \omega(p)(v) \in \mathfrak{g}$, then there exists $(\beta_1^p(v), \dots, \beta_r^p(v))$ such that $\omega(p)(v) = \sum_{1 \leq k \leq r} \beta_k^p(v) e_k$. We have also

$$\begin{aligned} \Omega &= \sum_{1 \leq k \leq r} \Omega_k e_k \\ \omega &= \sum_{1 \leq i \leq r} \omega_i e_i \end{aligned}$$

where $\omega_i : p \mapsto \beta_i^p$.

By

$$\omega(e_k^\dagger) = e_k \quad (1 \leq k \leq r)$$

Then for every $1 \leq j, k \leq r$, $\omega_k(e_j^\dagger) = \delta_{k,j}$.

Thanks to this, $\{\omega_1, \dots, \omega_r\}$ is an independent family of the one-forms on P of dimension r ; This is a basis for the space of one-forms on P .

Every element $\varphi \in \mathcal{L}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$ is defined by a set of structure constants $c_{i,j}^k$ ($1 \leq i, j \leq r-s+1, 1 \leq k \leq r$), $c_{i,j}^k = -c_{j,i}^k$, where

$$\varphi(e_i + \mathfrak{h}, e_j + \mathfrak{h}) = \sum_{1 \leq k \leq r} c_{i,j}^k e_k \quad (1 \leq i, j \leq r-s+1)$$

Then, $\{\omega_i \wedge \omega_j \mid i < j\}$ constitute a basis of the 2-forms on P , and

$$\Omega_k = \sum_{1 \leq i < j \leq r-s+1} d_{i,j}^k \omega_i \wedge \omega_j \quad (1 \leq k \leq r)$$

$$\Omega_k(\widehat{e}_i, \widehat{e}_j) = \sum_{1 \leq i < j \leq r-s+1} c_{i,j}^k e_k \quad (1 \leq k \leq r)$$

We find $d_{i,j}^k = c_{i,j}^k$.

Finally,

$$\Omega = \sum_{1 \leq k \leq r} \Omega_k e_k$$

with

$$\Omega_k = \sum_{1 \leq i < j \leq r-s+1} c_{i,j}^k \omega_i \wedge \omega_j \quad (1 \leq k \leq r)$$

Jets spaces and transformations

1. Transformations and functions

The equivalence problem is the question of determining, if two differential structures are equivalent under a type of transformation.

Before talking about the Cartan equivalence to solve the equivalence problem, we will discuss about symmetry groups of differential equations.

They allow us to understand differential equations.

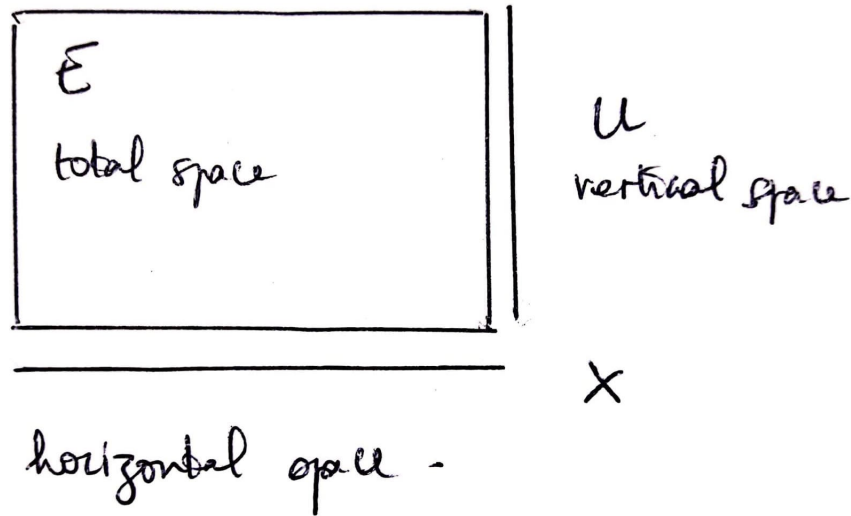
In this chapter, we will talk about the space where the equivalence problem will play, and submitting the action of a Lie group. Moreover, we will see some type of transformations.

Remark 1.1. For the equivalence problem, we talk about the local equivalence problem. So, all results are local.

Let G a Lie group of dimension r .

A general system of differential equations involves p independent variables $x = (x_1, \dots, x_p)$, which we can view as local coordinates on the euclidean space $X \simeq \mathbb{R}^p$, and q dependent variables $u = (u_1, \dots, u_q)$, coordinates on $U \simeq \mathbb{R}^q$.

Definition 1.2. The total space is the euclidean space $E = X \times U \simeq \mathbb{R}^{p+q}$ coordinated by the independent and dependent variables.



The most basic type of transformation is the point transformation.

Definition 1.3. A point transformation is a local diffeomorphism Φ defined by

$$\Phi : (u, x) \longmapsto (\chi(x, u), \psi(x, u)) =: (\bar{x}, \bar{u}).$$

Example 1.4. If G acts on E , then

$$g : (x, u) \mapsto g \cdot (x, u) = (\chi(x, u), \psi(x, u)).$$

is a point transformation.

Thus, a group G acting on E is said a group of point transformations.

There are others examples :

- (1) Base transformation ; it is a point transformation with $\bar{x} = \chi(x)$ and $\bar{u} = u$.
- (2) Fiber-preserving transformation ; it is a point transformation with $\bar{x} = \chi(x)$ and $\bar{u} = \psi(x, u)$.

In the case of group action from a connected Lie group G on E , a general vector field of the associated Lie algebra takes the form

$$v = \sum_{i=1}^p \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \varphi_\alpha(x, u) \frac{\partial}{\partial u_\alpha}. \quad (1.5)$$

Remark 1.6. Recall that we don't distinguish the Lie algebra of G , $\mathfrak{g}$ and the Lie algebra of vector field $\mathfrak{g}^\dagger$ on E by the action of G .

The vector field will generate a one-parameter group of fiber-preserving transformation if and only if for every $1 \leq i \leq p$, $\xi_i = \xi_i(x)$.

Definition 1.7. A vector field v of the form (1.4) is said vertical if the ξ_i are equal to zero. A vector field v of the form (1.4) is said horizontal if the ψ_i are equal to zero.

A classical solution to a system of differential equations will be described by a smooth function $u = f(x)$.

The graph of the function, $\Gamma_f = \{x, f(x)\}$, determines a regular p -dimensional submanifold of E . However, not every, regular p -dimensional submanifold of E defines the graph of a smooth function.

Definition 1.8. A submanifold Γ of E is said transverse if its tangent space contains no vertical tangent directions.

A group G acts on E transversally if for every function $u = f(x)$, its graph Γ_f is transverse.

Proposition 1.9. A regular p -dimensional submanifold Γ of E which is transverse at a point $z_0 = (x_0, u_0) \in \Gamma$ coincide in a neighborhood of x_0 with the graph of a smooth function $u = f(x)$.

Then in the case where the domain of f is compact, any point transformation, sufficiently closed to the identity transformation, given by a group G will act on a function $u = f(x)$ by point wise transforming its graph; if $\Gamma_f = \{(x, f(x))\}$ denotes the graph of f , then the transformed function $\bar{f} = g \cdot f$ will have the graph

$$\Gamma_{\bar{f}} = \{(\bar{x}, \bar{f}(\bar{x}))\} = g \cdot \Gamma_f = \{g \cdot (x, f(x))\}$$

Example 1.10. Consider the one-parameter group of rotations

$$g_t : (x, u) \in E \simeq \mathbb{R} \times \mathbb{R} \mapsto (x \cos(t) - u \sin(t), x \sin(t) + u \cos(t)) \quad (t \in \mathbb{R})$$

For a t not too large, a rotation transforms a function $u = f(x)$ by rotating its graph. The equation for the transformed function $\bar{f} := g_t \cdot f$ is given in implicitly form

$$\bar{x} = x \cos(t) - f(x) \sin(t), \quad \bar{u} = x \sin(t) + f(x) \cos(t)$$

so that $\bar{u} = \bar{f}(\bar{x})$ is found by eliminating x from these two equations.

For example, if $u = ax + b$, then the transformed function is also affine, and given by

$$\bar{u} = \frac{\sin(t) + a \cos(t)}{\cos(t) - a \sin(t)} \bar{x} + \frac{b}{\cos(t) - a \sin(t)}$$

which is defined provided the graph of f has not been rotated to be vertical.

2. Invariant functions

Let G a group of point transformation acting on $E \simeq X \times U$.

Now, we talk about the important notion of invariant functions of the bundle E .

Definition 2.1. A function $u = f(x)$ is said to be invariant under the transformation group G if its graph Γ_f is a locally G -invariant subset.

Example 2.2. The graph of any invariant function for the rotation group of the previous example, must be an arc of a circle centered at the origin so $u = \pm\sqrt{c^2 - x^2}$. Moreover, there are no globally defined invariant functions in this case.

Example 2.3. It is important to distinguish between an "invariant function" the section of the bundle E and the invariant function $I(x, u)$ defined locally on E .

For $f : X \rightarrow U$ an invariant function of the bundle E , its graph Γ_f is a G -invariant subset.

Then there is an action of G on Γ_f denoted $\mathcal{O}$, thus Γ_f locally a union of orbits in X . Therefore

$$\dim(T_z \mathcal{O}) \leq \dim(X) = p \quad (z=(x,u) \in E)$$

Since for every $z \in E$, $T_z \mathcal{O} = \mathfrak{g}|_z$, if Γ_f is transversal then for every $z \in E$, $\mathfrak{g}|_z$ contain no vertical tangent vectors.

Example 2.4. The infinitesimal generator of the rotation group $v = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$ is vertical at $u = 0$. Thus, the rotation group fails the transversality criterion on the x -axis and there is no smooth, rotationally invariants functions $u = f(x)$ passing through such points.

However, in the case of the group acts semi-regularly and transversally, then we can characterize the invariant function by the use the functionally independent real-valued invariant of the group action and the theorem of characterization of submanifold G -invariant (theorem 5.15 of the chapter Lie groups).

Theorem 2.5. Let G be a transformation group acting semi-regularly and transversally on $E \simeq X \times U$ with r -dimensional orbits.

Let $I_1(x, u), \dots, I_{p-s}(x, u), J_1(x, u), \dots, J_q(x, u)$ a complete set of functionally independent invariants function for G .

Every G -invariant function $u = f(x)$ can locally be written in the implicit form

$$(I_1(x, u), \dots, I_{p-s}(x, u)) = h(J_1(x, u), \dots, J_q(x, u))$$

As usual, the most convenient characterization of the invariant functions is based on an infinitesimal condition.

Since the graph of a function $u = f(x)$ is defined by the vanishing of its components $u_\alpha - f_\alpha(x)$, the theorem of the characterization of symmetry group of regular systems of equations (the theorem 12.1 of the chapter Lie groups) imposes

$$0 = v(u_\alpha - f_\alpha(x)) = \varphi_\alpha(x, u) - \sum_{i=1}^p \xi_i(x, u) \frac{\partial f_\alpha}{\partial x_i} \quad (1 \leq \alpha \leq q, v \in \mathfrak{g})$$

Then we define a new notion.

Definition 2.6. The characteristic of a vector field v given by (1.4) is the q -uple of functions defined by

$$Q_\alpha(x, u^{(1)}) = \varphi_\alpha(x, u) - \sum_{i=1}^p \xi_i(x, u) \frac{\partial u_\alpha}{\partial x_i} \quad (1 \leq \alpha \leq q)$$

Then, we have the following theorem.

Theorem 2.7. A function $u = f(x)$ is invariant under a connected group of point transformations if and only if it is a solution to

$$Q_\alpha(x, u^{(1)}) = 0 \quad (1 \leq \alpha \leq q)$$

determined by the set of infinitesimal generator $v \in \mathfrak{g}$.

Example 2.8. The characteristic of the rotation vector field $v = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$ is $Q = x + uu_x$, where $u_x := \frac{\partial u}{\partial x}$.

Then any rotationally invariant function must satisfy the differential equation $x + uu_x = 0$.

3. Jets and prolongations

We keep the same notation for X , U and E as in the previous section.

Since we are interested in studying the symmetries of differential equations, we need to know how the group of transformation act on E and the derivatives of the dependent variables.

To do this we need a space called the jet space of E , that we will introduce.

A smooth scalar-valued function $f : X \rightarrow U$ has $\binom{p+k-1}{k}$ different k^{th} order partial derivatives $\partial_J f = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$ where $J = (j_1, \dots, j_k)$, for every $1 \leq l \leq k$, $1 \leq j_l \leq p$, and $k = \#J$.

Definition 3.1. For the total space $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$, we define the n^{th} jet space $J^n := J^n E = X \times U^{(n)}$ is the euclidean space of dimension $p + q^{(n)} := p + q \binom{p+n}{n}$, whose coordinate consist of the p independent variables x_i , the q dependent variables u_α and the derivative $u_{\alpha;J}$, $1 \leq \alpha \leq q$ of orders $1 \leq \#J \leq n$.

Remark 3.2. There is a natural projection $\pi_n^{n+k} : J^{n+k} \rightarrow J^n$ such that $\pi_n^{n+k}(x, u^{(n+k)}) = (x, u^{(n)})$.

Remark 3.3. If $M \subset E$ is an open subset, then $J^n M = (\pi_0^n)^{-1}(M) \subset J^n E$ is the open subset of the n^{th} jet space which projects back down to M .

Definition 3.4. A smooth function $u = f(x)$, with $f : X \rightarrow U$ has n^{th} prolongation $u^{(n)} = f^{(n)}(x)$ which is the function $f^{(n)} : X \rightarrow U^{(u)}$ defined by all the partial derivatives of f up to order n .

Remark 3.5. The graph of the prolonged function $f^{(n)}$, namely $\Gamma_f^{(n)} = \{(x, f^{(n)}(x))\}$ is a p -dimensional submanifold of J^n .

Proposition 3.6. At a point $x \in X$, two functions have the same n^{th} order prolongation, and so determine the same point of J^n , if and only if they and their first n derivatives agree at the point x .

Note that it is the same they have the same n^{th} order Taylor polynomial at the point x .

Definition 3.7. We can define J^n as the set of equivalence class of smooth functions using the equivalence relation given in the previous proposition.

Remark 3.8. For $f : X \rightarrow U$, $\pi_n^{n+k} \circ f^{(n+k)} = f^{(n)}$.

If $g : (x, u) \mapsto g \cdot (x, u)$ is a point transformation, then g acts on functions by transforming their graph and hence on the derivatives of the functions. Then we can define the induced prolonged transformation on J^n by

$$g^{(n)} : (x, u^{(n)}) \mapsto g^{(n)} \cdot (x, u^{(n)}).$$

Remark 3.9. For $f : X \rightarrow U$ and a point transformation g ,

$$g^{(n)} \cdot \Gamma_f^{(n)} = \Gamma_{g \cdot f}^{(n)}$$

Example 3.10. Consider the one-parameter of rotations. We will compute the first prolongation $g_t^{(1)}$ in the case of linear function.

Let (x_0, u_0, p_0) where $p = u_x$ such that $u = f(x) = p_0(x - x_0) + u_0$.

Note that $u_0 = f(x_0)$, $f'(x_0) = p_0$ and

$$\bar{f}(\bar{x}) = \frac{\sin(t) + p_0 \cos(t)}{\cos(t) - p_0 \sin(t)} \bar{x} + \frac{u_0 - p_0 x_0}{\cos(t) - p_0 \sin(t)}$$

Then $\bar{f}'(\bar{x}) = \frac{\sin(t) + p\cos(t)}{\cos(t) - p\sin(t)}$.

Then

$$g_t^{(1)} : (x, u, p) \mapsto (x\cos(t) - u\sin(t), x\sin(t) + u\cos(t), \frac{\sin(t) + p\cos(t)}{\cos(t) - p\sin(t)})$$

such that $p \neq \cos(t)$.

4. Total derivatives

To simplify the computations of prolongations, we introduce the concept of total derivative.

Definition 4.1. A smooth real-valued function $F : J^n \rightarrow \mathbb{R}$ defined on an open subset of J^n is called a differential function of order n .

Remark 4.2. Any n^{th} order differential equation will be the vanishing of a differential function of order n .

For example, the planar Laplace equation is given by the vanishing of $F : (x, y, u^{(2)}) \mapsto u_{xx} + u_{yy}$ where $E = \mathbb{R}^2 \times \mathbb{R}$.

Moreover, F is defined on any jet space J^k for $k \geq 2$.

Remark 4.3. We won't distinguish between F and its composition $F \circ \pi_n^{n+k}$.

Then, we define

Definition 4.4. The order of a differential function refers to the maximal order derivative coordinate upon which it depends.

Definition 4.5. Let F be a differential function of order n .

The total derivative F with respect to x_i is the $(n+1)^{st}$ order differential function $D_i F$ satisfying

$$D_i F(x, f^{(n+1)}(x)) = \frac{\partial}{\partial x_i} F(x, f^{(n)}(x))$$

for any function $u = f(x)$.

Example 4.6. For $E = \mathbb{R} \times \mathbb{R}$,

$$D_x F(x, f^{(n+1)}(x)) = \sum_{k=0}^n \frac{\partial F}{\partial y_k}(x, u^{(n)}) \frac{\partial^k f}{\partial x^k}(x)$$

where $y_k = \begin{cases} x & \text{if } k = 0 \\ u & \text{if } k = 1 \\ u^{(j)} & \text{if } k = j \end{cases}$

Then

$$D_x F(x, u^{(n)}) = \sum_{k=0}^n \frac{\partial F}{\partial y_k}(x, u^{(n)}) y_{k+1}$$

Example 4.7.

$$D_x(xuu_{xx}) = uu_{xx} + xuu_{xxx} + xu_x u_{xx}$$

In the general framework, the total derivative with respect to the i^{th} independent variable x_i is the first order differential operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \sum_J u_{\alpha;J} \frac{\partial}{\partial u_{\alpha;J}}$$

where $u_{\alpha;J} = D_i(u_{\alpha;J}) = u_{\alpha;j_1, \dots, j_k, i}$ and the sum is over all symmetric multi-indices J of arbitrary order.

Even though D_i involves an infinite summation, when applying the total derivative to any differential function, only finitely many term are needed; if we apply F a differential function of order n , then $\#J \leq n$.

Definition 4.8. Higher order total derivatives are defined by

$$D_J = D_{j_1} \dots D_{j_k},$$

for any multi-index $J = (j_1, \dots, j_k)$, with $1 \leq j_l \leq p$.

5. Prolongation of vector fields

Definition 5.1. Let v be a vector field generating a one-parameter group of point transformation $\exp(tv)$ on $E = X \times U$.

The associated n^{th} order prolonged vector field $v^{(n)}$ is the vector field on J^n which is the infinitesimal generator of the prolonged one-parameter group $\exp(tv)^{(n)}$.

We have definition of the vector field generated by a flow :

$$v_{|(x, u^{(n)})}^{(n)} = \frac{d}{dt} \exp(tv)^{(n)} \cdot (x, u^{(n)})|_{t=0}.$$

However, we see a new formula based on the contact structure of the jet space with the following theorem.

Theorem 5.2. Let v be a vector field given by (1.4), and let $Q = (Q_1, \dots, Q_q)$ be its characteristic.

The prolongation of v is given by

$$v^{(n)} = \sum_{i=1}^p \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \sum_{j=\#J=0}^n \varphi_{\alpha;J}(x, u^{(j)}) \frac{\partial}{\partial u_{\alpha;J}}$$

with coefficients

$$\varphi_{\alpha;J} = D_J Q_\alpha + \sum_{i=1}^p \xi_i u_{\alpha;J}$$

Proof. To see the theorem 2.36 page 110 of "Applications of Lie Groups to Differential Equations" of Olver. □

Example 5.3. Assume that $E \simeq \mathbb{R} \times \mathbb{R}$ and let $v = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u}$ with its characteristic

$$Q(x, u, u_x) = \varphi(x, u) - \xi(x, u)u_x$$

The second prolongation of v is

$$v^{(2)} = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} + \varphi_{1;1}(x, u^{(1)}) \frac{\partial}{\partial u_x} + \varphi_{1;2}(x, u^{(2)}) \frac{\partial}{\partial u_{xx}}$$

on J^2 , whose the coefficients $\varphi_{1;1}$ and $\varphi_{1;2}$ are

$$\begin{aligned} \varphi_{1;1} &= D_x Q + \xi u_{xx} = \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2 \\ \varphi_{1;2} &= D_x^2 Q + \xi u_{xxx} = \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 \\ &\quad + (\varphi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx} \end{aligned}$$

Example 5.4. The second prolongation of the generator $v = -u \frac{\partial x}{\partial x} + x \frac{\partial}{\partial u}$ of the rotation group is given by

$$v^{(2)} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}}$$

An alternative approach to the prolongation formula for vector field is to introduce the evolutionary vector field.

Definition 5.5. Let v be a vector field and its characteristic Q . We define the evolutionary vector field associated to v by

$$v_Q = \sum_{\alpha=1}^q Q_\alpha(x, u^{(1)}) \frac{\partial}{\partial u_\alpha}$$

Then its n^{th} prolongation is

$$v_Q^{(n)} = \sum_{\alpha=1}^q \sum_{\#J \geq 0} D_J Q_\alpha(x, u^{(1)}) \frac{\partial}{\partial u_{\alpha;J}}$$

we see that

$$v^{(n)} = v_Q^{(n)} + \sum_{i=1}^p \xi_i D_i$$

Remark 5.6. Since for every $v, w \in \mathfrak{g}$,

$$[v, w]^{(n)} = [v^{(n)}, w^{(n)}]$$

then the prolongation process induces is a Lie algebra homomorphism from the Lie algebra of vector fields on E to the Lie algebra of prolonged vector fields on $J^n E$.

Thus, for $\{v^1, \dots, v^r\}$ a basis of $\mathfrak{g}$, a basis of $\mathfrak{g}^{(n)}$, the Lie algebra of vector field on $J^n E$ is given by $\{(v^1)^{(n)}, \dots, (v^r)^{(n)}\}$.

6. Contact forms

we now investigate the contact structure of jet spaces, determining a remarkable system of differential forms - the contact forms - which serves to characterize precisely prolonged functions and transformations.

One immediate consequence of this approach is a simple proof of the prolongation formula of a vector field.

Contact forms also prove to be of fundamental importance in setting up and solving equivalence problems for differential equations. by the Cartan method; we will, in two chapters, formulate equivalence problems of differential equations in terms of contact forms.

Definition 6.1. A differential one-form θ on J^n is called a contact form if for every smooth function $f : X \rightarrow U$, $(f^{(n)})^* \theta = 0$.

Example 6.2. Consider the case of $E \simeq \mathbb{R} \times \mathbb{R}$. On J^1 , with coordinates $x, u, p = u_x$, a general one-form takes the form $\theta = adx + bdu + cdp$, where a, b, c are functions of x, u, p . A function $f : X \rightarrow U$, $u = f(x)$, has first prolongation $p = f'(x)$, hence

$$(f^{(1)})^* \theta = [a(x, f(x), f'(x)) + b(x, f(x), f'(x))f'(x) + c(x, f(x), f'(x))f''(x)]dx$$

This will vanish for all function f if and only if $c = 0$ and $a = -bp$, hence $\theta = b(x, u, p)\theta_0$ where $\theta_0 = du - pdx$.

For a contact form θ on J^2 , we find

$$\theta = b\theta_0 + c\theta_1$$

where $\theta_0 = du - pdx$ and $\theta_1 = dp - qdx$.

In general, the general contact form can be written as a linear combination of the basic contact forms $\theta_k = du_k - u_{k+1}dx$ for $1 \leq k \leq n-1$ where $u_k = D_x^k u$ is the k^{th} order

derivative of u .

More generally,

Theorem 6.3. *Every contact form on J^n can be written as a linear combination, $\theta = \sum_{\alpha=1}^q \sum_{\#J \leq n-1} P_{\alpha;J} \theta_{\alpha;J}$, with smooth coefficient functions $P_{\alpha;J} : J^n \rightarrow \mathbb{R}$, of the basic contacts forms*

$$\theta_{\alpha;J} = du_{\alpha;J} - \sum_{i=1}^p u_{\alpha;J,i} dx_i. \quad (1 \leq \alpha \leq q, 0 \leq \#J \leq n-1)$$

We call $\#J$ the order of the contact form $\theta_{\alpha;J}$.

Then we have

Theorem 6.4. *A section $u^{(n)} = F(x)$ of the jet space J^n is the prolongation of some function $u = f(x)$, meaning $f^{(n)} = F$, if and only if*

$$F^* \theta_{\alpha;J} = 0 \quad (1 \leq \alpha \leq q, 0 \leq \#J \leq n-1)$$

In other words, a transverse p -dimensional submanifold $\Gamma^{(n)} \subset J^n$ is locally the graph of a prolonged function $u = f(x)$ if and only if $\theta_{\alpha;J}(\Gamma^{(n)}) = 0$.

The contact forms also provide us with a nice intrinsic characterization of the total derivative operators defined earlier.

Definition 6.5. A one-form ω is called horizontal if it annihilates all the vertical tangent directions in J^n .

In local coordinates, the horizontal one-forms are just linear combination, $\omega = \sum_{i=1}^p Q_i(x, u^{(n)}) dx_i$.

To any one-form ω on J^n , there is an intrinsically defined horizontal form ω_H on J^{n+1} called the **horizontal component** of ω , which is defined such that $\omega = \omega_H + \theta$ where θ is a contact form of order n .

In coordinates, we find

$$\begin{aligned} \omega &= \sum_{i=1}^p Q_i(x, u^{(n)}) dx_i + \sum_{\alpha=1}^q \sum_{\#J \leq n} P_{\alpha;J}(x, u^{(n)}) du_{\alpha;J} \\ &= \sum_{i=1}^p \left(Q_i + \sum_{\alpha=1}^q \sum_{\#J \leq n-1} u_{\alpha;J,i} P_{\alpha;J} \right) dx_i + \sum_{\alpha=1}^q \sum_{\#J \leq n-1} P_{\alpha;J} \theta_{\alpha;J} \end{aligned}$$

Then

$$\omega_H = \sum_{i=1}^p \left(Q_i(x, u^{(n)}) + \sum_{\alpha=1}^q \sum_{\#J \leq n-1} u_{\alpha;J,i} P_{\alpha;J}(x, u^{(n)}) \right) dx_i$$

Remark 6.6. The horizontal component ω_H of a form ω on J^n depends on $(n+1)^{st}$ order derivatives.

For example, if $p = q = 1$, the horizontal component of $\omega = Qdx + Pdu$ is $\omega_H = (Q + u_x P)dx$.

Definition 6.7. Let $F : J^n \rightarrow R$, be a differential function of order n . The total differential of F is the horizontal component of its ordinary differential dF , written $DF = (dF)_H$.

To compute the total differential DF , note that

$$dF = \sum_{i=1}^p \frac{\partial F}{\partial x_i} dx_i + \sum_{\alpha=1}^q \sum_{\#J \leq n-1} \frac{\partial F}{\partial u_{\alpha;J}} du_{\alpha;J}$$

Thus,

$$DF = \sum_{i=1}^p D_i F dx_i.$$

Example 6.8. Let $F : J^2 \rightarrow \mathbb{R}$ be a differential function of order 2, $p = u_x$, and $q = u_{xx}$. We have

$$\begin{aligned} dF &= F_x dx + F_u dx + F_p dp + F_q dq \\ &= (F_x + pF_u + qF_p + rF_q)dx + F_u(du - pdx) + F_p(dp - qdx) + F_q(dq - rdx), \end{aligned}$$

where $r = u_{xxx}$.

7. Contact transformations

The n^{th} prolongation $g^{(n)} : J^n \rightarrow J^n$ of any point transformation g can be characterized by that it maps the prolonged graph of any function to the prolonged graph of the prolonged function.

By the theorem 6.4, $g^{(n)}$ maps contact forms to contact forms.

Conversely, let a local diffeomorphism $\Phi : J^n \rightarrow J^n$ which maps contact forms to contact forms.

Since the graphs of prolonged functions are characterized by the vanishing of the contact forms theorem, the image of $\Phi[\Gamma_f^{(n)}]$ of the prolonged graph of a function $u = f(x)$ is, provided it remains transverse, the graph of the n^{th} prolongation of some transformed functions $\bar{u} = \bar{f}(\bar{x})$, so $\Phi[\Gamma_f^{(n)}] = \Gamma_{\bar{f}}^{(n)}$ serve to define an action of Φ on functions $u = f(x)$.

Finally, maps which preserve the set of contact forms are candidates for symmetries of differential equations.

Definition 7.1. A local diffeomorphism $\Phi : J^n \rightarrow J^n$ defines a contact transformation of order n if for every contact form θ on J^n , $(\Phi)^*\theta$ is a contact form.

Example 7.2. The Legendre transformation is the first order contact transformation $\Phi : (x, u, p = u_x) \mapsto (\bar{x}, \bar{u}, \bar{p})$ given by

$$\bar{x} = p, \quad \bar{u} = u - xp, \quad \bar{p} = -x,$$

defined on J^1 .

We can see that there are relations between point transformations and contact transformations.

Theorem 7.3. (of Bäcklund)

Let $E \simeq \mathbb{R}^p \times \mathbb{R}^q$.

If $q > 1$, then every contact transformation of order n is the n^{th} prolongation of a point transformation.

If $q = 1$, then there are first order contact transformation which do not come from point transformation, but every contact transformation of order n is the $(n - 1)^{\text{th}}$ prolongation of a contact transformation on J^1 .

Proof. To see "Equivalence, Invariance and Symmetries" page 128-132 of Olver. □

Remark 7.4. The proof of the theorem allow us to prove by another way the preceding theorem 5.2.

8. Classification of groups of contact transformations

Lie found the complete classification of all-finite dimensional Lie groups of contact transformations acting on a space of one dependent and one independent complex variable. These include the groups of points transformations already discussed (section 11 of chapter Lie groups).

We mention an important theorem of Lie.

Theorem 8.1. Every connected, finite-dimensional Lie group of contact transformation action on a complex two-dimensional manifold without fixes points is locally equivalent, under a contact transformation, to either one of the Lie group of point transformations listed on Tables 1 – 3 (page 51), or to one of the three contact transformation groups listed below.

Lie algebras of contact transformation in $\mathbb{C}^2$

Generators	Dim.	Structure
4.1	$1, x, x^2, u_x, xu_x, u_x^2$	6 $\mathfrak{sa}(2) \rtimes \mathbb{C}$
4.2	$1, x, x^2, u, u_x, xu_x, u_x^2$	7 $\mathfrak{a}(2) \rtimes \mathbb{C}$
4.3	$1, x, x^2, u, u_x, xu_x, x^2u_x - 2xu,$ $u_x^2, 2uu_x - xu_x^2, 4xuu_x - 4u^2 - x^2u_x^2$	10 $\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$

Symmetries of differential equations

1. Symmetry groups and differential equations

In this chapter we discuss the foundations and some applications of Lie's theory of symmetry groups of differential equations. The base infinitesimal method for calculation symmetry group is presented, and used to determine the general symmetry group of some particular differential equations of interest.

Let $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ the total space.

Consider a general n^{th} order system of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0 \quad (1 \leq \nu \leq m)$$

in p independent variables $x = (x_1, \dots, x_p)$ and q dependent variables $u = (u_1, \dots, u_q)$. The system of differential equations which we often abbreviate $\Delta = 0$, is defined by the vanishing of a collection of differential function $\Delta_\nu : J^n \rightarrow \mathbb{R}$.

Definition 1.1. For a system of differential equations $\Delta = 0$, we define the associated variety

$$S_\Delta = \{(x, u^{(n)}) \mid \Delta_\nu(x, u^{(n)}) = 0 \ (1 \leq \nu \leq m)\}$$

contained in the n^{th} order jet space.

The defining functions Δ_ν are assumed to be regular in a neighborhood of S_Δ . By the theorem 3.10 of the chapter Geometric foundations part 1, we have that S_Δ is a submanifold of J^n of dimension $p + q \binom{p+n}{n} - r$ where r is the rank of the system.

Definition 1.2. A function $u = f(x)$, where $f : X \rightarrow U$, defines a solution to the system $\Delta = 0$ if for every $1 \leq \nu \leq m$, $\Delta_\nu(x, f^{(n)}) = 0$.

Then,

Proposition 1.3. A function $u = f(x)$, where $f : X \rightarrow U$, defines a solution to the system $\Delta = 0$ if and only if the graph $\Gamma_f^{(n)} = \{(x, f^{(n)})\}$ is contained in S_Δ .

Definition 1.4. A symmetry of a system of differential equations $\Delta = 0$ is a transformation which maps smooth solutions to smooth solutions.

Example 1.5. The most basic type of symmetry is a group of point transformations on E .

We will concentrate on studying on the basic point transformation symmetry groups.

Definition 1.6. Let $\Delta = 0$ a system of differential equations.

A point transformation $g : E \rightarrow E$, acting on E , is called a symmetry of the system of differential equations $\Delta = 0$ if, whenever $u = f(x)$ is a solution and the transformed function $\bar{f} = g \cdot f$ is well-defined, then $\bar{u} = \bar{f}(\bar{x})$ is also a solution to $\Delta = 0$.

Now, trying to give a geometric formulation of this definition.

Proposition 1.7. Let $\Delta = 0$ a system of differential equations, with the associated variety $S_\Delta \subset J^n$.

If the n^{th} prolongation $g^{(n)} : J^n \rightarrow J^n$ of a point transformation $g : E \rightarrow E$, satisfies $g^{(n)}(S_\Delta) \subset S_\Delta$ then g is a symmetry of the system $\Delta = 0$.

Proof. Note that $u = f(x)$ is a solution of $\Delta = 0$ if and only if $\Gamma_f^{(n)}$ is a subset of S_Δ . Now the invariance of S_Δ implies that $g^{(n)} \cdot \Gamma_f^{(n)} = \Gamma_{g \cdot f}^{(n)}$ is also contained in S_Δ , hence it is also a solution. □

Remark 1.8. The proposition holds for the case of contact transformations ;

A contact map $g^{(n)} : J^n \rightarrow J^n$ will act on a solution $u = f(x)$ by pointwise transforming its prolonged graph $\Gamma_f^{(n)}$ which, in view of the contact conditions, forms the prolonged graph of $\bar{u} = \bar{f}(\bar{x})$. The map $g^{(n)}$ will determine a symmetry to the system Δ if it leaves the variety S_Δ invariant.

Remark 1.9. The converse is not necessary true, whereas for a system of algebraic equations, every point in the associated variety is tautologically a solution. This is no longer the case for systems of differential equations.

Only those point in S_Δ which to actual solutions are required to be mapped to each other by the prolonged symmetry transformation.

Then, we formalize the required notion.

Definition 1.10. A system of differential equations $\Delta = 0$ is called locally solvable at a point $(x_0, u_0^{(n)}) \in S_\Delta$ if there exists a smooth solution $u = f(x)$ defined in a neighborhood of x_0 , which satisfies $u_0^{(n)} = f^{(n)}(x_0)$.

A system of differential equation $\Delta = 0$ is called fully regular if the system is regular and locally solvable at every point in S_Δ .

Remark 1.11. The local solvability of a system of ordinary differential equations is the same as the property of existence of solutions to the standard initial valued problem (the

Cauchy problem).

Then,

Theorem 1.12. *A transformation $g : E \rightarrow E$ is a symmetry of a locally solvable system of differential equation $\Delta = 0$ if and only if $g^{(n)}(S_\Delta) \subset S_\delta$.*

Proof. Let $z_0 = (x_0, u_0^{(n)}) \in S_\Delta$ such that $\bar{z}_0 = g^{(n)} \cdot z_0$ is defined.

By the local solvability of S_Δ to a smooth solution $u = f(x)$ defined near x_0 with $u_0^{(n)} = f^{(n)}(x_0)$. Then $\bar{f} = g \dots f$ is also a solution at least near the $(\bar{x}_0, \bar{u}_0)$, hence $\bar{z}_0 = (\bar{x}_0, \bar{f}^{(n)}(\bar{x}_0))$, as desired. □

2. Infinitesimal methods

We will assumed that we have a connected Lie group of point (or contact) transformations G .

In this case, the infinitesimal generators of Lie algebra of vector fields on E take the form

$$v = \sum_{i=1}^p \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \varphi_\alpha(x, u) \frac{\partial}{\partial u_\alpha}.$$

Definition 2.1. A group G of transformations acting on E is called a symmetry group of a fully regular and locally solvable system of differential equations $\Delta = 0$ if every $g : E \rightarrow E$ is a symmetry of $\Delta = 0$.

By the theorem 12.1 of the chapter Lie groups, we have

Theorem 2.2. *A connected Lie group G of transformations is a symmetry group of a fully regular and locally solvable system of differential equations $\Delta = 0$ if and only if*

$$v^{(n)}(\Delta_\nu) = 0 \tag{2.3}$$

for every point $(x, u^{(n)}) \in \{\Delta = 0\}$, $1 \leq \nu \leq m$, and $v \in \mathfrak{g}$.

Remark 2.4. The condition (2.3) is the determining equations of the symmetry group for the system $\Delta = 0$.

For a the special case of contact transformations, we have

$$v_Q^{(n)}(\Delta_\nu) = 0, \tag{2.5}$$

for every point $(x, u^{(n)}) \in \{\Delta = 0\}$, $1 \leq \nu \leq m$, and $v \in \mathfrak{g}$.

Example 2.6. Consider the second order ordinary differential equation

$$u_{xx} = 0. \quad (2.7)$$

An infinitesimal point symmetry of (2.7) is a vector field on $E = \mathbb{R}^2$

$$v = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u}.$$

Computing the second prolongation, we find

$$v^{(2)} = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} + \varphi_{1;1}(x, u) \frac{\partial}{\partial u_x} + \varphi_{1;2} \frac{\partial}{\partial u_{xx}}$$

on J^2 , whose the coefficients $\varphi_{1;1}$ and $\varphi_{1;2}$ are

$$\begin{aligned} \varphi_{1;1} &= D_x Q + \xi u_{xx} = \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2 \\ \varphi_{1;2} &= D_x^2 Q + \xi u_{xxx} = \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 \\ &\quad + (\varphi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx} \end{aligned}$$

The infinitesimal criterion (2.3) means that if $u_{xx} = 0$, then

$$\varphi_{1;2} = 0.$$

Substituting the formula of $\varphi_{1;2}$ equal to 0 gives, with $u_{xx} = 0$,

$$\varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 = 0.$$

Since we can prescribe x, u and u_x arbitrarily, it is equivalent to

$$\varphi_{xx} = 0, \quad 2\varphi_{xu} = \xi_{xx}, \quad \varphi_{uu} = 2\xi_{xu}, \quad \xi_{uu} = 0$$

Then the general solution given by

$$\xi(x, u)[c] = c_1 x^2 + c_2 x u + c_3 x + c_4 u + c_5, \quad \varphi(x, u)[c] = c_1 x u + c_2 u^2 + c_6 x + c_7 u + c_8,$$

forming

$$v^{(2)}[c] = \xi(x, u)[c] \frac{\partial}{\partial x} + \varphi(x, u)[c] \frac{\partial}{\partial u}$$

where $c = \begin{pmatrix} c_1 \\ \vdots \\ c_8 \end{pmatrix} \in \mathbb{R}^8$.

Therefore, the equation (2.7) admits an eight-dimensional Lie algebra of infinitesimal symmetries, spanned by $\{v^{(2)}[e_i] \mid 1 \leq i \leq 8\}$;

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x \frac{\partial}{\partial x}, \quad u \frac{\partial}{\partial u}, \quad u \frac{\partial}{\partial u}, \quad x^2 \frac{\partial}{\partial x} + x u \frac{\partial}{\partial u}, \quad x u \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u}.$$

Note that the Lie algebras appear in Case 2.3 of the Tables of classification (page 50). The corresponding Lie group is $SL(3)$ acting via linear fractional transformations

$$\left\{ \begin{aligned} &SL(3) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ &\left(\begin{pmatrix} a & b & c \\ d & e & f \\ h & j & k \end{pmatrix}, x, u \right) \longmapsto \left(\frac{ax + bu + c}{hx + ju + k}, \frac{dx + eu + f}{hx + ju + k} \right) \end{aligned} \right.$$

Example 2.8. Consider the second order ordinary differential equation

$$u_{xx} = 0. \quad (2.9)$$

This equation admits under a fiber-preserving transformation a six-dimensional symmetry group.

Example 2.10. The situation is different for contact symmetries.

With $v_Q = Q(x, u, u_x) \frac{\partial}{\partial u}$, the $v_Q^{(2)} = D_x^2 Q$.

The infinitesimal criterion (2.5) is equivalent to

$$\frac{\partial^2 Q}{\partial x^2} + 2p \frac{\partial^2 Q}{\partial x \partial u} + p^2 \frac{\partial^2 Q}{\partial u^2} = 0$$

where $p = u_{xx}$, and is also equivalent to

$$\left(\frac{\partial}{\partial x} + p \frac{\partial}{\partial u} \right)^2 Q = 0$$

Therefore,

$$Q(x, u, p) = A(x, u, p) + xB(x, u, p)$$

Therefore, the contact transformation group of $u_{xx} = 0$ is infinite-dimensional.

Example 2.11. Consider the differential equation on $E = \mathbb{R}^2 \times \mathbb{R}$

$$u_t = u_{xx}$$

Its symmetry group is infinite-dimensional with its Lie algebra of vector field generated by

$$\begin{aligned} v^1 &= \frac{\partial}{\partial x}, \quad v^2 = \frac{\partial}{\partial t}, \quad v^3 = u \frac{\partial}{\partial u}, \quad v^4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}, \\ v^5 &= 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}, \quad v^6 = 4xt \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} - (x^2 + 2t)u \frac{\partial}{\partial u}, \\ v_\alpha &= \alpha(x, t) \frac{\partial}{\partial u} \text{ where } \alpha_t = \alpha_{xx} \end{aligned}$$

Now, we can see a partial solution of the equivalence problem of differential equation. Before, define the notion of equivalence.

Definition 2.12. Let (Δ_1) and (Δ_2) be two system of differential equations defined on $J^n E$.

We say that Δ_1 and Δ_2 are equivalent if there is a local diffeomorphism $\Phi : E \rightarrow E$ such that $\Phi^{(n)} : S_{\Delta_1} \rightarrow S_{\Delta_2}$ is a local diffeomorphism.

It is also equivalent to $(\Phi^{(n)})^*(\Delta_1) = \Delta_2$.

Then,

Theorem 2.13. Two equivalent system differential equations have isomorphic symmetry groups.

Proof. If $\Phi : E \rightarrow E$ is a transformation, and g a symmetry of the first system, then $\bar{g}^{(n)} = \Phi^{(n)} \cdot g^{(n)} \cdot (\Phi^{(n)})^{-1}$ is a symmetry of the second system. $\square$

Remark 2.14. If we make a change of variables outside the class of diffeomorphisms, then there is no guarantee of the isomorphism between the both symmetry groups ; Consider

$$v_t = v_{xx} + 2vv_x$$

Its symmetry group is five-dimensional.

Moreover, the system

$$w_t = w_{xx}$$

has an infinite-dimensional symmetry group.

However, by the map $w \mapsto \frac{\partial \log(v)}{\partial x}$, we can map the second to the first system.

We can mention a general theorem for the second order ordinary differential equation.

Theorem 2.15. *The point transformation symmetry group of a second order ordinary differential equation has dimension at most 8.*

Moreover, the equation admits an 8-dimensional symmetry group if and only if it can be mapped by a point transformation to the linear equation $u_{xx} = 0$ which has symmetry group $SL(3)$.

Symmetry classification of order ordinary differential equations under point transformations :

Symmetry group	Dim.	Type	Invariant Equation
$\frac{\partial}{\partial u}$	1	3.1	$u_{xx} = F(x, u_x)$
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}$	2	1.5	$u_{xx} = F(u_x)$
$\frac{\partial}{\partial x}, e^x \frac{\partial}{\partial u}$	2	1.5	$u_{xx} - u_x = F(u_x - u)$
$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} - 2xu \frac{\partial}{\partial u}$	3	1.1	$u_{xx} = \frac{3u_x^2}{u} + cu^3$
$\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} - (2xu + 1) \frac{\partial}{\partial u}$	3	1.2	$u_{xx} = 6uu_x - 4u^3 + c(u_x - u^2)^{3/2}$
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u}, \alpha \notin \{0, \frac{1}{2}, 1, 2\}$	3	1.7	$u_{xx} = cu_x^{\frac{\alpha-2}{\alpha-1}}$
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x} + (x + u) \frac{\partial}{\partial u}$	3	1.8	$u_{xx} = ce^{-u_x}$
$\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, x \frac{\partial}{\partial x}, u \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}, u \frac{\partial}{\partial u}, x^2 \frac{\partial}{\partial x} + xu \frac{\partial}{\partial u}, xu \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u}$	8	2.3	$u_{xx} = 0$

Equivalence of coframes

1. Frames and coframes

Definition 1.1. Two coframes are said to be equivalent if they are mapped to each other by a diffeomorphism.

The equivalence problem for coframes is the most important the equivalence problems that we are to treat, because it includes all the others as special cases.

The Cartan equivalence method provides a explicit, practical algorithm for reducing most other equivalence problems to an equivalence problem for a suitable coframe. The exceptions are those problems admitting an infinite dimensional symmetry group.

Therefore, it is crucial to learn to equivalence problem for coframes.

In this chapter, we will describe the basic constructions required to solve the fundamental equivalence problem for coframes.

Let M be a smooth manifold of dimension m .

We recall what is a frame and a coframe of M .

Definition 1.2. A frame on M is an ordered set of vector fields $\mathcal{V} = \{v_1, \dots, v_m\}$ having the property that they form a basis for $T_x M$ at each point $x \in M$.

A coframe on M is an ordered set of one-forms $\theta = \{\theta^1, \dots, \theta^m\}$ having the property that they form a basis for $T_x^* M$ at each point $x \in M$.

Remark 1.3. A set of one-forms $\theta = \{\theta^1, \dots, \theta^m\}$ defines a coframe on M if and only if $\theta^1 \wedge \dots \wedge \theta^m$ doesn't vanish.

Definition 1.4. A frame and a coframe are dual to each other if and only if they form dual bases for the tangent and cotangent space to M at each point.

Given a coframe $\{\theta^1, \dots, \theta^m\}$, we denote the dual frame vector fields by $\{\frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^m}\}$, so that

$$\theta^i(\frac{\partial}{\partial \theta^j}) = \delta_{i,j}. \quad (1 \leq i, j \leq m)$$

Remark 1.5. If we introduce a local coordinate $x = (x_1, \dots, x_m)$ on M , then the coframe can be written in terms of the coordinate coframe; for every $1 \leq i \leq m$, $\theta^i = \sum_{j=1}^m a_{i,j}(x) dx_j$, where $A = (a_{i,j})$ is a smooth matrix.

The dual frame is given by $\frac{\partial}{\partial \theta^i} = \sum_{j=1}^m b_{i,j}(x) \frac{\partial}{\partial x_j}$ where $B = A^{-1}$.

Then,

Proposition 1.6. For $f : M \rightarrow \mathbb{R}$ a smooth function and a coframe Θ on M , we have

$$df = \sum_{j=1}^m \frac{\partial f}{\partial \theta^j} \theta^j$$

where $\frac{\partial f}{\partial \theta^j} = \sum_{i=1}^m b_{j,i}(x) \frac{\partial f}{\partial x_i}$.

If $\theta = \{\theta^1, \dots, \theta^m\}$ is a coframe on M , then any one-form ω can be written as a linear combination

$$\omega = \sum_{i=1}^m h_i \theta^i$$

where $h_i : M \rightarrow \mathbb{R}$ are smooth functions.

More generally, if Ω is a k -form on M , then

$$\Omega = \sum_{1 \leq i_1 < \dots < i_k \leq m} h_I \theta^{i_1} \wedge \dots \wedge \theta^{i_k}$$

where $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq m$.

2. The structure functions

Let $M, \bar{M}$ be two m -dimensional manifolds.

Suppose we are given two coframes, $\theta = \{\theta^1, \dots, \theta^m\}$ on M and $\bar{\theta} = \{\bar{\theta}^1, \dots, \bar{\theta}^m\}$ on $\bar{M}$.

The basic problem, but important, for coframe is to determine whether the two coframes can be mapped to each other by a diffeomorphism $\Psi : M \rightarrow \bar{M}$, such that

$$\Phi^* \bar{\theta}^i = \theta^i \quad (2.1)$$

for $1 \leq i \leq m$.

If (2.1) holds, then

$$\Phi^*(d\bar{\theta}^i) = d\theta^i. \quad (1 \leq i \leq m)$$

Since θ is a coframe on M , we have **the fundamental structure equations** for the coframe θ given by

$$d\theta^i = \sum_{\substack{j,k \\ j < k}}^m T_{j,k}^i \theta^j \wedge \theta^k \quad (1 \leq i \leq m)$$

Remark 2.2. If θ is the Maurer-Cartan coframe for a Lie group G relative to a some choice of basis for the Lie algebra $\mathfrak{g}$, then the structure equations coincide with the Maurer-Cartan structure equations.

Thus, in this case, $T_{j,k}^i = -c_{j,k}^i$.

Then, if θ and $\bar{\theta}$ are equivalent, we deduce that

$$\sum_{\substack{j,k \\ j < k}}^m T_{j,k}^i(x) \theta^j \wedge \theta^k = \sum_{\substack{j,k \\ j < k}}^m \bar{T}_{j,k}^i(\bar{x}) \theta^j \wedge \theta^k, \quad (1 \leq i \leq m)$$

where $\bar{x} = \Phi(x)$.

Since the θ^i are linearly independent one-forms, we have

$$\bar{T}_{j,k}^i(\bar{x}) = T_{j,k}^i(x) \quad (1 \leq i \leq m, 1 \leq j < k \leq m)$$

We deduce invariants functions $T_{j,k}^i : M \rightarrow \mathbb{R}$.

Remark 2.3. If a structure function $T_{j,k}^i$ for the coframe θ is constant, then $\bar{T}_{j,k}^i$ is constant for any equivalent coframe $\bar{\theta}$.

However, more generally, the expression of $T_{j,k}^i$ and $\bar{T}_{j,k}^i$ depend on the expression of Ψ . To answer to the question of equivalence of coframes we want necessary and sufficient conditions independent to the expression of Ψ that we don't know. Then, we will work on the relations between these invariants functions which are independent of Ψ .

Example 2.4. A coframe on $M \subset \mathbb{R}^2$ consists of a pair of one-forms

$$\theta^1 = A dx + B dy, \quad \theta^2 = C dx + D dy,$$

where A, B, C, D are smooth functions of x, y satisfying $AD - BC$ doesn't vanish such that $\theta^1 \wedge \theta^2 = (AD - BC) dx \wedge dy$ doesn't vanish.

The structure functions are

$$d\theta^1 = J\theta^1 \wedge \theta^2, \quad d\theta^2 = K\theta^1 \wedge \theta^2,$$

where

$$T_{1,2}^1 = J = \frac{B_x - A_y}{AD - BC}, \quad T_{1,2}^2 = K = \frac{D_x - C_y}{AD - BC}$$

3. Derived invariants

Let $M, \bar{M}$ be two m -dimensional manifolds.

Suppose we are given two coframes, $\theta = \{\theta^1, \dots, \theta^m\}$ on M and $\bar{\theta} = \{\bar{\theta}^1, \dots, \bar{\theta}^m\}$ on $\bar{M}$.

The structure functions associated with a coframe provided us $\frac{1}{2}m^2(m-1)$ invariant functions, but that is not all.

Let I any invariant function, which is mapped to a corresponding invariant $\bar{I}$ under the

given changes of variables : $\bar{I}(\bar{x}) = \bar{I}(\Phi(x)) = I(x)$.

Then $\Phi^*(d\bar{I}) = dI$, and by re-expressing them in terms of the respective coframes θ and $\bar{\theta}$

$$\sum_{j=1}^m \frac{\partial I}{\partial \theta^j}(x) \theta^j = dI(x) = \Phi^*(d\bar{I}(\bar{x})) = \sum_{j=1}^m \frac{\partial \bar{I}}{\partial \bar{\theta}^j}(\bar{x}) \bar{\theta}^j$$

Then,

$$\frac{\partial \bar{I}}{\partial \bar{\theta}^j}(\bar{x}) = \frac{\partial I}{\partial \theta^j}(x). \quad (1 \leq j \leq m)$$

We can repeat the process to find an infinite collection of potentially different invariant functions with the convention

$$\frac{\partial^2 I}{\partial \theta^j \partial \theta^k} = \frac{\partial}{\partial \theta^j} \left(\frac{\partial I}{\partial \theta^k} \right)$$

Example 3.1. Consider the general coframe on $M \subset \mathbb{R}^2$. The corresponding coframe derivatives are, for a function $F : M \rightarrow \mathbb{R}$,

$$\begin{aligned} \frac{\partial F}{\partial \theta^1} &= \frac{D}{AD - BC} \frac{\partial F}{\partial x} - \frac{C}{AD - BC} \frac{\partial F}{\partial y}, \\ \frac{\partial F}{\partial \theta^2} &= -\frac{B}{AD - BC} \frac{\partial F}{\partial x} + \frac{A}{AD - BC} \frac{\partial F}{\partial y} \end{aligned}$$

Remark 3.2.

$$\frac{\partial^2 I}{\partial \theta^j \partial \theta^k} - \frac{\partial^2 I}{\partial \theta^k \partial \theta^j} = \sum_{i=1}^p T_{j,k}^i \frac{\partial I}{\partial \theta^i}$$

Then by these formulas we can compute lots of invariant functions of this coframe.

4. Classifying functions

Let $M, \bar{M}$ be two m -dimensional manifolds.

Suppose we are given two coframes, $\theta = \{\theta^1, \dots, \theta^m\}$ on M and $\bar{\theta} = \{\bar{\theta}^1, \dots, \bar{\theta}^m\}$ on $\bar{M}$.

Now we are confronted with an overabundance of invariant functions.

How do we make effectively use this plethora of invariants ?

Much more fundamental are the relationships among the invariants.

Suppose that I_1, I_2, I_3 are three invariants and $\bar{I}_1, \bar{I}_2, \bar{I}_3$ are the corresponding in the barred coordinates.

If the original invariants are functionally dependent, say $I_3 = H(I_1, I_2)$ for some function H , then by invariance, $\bar{I}_3 = H(\bar{I}_1, \bar{I}_2)$. Thus, the function H is an invariant of the problem

which is completely independent of the coordinates !

We will call the collection of all such functional relationships the **classifying functions** associated with the given coframe.

We can note that if two coframes are equivalent, then they have the same classifying functions.

To explain more formally the notion of classifying functions, continue with our coframe Θ .

Definition 4.1. For a coframe θ on M , a structure invariant is a function defined by

$$T_\sigma = \frac{\partial^s T_{j,k}^i}{\partial \theta^{l_s} \partial \theta^{l_{s-1}} \dots \partial \theta^{l_1}}, \quad (\sigma = (i, j, k, l_1, \dots, l_s), j < k, 1 \leq i, j, k, l_1, \dots, l_s \leq m)$$

where $s = \text{order}(\sigma)$ is defined as the order of T_σ .

Proposition 4.2. If θ and $\bar{\theta}$ are two coframes equivalents, then for $\bar{x} = \Phi(x)$, we have **the complete system of invariance equations**

$$\bar{T}_\sigma(\bar{x}) = T_\sigma(x) \quad (\text{order}(\sigma) \geq 0)$$

The structure invariants serves to define the components of the general structure map.

Definition 4.3. The s^{th} order classifying space $\mathbb{K}^{(s)} = \mathbb{K}^{(s)}(m)$ associated with an m -dimensional manifold M is the euclidean space of dimension $q_s(m) = \frac{1}{2}m^2(m-1)\binom{m+s}{m}$, which is coordinated by $z^{(s)} = (\dots, z_\sigma, \dots)$.

The entries of $z^{(s)}$ are labeled by nondecreasing multi-indices $\sigma = (i, j, k, l_1, \dots, l_r)$, $j < k, 1 \leq l_1 \leq \dots \leq l_r \leq m, 0 \leq r \leq s$.

The s^{th} order structure map associated with a coframe θ on M is the map $\mathbb{T}^{(s)} : M \rightarrow \mathbb{K}^{(s)}$ whose the components are the structure invariants $z_\sigma = T_\sigma(x)$ for order $\sigma \leq s$.

If θ and $\bar{\theta}$ are equivalent coframes, then for all $s \geq 0$,

$$\bar{\mathbb{T}}^{(s)}(\bar{x}) = \mathbb{T}^{(s)}(x)$$

where $\bar{x} = \Psi(x)$.

Definition 4.4. The s^{th} order classifying set $C^{(s)} := C^{(s)}(\theta, U)$ associated with a coframe θ on an open subset $U \subset M$ is defined as the image of the structure map $\mathbb{T}^{(s)}$

$$C^{(s)}(\theta, U) = \{\mathbb{T}^{(s)}(x) \mid x \in U\} \subset \mathbb{K}^{(s)}$$

Then,

Proposition 4.5. Suppose θ and $\bar{\theta}$ are equivalent coframes under $\Phi : M \rightarrow \bar{M}$.

For each $s \geq 0$, the s^{th} order classifying sets are the same.

Thus $C^{(s)}(\bar{\theta}, \bar{U}) = C^{(s)}(\theta, U)$, where $U \subset M$ is the domaine and $\bar{U} = \Psi(U)$.

Example 4.6. Consider a coframe on $M \subset \mathbb{R}^2$ already discussed.

$C^{(0)}$ is considered with coordinate (z, w) and $C^{(1)}$ with

$$z^{(1)} = (z, w, z_1, z_2, w_1, w_2) = (J, K, \frac{\partial J}{\partial \theta^1}, \frac{\partial J}{\partial \theta^2}, \frac{\partial K}{\partial \theta^1}, \frac{\partial K}{\partial \theta^2})$$

If J, K are functionally independent, then $C^{(0)}$ will be an open subset of $\mathbb{K}^{(0)}$.

If J, K are functionally dependent, then $C^{(0)}$ will be either a curve or a point.

If there zero functionally independent first order structure invariants, then $C^{(1)}$ will be a point in $\mathbb{K}^{(1)}$.

If there one functionally independent first order structure invariants, then $C^{(1)}$ will be a one-dimensional curve in $\mathbb{K}^{(1)}$.

If there two functionally independent first order structure invariants, then $C^{(1)}$ will be a two-dimensional curve in $\mathbb{K}^{(1)}$.

Example 4.7. Consider the particular coframe

$$\theta^1 = \frac{\lambda}{x+y} dx, \quad \theta^2 = \frac{\mu}{x+y} dy$$

where μ, λ are nonzero constants, on $M = \{x+y > 0\} \subset \mathbb{R}^2$.

Then,

$$C^{(0)} = \{\mu^{-1}, -\lambda^{-1}\}.$$

$$C^{(1)} = (\mu^{-1}, -\lambda^{-1}, 0, 0, 0, 0).$$

5. The classifying manifolds

Now our goal is to determine in what sense our necessary conditions for equivalence problem for coframes are also sufficient.

Let M be a m -dimensional manifold.

Definition 5.1. A coframe θ is called full regular if, for each $s \geq 0$, the s^{th} order structure map $\mathbb{T}^{(s)} : M \rightarrow \mathbb{K}^{(s)}$ is regular.

In this case, we let $\rho_s := \text{rank}(\mathbb{T}^{(s)})$, which is also the dimension of $C^{(s)}$.

Then, we call $C^{(s)}$ the s^{th} order classifying manifold associated with the coframe θ .

Moreover, we define

$$\mathcal{F}^{(s)} := \mathcal{F}^{(s)}(\theta) := \{T_\sigma \mid \text{order}(\sigma) \leq s\} \quad (0 \leq s)$$

Locally, we can choose ρ_s functionally independent structure invariants $I_\nu = T_{\sigma_\nu}$, $1 \leq \nu \leq \rho_s$, which generate the full set s^{th} order structure invariant $\mathcal{F}^{(s)}$, in the sense that every other invariant T_σ of order inferior to s can be expressed as functions of the basic invariants :

$$T_\sigma = H_\sigma(I_1, \dots, I_{\rho_s}) \quad (\text{order}(\sigma) \leq s)$$

In this way, the invariants $I_1, \dots, I_{\rho_s}$ furnish local coordinates on the ρ_s -dimensional classifying manifold $C^{(s)}$, which can be locally identified with an open subset of the graph of the function $\mathbb{H}^{(s)} = (\dots, H_\sigma, \dots)$.

The components H_σ , which express the functional relation among the structure invariants, are the **classifying functions** associated with the coframe.

Since we are interested in local equivalence of coframes, the classifying manifolds need only agree locally, on the domain of the equivalence map Ψ . Therefore, by the proposition 4.5, necessary conditions for the equivalence problem of coframes are that, for each $s \geq 0$, their s^{th} order classifying manifolds overlap, in the sense of the definition 3.5 of the chapter "Geometric foundation part 1".

Then, we can express the fundamental theorem for the equivalence problem for coframes.

Theorem 5.2. *(of equivalence problem for coframes)*

Let θ and $\bar{\theta}$ be smooth, fully regular coframes defined, respectively, on m -dimensional manifolds M and $\bar{M}$.

There exists a local diffeomorphism $\Phi : M \rightarrow \bar{M}$ mapping the coframes to each other $\Phi^*\bar{\theta} = \theta$, if and only if for each $s \geq 0$, their s^{th} order classifying manifolds $C^{(s)}(\theta)$ and $C^{(s)}(\bar{\theta})$ overlap.

Proof. To see the chapter 14 of "Equivalence, Invariance and Symmetries" of Olver. □

Example 5.3. Consider the case of a coframe with constant structure functions.

In this case, $C^{(0)}$ is a single point, defined by the constant values the invariants $T_{j,k}^i = -c_{j,k}^i$, $1 \leq i \leq m$, $1 \leq j < k \leq m$.

We have for each $s \geq 0$, $\rho_s = 0$.

We call a such coframe as having **rank zero**.

Therefore, any two rank zero coframes with the same constant functions are locally equivalent.

Therefore, the theorem and the example give us the following result.

Corollary 5.4. Let Θ be a rank zero coframe on M , with constant structure functions $T_{j,k}^i = -c_{j,k}^i$, $1 \leq i \leq m$, $1 \leq j < k \leq m$.

Let G be a m dimensional Lie group whose Lie algebra $\mathfrak{g}$ has structure constants $c_{j,k}^i$ relative to a basis $\{v_1, \dots, v_m\}$ and let $\{\alpha^1, \dots, \alpha^m\}$ denote the dual basis of Maurer-Cartan

forms.

Then there exists a local diffeomorphism $\Phi : M \rightarrow G$ such that for every $1 \leq i \leq m$, $\Phi^* \alpha^i = \theta^i$.

Corollary 5.5. *All coframe of rank 0 on M with m constant structure functions are equivalent and their symmetry group is a m -dimensional manifold.*

The one remaining difficulty in applying the theorem of equivalence problem for coframes is that it requires us to verify an infinite number of conditions; one for each $s \geq 0$.

The key is the fact that for each $s \geq 0$, $\rho_s \leq \rho_{s+1}$ and $\rho_s \leq m$, then

Proposition 5.6. *Let θ be a fully regular coframe, and let ρ_s denote then rank of the s^{th} order structure map $\mathbb{T}^{(s)}$.*

The smallest s for which $\rho_s = \rho_{s+1}$ is called the order of the coframe and we have

$$0 \leq \rho_0 < \rho_1 < \cdots < \rho_s = \rho_{s+1} = \cdots = r \leq m$$

The stabilizing rank r will be referred to as rank of the coframe.

By this proposition, we can compute the order and the rank of a coframe ;

1) Let $\mathcal{F}^{(0)} = \{T_{j,k}^i\}$, and let ρ_0 the number of functionally independent structure functions $T_{j,k}^i$.

2) Next, compute $\rho_1 \geq \rho_0$,

* If $\rho_1 = \rho_0$, then the order of the coframe is $s = 0$ and of rank $r = \rho_0$.

* Otherwise, $\rho_0 < \rho_1$, then we compute ρ_2 .

3) We repeat the processes up to find the order s and the rank r .

Note that by the proposition, the algorithm will end.

Finally, we have the more refined version of the theorem of equivalence problem for coframes, proved in the chapter 14 of "Equivalence, Invariance and Symmetries" of Olver.

Theorem 5.7. *(of equivalence problem for coframes)*

Let θ and $\bar{\theta}$ be smooth, fully regular coframes defined, respectively, on m -dimensional manifolds M and $\bar{M}$. There exists a local diffeomorphism $\Phi : M \rightarrow \bar{M}$ such that $\Phi^* \bar{\theta} = \theta$ if and only if they have the same order $\bar{s} = s$, and their $(s+1)^{st}$ order classifying manifolds $C^{(s+1)}(\theta)$ and $C^{(s+1)}(\bar{\theta})$ overlap.

Example 5.8. Consider again the coframe on $\mathbb{R}^2$.

The coframe can have rank 0, 1 or 2 at each point in the regular case.

- (1) If $r = 0$, then $s = 0$.
- (2) If $r = 1$, then $s = 0$.
- (3) If $r = 2$, $\rho_0 = 2$, then $s = 0$.
- (4) If $r = 2$, $\rho_0 = 1 < \rho_1 = 2$, then $s = 1$.

6. Symmetries of a coframe

Let M be a smooth manifold of dimension m .

Now, we talk about symmetry group for a coframe θ defined on M .

Definition 6.1. Let $\theta = \{\theta^1, \dots, \theta^m\}$ be a coframe defined on a M .

The symmetry group of θ is the group of self-equivalences, meaning local diffeomorphisms $\Phi : M \rightarrow M$ such that for every $1 \leq i \leq m$, $\Phi^* \theta^i = \theta^i$.

The main result, proved again in the chapter 14 of "Equivalence, Invariance and Symmetries" of Olver, is

Theorem 6.2. *If θ is a coframe of rank r on m -dimensional manifold M , then the symmetry group of θ is a finite-dimensional Lie group of dimension $m - r$.*

Example 6.3. Consider again a coframe on $\mathbb{R}^2$. Its rank is either 0, 1, or 2, hence its symmetry group can have dimension 2, 1, or 0.

Example 6.4. Consider a coframe on M with rank zero; all the structure functions are constant.

There is a local diffeomorphism $\Phi : M \rightarrow G$, mapping the coframe to the Maurer-Cartan coframe on the Lie group G whose structure constants coincide with the constant structure functions (up to sign).

By right-invariance of the Maurer-Cartan forms, the Lie group G itself, acting via right multiplication, provide the symmetry group of the coframe.

Formulation of equivalence problems

1. Equivalence problems using differential forms

Now, we consider more general problems of equivalence, in which the equivalence conditions can be rewritten using differential forms, but a fully regular invariant coframe is not so immediately apparent.

However, the Cartan's process, with the equivalent problem rewritten in differential forms, will produce either an equivalence problem for a suitable coframes, either a direct solution (transitive and intransitive cases) for the equivalence problem using differential forms.

In Cartan's approach, the conditions of equivalence of two objects (differential equations, polynomials, differential operators, ...) must be reformulated in terms of differential forms, more specifically, of one-forms.

In this chapter, we explain how a variety of equivalence problem can be reformulated in language of differential forms.

The implementation of the Cartan's algorithm will presented in the next chapter.

Example 1.1. The simplest example is the problem of equivalence of one-forms, which is already in the proper form.

Let M and $\overline{M}$ be two m -dimensional manifolds.

Two one-forms ω and $\overline{\omega}$ defined on M and $\overline{M}$ are equivalent if there is a smooth local diffeomorphism $\Phi : M \rightarrow \overline{M}$ such that $\Phi^*\overline{\omega} = \omega$.

Example 1.2. Consider the equivalence problem of Riemannian metrics.

Let M and $\overline{M}$ be two m -dimensional Riemannian manifolds.

Let ds^2 and $d\overline{s}^2$ two metrics on M and $\overline{M}$.

The metrics are equivalent if there is a smooth local diffeomorphism $\Psi : M \rightarrow \overline{M}$ such that $\Psi^*d\overline{s}^2 = ds^2$.

We can introduce coframes ω on M and $\overline{\omega}$ on $\overline{M}$, which diagonalize the metric at each point :

$$ds^2 = \sum_{i=1}^m (\omega^i)^2 \quad d\overline{s}^2 = \sum_{i=1}^m (\overline{\omega}^i)^2$$

Then, the two metrics will be equivalent if and only if their diagonalizing coframes satisfy

$$\Psi^*\overline{\omega}^i = \sum_{j=1}^m g_{i,j}(x) \omega^j \quad (1 \leq i \leq m)$$

where $g = (g_{i,j}) \in O(n)$.

Example 1.3. Consider the general equivalence problem of second order ordinary differential equation (Δ)

$$u_{xx} = Q(x, u, u_x) \quad (1.4)$$

where $Q : J^1 \rightarrow \mathbb{R}$ is a differential function of order 1.

There are number of possible classes of admissible changes of variables, and each class will lead to a different equivalence problem with a different formulation in terms of differential forms.

Here we explain how to set up the three most basic of these problems - equivalence problem under fiber-preserving, point, and contact transformations.

Let (Δ) and $(\bar{\Delta})$ be two second order ordinary differential equation.

Let $\Phi : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x, u) \longmapsto (\bar{x}, \bar{u}) \end{cases}$ a local diffeomorphism such that $(\Phi^{(1)})^*(\bar{u}_{\bar{x}\bar{x}}) = u_{xx}$.

We know that $\Phi^{(1)}$ is a contact transformation.

Firstly, we have, by the property of the first order contact forms :

$$(\Phi^{(1)})^*(d\bar{u} - \bar{p}d\bar{x}) = \lambda(du - pdx)$$

where $\lambda : J^1 \rightarrow \mathbb{R}$ is a differential function of order 1.

By omitting the pull-back, we have

$$d\bar{u} - \bar{p}d\bar{x} = \lambda(du - pdx) \quad (1.5)$$

Similarly, by the property of the second order contact transformations :

$$d\bar{p} - \bar{q}d\bar{x} = \mu(du - pdx) + \nu(dp - qdx) \quad (1.6)$$

where $\mu, \nu : J^2 \rightarrow \mathbb{R}$ are differential functions of order 2.

Moreover, replacing $q = u_{xx}$ by (1.4) in (1.6) we obtain

$$d\bar{p} - \bar{q}d\bar{x} = \mu(du - pdx) + \nu(dp - Q(x, u, p)dx) \quad (1.7)$$

Then,

Theorem 1.8. Let (Δ) and $(\bar{\Delta})$ be two second order ordinary differential equation.

(Δ) and $(\bar{\Delta})$ are equivalent under contact transformations if and only if there exists $\Phi :$

$\begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x, u) \longmapsto (\bar{x}, \bar{u}) \end{cases}$ a local diffeomorphism satisfying

$$(1) \quad d\bar{u} - \bar{p}d\bar{x} = \lambda(du - pdx),$$

$$(2) \quad d\bar{p} - \bar{q}d\bar{x} = \mu(du - pdx) + \nu(dp - Q(x, u, p)dx).$$

where $\mu, \nu : J^2 \rightarrow \mathbb{R}$ are differential functions of order 2.

In the case of fiber-preserving transformation $\Phi(x, u) = (\varphi(x), \psi(x, u))$, then we have an additional conditional

$$d\bar{x} = \alpha dx \quad (1.9)$$

where $\alpha : J^1 \rightarrow \mathbb{R}$ is a differential function of order 1.

Then,

Theorem 1.10. *Let (Δ) and $(\bar{\Delta})$ be two second order ordinary differential equation. (Δ) and $(\bar{\Delta})$ are equivalent under fiber-preserving transformations if and only if there exists $\Phi : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x, u) \longmapsto (\bar{x}, \bar{u}) \end{cases}$ a local diffeomorphism satisfying*

- (1) $d\bar{x} = \alpha dx$
- (2) $d\bar{u} - \bar{p}d\bar{x} = \lambda(du - pdx),$
- (3) $d\bar{p} - \bar{q}d\bar{x} = \mu(du - pdx) + \nu(dp - Q(x, u, p)dx).$

where $\alpha : J^1 \rightarrow \mathbb{R}$ is a differential function of order 1 and $\mu, \nu : J^2 \rightarrow \mathbb{R}$ are differential functions of order 2.

In the case of point transformation, we have an additional condition, compared to contact transformations, which is

$$d\bar{x} = \alpha dx + \beta du \quad (1.11)$$

where $\alpha, \beta : J^1 \rightarrow \mathbb{R}$ are differential functions of order 1.

Then,

Theorem 1.12. *Let (Δ) and $(\bar{\Delta})$ be two second order ordinary differential equation. (Δ) and $(\bar{\Delta})$ are equivalent under point transformations if and only if there exists $\Phi : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x, u) \longmapsto (\bar{x}, \bar{u}) \end{cases}$ a local diffeomorphism satisfying*

- (1) $d\bar{x} = \alpha dx + \beta du$
- (2) $d\bar{u} - \bar{p}d\bar{x} = \lambda(du - pdx),$
- (3) $d\bar{p} - \bar{q}d\bar{x} = \mu(du - pdx) + \nu(dp - Q(x, u, p)dx).$

where $\alpha, \beta : J^1 \rightarrow \mathbb{R}$ are differential functions of order 1 and $\mu, \nu : J^2 \rightarrow \mathbb{R}$ are differential functions of order 2.

Remark 1.13. For the equivalence problem of second order ordinary differential equations, since we begin with $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\Phi^{(2)}$ is a contact transformation. Then, in this case, the equivalence problem under contact transformations is a generalization of point or fiber-preserving transformations.

2. Coframes and structure groups

In the preceding section, a class of equivalence problems can be completely reformulated as a system of differential equations involving differential one-forms.

The goal of Cartan equivalence method is to reduce the original one-form formulation of the problem to an equivalence problem involving a full invariant coframe, to which the results of the previous chapter can be applied.

In order to begin the Cartan's algorithm, the problem must first be placed into a more amenable formulation based on coframe on the underlying manifold.

Example 2.1. * Since the preceding problems, such as the equivalence problem for Riemannian metrics are already based on coframe.

* Other are "underdetermined" equivalence problems in the that the equivalence condition can be formulated using fewer one-forms than would be needed. However, provided the original one-forms are linearly independent at each point, it is easy to convert that underdetermined equivalence problems into the proper form merely by adjoining one or more additional one-forms so as to form a coframe.

* A third possibility, that occasionally arises it that the most natural formulation of the requisite equivalence conditions uses one-forms which are point wise linearly dependent, and so cannot form a coframe ("overdetermined" equivalence problem). But, we won't talk about it.

Now, we describe what it is the "proper form" for an equivalence problem.

Let $\Omega = \{\omega^1, \dots, \omega^m\}$ and $\bar{\Omega} = \{\bar{\omega}^1, \dots, \bar{\omega}^m\}$ be two coframe on m -dimensional manifold M and $\bar{M}$.

We use the identification between a coframe and its column vector.

$$\omega = \begin{pmatrix} \omega^1 \\ \dots \\ \omega^m \end{pmatrix}$$

Unlike the coframe equivalence problem, the one-forms are not directly mapped to each other; rather the pull-back $\Phi^*\bar{\omega}^i$ of each one-form on $\bar{M}$ is a linear combination of the ω^i 's.

In fact, talking of equivalence problem means G -equivalence problem for a some G Lie group.

Definition 2.2. Let $G \subset GL(m)$ be a Lie group.

Let ω and $\bar{\omega}$ be two coframes defined, respectively, on M and $\bar{M}$.

The G -valued equivalence problem for these coframes is to determine whether or not there exists a local diffeomorphism $\Phi : M \rightarrow \bar{M}$ and a G -valued function $g : M \rightarrow G$ such that

$$\Phi^*\bar{\omega} = g \cdot \omega \tag{2.3}$$

In others words,

$$\Phi^*(\bar{\omega}^i) = \sum_{j=1}^m g_{i,j} \omega^j \tag{1 \leq i \leq m}$$

where $g = (g_{i,j})$.

We call G the structure group of the equivalence problem.

Remark 2.4. Two equivalent coframes have isomorphic symmetry group.

Example 2.5. * The simplest example is when $G = \{e\}$. The $\{e\}$ -equivalence problem is the equivalence problem for coframes treated in the previous chapter.

* At the extreme, if $G = GL(m)$, then any local diffeomorphism Ψ satisfies the property for some $g \in GL(m)$, and so two coframes are $GL(m)$ -equivalent.

The goal of the equivalence method is through a series of "invariant operations", to reduce the structure G to the trivial structure $\{e\}$, at which point the solution to the equivalence problem for coframes can be used to solve the problem completely.

The reduction procedure provides an invariant specification of the entries of the unknown matrix-valued function g , thereby determining an invariant coframe for the equivalence problem.

This may not always possible, and subsequent chapter will describe other possible outcomes of the equivalence procedure.

First, we show how the previously described equivalence problem examples can be into the proper form of G -valued equivalence problem for some matrix Lie group G .

Remark 2.6. For simplicity, we can omit the diffeomorphism Ψ in the equivalence problem.

Example 2.7. Consider the equivalence problem for second order ordinary differential equation.

This problem is already in the proper form ;

The coframe on $M = J^1$ with $(x, u, p = u_x)$ is

$$\omega^1 = du - p dx, \quad \omega^2 = dp - Q(x, u, p) dx, \quad \omega^3 = dx$$

* For the equivalence problem under contact transformations, the equivalence conditions seen can be written

$$\bar{\omega}^1 = a_1 \omega^1, \quad \bar{\omega}^2 = a_2 \omega^1 + a_3 \omega^2, \quad \bar{\omega}^3 = a_4 \omega^1 + a_5 \omega^2 + a_6 \omega^3$$

for suitable function $a_i = a_i(x, u, p)$, $1 \leq i \leq 6$.

The equivalence problem have the required form

$$\bar{\omega} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \cdot \omega$$

where the structure group is

$$G_c = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 6.

* Under point transformations, the coframe satisfies

$$\bar{\omega}^1 = a_1 \omega^1, \quad \bar{\omega}^2 = a_2 \omega^1 + a_3 \omega^2, \quad \bar{\omega}^3 = a_4 \omega^1 + a_6 \omega^3$$

where the structure group is

$$G_p = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & 0 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 5.

* Under fiber-preserving transformations, the coframe satisfies

$$\bar{\omega}^1 = a_1 \omega^1, \quad \bar{\omega}^2 = a_2 \omega^1 + a_3 \omega^2, \quad \bar{\omega}^3 = a_6 \omega^3$$

where the structure group is

$$G_f = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 4.

Definition 2.8. The symmetry group of an differential equation is the symmetry group of the associated coframe ω or lifted coframe $\theta = g \cdot \omega$ describing the equivalence problem.

Example 2.9. The problem for equivalence of Riemannian metrics is already of the required form with $G = O(n)$.

3. Normalization

Once we have the property formulated an equivalence problem in terms of a coframe ω and structure group G , we are in a position to apply Cartan's equivalence algorithm.

The preliminary step is to modify the G -equivalence problem si as to to make the two of variables appear more symmetric.

We use the fact that for $g : M \rightarrow G$, omitting the pull-back ;

$$\boxed{\bar{\omega} = g \cdot \omega \iff \exists (h, \bar{h}) : M \times \bar{M} \rightarrow G \mid \bar{h} \cdot \bar{\omega} = h \cdot \omega} \quad (3.1)$$

Proof. ($\implies$) We use $h = g$ and $\bar{h} = id_{M,G}$

($\impliedby$) We set $g = \bar{h}^{-1} \cdot h$.

□

For $\theta_h = h \cdot \omega$, then,
 ω and $\bar{\omega}$ are G -equivalent if and only if there exist $h : M \rightarrow G$, and $\bar{h} : \bar{M} \rightarrow G$ such that $\theta_{\bar{h}} = \theta_h$.

Then we want to reduce the G -equivalence problem to the equivalence problem for coframes. To obtain this equivalence, we need to specify the matrix entries h and $\bar{h}$.
 If we succeed, then ω and $\bar{\omega}$ are G -equivalent if and only if θ and $\bar{\theta}$ are $\{e\}$ -equivalent.

The question is :

How do we determine the functions h and $\bar{h}$ in an algorithm manner ?

The key to answering this question is to find combination of the group parameters $g_{i,j}$ and possibly the coordinates x_k , which are unchanged by the coordinate transformations, meaning that they retain the same formula in the transformation coordinates $\bar{g}_{i,j}, \bar{x}_k$.
 Each such combination will enable us to normalize one of the parameters and thereby reduce the dimension of the structure group by one.
 If we can find enough invariant combinations, we can effectively eliminate all the group parameters, reducing G to $\{e\}$.
 Hence the equivalence problem will be reduce to the coframe equivalence problem.

Suppose that for any coframe of the given form, we can find a function $H : (g, \omega) \mapsto H(g, \omega)$ depending on the coordinates and the coframe elements which is invariant under any diffeomorphism Ψ , meaning that if $\bar{x} = \Phi(x)$, $\Phi^*(\bar{g} \cdot \bar{\omega})(x) = (g \cdot \omega)(x)$, then

$$\Phi^* H(\bar{g}, \bar{\omega}) = H(g(x), \omega(x))$$

Cartan observed that if H explicitly depends on the group parameters, we can normally H to take any convenient constant values, e.g., $H = 0$ or $H = 1$, by specifying one of the group parameters.

The effects is to eliminate one of the unspecified group parameters and thus reduce the equivalence problem to an equivalence problem whose the structure group has dimension one less than the original.

Example 3.2. Consider the equivalence problem for second order ordinary differential equation under fiber-preserving transformations.

The Cartan formulation is given by

$$\bar{\omega} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \cdot \omega,$$

where $a_1, a_2, a_3, a_6 : J^1 \rightarrow \mathbb{R}$, or equivalently

$$\begin{pmatrix} \bar{a}_1 & 0 & 0 \\ \bar{a}_2 & \bar{a}_3 & 0 \\ 0 & 0 & \bar{a}_6 \end{pmatrix} \cdot \bar{\omega} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \cdot \omega,$$

or,

$$\bar{a}_1 \bar{\omega}^1 = a_1 \omega^1, \quad \bar{a}_2 \bar{\omega}^1 + \bar{a}_3 \bar{\omega}^2 = a_2 \omega^1 + a_3 \omega^2, \quad \bar{a}_6 \bar{\omega}^3 = a_6 \omega^3,$$

and where for $\Phi : (x, u) \mapsto (\varphi(x), \psi(x, u)) =: (\bar{x}, \bar{u})$

$$\Phi^{(1)} : (x, u, p) \mapsto (\bar{x}, \bar{u}, \frac{\psi_u p + \psi_x}{\varphi_x})$$

Thanks to the coframe on $M = J^1$ replacing in the equivalence conditions, we find

$$\bar{a}_1 \psi_u = a_1 \quad \bar{a}_3 \frac{\psi_u}{\varphi_x} = a_3 \quad \bar{a}_6 \varphi_x = a_6$$

Then,

$$H(g, \omega) = \frac{a_3 a_6}{a_1}$$

is an invariant function, since we see that $\frac{\bar{a}_3 \bar{a}_6}{\bar{a}_1} = \frac{a_3 a_6}{a_1}$.

We can equal both ratios to 1 without changing the invariant structure nature of the equivalence conditions.

Then the new equivalence problem is given by $a_3 a_6 = a_1$ and $\tilde{G} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_6^{-1} a_1 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$.

$$\bar{g} \cdot \bar{\omega} = g \cdot \omega, \quad g \in G \iff \exists (h, \bar{h}) : M \times \bar{M} \rightarrow \tilde{G} \mid \bar{h} \cdot \bar{\omega} = h \cdot \omega$$

Why is it clearly an equivalence between these both equivalence problems ?

In fact, H of the example is a special case, characterized of constant type for the coframe ω .

Let ω and $\bar{\omega}$ be two coframes which are G -equivalent based on M and $\bar{M}$ respectively. Let $N = \text{Im}(H)$.

Note that, for $g, g^* : M \rightarrow G$ and $\bar{g}, \bar{g}^* : \bar{M} \rightarrow G$, if $\bar{g}^* = g^*$ then $\bar{g}^* \cdot g \cdot \omega = g^* \cdot g \cdot \omega$. Then,

$$H(\bar{g}^* \cdot g, \omega) = H(g^* \cdot g, \omega)$$

Consequently, there is an induced action of G on N ,

$$\begin{cases} G \times N \longrightarrow N \\ (h, z) \longmapsto h \cdot z = H(h^* \cdot g, \omega) \end{cases}$$

where $z = H(g, \omega)$.

Then,

ω and $\bar{\omega}$ are G -equivalent if and only if for all $x \in M$, $H(e, \omega(x))$ and $H(e, \bar{\omega}(x))$ are in the same orbit of G on N .

Definition 3.3. An invariant function H is said to define a normalization of constant type for the coframe if, for each point $x \in M$, $H(e, \omega(x))$ lie in a single orbit $\mathcal{O}_\omega \subset N = \text{Im}(H)$ of the induced action of G on N .

To answer the question why we can equivalently normalize an invariant function of constant type is justified by

Proposition 3.4. *Let $\bar{\omega}$ and ω be of constant type for the invariant function H .*

If ω and $\bar{\omega}$ are G -equivalent then their associated orbits in $\text{Im}(H) = N$ are the same ; $\mathcal{O}_\omega = \mathcal{O}_{\bar{\omega}}$.

Moreover, for a point $z \in \mathcal{O}_\omega$, we let $\tilde{G} = G_z \subset G$ its isotropy group.

We can construct smooth G -valued functions $\bar{g}_0$ and g_0 such that $H(\bar{g}_0, \bar{\omega}) = z = H(g_0, \omega)$, which we use to define the modified coframes

$$\varpi = g_0 \cdot \omega \quad \bar{\varpi} = \bar{g}_0 \cdot \bar{\omega}$$

Then,

$\bar{g} \cdot \bar{\omega} = g \cdot \omega$ for some $(g, \bar{g}) : M \times \bar{M} \rightarrow G$ if and only if $\bar{h} \cdot \bar{\varpi} = h \cdot \varpi$ for some $(h, \bar{h}) : M \times \bar{M} \rightarrow \tilde{G}$.

Proof. Since $H(g, \omega) = H(\bar{g}, \bar{\omega})$ are assumed to both lie in the same orbit $\mathcal{O}$ of G , then by regularity and invariance of H , we can choose a smooth group-valued function $g, g^* : M \rightarrow G$ such that for fixed $z \in \mathcal{O}$, $H(\bar{g}^* \cdot \bar{g}, \bar{\omega}) = z = H(g^* \cdot g, \omega)$.

Replacing the original equivalence condition $\bar{g} \cdot \bar{\omega} = g \cdot \omega$ by $\bar{g}^* \cdot \bar{g} \cdot \bar{\omega} = g^* \cdot g \cdot \omega$, we see that if two coframes are equivalent, they are equivalent for some specification of group parameter that fixes $H = z$.

Define the G -valued function h such that $g^* \cdot g = h \cdot g_0$.

Since $H(h \cdot g_0, \omega) = z = H(g^* \cdot g, \omega)$, we deduce that h takes values in $G_z = \tilde{G}$.

Similarly define $\bar{h} : \bar{M} \rightarrow G$ such that $\bar{g}^* \cdot \bar{g} = \bar{h} \cdot \bar{g}_0$.

We deduce the equivalence of the modified coframes.

□

Remark 3.5. The reduced structure group $\tilde{G}$ can be identified with the isotropy subgroup of the chosen canonical form for the function H .

Remark 3.6. Other type of such function H is called non-constant type. This type is more difficult, but we can see "The Method of Equivalence and Its Applications" of Gardner.

Cartan's equivalence method

1. Introduction

Once an equivalence problem has been reformulated in the proper Cartan form, in terms of a coframe ω on the m -dimensional base manifold M , along with a structure group $G \subset GL(m)$.

The Cartan's equivalence problem consist to give an answer to the local equivalence problem. A problem formulated in terms of differential one-forms.

We are ready to apply the Cartan equivalence method. The goal is to normalize the structure group valued coefficients in a suitable invariant manner, and this is accomplished through the determination of a sufficient number of invariant combinations.

The Cartan method provides an algorithmic approach for finding such invariant combinations.

Each group dependent invariant combinations allow us to normalize one group parameter.

In this chapter, we will describe the basic steps on the Cartan equivalence method, and illustrate its applications to the equivalence problem of second order ordinary differential equations.

2. The structure equations

Consider coframes $\omega = \{\omega^1, \dots, \omega^m\}$ and $\bar{\omega} = \{\bar{\omega}^1, \dots, \bar{\omega}^m\}$, respectively, on a m -dimensional manifold M and $\bar{M}$.

The G -equivalence condition has the form in a more symmetrical form

$$\bar{g} \cdot \bar{\omega} = g \cdot \omega$$

for $(g, \bar{g}) : M \times \bar{M} \rightarrow G$.

This motivates the preliminary step in the Cartan method which is to introduce the **lifted coframe**

$$\theta = g \cdot \omega, \tag{2.1}$$

or in full-detail

$$\theta^i = \sum_{j=1}^m g_{i,j} \omega^j \quad (1 \leq i \leq m)$$

where $\theta^i : M \times G \rightarrow \mathbb{R}$.

Technically speaking, the θ^i do not form a coframe on $M \times G$.

Note that, $\Phi^*\bar{\theta} = \theta$ if and only if $\Phi^*\bar{\theta}^i = \theta^i$, ($1 \leq i \leq m$).

The key to find invariant combinations is the exterior derivative.

We differentiate the lifted coframe.

For $1 \leq i \leq m$,

$$d\theta^i = d\left(\sum_{j=1}^m g_{i,j}\omega^j\right) = \sum_{j=1}^m \left(dg_{i,j} \wedge \omega^j + g_{i,j}d\omega^j\right),$$

with $\omega^i = \sum_{j=1}^m g_{i,j}^{-1}\theta^j$, $\gamma = dg \cdot g^{-1} = (\gamma_{i,j})$ the Maurer-Cartan form.

We obtain

$$d\theta^i = \sum_{j=1}^m \gamma_{i,j} \wedge \theta^j + \sum_{j=1}^m g_{i,j}d\omega^j.$$

Moreover, for $1 \leq i \leq m$, $d\omega^i = \sum_{j,k=1}^m a_{j,k}\omega^j \wedge \omega^k$ and $\theta^i = \sum_{j=1}^m g_{i,j}\omega^j$, then we can express $d\omega^i$ in terms of sums of wedge of the θ^i 's.

Then

$$d\theta^i = \sum_{j=1}^m \gamma_{i,j} \wedge \theta^j + \sum_{j=1}^m T_{j,k}^i(x, g)\theta^j \wedge \theta^k,$$

where $T_{j,k}^i$ are the **torsion coefficients**, which may be constant, or may depend on the variables x and the group parameters g .

If $\{\alpha^1, \dots, \alpha^r\}$ is a basis for the space of Maurer-Cartan forms, then

$$\gamma_{i,j} = \sum_{l=1}^r A_{j,l}^i \alpha^l \quad (1 \leq i, j \leq m)$$

where $A_{i,j}^i$ are constant.

Then, we have the **structure equation** for the lifted coframe θ , for every $1 \leq i \leq m$

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \alpha^l \wedge \theta^j + \sum_{j=1}^m T_{j,k}^i(x, g)\theta^j \wedge \theta^k. \quad (2.2)$$

For the barred lifted coframe, we have the corresponding structure equation

$$d\bar{\theta}^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \alpha^l \wedge \bar{\theta}^j + \sum_{j=1}^m T_{j,k}^i(x, g)\bar{\theta}^j \wedge \bar{\theta}^k. \quad (1 \leq i \leq m)$$

Remark 2.3. Note that the coefficients $A_{j,l}^i$ are the same as in the barred case.

3. Absorption

The important idea is that any consistent specification of the group parameters as function $g : M \rightarrow G$ will reduce the Maurer-Cartan forms α^l on G to one forms $\tilde{\alpha}^l = \sum_{i=1}^m v_{l,i} dx_i$ on M , which we can re-express in terms of the coframe and even in the coframe θ ; $\tilde{\alpha}^l = \sum_{j=1}^m z_j^l \theta^j$.

The problem is that we don't know the $g_{i,j}$'s.

However, we will observe the behavior of the structure equation if the Maurer-Cartan forms α^l 's are modified.

This will allow us to determine which of the associated torsion coefficients $T_{j,k}^i$ are invariants.

Then we will have candidate for invariant functions to reduce the equivalence problem.

Replace the Maurer-Cartan forms α^l 's by a general combination of lifted coframe elements ;

$$\alpha^l \rightarrow \sum_{j=1}^m z_j^l \theta^j \quad (1 \leq l \leq r) \quad (1)$$

where the z_j^l 's are unspecified coefficients.

Substituting this into the structure equation, we have

$$d\theta^i = \sum_{\substack{j,k=1 \\ j < k}}^m \left(B_{j,k}^i(z) + T_{j,k}^i(x, g) \right) \theta^j \wedge \theta^k \quad (1 \leq i \leq m) \quad (2)$$

where $B_{j,k}^i(z) := \sum_{l=1}^r (A_{k,l}^i z_j^l - A_{j,l}^i z_k^l)$.

Similarly we have the same form for the barred lifted coframe.

If $\Phi^* \bar{\theta} = \theta$, then for some unspecified coefficient $z = z(x)$, $\bar{z} = \bar{z}(\bar{x})$, for each $1 \leq i \leq m$, and $\Phi^* d\bar{\theta}^i = d\theta^i$, then

$$B_{j,k}^l(\bar{z}) + \bar{T}_{j,k}^l(\bar{x}, \bar{g}) = B_{j,k}^l(z) + T_{j,k}^l(x, g) \quad (\bar{x} = \Psi(x)) \quad (3)$$

If we find z such that $B_{j,k}^i(z) = 0 = B_{j,k}^i(\bar{z})$ for some j, k, i then for any specification group parameters $g, \bar{g}$,

$$\bar{T}_{j,k}^i(\bar{x}, \bar{g}) = T_{j,k}^i(x, g). \quad (1 \leq i, j, k \leq m) \quad (4)$$

A such torsion coefficients are called **essential torsion coefficients** for the structure equation.

Remark 3.1. If an essential torsion coefficient does explicitly depend on the group parameters, then it provides an invariant combination which can be normalized and thereby reducing the structure group.

This process is called **absorption of torsion** for the lifted coframe.

This process for a lifted coframe θ is the following :

1) Compute the structure equation given by

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \alpha^l \wedge \theta^j + \sum_{j=1}^m T_{j,k}^i(x, g) \theta^j \wedge \theta^k, \quad (1 \leq i \leq m)$$

where $\{\alpha^l\}_{1 \leq l \leq r}$ are the Maurer-Cartan forms on G .

2) Substitute the α^l 's by

$$\pi^l = \alpha^l - \sum_{i=1}^m z_i^l \theta^i \quad (1 \leq l \leq r)$$

Then

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \pi^l \wedge \theta^j + \sum_{j=1}^m \tilde{U}_{j,k}^i(x, g) \theta^j \wedge \theta^k, \quad (1 \leq i \leq m)$$

3) Find the solutions $(z_i^l)_{\substack{1 \leq i \leq m \\ 1 \leq l \leq r}}$ which reduce to maximum the number of torsion coefficients.

4) Obtain

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \pi^l \wedge \theta^j + \sum_{j=1}^m U_{j,k}^i(x, g) \theta^j \wedge \theta^k, \quad (1 \leq i \leq m)$$

where $U_{j,k}^i$ are the essential torsion coefficient for the left coframe, and the π^l 's are the modified Maurer-Cartan forms on G .

5) Apply the normalization process (see in the previous chapter).

6) Repeat the steps 1) – 5) up to if we cannot obtain one essential torsion coefficient depending of the group parameters or if the equivalence problem is reduced to the $\{e\}$ -equivalence problem.

Example 3.2. Take again the equivalence problem for second order ordinary differential equations $u_{xx} = Q(x, u, u_x)$ with the coframe on $J^1 \ni (x, u, u_x = p)$

$$\omega^1 = du - p dx, \quad \omega^2 = dp - Q(x, u, p) dx, \quad \omega^3 = dx.$$

For the equivalence problem under contact transformations, the lifted coframe is

$$\theta^1 = a_1 \omega^1, \quad \theta^2 = a_2 \omega^1 + a_3 \omega^2, \quad \theta^3 = a_4 \omega^1 + a_5 \omega^2 + a_6 \omega^3,$$

where the structure group is

$$G_c = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 6.

Then compute the differentials

$$d\theta^1 = da_1\omega^1 + a_1d\omega^1$$

$$d\theta^1 = \frac{da_1}{a_1}\theta^1 - a_1(dp \wedge dx).$$

$$\begin{aligned} d\theta^2 &= da_2\omega^1 + a_2d\omega^1 + da_3\omega^2 + a_3d\omega^2 \\ &= \frac{da_2}{a_1}\theta^1 - a_2(dp \wedge dx) + da_3\left(\frac{\theta^2}{a_3} - \frac{a_2}{a_3}\omega^1\right) - a_3dQ \wedge dx \end{aligned}$$

$$d\theta^2 = \left(\frac{da_2}{a_1} - \frac{a_2}{a_1a_3}da_3\right)\theta^1 + \left(-a_2 - a_3Q_p\right)dp \wedge dx + \frac{da_3}{a_3}\theta^2 - a_3Q_u du \wedge dx.$$

$$\begin{aligned} d\theta^3 &= da_4\omega^1 + a_4d\omega^1 + a_5\omega^2 + a_5d\omega^2 + \frac{da_6}{a_6}\theta^3 \\ &= \frac{da_4}{a_1}\theta^1 - a_4dp \wedge dx + da_5\left(\frac{\theta^2}{a_3} - \frac{a_2}{a_3}\omega^1\right) - a_5dQ \wedge dx + \frac{da_6}{a_6}\theta^3 \\ &= \frac{da_4}{a_1}\theta^1 - a_4dp \wedge dx + \frac{da_5}{a_3}\theta^2 - \frac{a_2}{a_3a_1}da_5 \wedge \theta^1 - a_5(Q_u du + Q_p dp) \wedge dx + \frac{da_6}{a_6}\theta^3 \end{aligned}$$

$$d\theta^3 = \left(\frac{da_4}{a_1} - \frac{a_2}{a_1a_3}da_5\right)\theta^1 + (-a_4 - a_5Q_p)dp \wedge dx + \frac{da_5}{a_3}\theta^2 - a_5Q_u du \wedge dx + \frac{da_6}{a_6}\theta^3.$$

Then express $dp \wedge dx$, $du \wedge dx$, and $dp \wedge du$ in terms of $\theta^1 \wedge \theta^2$, $\theta^1 \wedge \theta^3$ and $\theta^2 \wedge \theta^3$;

$$\begin{aligned} \theta^1 \wedge \theta^2 &= a_1\omega^1 \wedge (a_2\omega^1 + a_3\omega^2) = a_1a_3\omega^1 \wedge \omega^2 \\ &= a_1a_3(du \wedge dp - Qdu \wedge -pdx \wedge dp). \end{aligned}$$

$$\theta^1 \wedge \theta^2 = -a_1a_3dp \wedge du - Qa_1a_3du \wedge dx + a_1a_3pdp \wedge dx.$$

$$\theta^1 \wedge \theta^3 = a_1\omega^1 \wedge a_6\omega^3 = a_1a_6(du - pdx) \wedge dx = a_1a_6du \wedge dx.$$

Then

$$du \wedge dx = \frac{\theta^1 \wedge \theta^3}{a_1a_6} \quad (3.3)$$

$$\begin{aligned} \theta^2 \wedge \theta^3 &= (a_2\omega^1 + a_3\omega^2) \wedge a_6\omega^3 = a_2a_6\omega^1 \wedge \omega^3 + a_3a_6\omega^2 \wedge \omega^3 \\ &= a_2a_6du \wedge dx + a_3a_6(dp - Qdx) \wedge dx \\ &= \frac{a_2}{a_1}\theta^1\theta^3 + a_3a_6dp \wedge dx. \end{aligned}$$

Then

$$dp \wedge dx = \frac{\theta^2 \wedge \theta^3}{a_3a_6} - \frac{a_2}{a_1a_3a_6}\theta^1 \wedge \theta^3. \quad (3.4)$$

Then

$$\begin{aligned}\theta^1 \wedge \theta^3 &= -a_1 a_3 dp \wedge du - Q a_1 a_3 \frac{\theta^1 \wedge \theta^3}{a_1 a_6} + a_1 a_3 p \left(\frac{\theta^2 \wedge \theta^3}{a_3 a_6} - \frac{a_2}{a_1 a_3 a_6} \right) \theta^1 \wedge \theta^3 \\ &= -a_1 a_3 dp \wedge du - \frac{a_1 a_3}{a_1 a_6} Q \theta^1 \wedge \theta^3 + \frac{a_1}{a_6} p \theta^2 \wedge \theta^3 - \frac{a_2}{a_6} p \theta^1 \wedge \theta^3.\end{aligned}$$

Then

$$\boxed{dp \wedge du = -\frac{\theta^1 \wedge \theta^2}{a_1 a_3} - \left(\frac{Q}{a_1 a_6} + \frac{a_2}{a_1 a_3 a_6} p \right) \theta^1 \wedge \theta^3 + \frac{1}{a_3 a_6} p \theta^2 \wedge \theta^3.} \quad (3.5)$$

Then we can express $d\theta^1$, $d\theta^2$, and $d\theta^3$ in terms of $\{\theta^1, \theta^2, \theta^3\}$:

$$\begin{aligned}d\theta^1 &= \frac{da_1}{a_1} \theta^1 - a_1 \left(\frac{\theta^2 \wedge \theta^3}{a_3 a_6} - \frac{a_2}{a_1 a_3 a_6} \theta^1 \wedge \theta^3 \right) \\ &= \frac{da_1}{a_1} \theta^1 - \frac{a_1}{a_3 a_6} \theta^2 \wedge \theta^3 + \frac{a_2}{a_3 a_6} \theta^1 \wedge \theta^3 \\ &= \left(\frac{da_1}{a_1} - \frac{a_2}{a_3 a_6} \theta^3 \right) \wedge \theta^1 - \frac{a_1}{a_3 a_6} \theta^2 \wedge \theta^3\end{aligned}$$

$$\boxed{d\theta^1 = \alpha^1 \wedge \theta^1 + \frac{a_2}{a_3 a_6} \theta^1 \wedge \theta^3 - \frac{a_1}{a_3 a_6} \theta^2 \wedge \theta^3,}$$

where $\alpha^1 = \frac{da_1}{a_1}$.

$$\begin{aligned}d\theta^2 &= \left(\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3 \right) \theta^1 + \left(-a_2 - a_3 Q_p \right) dp \wedge dx + \frac{da_3}{a_3} \theta^2 - a_3 Q_u du \wedge dx \\ &= \left(\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3 \right) \theta^1 + \left(-a_2 - a_3 Q_p \right) \left(\frac{\theta^2 \wedge \theta^3}{a_3 a_6} - \frac{a_2}{a_1 a_3 a_6} \theta^1 \wedge \theta^3 \right) + \frac{da_3}{a_3} \theta^3 - a_3 Q_u \frac{\theta^1 \wedge \theta^3}{a_1 a_6} \\ &= \left(\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3 \right) \theta^1 - \frac{a_2}{a_3 a_6} \theta^2 \wedge \theta^3 + \frac{a_2^2}{a_1 a_3 a_6} \theta^1 \wedge \theta^3 - \frac{Q_p}{a_6} \theta^2 \wedge \theta^3 + \frac{a_2}{a_1 a_6} Q_p \theta^1 \wedge \theta^3 + \frac{da_3}{a_3} \theta^3 \\ &\quad - \frac{a_3}{a_1 a_6} Q_u \theta^1 \wedge \theta^3 \\ &= \left(\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3 - \frac{a_2^2}{a_1 a_3 a_6} \theta^3 - \frac{a_2}{a_1 a_6} Q_p \theta^3 + \frac{a_3}{a_1 a_6} Q_u \theta^3 \right) \wedge \theta^1 + \left(\frac{da_3}{a_3} + \frac{a_2}{a_3 a_6} \theta^3 + \frac{Q_p}{a_6} \theta^3 \right) \wedge \theta^2,\end{aligned}$$

$$\boxed{d\theta^2 = \alpha^2 \wedge \theta^1 - \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2} \theta^1 \wedge \theta^3 + \alpha^3 \wedge \theta^2 - \frac{a_2 + Q_p a_3}{a_3 a_6} \theta^2 \wedge \theta^3.}$$

where $\alpha^2 = \frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3$ and $\alpha^3 = \frac{da_3}{a_3}$.

$$\begin{aligned}d\theta^3 &= \left(\frac{da_4}{a_1} - \frac{a_2}{a_1 a_3} da_5 \right) \wedge \theta^1 + \left(-a_4 - a_5 Q_p \right) \left(\frac{\theta^2 \wedge \theta^3}{a_3 a_6} - \frac{a_2}{a_1 a_3 a_6} \theta^1 \wedge \theta^3 \right) \\ &\quad + \frac{da_5}{a_3} \theta^2 - a_5 Q_u \frac{\theta^1 \wedge \theta^3}{a_1 a_6} + \frac{da_6}{a_6} \theta^3,\end{aligned}$$

$$\boxed{d\theta^3 = \alpha^4 \wedge \theta^1 + \left[(a_4 + a_5 Q_p) \frac{a_2}{a_1 a_3 a_6} - \frac{a_5 Q_u}{a_1 a_6} \right] \wedge \theta^1 \wedge \theta^3 + \alpha^5 \wedge \theta^2 - \frac{a_4 + a_5 Q_p}{a_3 a_6} \theta^2 \wedge \theta^3 + \alpha^6 \wedge \theta^3}$$

where $\alpha^4 = \frac{da_4}{a_1} - \frac{a_2}{a_1 a_3} da_5$, $\alpha^5 = \frac{da_5}{a_3}$ and $\alpha^6 = \frac{da_6}{a_6}$.

Then we have the **structure equations** :

$$\begin{aligned} d\theta^1 &= \alpha^1 \wedge \theta^1 + \frac{a_2}{a_3 a_6} \theta^1 \wedge \theta^3 - \frac{a_1}{a_3 a_6} \theta^2 \wedge \theta^3, \\ d\theta^2 &= \alpha^2 \wedge \theta^1 - \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2} \theta^1 \wedge \theta^3 + \alpha^3 \wedge \theta^2 - \frac{a_2 + Q_p a_3}{a_3 a_6} \theta^2 \wedge \theta^3, \\ d\theta^3 &= \alpha^4 \wedge \theta^1 + [(a_4 + a_5 Q_p) \frac{a_2}{a_1 a_3 a_6} - \frac{a_5 Q_u}{a_1 a_6}] \wedge \theta^1 \wedge \theta^3 + \alpha^5 \wedge \theta^2 \\ &\quad - \frac{a_4 + a_5 Q_p}{a_3 a_6} \theta^2 \wedge \theta^3 + \alpha^6 \wedge \theta^3. \end{aligned}$$

We do the absorption process with

$$\begin{aligned} \pi^1 &= \alpha^1 - \frac{a_2}{a_3 a_6} \theta^3, \\ \pi^2 &= \alpha^2 + \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2} \theta^3, \\ \pi^3 &= \alpha^3 + \frac{a_2 + Q_p a_3}{a_3 a_6} \theta^3, \\ \pi^4 &= \alpha^4 - \left(\frac{a_2(a_4 + a_5 Q_p)}{a_1 a_3 a_6} - \frac{a_5 Q_u}{a_1 a_6} \right) \theta^3, \\ \pi^5 &= \alpha^5 + \frac{a_4 + a_5 Q_p}{a_3 a_6} \theta^3, \\ \pi^6 &= \alpha^6. \end{aligned}$$

Then the structure equations give us to essential torsion coefficient ;

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 + T_{2,3}^1 \theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + \pi^3 \wedge \theta^2, \\ d\theta^3 &= \pi^4 \wedge \theta^1 + \pi^5 \wedge \theta^2 + \pi^6 \wedge \theta^3. \end{aligned}$$

where $T_{2,3}^1 = -\frac{a_1}{a_3 a_6}$ is the only essential torsion coefficient.

We can normalize with

$$T_{2,3}^1 = -1$$

Then

$$a_1 = a_3 a_6.$$

The G_c -equivalence problem is equivalent to the $\tilde{G}_c$ -equivalence problem, where

$$\tilde{G}_c = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_1 a_6^{-1} & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1 a_6 \neq 0 \right\}$$

To be clearly equivalent, we need to express the structure equation with 5 Maurer-Cartan forms. For this we re-compute the structure equation with the normalization $a_1 = a_3 a_6$, we find

$$\begin{aligned} d\theta^1 &= \varpi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \varpi^2 \wedge \theta^1 + \varpi^3 \wedge \theta^2, \\ d\theta^3 &= \varpi^4 \wedge \theta^1 + \varpi^5 \wedge \theta^2 + \varpi^6 \wedge \theta^3. \end{aligned}$$

where

$$\begin{aligned} \varpi^1 &= \alpha^1 - \frac{a_2}{a_1} \theta^3, \\ \varpi^2 &= \alpha^2 + \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2} \theta^3, \\ \varpi^3 &= \alpha^3 + \frac{a_2 a_6 + Q_p a_1}{a_1 a_6} \theta^3, \\ \varpi^4 &= \alpha^4 - \left(\frac{a_2(a_4 + a_5 Q_p)}{a_1 a_4 a_6} - \frac{a_5 Q_u}{a_1 a_6} \right) \theta^3, \\ \varpi^5 &= \alpha^5 + \frac{a_4 + a_5 Q_p}{a_3 a_6} \theta^3, \\ \varpi^6 &= \alpha^6. \end{aligned}$$

where

$$\begin{aligned} \alpha^1 &= \frac{da_1}{a_1}, \\ \alpha^2 &= \frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1 + \frac{a_2}{a_6 a_1} da_6, \\ \alpha^3 &= \frac{da_3}{a_3}, \\ \alpha^4 &= \frac{da_4}{a_1} - \frac{a_2 a_6}{a_1^2} da_5, \\ \alpha^5 &= \frac{a_6}{a_1} da_5, \\ \alpha^6 &= \frac{da_6}{a_6}. \end{aligned}$$

However, since

$$\begin{aligned} \varpi^1 - \varpi^6 &= \frac{da_1}{a_1} - \frac{a_2}{a_1} \theta^3 - \frac{da_6}{a_6} \\ &= \varpi^3 - \left(\frac{a_2}{a_1} + \frac{Q_p}{a_6} \right) \theta^3 - \frac{a_2}{a_1} \theta^3 \end{aligned}$$

and since, for $\widetilde{\varpi}^6 := \varpi^6 - \frac{2a_2}{a_1} \theta^3 - \frac{Q_p}{a_6} \theta^3$, we have $\widetilde{\varpi}^6 \wedge \theta^3 = \varpi^6 \wedge \theta^3$, then we can have a new Maurer-Cartan form

$$\varpi^6 \leftarrow \widetilde{\varpi}^6$$

such that

$$\begin{aligned} d\theta^1 &= \varpi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \varpi^2 \wedge \theta^1 + (\varpi^1 - \varpi^6) \wedge \theta^2, \\ d\theta^3 &= \varpi^4 \wedge \theta^1 + \varpi^5 \wedge \theta^2 + \varpi^6 \wedge \theta^3. \end{aligned}$$

where there are 5 modified Maurer-Cartan forms.

We note that there is no longer essential torsion coefficient depending of group parameters. We need to keep on the Cartan's algorithm.

4. Involution

We now turn to the cases when the Cartan equivalence procedure doesn't lead to a complete reduction of the structure just with normalization and absorption process.

We may assume that, after perhaps one or more loops through the absorption and reduction algorithm, we are left with a lifted coframe that still involves one or more of the original group parameters, yet none of the remaining essential torsion coefficient explicitly on the group parameters.

What do we do ?

Consider the possible symmetry group G of dimension r of an equivalence problem on M (of dimension m). If the Cartan method leads to a complete reduction resulting in an invariant coframe θ , then the associated symmetry group is an $(m - r)$ dimensional manifold.

Consequently, the dimension of the symmetry group can never exceed the dimension of the manifold on which the problem is formulated as a Cartan equivalence problem. However, we have already encounter equivalence problems where the symmetry group has a larger dimension than the space on which the equivalence problem is formulated.

Example 4.1. The equivalence problem for second order ordinary differential equations is formulated on the 3-dimensional jet space J^1 , but the dimension of under the point transformation symmetry group can be as large as 8, e.g., for $u_{xx} = 0$. Then we cannot except that this equivalence problems could be reduce to an equivalence of coframe an any manifold having dimension less than 8, and so we will be unable to obtain an invariant coframe on J^1 .

Example 4.2. Every second order ordinary differential equations admits an infinite-dimensional group of contact transformation, then we can never realize the equivalence problem as an equivalence problem of coframes on any finite-dimensional manifold.

Fortunately, Cartan figured out how to deal with these types of problems as well.

There are 2 types of problems :

The first, in which the symmetry group is finite-dimensional, but of larger dimension than the underlying space, is solved by the process of "prolongation".

The second type of problem is when the symmetry group is infinite-dimensional. The existence of an infinite-dimensional symmetry group is a consequence of the "involutivity" of a certain differential system prescribed by the structure equations for the lifted coframe.

The decision as to whether we need to prolong an equivalence problem (finite-dimensional symmetry group) or not (infinite-dimensional symmetry group) is handled by an arithmetic test for involutivity due to Cartan.

In the next section, we will discuss how to implement the Cartan's test practically, and in the case of an involutive system, how the infinite-dimensional symmetry group and canonical forms are determined.

Moreover, we will see further the prolongation algorithm in the case where the system is not involutive.

Example 4.3. To illustrate our discussion, we consider the three basic equivalence problems for a second order ordinary differential equations $u_{xx} = Q(x, u, u_x)$, $p = u_x$, under respectively fiber-preserving, point, and contact transformation.

We know that the coframe on $J^1 \ni (x, u, u_x = p)$ is

$$\omega^1 = du - p dx, \quad \omega^2 = dp - Q(x, u, p) dx, \quad \omega^3 = dx.$$

For the equivalence problem under contact transformations, the lifted coframe is

$$\theta^1 = a_1 \omega^1, \quad \theta^2 = a_2 \omega^1 + a_3 \omega^2, \quad \theta^3 = a_4 \omega^1 + a_5 \omega^2 + a_6 \omega^3,$$

where the structure group is

$$G_c = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 6.

We have seen after absorption and normalization that we obtain the structure equations

$$(*) : \begin{cases} d\theta^1 = \varpi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 = \varpi^2 \wedge \theta^1 + (\varpi^1 - \varpi^6) \wedge \theta^2, \\ d\theta^3 = \varpi^4 \wedge \theta^1 + \varpi^5 \wedge \theta^2 + \varpi^6 \wedge \theta^3. \end{cases}$$

with $\varpi^1, \varpi^2, \varpi^4, \varpi^5, \varpi^6$ the modified Maurer-Cartan forms on $\widetilde{G}_c$, where

$$\widetilde{G}_c = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_1 a_6^{-1} & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1 a_6 \neq 0 \right\}$$

* The case of point transformation is given by

$$\pi^4 = 0$$

* The case of fiber-preserving transformation is given by

$$\pi^4 = \pi^5 = 0$$

Now, we must figure out whether the symmetry group of the equivalence problem is infinite-dimensional, which is indicated by the previous system being in involution, or finite-dimensional, so that the previous system must be prolonged.

5. Cartan's test

The Cartan's test distinguish between the two possibilities of involution versus prolongation.

Let assume that we are given an equivalence problem based on the regular lifted coframe $\theta = \{\theta^1, \dots, \theta^m\}$ which are one-forms on the $(m + r)$ -dimensional product space $M \times G$, since their coefficients depend on the group parameters.

Let $r = \dim(G)$, the dimension of the (reduced) structure group G . The structure equations for the coframe have the form

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \alpha^l \wedge \theta^j + \sum_{\substack{j,k=1 \\ j < k}}^m T_{j,k}^i \theta^j \wedge \theta^k \quad (1 \leq i \leq m)$$

where the α^l is a basis for the Maurer-Cartan forms on G , $A_{j,k}^i$ are constant.

Note that replacing each Maurer-Cartan form α^l by an arbitrary combination of the coframe elements $\sum_{j=1}^m z_j^l \theta^j$ results in the linear absorption equations :

$$\sum_{l=1}^r (A_{j,l}^i z_k^l - A_{k,l}^i z_j^l) = T_{j,k}^i, \quad (1 \leq i \leq m, 1 \leq j < k \leq m)$$

for the unknown coefficient z_j^l .

In solving the previous linear system, some of the variables z_j^l will be specified (called "basic variables") and other can be given arbitrary values (called "free variables").

Definition 5.1. The degree of indeterminacy $r^{(1)}$ of a lifted coframe is the number of free variable in the solution of the associated linear absorption system.

Remark 5.2. In other words,

$$r^{(1)} = \dim\{(z_j^l)_{\substack{1 \leq l \leq r \\ 1 \leq j \leq m}} \mid \sum_{l=1}^r (A_{j,l}^i z_k^l - A_{k,l}^i z_j^l) = 0\} \quad (1 \leq i \leq m, 1 \leq j < k \leq m)$$

Remark 5.3. For θ and $\bar{\theta}$ two lifted coframes. If θ and $\bar{\theta}$ are equivalent by Ψ , then $r^{(1)}$ is the degree of indeterminacy of the both lifted coframes.

The degree of indeterminacy $r^{(1)}$ of the lifted coframe θ plays an important role for the Cartan's test.

Remark 5.4. $r^{(1)}$ only depends on how the (reduced) structure group G is realized as a subgroup of $GL(m)$ and not on the particular formulae for the base coframe elements.

Example 5.5. Consider the structure equations $(*)$ for the equivalence of second order ordinary differential equations.

To compute $r^{(1)}$, we replace for each $1 \leq l \leq r$, ϖ^l by $z_1^l \theta^1 + z_2^l \theta^2 + z_3^l \theta^3$.

Equating the resulting coefficients of the basis two-forms $\theta^j \wedge \theta^k$, to zero leads to the homogeneous linear system

$$\begin{aligned} z_2^1 &= z_3^1 = 0, \quad z_1^3 - z_2^2 = z_3^2 = z_3^3 = 0, \\ z_1^5 - z_2^4 &= z_1^1 - z_1^3 - z_1^4 = z_2^1 - z_2^3 - z_3^5 = 0. \end{aligned}$$

The general solution is

$$\begin{aligned} z_2^1 &= z_3^1 = z_3^2 = z_3^3 = 0, \\ z_1^1 &= z_1^3 + z_3^4, \quad z_2^2 = z_1^3, \quad z_2^3 = -z_3^5, \quad z_2^4 = z_1^5. \end{aligned}$$

The seven parameters $z_1^2, z_1^3, z_1^4, z_1^5, z_2^5, z_3^4, z_3^5$ can be chosen arbitrary.

Therefore $r^{(1)} = 7$ for contact transformations.

For point transformations, $\pi^5 = 0$, so $r^{(1)} = 4$.

For fiber-preserving transformations, $\pi^4 = \pi^5 = 0$, so $r^{(1)} = 2$ (since z_j^4, z_j^5 do not appear).

To describe the Cartan's test, we need to introduce the reduced Cartan characters associated with the structure equation of the coframe θ .

Definition 5.6. For $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^m$, we define

$$L_l^i[v] = \sum_{j=1}^m A_{j,l}^i v_j \quad (1 \leq i \leq m, 1 \leq l \leq r)$$

Thus $L[v]$ is the coefficient matrix for the Maurer-Cartan forms $\{\alpha^l\}_{1 \leq l \leq r}$ in the structure equation upon replacing each coframe element θ^i by the corresponding entry v_i .

Definition 5.7. We call the first reduced character for the lifted coframe, denoted by

$$s'_1 = \max\{\text{rank}(L[v]) \mid v \in \mathbb{R}^m\},$$

and the second character s'_2 obtained by

$$s'_1 + s'_2 = \max\left\{\text{rank} \begin{pmatrix} L[v^1] \\ L[v^2] \end{pmatrix} \mid v^1, v^2 \in \mathbb{R}^m\right\}$$

More generally,

Definition 5.8. Let θ be a lifted coframe on a m -dimensional manifold with the r -dimensional structure group G .

The first $m - 1$ reduced characters $s'_1, \dots, s'_{m-1}$ of the coframe are defined as

$$s'_1 + \dots + s'_k = \max\left\{\text{rank} \begin{pmatrix} L[v^1] \\ \vdots \\ L[v^k] \end{pmatrix} \mid v^1, \dots, v^k \in \mathbb{R}^m\right\} \quad (1 \leq k \leq m-1)$$

where $L[v]$ is the $m \times r$ -matrix defined in the definition 5.6 and s'_m is defined by

$$s'_1 + \dots + s'_{m-1} + s'_m = r$$

Example 5.9. Consider our example under contact transformations.

The corresponding 3×5 -matrix used to defines the reduced characters is

$$L[v] = \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 \\ 0 & v_1 & v_2 & 0 & 0 \\ v_3 & 0 & -v_3 & v_1 & v_2 \end{pmatrix}. \quad (v=(v_1, v_2, v_3) \in \mathbb{R}^3)$$

We have $s'_1 = 3$; for example this achieved by $v = (1, 0, 0)$.

The second Cartan character is defined by

$$s'_1 + s'_2 = \max\left\{\text{rank} \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 \\ 0 & v_1 & v_2 & 0 & 0 \\ v_3 & 0 & -v_3 & v_1 & v_2 \\ \widehat{v}_1 & 0 & 0 & 0 & 0 \\ 0 & \widehat{v}_1 & \widehat{v}_2 & 0 & 0 \\ \widehat{v}_3 & 0 & -\widehat{v}_3 & \widehat{v}_1 & \widehat{v}_2 \end{pmatrix} \mid v, \widehat{v} \in \mathbb{R}^3\right\} = 5$$

achieved by $v = (1, 0, 0)$ and $\widehat{v} = (0, 1, 0)$.

The final reduced character is $s'_3 = 5 - s'_1 - s'_2 = 0$.

The key tool for the Cartan's test is started in the following definition.

Definition 5.10. Let θ be a lifted coframe with reduced characters $s'_1, \dots, s'_m$ and degree of indeterminacy $r^{(1)}$.

We say that θ is involutive if it satisfies the Cartan's test :

$$s'_1 + 2s'_2 + \dots + ms'_m = r^{(1)}$$

6. The transitive case

The equivalence problem for involutive coframe has a complete solution.

There are two type of involutive coframe ; thus of transitive structure equations and intransitive structure equations.

Definition 6.1. An involutive lifted coframe θ is said transitive or its structure equations is transitive if their essential torsion coefficients of the structure equations are all constants. Otherwise, their essential torsion coefficients contain a non-constant essential torsion coefficient (not depending of any remaining group parameters), then we say θ is intransitive or the structure equations are intransitive.

We mention the following theorem to complete the process of the transitive case.

Theorem 6.2. (of transitive case)

Let θ and $\bar{\theta}$ be regular lifted coframe on m -dimensional M and $\bar{M}$ having the same structure group $G \subset GL(m)$.

Assume that both θ and $\bar{\theta}$ are involutive, and that all the essential torsion coefficients in the two sets of structure equations are constants. Then θ is locally equivalent to $\bar{\theta}$ if and only if they have the same constant essential torsion.

Moreover, if $s'_k > 0$ is the last nonzero character, so for $l > k$, $s'_l = 0$, then the set of equivalences depends on s'_k arbitrary functions, each depending on k variables.

Corollary 6.3. An involutive lifted coframe has an infinite-dimensional symmetry group depending on s'_k arbitrary functions of k variables.

Example 6.4. For the equivalence problem of second order ordinary differential equations, see for different transformations.

* For contact transformations, $r^{(1)} = 7$ and $s'_1 = 3, s'_2 = 2, s'_3 = 0$ so the lifted coframe satisfies the Cartan's test.

Therefore, the structure equation for the contact transformations of second order ordinary differential equations are involutive.

* For point transformations, $r^{(1)} = 4$ and $s'_1 = 3, s'_2 = 1, s'_3 = 0$, so the lifted coframe doesn't satisfy the Cartan's test.

Therefore, the structure equation for the point transformations of second order ordinary differential equations are involutive.

* For point transformations, $r^{(1)} = 4$ and $s'_1 = 3, s'_2 = 1, s'_3 = 0$, so the lifted coframe doesn't satisfy the Cartan's test.

Therefore, the structure equation for the point transformations of second order ordinary differential equations are not involutive.

* For fiber-preserving transformations, $r^{(1)} = 2$ and $s'_1 = 3, s'_2 = 1, s'_3 = 0$, so the lifted coframe doesn't satisfy the Cartan's test.

Therefore, the structure equation for the point transformations of second order ordinary differential equations are not involutive.

Finally, we mention the following theorem.

Theorem 6.5. *All nonsingular, second order ordinary differential equations are locally equivalent under contact transformations.*

Moreover, every second order ordinary differential equation admits an infinite-dimensional symmetry group of contact transformations depending on two arbitrary functions of two variables, hence can be mapped to any other one in infinitely many ways.

Proof. See the chapter 15 of "Equivalence, Invariance and Symmetries" of Olver. □

7. The intransitive case

We briefly explain this case.

The intransitive case, which arises when there are non constant essential torsion coefficients is more complicated. The essential torsion coefficients will define invariants of the problem, which must be included into the equivalence conditions.

To solve this problem, we mention the using of the following theorem.

Theorem 7.1. *(of intransitive case)*

Let θ and $\bar{\theta}$ be regular lifted coframes on m -dimensional M and $\bar{M}$ having the same structure group $G \subset GL(m)$.

Assume that both θ and $\bar{\theta}$ are involutive, and that all the essential torsion coefficients and their coframe derivatives are independent of the group parameters, forming a regular family of functions, and hence parametrizing the classifying manifolds for each coframe.

The coframes θ and $\bar{\theta}$ are locally equivalent if and only if they have the same order $\bar{s} = s$, and their $(s + 1)^{st}$ order classifying manifold $C^{(s+1)}(\theta)$ and $C^{(s+1)}(\bar{\theta})$ overlap.

Moreover, the symmetry group of a such coframe is infinite-dimensional, depending on s'_k arbitrary functions of k variables, where s'_k denotes the last nonzero reduced character of the structure equations.

Proof. See the chapter 15 of "Equivalence, Invariance and Symmetries" of Olver. □

An example of this case is the equivalence of non-closed two forms proved in the chapter 11 of "Equivalence, Invariance and Symmetries" of Olver.

Theorem 7.2. *All non-closed two-forms on a three dimensional manifold are locally equivalent.*

8. The prolongation case

Finally, we must tackle the problem in which the Cartan equivalence procedure does not lead to a complete reduction of the structure group, moreover, the structure equations are not involutive, so that we cannot deduce the existence of an infinite-dimensional symmetry group.

The reduction of the difficulty is to reformulate the problem on a suitably larger dimension manifold and then, if necessary, reapply the Cartan reduction algorithm (absorption and normalization).

How to construct the required higher dimensional equivalence problem ?

The solution is the prolongation method of Cartan ;

We assume that, after perhaps one or more loop through the absorption and reduction algorithm, we have produced a coframe that still involves one or more of the original group parameters, yet, none of the remaining essential torsion coefficients, nor any of their coframe derivative depends explicitly on the group parameters.

For simplicity, let G denote the reduced structure group at the point, and $r = \dim(G)$. Moreover, we are assuming that our lifted coframe θ is not involutive :

$$r^{(1)} < \sum_{i=1}^m i s'_i$$

Cartan prolonged the natural lifted coframe with new variables to be appended are the group parameters or coordinates on the remaining structure group G itself.

In other words, the prolonged equivalence problem will be formulated in terms of a lifted coframe on the $(m + r)$ -dimensional manifold $M^{(1)} := M \times G$.

The prolonged lifted coframe will contain $r^{(1)}$ unspecified parameters to parametrize the new structure group.

After this construction, the goal is, to using the same basic absorption and normalization algorithm, to reduce the equivalence problem to one that we know how to solve.

The prolongation case is distinguished in between two subclasses ; the determinate case and the indeterminate case.

9. The determinate case

Let θ be a lifted coframe not involutive on a m -dimensional manifold M with the structure group G (r -dimensional Lie group).

Suppose that after absorption and reductions, the lifted coframe θ satisfies $r^{(1)} = 0$.

Definition 9.1. Let θ be a lifted coframe not involutive on M with the structure group G . We say that θ is determinate if $r^{(1)} = 0$. Otherwise, $r^{(1)} > 0$, we say that θ is indeterminate.

Then, we use the absorption process. The structure equations are :

$$d\theta^i = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \pi^l \wedge \theta^j + \sum_{j=1}^m U_{j,k}^i(x, g) \theta^j \wedge \theta^k, \quad (1 \leq i \leq m)$$

where $U_{j,k}^i$ are the essential torsion coefficients, which do not depend on any group parameters and π^l the modified Maurer-Cartan forms.

As before, if two lifted coframes are equivalent, then essential torsion coefficients are invariant.

The one-forms $\{\pi^1, \dots, \pi^r\}$ are equal, modulo the θ 's, to Maurer-Cartan forms $\{\alpha^1, \dots, \alpha^r\}$ on G .

More precisely,

$$\pi^l = \alpha^l - \sum_{j=1}^m z_j^l \theta^j, \quad (1 \leq l \leq r)$$

where $(z_j^l)_{\substack{1 \leq j \leq m \\ 1 \leq l \leq r}}$ are the solution of the linear absorption equations

$$\sum_{l=1}^r (A_{j,l}^i z_k^l - A_{k,l}^i z_j^l) = U_{j,k}^i. \quad (1 \leq i \leq m, 1 \leq j < k \leq m)$$

Since $r^{(1)} = 0$, the coefficients $z_j^l = S_j^l(x, g)$ are all uniquely specified and the only solution to

$$\sum_{l=1}^r \sum_{j=1}^m A_{j,k}^i \pi^l \wedge \theta^j = 0, \quad (1 \leq i \leq m)$$

is the trivial one $\pi^l = 0$ ($1 \leq l \leq r$).

Now, suppose we have an equivalence $\Psi : M \rightarrow \overline{M}$, so that for every $1 \leq i \leq m$, $\Psi^* \bar{\theta}^i = \theta^i$ preserves the lifted coframes for some choice of the group parameters $g = g(x)$ and $\bar{g} = \bar{g}(\bar{x})$.

Then for every $1 \leq i \leq m$, $\Psi^*(d\bar{\theta}^i) = d\theta^i$.

Then,

$$\sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \bar{\pi}^l \wedge \theta^j + \sum_{j=1}^m \bar{U}_{j,k}^i(x, g) \theta^j \wedge \theta^k = \sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i \pi^l \wedge \theta^j + \sum_{j=1}^m U_{j,k}^i(x, g) \theta^j \wedge \theta^k.$$

Since the essential torsion coefficients are equivalents, we have

$$\sum_{l=1}^r \sum_{j=1}^m A_{j,l}^i (\bar{\pi}^l - \pi^l) \wedge \theta^l = 0. \quad (1 \leq i \leq m)$$

Thus, if the equivalence problem is determinate, then for every $1 \leq l \leq r$, $\bar{\pi}^l = \pi^l$, or more precisely

$$\Phi^*(\bar{\pi}^l) = \pi^l. \quad (1 \leq l \leq r)$$

Then we have candidate to complete the lifted coframe on $M^{(1)} := M \times G$.

A prolonged invariant lifted coframe on $M^{(1)}$ is constituted of the original lifted coframe on M $\theta = g \cdot \omega$ and the modified Maurer-Cartan forms $\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^r \end{pmatrix}$.

Similarly, we construct a prolonged invariant lifted coframe on $\bar{M}^{(1)}$ by $(\bar{\theta}, \bar{\pi})$.

The key point is that every equivalence between the original lifted coframe extends to an equivalence between the prolonged lifted coframes and conversely.

Theorem 9.2. (of the determinate case)

Let $\theta = g \cdot \omega$ and $\bar{\theta} = \bar{g} \cdot \bar{\omega}$ be two regular lifted coframes defined respectively on M and $\bar{M}$ having the same structure group G , no group dependent essential torsion coefficient, and degree of indeterminacy $r^{(1)} = 0$.

Let π and $\bar{\pi}$ be the modified Maurer-Cartan forms obtained by solving the absorption equations.

Then there exists a diffeomorphism $\Phi : M \rightarrow \bar{M}$ mapping $\bar{\theta}$ to θ for some choice of group parameters $g = g(x)$ and $\bar{g} = \bar{g}(\bar{x})$ if and only if there is a diffeomorphism $\Psi : M \times G \rightarrow \bar{M} \times G$ mapping $(\bar{\theta}, \bar{\pi})$ to (θ, π) .

Proof. First assume that the original equivalence problem has a solution, so there exists a map $\Phi : M \rightarrow \bar{M}$ and a G -valued matrix of function g_0 such that $\Phi^* \bar{\omega} = g_0 \cdot \omega$.

The prolonged lifted coframes $\theta = g \cdot \omega$ and $\bar{\theta} = \bar{g} \cdot \bar{\omega}$, where g and $\bar{g}$ new denote coordinates on the group G , will then be equivalent if and only if $\bar{g} = g \cdot g_0^{-1}$.

This serves to define the prolonged map $\Psi : M \times G \rightarrow \bar{M} \times G$, which is given by

$$\Psi : (x, g) \mapsto (\Phi(x), g \cdot g_0^{-1}) \quad (9.3)$$

Our earlier analysis of the structure equations implies that Ψ preserves both arts of the prolonged lifted coframes :

$$\Psi^* \bar{\theta} = \theta, \quad \Psi^* \bar{\pi} = \pi \quad (9.4)$$

Conversely, consider a map $\Psi : M \times G \rightarrow \overline{M} \times G$ which preserves the prolonged lifted coframes (9.4). We must show that Ψ is necessary of the form (9.3).

The first set of conditions in (9.3) implies that each $\Psi^* d\overline{x}^i$ is a linear combination of the dx^j 's.

Therefore Ψ takes the projectable form $\Psi(x, g) = (\Phi(x), \Lambda(x, g))$.

Write $\Lambda(x, g) = (g \cdot h)(x, g)$ for some G -valued function h .

Then,

$$\Psi^*(d\overline{g} \cdot \overline{g}^{-1}) = dg \cdot g^{-1} + g \cdot dh \cdot (gh)^{-1}. \quad (9.5)$$

But, according to (9.3), the Maurer-Cartan forms must match up to combinations of the dx 's, so for every $1 \leq l \leq r$, $\Psi^*(\overline{\alpha}^l) = \alpha^l + \sum_{i=1}^m w_i^l dx^i$.

Since the Maurer-Cartan forms are the independent entries of $dg \cdot g^{-1}$ and $d\overline{g} \cdot \overline{g}^{-1}$, (9.5) implies that the entries of dh can only depend on the dx 's, and therefore $h = h(x)$ depends only on x . This means that the equivalence map must be of the previous form (9.3), with $g_0 = h^{-1}$. It is now a simple matter to prove that, for this choice of g_0 , the projection Φ of Ψ to M satisfies $\Phi^* \overline{\omega} = g_0 \cdot \omega$, and hence we obtain the desired equivalence. $\square$

We can reformulate the theorem as follows, for $M = \overline{M}$.

Corollary 9.6. *Let $\theta = g \cdot \omega$ be coframe on M having G as structure group.*

There is a self-equivalence $\Phi : M \rightarrow M$ if, and only if, there is a self-equivalence $\Psi : M \times G \rightarrow M \times G$.

Equivalently, the symmetry group of θ is isomorphic to the symmetry group of a prolonged coframe of θ .

After this prolongation process, the goal is the same, to use the same basic absorption and normalization procedure in order to reduce the equivalence problem to one we know to solve.

Let us give an example of the determinate case ; the equivalence of surfaces proved in the chapter 12 of "Equivalence, Invariance and Symmetries" of Olver.

Theorem 9.7. *Let S be a surface with first fundamental form ds^2 and Gaussian curvature κ . The group of isometries of S is a Lie group of dimension at most 3. Moreover, the isometry group is 3-dimensional if and only if the surface has constant Gaussian curvature, in which case :*

- (1) *If $\kappa < 0$ (hyperbolic geometry), the connected component of the symmetry group has the structure $SL(2, \mathbb{R})$.*
- (2) *If $\kappa = 0$ (euclidean geometry), the connected component of the symmetry group has the structure $SE(2, \mathbb{R})$.*
- (3) *If $\kappa > 0$ (hyperbolic geometry), the connected component of the symmetry group has the structure $SO(2, \mathbb{R})$.*

10. The indeterminate case

The only case yet to be treated occurs when the problem reduces to a lifted coframe satisfying $r^{(1)} > 0$, the indeterminate case.

The prolongation process is essentially the same as before, but now every free parameter in the solution of the linear absorption equations provides a new group parameter which must essentially be normalized if we are to solve the prolonged problem.

The modified Maurer-Cartan forces are defined by the same formula in the determinate case. In the determine case, this completely specified the z_j^l 's in terms of the base variables and group parameters.

In the indeterminate case, $r^{(1)}$ of the z_j^l 's are free variables. Let us denote the free variables by w^μ , $1 \leq \mu \leq r^{(1)}$. So the general solution to the linear absorption equations take the form

$$z = K[w] + S,$$

where $K : \mathbb{R}^{r^{(1)}} \rightarrow R^{r^{(1)}}$ is a linear map and S a vector formed by a particular solution of the linear absorption equations, depending on x and g .

Thus, the modified Maurer-Cartan forms are defined by

$$\pi = \alpha - (K[w] + S)\theta$$

Consequently, our prolongation procedure lead us to introduce a lifted coframe on the prolonged manifold $M^{(1)} := M \times G$ which is (θ, π) where θ is the original lifted coframe on M and $\pi = \alpha - S\theta$.

The new structure group is

$$G^{(1)} = \left\{ \begin{pmatrix} I_m & 0_r \\ K[w] & I_m \end{pmatrix} \mid w \in \mathbb{R}^{r^{(1)}} \right\},$$

with $r^{(1)} = \dim(G^{(1)})$.

We can prove the isomorphism between the original equivalence problem on M and $\bar{M}$ with structure group G and the new equivalence problem on $M^{(1)}$ and $\bar{M}^{(1)}$ with structure group $G^{(1)}$ by the following result, which is proved by the same way than the determinate case.

Theorem 10.1. *(of the indeterminate case)*

Let (θ, π) and $(\bar{\theta}, \bar{\pi})$ be lifted prolonged coframes constructed above.

There exists a diffeomorphism $\Phi : M \rightarrow \bar{M}$ mapping $\bar{\theta}$ to θ for some choice of group parameters g and $\bar{g}$ if and only if there is a diffeomorphism $\Psi : M \times G \rightarrow \bar{M} \times G$ mapping $(\bar{\theta}, \bar{\pi})$ to (θ, π) for some choice of prolonged group parameters w and $\bar{w}$.

Corollary 10.2. *Let $\theta = g \cdot \omega$ be lifted coframe on M having G as structure group. There is a self-equivalence $\Phi : M \rightarrow M$ if, and only if, there is a self-equivalence $\Psi : M \times G \rightarrow M \times G$. Equivalently, the symmetry group of θ is isomorphic to the symmetry group of a prolonged coframe of θ .*

After this prolongation process, the goal is the same, to use the same basic absorption and normalization procedure in order to reduce the equivalence problem to one we know to solve.

Example 10.3. Consider the three basic equivalence problems for a second order ordinary differential equations $u_{xx} = Q(x, u, u_x)$, $p = u_x$, under fiber-preserving transformations.

Etape 1 :

We know that the coframe on $J^1 \ni (x, u, u_x = p)$ is

$$\omega^1 = du - p dx, \quad \omega^2 = dp - Q(x, u, p) dx, \quad \omega^3 = dx.$$

The associated lifted coframe is

$$\theta^1 = a_1 \omega^1, \quad \theta^2 = a_2 \omega^1 + a_3 \omega^2, \quad \theta^3 = a_6 \omega^3,$$

where the structure group is

$$G_f = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_6 \end{pmatrix} \mid a_1 a_3 a_6 \neq 0 \right\}$$

of dimension 6.

Etape 2 :

We have seen after absorption and normalization that we obtain the structure equations :

$$\begin{aligned} d\theta^1 &= \varpi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \varpi^2 \wedge \theta^1 + (\varpi^1 - \varpi^6) \wedge \theta^2, \\ d\theta^3 &= \varpi^6 \wedge \theta^3. \end{aligned}$$

with $\varpi^1, \varpi^2, \varpi^6$ the modified Maurer-Cartan forms on $\widetilde{G}_c$, where

$$\widetilde{G}_f = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_1 a_6^{-1} & 0 \\ 0 & 0 & a_6 \end{pmatrix} \mid a_1 a_6 \neq 0 \right\}$$

and

$$\begin{aligned}\varpi^1 &= \alpha^1 - \frac{a_2}{a_1}\theta^3, \\ \varpi^2 &= \alpha^2 + \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2} \theta^3, \\ \varpi^3 &= \alpha^3 + \frac{a_2 a_6 + Q_p a_1}{a_1 a_6} \theta^3, \\ \varpi^6 &= \alpha^6 - \frac{a_1 Q_p + 2a_2 a_6}{a_1 a_6} \theta^3\end{aligned}$$

where

$$\begin{aligned}\alpha^1 &= \frac{da_1}{a_1}, \\ \alpha^2 &= \frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1 + \frac{a_2}{a_6 a_1} da_6, \\ \alpha^3 &= \frac{da_3}{a_3}, \\ \alpha^6 &= \frac{da_6}{a_6},\end{aligned}$$

and by the normalization the lifted coframe becomes :

$$\theta^1 = a_1 \omega^1, \quad \theta^2 = a_2 \omega^1 + \frac{a_1}{a_6} \omega^2, \quad \theta^3 = a_6 \omega^3,$$

The degree of indeterminacy is $r^{(1)} = 2$, then with $z_1^1 = z_2^2 = r \in \mathbb{R}$ and $z_1^2 = s \in \mathbb{R}$. Then

$$\begin{aligned}\pi^1 &= \varpi^1 + r\theta^1, \\ \pi^2 &= \varpi^2 + s\theta^1 + r\theta^2, \\ \pi^6 &= \varpi^6,\end{aligned}$$

are the modified Maurer-Cartan forms.

Etape 3 :

Now, we can extend the equivalence problem to

$$\widetilde{G}_f^{(1)} = \left\{ \begin{pmatrix} & I_3 & & 0_3 \\ r & 0 & 0 & \\ s & r & 0 & I_3 \\ 0 & 0 & 0 & \end{pmatrix} \mid r, s \in \mathbb{R} \right\}$$

with the lifted coframe $\{\theta^1, \theta^2, \theta^3, \pi^1, \pi^2, \pi^6\}$ based on $J^1 \times \widetilde{G}_f \ni (x, u, p, a_1, a_2, a_6)$.

Etape 4 :

Then, we apply the Cartan's process ;

We need to compute the structure equations.
 Since, π is the general solution of the linear absorption equation, we have :

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2, \\ d\theta^3 &= \pi^6 \wedge \theta^3. \end{aligned}$$

We need to compute for every $i \in \{1, 2, 6\}$, $d\pi^i$:

Remark 10.4. Note the computing of d^2 for every π ,

* For $d^2\theta^1$:

$$0 = d\pi^1 \wedge \theta^1 + \pi^1 \wedge \theta^2 \wedge \theta^3 - \pi^2 \wedge \theta^1 \wedge \theta^3 + (\pi^1 - \pi^6) \wedge \theta^2 \wedge \theta^3 + \theta^2 \wedge \pi^6 \wedge \theta^3. \quad (10.5)$$

* For $d^2\theta^2 = 0$ we obtain :

$$0 = d\pi^2 \wedge \theta^1 + d\pi^1 \wedge \theta^2 + \pi^2 \wedge \theta^2 \wedge \theta^3 - d\pi^6 \wedge \theta^3 + \pi^6 \wedge \pi^2 \wedge \theta^1. \quad (10.6)$$

* For $d^2\theta^3 = 0$ we obtain :

$$0 = d\pi^6 \wedge \theta^3. \quad (10.7)$$

By the Cartan's lemma applied to (10.5) , there exists $(b_1^6, b_2^6, \dots, b_5^6) : J^1 \times \widetilde{G}_f \rightarrow \mathbb{R}$ such that

$$d\pi^6 = b_1^6 \theta^3 \wedge \theta^2 + b_2^6 \theta^3 \wedge \theta^1 + b_3^6 \theta^3 \wedge da_1 + b_4^6 \theta^3 \wedge da_2 + b_5^6 \theta^3 \wedge da_3.$$

Then substituting this expressing in (10.4) we find again by Cartan's lemma $(b_1^2, b_2^2, \dots, b_5^2) : J^1 \times \widetilde{G}_f \rightarrow \mathbb{R}$ such that

$$d\pi^2 = b_1^2 \theta^3 \wedge \theta^2 + b_2^2 \theta^3 \wedge \theta^1 + b_3^2 \theta^3 \wedge da_1 + b_4^2 \theta^3 \wedge da_2 + b_5^2 \theta^3 \wedge da_3.$$

Finally, by the same way, there exist $(b_1^1, b_2^1, \dots, b_5^1) : J^1 \times \widetilde{G}_f \rightarrow \mathbb{R}$ such that

$$d\pi^1 = b_1^1 \theta^3 \wedge \theta^2 + b_2^1 \theta^3 \wedge \theta^1 + b_3^1 \theta^3 \wedge da_1 + b_4^1 \theta^3 \wedge da_2 + b_5^1 \theta^3 \wedge da_3.$$

Now, we will be more precise on the form of $d\pi^i$, for $i \in \{1, 2, 6\}$.

Compute $d\pi^1$:

$$\begin{aligned}
d\pi^1 &= d(\varpi^1) + rd\theta^1 \\
&= d\alpha^1 - d\left(\frac{a_2}{a_1}\theta^3\right) + r\pi^1 \wedge \theta^1 - r\theta^2 \wedge \theta^3 \\
&= 0 - d\left(\frac{a_2}{a_1}\right)\theta^3 - \frac{a_2}{a_1}d\theta^3 + r\pi^1 \wedge \theta^1 - r\theta^2 \wedge \theta^3 \\
&= -\frac{(da_2)a_1 - (da_1)a_2}{a_1^2}\theta^3 - \frac{a_2}{a_1}\varpi^6 \wedge \theta^3 + r\pi^1 \wedge \theta^1 - r\theta^2 \wedge \theta^3 \\
&= -\frac{da_2}{a_1}\theta^3 + \frac{a_2}{a_1^2}da_1\theta^3 - \frac{a_2}{a_1}\frac{da_6}{a_6}\theta^3 + \frac{a_1a_2Q_p + 2a_2^2a_6}{a_1^2a_6}\theta^3 \wedge \theta^3 + r\pi \wedge \theta^3 - r\theta^2 \wedge \theta^3 \\
&= \left(r\varpi^1 + r^2\theta^1\right) \wedge \theta^1 + \left(-\frac{da_2}{a_1} + \frac{a_2}{a_1^2}da_1 - \frac{a_2}{a_1}\frac{da_6}{a_6} - r\theta^2\right) \wedge \theta^3 \\
&= \left(r\frac{da_1}{a_1} - r\frac{a_2}{a_1}\theta^3\right) \wedge \theta^1 - \left(\varpi^2 + s\theta^1 + r\theta^2 - s\theta^1\right) \wedge \theta^3 \\
&= \left(r\frac{da_1}{a_1} - (s + r\frac{a_2}{a_1})\theta^2\right) \wedge \theta^1 - \pi^2 \wedge \theta^3,
\end{aligned}$$

$$d\pi^1 = \rho \wedge \theta^1 - \pi^2 \wedge \theta^3,$$

where

$$\rho = r\frac{da_1}{a_1} - (s + r\frac{a_2}{a_1})\theta^3$$

Compute $d\pi^2$:

$$\begin{aligned}
d\pi^2 &= d\varpi^2 + sd\theta^1 + rd\theta^2 \\
d\pi^2 &= d\alpha^2 + d\left(\frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2}{a_1^2 a_6^2}\right)\theta^3 + \frac{Q_u}{a_6^2}d\theta^3 - \frac{a_2}{a_1 a_6}Q_p d\theta^3 - \frac{a_2^2}{a_1^2}d\theta^3 \\
&\quad + s(\pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3) + r(\pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2) \\
d\pi^2 &= d\left(\frac{da_2}{a_1}\right) - d\left(\frac{a_2}{a_1^2}\right)da_1 + d\left(\frac{a_2}{a_6 a_1}\right)da_6 + \left[d\left(\frac{Q_u}{a_6^2}\right) - d\left(\frac{a_2}{a_1 a_6}Q_p\right) - d\left(\frac{a_2^2}{a_1^2}\right)\right] \wedge \theta^3 \\
&\quad + \frac{Q_u}{a_6^2}\pi^6 \wedge \theta^3 - \frac{a_2}{a_1 a_6}Q_p \pi^6 \wedge \theta^3 - \frac{a_2^2}{a_1^2}\pi^6 \wedge \theta^3 + s(\varpi^1 + r\theta^1) \wedge \theta^1 - s\theta^2 \wedge \theta^3 \\
&\quad + r(\varpi^2 + s\theta^1 + r\theta^2) \wedge \theta^1 + r(\alpha^1 - \alpha^6 - \frac{a_2}{a_1}\theta^3 + \frac{Q_p}{a_6}\theta^3 + \frac{2a_2}{a_1}\theta^3 + r\theta^1) \wedge \theta^2 \\
d\pi^2 &= \frac{0 \times a_1 - da_1 da_2}{a_1^2} - \frac{(da_2)a_1^2 - 2a_1(da_1)a_2}{a_1^4}da_1 + \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2}a_2 da_6 \\
&\quad + \frac{(dQ_u)a_6^2 - 2a_6(da_6)Q_u}{a_6^4}\theta^3 - \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2}a_2 Q_p \theta^3 - \frac{a_2}{a_1 a_6}dQ_p \theta^3 \\
&\quad - \frac{2a_2(da_2)a_1^2 - 2a_1(da_1)a_2^2}{a_1^4}\theta^3 + \frac{Q_u}{a_6^2}\left(\frac{da_6}{a_6} - \frac{Q_p}{a_6}\theta^3 - \frac{2a_2}{a_1}\theta^3\right) \wedge \theta^3 \\
&\quad - \frac{a_2}{a_1 a_6}Q_p\left(\frac{da_6}{a_6} - \frac{Q_p}{a_6}\theta^3 - \frac{2a_2}{a_1}\theta^3\right) \wedge \theta^3 - \frac{a_2^2}{a_1^2}\left(\frac{da_6}{a_6} - \frac{Q_p}{a_6}\theta^3 - \frac{2a_2}{a_1}\theta^3\right) \wedge \theta^3 \\
&\quad + s\left(\frac{da_1}{a_1} - \frac{a_2}{a_1}\theta^3\right) \wedge \theta^1 - s\theta^1 \wedge \theta^3 + r\varpi^2 \wedge \theta^1 + r^2\theta^2 \wedge \theta^1 + r\left(\frac{da_1}{a_1} - \frac{da_6}{a_6}\right)\theta^2 + \frac{ra_2}{a_1}\theta^2 \wedge \theta^3 \\
&\quad - \frac{rQ_p}{a_6}\theta^2 \wedge \theta^3 - \frac{2ra_2}{a_1}\theta^2 \wedge \theta^3 - r^2\theta^2 \wedge \theta^1
\end{aligned}$$

$$\begin{aligned}
d\pi^2 = & -\frac{da_1 da_2}{a_1^2} - \frac{(da_2)a_1^2 - 2a_1(da_1)a_2}{a_1^4} da_1 + \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2} a_2 da_6 \\
& + \frac{(dQ_u)a_6^2 - 2a_6(da_6)Q_u}{a_6^4} \theta^3 - \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2} a_2 Q_p \theta^3 - \frac{a_2}{a_1 a_6} dQ_p \theta^3 \\
& - \frac{2a_2(da_2)a_1^2 - 2a_1(da_1)a_2^2}{a_1^4} \theta^3 + \frac{Q_u}{a_6^2} \left(\frac{da_6}{a_6}\right) \wedge \theta^3 - \frac{a_2}{a_1 a_6} Q_p \left(\frac{da_6}{a_6}\right) \wedge \theta^3 - \frac{a_2^2}{a_1^2} \left(\frac{da_6}{a_6}\right) \wedge \theta^3 \\
& + s \left(\frac{da_1}{a_1} - \frac{a_2}{a_1} \theta^3\right) \wedge \theta^1 - s \theta^1 \wedge \theta^3 + r \varpi^2 \wedge \theta^1 + r \left(\frac{da_1}{a_1} - \frac{da_6}{a_6}\right) \theta^2 + \frac{ra_2}{a_1} \theta^2 \wedge \theta^3 \\
& - \frac{rQ_p}{a_6} \theta^2 \wedge \theta^3 - \frac{2ra_2}{a_1} \theta^2 \wedge \theta^3 \\
d\pi^2 = & -\frac{da_1 da_2}{a_1^2} - \frac{(da_2)a_1^2 - 2a_1(da_1)a_2}{a_1^4} da_1 + \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2} a_2 da_6 \\
& + \frac{(dQ_u)a_6^2 - 2a_6(da_6)Q_u}{a_6^4} \theta^3 - \frac{(da_2)a_1 a_6 - [(da_1)a_6 + (da_6)a_1]}{a_1^2 a_6^2} a_2 Q_p \theta^3 - \frac{a_2}{a_1 a_6} dQ_p \theta^3 \\
& - \frac{2a_2(da_2)a_1^2 - 2a_1(da_1)a_2^2}{a_1^4} \theta^3 + \frac{Q_u}{a_6^2} \left(\frac{da_6}{a_6}\right) \wedge \theta^3 - \frac{a_2}{a_1 a_6} Q_p \left(\frac{da_6}{a_6}\right) \wedge \theta^3 - \frac{a_2^2}{a_1^2} \left(\frac{da_6}{a_6}\right) \wedge \theta^3 \\
& + s \left(\frac{da_1}{a_1} - \frac{a_2}{a_1} \theta^3\right) \wedge \theta^1 - s \theta^1 \wedge \theta^3 + R,
\end{aligned}$$

where

$$R = r \varpi^2 \wedge \theta^1 + r \left(\frac{da_1}{a_1} - \frac{da_6}{a_6}\right) \theta^2 + \frac{ra_2}{a_1} \theta^2 \wedge \theta^3 - \frac{rQ_p}{a_6} \theta^2 \wedge \theta^3 - \frac{2ra_2}{a_1} \theta^2 \wedge \theta^3.$$

Then,

$$\begin{aligned}
d\pi^2 = & -\frac{da_1 da_2}{a_1^2} - \frac{da_2 da_1}{a_1^2} + \frac{2a_2}{a_1^3} (da_1)^2 + \frac{da_2 da_6}{a_1 a_6} - \frac{a_2}{a_1^2 a_6} da_1 da_6 - \frac{a_2}{a_1 a_6} (da_6)^2 \\
& + \frac{1}{a_6^2} (Q_{ux} dx + Q_{uu} du + Q_{pu} dp) \wedge \theta^3 - \frac{2Q_u}{a_6^3} da_6 \wedge \theta^3 - \frac{Q_p}{a_1 a_6} da_2 \theta^3 + \frac{a_2}{a_1^2 a_6} Q_p da_1 \theta^1 \\
& + \frac{a_2}{a_1 a_6^2} Q_p da_6 \theta^3 - \frac{a_2}{a_1 a_6} (Q_{px} dx + Q_{pu} du + Q_{pp} dp) \wedge \theta^3 - \frac{2a_2}{a_1^2} da_2 \theta^3 + \frac{2a_2^2}{a_1^3} da_1 \theta^3 \\
& + \frac{Q_u}{a_6^3} da_6 \theta^3 - \frac{a_2}{a_1 a_6^2} Q_p da_6 \theta^3 - \frac{a_2^2}{a_1^2 a_6} da_6 \theta^3 + s \frac{da_1}{a_1} \theta^1 - \frac{sa_2}{a_1} \theta^3 \wedge \theta^1 - s \theta^2 \wedge \theta^1 + R \\
d\pi^2 = & \frac{da_2 da_6}{a_1 a_6} - \frac{a_2}{a_1^2 a_6} da_1 da_6 + \frac{1}{a_6} Q_{uu} du \wedge dx + \frac{1}{a_6} Q_{up} dp \wedge dx + \Delta - \frac{a_2}{a_1} Q_{pu} du \wedge dx \\
& - \frac{a_2}{a_1} Q_{pp} dp \wedge dx + s \frac{da_1}{a_1} \theta^1 - \frac{sa_2}{a_1} \theta^3 \wedge \theta^1 + s \theta^2 \wedge \theta^1 + R
\end{aligned}$$

where

$$\Delta = -\frac{2Q_u}{a_6^3} da_6 \theta^3 - \frac{Q_p}{a_1 a_6} da_2 \theta^3 + \frac{a_2}{a_1^2 a_6} Q_p da_1 \theta^3 - \frac{2a_2}{a_1^2} da_2 \theta^3 + \frac{2a_2^2}{a_1^3} da_1 \theta^3 + \frac{Q_u}{a_6^3} da_6 \theta^3 - \frac{a_2^2}{a_1^2 a_6} da_6 \theta^3.$$

Then,

$$\begin{aligned} d\pi^2 = & \frac{da_2 da_6}{a_1 a_6} - \frac{a_2}{a_1^2 a_6} da_1 da_6 + \frac{Q_{uu}}{a_6^2} \frac{\theta^1 \wedge \theta^3}{a_1} + \frac{Q_{up}}{a_6} \frac{\theta^2 \wedge \theta^3}{a_1} - \frac{a_2 Q_{up}}{a_1^2 a_6} \theta^1 \wedge \theta^3 \\ & + \Delta - \frac{a_2}{a_1^2} Q_{pu} \frac{\theta^1 \wedge \theta^3}{a_6} - \frac{a_2}{a_1^2} Q_{pp} \theta^2 \wedge \theta^3 + \frac{a_2^2}{a_1^3} Q_{pp} \theta^1 \wedge \theta^3 + s \frac{da_1}{a_1} \theta^1 \\ & - \frac{sa_2}{a_1} \theta^3 \wedge \theta^1 - s \theta^2 \wedge \theta^3 + R. \end{aligned}$$

Now, we compute $\pi^2 \wedge \pi^6$:

$$\begin{aligned} \pi^2 \wedge \pi^6 &= (\varpi^2 + s\theta^1 + r\theta^2) \wedge \varpi^6 \\ \pi^2 \wedge \pi^6 &= \varpi^2 \wedge \varpi^6 + s\theta^1 \wedge \varpi^6 + r\theta^2 \wedge \varpi^6 \\ \pi^2 \wedge \pi^6 &= \left(\alpha^2 + \frac{a_1^2 Q_u - a_1 a_2 a_6 Q_p - a_2^2 a_6^2 \theta^3}{a_1^2 a_6^2} \right) \wedge \left(\alpha^6 - \frac{a_1 Q_p + 2a_2 a_6 \theta^3}{a_1 a_6} \right) \\ &+ s\theta^1 \wedge \left(\alpha^6 - \frac{Q_p \theta^3}{a_6} - \frac{2a_2 \theta^3}{a_1} \right) + r\theta^2 \wedge \left(\alpha^6 - \frac{Q_p \theta^3}{a_6} - \frac{2a_2 \theta^3}{a_1} \right) \\ \pi^2 \wedge \pi^6 &= \alpha^2 \wedge \alpha^6 - \alpha^2 \wedge \left(\frac{Q_p \theta^3}{a_6} + \frac{2a_2 \theta^3}{a_1} \right) + \left(\frac{Q_u \theta^3}{a_6^2} - \frac{a_2}{a_1 a_6} Q_p \theta^3 - \frac{a_2^2}{a_1^2} \theta^3 \right) \wedge \alpha^6 \\ &+ s\theta^1 \wedge \frac{da_6}{a_6} - \frac{sQ_p}{a_6} \theta^1 \wedge \theta^3 - \frac{2sa_2}{a_1} \theta^1 \wedge \theta^3 + r\theta^2 \wedge \frac{da_6}{a_6} - r \frac{Q_p}{a_6} \theta^2 \wedge \theta^3 - 2r \frac{a_2}{a_1} \theta^2 \wedge \theta^3 \\ \pi^2 \wedge \pi^6 &= \left(\frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1 + \frac{a_2}{a_6 a_1} da_6 \right) \wedge \frac{da_6}{a_6} - \left(\frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1 + \frac{a_2}{a_6 a_1} da_6 \right) \wedge \left(\frac{Q_p \theta^3}{a_6} + \frac{2a_2 \theta^3}{a_1} \right) \\ &+ \frac{Q_u}{a_6^3} \theta^3 \wedge da_6 - \frac{a_2}{a_1 a_6^2} Q_p \theta^3 \wedge da_6 - \frac{a_2^2}{a_1^2 a_6} \theta^3 \wedge da_6 - \frac{s}{a_6} da_6 \wedge \theta^1 + \frac{sQ_p}{a_6} \theta^3 \wedge \theta^1 \\ &+ \frac{2sa_2}{a_1} \theta^3 \wedge \theta^1 - \frac{r}{a_6} da_6 \wedge \theta^2 - r \frac{Q_p}{a_6} \theta^2 \wedge \theta^3 - 2r \frac{a_2}{a_1} \theta^2 \wedge \theta^3 \\ \pi^2 \wedge \pi^6 &= \frac{da_2 da_6}{a_1 a_6} - \frac{a_2}{a_1^2 a_6} da_1 \wedge da_6 - \frac{Q_p}{a_1 a_6} da_2 \wedge \theta^3 - \frac{2a_2}{a_1^2} da_2 \wedge \theta^3 + \frac{a_2 Q_p}{a_1^2 a_6} da_1 \wedge \theta^3 \\ &+ \frac{2a_2^2}{a_1^3} da_1 \wedge \theta^3 - \frac{2a_2^2}{a_1^2 a_6} da_6 \wedge \theta^3 - \frac{Q_u}{a_6^3} da_6 \wedge \theta^3 + \frac{a_2}{a_1 a_6^2} Q_p da_6 \wedge \theta^3 + \frac{a_2^2}{a_1^2 a_6} da_6 \wedge \theta^3 \\ &- \frac{a_2}{a_1 a_6^2} Q_p da_6 \wedge \theta^3 - s \frac{da_6}{a_6} \theta^1 + s \frac{Q_p}{a_6} \theta^3 \wedge \theta^1 + 2s \frac{a_2}{a_1} \theta^3 \wedge \theta^1 - r \frac{da_6}{a_6} \wedge \theta^2 - \frac{rQ_p}{a_6} \theta^2 \wedge \theta^3 \\ &- 2r \frac{a_2}{a_1} \theta^2 \wedge \theta^3. \end{aligned}$$

Then,

$$\begin{aligned} \pi^2 \wedge \pi^6 &= \frac{da_2 \wedge da_6}{a_1 a_6} - \frac{a_2}{a_1^2 a_6} da_1 \wedge da_6 + \Delta + \left[-s \frac{da_6}{a_6} + \frac{sQ_p}{a_6} \theta^3 + \frac{2sa_2}{a_1} \theta^3 \right] \wedge \theta^1 \\ &+ \left(-r \frac{da_6}{a_6} \right) \wedge \theta^2 - \left(\frac{rQ_p}{a_6} + \frac{2ra_2}{a_1} \right) \theta^2 \wedge \theta^3. \end{aligned}$$

Finally,

$$\begin{aligned}
d\pi^2 &= \pi^2 \wedge \pi^6 - \left(-s \frac{da_6}{a_6} + \frac{sQ_p}{a_6} \theta^3 + \frac{2sa_2}{a_1} \theta^3 \right) \wedge \theta^1 + \frac{rda_6}{a_6} \wedge \theta^2 + \left(\frac{rQ_p}{a_6} + \frac{2ra_2}{a_1} \right) \theta^2 \wedge \theta^3 \\
&\quad + \left(-\frac{sa_2}{a_1} \theta^3 - \frac{Q_{uu}}{a_1 a_6^2} \theta^3 + \frac{2a_2 Q_{pu}}{a_1^2 a_6} \theta^3 - \frac{a_2^2 Q_{pp}}{a_1^3} \theta^3 + s \frac{da_1}{a_1} \right) \wedge \theta^1 + \left(-\frac{a_2 Q_{pp}}{a_1^2} + \frac{Q_{pu}}{a_1 a_6} \right) \theta^2 \wedge \theta^3 \\
&\quad - s \theta^2 \wedge \theta^3 + R \\
d\pi^2 &= \pi^2 \wedge \pi^6 + \left(s \frac{da_6}{a_6} - \frac{sQ_p}{a_6} \theta^3 - \frac{2sa_2}{a_1} \theta^3 - \frac{sa_2}{a_1} - \frac{Q_{uu}}{a_1 a_6^2} \theta^3 + \frac{2a_2 Q_{pu}}{a_1^2 a_6} \theta^3 - \frac{a_2^2}{a_1^3} Q_{pp} \theta^3 \right. \\
&\quad \left. + \frac{sda_1}{a_1} \right) \wedge \theta^1 + \left(r \frac{da_6}{a_6} - r \frac{da_6}{a_6} + r \frac{da_1}{a_1} \right) \wedge \theta^2 + \left(\frac{rQ_p}{a_6} + \frac{2ra_2}{a_1} - \frac{a_2}{a_1^2} Q_{pp} - s - \frac{ra_2}{a_1} \right. \\
&\quad \left. + \frac{Q_{pu}}{a_1 a_6} - \frac{rQ_p}{a_6} \right) \theta^2 \wedge \theta^3 + r \left(\frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1 + \frac{a_2}{a_6 a_1} da_6 \right) \wedge \theta^1 + r \left(\frac{Q_u}{a_6^2} \theta^3 \wedge \theta^1 \right. \\
&\quad \left. - \frac{a_2}{a_1 a_6} Q_p \theta^3 \wedge \theta^1 - \frac{a_2^2}{a_1^2} \theta^3 \wedge \theta^1 \right).
\end{aligned}$$

Then,

$$d\pi^2 = \sigma \wedge \theta^1 + \rho \wedge \theta^2 - A \theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^6,$$

where

$$\begin{aligned}
\sigma &= \left(s \frac{da_6}{a_6} + s \frac{da_1}{a_1} + r \frac{da_2}{a_1} - \frac{ra_2}{a_1^2} da_1 + \frac{ra_2}{a_1 a_6} da_6 \right) + \left(-3 \frac{sa_2}{a_1} - s \frac{Q_p}{a_6} \right. \\
&\quad \left. - \frac{Q_{uu}}{a_1 a_6^2} + \frac{2a_2 Q_{pu}}{a_1^2 a_6} - \frac{a_2^2}{a_1^3} Q_{pp} + \frac{rQ_u}{a_6^2} - \frac{ra_2}{a_1 a_6} Q_p - \frac{ra_2^2}{a_1^2} \right) \theta^3, \\
A &= 2s + \frac{a_2 a_6 Q_{pp} - a_1 Q_{pu}}{a_1^2 a_6}.
\end{aligned}$$

Compute $d\pi^6$:

$$d\pi^6 = d\varpi^6$$

$$d\pi^6 = d\alpha^6 - d\left(\frac{a_1 Q_p + 2a_2 a_6}{a_1 a_6} \theta^3\right)$$

$$d\pi^6 = d\left(\frac{da_6}{a_6}\right) - \left[d\left(\frac{Q_p}{a_6} \theta^3\right) + d\left(\frac{2a_2}{a_1} \theta^3\right)\right]$$

$$d\pi^6 = -d\left(\frac{Q_p}{a_6}\right) \theta^3 - \frac{Q_p}{a_6} d\theta^3 - d\left(\frac{2a_2}{a_1}\right) \theta^3 - \frac{2a_2}{a_1} d\theta^3$$

$$d\pi^6 = -\frac{(dQ_p)a_6 - (da_6)Q_p}{a_6^2} \theta^3 - \frac{Q_p}{a_6} (\pi^6 \wedge \theta^3) - \frac{2(da_2)a_1 - 2(da_1)a_2}{a_1^2} \theta^3 - \frac{2a_2}{a_1} \pi^6 \wedge \theta^3$$

$$\begin{aligned} d\pi^6 = & -\frac{Q_{px}}{a_6} dx \wedge \theta^3 - \frac{Q_{pu}}{a_6} du \wedge \theta^3 - \frac{Q_{pp}}{a_6} dp \wedge \theta^3 + \frac{Q_p}{a_6^2} da_6 \wedge \theta^3 \\ & - \frac{Q_p}{a_6} \left(\alpha^6 - \frac{Q_p}{a_6} \theta^3 - \frac{2a_2}{a_1} \theta^3\right) \wedge \theta^3 - \frac{2da_2}{a_1} \wedge \theta^3 + \frac{2a_2}{a_1} da_1 \wedge \theta^3 \\ & - \frac{2a_2}{a_1} \left(\alpha^6 - \frac{Q_p}{a_6} \theta^3 - \frac{2a_2}{a_1} \theta^3\right) \wedge \theta^3 \end{aligned}$$

$$\begin{aligned} d\pi^6 = & -Q_{pu} du \wedge dx - Q_{pp} dp \wedge dx + \frac{Q_p}{a_6^2} da_6 \wedge \theta^3 - \frac{Q_p}{a_6^2} da_6 \wedge \theta^3 - \frac{2da_2}{a_1} \theta^3 \\ & + \frac{2a_2}{a_1^2} da_1 \theta^3 - \frac{2a_2}{a_1^2} da_6 \wedge \theta^3 \end{aligned}$$

$$d\pi^6 = -\frac{Q_{pu}}{a_1 a_6} \theta^1 \wedge \theta^3 - \frac{Q_{pp}}{a_1} \theta^2 \wedge \theta^3 + \frac{Q_{pp}}{a_1^2} a_2 \theta^1 \wedge \theta^3 + \left(-\frac{2da_2}{a_1} + \frac{2a_2}{a_1^2} da_1 - \frac{2a_2}{a_1^2} da_6\right) \wedge \theta^3$$

$$\begin{aligned} d\pi^6 = & A\theta^1 \wedge \theta^3 - 2s\theta^1 \wedge \theta^3 + \left(2r - \frac{Q_{pp}}{a_1^2}\right) \theta^2 \wedge \theta^3 - 2r\theta^2 \wedge \theta^3 \\ & + \left(-\frac{2da_2}{a_1} - \frac{2a_2}{a_1^2} da_1 - \frac{2a_2}{a_1^2} da_6\right) \wedge \theta^3. \end{aligned}$$

Then,

$$d\pi^6 = A\theta^1 \wedge \theta^3 + B\theta^2 \wedge \theta^3 - 2\pi^2 \wedge \theta^3,$$

where

$$B = 2r - \frac{Q_{pp}}{a_1^2}$$

Then we have

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2, \\ d\theta^3 &= \pi^6 \wedge \theta^3, \\ d\pi^1 &= \rho \wedge \theta^1 - \pi^2 \wedge \theta^3, \\ d\pi^2 &= \sigma \wedge \theta^1 + \rho \wedge \theta^2 - A\theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^6, \\ d\pi^6 &= A\theta^1 \wedge \theta^3 + B\theta^2 \wedge \theta^3 - 2\pi^2 \wedge \theta^3, \end{aligned}$$

where

$$\begin{aligned} \rho &= r \frac{da_1}{a_1} - (s + r \frac{a_2}{a_1})\theta^3, \\ \sigma &= \left(s \frac{da_6}{a_6} + s \frac{da_1}{a_1} + r \frac{da_2}{a_1} - \frac{ra_2}{a_1^2} da_1 + \frac{ra_2}{a_1 a_6} da_6 \right) + \left(-3 \frac{sa_2}{a_1} - s \frac{Q_p}{a_6} \right. \\ &\quad \left. - \frac{Q_{uu}}{a_1 a_6^2} + \frac{2a_2 Q_{pu}}{a_1^2 a_6} - \frac{a_2^2}{a_1^3} Q_{pp} + \frac{rQ_u}{a_6^2} - \frac{ra_2}{a_1 a_6} Q_p - \frac{ra_2^2}{a_1^2} \right) \theta^3, \\ A &= 2s + \frac{a_2 a_6 Q_{pp} - a_1 Q_{pu}}{a_1^2 a_6}, \\ B &= 2r - \frac{Q_{pp}}{a_1^2}. \end{aligned}$$

By the process of normalization we set $A = B = 0$, then

$$\begin{aligned} r &= \frac{Q_{pp}}{2a_1} \\ s &= \frac{a_1 Q_{pu} - a_2 a_6 Q_{pp}}{2a_1^2 a_6}. \end{aligned}$$

The parameter r, s of the group $\widetilde{G}_f^{(1)}$ are determined.

Then, for the lifted coframe (θ, π) the $\widetilde{G}_f^{(1)}$ -equivalence problem is reduced to the $\{e\}$ -equivalence problem.

Remark 10.8. We have an invariant coframe on $J^1 \times \widetilde{G}_f$. However, do not apply the theorem of equivalence for this coframe. Our equivalence problem is about θ and not the pair (θ, π) .

Therefore, we need to determine the group parameters a_1, a_2, a_6 of $\widetilde{G}_f$.

To do this, we use the structure equation of the coframe (θ, π) on $J^1 \times \widetilde{G}_f$ and extract information.

Etape 5 :

Determine the structure equations of the coframe (θ, π) on $J^1 \times \widetilde{G}_f$.

* For every $1 \leq i \leq 3$, $d\theta^i$ doesn't change.

* Compute the differentiate of the π^i 's.

Compute $d\pi^1$:

$$d\pi^1 = \left(d\varpi^1 + \frac{Q_{pp}}{2a_1} \theta^1 \right) = d\varpi^1 + d\left(\frac{Q_{pp}}{2a_1} \theta^1 \right) + \frac{Q_{pp}}{2a_1} d\theta^1$$

$$d\pi^1 = d\pi_{\text{previous}}^1 + d\left(\frac{Q_{pp}}{2a_1} \right) \wedge \theta^1$$

$$d\pi^1 = \rho \wedge \theta^1 - \pi^2 \wedge \theta^3 + \frac{dQ_{pp}}{2a_1} \wedge \theta^1 - \frac{Q_{pp}}{2a_1} da_1 \wedge \theta^1,$$

where $\rho = \rho(r = \frac{Q_{pp}}{2a_1})$.

Calculus of $\frac{dQ_{pp}}{2a_1} \wedge \theta^1$:

$$\frac{dQ_{pp}}{2a_1} \wedge \theta^1 = \frac{1}{2a_1} \left(Q_{ppx} dx + Q_{ppu} du + Q_{ppp} dp \right) \wedge \theta^1$$

$$\begin{aligned} \frac{dQ_{pp}}{2a_1} \wedge \theta^1 &= \frac{1}{2a_1} Q_{ppx} dx \wedge a_1 (du - p dx) + \frac{1}{2a_1} Q_{ppu} du \wedge a_1 (du - p dx) \\ &\quad + \frac{1}{2a_1} Q_{ppp} dp \wedge a_1 (du - p dx) \end{aligned}$$

$$\frac{dQ_{pp}}{2a_1} \wedge \theta^1 = \left(-\frac{1}{2} Q_{ppx} - \frac{1}{2} p Q_{ppu} \right) du \wedge dx + \frac{1}{2} Q_{ppp} dp \wedge du - \frac{1}{2} Q_{ppp} p dp \wedge dx$$

$$\begin{aligned} \frac{dQ_{pp}}{2a_1} \wedge \theta^1 &= \left(-\frac{Q_{ppx}}{2} - \frac{p}{2} Q_{ppu} \right) \frac{\theta^1 \wedge \theta^3}{a_1 a_6} + \frac{1}{2} Q_{ppp} \left(-\frac{a_6}{a_1^2} \theta^1 \wedge \theta^2 - \left(\frac{Q}{a_1 a_6} + \frac{a_2}{a_1^2} p \right) \theta^1 \wedge \theta^3 \right. \\ &\quad \left. + \frac{p}{a_1} \theta^2 \wedge \theta^3 \right) - \frac{1}{2} p Q_{ppp} \left(\frac{\theta^1 \wedge \theta^3}{a_1} - \frac{a_2}{a_1^2} \theta^1 \wedge \theta^3 \right) \end{aligned}$$

$$\begin{aligned} \frac{dQ_{pp}}{2a_1} \wedge \theta^1 &= \left(-\frac{Q_{ppx}}{2a_1 a_6} - \frac{p Q_{ppu}}{2a_1 a_6} - \frac{Q_{ppp} Q}{2a_1 a_6} - \frac{a_2 Q_{ppp}}{2a_1^2} p + \frac{a_2 Q_{ppp}}{2a_1^2} p \right) \theta^1 \wedge \theta^3 \\ &\quad + \left(-\frac{a_6 Q_{ppp}}{2a_1^2} \right) \theta^1 \wedge \theta^2 + \left(\frac{Q_{ppp}}{2a_1} p - \frac{Q_{ppp}}{2a_1} p \right) \theta^2 \wedge \theta^3. \end{aligned}$$

Then

$$\boxed{\frac{dQ_{pp}}{2a_1} \wedge \theta^1 = \left(-\frac{Q_{ppx}}{2a_1 a_6} - \frac{p Q_{ppu}}{2a_1 a_6} - \frac{Q_{ppp} Q}{2a_1 a_6} \right) \theta^1 \wedge \theta^3 + \left(-\frac{a_6 Q_{ppp}}{2a_1^2} \right) \theta^1 \wedge \theta^2.}$$

Then

$$d\pi^1 = -\frac{a_6}{2a_1^2} Q_{ppp} \theta^1 \wedge \theta^2 + \left(\frac{Q_{pu}}{2a_1 a_6} - \frac{Q_{ppx}}{2a_1 a_6} - p \frac{Q_{ppu}}{2a_1 a_6} - \frac{Q_{ppp} Q}{2a_1 a_6} \right) \theta^1 \wedge \theta^3 - \pi^2 \wedge \theta^3$$

$$\boxed{d\pi^1 = -J_1 \theta^1 \wedge \theta^2 + J_2 \theta^1 \wedge \theta^3 - \pi^2 \wedge \theta^3,}$$

where

$$J_1 = \frac{a_6}{2a_1^2} Q_{ppp}$$

$$J_2 = \frac{1}{2a_1 a_6} (Q_{pu} - \hat{D}_x Q_{pp}) = \frac{Q_{pu}}{2a_1 a_6} - \frac{Q_{ppx}}{2a_1 a_6} - p \frac{Q_{ppu}}{2a_1 a_6} - \frac{Q_{ppp} Q}{2a_1 a_6}.$$

where

$$\hat{D}_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + Q(x, u, p) \frac{\partial}{\partial p}.$$

Compute $d\pi^2$:

$$\begin{aligned} d\pi^2 &= d(\varpi^2 + s\theta^1 + r\theta^2) \\ d\pi^2 &= d\varpi^2 + ds \wedge \theta^1 + sd\theta^1 + dr \wedge \theta^2 + rd\theta^2 \\ d\pi^2 &= d\pi_{\text{previous}}^2 + ds \wedge \theta^1 + dr \wedge \theta^2 \\ d\pi^2 &= \sigma \wedge \theta^1 + \rho \wedge \theta^2 + \pi^2 \wedge \pi^6 + ds \wedge \theta^1 + d\left(\frac{Q_{pp}}{2a_1}\right) \wedge \theta^2 \\ d\pi^2 &= \left[\left(\frac{Q_{pu}}{2a_1 a_6^2} da_6 + \frac{Q_{pu}}{2a_1^2 a_6} da_1 - \frac{a_2}{a_1^3} Q_{pp} da_1 + \frac{Q_{pp}}{2a_1^2} da_2 \right) \right. \\ &\quad \left. + \left(\frac{a_2 Q_{pu}}{2a_1^2 a_6} - \frac{Q_{pu} Q_p}{2a_1 a_6^2} - \frac{Q_{uu}}{a_1 a_6^2} + \frac{Q_{pp} Q_u}{2a_1 a_6^2} \right) \theta^3 \right] \wedge \theta^1 \\ &\quad + \frac{Q_{pp}}{2a_1^2} da_1 \wedge \theta^2 - \frac{Q_{pu}}{2a_1 a_6} \theta^3 \wedge \theta^2 + \pi^2 \wedge \pi^6 + ds \wedge \theta^1 \\ &\quad + d\left(\frac{Q_{pp}}{2a_1}\right) \wedge \theta^2 \end{aligned}$$

Calculus of $ds \wedge \theta^1$:

$$\begin{aligned} ds \wedge \theta^1 &= d\left(\frac{Q_{pu}}{2a_1 a_6}\right) \wedge \theta^1 - d\left(\frac{a_2}{2a_1^2} Q_{pp}\right) \wedge \theta^1 \\ ds \wedge \theta^1 &= \frac{(dQ_{pu})(2a_1 a_6) - 2[(da_1)a_6 + (da_6)a_1]Q_{pu}}{4a_1^2 a_6^2} \wedge \theta^1 - d\left(\frac{a_2}{2a_1^2}\right) \wedge Q_{pp} \theta^1 \\ &\quad - \frac{a_2}{2a_1^2} dQ_{pp} \wedge \theta^1 \\ ds \wedge \theta^1 &= \frac{dQ_{pu}}{2a_1 a_6} \wedge \theta^1 - \frac{Q_{pu}}{2a_1^2 a_6} da_6 \wedge \theta^1 - \frac{Q_{pu}}{2a_1 a_6^2} da_6 \wedge \theta^1 \\ &\quad - \left(\frac{(da_2)(2a_1^2) - 4(da_1)a_1 a_2}{4a_1^4} \right) \wedge Q_{pp} \theta^1 - \frac{a_2}{2a_1^2} dQ_{pp} \wedge \theta^1 \\ ds \wedge \theta^1 &= \delta - \frac{Q_{pu}}{2a_1^2 a_6} da_1 \wedge \theta^1 - \frac{Q_{pu}}{2a_1 a_6^2} da_6 \wedge \theta^1 - \frac{Q_{pp}}{2a_1^2} da_2 \wedge \theta^1 \\ &\quad + \frac{a_2}{a_1^3} Q_{pp} da_1 \wedge \theta^1 + \mu, \end{aligned}$$

where

$$\delta = \frac{dQ_{pu}}{2a_1a_6} \wedge \theta^1,$$

$$\mu = -\frac{a_2}{2a_1^2} dQ_{pp} \wedge \theta^1.$$

$$\delta = \frac{1}{2a_1a_6} (Q_{pux}dx + Q_{puu}du + Q_{ppu}dp) \wedge a_1(du - pdx)$$

$$\delta = -\frac{Q_{ppu}}{2a_1^2} \theta^1 \wedge \theta^2 + \left(-\frac{Q_{pux}}{2a_1a_6^2} - \frac{pQ_{puu}}{2a_1a_6^2} - \frac{Q_{ppu}}{2a_1a_6^2} \right) \theta^1 \wedge \theta^3,$$

$$\mu = \left(\frac{a_2}{2a_1^2a_6} Q_{ppx} + \frac{a_2p}{2a_1^2a_6} Q_{ppu} + \frac{a_2}{2a_1^2a_6} Q_{ppp} \right) \theta^1 \wedge \theta^3 + \frac{a_2a_6}{2a_1^3} Q_{ppp} \theta^1 \wedge \theta^2.$$

Then

$$d\pi^2 = \left(-\frac{a_2}{2a_1^2a_6} Q_{pu} + \frac{Q_{pu}Q_p}{2a_1a_6^2} + \frac{Q_{uu}}{a_1a_6^2} - \frac{Q_{pp}Q_u}{2a_1a_6^2} \right) \theta^1 \wedge \theta^3$$

$$+ \frac{Q_{pp}}{2a_1^2} da_1 \wedge \theta^2 - \frac{Q_{pu}}{2a_1a_6} \theta^3 \wedge \theta^2 + \pi^2 \wedge \pi^6 - \frac{Q_{ppu}}{2a_1^2} \theta^1 \wedge \theta^2 + \left(-\frac{Q_{pux}}{2a_1a_6^2} - p\frac{Q_{puu}}{2a_1a_6^2} \right.$$

$$\left. - \frac{Q_{ppu}}{2a_1a_6^2} \right) \theta^1 \wedge \theta^3 + \left(\frac{a_2Q_{ppx}}{2a_1^2a_6} + \frac{a_2pQ_{ppu}}{2a_1^2a_6} + \frac{a_2Q_{ppp}}{2a_1^2a_6} \right) \theta^1 \wedge \theta^3$$

$$+ \frac{a_2a_6}{2a_1^3} Q_{ppp} \theta^1 \wedge \theta^3 + d\left(\frac{Q_{pp}}{2a_1} \right) \wedge \theta^2.$$

Calculus of $d\left(\frac{Q_{pp}}{2a_1} \right) \wedge \theta^2$:

After a long calculus, we find

$$d\left(\frac{Q_{pp}}{2a_1} \right) \wedge \theta^2 = -\frac{Q_{ppx}}{2a_1a_6} \theta^2 \wedge \theta^3 + \frac{Q_{ppu}}{2a_1^2} \theta^1 \wedge \theta^2 - \frac{p}{2a_1a_6} Q_{ppu} \theta^2 \wedge \theta^3$$

$$- \frac{a_2a_6}{2a_1^3} Q_{ppp} \theta^1 \wedge \theta^2 - \frac{Q_{ppp}}{2a_1a_6} \theta \wedge \theta^3 - \frac{Q_{pp}}{2a_1^2} da_1 \wedge \theta^2.$$

Then

$$d\pi^2 = \left[\frac{1}{a_1a_6^2} \left(2Q_{uu} + Q_pQ_{pu} - Q_{pp}Q_u - Q_{pux} - pQ_{puu} \right. \right.$$

$$\left. - Q_{ppu} \right) - \frac{a_2}{a_1} \left(\frac{Q_{pu}}{2a_1a_6} - \frac{Q_{ppx}}{2a_1a_6} - p\frac{Q_{ppu}}{2a_1a_6} - \frac{Q_{ppp}}{2a_1a_6} \right) \right] \theta^1 \wedge \theta^3$$

$$+ \left(\frac{Q_{pu}}{2a_1a_6} - \frac{Q_{ppx}}{2a_1a_6} - p\frac{Q_{ppu}}{2a_1a_6} - \frac{Q_{ppp}}{2a_1a_6} \right) \theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^6$$

By

$$J_3 = \frac{1}{a_1a_6^2} \left(2Q_{uu} + Q_pQ_{pu} - Q_{pp}Q_u - Q_{pux} - pQ_{puu} \right.$$

$$\left. - Q_{ppu} \right) - \frac{a_2}{a_1} \left(\frac{Q_{pu}}{2a_1a_6} - \frac{Q_{ppx}}{2a_1a_6} - p\frac{Q_{ppu}}{2a_1a_6} - \frac{Q_{ppp}}{2a_1a_6} \right)$$

Finally, we obtain the structure equations of the prolonged coframe $(*)$:

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2, \\ d\theta^3 &= \pi^6 \wedge \theta^3, \\ d\pi^1 &= -J_1\theta^1 \wedge \theta^2 + J_2\theta^1 \wedge \theta^3 - \pi^2 \wedge \theta^3, \\ d\pi^2 &= J_3\theta^1 \wedge \theta^3 + J_2\theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^6, \\ d\pi^6 &= -2\pi^2 \wedge \theta^3, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \frac{a_6}{2a_1^2} Q_{ppp}, \\ J_2 &= \frac{1}{2a_1 a_6} (Q_{pu} - \hat{D}_x Q_{pp}), \\ J_3 &= \frac{1}{2a_1 a_6^2} (2Q_{uu} + Q_p Q_{pu} - \hat{D}_x Q_{pu} - Q_u Q_{pp}) - \frac{a_2}{a_1} J_2, \end{aligned}$$

where

$$\hat{D}_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + Q(x, u, p) \frac{\partial}{\partial p}.$$

Further reduction of the original group parameter depends on the precise value of the invariant J_1, J_2, J_3 . In fact, for $J_1 = J_2 = J_3 = 0$ we obtain the following theorem.

Theorem 10.9. *Let $u_{xx} = Q(x, u, u_x)$ be a second order ordinary differential equation. The following conditions are equivalent :*

- (1) *The equation admits a six-dimensional fiber-preserving symmetry group.*
- (2) *The equation can be mapped to $u_{xx} = 0$ by a fiber-preserving transformation.*
- (3) *Q satisfies $Q : (x, u, u_x) \mapsto \frac{1}{2}M_u u_x^2 + M_x u_x + N$ with $M_{xx} + (NM_u)_u - M_x M_{xu} - 2N_{uu} = 0$, where $M, N : E \rightarrow \mathbb{R}$.*

Proof. Assume (3), then $J_1 = J_2 = J_3 = 0$.

By computing, we see that it is equivalent.

In this case, the prolonged coframe has rank zero, so the coframe is equivalent under fiber-preserving transformation to the dual basis of Maurer-Cartan forms on G with $6 = \dim(G)$.

But we know that, for $u_{xx} = 0$ we obtain by the Cartan method ($Q = 0$), the prolonged coframe as rank 0.

Then, we have a fiber-preserving equivalence between the prolonged coframes (θ, π) and $(\theta_{\{u_{xx}=0\}}, \pi_{\{u_{xx}=0\}})$.

By the theorem of the indeterminate case, equivalently, we deduce that θ and $\theta_{\{u_{xx}=0\}}$ are equivalent under fiber preserving transformation, which is equivalent to, that is the equation is equivalent to $u_{xx} = 0$ by a fiber-preserving transformation.

Moreover by the corollary of the theorem of the indeterminate case, we deduce equivalently that the fiber-preserving group of θ and those of $\theta_{\{u_{xx}=0\}}$ are isomorphic and the

fiber-preserving symmetry group of $u_{xx} = 0$ is 6-dimensional. □

Remark 10.10. If the invariants functions J_1, J_2, J_3 are not all zero, then there are various possible reductions, but in all cases, the equivalence problem reduces back to an invariant coframe on the base manifold J^1 .

Consequently, a second order ordinary differential equation admits a fiber-preserving symmetry-group of dimension either 0, 1, 2, 3 or 6.
(See "Classification of second-order ordinary differential equations admitting Lie groups of fiber-preserving" of Hsu, L. and Kamran, N.).

Example 10.11. Assume $J_1 J_2 \neq 0, J_3 = 0$.

By these assumption, we find expression for a_1, a_2, a_6 so substituting these values into the original lifted coframe θ results in an invariant coframe on J^1 , with structure group $\{e\}$ and structure equations of the form

$$\begin{aligned} d\theta^1 &= (I_1 + I_2 - I_3)\theta^1 \wedge \theta^2 - I_1\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^3, \\ d\theta^2 &= -2I_4\theta^1 \wedge \theta^2 + I_5\theta^1 \wedge \theta^3 + I_3\theta^2 \wedge \theta^3, \\ d\theta^3 &= I_4\theta^1 \wedge \theta^3 + I_2\theta^2 \wedge \theta^3, \end{aligned}$$

where the structure invariant $I_1, \dots, I_5$ are complicated expressions in Q and its derivatives.

Here, we can apply the equivalence theorem for invariant coframes !

In particular since $\dim(J^1) = 3$, the equation admits a three dimensional symmetry group if and only if all five structure functions are constant.

Moreover, we mention the equivalence problem of second order ordinary differential equations under point transformation, which by the same way, give the following theorem.

Theorem 10.12. Let $u_{xx} = Q(x, u, u_x)$ be a second order ordinary differential equation. The following conditions are equivalent :

- (1) The equation admits a eight-dimensional point symmetry group.
- (2) The equation can be mapped to $u_{xx} = 0$ by a point transformation.
- (3) Q satisfies $Q : (x, u, u_x) \mapsto Ku_x^3 + 3Lu_x^2 + 3Mu_x + N$ with $K_{xx} - 2L_{xu} + M_{uu} - NK_u - 2KN_u = 3(KM)_x + 3LM_u - 6LL_x = 0$, and $L_{xx} - 2M_{xu} + N_{uu} + KN_x + 2NK_x - 3(LN)_u - 3ML_x + 6MM_u = 0$ where $K, L, M, N : E \rightarrow \mathbb{R}$.

11. Summary

Now, we summarize the Cartan's method.

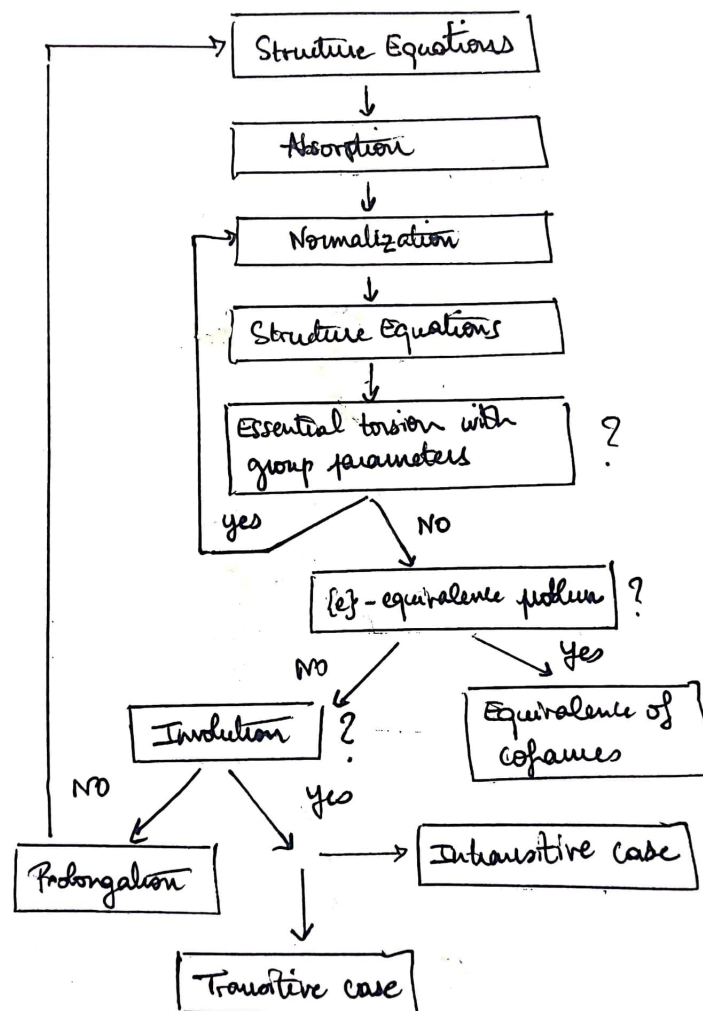
Let an equivalence problem describes in terms of one-forms $\omega, \bar{\omega}$ on a m -dimensional manifold, with a structure Lie group G .

$$\bar{\omega}, \omega \text{ are } G\text{-equivalent} \iff \exists (g, \bar{g}) : M \times \bar{M} \rightarrow G \mid \bar{g} \cdot \bar{\omega} = g \cdot \omega$$

We want to specified the entries of g and $\bar{g}$.

Then, set $\theta = g \cdot \omega$ and $\bar{\theta} = \bar{g} \cdot \bar{\omega}$ the lifted coframes associated to ω and $\bar{\omega}$.

We use the Cartan's algorithm summarized in this schema.



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Finally, it remains the fact if for an equivalence problem given, the Cartan's algorithm will end.

Cartan noted that in lot of examples, the answer is true.

More formally, **Malgrange** showed the ending of the Cartan's algorithm in the paper "Sur le problème d'équivalence de Cartan" Ann. Fac. Toulouse, XXVI - 5, 1977, pages

1087 – 1136.

Remark 11.1. Note that this algorithm doesn't give us the synthesis of the equivalence problem ; in the case where there is an equivalence problem, there is no information for a such Φ which allow the equivalence problem.

Equivalence problem of second order ordinary differential equation by Cartan's connections

1. Second order ordinary equations and pairs of direction fields

Now, we give the equivalence problem for second order ordinary differential equations under point transformations to $y'' = 0$ by the Cartan connections.

Give a definition of the jet space in terms of tangent space.

Definition 1.1. The jet space J^1 of $\mathbb{R}^2$, denoted $J^1(\mathbb{R}^2)$, is the set

$$J^1(\mathbb{R}^2) = \{l_p \mid l_p \text{ is a one-dimensional subspace in } T_p\mathbb{R}^2\}$$

Let $\pi : J^1(\mathbb{R}^2) \rightarrow \mathbb{R}^2$ denote the natural projection which takes l_p to the corresponding point p of the plan $\mathbb{R}^2$.

Let (x, y, z_1) and (x, y, z_2) two coordinate systems in $J^1(\mathbb{R}^2)$. For $p = (x, y) \in \mathbb{R}^2$, we set $l_p = \langle \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \rangle$.

Let

$$U_1 = \{l_p = \langle \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \rangle \mid \alpha \neq 0\}$$

$$U_2 = \{l_p = \langle \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \rangle \mid \beta \neq 0\}$$

Then $(x, y, z_1 = \frac{\beta}{\alpha})$ and $(x, y, z_2 = \frac{\alpha}{\beta})$ will be respectively coordinates in U_1 and U_2 .

If we identify $T_p\mathbb{R}^2$ with $\mathbb{R}^2$, then U_1 will contain all direction which are not parallel to the y -axis, while U_2 will consist of all those which are not parallel to the x -axis.

The transition function from U_1 to U_2 is

$$\varphi_{12} : (x, y, z_1) \mapsto (x, y, \frac{1}{z_1})$$

These coordinate charts make $J^1(\mathbb{R}^2)$ into a three-dimensional smooth manifold. Then we note (x, y, z) the coordinate system of $J^1(\mathbb{R}^2)$.

Definition 1.2. Let N be a 1-dimensional submanifold in the plane. The prolongation $N^{(1)}$ of N is a curve in $J^1(\mathbb{R}^2)$ which has the form

$$N^{(1)} = \{T_p N \mid p \in N\}$$

Example 1.3. If N is the graph of some function $x \mapsto y(x)$; $N = \{(x, y(x)) \mid x \in \mathbb{R}\}$ then

$$N^{(1)} = \{(x, y(x), y'(x)) \mid x \in \mathbb{R}\}$$

The tangent space $T_{(x, y(x), y'(x))} N^{(1)}$ is generated by

$$\frac{\partial}{\partial x} + y'(x) \frac{\partial}{\partial y} + y''(x) \frac{\partial}{\partial z} = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + y''(x) \frac{\partial}{\partial z}$$

Definition 1.4. The contact distribution C on $J^1(\mathbb{R}^2)$ is the family

$$C = \{C_q \mid q \in J^1(\mathbb{R}^2)\}$$

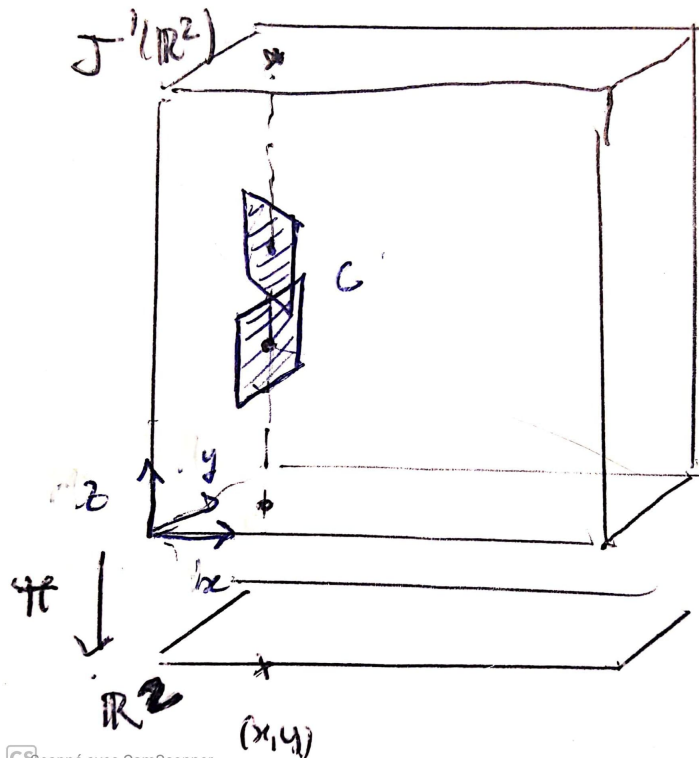
and

$$C_q = \{\langle T_q N^{(1)} \rangle \mid N^{(1)} \ni q, N \text{ one-dimensional submanifold}\}$$

Remark 1.5. In local coordinates, for $q \in J^1(\mathbb{R}^2)$

$$C_q = \left\langle \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right\rangle = \ker(dy - zdx)$$

$$\ker d\pi_q \subset C_q$$



Definition 1.6. A map $\varphi : M^m \rightarrow N^n$ has maximal rank if for all $x \in M$, $\text{rank}(d\varphi_x) = \min\{m, n\}$.

Lemma 1.7. If M is a curve in $J^1(\mathbb{R}^2)$ tangent to the contact distribution C and such that $\pi|_M : M \rightarrow \mathbb{R}^2$ has maximal rank, then locally, M is the prolongation of some curve in the plane.

Proof. Without loss of generality we may assume that the projection of M onto $\mathbb{R}^2$, locally, is the graph of some function y .

Then

$$M = \{(x, y(x), y'(x)) \mid x \in \mathbb{R}\}$$

But the vector $\frac{\partial}{\partial x} + y'(x)\frac{\partial}{\partial y} + z'(x)\frac{\partial}{\partial z}$ lies in $C_{(x, y(x), z'(x))}$ if and only if $z = y'$ and M is the prolongation of the curve $N = \{(x, y(x)) \mid x \in \mathbb{R}\}$. □

Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ an arbitrary diffeomorphism of the plane.

Definition 1.8. The diffeomorphism $\varphi^{(1)} : J^1(\mathbb{R}^2) \rightarrow J^1(\mathbb{R}^2)$ defined by

$$\varphi^{(1)} : l_p \mapsto (d\varphi)_p(l_p)$$

is said to be the prolongation of φ .

Let

$$y'' = Q(x, y, y') \tag{1.9}$$

be an arbitrary second order ordinary differential equation.

Consider the direction field E in the chart U_1 with coordinate (x, y, z) defined by

$$E_{(x, y, z)} = \left\langle \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + Q(x, y, z) \frac{\partial}{\partial z} \right\rangle$$

Lemma 1.10. Assume that N is the graph of the function $x \mapsto y(x)$.

y is a solution of the equation (1.7) if and only if $N^{(1)}$ is an integral curve of the direction field E .

Proof. Suppose y is a solution of (1.7). Then $N^{(1)} = \{(x, y(x), y'(x)) \mid x \in \mathbb{R}\}$, and the tangent space to $N^{(1)}$ is generated by

$$\frac{\partial}{\partial x} + y'(x) \frac{\partial}{\partial y} + y''(x) \frac{\partial}{\partial z} = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + Q(x, y, y') \frac{\partial}{\partial z}$$

Now since the coordinate $z = y'$ for all point on $N^{(1)}$, we see that $N^{(1)}$ is an integral manifold for E .

The converse is proved in a similar manner. □

Note that E is contained in the contact distribution C and that the projection of E onto $\mathbb{R}^2$ by means of π has maximal rank.

Any integral curve of the direction field E is therefor a lift of the graph of a solution of equation (1.7).

Then, we have the following theorem.

Theorem 1.11. *There is a one-to-one correspondence between the solution of equation (1.7) and the integral curves of the direction field E ; the graphs of solutions are projections of the integral curves, while the integral curves are of the graphs of the solutions.*

Let us now consider an arbitrary direction field E in $J^1(\mathbb{R}^2)$ such

- (1) E is contained in the contact distribution C .
- (2) At each point $q \in J^1(\mathbb{R}^2)$, E_q does not coincide with the vertical field $V_q = \ker(d\pi_q)$.

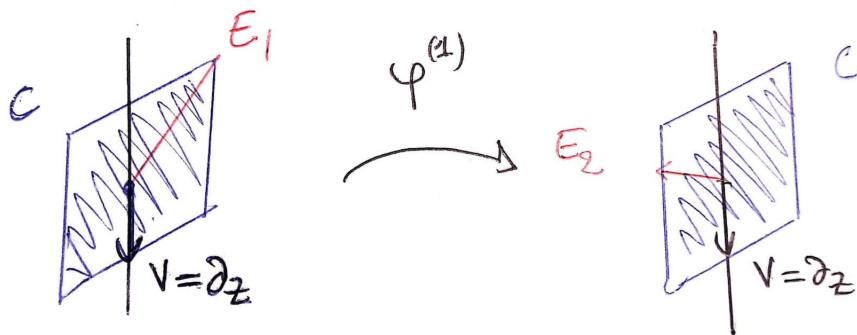
Remark 1.12. E may be seen as a second order ordinary differential equation whose solution are curves in the plane : a curve $N \subset \mathbb{R}^2$ is by definition a solution of this equation if $N^{(1)}$ is an integral curve of the vector field E .

Thus, geometrically, the second order ordinary differential equations may be identified with the direction fields in $J^1(\mathbb{R}^2)$ satisfying (1) and (2).

Finally, we define the equivalence problem of second order ordinary differential equations under point transformations.

Definition 1.13. Let E_1, E_2 be two direction fields contained in the contact distribution C corresponding to two second order ordinary differential equations.

We say that these equations are locally equivalent if there exists a local diffeomorphism φ of the plane such that $\varphi^{(1)}(E_1) = E_2$.



2. Cartan connections and pairs of direction fields

Let $G = SL(3)$ the Lie group which is the symmetry group of $y'' = 0$ under point transformations (already seen in the chapter "Symmetries of differential equations").

Let H the group of all upper-triangular matrices in G , and $M_0 = G/H$.
 H is a sub Lie group of G .

The corresponding Lie algebras has the form

$$\mathfrak{g} = \mathfrak{sl}(3), \quad \mathfrak{h} = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \mid x_{11} + x_{22} + x_{33} = 0 \right\}$$

Any differential form ω with values in $\mathfrak{g}$ can be uniquely written in the form

$$\omega = (\omega_{i,j})_{1 \leq i,j \leq 3}, \quad \omega_{11} + \omega_{22} + \omega_{33} = 0$$

We recall an important definition.

Definition 2.1. The curvature function of a Cartan connection $\omega : TP \rightarrow \mathfrak{g}$ is the map $c \in \mathcal{C}^\infty(P, \mathcal{L}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}))$ such that

$$c(p) : \begin{cases} \mathfrak{g}/\mathfrak{h} \wedge \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g} \\ (x' + \mathfrak{h}, x'' + \mathfrak{h}) \longmapsto \Omega(p)(\widehat{x'}_{|p}, \widehat{x''}_{|p}) \end{cases}$$

Definition 2.2. Let (E_1, E_2) be a pair of directions fields in $J^1(\mathbb{R}^2)$ and P a principal fiber bundle with Lie group G .

We say that a Cartan connection $\omega : TP \rightarrow \mathfrak{g}$ is associated with the pair (E_1, E_2) if there exists $s : \mathbb{R}^3 \rightarrow P$ a section such that

$$\begin{aligned} E_1 &= \langle s^* \omega_{21}, s^* \omega_{31} \rangle^\perp \\ E_2 &= \langle s^* \omega_{31}, s^* \omega_{32} \rangle^\perp \end{aligned}$$

We will prove the following important theorem.

Theorem 2.3. Let E a direction field in $J^1(\mathbb{R}^2)$ corresponding to a second order ordinary differential equation $y'' = Q(x, y, y')$ and $V = \frac{\partial}{\partial z}$ the vertical vector field.

Up to isomorphism there exist a unique principal fiber bundle P , and a unique Cartan connection ω on P satisfying the following conditions :

- (1) ω is associated with the pair (E, V) .
- (2) For all $p \in P$, $c(p) \in W$.

where W is generated by

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

with $(a, b, c, d) \in \mathbb{R}^4$.

Remark 2.4.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{h}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{h}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{h}.$$

is a basis of $\mathfrak{g}/\mathfrak{h}$.

Remark 2.5. W is a G -invariant subset of $\mathcal{L}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$.

Indeed :

Let

$$g = \begin{pmatrix} x & t & v \\ 0 & y & u \\ 0 & 0 & z \end{pmatrix} \in G,$$

where $xyz = 1$.

Let $\varphi \in W$.

Then by computation,

$$\begin{aligned} g \cdot \varphi : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto \frac{x^3}{y^2 z^2} \begin{pmatrix} 0 & az & -au + by \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ g \cdot \varphi : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto \frac{xy}{z^3} \begin{pmatrix} 0 & 0 & cx + dt \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}, \\ g \cdot \varphi : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{h} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{h} &\mapsto 0_3. \end{aligned}$$

Then $g \circ \varphi \in W$.

Proof. Proceeding by analysis,

We begin with the local case.

Assume $P = \mathbb{R}^3 \times G$.

Let $s : \mathbb{R}^3 \rightarrow P$ the trivial section given by $a \mapsto (a, e)$ with e the trivial element of G .

Let ω a Cartan connection on P with values in $\mathfrak{g}$ and its curvature form Ω .

Let $\tilde{\omega} = s^*\omega$ a form on $\mathbb{R}^3$ with values in $\mathfrak{g}$.

Note that $\tilde{\omega}$ uniquely determines ω .

Indeed :

For point $p = (a, e) \in P$, with $a \in \mathbb{R}^3$, we have

$$\omega(p)|_{T_a\mathbb{R}^3} = \tilde{\omega}(a), \quad \omega(p)|_{T_eG} = id_{\mathfrak{g}},$$

and from the definition of Cartan connection it follows that for any point $p = (a, g)$,

$$\omega(p) = (Ad(g)) \circ \omega(a, e) \circ d(L_{g^{-1}})_{(a,e)}$$

Let $\tilde{\Omega} = s^*\Omega$ and then in just the same way, the form $\tilde{\Omega}$ uniquely determines the curvature form Ω .

Condition (1) of the theorem is equivalent to the condition (1')

$$\begin{aligned} \tilde{\omega}_{21} &= \alpha dx + \delta(dy - zdx), \\ \tilde{\omega}_{31} &= \gamma(dy - zdx), \\ \tilde{\omega}_{32} &= \lambda(dy - zdx) + \mu(dz - Qdx). \end{aligned}$$

Indeed :

For

$$\begin{aligned} \tilde{\omega}_{21} &= \alpha^1 dx + \beta^1 dy + \gamma^1 dz, \\ \tilde{\omega}_{31} &= \alpha^2 dx + \beta^2 dy + \gamma^2 dz, \\ \tilde{\omega}_{32} &= \alpha^3 dx + \beta^3 dy + \gamma^3 dz, \end{aligned}$$

we use the conditions

$$\begin{aligned} 0 &= \tilde{\omega}_{21}\left(\frac{\partial}{\partial z}\right), \\ 0 &= \tilde{\omega}_{31}\left(\frac{\partial}{\partial z}\right), \\ 0 &= \tilde{\omega}_{31}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}\right), \\ 0 &= \tilde{\omega}_{32}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}\right), \end{aligned}$$

to find the previous relations.

Since W is a G -invariant subset of $\mathcal{L}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$, condition (2) need to be verified only for $p = s(a)$, for all $a \in \mathbb{R}^3$.

Indeed :

Let $p \in P$, for $a \neq 0$ such that $a^{-1}p = (*, e) = s(*)$.

Condition (2) for $s(*)$ gives $c(s(*)) \in W$ and $c(s(*)) = c(a^{-1}p) = ac(p)$.

By G -invariance of W , $c(p) = a^{-1}ac(p) \in W$.

Thus, the condition (2) is equivalent to the condition (2')

$$\tilde{\Omega} = \begin{pmatrix} 0 & a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} & b\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + c\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} \\ 0 & 0 & d\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we need to show the existence of a unique ω on $P = \mathbb{R}^3 \times G$ satisfying (1') and (2').

Note that the identification $P = \mathbb{R}^3 \times G$ is not canonical and that all such identification are in one-to-one correspondence with the section $s' : \mathbb{R}^3 \rightarrow P$ of the form $s' = s\varphi$ where $\varphi : \mathbb{R}^3 \rightarrow G$ is a smooth function.

Then the form $\tilde{\omega}$ becomes

$$\tilde{\omega}' = (s')^*\omega = (Ad\varphi^{-1})\tilde{\omega} + \varphi^*\theta,$$

where θ is the Maurer-Cartan form on G .

With a suitable choice of s , we can always ensure that

$$\begin{aligned} \tilde{\omega}_{21} &= dx, \\ \tilde{\omega}_{31} &= dy - zdx, \\ \tilde{\omega}_{32} &= k(dz - Qdx). \end{aligned}$$

Now, we will find the entries of $\tilde{\omega}$ by the condition (2') :

* We have $\tilde{\Omega}_{31} = 0$:

This is equivalent to

$$\begin{aligned} 0 &= d\tilde{\omega}_{31} + \tilde{\omega}_{31} \wedge \tilde{\omega}_{11} + \tilde{\omega}_{32} \wedge \tilde{\omega}_{21} + \tilde{\omega}_{33} \wedge \tilde{\omega}_{31} \\ 0 &= -dz \wedge dx + (dy - zdx) \wedge (\tilde{\omega}_{11} - \tilde{\omega}_{33}) + k(dz - Qdx) \wedge dx \\ 0 &= -dz \wedge dx + (\tilde{\omega}_{33} - \tilde{\omega}_{11}) \wedge (dy - zdx) + kdy \wedge dx \\ 0 &= [(k-1)dz] \wedge dx + [(\tilde{\omega}_{33} - \tilde{\omega}_{11})] \wedge (dy - zdx). \end{aligned}$$

By the coframe $\{dx, dy - zdx, dz - Qdx\}$ and Cartan's lemma :

$$\begin{aligned} (k-1)dz &= gdx + j(dy - zdx) + v(dz - Qdx), \\ \tilde{\omega}_{33} - \tilde{\omega}_{11} &= hdx + u(dy - zdx) + w(dz - Qdx), \end{aligned}$$

where $j = A_{12} = A_{21} = h$, $v = A_{13} = A_{31}$ and $g = A_{11}$ (notation of Cartan's lemma).

Substituting the values of $(k-1)dz$ and $\tilde{\omega}_{33} - \tilde{\omega}_{11}$ in

$$0 = [(k-1)dz] \wedge dx + [(\tilde{\omega}_{33} - \tilde{\omega}_{11})] \wedge (dy - zdx).$$

and expressing in terms of $\{dx \wedge dy, dy \wedge dz, dx \wedge dz\}$ give us $h = j + Qw$, $v = zw$ and $w = 0$.

Then $v = w = 0$, and $j = h$.

Then

$$\begin{aligned} (k-1)dz &= gdx + j(dy - zdx), \\ \tilde{\omega}_{33} - \tilde{\omega}_{11} &= hdx + u(dy - zdx). \end{aligned}$$

But $(k-1)dz = gdx + jdy - zjdx$, and by the coframe $\{dx, dy, dz\}$ we have

$$k = 1, \quad j = g = 0.$$

Then $\boxed{k = 1}$ and

$$\boxed{\tilde{\omega}_{33} - \tilde{\omega}_{11} = u(dy - zdx),}$$

where $u \in \mathcal{C}^\infty(\mathbb{R}^3)$.

* We have $\tilde{\Omega}_{21} = 0$:

Similarly, we find with this condition that

$$\begin{aligned} \tilde{\omega}_{22} - \tilde{\omega}_{11} &= vdx + t(dy - zdx), \\ \tilde{\omega}_{23} &= tdx + w(dy - zdx), \end{aligned}$$

where $v, w, t \in \mathcal{C}^\infty(\mathbb{R}^3)$.

By a suitable choice of a section s , t can be made equal to zero thus,

$$\boxed{\tilde{\omega}_{22} - \tilde{\omega}_{11} = vdx,}$$

$$\boxed{\tilde{\omega}_{23} = w(dy - zdx),}$$

where $v, w \in \mathcal{C}^\infty(\mathbb{R}^3)$.

* We have $\tilde{\Omega}_{32} = 0$:

Similarly than $\tilde{\Omega}_{31} = 0$, we can obtain from $\tilde{\Omega}_{32} = 0$ that $\boxed{v = Q_z}$ and

$$\boxed{\tilde{\omega}_{12} = Q_y dx - u(dz - Qdx) + \lambda(dy - zdx),}$$

where $\lambda \in \mathcal{C}^\infty(\mathbb{R}^3)$.

* Similarly from $\tilde{\Omega}_{22} + \tilde{\Omega}_{33} - 2\tilde{\Omega}_{11} = 0$, we find $\boxed{u = -\frac{1}{2}Q_{zz}}$ and

$$\boxed{\tilde{\omega}_{13} = \lambda dx - \frac{1}{3}Q_{yz}dx - \frac{1}{6}dQ_{zz} + \mu(dy - zdx),}$$

where $\mu \in \mathcal{C}^\infty(\mathbb{R}^3)$.

* Similarly from $\tilde{\Omega}_{22} - \tilde{\Omega}_{11} = 0$ we find

$$\boxed{w = \frac{1}{6}Q_{zzz},}$$

$$\boxed{\lambda = \frac{2}{3}Q_{yz} - \frac{1}{6}\hat{D}_x(Q_{zz}).}$$

Then we find

$$\begin{aligned}
\tilde{\omega}_{21} &= dx, \\
\tilde{\omega}_{31} &= dy - zdx, \\
\tilde{\omega}_{32} &= (dz - Qdx), \\
\tilde{\omega}_{22} - \tilde{\omega}_{11} &= Q_z dx, \\
\tilde{\omega}_{33} - \tilde{\omega}_{11} &= -\frac{1}{2}Q_{zz}(dy - zdx), \\
\tilde{\omega}_{12} &= Q_y dx + \frac{1}{2}Q_{zz}(dz - Qdx) + \left(\frac{2}{3}Q_{yz} - \frac{1}{6}\hat{D}_x Q_{zz}\right)(dy - zdx), \\
\tilde{\omega}_{13} &= \left(\frac{1}{3}Q_{yz} - \frac{1}{6}\hat{D}_x Q_{zz}\right)dx - \frac{1}{6}d(Q_{zz}) + \mu(dy - zdx), \\
\tilde{\omega}_{23} &= \frac{1}{6}Q_{zzz}(dy - zdx),
\end{aligned}$$

where $\mu \in \mathcal{C}^\infty(\mathbb{R}^3)$.

The only arbitrary coefficient here is μ .

We will use the last information of (2') to find μ .

The last information are the form of $\tilde{\Omega}_{12}$, $\tilde{\Omega}_{13}$, and $\tilde{\Omega}_{23}$.

By $\tilde{\Omega} = \begin{pmatrix} 0 & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} \\ 0 & 0 & \tilde{\Omega}_{23} \\ 0 & 0 & 0 \end{pmatrix}$ and the fact that $\tilde{\Omega} \wedge_{\mathfrak{g}} \tilde{\omega} = \tilde{\omega} \wedge_{\mathfrak{g}} \tilde{\Omega}$, we find

$$\begin{aligned}
0_3 &= \tilde{\Omega} \wedge_{\mathfrak{g}} \tilde{\omega} - \tilde{\omega} \wedge_{\mathfrak{g}} \tilde{\Omega} \\
0_3 &= \begin{pmatrix} \tilde{\Omega}_{12} \wedge \tilde{\omega}_{21} + \tilde{\Omega}_{13} \wedge \tilde{\omega}_{31} & * & * \\ \tilde{\Omega}_{23} \wedge \tilde{\omega}_{31} & \tilde{\Omega}_{23} \wedge \tilde{\omega}_{32} - \tilde{\omega}_{21} \wedge \tilde{\Omega}_{12} & * \\ 0 & -\tilde{\omega}_{31} \wedge \tilde{\Omega}_{12} & * \end{pmatrix}
\end{aligned}$$

Then

$$\tilde{\Omega}_{12} \wedge \tilde{\omega}_{21} + \tilde{\Omega}_{13} \wedge \tilde{\omega}_{31} = 0, \quad (1)$$

$$\tilde{\Omega}_{23} \wedge \tilde{\omega}_{31} = 0, \quad (2)$$

$$\tilde{\Omega}_{12} \wedge \tilde{\omega}_{21} + \tilde{\Omega}_{23} \wedge \tilde{\omega}_{32} = 0, \quad (3)$$

$$\tilde{\Omega}_{12} \wedge \tilde{\omega}_{31} = 0. \quad (4)$$

By Cartan's lemma for (2), we have

$$\tilde{\Omega}_{23} = d\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} + e\tilde{\omega}_{21} \wedge \tilde{\omega}_{31}.$$

Substituting this in (3) gives

$$\tilde{\Omega}_{21} \wedge \tilde{\omega}_{21} + e\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} \wedge \tilde{\omega}_{31} = 0. \quad (2.6)$$

By Cartan's lemma and the coframe $\{\tilde{\omega}_{21} \wedge \tilde{\omega}_{31}, \tilde{\omega}_{21} \wedge \tilde{\omega}_{32}, \tilde{\omega}_{31} \wedge \tilde{\omega}_{32}\}$, we have

$$\tilde{\Omega}_{12} = a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + b\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} + c\tilde{\omega}_{21} \wedge \tilde{\omega}_{32},$$

and in (2.6), we find

$$b = -e,$$

and in (4), we find

$$c = 0$$

Then

$$\tilde{\Omega}_{12} = a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} - e\tilde{\omega}_{31} \wedge \tilde{\omega}_{32}.$$

$\tilde{\Omega}_{21}$ in (1) gives,

$$-e\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} \wedge \tilde{\omega}_{32} + \tilde{\Omega}_{13} \wedge \tilde{\omega}_{31} = 0. \quad (2.7)$$

For

$$\tilde{\Omega}_{13} = \beta\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + \delta\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} + \gamma\tilde{\omega}_{21} \wedge \tilde{\omega}_{32}.$$

in (2.7) gives

$$\gamma = -e$$

Then

$$\tilde{\Omega}_{13} = \beta\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + \delta\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} - e\tilde{\omega}_{21} \wedge \tilde{\omega}_{32}.$$

Then, renaming the coefficient functions in $a, b, c, d, e \in \mathcal{C}^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} \tilde{\Omega}_{12} &= a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} - e\tilde{\omega}_{31} \wedge \tilde{\omega}_{32}, \\ \tilde{\Omega}_{13} &= a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + c\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} - e\tilde{\omega}_{21} \wedge \tilde{\omega}_{32}, \\ \tilde{\Omega}_{23} &= d\tilde{\omega}_{31} \wedge \tilde{\omega}_{32} + e\tilde{\omega}_{21} \wedge \tilde{\omega}_{31}. \end{aligned}$$

Now, we use the condition (2') :

$\tilde{\Omega}$ is of the form (2') if and only if $e = 0$.

Then, we use the expression for $e = 0$ in $\tilde{\Omega}_{23}$:

$$a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} = d\tilde{\Omega}_{23} = d\tilde{\omega}_{23} + \tilde{\omega}_{21} \wedge \tilde{\omega}_{13} + \tilde{\omega}_{22} \wedge \tilde{\omega}_{23} + \tilde{\omega}_{23} \wedge \tilde{\omega}_{33}$$

then expressing in terms of $\{dx \wedge dy, dy \wedge dz, dx \wedge dz\}$, we find

$$\begin{aligned} d &= -\frac{1}{6}Q_{zzzz}, \\ \mu &= \frac{1}{6}Q_{yzz} - \frac{1}{6}Q_z Q_{zzz} - \frac{1}{6}\hat{D}_x Q_{zzz}. \end{aligned}$$

Remark 2.8. Doing the same for $\tilde{\Omega}_{12}$ and $\tilde{\Omega}_{13}$ allow us to find

$$\begin{aligned} a &= -Q_{yy} + \frac{1}{2}Q Q_{yzz} + \frac{1}{2}Q_y Q_{zz} + \frac{2}{3}Q_{xyz} - \frac{1}{6}Q_{xxzz} \\ &\quad - \frac{1}{3}z Q_{xyzz} - \frac{1}{6}Q_x Q_{zzz} - \frac{1}{3}Q Q_{xzzz} + \frac{2}{3}z Q_{yyz} \\ &\quad - \frac{1}{6}z^2 Q_{yyzz} - \frac{1}{6}z Q_y Q_{zzz} - \frac{1}{3}z Q Q_{yzzz} - \frac{2}{3}Q_z Q_{yz} \\ &\quad + \frac{1}{6}Q_z Q_{xzz} + \frac{1}{6}z Q_z Q_{yzz} - \frac{1}{6}Q^2 Q_{zzzz}. \end{aligned}$$

From the structure equation of $\tilde{\Omega}$; $d\tilde{\Omega} = \frac{1}{2}\tilde{\Omega} \wedge_{\mathfrak{g}} \tilde{\omega}$, we find

$$\begin{aligned} b &= a_z, \\ c &= -\hat{D}_x(d) - 2Q_z d. \end{aligned}$$

Conversely, for this value μ , we have the wished form for $\tilde{\Omega}$:

$$\begin{aligned} \tilde{\Omega}_{12} &= a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31}, \\ \tilde{\Omega}_{13} &= a\tilde{\omega}_{21} \wedge \tilde{\omega}_{31} + c\tilde{\omega}_{31} \wedge \tilde{\omega}_{32}, \\ \tilde{\Omega}_{23} &= d\tilde{\omega}_{31} \wedge \tilde{\omega}_{32}. \end{aligned}$$

where

$$\begin{aligned} \tilde{\omega}_{21} &= dx, \\ \tilde{\omega}_{31} &= dy - zdx, \\ \tilde{\omega}_{32} &= (dz - Qdx), \\ \tilde{\omega}_{22} - \tilde{\omega}_{11} &= Q_z dx, \\ \tilde{\omega}_{33} - \tilde{\omega}_{11} &= -\frac{1}{2}Q_{zz}(dy - zdx), \\ \tilde{\omega}_{12} &= Q_y dx + \frac{1}{2}Q_{zz}(dz - Qdx) + (\frac{2}{3}Q_{yz} - \frac{1}{6}\hat{D}_x Q_{zz})(dy - zdx), \\ \tilde{\omega}_{13} &= (\frac{1}{3}Q_{yz} - \frac{1}{6}\hat{D}_x Q_{zz})dx - \frac{1}{6}d(Q_{zz}) + \mu(dy - zdx), \\ \tilde{\omega}_{23} &= \frac{1}{6}Q_{zzz}(dy - zdx), \\ \text{where } \mu &= \frac{1}{6}Q_{yzz} - \frac{1}{6}Q_z Q_{zzz} - \frac{1}{6}\hat{D}_x Q_{zzz}. \end{aligned}$$

Finally, for this choice of Cartan connection ω on $P = \mathbb{R}^3 \times G$, the theorem is proved.

Now, prove the theorem globally.

Let a covering $\{U_\alpha\}_{\alpha \in I}$ of the three-dimensional space $J^1(\mathbb{R}^2)$, and using the local consideration, we can construct a unique Cartan connection ω_α on each trivial fiber bundle $U_\alpha \times G$.

Let s_α, s_β be the trivial section of the fiber bundle $U_\alpha \times G$ and $U_\beta \times G$, and let $\tilde{\omega}_\alpha = s_\alpha^* \omega_\alpha, \tilde{\omega}_\beta = s_\beta^* \omega_\beta$.

Now, show that for any two disjoint domains U_α and U_β , that there exists a unique function

$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$$

such that

$$\tilde{\omega}_\beta = (Ad\varphi_{\alpha,\beta}^{-1}) \circ \tilde{\omega}_\alpha + \varphi_{\alpha,\beta}^* \theta. \quad (2.9)$$

Without loss of generality, we can assume that the form $\tilde{\omega}_\alpha$ has the form started above. The form $\tilde{\omega}_\beta$ on $U_\alpha \cap U_\beta$ satisfies the conditions (1) and (2) of the theorem, therefore can be made $\tilde{\omega}_\alpha$ by a suitable choice of section. But, by the above, the section is uniquely defined.

Thus there exists a unique function $\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow G$ satisfying (2.8).

In its turn, the family $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in I}$ uniquely (up to isomorphism) determines a principal fiber bundle P with structural group G and a Cartan connection ω on P .

□

Corollary 2.10. *Two second order ordinary differential equations are locally equivalent if and only if so are the corresponding Cartan connections.*

Corollary 2.11. *A second order ordinary differential equation is locally equivalent to $y'' = 0$ if and only if the structure function of the corresponding Cartan connection is identically equal to zero.*

Proof. The Cartan connection corresponding to the equation $y'' = 0$ is equal to zero.

Thus, our original equation is equivalent to $y'' = 0$, if and only if, by the previous theorem, its curvature form is zero if and only if its structure function is zero.

□

The geometry of marked contact Engel structures

In General Relativity, we study the massless particles in the conformal Lorentzian geometry in 4-dimensions $(M^4, [g])$.

Definition 0.12. A congruence of $(M^4, [g])$ is a set of integral curves of a nowhere vanishing vector field in $(M^4, [g])$.

A special notion in $(M^4, [g])$ is the **null congruences** i.e. foliations of $(M^4, [g])$ by null rays (a type of ray containing for example light rays).

Assume that $(M^4, [g])$ has a null congruence. Let $K \subset TM^4$ a line subbundle such that any section $s : M^4 \rightarrow K$ be tangent to the congruence.

Then, $(M^4, [g], K)$ is called a **a null congruence structure**.

It is good structure to study the massless particules.

Moreover, study particules in a space with lot of symmetries simplify the study.

In the physic literature, we saw that the null congruence structure admits space with lot of symmetries : homogeneous models associated to this structure.

In fact, the marked contact twisted cubic structures in mathematics are analogue of the null congruence structures in physics.

In these next chapters, we will study the marked contact twisted cubic structures and we will classify all non-equivalent homogeneous models of the marked contact Engel structures.

The marked contact Engel structures are a subclass of the marked contact twisted cubic structures.

Indeed, this leads us to a more general problem : **The classification of all non-equivalent homogeneous models of the marked contact twisted cubic structures.**

Notions on Legendrian twisted cubics

Let us give some notions about twisted cubics.

Definition 0.13. The twisted cubic is the image of the Veronese map

$$v : \begin{cases} \mathbb{P}^1 \longrightarrow \mathbb{P}^3 = \mathbb{P}(\mathbb{R}^4) \\ [X_0 : X_1] \longmapsto [X_0^3 : X_0^2 X_1 : X_0 X_1^2 : X_1^3] \end{cases}$$

denoted by $\gamma := \text{Im}(v) \subset \mathbb{P}(\mathbb{R}^4)$.

Proposition 0.14.

$$\gamma = \{F_1 = F_2 = F_3 = 0\}$$

where

$$\begin{aligned} F_1(X_0, X_1, X_2, X_3) &= X_0 X_2 - (X_1)^2 \\ F_2(X_0, X_1, X_2, X_3) &= X_1 X_2 - (X_2)^2 \\ F_3(X_0, X_1, X_2, X_3) &= X_1 X_2 - X_0 X_3 \end{aligned}$$

Proof. The inclusion $\subset$ is clear.

Conversely, let $p = [X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3$ such that

$$\begin{cases} X_0 X_2 = (X_1)^2 \\ X_1 X_3 = (X_2)^2 \\ X_1 X_2 = X_0 X_3 \end{cases}$$

Either X_0 or X_3 must be nonzero.

If $X_0 \neq 0$, then $p = v([X_0 : X_1])$, else ($X_3 \neq 0$) then $p = v([X_2 : X_3])$. □

Note that $\mathbb{R}^4 = \odot^3 \mathbb{R}^2$ then we can define the twisted cubic on this space.

Definition 0.15. The twisted cubic $\gamma \subset \mathbb{P}(\odot^3 \mathbb{R}^2)$ is the image of the Veronese map

$$v : \begin{cases} \mathbb{P}(\mathbb{R}^2) \longrightarrow \mathbb{P}(\odot^3 \mathbb{R}^2) \\ [w] \longmapsto [w \odot w \odot w] \end{cases}$$

Indeed, let (e_1, e_2) be a basis of $\mathbb{R}^2$.
Let $w = se_1 + te_2, (s, t) \in \mathbb{R}^2$.

$$w \odot w \odot w = s^3 E_1 + s^2 t E_2 + s t^2 E_3 + t^3 E_4$$

Then $v([w] = [s : t]) = [s^3 : s^2 t : s t^2 : t^3]$ where

$$\begin{aligned} E_1 e_1 \odot e_1 \odot e_1 &=: e_1^3, \\ E_2 &= 3e_1 \odot e_1 \odot e_2 =: 3e_1^2 e_2, \\ E_3 &= 3e_1 \odot e_2 \odot e_2 =: 3e_1 e_2^2, \\ E_4 &= e_2^3, \end{aligned}$$

form a basis of $\odot^3 \mathbb{R}^2$, denoted E_* .

Let $\{E^1, E^2, E^3, E^4\}$ its dual basis.

By the previous proposition, we can say

$$\gamma = \{g_1 = g_2 = g_3\}$$

where

$$\begin{aligned} g_1 &= E^1 E^3 - (E^2)^2, \\ g_2 &= E^2 E^4 - (E^3)^2, \\ g_3 &= E^2 E^3 - E^1 E^4, \end{aligned}$$

are three quadratic forms on $\odot^3 \mathbb{R}^2$.

With respect to the introduced basis, let ρ be an irreducible representation of $GL(2, \mathbb{R})$ on $\odot^3 \mathbb{R}^2$ given by

$$\rho : A := \begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix} \mapsto Mat_{E_*}(A^3) = \begin{pmatrix} \alpha^3 & 3\alpha^2\beta & 3\alpha\beta^2 & \beta^3 \\ \alpha^2\epsilon & \alpha^2\delta + 2\alpha\beta\epsilon & 2\alpha\beta\delta & \beta^2\delta \\ \alpha\epsilon^2 & 2\alpha\delta\epsilon + \beta\epsilon^2 & \alpha\delta^2 + 2\beta\delta\epsilon & \beta\delta^2 \\ \epsilon^3 & 3\delta\epsilon^2 & 3\delta^2\epsilon & \delta^3 \end{pmatrix} \quad (0.16)$$

such that $\rho(A)E_1 = \alpha^3 E_1 + \alpha^2 \epsilon E_2 + \alpha \epsilon^2 E_3 + \epsilon^3 E_4$.

Then the tangent map at the identity of ρ defines an irreducible Lie algebra representation

$$\rho' = T_e \rho : \mathfrak{gl}(2, \mathbb{R}) \longrightarrow End(\odot^3 \mathbb{R}^2).$$

Then we have

Proposition 0.17. *The subalgebra of $End(\odot^3 \mathbb{R}^2)$ preserving $\gamma \subset \mathbb{P}(\odot^3 \mathbb{R}^2)$ is precisely $\rho'(\mathfrak{gl}(2, \mathbb{R}))$.*

Remark 0.18.

$$\det(\rho(A)) = (\delta\alpha - \beta\epsilon)^6$$

Definition 0.19. Let $\rho : G \rightarrow GL(n, \mathbb{R})$ be a representation of G .
A multilinear form $F : \mathbb{R}^k \rightarrow \mathbb{R}$ is called (G, ρ) -invariant if

$$F(\rho(a)X_1, \dots, \rho(a)X_k) = \det(\rho(a))^{\frac{k}{n}} F(X_1, \dots, X_k) \quad (a \in G; X_1, \dots, X_k \in \mathbb{R}^k)$$

There is a skew-symmetric form ω on $\odot^3 \mathbb{R}^2$ preserved by the $GL(2, \mathbb{R})$ -action ρ up to scalar :

Let $(F^1, \dots, F^4)$ be a basis of $(\odot^3 \mathbb{R}^2)^*$ such that $F = \rho(A)\dot{E}$:

$$\begin{aligned} F^1 &= \alpha^3 E^1 + \alpha^2 \epsilon E^2 + \alpha \epsilon^2 E^3 + \epsilon^3 E^4, \\ F^2 &= 3\alpha^2 \beta E^1 + (\alpha^2 \delta + 2\alpha \beta \epsilon) E^2 + (2\alpha \beta \delta + \beta \epsilon^2) E^3 + 3\delta \epsilon^2 E^4, \\ F^3 &= 3\alpha \beta^2 E^1 + (2\alpha \beta \delta + \beta^2 \epsilon) E^2 + (\alpha \delta^2 + 2\beta \delta \epsilon) E^3 + 3\delta^2 \epsilon E^4, \\ F^4 &= \beta^3 E^1 + \beta^2 \delta E^2 + \beta \delta^2 E^3 + \delta^3 E^4. \end{aligned}$$

The we compute

$$F^1 \wedge F^4 - 3F^2 \wedge F^3 = (\alpha \delta - \beta \epsilon)^3 (E^1 \wedge E^4 - 3E^2 \wedge E^3)$$

Definition 0.20. A symplectic form on V , a n -dimensional space over $\mathbb{R}$, is a map $\omega : V \times V \rightarrow \mathbb{R}$ such that

- (1) ω is bilinear,
- (2) skew-symmetric,
- (3) nondegenerate ; $\forall x \in V, \exists y \in V, \omega(x, y) \neq 0$.

Remark 0.21. For $\{e_i\}_{1 \leq i \leq n}$ a basis of V a n -dimensional vector space and its dual basis $\{e^{*i}\}_{1 \leq i \leq n}$.

A symplectic form ω has the form $\omega = \frac{1}{2} \sum_{1 \leq i < j \leq n} \omega_{i,j} e^{*i} \wedge e^{*j}, \omega_{i,j} \in \mathbb{R}$.

Proposition 0.22. Let ω be a symplectic form on V an n -dimensional vector space over $\mathbb{R}$.
There exists $\eta \in \text{End}(V)$ such that

$$\omega(x, y) = {}^t X H Y \quad (x, y \in V)$$

and $\eta^* = -\eta$, where for a basis e of V

$$\begin{aligned} X &= \text{Mat}_e(x), \\ Y &= \text{Mat}_e(y), \\ H &= \text{Mat}_e(\eta). \end{aligned}$$

Our symplectic form ω has the following form in $E = \{E_1, E_2, E_3, E_4\}$

$$\text{Mat}_{E^*}(\omega) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Definition 0.23. Let V be a n -dimensional vector space over $\mathbb{R}$ and $X \subset \mathbb{P}(V)$.

We denote $\hat{X} = \{(\lambda x) \mid [x] \in X, \lambda \in \mathbb{R}\}$ the affine cone of X .

Definition 0.24. Let ω be a symplectic form on V a n -dimensional vector space over $\mathbb{R}$. A linear subspace $W \subset V$ is a Lagrangian subspace with respect to ω if it is a maximal subspace such that $\omega|_W \equiv 0$.
(So $\dim(W) = \frac{1}{2}\dim(V)$)

Definition 0.25. Let V be a n -dimensional vector space over $\mathbb{R}$. A sub variety $X \subset \mathbb{P}(V)$ is Legendrian if for each point of its affine cone $\widehat{X}$, the tangent space is Lagrangian.

Definition 0.26. Let ω be a symplectic form on $\odot^3\mathbb{R}^3$. The conformal class of $[\omega]$ is the set of ω' symplectic form on $\odot^3\mathbb{R}^2$ such that there is $\alpha \in \mathbb{R}^*$ such that $\omega' = \alpha\omega$.

Proposition 0.27. The conformal symplectic structure $[\omega]$ generated by $\omega = E^1 \wedge E^4 - 3E^2 \wedge E^3$ is the unique conformal structure such that the twisted cubic γ is Legendrian with respect to ω .

Proof. For $\widehat{p} = s^3E_1 + s^2tE_2 + st^2E_3 + t^3E_4 \in \widehat{\gamma}$,

$$T_{\widehat{p}}\widehat{\gamma} = \left\langle \frac{\partial \widehat{v}}{\partial s}(s, t), \frac{\partial \widehat{v}}{\partial t}(s, t) \right\rangle,$$

where $\widehat{v} : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^4 \\ (s, t) \longmapsto (s^3, s^2t, st^2, t^3) \end{cases}$. Then

$$\begin{aligned} \frac{\partial \widehat{v}}{\partial s}(s, t) &= 3s^2E_1 + 2stE_2 + t^3E^3 =: x, \\ \frac{\partial \widehat{v}}{\partial t}(s, t) &= s^2E_2 + 2stE_3 + 3t^2E_4 =: y. \end{aligned}$$

Let $\omega = \frac{1}{2} \sum_{1 \leq i < j \leq 4} \omega_{i,j} E^i \wedge E^j$ be a symplectic form on $\odot^3\mathbb{R}^2$. Then,

$$\omega(x, y) = 3s^4\omega_{1,2} + 6s^3t\omega_{1,3} + 3s^2t^2(3\omega_{1,4} + \omega_{2,3}) + 6st^3\omega_{2,4} + 3t^4\omega_{3,4}$$

Thus γ is Legendrian at $\widehat{p}$ with respect to ω if and only if $\omega_{T_{\widehat{p}}\widehat{\gamma}} \equiv 0$ if and only if

$$\begin{cases} \omega_{1,2} = \omega_{1,3} = \omega_{2,4} = \omega_{3,4} = 0, \\ \omega_{1,4} = -\frac{1}{3}\omega_{2,3} = \alpha. \end{cases}$$

Then

$$Mat_e(\omega) = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & -3\alpha & 0 \\ 0 & 3\alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix}$$

Then

$$[Mat_e(\omega)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

□

Now we introduce a new notion.

Definition 0.28. The twisted cubic is marked if there is $p \in \gamma \subset \mathbb{P}(\odot^3 \mathbb{R}^2)$ distinguished i.e. it is an origine of γ .

Now we assume that the twisted cubic γ is marked.

Let $p = [\eta \odot \eta \odot \eta]$ a distinguished point in γ and $L = (\eta)$ be a one-dimensional vector space.

Let $a : \begin{cases} \widehat{\gamma} \longrightarrow \gamma \\ (w \odot w \odot w) \longmapsto [w \odot w \odot w] \end{cases}$.

We have $a(Vect(\{w \odot w \odot w \mid w \in L\})) = p$, then we set

$$p = Vect(\{w \odot w \odot w \mid w \in L\}) = l \in \widehat{\gamma}$$

We can see that

Proposition 0.29. The $GL(2, \mathbb{R})$ -action ρ acts transitively on γ .

Proof. we have

$$\begin{cases} GL(2, \mathbb{R}) \times \gamma \longrightarrow \gamma \\ (T, [w \odot w \odot w]) \longmapsto [T(w) \odot T(w) \odot T(w)] \end{cases}$$

Let $x = [w \odot w \odot w]$ and $y = [v \odot v \odot v]$.

Assume there is $T = \begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix} \in GL(2, \mathbb{R})$ such that $y = T \cdot x$.

It is sufficient to have T such that $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T(w)$.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \iff \begin{cases} \alpha w_1 + \beta w_2 = v_1 \\ \epsilon w_1 + \delta w_2 = v_2 \end{cases}$$

(1) Case : $(w_1, w_2) = (0, 0)$, we take $T = 0_2$.

(2) Case : $w_1 \neq 0, w_2 = 0$, then we take $T = \begin{pmatrix} \frac{v_1}{w_1} & \beta \\ \frac{v_2}{w_1} & \delta \end{pmatrix}$ with $\beta, \delta \in \mathbb{R}$ such that $v_1 \delta - \beta v_2 \neq 0$.

(3) Case : $w_1 = 0, w_2 \neq 0$, it is the previous case by changing the roles.

(4) Case : $w_1, w_2 \neq 0$, then we take $T = \begin{pmatrix} \frac{v_1}{w_1} - \beta \frac{w_2}{w_1} & \beta \\ \frac{v_2}{w_1} - \beta \frac{w_2}{w_1} & \delta \end{pmatrix}$ with $\beta, \delta \in \mathbb{R}$ such that $\delta v_1 - \beta v_2 \neq 0$.

□

By this proposition there is just one orbit by this action :

$$\gamma = \mathcal{O}(x),$$

where $x = [w \odot w \odot w]$.

we can choose $x = [e_2 \odot e_2 \odot e_2]$.

Thus we can choose l as follow

$$l = Vect(e_2 \odot e_2 \odot e_2).$$

Definition 0.30. A Borel subgroup B of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup.

Example 0.31. For $G = GL(2, \mathbb{R})$, B is a group of upper or lower triangular matrices.

Let B a Borel subgroup of $GL(2, \mathbb{R})$ defined as follow

$$B = \{A \in GL(2, \mathbb{R}) \mid \rho(A)(l) \subset l\} = \{A \mid A(L) \subset L\}.$$

Indeed :

For $l = Vect(e_2 \odot e_2 \odot e_2)$, $L = (e_2)$ and $A = \begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix}$

We have

$$\rho(A)(l) \subset l \iff Ae_2 \in L$$

Then $\beta = 0$.

Then

$$B = \{A \in GL(2, \mathbb{R}) \mid \begin{pmatrix} \alpha^3 & 0 & 0 & 0 \\ \alpha^2\epsilon & \alpha^2\delta & 0 & 0 \\ \alpha\epsilon^2 & 2\alpha\delta\epsilon & \alpha\delta^2 & 0 \\ \epsilon^3 & 3\delta\epsilon^2 & 3\delta^2\epsilon & \delta^3 \end{pmatrix}\} = \left\{ \begin{pmatrix} \alpha & 0 \\ \epsilon & \delta \end{pmatrix} \right\} \quad (0.32)$$

By its shape B preserves filtrations of $\mathbb{R}^4$.

Then we have :

Lemma 0.33. A distinguished point $p \in \gamma$ determine a filtration by subspaces

$$\{p\} \subset l \subset D \subset H \subset \mathbb{R}^4.$$

Moreover, if γ is Legendrian, then D is a Lagrangian subspace and H is the symplectic orthogonal to l .

Proof. For p distinguished point in γ , there is a line $l = Vect(\{t \odot t \odot t \mid t \in L\})$ for a unique one-dimensional subspace $L \subset \mathbb{R}^2$.

Then we set $D = Vect(\{t \odot t \odot e \mid t \in L, e \in \mathbb{R}^2\})$ the de-projectivized tangent line to γ at p and $H = Vect(\{t \odot e \odot f \mid t \in L, e, f \in \mathbb{R}^2\})$ the de-projectivized osculating plan to γ at p .

□

Definitions and descriptions of (marked) contact twisted cubic structures

1. Notions on graded Lie algebras

We begin with definitions.

Definition 1.1. A filtered vector space is a vector space V together with a sequence $\{V^i \mid i \in \mathbb{Z}\}$ of subspaces $V^i \subset V$ such that

$$\begin{aligned} V^{i+1} &\subset V^i, & (i \in \mathbb{Z}) \\ V &= \bigcup_{i \in \mathbb{Z}} V^i, \\ \bigcap_{i \in \mathbb{Z}} V^i &= \{0\}. \end{aligned}$$

We will deal with the case of finite-dimensional spaces and finite filtrations, which we will write

$$V = V^j \supset V^{j+1} \supset \dots \supset V^k$$

It is always assumed that for all $l < j$, $V^l = V$, and for all $l > k$, $V^l = \{0\}$.

Example 1.2. The trivial filtration on a vector space V is given by for all $i \leq 0$, $V^i = V$ and for $i > 0$, $V^i = 0$.

From a filtration $\{V^i\}_{i \in \mathbb{Z}}$ of a vector space V , we can canonically construct a **graded vector space** noted $gr(V)$ as follows

$$gr(V) = \bigoplus_{i \in \mathbb{Z}} gr_i(V),$$

where

$$gr_i(V) = V^i / V^{i+1}. \quad (i \in \mathbb{Z})$$

It is called **the associated graded to the filtered vector space V** .

In the case of a finite filtration $V = V^j \supset V^{j+1} \supset \dots \supset V^k$, we obtain a finite grading

$$gr(V) = gr_j(V) \oplus \dots \oplus gr_k(V),$$

by omitting the zero summands $gr_i(V)$ for $i < j$ and $i > k$.

Definition 1.3. A filtered vector bundle over a smooth manifold M is a smooth vector bundle $p : E \rightarrow M$ together with a sequence $\{E^i \mid i \in \mathbb{Z}\}$ of smooth sub bundles such that

$$E^{i+1} \subset E^i \quad (i \in \mathbb{Z})$$

and there is $i_0 < j_0 \in \mathbb{Z}$ such that

$$E^i = E \quad (i \leq i_0)$$

$$E^i = M \quad (i \geq j_0)$$

Given a such filtered bundle, we may form the quotient bundles

$$gr_i(E) = E^i / E^{i+1} \quad (i \in \mathbb{Z})$$

and the **associated graded vector bundle to E**

$$gr(E) = \bigoplus_{i \in \mathbb{Z}} gr_i(E)$$

Let $\mathfrak{g}$ be a Lie algebra.

Definition 1.4. A Lie sub algebra $\mathfrak{h}$ of $\mathfrak{g}$ is a linear subspace such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$$

Then we write $\mathfrak{h} \leq \mathfrak{g}$.

Remark 1.5. * Any intersection of arbitrary family of Lie subalgebras of $\mathfrak{g}$ is a subalgebra.
* For a subset $A \subset \mathfrak{g}$, we call the Lie subalgebra generated by A , the Lie algebra $\langle A \rangle = \bigcap_{\mathfrak{h} \supset A} \mathfrak{h}$.

Definition 1.6. We say that a Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is an ideal if

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$$

We write $\mathfrak{h} \triangleleft \mathfrak{g}$.

Remark 1.7. $[\mathfrak{g}, \mathfrak{g}]$ is called the commutator ideal.

$[\mathfrak{g}, \mathfrak{g}]$ gives to rise two sequences of ideals of $\mathfrak{g}$ in $\mathfrak{g}$;

(1) Let us define $\mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}]$ and

$$\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k] \quad (k \in \mathbb{N}^*)$$

We see that

$$\mathfrak{g}^k \leq \mathfrak{g} \quad (k \in \mathbb{N})$$

and we have the sequence called **the lower central series of $\mathfrak{g}$** :

$$\mathfrak{g} \supset \mathfrak{g}^2 \supset \dots \supset \mathfrak{g}^k \supset \mathfrak{g}^{k+1} \supset \dots$$

Definition 1.8. The Lie algebra $\mathfrak{g}$ is called nilpotent if there is $k \in \mathbb{N}$ such that

$$\mathfrak{g}^k = 0$$

(2) Let us define $\mathfrak{g}^{(1)} := g$ and

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \quad (k \in \mathbb{N}^*)$$

We see that

$$\mathfrak{g}^{(k)} \leq g \quad (k \in \mathbb{N})$$

and we have the sequence called **the derived series of g** :

$$g \supset \mathfrak{g}^{(2)} \supset \dots \supset \mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)} \supset \dots$$

Definition 1.9. The Lie algebra g is called solvable if there is $k \in \mathbb{N}$ such that

$$\mathfrak{g}^{(k)} = 0$$

Definition 1.10. A Lie algebra g is said semisimple if it has no nonzero solvable ideal.

A Lie algebra g is said simple if $g = [g, g]$ and the only ideals in g are $\{0\}$ and g .

Remark 1.11. g simple implies g semisimple.

Definition 1.12. Let g be a semi simple Lie algebra and $k \in \mathbb{N}^*$.

A k -grading on g is a decomposition

$$g = g_{-k} \oplus \dots \oplus g_k,$$

such that

$$[g_i, g_j] \subset g_{i+j} \quad (i, j \in \mathbb{Z})$$

$$g_- := g_{-k} \oplus \dots \oplus g_{-1}$$

is generated as a Lie algebra by g_{-1} and

$$g_{-k} \neq \{0\}, \quad g_k \neq \{0\}$$

By definition, if $g = g_{-k} \oplus \dots \oplus g_k$ is a k -grading, then $\mathfrak{p} = g_0 \oplus \dots \oplus g_k$ is a subalgebra of g , and $\mathfrak{p}_+ := g_1 \oplus \dots \oplus g_k$ is a nilpotent ideal of $\mathfrak{p}$.

Similarly, the subalgebra g_- is nilpotent.

Remark 1.13. Given a such that k -grading on g , we can form the associated filtration

$$g = g^{-k} \supset g^{-k+1} \supset \dots \supset g^k$$

defined by

$$g^i := \bigoplus_{j \geq i} g_j$$

and **the associated graded Lie algebra** :

$$gr(g) = \bigoplus_{i \in \mathbb{Z}} gr_i(g),$$

where

$$gr_i(g) = g^i / g^{i+1} \quad (i \in \mathbb{Z})$$

Example 1.14. For $\mathbb{K}$ a field, $\mathbb{K}^n = \mathbb{K} \oplus \dots \oplus \mathbb{K}$ is a n -grading of $\mathbb{K}^n$ with the filtration $\mathbb{K}^n \supset \mathbb{K}^{n-1} \supset \dots \supset \mathbb{K}^2 \supset \mathbb{K}$.

Consider a k -grading semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-k}$ and a Lie group G with Lie algebra $\mathfrak{g}$.

For a subset $A \subset \mathfrak{g}$, the **normalizer** $N_G(A)$ of A in G is defined as

$$N_G(A) = \{g \in G \mid \text{Ad}(g)(A) \subset A\}$$

As before, we denote by $\{\mathfrak{g}^i\}_{i \in \mathbb{Z}}$ the filtration induced by the k -grading.

Lemma 1.15. (1) *The Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$ can be characterized as*

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \text{ad}_X(\mathfrak{g}_i) \subset \mathfrak{g}_i, -k \leq i \leq k\},$$

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \text{ad}_X(\mathfrak{g}^i) \subset \mathfrak{g}^i, -k \leq i \leq k\},$$

where $\text{ad}_X(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

(2) *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

$P := \bigcap_{i=-k}^k N_G(\mathfrak{g}^i) \subset G$ is a closed subgroup with Lie algebra $\mathfrak{p}$.

Proof. (1) In both statement, $\subset$ is clear.

Conversely, take $X \in \mathfrak{g}$,

$$X = X_{-k} + \cdots + X_k.$$

Let $E \in \mathfrak{g}_0 \subset \mathfrak{p}$ be the grading element; it is the unique element $E \in \mathfrak{g}$ such that

$$[E, X] = jX \quad (X \in \mathfrak{g}_j, -k \leq j \leq k)$$

Indeed : E exists and lies in $\mathfrak{g}_0$;

Consider $D : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $D(X) = jX$ for $X \in \mathfrak{g}_j$ and $-k \leq j \leq k$.

Since for all $i, j \in \mathbb{Z}$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ then

$$D([X, Y]) = [D(X), Y] + [X, D(Y)] \quad (X, Y \in \mathfrak{g})$$

Semisimplicity of $\mathfrak{g}$ implies there is a unique $E \in \mathfrak{g}$ such that

$$D(X) = [E, X] \quad (X \in \mathfrak{g})$$

Decomposing $E = E_{-k} + \cdots + E_k$ with $E_i \in \mathfrak{g}_i$, we get

$$0 = [E, E] = \sum_{j=-k}^k jE_j$$

Hence $E = E_0 \in \mathfrak{g}$.

Then we have

$$\text{ad}_X(E) = kX_{-k} + \cdots + X_{-1} - X_1 - \cdots - kX_k$$

Conversely, if $\text{ad}_X(\mathfrak{g}_0) \subset \mathfrak{g}_0$, then for $i \neq 0$, $X_i = 0$ then $X \in \mathfrak{g}$.

Likewise, if $\text{ad}_X(\mathfrak{p}) \subset \mathfrak{p}$ we must have for all $i < 0$ $X_i = 0$, then $X \in \mathfrak{p}$.

(2) For each i , the subset $\mathfrak{g}^i \subset \mathfrak{g}$ is closed, and $N_G(\mathfrak{g}^i)$ is a closed subgroup of G , so $P \subset G$ is a closed subgroup.

Since for $X \in \mathfrak{g}$, $\text{Ad}(\exp(X)) = e^{\text{ad}(X)}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \text{ad}_X(\mathfrak{g}^i) \subset \mathfrak{g}^i, -k \leq i \leq k\}$ is the Lie algebra of P .

□

Definition 1.16. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a k -grading semi simple Lie algebra and let G be a Lie group with Lie algebra $\mathfrak{g}$.

(1) A parabolic subgroup G correspond to the given k -grading is a subgroup $P \subset G$ which lies between $\bigcap_{i=-k}^k N_g(\mathfrak{g}^i)$ and its connected component of the identity.

(2) Given a parabolic subgroup $P \subset G$, we define the Levi subgroup $G_0 \subset P$ by

$$G_0 := \{g \in P \mid \text{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i, -k \leq i \leq k\}$$

Remark 1.17. Any parabolic subgroup $P \subset G$ is closed and has Lie algebra $\mathfrak{p}$. Further, $G_0 \subset P$ is an intersection of normalizers of closed subsets, and hence a closed subgroup of P . From the lemma and its proof, we see that $G_0 \subset P$ corresponds to the Lie algebra $\mathfrak{g}_0 \subset \mathfrak{p}$.

Definition 1.18. A parabolic geometry is a Cartan geometry of type (G, P) where G is a semisimple Lie group (a Lie group which doesn't have nontrivial abelian norma subgroup) and P is a parabolic subgroup corresponding to some k -grading of the Lie algebra $\mathfrak{g}$ of G .

2. (Marked) contact twisted cubic structures

Now we come to the contact structures.

Definition 2.1. A contact form ω on a smooth manifold M of dimension $2n + 1$ is a form such that $\omega \wedge (d\omega)^n$ is a volume form on M .

If ω is a contact form on M then ω doesn't vanish on M . Then $C = \ker(d\omega)$ is a codimension one-subbundle of TM called **the contact subbundle**.

Moreover, $d\omega : C \times C \rightarrow \mathbb{R}$ is a non degenerate skew-symmetric bilinear form i.e. a non degenerate symplectic form on C .

Let $T^{-2}M = TM$, and $T^{-1}M = C$ a filtration of TM with the associated graded $gr(TM) = TM/C \oplus C$.

The Levi bracket

$$\mathcal{L} : \begin{cases} \Lambda^2 C \longrightarrow TM/C \\ \xi_x \wedge \nu_x \longmapsto [\xi, \nu]_x \text{mod}(C_x) \end{cases}$$

is non degenerate and

$$gr(T_x M) \simeq \mathbb{R} \oplus \mathbb{R}^{2n} \quad (x \in M)$$

with $[\cdot, \cdot] : \Lambda^2 \mathbb{R}^{2n} \rightarrow \mathbb{R}$ non degenerate.

Then $gr(TM)$ is called **the real Heisenberg algebra** $\mathfrak{h}_{2n+1}$.

Now we define a contact distribution structure on M .

Definition 2.2. A contact distribution structure on a smooth manifold M of dimension $2n + 1$ as a subbundle $C \subset TM$ of rank $2n$ such that the Levi bracket

$$\mathcal{L} : \begin{cases} \Lambda^2 C \longrightarrow TM/C \\ \xi_x \wedge \nu_x \longmapsto [\xi, \nu]_x \text{mod}(C_x) \end{cases}$$

is a non degenerate symplectic form.

For each point $x \in M$, $\mathcal{L}_x$ endows each fiber C_x with the structure of conformal symplectic vector space. Locally C is the kernel of a contact form i.e. $C = \ker(d\alpha)$ where α is a differential 1-one form such that $\alpha \wedge (d\alpha)^n \neq 0$ and the conformal symplectic structure on C_x is generated by $d\alpha|_{C_x}$.

We can associate to $gr(TM) = TM/C \oplus C$ a natural graded frame bundle $\mathcal{F}$ with structure bundle $Aut_{gr}(\mathfrak{h}_{2n+1})$. Precisely,

$$\mathcal{F}_x = \{\varphi : gr(T_x M) \rightarrow \mathfrak{h}_{2n+1}\} \quad (x \in M)$$

Lemma 2.3. (1) Any automorphism of $\mathfrak{h}_{2n+1}$ is uniquely determined by its restriction $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$.

(2) Viewed as a subgroup of $GL(2n, \mathbb{R})$, the group $Aut_{gr}(\mathfrak{h}_{2n+1})$ is generated by $Sp(2n, \mathbb{R})$, multiple of I_{2n} , and the diagonal matrice $I_{n,n}$; the first n entries equal to 1 and the other n equal to -1 .

(3) The subgroup of those automorphisms, which in addition preserve an orientation on $\mathbb{R} \subset \mathfrak{h}_{2n+1}$, is the conformal symplectic group $CSp(2n, \mathbb{R})$ generated by $Sp(2n, \mathbb{R})$ and multiples of I_{2n} .

Proof. Let $[\cdot, \cdot] : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be the bracket in $\mathfrak{h}_{2n+1}$ given by the standard symplectic form on $\mathbb{R}^{2n}$.

We have (1) by subjectivity of $[\cdot, \cdot]$.

For $A \in Sp(2n, \mathbb{R})$ and for all $x, y \in \mathbb{R}^{2n}$, $[Ax, Ay] = [x, y]$. This gives an automorphism. Likewise, for $a \in \mathbb{R}^*$, we get $[ax, ay] = a^2[x, y]$.

So multiples of I_{2n} are in $Aut_{gr}(\mathfrak{h}_{2n+1})$ too.

Finally,

$$[I_{n,n}x, I_{n,n}y] = [x, y] \quad (x, y \in \mathbb{R}^{2n})$$

Conversely, we assume $A \in GL(2n, \mathbb{R})$, $a \in \mathbb{R}^*$ such that

$$[Ax, Ay] = [x, y] \quad (x, y \in \mathbb{R}^{2n})$$

If $a < 0$, replace A by $I_{n,n}A$ to get an element for which $a > 0$.

Dividing then by $\sqrt{a}$, we obtain an element which preserve $[\cdot, \cdot]$ and thus lies in $Sp(2n, \mathbb{R})$.

So we have (2).

The statement (3) is clear. □

Now we are in position to define the central objects.

Definition 2.4. A contact twisted cubic structure on a 5-dimensional smooth manifold M and a contact distribution $C \subset TM$ together with a subbundle $\gamma \subset \mathbb{P}(C)$ whose at each $x \in M$, γ_x is a Legendrian twisted cubic with respect to the conformal symplectic structure $\mathcal{L}_x$ on C_x .

An equivalence between contact cubic structures (M, C, γ) and $(\widetilde{M}, \widetilde{C}, \widetilde{\gamma})$ is a diffeomorphism $f : M \rightarrow \widetilde{M}$ such that for all $x \in M$,

$$\begin{aligned} f_*(C_x) &= \widetilde{C}_{f(x)}, \\ f_*(\gamma_x) &= \widetilde{\gamma}_{f(x)}. \end{aligned}$$

A self equivalence is called an automorphism, or a symmetry.

Definition 2.5. A marked contact twisted cubic structure is a contact twisted cubic structure equipped with smooth section $\sigma : M \rightarrow \gamma$.

An equivalence between marked contact twisted cubic structures (M, C, γ, σ) and $(\widetilde{M}, \widetilde{C}, \widetilde{\gamma}, \widetilde{\sigma})$ is an equivalence f between the underlying contact twisted cubic structures (M, C, γ) and $(\widetilde{M}, \widetilde{C}, \widetilde{\gamma})$ such that

$$f_*(\sigma_x) = \widetilde{\sigma}_{f(x)} \quad (x \in M)$$

A self equivalence is called an automorphism or a symmetry.

Then we see various viewpoints on (marked) contact twisted cubic structure.

Proposition 2.6. A contact twisted cubic structure on a 5-dimensional manifold M , admits the following locally equivalent descriptions :

- (1) It is given by a contact distribution $C \subset TM$, an auxiliary rank 2 bundle $\mathcal{E} \rightarrow M$ and a vector bundle isomorphism

$$\Psi : \odot^3 \mathcal{E} \rightarrow C \quad (2.7)$$

compatible in the sense that for all $x \in M$, it pulls back the conformal symplectic structure $\mathcal{L}_x$ $GL(\mathcal{E}_x)$ -invariant on C_x one on $\odot^3 \mathcal{E}_x$.

- (2) It is given by a reduction of the graded frame bundle $\mathcal{F} \rightarrow M$ of a contact structure to the structure group $\rho(GL(2, \mathbb{R}))$ with respect to an irreducible representation

$$\rho : GL(2, \mathbb{R}) \rightarrow CSp(2, \mathbb{R})$$

- (3) It is given by a contact distribution $C = \ker(\alpha)$ on M and a reduction of the structure group of the frame bundle of C from $GL(4, \mathbb{R})$ to the irreducible $GL(2, \mathbb{R}) \subset CSp(d\alpha)$.

Proof. Given an isomorphism (2.7), for $x \in M$, the image of the map

$$\iota : \begin{cases} \mathbb{P}(\mathcal{E}_x) \longrightarrow \mathbb{P}(\odot^3 \mathcal{E}_x) \simeq \mathbb{P}(C_x) \\ \lambda \longmapsto [\Psi(\lambda \odot \lambda \odot \lambda)] \end{cases}$$

is a twisted cubic γ_x . By the compatibility requirement of the conformal symplectic structures and result about filtration of $\mathbb{R}^4$, the twisted cubic is Legendrian.

Conversely, given a subbundle $\gamma \subset \mathbb{P}(C)$ of twisted cubics, then in a neighborhood of each point there exists a rank 2 bundle $\mathcal{E}$ and a vector bundle isomorphism $\Psi : \odot^3 \mathcal{E} \simeq C$. The compatibility of the conformal symplectic structures follows from the fact that the twisted cubic is Legendrian and by the result of filtration of $\mathbb{R}^4$.

We omit the remaining equivalences. □

Proposition 2.8. *A marked contact twisted cubic structure on a 5-dimensional manifold M , admits the following locally equivalent descriptions :*

- (1) *It is given by a contact distribution $C \subset TM$, an auxiliary rank 2 bundle $\mathcal{E} \rightarrow M$ and a vector bundle isomorphism*

$$\Psi : \odot^3 \mathcal{E} \rightarrow C \quad (2.9)$$

compatible with the conformal symplectic structure and, in addition, a line bundle $L \subset \mathcal{E}$.

- (2) *It is given by a reduction of the graded frame bundle $\mathcal{F} \rightarrow M$ of a contact structure to the structure group $\rho(GL(2, \mathbb{R}))$ with respect to the irreducible representation*

$$\rho : B \rightarrow CSp(2, \mathbb{R})$$

of an irreducible $GL(2, \mathbb{R})$ -representation ρ to a Borel subgroup $B \subset GL(2, \mathbb{R})$.

- (3) *It is given by a contact twisted cubic structure equipped with a γ -congruence, that is, a foliation of M by curves whose tangent directions are everywhere contained in $\gamma \subset \mathbb{P}(C)$.*

Proposition 2.10. *A marked contact twisted cubic structure (M, C, γ, σ) is equipped with a flag of distributions*

$$l^\sigma \subset D^\sigma \subset H^\sigma \subset C \subset TM$$

where the 2-rank distribution $D^\sigma \subset C$ is Legendrian and the 3-rank distribution H^σ is the symplectic orthogonal to l^σ .

Definition 2.11. We say that a marked contact twisted cubic structure is integrable if the distribution D^σ is integrable.

In this case, the section $\sigma : M \rightarrow \gamma$ is called an integrable section.

Complex semisimple Lie algebras and their representations

In this section we will talk about the structure of complex semisimple Lie algebra and their complex representation ; presentation of Cartan subalgebras and the corresponding roots, or orderings, the resulting concepts of positive and simple roots, and the Dynkin diagram.

After this, we will talk about the classification of connected Dynkin diagrams and the its relation to the classification of complex simple Lie algebra. However, before we begin with the notion of the Killing form of a Lie algebra.

1. The Killing form

Let $\mathfrak{g}$ be a Lie algebra and V a finite dimensional vector space over $\mathbb{K}$ a field. Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$.

Definition 1.1. We define a bilinear form called the trace form associated to ρ .

$$B_V = B_\rho : \begin{cases} \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K} \\ (X, Y) \longmapsto \text{tr}(\rho(X) \circ \rho(Y)) \end{cases}$$

We can see that B_ρ is invariant ;

$$B_\rho([X, Y], Z) = B_\rho(X, [Y, Z]) \quad (X, Y \in \mathfrak{g})$$

Definition 1.2. We call the Killing form B of the Lie algebra $\mathfrak{g}$, the trace form associated to the adjoint representation ad where $ad(X)(Y) = ad_X(Y) = [X, Y]$, $(X, Y) \in \mathfrak{g}^2$.

Remark 1.3. Now we use the Killing form B .

Remark 1.4. For all $\phi \in \text{Aut}(\mathfrak{g})$,

$$B(\phi(X), \phi(Y)) = B(X, Y) \quad (X, Y \in \mathfrak{g})$$

i.e. B is invariant under arbitrary automorphisms of $\mathfrak{g}$.

Now we will characterize the Lie algebra $\mathfrak{g}$ in terms of the Killing form.

Lemma 1.5. Let V be a finite dimensional vector space and let $X \in \mathfrak{gl}(V)$.

We write $X = X_s + X_n$ the Jordan decomposition of X where X_s is the diagonalizable part and X_n the nilpotent part. Then, the Jordan decomposition of $ad(X)$ is given by

$$ad(X) = ad(X_s) + ad(X_n)$$

Proof. X_s and X_n commute then $ad(X_s)$ and $ad(X_n)$ too.

Thus, it suffices to show that $ad(X_s)$ is diagonalizable and $ad(X_n)$ nilpotent.

Note for $X \in \mathfrak{g}$,

$$ad(X_n)^k(Y) = \sum_{i=0}^k \alpha_i X_n^i Y_n^{k-i}$$

Thus, if $X_n^N = 0$ then $ad(X_n)^{2N} = 0$.

Take a basis of V with respect to which X_s is diagonale.

Consider $E_{i,j}$ the elementary matrix, $E_{i,j}$ is an eigenvector for $ad(X_s)$ with eigenvalue given by the difference of the i^{th} and the j^{th} diagonal entries of X_s .

□

We can talk about the Cartan's criteria.

Theorem 1.6. (1) Let V be a vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra.

If B_V is a zero on $\mathfrak{g}$ then $\mathfrak{g}$ is solvable.

Cartan's criteria : (For solvability and simplicity)

(2) A Lie algebra $\mathfrak{g}$ is solvable if, and only if, $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

(3) A Lie algebra $\mathfrak{g}$ is semisimple if, and only if, B is non degenerate.

Proof. (1) By complexification of V , we can view $\mathfrak{g} \subset \mathfrak{gl}(V_{\mathbb{C}})$ and $\mathfrak{g}_{|\mathbb{C}} \subset \mathfrak{gl}(V_{\mathbb{C}})$.

If $\mathfrak{g}_{|\mathbb{C}}$ is solvable then $\mathfrak{g}$ is solvable.

We may assume V is a complex vector space and $\mathfrak{g}$ is a complex Lie subalgebra of $\mathfrak{gl}(V)$.

Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, it is sufficient to show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Let $X \in [\mathfrak{g}, \mathfrak{g}]$, $X = X_s + X_n$.

Let $\widetilde{X}_s$ be the linear map which has the same eigenspaces as X_s but complex conjugate eigenvalues.

Using a basis of V such that X is its Jordan form, we see $tr(\widetilde{X}_s \circ X) = \sum_{j=1}^n |\lambda_j|^2$, where the λ_j are the eigenvalues of X .

If this vanishes, then X is nilpotent.

We mention the Engel's theorem which says that : If each element $X \in \mathfrak{g}$ acts as a nilpotent endomorphism of V , then there is $v \in V \setminus \{0\}$ such that for all $X \in \mathfrak{g}$, $X(v) = 0$.

This implies that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

We show that $tr(\widetilde{X}_s \circ X) = 0$;

We have

$$ad(X) = ad(X_s) + ad(X_n)$$

and $ad(X_s)$ can be written as a polynomial in $ad(X)$.

Then $ad(X_s)(\mathfrak{g}) \subset \mathfrak{g}$.

On the other hand, from the previous proof, $ad(\widetilde{X}_s)$ has the same eigenspaces as $ad(X_s)$.

Since the projections onto the eigenspaces of $ad(X_s)$ can be written as polynomial in $ad(X_s)$, we see that $ad(\widetilde{X}_s)$ is a polynomial in $ad(X_s)$.

So $ad(\widetilde{X}_s)(\mathfrak{g}) \subset \mathfrak{g}$.

Now since $X \in [\mathfrak{g}, \mathfrak{g}]$, $X = \sum_{finite} [Y_i, Z_i]$, for $(Y_i, Z_i) \in \mathfrak{g}^2$.

Then

$$\begin{aligned} tr(\tilde{X}_s \circ X) &= \sum_{finite} tr(\tilde{X}_s \circ [Y_i, Z_i]) = \sum_{finite} tr([\tilde{X}_i, Y_i] \circ Z_i) \\ &= \sum_{finite} B_V(ad(\tilde{X}_s)(Y_i), Z_i) \\ &= 0 \end{aligned}$$

(1) The necessity of the condition is clear.

Conversely, we assume $B([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]) = 0$ (which implies $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$).

By (1) the image of $[\mathfrak{g}, \mathfrak{g}]$ under the adjoint representation is solvable since $\ker(ad([\mathfrak{g}, \mathfrak{g}]))$ is the center of $[\mathfrak{g}, \mathfrak{g}]$ which is an abelian ideal, $[\mathfrak{g}, \mathfrak{g}]$ is solvable.

Since the quotient of $\mathfrak{g}$ by $[\mathfrak{g}, \mathfrak{g}]$ is abelian, then $\mathfrak{g}$ is solvable.

(3) For $\mathfrak{g}$ semisimple, consider

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid B(X, Y) = 0, \forall Y \in \mathfrak{g}\}$$

By invariance of the Killing form, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and by (1) $ad(\mathfrak{h}) \subset ad(\mathfrak{g})$ is solvable.

Since $\ker(ad(\mathfrak{h})) = Z(\mathfrak{g}) \cap \mathfrak{h}$ is abelian, then $\mathfrak{h}$ is solvable, thus $\mathfrak{h} = \{0\}$.

Conversely, we assume that B is non degenerate and $\mathfrak{h} \subset \mathfrak{g}$ is an abelian ideal.

For $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$, $ad(Y) \circ ad(X)$ maps $\mathfrak{g}$ to $\mathfrak{h}$ and $\mathfrak{h}$ to $\{0\}$.

Then this map is nilpotent then is tracefree.

Hence X is the null space (= the kernel) of B , so $X = 0$.

This show that $\mathfrak{g}$ has no trivial abelian ideal.

□

Next, we note an important consequence.

Corollary 1.7. (1) If $\mathfrak{g}$ is a semisimple Lie algebra, then there are simple ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ as a Lie algebra.

Moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and any ideal in $\mathfrak{g}$ as well as any homomorphic image of $\mathfrak{g}$ is simple.

(2) A real Lie algebra $\mathfrak{g}$ is semisimple if, and only if, its complexification $\mathfrak{g}_{\mathbb{C}}$ is semisimple.

(3) If $\mathfrak{g}$ is a complex simple Lie algebra and $\Phi : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is a $\mathfrak{g}$ -invariant complex bilinear form, then Φ is a multiple of the Killing form.

Proof. (1) If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{h}^{\perp}$, with respect to the Killing form, is an ideal by invariance of the Killing form.

We have $K|_{\mathfrak{h} \cap \mathfrak{h}^{\perp} \times \mathfrak{h} \cap \mathfrak{h}^{\perp}} = 0$ then $\mathfrak{h}^{\perp}$ is solvable.

By semisimplicity, $\mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$, hence $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and $[\mathfrak{h}, \mathfrak{h}^{\perp}] = \{0\}$.

Then the Killing form of $\mathfrak{h}$ is the restriction of the Killing form of $\mathfrak{g}$ and is non degenerate.

Thus, $\mathfrak{h}$ is semisimple and cannot be abelian, so the decomposition of $\mathfrak{g}$ into a sum of simple ideals, $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$ then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Finally, let $\mathfrak{a}$ be a Lie algebra and $\phi : \mathfrak{g} \rightarrow \mathfrak{a}$ be a homomorphism.

Then $\phi(\mathfrak{g}) \simeq \mathfrak{g}/\ker(\phi) \simeq \ker(\phi)^\perp$ which is semisimple.

(2) We know that if $g_{\mathbb{C}}$ is semisimple then $\mathfrak{g}$ is semisimple.

Conversely, we use the point (3) of the previous theorem and the fact that the Killing form of $g_{\mathbb{C}}$ is the complex bilinear extension of the Killing form of $\mathfrak{g}$.

(3) A $\mathfrak{g}$ -invariant subspace of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ then it follows that the adjoint representation of $\mathfrak{g}$ is a complex representation which is irreducible.

A complex bilinear form $\Phi : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ induces a map

$$\Phi^v : \begin{cases} \mathfrak{g} \longrightarrow \mathfrak{g}^* = \mathcal{L}(\mathfrak{g}, \mathbb{C}) \\ X \longmapsto (Y \mapsto \Phi(X, Y)) \end{cases}$$

Since $\Phi([Z, X], Y) = -\Phi(X, [Z, Y])$ for $(X, Y, Z) \in \mathfrak{g}^3$, then Φ^v is a homomorphism of $\mathfrak{g}$ -modules.

By (3) of the previous theorem, the map B^v induced by the Killing form is an isomorphism (since B is non degenerate). Thus, $(B^v)^{-1} \circ \Phi$ is an endomorphism of the irreducible representation $\mathfrak{g}$ of $\mathfrak{g}$.

Now, we can have the result by Schur's lemma.

□

2. Cartan subalgebras

We mention an important lemma.

Lemma 2.1. (Jordan decomposition) *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

(1) *Assume that $\mathfrak{g} \leq \mathfrak{gl}(V)$ for a finite-dimensional complex vector space V . For $X \in \mathfrak{g}$, let $X = S + N$ be the Jordan decomposition of the linear map $X : V \rightarrow V$.*

Then $S, N \in \mathfrak{g}$.

(2) *For any $X \in \mathfrak{g}$, there are unique $X_s, X_n \in \mathfrak{g}$ such that $ad(X) = ad(X_s) + ad(X_n)$ is the Jordan decomposition of $ad(X) : \mathfrak{g} \rightarrow \mathfrak{g}$.*

(3) *If $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a finite-dimensional complex representation, then for any element $X \in \mathfrak{g}$, the Jordan decomposition of $\rho(X) : V \rightarrow V$ is given by $\rho(X) = \rho(X_s) + \rho(X_n)$.*

Definition 2.2. An element $X \in \mathfrak{g}$ is called semisimple if the linear map $ad(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ is diagonalizable.

Remark 2.3. A such nonzero element always exists; if not by the lemma for any $X \in \mathfrak{g}$, $X_s = 0$. So for any $X \in \mathfrak{g}$, $ad(X)$ is nilpotent.

However by the Engel's theorem, this implies that $\mathfrak{g}$ is nilpotent and it is a contradiction.

To understand the theory of complex semisimple Lie algebra, we study the decomposition of $\mathfrak{g}$ onto eigenspaces for $ad(X)$ and $X \in \mathfrak{g}$ a semisimple element.

Let $X \in \mathfrak{g}$ be a semisimple element and $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}(X)$ be the decomposition onto eigenspaces for $ad(X)$.

By the Jacobi identity, for every λ, μ eigenvalues, $[\mathfrak{g}_{\lambda}(X), \mathfrak{g}_{\mu}(X)] \subset \mathfrak{g}_{\lambda+\mu}(X)$.

By the part (3) of the proviso lemma, for any finite-dimensional complex representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ the map $\rho(X) : V \rightarrow V$ is diagonalizable.

Let $V = \bigoplus_{\mu} V_{\mu}(X)$ be the decomposition into eigenspaces for the linear map $\rho(X)$.

Since for every $v \in V$, $(X, Y) \in \mathfrak{g}^2$, $X \cdot Y \cdot v = [X, Y] \cdot v + Y \cdot X \cdot v$, then for $Y \in \mathfrak{g}_{\lambda}(X)$, $v \in V_{\mu}(X)$, we have $Y \cdot v \in V_{\lambda+\mu}(X)$.

So we get information on the possible eigenvalues of $\mathfrak{g}$.

Next we will refine this decomposition in using the notion of simultaneous diagonalization with a type of subalgebra.

Definition 2.4. A Cartan subalgebra of complex semisimple Lie algebra is a maximal commutative subalgebra which consists of semisimple elements.

Is there Cartan subalgebra in a complex semisimple Lie algebra ?

To answer we need to introduce the centralizer.

Definition 2.5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $H \in \mathfrak{g}$. We define the centralizer of H in $\mathfrak{g}$ by

$$\mathfrak{c}(H) = \{X \in \mathfrak{g} \mid [X, H] = 0\}$$

For every $H \in \mathfrak{g}$, $\mathfrak{c}(H) \subset \mathfrak{g}$ is a Lie subalgebra.

Remark 2.6. We only are interested in the case of H a semisimple element of $\mathfrak{g}$.

Definition 2.7. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

We define the rank of $\mathfrak{g}$:

$$\text{rank}(\mathfrak{g}) = \min_{H \text{ semisimple element of } \mathfrak{g}} \dim(\mathfrak{c}(H))$$

Moreover, a H_0 such that $\dim(\mathfrak{c}(H_0)) = \text{rank}(\mathfrak{g})$ is called a regular semisimple element of $\mathfrak{g}$.

Example 2.8. Let $\mathfrak{g} = \mathfrak{sl}(\mathbb{C})$, and $H = \text{diag}(a_1, \dots, a_n)$ such that $a_1 + \dots + a_n = 0$.

We have for $i \neq j$, $[H, E_{i,j}] = (a_i - a_j)E_{i,j}$.

Then $\mathfrak{h} = \{H \in \mathfrak{g} \mid \text{tr}(H) = 0\}$ is a Cartan subalgebra and $\text{rank}(\mathfrak{g}) = n - 1$.

Moreover, we need to introduce some notions about automorphisms of a Lie algebra $\mathfrak{g}$. Since a $\Phi \in \text{Aut}(\mathfrak{g})$ satisfies by definition

$$\Phi([X, Y]) = [\Phi(X), \Phi(Y)] \quad (X, Y \in \mathfrak{g})$$

Then $\text{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$ and thus a Lie subgroup.

The Lie algebra of this Lie subgroup is given by all maps $A : \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies

$$A([X, Y]) - [A(X), Y] - [X, A(Y)] = 0, \quad (X, Y \in \mathfrak{g})$$

and this Lie algebra is noted $\mathfrak{der}(\mathfrak{g})$.

In the semisimple case, $Aut(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$.
The connected component of the identity in $Aut(\mathfrak{g})$ is called the group of inner automorphisms of $\mathfrak{g}$, noted $Int(\mathfrak{g})$.

Now we mention the theorem answering to the question.

Theorem 2.9. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

- (1) *If $H \in \mathfrak{g}$ is a regular semisimple element, then $\mathfrak{c}(H) \leq \mathfrak{g}$ is a Cartan subalgebra.*
- (2) *Any two Cartan subalgebra in $\mathfrak{g}$ are conjugate by an inner automorphism of $\mathfrak{g}$.*

With the notion of Cartan subalgebra we can refine the decomposition $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}(X)$ with $X \in \mathfrak{g}$ a semisimple element.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. For any finite-dimensional complex representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ the operators $\rho(H) : V \rightarrow V$ for $H \in \mathfrak{g}$ are simultaneously diagonalizable. Then the eigenvalues describe a linear functional

$$\lambda : \mathfrak{h} \rightarrow \mathbb{C}.$$

Definition 2.10. The functionals corresponding to nontrivial eigenspaces are called the weights of the representation ρ .

Let λ a weight, we call the weight space the space

$$V_{\lambda} := \{v \in V \mid \forall H \in \mathfrak{h}, \rho(H)(v) = \lambda(H)v\}.$$

Remark 2.11. $wt(V) = \{\lambda \in \mathfrak{h}^* \mid \lambda \text{ weight}\}$ is a finite subset of $\mathfrak{h}^*$.

Consider the adjoint representation.

The nonzero weight are called **the roots** of the Lie algebra $\mathfrak{g}$ and the weight space corresponding to root is called **the root space**.

We will denote the set of all roots of $\mathfrak{g}$ by $\Delta \subset \mathfrak{h}^*$ (for $\rho = adj$).

The weight space corresponding to the weight zero is the Cartan subalgebra, so we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

By the Jacobi identity

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta}. \quad (\alpha, \beta \in \Delta)$$

$$\mathfrak{g}_{\gamma} = \{0\}. \quad (\gamma \notin \Delta)$$

Remark 2.12.

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\alpha}] \subset \mathfrak{g}_0 = \mathfrak{h} \quad (\alpha \in \Delta)$$

Example 2.13. For $\mathfrak{g} = \mathfrak{sl}(\mathbb{C})$, for $1 \leq i \leq n$, $e_i : \begin{cases} \mathfrak{h} \longrightarrow \mathbb{C} \\ H \longmapsto H_{i,i} \end{cases}$.

$$\Delta = \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \subset \mathfrak{h}^*,$$

$$\mathfrak{h}_0^* = \{H \in \mathfrak{g} \mid \text{tr}(H) = 1, H \in M(n, \mathbb{R})\},$$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}.$$

3. The root system of a semisimple Lie algebra

Let a semisimple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$ and the corresponding root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

We can observe that $\alpha \in \Delta$ implies $-\alpha \in \Delta$ and that the Killing form induces a duality between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

Indeed, for every $\alpha, \beta \in \Delta$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ then for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ $ad(X) \circ ad(Y)(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{\alpha+\beta+\gamma}$.

so $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$.

The non degeneracy of the Killing form implies that for $\alpha \in \Delta$ also $-\alpha \in \Delta$ and the duality between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

We can see that $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate.

For each $\lambda \in \mathfrak{h}^*$, there is a unique $H_\lambda \in \mathfrak{h}$ such that

$$\lambda(H) = B(H_\lambda, H). \quad (H \in \mathfrak{h})$$

Then we define a nondegenerate complex bilinear form on $\mathfrak{h}^*$ by

$$\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu).$$

Thus, we can have the following proposition.

Proposition 3.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, $\Delta \leq \mathfrak{h}^*$ the corresponding set of roots, and $\langle, \rangle$ the complex bilinear form on $\mathfrak{h}^*$ induced by the Killing form.*

Then we have

- (1) *For any $\alpha \in \Delta$, also $-\alpha \in \Delta$ and there are the only complex multiples of α which are roots.*
- (2) *For any $\alpha \in \Delta$, the root space $\mathfrak{g}_\alpha$ is one-dimensional and the space $\mathfrak{s}_\alpha = \text{Vect}(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha})$ with $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*
- (3) *For $\alpha, \beta \in \Delta$, with $\beta \neq -\alpha$ we have*

$$\begin{cases} [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta \\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}, & \text{otherwise} \end{cases}.$$

- (4) *For $\alpha, \beta \in \Delta$ with $\beta \neq \pm\alpha$, and $z \in \mathbb{C}$, a functional of the form $\beta + z\alpha$ is a root if, and only if, $z \in \mathbb{Z}$.*

The roots of this form are an unbroken string

$$\beta - p\alpha; \beta - (p-1)\alpha; \dots; \beta + (q-1)\alpha; \beta + q\alpha,$$

where $p, q \geq 0$ and $p - q = \frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}$.

Next, we will talk about the Dynkin diagram. Then, we will work in the real space. More precisely, let $V \subset \mathfrak{h}^*$ be the real span of the set Δ of roots and let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the real subspace spanned by the H_α , for $\alpha \in \Delta$.

We will work on the real Euclidian space $\mathfrak{h}_0^*$ equipped by the complex bilinear form $\langle, \rangle$.

We claim that $\mathfrak{h}_0$ is the real form of $\mathfrak{h}$, its dual is $V = \mathfrak{h}^*$, the form $B|_{\mathfrak{h}_0 \times \mathfrak{h}_0}$, and $\langle, \rangle|_{\mathfrak{h}^* \times \mathfrak{h}^*}$ are positive definite.

Indeed :

Let $\alpha \in \Delta$, $\langle\alpha, \alpha\rangle = B(H_\alpha, H_\alpha) = \text{tr}(ad(H_\alpha) \circ ad(H_\alpha))$.

$ad(H_\alpha)$ acts trivially on $\mathfrak{h}^*$ and by multiplication by $\beta(H_\alpha)$ on $\mathfrak{g}_\beta$. Thus, $\langle\alpha, \alpha\rangle = \sum_{\beta \in \Delta} \beta(H_\alpha)^2$.

Since $\beta(H_\alpha) = \langle\alpha, \beta\rangle$,

$$\langle\alpha, \alpha\rangle = 2\langle\alpha, \alpha\rangle^2 + \sum_{\substack{\beta \in \Delta \\ \beta \neq \pm\alpha}} \langle\beta, \alpha\rangle^2. \quad (3.2)$$

By the part (4) of the proposition, $\frac{\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Q}$.

Dividing (3.2) by $\langle\alpha, \alpha\rangle^2$, we have $\langle\alpha, \alpha\rangle = \alpha(H_\alpha) \in \mathbb{Q}$.

Then, for every $\beta \in \Delta$, $\beta(H_\alpha) \in \mathbb{Q}$.

Again by (3.2), for all $\alpha \in \Delta$, $\langle\alpha, \alpha\rangle > 0$.

Hence, the reduction of $\langle, \rangle$ to $\mathfrak{h}_0$ is positive definite.

Moreover, the restriction of all roots to $\mathfrak{h}_0$ are real valued.

Thus, the real dimension of $\mathfrak{h}_0$ can be at most the complex dimension of $\mathfrak{h}$, since the roots span $\mathfrak{h}^*$, it cannot be smaller than that either. The remaining claim follows from dimensional consideration.

By these results, we consider the reflections of $\mathfrak{h}_0^*$.

For $\alpha \in \Delta$, the **root reflection** $s_\alpha : \mathfrak{h}_0^* \rightarrow \mathfrak{h}_0^*$ on the hyperplan $\alpha^\perp$ is given by

$$s_\alpha(\phi) = \phi - \frac{2\langle\phi, \alpha\rangle}{\langle\alpha, \alpha\rangle} \alpha. \quad (\phi \in \mathfrak{h}_0^*)$$

We have for every $\alpha \in \Delta$, $s_\alpha(\alpha) = -\alpha \in \Delta$.

For $\beta \in \Delta$, and $\beta \neq \pm\alpha$, $s_\alpha(\beta) = \beta - (p - q)\alpha$ where the α -string through β has the form $\mathfrak{g}_{\beta-p\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta+q\alpha}$.

Since $-p \leq q - p \leq q$, then $s_\alpha(\beta) \in \Delta$.

By this, we can obtain the following theorem.

Theorem 3.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r , and let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ the corresponding set of roots, $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ the real span of Δ and let $\langle, \rangle$ be the complex bilinear form on $\mathfrak{h}^*$ induced by the Killing form. Then,*

- (1) $\mathfrak{h}_0^*$ has real dimension r , and $\langle, \rangle|_{\mathfrak{h}_0^* \times \mathfrak{h}_0^*}$ is positive definite.

- (2) $\Delta \subset \mathfrak{h}_0^*$ spans $\mathfrak{h}_0^*$, any root reflection s_α for $\alpha \in \Delta$ maps Δ to itself and for $\alpha, \beta \in \Delta$ we have $2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.
- (3) For $\alpha \in \Delta$, $2\alpha \notin \Delta$.

Remark 3.4. A finite subset in a Euclidian vector space which satisfies the condition (2) is called an **abstract root system**.

If in addition, the condition (3) is satisfied, then this abstract root system is called **reduced**.

4. Positive and simple roots ; Dynkin diagram

Now, we will talk about Dynkin diagrams.

Since an abstract root system Δ on a Euclidian vector space $\mathfrak{h}^*$ is symmetric under root reflections, it follows that $\alpha \in \Delta$ implies $-\alpha \in \Delta$.

A natural idea is to split the set of roots into positive and negative roots.

Definition 4.1. Let V be a set. We define the positive V noted V^+ such that

$$V = -V^+ \cup \{0\} \cup V^+.$$

and V^+ is stable under addition and multiplication by positive scalars.

Remark 4.2. * A such decomposition gives a total ordering on V , defined by

$$v \leq w \iff v = w \text{ or } w - v \in V^+.$$

* We can get an ordering of V with a basis of V^* ;

Let $\phi_1, \dots, \phi_n$ be a basis of V^* and then

$$v \in V^+ \iff \exists j \in [1, n], \begin{cases} \phi_i(v) = 0, \text{ if } i < j \\ \phi_j(v) > 0 \end{cases}.$$

Let $V = \mathfrak{h}^*$ and V^+ be a positive set chosen, we define the set of **positive roots** $\Delta^+ \subset V^+$ by

$$\Delta^+ = \Delta \cap V^+.$$

Then $\Delta = -\Delta^+ \cup \{0\} \cup \Delta^+$.

Notation : $\alpha > 0$ if $\alpha \in \Delta^+$.

Moreover, we define the set of **simple roots** $\Delta^0 \subset \Delta^+$ as the set of those positive roots, which cannot be written as the sum of two positive roots.

Remark 4.3. * $Vect(\Delta^0) = \mathfrak{h}^*$, and any root can be written as a linear combination of simple roots, in which the coefficient are either all positives or all negatives.

Example 4.4. With the notation of the previous example, we let $\{E_{1,1} - E_{n,n}; \dots; E_{n-1,n-1} - E_{n,n}\}$ as the ordered basis for $\mathfrak{h}_0^*$ to construct a positive subset in $\mathfrak{h}_0^*$, then we obtain $\Delta^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ and $\Delta^0 = \{\alpha_1, \dots, \alpha_{n-1}\}$, where for all $1 \leq i \leq n-1$, $\alpha_i = e_i - e_{i+1}$.

Now, we look the map s_α for $\alpha \in \Delta$.

For $\alpha, \beta \in \Delta$, the Cauchy-Schwarz shows that $|\frac{2\langle\beta, \alpha\rangle\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle\langle\beta, \beta\rangle}| \leq 4$ and equality if, and only if, α and β are propositional.

Hence, $\frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in [-4, 4]$.

Moreover, for $\alpha \neq \beta$, and a reduced abstract root system, the borders cannot be attempted. For $\alpha, \beta \in \Delta^0$, $\alpha - \beta \notin \Delta$, and then $\langle\alpha, \beta\rangle \leq 0$.

Finally, for a reduced abstract root system and $\alpha, \beta \in \Delta$ with $\alpha \neq \pm\beta$. The elements of Δ of the form $\beta + n\alpha$ form an unbroken string of the form $\beta - p\alpha; \dots; \beta + q\alpha$ with $p - q = \frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}$.

Replacing β by $\beta - p\alpha$, we obtain that a such string has length at most four.

Definition 4.5. Let Δ be an abstract root system on $\mathfrak{h}^*$ and $\Delta^0 = \{\alpha_1; \dots; \alpha_l\}$ be the corresponding set of simple roots.

We define the Cartan matrix of Δ , the $l \times l$ -matrix $A = (a_{i,j})$ defined by

$$a_{i,j} = 2 \frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} = c_{\alpha_i, \alpha_j}. \quad (1 \leq i, j \leq l)$$

Clearly, the matrix depend on $\Delta^0 = \{\alpha_i \mid 1 \leq i \leq l\}$.

For all $1 \leq i \leq l$, $a_{i,i} = 2$.

For all $1 \leq i, j \leq l$, $a_{i,j} \in \mathbb{Z}$.

For all $1 \leq i < j \leq l$, $a_{i,j} = 0 \implies a_{j,i} = 0$.

Remark 4.6. * For all $1 \leq i \neq j \leq l$, $|c_{\alpha_i, \alpha_j}| = 1$ or $|c_{\alpha_j, \alpha_i}| = 1$.

* For all $1 \leq i, j \leq l$, $s_{\alpha_j}(\alpha_i) = \alpha_i - a_{i,j}\alpha_j = \alpha_i - c_{\alpha_i, \alpha_j}\alpha_j$.

Definition 4.7. we say that two Cartan matrix of Δ and Δ' are equivalent if, and only if, there is a bijection $\alpha \mapsto \alpha'$ from Δ to Δ' such that for every $1 \leq i, j \leq l = l(\Delta) = l(\Delta')$, $a_{i,j} = a'_{i,j}$.

Example 4.8. (1) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with the Cartan subalgebra.

$\mathfrak{h} = \{H \in \mathfrak{g} \mid \text{tr}(H) = 0\}$.

For $1 \leq i \leq n$, let $e_i : H \in \mathfrak{h} \mapsto H_{i,i} \in \mathbb{C}$.

We have $\Delta = \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \subset \mathfrak{h}^*$ and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}$, where $\mathfrak{g}_{e_i - e_j}$ is the one-dimensional subspace spanned by the $E_{i,j}$.

Moreover, for $\{E_{1,1} - E_{n,n}; \dots; E_{n-1,n-1} - E_{n,n}\}$ as an ordered basis of $\mathfrak{h}_0$, we have

$$\begin{aligned} \Delta &= \{e_i - e_j \mid 1 \leq i < j \leq n\}, \\ \Delta^0 &= \{\alpha_1; \dots; \alpha_{n-1}\}, \end{aligned}$$

where for $1 \leq i \leq n-1$, $\alpha_i = e_i - e_{i+1}$.

Let $H = \text{diag}(a_1; \dots; a_n)$ and $H_2 = \text{diag}(b_1; \dots; b_n)$, we have

$$\begin{aligned} B(H_1, H_2) &= \sum_{1 \leq i \neq j \leq n} (a_i - a_j)(b_i - b_j) \\ &= \sum_{1 \leq i \leq n} \sum_{i \neq j} (a_i b_i + a_j b_j - a_i b_j - a_j b_i). \end{aligned}$$

With $\sum_{i \neq j} a_j = -a_i$ we have

$$B(H_1, H_2) = 2n \sum_{1 \leq i \leq n} a_i b_i = 2n \left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle.$$

In particular, for $1 \leq i, j \leq n$, $H_{e_i - e_j} = \frac{1}{2n}(E_{ii} - E_{jj})$.

Hence, for all $1 \leq i \leq n$ $\langle \alpha_i, \alpha_j \rangle = \frac{1}{n}$, for all $1 \leq i \leq n-2$ $\langle \alpha_i, \alpha_{i+1} \rangle = -\frac{1}{2n}$, and for all $|i - j| > 1$ then $\langle \alpha_i, \alpha_j \rangle = 0$.

Then, the associated Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \dots & \dots & \dots & \\ & & \dots & \dots & \dots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

Having a Cartan matrix of Δ , we can associate a Dynkin diagram as follow :

Let $A = (a_{i,j})$ the Cartan matrix.

We define a partial oriented graph ;

– Δ is the set of vertices.

– For $\alpha_i \neq \alpha_j \in \Delta$, the number of edges between α_i and α_j is $n_{\alpha_i, \alpha_j} := c_{\alpha_i, \alpha_j} c_{\alpha_j, \alpha_i}$.

If $n_{\alpha_i, \alpha_j} > 1$ and $|c_{\alpha_i, \alpha_j}| = 1$ then we use the orientation

$$\alpha_i \longrightarrow \alpha_j$$

Conversely, having a Dynkin diagram Γ , we can associate a Cartan matrix $C(\Gamma) = C = (c_{\alpha_i, \alpha_j})$ as follow :

– $c_{\alpha_i, \alpha_i} = 2$.

– $c_{\alpha_i, \alpha_j} = \begin{cases} -n_{\alpha_i, \alpha_j}, & \text{if } n_{\alpha_i, \alpha_j} < 2 \text{ or } \alpha_j \longrightarrow \alpha_i \\ -1, & \text{if } n_{\alpha_i, \alpha_j} \geq 2 \text{ or } \alpha_i \longrightarrow \alpha_j \end{cases}.$

Example 4.9. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the Dynkin diagram is



With $n-1$ vertices.

5. The Weyl group

The classification of finite-dimensional simple Lie algebra is reduce to the classification of Dynkin diagrams.

Now, we have seen how to associate to a semisimple Lie algebra a Dynkin diagram making two choices ; we need to choose a Cartan subalgebra and a subset $\Delta^+ \subset \Delta$ of positive roots.

We have seen that any two Cartan subalgebras are conjugate under an inner automorphism of $\mathfrak{g}$, then the Dynkin diagram is independent of the chosen Cartan subalgebra.

To see that it doesn't depend on the subset Δ^+ , we need to introduce the Weyl group of an abstract root system.

Definition 5.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra.

We define the Weyl group $W = W(\Delta)$ of the corresponding abstract root system is called the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Let $\Delta \subset V$ is an abstract root system, then for every $\alpha \in \Delta$, $s_\alpha(\Delta) \subset \Delta$.

Then the Weyl group is

$$W(\Delta) = Vect(s_\alpha)_{\alpha \in \Delta} \subset O(\mathfrak{h}^*).$$

Remark 5.2. We have an associated action

$$\begin{cases} W \times \Delta \longrightarrow \Delta \\ (w, \alpha) \longmapsto w(\alpha) \end{cases}.$$

Then, $W \subset \mathfrak{S}_{|\Delta|}$ is finite.

Definition 5.3. Let W be the Weyl group of Δ .

We define the sign of $w \in W$ by

$$sign(w) = det(w).$$

Definition 5.4. We say that $\Delta^0 = \{\alpha_1, \dots, \alpha_n\} \subset \Delta$ is a simple subsystem of Δ if it is defined to be a basis of V consisting of reduced roots (i.e. for all $1 \leq j \leq n$, $2\alpha_j \notin \Delta$) such that any element of Δ can be written as a linear combination of the α_j with all coefficients of the same sign.

Having a such simple subsystem, we defines the corresponding positive roots $\Delta^+ \subset \Delta$ as consisting of those roots for which all coefficient are positives. Then, to choose a simple subsystem Δ^0 is equivalent to choose a set of positive roots Δ^+ .

Definition 5.5. Let $\Delta^0 = \{\alpha_1, \dots, \delta_n\}$ be a simple subsystem. We define the **height** of a root by the map

$$ht : \begin{cases} \Delta \longrightarrow \mathbb{Z} \\ \sum_{i=1}^n a_i \alpha_i \longmapsto \sum_{i=1}^n a_i \end{cases}.$$

Then we mention the following proposition.

Proposition 5.6. *Let Δ be an abstract root system and let $\Delta^0 = \{\alpha_1, \dots, \alpha_n\}$ be a simple subsystem.*

(1) $W(\Delta) = Vect(s_{\alpha_i})_{1 \leq i \leq n}$.

(2) *There is a bijection*

$$\begin{cases} W(\Delta) \longrightarrow \{\text{Simple subsystem of } \Delta\} \\ w \longmapsto w(\Delta^0) \end{cases}.$$

We see that any two simple subsystem are related by an orthogonal isomorphism ($w \in W(\Delta)$).

Then, the associated Cartan matrix which depends on inner products, do not change.

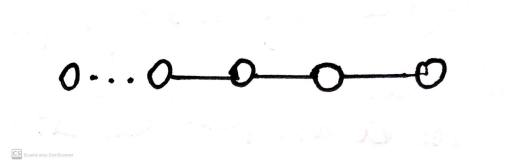
Thus, the associated Dynkin diagram do not change.

6. The classification of Dynkin diagram and of complex simple Lie algebra

We mention the main theorem.

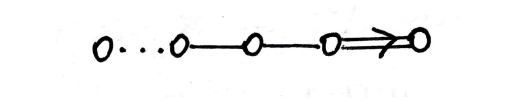
Theorem 6.1. *Any connected Dynkin diagram is of the following form.*

(1) A_n



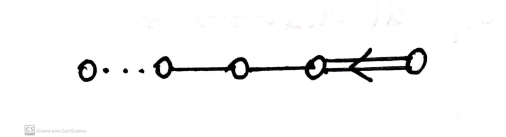
with n vertices.

(2) B_n



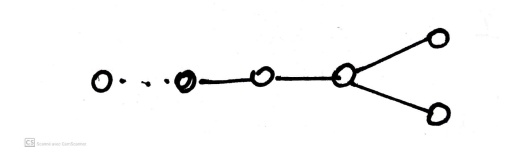
with n vertices.

(3) C_n



with n vertices.

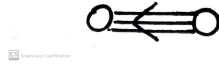
(4) D_n



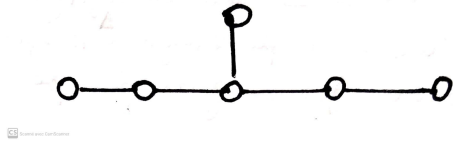
with n vertices.

Or of the "exceptional" form :

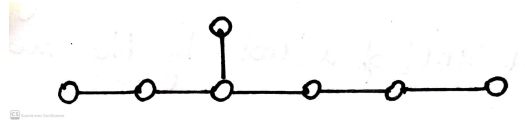
(1) G_2



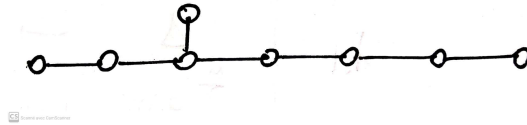
(2) E_6



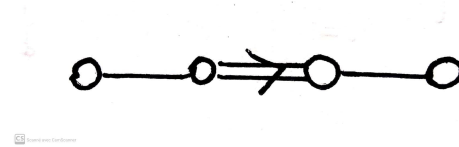
(3) E_7



(4) E_8



(5) F_4



Then, this implies that isomorphic simple Lie algebras lead to the same Dynkin diagram.

Conversely,

Theorem 6.2. (1) Any reduced abstract root system is isomorphic to the root system of some finite-dimensional complex simple Lie algebra.

(2) Two complex simple Lie algebra are isomorphic if, and only if root systems are isomorphism i.e. if, and only if they have the same Dynkin diagram, if and only if their root systems are isomorphic.

Example 6.3. We describe the root system of G_2 .

Its Dynkin diagram is given by

$$\Delta^0 = \{\alpha_1, \alpha_2\}, n_{\alpha_1, \alpha_2} = 3.$$

Then

$$C(G_2) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

and $\|\alpha_1\alpha_2\| = \sqrt{3}$.

Since $c_{\alpha_1, \alpha_2} < 0$, then $\cos(\widehat{\alpha_1\alpha_2}) = -\frac{\|\alpha_1\alpha_2\|}{\|\alpha_1\|\|\alpha_2\|} = -\frac{\sqrt{3}}{2}$.

Then $\widehat{\alpha_1\alpha_2} = \frac{\pi}{3}$.

We know that $\alpha, \beta \in \Delta$ implies $s_\alpha(\beta) \in \Delta$ and for $\beta_1, \dots, \beta_r \in \Delta^0$

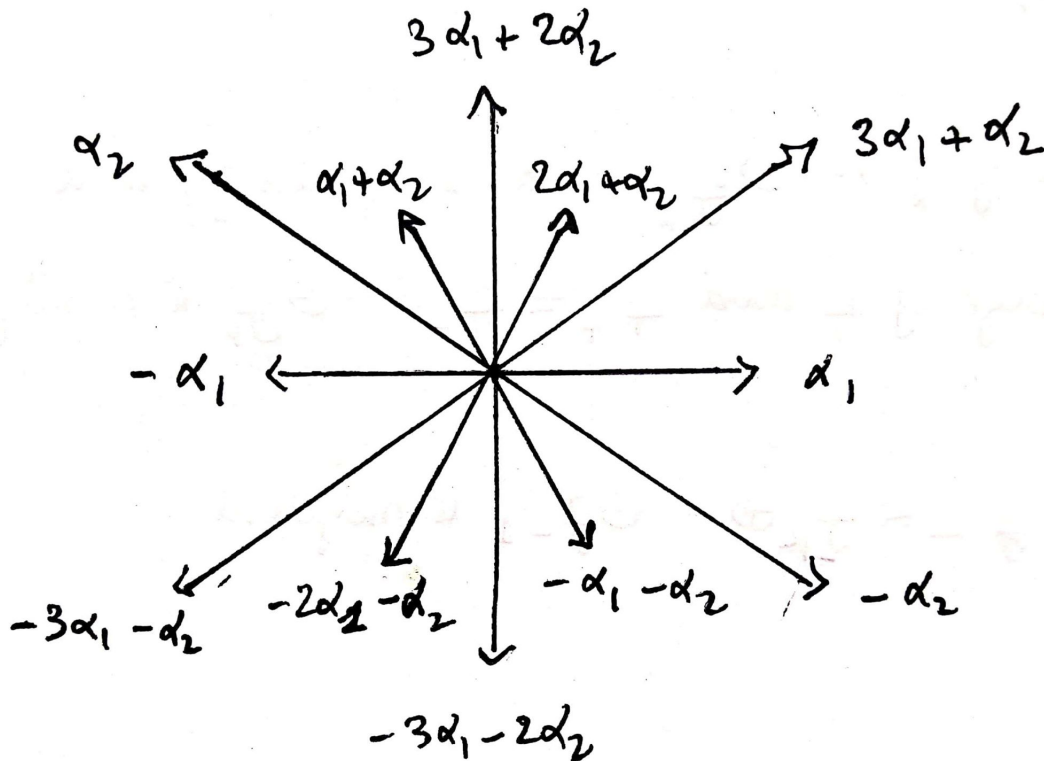
$$s_{\beta_1} \circ \dots \circ s_{\beta_r}(\alpha_1) \in \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\},$$

$$s_{\beta_1} \circ \dots \circ s_{\beta_r}(\alpha_2) \in \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}.$$

Then

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}.$$

Finally, the root diagram is



(Marked) contact Engel structures and the exponential Lie group G_2

1. Example of Lie group of type G_2

Now, we construct G_2 in terms of complex simple Lie algebra.
Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Proposition 1.1. *We define*

$$L = \{M_{(a,x,y)} = \begin{pmatrix} 0 & -2y^t & -2x^t \\ x & a & l_y \\ y & l_x & -a^t \end{pmatrix} \mid a \in \mathfrak{sl}(3, \mathbb{F}), x, y \in \mathbb{F}^3\}$$

a Lie algebra of $\mathfrak{gl}(7, \mathbb{F})$ of dimension 14, where

$$l_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3)$$

Let $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{F}^3$, and $(E_{i,j})_{1 \leq i, j \leq 3}$ the elementary matrix of $Mat(3, \mathbb{F})$.

Let $\mathfrak{h} = \{M_{(a,0,0)} \text{ diagonal} \mid a \in \mathfrak{sl}(3, \mathbb{F})\}$ be an abelian subalgebra of L .

We define for $1 \leq i \leq 3$, $\epsilon_i : \mathfrak{h} \rightarrow \mathbb{F}$ by

$$\epsilon_i(\text{diag}(0; s_1; s_2; s_3; -s_1; -s_2; -s_3)) = s_i,$$

where $\sum_{i=1}^3 s_i = 0$.

Then $\{\epsilon_1, \epsilon_2\}$ is a basis of $\mathfrak{h}^*$, since $\sum_{i=1}^3 \epsilon_i = 0$.

Now, we compute a basis of L of eigenvalues for $ad(h)$ where $h = \text{diag}(0; s_1; s_2; s_3; -s_1; -s_2; -s_3)$.

We have

$$[h, M_{(E_{i,j}, 0, 0)}] = M_{([\sum_{k=1}^n \delta_k E_{k,k}, E_{i,j}], 0, 0)} = (s_i - s_j)M_{(E_{i,j}, 0, 0)} = (\epsilon_i - \epsilon_j)(h)M_{(E_{i,j}, 0, 0)},$$

$$[h, M_{(0, e_i, 0)}] = \epsilon_i(h)M_{(0, e_i, 0)},$$

$$[h, M_{(0, 0, e_i)}] = -\epsilon_i(h)M_{(0, 0, e_i)};$$

Hence $\mathfrak{h}$ is a Cartan subalgebra and

$$L = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} L_\alpha \right),$$

where

$$\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm \epsilon_i \mid 1 \leq i \leq 3\} \subset \mathfrak{h}^*.$$

We have

$$L_{\epsilon_i - \epsilon_j} = \mathbb{F}M_{(E_{i,j}, 0, 0)}, \quad (1 \leq i \neq j \leq 3)$$

$$L_{\epsilon_i} = \mathbb{F}M_{(0,e_i,0)}, \quad (1 \leq i \leq 3)$$

$$L_{-\epsilon_i} = \mathbb{F}M_{(0,0,e_i)}. \quad (1 \leq i \leq 3)$$

Proposition 1.2. *L is a simple Lie algebra of type G_2 .*

Proof. Assume L admits an ideal $I \neq (0)$ and L .

Since $\mathfrak{h} \subset L$, $[\mathfrak{h}, I] \subset I$ then,

$$I = I \cap \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} I \cap L_\alpha \right).$$

Let $S = \{M_{(a,0,0)} \mid a \in \mathfrak{sl}(3, \mathbb{F})\}$ be a Lie algebra of L isomorphic to $\mathfrak{sl}(3, \mathbb{F})$.

If $I \cap S \neq (0)$, by simplicity of S , then $I \cap S = S$, then $\mathfrak{h} \subset S \subset I$, hence $L = \mathfrak{h} + [\mathfrak{h}, L] \subset I + [I, L] \subset I$ and $I = L$ (contradiction).

Otherwise, $I \cap S = (0)$, there is a some $\alpha \in \{\pm\epsilon_i \mid 1 \leq i \leq 3\}$ such that $I \cap L_\alpha \neq (0)$.

Since $\dim(L_\alpha) = 1$, then $L_\alpha \subset I$ and so

$$[L_\alpha, L_{-\alpha}] \subset [I, L] \subset I.$$

But by property of root decomposition of simple Lie algebra

$$0 \neq [L_\alpha, L_{-\alpha}] \subset L_0 = \mathfrak{h} \subset S$$

But $I \cap S = (0)$, so it is a contradiction.

Thus, L is simple. □

Now, we call $\alpha_1 := \epsilon_2$ and $\alpha_2 := \epsilon_1 - \epsilon_2$.

Then

$$\alpha_1 + \alpha_2 \epsilon_1,$$

$$3\alpha_1 + \alpha_2 = \epsilon_2 - \epsilon_3,$$

$$2\alpha_1 + \alpha_2 = -\epsilon_3,$$

$$3\alpha_1 + 2\alpha_2 = \epsilon_1 - \epsilon_3;$$

so that $\Delta = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ is the root system of type G_2 .

Remark 1.3. For $\mathbb{F} = \mathbb{R}$, $L_{\mathbb{R}} =: \mathfrak{g}_{2,2}$.

For $\mathbb{F} = \mathbb{C}$, $L_{\mathbb{C}} =: \mathfrak{g}_2$.

2. Real forms of a complex semisimple Lie algebra

Definition 2.1. Let $\mathfrak{g}$ be a complex Lie algebra. A real form of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ (viewed as a $\mathbb{R}$ -vector space) such that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 + i\mathfrak{g}_0$.

Theorem 2.2. *Every complex semisimple Lie algebra $\mathfrak{g}$ has a real form which is compact.*

Remark 2.3. A compact real form of $\mathfrak{g}$ can be construct as follows; a real form of $\mathfrak{g}$, where the Killing form is strictly negative definite; is compact.

Definition 2.4. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, $\mathfrak{g}$ its complexification, and σ the conjugaison of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

A direct decomposition $\mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$, into a subalgebra $\mathfrak{t}_0$ and a vector space $\mathfrak{p}_0$, is called a Cartan decomposition if there exists a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_c) \subset \mathfrak{g}_c, \mathfrak{t}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_c, \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{g}_c).$$

Remark 2.5. Each semisimple real Lie algebra $\mathfrak{g}_0$ has a Cartan decomposition.

We can see that any two Cartan decomposition are conjugate under an inner automorphism.

Theorem 2.6. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra.

Suppose $\mathfrak{g}_0 = \mathfrak{t}_1 + \mathfrak{p}_1$, $\mathfrak{g}_0 = \mathfrak{t}_2 + \mathfrak{p}_2$ are two Cartan decompositions of $\mathfrak{g}_0$. Then there exists $\psi \in \text{Int}(\mathfrak{g}_0)$ such that

$$\psi(\mathfrak{t}_1) = \mathfrak{t}_2, \psi(\mathfrak{p}_1) = \mathfrak{p}_2.$$

Definition 2.7. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is called normal or split if in each Cartan decomposition $\mathfrak{g}_0 = \mathfrak{t} + \mathfrak{p}$, the space $\mathfrak{p}$ contains a Cartan subalgebra of $\mathfrak{g}_0$.

Theorem 2.8. Each complex semisimple Lie algebra has a split real form and this is unique up to isomorphism.

Remark 2.9. We mention the fact that from the notion of Stake diagram, we can classify the real simple Lie algebras.

We can also obtain the real forms of complex simple Lie algebra.

For example, $\mathfrak{g}_2$ has two real forms ; the split real form $\tilde{\mathfrak{g}}_2$ and the compact real form $\mathfrak{g}_2^c$.

Now we will discuss about $\tilde{\mathfrak{g}}_2$.

3. Example of Lie algebra of type $\tilde{\mathfrak{g}}_2$

Now, we give an example of a Lie group having a Lie algebra isomorphic to $\tilde{\mathfrak{g}}_2$ and let it $\tilde{G}_2$.

$\tilde{G}_2$ is related to the action of $GL(7, \mathbb{R})$ on 3-forms on $\mathbb{R}^7$.

Let $(\tau^1, \dots, \tau^4)$ be an orthonormal coframe for the flat Riemannian metric $\eta = \sum_{i=1}^4 (\tau^i)^2$ in $\mathbb{R}^4$.

Let $(\sigma^1, \sigma^2, \sigma^3)$ the standard basis of the self-dual 2-forms :

$$\begin{aligned}\sigma^1 &= \tau^1 \wedge \tau^3 + \tau^4 \wedge \tau^2, \\ \sigma^2 &= \tau^4 \wedge \tau^1 + \tau^3 \wedge \tau^2, \\ \sigma^3 &= \tau^1 \wedge \tau^2 + \tau^3 \wedge \tau^4,\end{aligned}$$

and we define two 3-forms on $\mathbb{R}^7$ by

$$\sigma^- = \sigma^1 \wedge \tau^5 + \sigma^2 \wedge \tau^6 + \sigma^3 \wedge \tau^6 - \tau^5 \wedge \tau^6 \wedge \tau^7,$$

where $(\tau^1, \dots, \tau^4, \tau^5, \dots, \tau^7)$ is a coframe on $\mathbb{R}^7$.

We can define an action of $GL(7, \mathbb{R})$ on the 3-forms via :

$$\left\{ \begin{array}{l} GL(7, \mathbb{R}) \times \{3\text{-forms on } \mathbb{R}^7\} \longrightarrow \{3\text{-forms on } \mathbb{R}^7\} \\ ((a_{i,j})_{i,j}, \sigma = \sum_{\mu < \nu < \rho} \frac{1}{6} \sigma_{\mu < \nu < \rho} \tau^\mu \wedge \tau^\nu \wedge \tau^\rho) \longmapsto \sigma' = \frac{1}{6} \sum_{\alpha < \beta < \gamma} a_{\mu, \alpha} a_{\nu, \beta} a_{\rho, \gamma} \sigma_{\mu, \nu, \rho} \tau^\alpha \wedge \tau^\beta \wedge \tau^\gamma. \end{array} \right.$$

We have

Proposition 3.1. *The stabilizer of this action*

$$S^- := \{a \in GL(7, \mathbb{R}) \mid \sum_{\mu < \nu < \rho} a_{\mu, \alpha} a_{\nu, \beta} a_{\rho, \gamma} \sigma_{\mu, \nu, \rho}^- = \sigma_{\alpha, \beta, \gamma}^-\}$$

has a Lie algebra isomorphic to $\tilde{\mathfrak{g}}_2$.

Then we set $S^- =: \tilde{G}_2$.

In the proof we see

$$\tilde{\mathfrak{g}}_2 = \{X \in Mat(7, \mathbb{R}) \mid X = A + B + C\},$$

where

$$A = \begin{pmatrix} 0 & \frac{4}{3}\alpha_2 & \frac{4}{3}\alpha_0 & \frac{4}{9}\alpha_1 - \alpha_3 & -\frac{4}{9}\alpha_1 - \alpha_3 & -\frac{4}{3}\alpha_0 & 0 \\ -\frac{4}{3}\alpha_2 & 0 & 2\alpha_3 & 0 & 0 & -2\alpha_3 & -\frac{4}{3}\alpha_2 \\ -\frac{4}{3}\alpha_0 & -2\alpha_3 & 0 & -\frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & \frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & 0 & -\frac{4}{3}\alpha_0 \\ -\frac{4}{9}\alpha_1 + \alpha_3 & 0 & \frac{2}{3}\alpha_2 - \frac{3}{2}\alpha_4 & 0 & 0 & -\frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & -\frac{4}{9}\alpha_1 + \alpha_3 \\ -\frac{4}{9}\alpha_1 - \alpha_3 & 0 & \frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & 0 & 0 & -\frac{2}{3}\alpha_2 - \frac{3}{2}\alpha_4 & -\frac{4}{9}\alpha_1 - \alpha_3 \\ -\frac{4}{3}\alpha_0 & -2\alpha_3 & 0 & -\frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & \frac{2}{3}\alpha_2 + \frac{3}{2}\alpha_4 & 0 & -\frac{4}{3}\alpha_0 \\ 0 & -\frac{4}{3}\alpha_2 & -\frac{4}{3}\alpha_0 & -\frac{4}{9}\alpha_1 + \alpha_3 & \frac{4}{9}\alpha_1 + \alpha_3 & \frac{4}{3}\alpha_0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \frac{3}{4}\beta_2 - \frac{1}{3}\beta_3 & 0 & 0 & -\frac{3}{4}\beta_2 - \frac{1}{3}\beta_3 & 3\beta_1 + \beta_4 \\ 0 & 0 & 0 & \frac{3}{2}\beta_2 - \frac{2}{3}\beta_3 & \frac{3}{2}\beta_2 + \frac{2}{3}\beta_3 & 0 & 0 \\ -\frac{3}{4}\beta_2 + \frac{1}{3}\beta_3 & 0 & 0 & 0 & 0 & -3\beta_1 + \beta_4 & \frac{3}{4}\beta_2 + \frac{1}{3}\beta_3 \\ 0 & -\frac{3}{2}\beta_2 + \frac{2}{3}\beta_3 & 0 & 0 & -2\beta_4 & 0 & 0 \\ 0 & \frac{3}{2}\beta_2 + \frac{2}{3}\beta_3 & 0 & -2\beta_4 & 0 & 0 & 0 \\ -\frac{3}{4}\beta_1 - \frac{1}{3}\beta_3 & 0 & -3\beta_1 + \beta_4 & 0 & 0 & 0 & \frac{3}{4}\beta_2 - \frac{1}{3}\beta_3 \\ 3\beta_1 + \beta_4 & 0 & \frac{3}{4}\beta_2 + \frac{1}{3}\beta_3 & 0 & 0 & -\frac{3}{4}\beta_2 + \frac{1}{3}\beta_3 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -\frac{3}{2}\gamma_3 & -\frac{9}{8}\gamma_0 & -\frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 & \frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 & -\frac{9}{8}\gamma_0 & 0 \\ \frac{3}{2}\gamma_3 & 0 & \gamma_2 & 0 & 0 & \gamma_2 & -\frac{3}{2}\gamma_3 \\ \frac{9}{8}\gamma_0 & -\gamma_2 & 0 & -\gamma_1 + \frac{3}{4}\gamma_3 & \gamma_1 + \frac{3}{4}\gamma_3 & 0 & -\frac{9}{8}\gamma_0 \\ \frac{1}{2}\gamma_2 - \frac{27}{8}\gamma_4 & 0 & \gamma_1 - \frac{3}{4}\gamma_3 & 0 & 0 & \gamma_1 - \frac{3}{4}\gamma_3 & -\frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 \\ \frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 & 0 & \gamma_1 + \frac{3}{4}\gamma_3 & 0 & 0 & \gamma_1 + \frac{3}{4}\gamma_3 & -\frac{1}{2}\gamma_2 - \frac{27}{8}\gamma_4 \\ -\frac{9}{8}\gamma_0 & \gamma_2 & 0 & \gamma_1 - \frac{3}{4}\gamma_3 & -\gamma_1 - \frac{3}{4}\gamma_3 & 0 & \frac{9}{8}\gamma_0 \\ 0 & -\frac{3}{2}\gamma_3 & -\frac{9}{8}\gamma_0 & -\frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 & \frac{1}{2}\gamma_2 + \frac{27}{8}\gamma_4 & -\frac{9}{8}\gamma_0 & 0 \end{pmatrix},$$

where $\alpha_0, \dots, \alpha_4, \beta_1, \dots, \beta_4, \gamma_0, \dots, \gamma_4$ are 14 arbitrary real constants.

Then

$$e_0 = \frac{dA}{d\alpha_0}, e_i = \frac{dA}{d\alpha_i}, e_{4+j} = \frac{dB}{d\beta_j}, e_{8+i} = \frac{dC}{d\gamma_i}, e_{13} = \frac{dC}{d\gamma_0},$$

where $1 \leq i, j \leq 4$, form a basis of $\tilde{\mathfrak{g}}_2$.

4. Complex parabolic subalgebras

Now, we talk about parabolic subalgebras of complex semisimple Lie algebras.

Definition 4.1. A Borel subalgebra of $\mathfrak{g}$ a complex semisimple Lie algebra is a maximal solvable subalgebra $\mathfrak{b} \leq \mathfrak{g}$.

Proposition 4.2. For $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ a complex semisimple Lie algebra, the associated Borel subalgebra is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and Δ^+ is the positive system associated to $\mathfrak{h}$.

We call $\mathfrak{b}$ the standard Borel subalgebra associated to $\mathfrak{h}$ and Δ^+ .

Proof. $\mathfrak{n}_+$ is a nilpotent subalgebra and $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}_+$.

So $\mathfrak{b}$ is solvable.

Any subalgebra of $\mathfrak{g}$, which strictly contains $\mathfrak{b}$; has to contain at least one negative root space, say $\mathfrak{g}_{-\alpha}$ for a some $\alpha \in \Delta^+$. But then $\mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \oplus \mathfrak{g}_\alpha$ is a subalgebra isomorphic

to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

Since any subalgebra of a solvable Lie algebra is solvable, $\mathfrak{b}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$.

□

Remark 4.3. The choice of Cartan subalgebra and positive roots are unique up to conjugaison is equivalent to the fact that any two Borel subalgebras of a complex semisimple Lie algebra are conjugate by an inner automorphism of $\mathfrak{g}$.

Definition 4.4. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie subalgebra containing a Borel subalgebra.

Lie subalgebra containing a standard Borel subalgebra are called standard parabolic subalgebras.

Since any Borel subalgebra is conjugate to $\mathfrak{b}$, any parabolic subalgebra of $\mathfrak{g}$ is conjugate to a standard one.

Therefore, to understand parabolic subalgebras suffices to deal with the standard parabolic subalgebras.

Remark 4.5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with $\mathfrak{h}$ a Cartan subalgebra and Δ the set of all roots.

α is a positive root if, and only if α cannot be written as the sum of two positive roots.

If $\alpha \in \Delta^+$ is not simple (i.e. $\alpha \in \Delta^0$), then there is a simple root $\alpha \in \Delta^0$ such that $\alpha - \alpha_i \in \Delta$.

With this remark, we can have the following proposition.

Proposition 4.6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, Δ the corresponding set of roots and Δ^0 the set of simple roots for some choice of a positive system.

Then standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ are in bijective correspondance with subsets $\Sigma \subset \Delta^0$.

Precisely, we associate to $\mathfrak{p}$ the subset $\Sigma_{\mathfrak{p}} = \{\alpha \in \Delta^0 \mid \mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}\}$.

Conversely, the standard parabolic subalgebra $\mathfrak{p}_{\Sigma}$ corresponding to a subset Σ is the sum of the standard Borel subalgebra $\mathfrak{b}$ and all negative root space corresponding to roots which can be written as a linear combination of elements of $\Delta^0 \setminus \Sigma$.

Remark 4.7.

$$\Sigma \subset \Sigma' \subset \Delta^0 \implies \mathfrak{p}_{\Sigma'} \subset \mathfrak{p}_{\Sigma}$$

The description of standard parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ suggest a relation to k -grading.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

Let $\Sigma \subset \Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ of simple roots and a root $\alpha \in \Delta$.

We define the Σ -height ht_Σ already defined.

For $i \in \mathbb{Z} \setminus \{0\}$, let $\mathfrak{g}_i := \bigoplus_{\alpha | ht_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$ and $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha | ht_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$.

There is a maximal root in this ordering Σ ; $k = \max_{\alpha \in \Sigma} ht_\Sigma(\alpha)$.

Then we have a k -grading of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k.$$

Theorem 4.8. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra with corresponding roots Δ , Δ^+ a set of positive roots and $\Delta^0 \subset \Delta^+$ the set of simple roots.*

- (1) *For any standard parabolic standard $\mathfrak{p} \leq \mathfrak{g}$ corresponding to the subset $\Sigma \subset \Delta^0$, the decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ according to Σ -height makes $\mathfrak{g}$ into a k -graded Lie algebra such that $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$.*

Moreover, the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive and the dimension of its center $z(\mathfrak{g}_0) = \{x \in \mathfrak{g}_0 \mid \forall y \in \mathfrak{g}_0, xy = yx\}$ coincides with the number of elements of Σ .

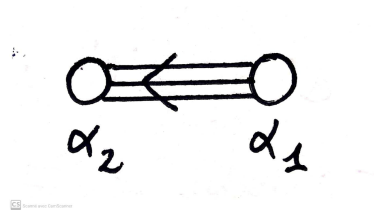
- (2) *Conversely, for any k -grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$; the subalgebra $\mathfrak{g}^0$ is parabolic, and choosing a Cartan subalgebra and positive roots in such a way that $\mathfrak{g}^0$ is a standard parabolic subalgebra $\mathfrak{p}_\Sigma$, the grading is given by the Σ -height.*

Definition 4.9. Let $\mathfrak{g}$ be a complex semisimple Lie algebra endowed with a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ and a set Δ^+ of positive roots.

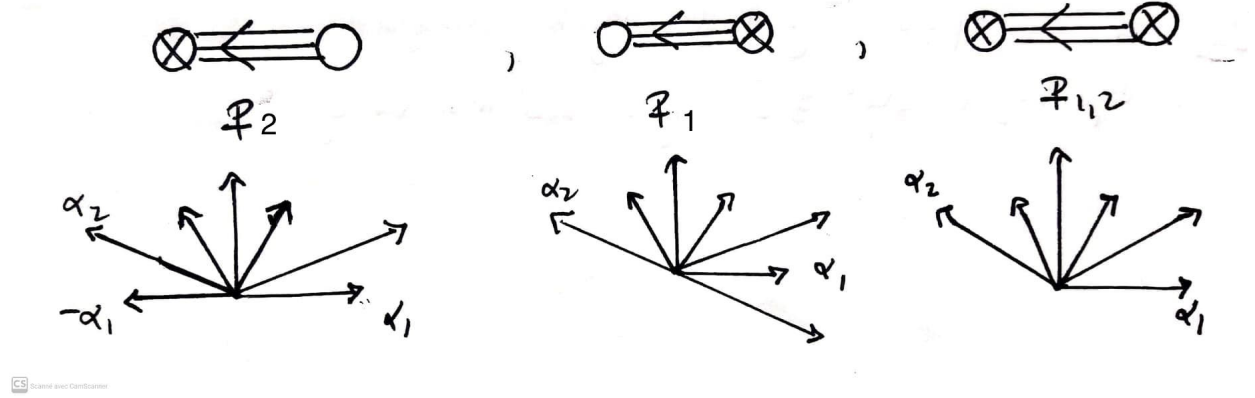
We denote the parabolic subalgebra $\mathfrak{p}_\Sigma \subset \mathfrak{g}$ corresponding to $\Sigma \subset \Delta^0$ by representing in the Dynkin diagram of $\mathfrak{g}$ the nodes corresponding to elements of Σ by a cross instead of a dot.

Example 4.10. (The parabolic subalgebras of $\mathfrak{g}_2$)

Its Dynkin diagram, with $\Delta^0 = \{\alpha_1, \alpha_2\}$.



The parabolic subalgebras are



Where

$$\begin{aligned} \mathfrak{p}_1 &= \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}, \\ \mathfrak{p}_2 &= \mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1}, \\ \mathfrak{p}_{1,2} &= \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1}. \end{aligned}$$

Remark 4.11. To talk about contact Engel structures, we interest in the case of the Lie algebra $\tilde{\mathfrak{g}}_2 =: \mathfrak{g}$ and its parabolic subalgebras.

Theorem 4.12. $\mathfrak{g}$ has, up to conjugacy, 3 parabolic subalgebras noted again $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_{1,2}$ with associated Lie groups P_1 , P_2 , and $P_{1,2}$.

Now, we describe P_2 .

Definition 4.13. Let g be a metric of a non-Riemannian signature on $\mathbb{R}^n$.

A real vector subspace $N \subset \mathbb{R}^n$ is called a totally null k -plane of $k = \dim(N)$ and for all $v_1, v_2 \in N$, $g(v_1, v_2) = 0$.

Definition 4.14. We note $\mathbb{R}^{(p,q)}$ the space $\mathbb{R}^{p+q}$ equipped with a signature metric of type (p, q) .

We will focus on totally null 2-planes in $\mathbb{R}^{(4,3)}$ equipped with signature metric $(4, 3)$

$$g = \sum_{i=1}^4 (\tau^i)^2 - \sum_{i=5}^7 (\tau^i)^2, \quad (4.15)$$

and the 3-form

$$\sigma^- = \sigma^1 \wedge \tau^5 + \sigma^2 \wedge \tau^6 + \sigma^3 \wedge \tau^6 - \tau^5 \wedge \tau^6 \wedge \tau^7, \quad (4.16)$$

where $(\tau^1, \dots, \tau^7)$ is a coframe on $\mathbb{R}^7$.

Let $N = Vect(v_1, v_2)$ be a totally null 2-plane in $(\mathbb{R}^{(4,3)}, g, \sigma^-)$.

Remark 4.17. We recall the inner product $\lrcorner$ of a vector $v \in \mathbb{R}^7$ and a p -form ω .

$$v \lrcorner \omega : (x_1; \dots; x_{p-1}) \longmapsto \omega(v; x_1; \dots; x_{p-1}).$$

We have two possibilities :

- (1) $v_1 \lrcorner v_2 \lrcorner (\sigma^-) \equiv 0$,
- (2) $v_1 \lrcorner v_2 \lrcorner (\sigma^-) \neq 0$.

In the case (2), $v_3 = g^{-1}(v_1 \lrcorner v_2 \lrcorner \sigma^-) \neq 0$ is yet another null vector in $\mathbb{R}^{(4,3)}$. Then v_3 is transversal to the 2-plane N and orthogonal to it. Thus, we have $N' = Vect(v_1, v_2, v_3)$ a null 3-plane (not interested here).

In the case (1), we call N a proper totally null 2-plane in $(\mathbb{R}^{(4,3)}, g, \sigma^-)$.

Given a proper totally null 2-plan N in $(\mathbb{R}^{(4,3)}, g, \sigma^-)$, we can chose a coframe $(\tau^i)_{1 \leq i \leq 7}$ on $\mathbb{R}^{(4,3)}$ such that σ^- is as in (4.14), g is as in (4.13), and N is given by

$$N = Vect(v_1, v_2),$$

with $v_1 = h_1 + h_7$, $v_2 = h_3 - h_6$, where $(h_i)_{1 \leq i \leq 7}$ the dual basis of $(\tau^i)_{1 \leq i \leq 7}$.

Then we have the following proposition.

Proposition 4.18. *The stabilizer P_2 in $\tilde{G}_2$ of a proper totally null 2-plane N in $(\mathbb{R}^{(4,3)}, g, \sigma^-)$ is a 9-dimensional parabolic subgroup of $\tilde{G}_2$ and the parabolic subalgebra is $\mathfrak{p}_2$.*

Remark 4.19. We can see that P_1 is the stabilizer of a null line in $\mathbb{R}^{(4,3)}$.

5. (Marked) contact Engel structures

We will construct a natural contact twisted cubic structure on G_2/P_2 , then we will define the notion of (marked) contact Engel structure.

Let P_2 , and $G_+ \subset P_2$ the unipotent radical of P_2 .

We recall that the radical of P_2 is the identity component of its maximal normal solvable subgroup.

Then the unipotent radical of P_2 is the set of unipotent elements in the radical of P_2 .

Let $G_0 := P_2/G_+$ be the reductive Levi factor ;

$$P_2 = G_0 \ltimes G_+.$$

Denote $\mathfrak{g}_+$ and $\mathfrak{g}_0 = \mathfrak{p}_2/\mathfrak{g}_+$ be the corresponding Lie algebras.

Via the adjoint action, P_2 preserve a filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \dots \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \dots \supset \mathfrak{g}^k, \quad (5.1)$$

where $\mathfrak{g}^1 := \mathfrak{g}_+$, for $j \geq 2$, $\mathfrak{g}^j := [\mathfrak{g}^{j-1}, \mathfrak{g}_+]$, and for $j \leq -1$ $\mathfrak{g}^{j+1} = (\mathfrak{g}^{-j})^\perp$ with respect to the Killing form.

In particular $\mathfrak{g}^0 = \mathfrak{p}_2$.

This determines an identification of the filtered Lie algebra $\mathfrak{g} := \widetilde{\mathfrak{g}}_2$ with its associated graded Lie algebra

$$gr(\mathfrak{g}) = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k.$$

We have $G_0 = GL(2, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{gl}(2, \mathbb{R})$.

Moreover, $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ and $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where

$$\mathfrak{p}_2 = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0. \quad (5.2)$$

Now, we can construct a Engel structure on (marked) contact twisted cubic structures.

Proposition 5.3. *The homogeneous space $\widetilde{G}_2/P_2$ is naturally equipped with a $\widetilde{G}_2$ -invariant contact twisted cubic structure.*

Proof. The tangent bundle of $\widetilde{G}_2/P_2$ is the associated bundle

$$T(\widetilde{G}_2/P_2) = \widetilde{G}_2 \times_{P_2} (\mathfrak{g}/\mathfrak{p}_2), \quad (5.4)$$

where $\mathfrak{g} := \widetilde{\mathfrak{g}}_2$.

The identification is induced by the trivialization of the tangent bundle of the Lie group $\widetilde{G}_2$ by left-invariant vector fields. Using the Maurer-Cartan form $\omega_{\widetilde{G}_2}$, it can be written as

$$\xi_x \longmapsto [g, \omega_{\widetilde{G}_2}(\xi_x) + \mathfrak{p}_2],$$

where $g \in \omega_{\widetilde{G}_2}$, $x = gP_2$, and $\xi_x \in T_x(\widetilde{G}_2/P_2)$.

The filtration (5.1) induces a P_2 -invariant filtration

$$C_o = \mathfrak{g}^{-1}/\mathfrak{p}_2 \subset \mathfrak{g}/\mathfrak{p}_2 = T_o(\widetilde{G}_2/P_2)$$

where $o = eP_2$ and, via the identification (5.3), a subbundle $C \subset T(G_2/P_2)$ of codimension one.

The Levi bracket $\mathcal{L} : \Lambda^2 C \rightarrow TM/C$ corresponds to the Lie bracket on $\mathfrak{g}_- = gr(\mathfrak{g}/\mathfrak{p}_2)$.

Since this is the 5-dimensional Heisenberg Lie algebra, C is a contact subbundle.

Moreover, since the unipotent radical acts trivially on $\mathfrak{g}^{-1}/\mathfrak{p}_2$, the P_2 action factors to a G_0 -action on $C_o = \mathfrak{g}^{-1}/\mathfrak{p}_2$. The latter action is irreducible, and the orbit through a highest weight line defines a G_0 -invariant Legendrian twisted cubic $\gamma_o \subset \mathbb{P}(C_o)$.

□

Definition 5.5. A contact twisted cubic structure is called a contact Engel structure if, and only if it locally equivalent to the $\widetilde{G}_2$ -invariant structure on $\widetilde{G}_2/P_2$.

Proposition 5.6. *Let $\gamma \subset \mathbb{P}(C)$ be the $\widetilde{G}_2$ -invariant contact twisted cubic structure on $\widetilde{G}_2/P_2$. Then*

$$\gamma = \widetilde{G}_2 \times_{P_2} P_2/P_{1,2} = \widetilde{G}_2/P_{1,2}.$$

Proof. The left $\tilde{G}_2$ -action on $\tilde{G}_2/P_2$ lifts to a $\tilde{G}_2$ -action on γ . Consider the fibre $\gamma_o \subset \mathbb{P}(\mathfrak{g}^{-1}/\mathfrak{p}_2)$ over the origin $o = eP_2$.

Then the $\tilde{G}_2$ -action on γ restricts to a P_2 -action on γ_o , which factors to an action of $G_0 = GL(2, \mathbb{R})$, since the unipotent radical acts trivially. The latter action is transitive on γ_o and the stabilizer of a point in γ_o (which is a height weight line in $\mathfrak{g}^{-1}/\mathfrak{p}_2$) is the Borel subgroup $B \subset GL(2, \mathbb{R})$ as in (0.31). Then the stabilizer in P_2 of the point is $B \rtimes \exp(\mathfrak{g}_+)$, which is the Borel subgroup $P_{1,2} \subset G_2$, and so

$$\gamma = \tilde{G}_2 \times_{P_2} \gamma_o = \tilde{G}_2 \times_{P_2} P_2/P_{1,2} = \tilde{G}_2/P_{1,2}.$$

□

Definition 5.7. A marked contact Engel structure is a marked contact twisted cubic structure whose underlying contact twisted cubic structure is flat.

Local invariant and homogeneous models of marked contact Engel structures via's Cartan method

In this chapter, we apply the Cartan method to the local equivalence problem of marked contact Engel structures.

1. Adapted coframes

We begin with the adapted coframes.

Definition 1.1. Let (M, C, γ) be a contact twisted cubic structure.

A local coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ where for all $1 \leq i \leq 4$, $\omega^i : U \mapsto C|_U$, on U an open set of M , is adapted to the contact twisted cubic structure if it terms of this coframe $C = \ker(\omega^0)$ and $\gamma \subset \mathbb{P}(C)$ is the projectivization of the set of all tangent vectors contained in C that are simultaneously null for the following three symmetric tensor fields

$$g_1 = \omega^1 \wedge \omega^3 - (\omega^2)^3, \quad g_2 = \omega^2 \wedge \omega^4 - (\omega^3)^2, \quad g_3 = \omega^2 \wedge \omega^3 - \omega^1 \wedge \omega^4.$$

Then we have the following proposition.

Proposition 1.2. – Two coframes $(\widehat{\omega}^0, \dots, \widehat{\omega}^4)$ and $(\omega^0, \dots, \omega^4)$ on U are adapted to the same twisted cubic structures if, and only if

$$\begin{pmatrix} \widehat{\omega}^0 \\ \widehat{\omega}^1 \\ \widehat{\omega}^2 \\ \widehat{\omega}^3 \\ \widehat{\omega}^4 \end{pmatrix} = \begin{pmatrix} s_0 & 0 & 0 & 0 & 0 \\ s_1 & s_5^3 & 3s_5^2s_6 & 3s_5s_6^2 & s_6^3 \\ s_2 & s_5^2s_7 & s_5(s_5s_8 + 2s_6s_7) & s_6(2s_5s_8 + s_6s_7) & s_6^2s_8 \\ s_3 & s_5s_7^2 & s_7(2s_5s_8 + s_6s_7) & s_8(s_5s_8 + 2s_6s_7) & s_6s_8^2 \\ s_4 & s_7^3 & 3s_7^2s_8 & 3s_7s_8^2 & s_8^3 \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix}, \quad (1.3)$$

where $s_0, \dots, s_8$ are smooth functions on U such that the determinant $s_0(s_6s_7 - s_5s_8)^6 \neq 0$.

– Two contact twisted cubic structures represented by $(\omega^0, \dots, \omega^4)$ on U and $(\widehat{\omega}^0, \dots, \widehat{\omega}^4)$ on V are locally equivalent if, and only if there is a local diffeomorphism $f : U \rightarrow V$ such that

$$\begin{pmatrix} f^*(\widehat{\omega}^0) \\ f^*(\widehat{\omega}^1) \\ f^*(\widehat{\omega}^2) \\ f^*(\widehat{\omega}^3) \\ f^*(\widehat{\omega}^4) \end{pmatrix} = M \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix},$$

where M is a matrix of the form as in (1.3).

Remark 1.4. The bottom 4×4 block in the transformation matrix form (1.3) is $GL(2, \mathbb{R})$ in the 4-dimensional irreducible representation viewed.

Definition 1.5. Let $\sigma : U \rightarrow \gamma \subset \mathbb{P}(C)$ be a section, and (M, C, γ, σ) be a marked contact twisted cubic structure on U .

A local coframe $(\omega^0, \dots, \omega^4)$ is adapted to the marked contact twisted cubic structure (M, C, γ, σ) if it is adapted to the underlying contact twisted and moreover the line field l^σ is given by $l^\sigma = \ker(\omega^0, \dots, \omega^4)$.

Then we have the following proposition.

Proposition 1.6. – Two coframes $(\hat{\omega}^0, \dots, \hat{\omega}^4)$ and $(\omega^0, \dots, \omega^4)$ on U are adapted to the same twisted cubic structures if, and only if

$$\begin{pmatrix} \hat{\omega}^0 \\ \hat{\omega}^1 \\ \hat{\omega}^2 \\ \hat{\omega}^3 \\ \hat{\omega}^4 \end{pmatrix} = \begin{pmatrix} s_0 & 0 & 0 & 0 & 0 \\ s_1 & s_5^3 & 0 & 0 & 0 \\ s_2 & s_5^2 s_7 & s_5^2 s_8 & 0 & 0 \\ s_3 & s_5 s_7^2 & 2s_7 s_5 s_8 & s_5 s_8^2 & 0 \\ s_4 & s_7^3 & 3s_7^2 s_8 & 3s_7 s_8^2 & s_8^3 \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix}, \quad (1.7)$$

where $s_0, \dots, s_8$ are smooth functions on U such that the determinant $s_0 s_5 s_8 \neq 0$.

– Two contact twisted cubic structures represented by $(\omega^0, \dots, \omega^4)$ on U and $(\hat{\omega}^0, \dots, \hat{\omega}^4)$ on V are locally equivalent if, and only if there is a local diffeomorphism $f : U \rightarrow V$ such that

$$\begin{pmatrix} f^*(\hat{\omega}^0) \\ f^*(\hat{\omega}^1) \\ f^*(\hat{\omega}^2) \\ f^*(\hat{\omega}^3) \\ f^*(\hat{\omega}^4) \end{pmatrix} = M \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix},$$

where M is a matrix of the form as in (1.3).

2. Structure equations for marked contact Engel structures

In this section, we will construct an adapted coframe for a marked contact Engel structure.

Let (M, C, γ, σ) a marked contact Engel structure.

Since the equivalence problem is local, we can assume

$$M = \tilde{G}_2 / P_2.$$

Now, we will determine the Maurer-Cartan equation of $\tilde{G}_2$ with respect to the basis $(e_0, \dots, e_{13})$ of $\mathfrak{g} := \tilde{\mathfrak{g}}_2$, which is adapted to the contact grading (see (5.2) in the previous chapter) :

$$\mathfrak{g} = gr(\mathfrak{g}) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

There is a basis of 1-forms $(\theta^i)_{0 \leq i \leq 13}$ on the Lie group $\tilde{G}_2$ consisting of the right-invariant forms $(\theta^i)_{0 \leq i \leq 13}$ such that the Maurer-Cartan form α in $\tilde{G}_2$ is

$$\alpha = \sum_{i=0}^{13} \theta^i e_i.$$

The θ^i satisfies the fundamental structure equations

$$d\theta^i = -\frac{1}{2} \sum_{0 \leq j < k \leq 13} c_{j,k}^i \theta^j \wedge \theta^k. \quad (0 \leq i \leq 13)$$

Then we obtain by computing

$$\begin{aligned} d\theta^0 &= -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\ d\theta^1 &= 6\theta^0 \wedge \theta^9 - 3\theta^1 \wedge \theta^5 - 3\theta^1 \wedge \theta^8 + 3\theta^2 \wedge \theta^7, \\ d\theta^2 &= 2\theta^0 \wedge \theta^{10} + \theta^1 \wedge \theta^6 - 3\theta^2 \wedge \theta^5 - \theta^2 \wedge \theta^8 + 2\theta^3 \wedge \theta^7, \\ d\theta^3 &= 2\theta^0 \wedge \theta^{11} + 2\theta^2 \wedge \theta^6 - 3\theta^3 \wedge \theta^5 + \theta^3 \wedge \theta^8 + \theta^4 \wedge \theta^7, \\ d\theta^4 &= 6\theta^0 \wedge \theta^{12} + 3\theta^3 \wedge \theta^6 - 3\theta^4 \wedge \theta^5 + 3\theta^4 \wedge \theta^8, \\ d\theta^5 &= 2\theta^0 \wedge \theta^{13} - \theta^1 \wedge \theta^{12} + \theta^2 \wedge \theta^{11} - \theta^3 \wedge \theta^{10} + \theta^4 \wedge \theta^9, \\ d\theta^6 &= 6\theta^2 \wedge \theta^{12} - 4\theta^3 \wedge \theta^{11} + 2\theta^4 \wedge \theta^{10} + 2\theta^6 \wedge \theta^8, \\ d\theta^7 &= -2\theta^1 \wedge \theta^{11} + 4\theta^2 \wedge \theta^{10} - 6\theta^3 \wedge \theta^9 - 2\theta^7 \wedge \theta^8, \\ d\theta^8 &= -3\theta^1 \wedge \theta^{12} + \theta^2 \wedge \theta^{11} + \theta^3 \wedge \theta^{10} - 3\theta^4 \wedge \theta^9 - \theta^6 \wedge \theta^7, \\ d\theta^9 &= -\theta^1 \wedge \theta^{13} - 3\theta^5 \wedge \theta^9 - \theta^7 \wedge \theta^{10} + 3\theta^8 \wedge \theta^9, \\ d\theta^{10} &= -3\theta^2 \wedge \theta^{13} - 3\theta^5 \wedge \theta^{10} - 3\theta^6 \wedge \theta^9 - 2\theta^7 \wedge \theta^{11} + \theta^8 \wedge \theta^{10}, \\ d\theta^{11} &= -3\theta^3 \wedge \theta^{13} - 3\theta^5 \wedge \theta^{11} - 2\theta^6 \wedge \theta^{10} - 3\theta^7 \wedge \theta^{12} - \theta^8 \wedge \theta^{11}, \\ d\theta^{12} &= -\theta^4 \wedge \theta^{13} - 3\theta^5 \wedge \theta^{12} - \theta^6 \wedge \theta^{11} - 3\theta^8 \wedge \theta^{12}, \\ d\theta^{13} &= -6\theta^5 \wedge \theta^{13} - 6\theta^9 \wedge \theta^{12} + 2\theta^{10} \wedge \theta^{11}. \end{aligned} \quad (2.1)$$

The kernel of the nine right-invariant forms $\theta^5, \theta^6, \dots, \theta^{13}$ defines an integrable distribution and the leaves of the corresponding foliation correspond to certain sections of $\tilde{G}_2 \rightarrow \tilde{G}_2/P_2$.

For any section of $\tilde{G}_2 \rightarrow \tilde{G}_2/P_2$ the pullbacks of the forms $\theta^5, \theta^6, \dots, \theta^{13}$ vanish. Moreover, the remaining forms $\theta^0, \theta^1, \dots, \theta^4$ define an adapted coframe $(\alpha^0, \dots, \alpha^4)$ for the contact Engel structure on $\tilde{G}_2 \rightarrow \tilde{G}_2/P_2$, which satisfies the system

$$d\alpha^0 = \alpha^1 \wedge \alpha^4 - 3\alpha^2 \wedge \alpha^3, d\alpha^1 = 0, d\alpha^2 = 0, d\alpha^3 = 0, d\alpha^4 = 0. \quad (2.2)$$

Integrating this system yields local coordinates $(x^0, x^1, x^2, x^3, x^4)$ such that

$$\alpha^0 = dx^0 + x^1 dx^4 - 3x^2 dx^3, \alpha^1 = dx^1, \alpha^2 = dx^2, \alpha^3 = dx^3, \alpha^4 = dx^4. \quad (2.3)$$

Then a such coframe $(\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4)$ is an adapted coframe for the contact Engel structure (M, C, γ) .

Now, we construct an adapted coframe for the marked contact Engel structure (M, C, γ, σ) .

We denote by $(X_0, X_1, X_2, X_3, X_4)$ the frame dual to the coframe $(\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4)$.

We may assume that the section $\sigma : U \rightarrow \gamma$, where U is an open set of $\tilde{G}_2/P_2$, is of the form

$$\sigma = [-t^3 X_1 + t^2 X_2 - t X_3 + X_4], \quad (2.4)$$

where $t = t(x^0, x^1, x^2, x^3, x^4)$ is smooth function on U .

Remark 2.5. The choice of t determines a marked contact Engel structure, and up to equivalence, all marked contact Engel structures can be obtained by this way. Different t 's can correspond to the same structure (up to local equivalence).

By the proposition 2.10 of the chapter "Definitions and descriptions of (marked) contact twisted cubic structures", the filtration is of the form

$$l^\sigma = Vect(\xi_4) \subset D^\sigma = Vect(\xi_4, \xi_3) \subset H^\sigma = Vect(\xi_4, \xi_3, \xi_2) \subset C^\sigma = Vect(\xi_4, \xi_3, \xi_2, \xi_1), \quad (2.6)$$

where

$$\begin{aligned} \xi_4 &:= -t^3 X_1 + t^2 X_2 - t X_3 + X_4 = -(x^1 + 3tx^2) \frac{\partial}{\partial x^0} - t^3 \frac{\partial}{\partial x^1} + t^2 \frac{\partial}{\partial x^2} - t \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}, \\ \xi_3 &:= 3t^2 X_1 - 2t X_2 + X_3 = 3x^2 \frac{\partial}{\partial x^0} + 3t^2 \frac{\partial}{\partial x^1} - 2t \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \\ \xi_2 &:= -3t X_1 + X_2 = -3t \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \\ \xi_1 &:= X_1 = \frac{\partial}{\partial x^0}, \\ \xi_0 &:= X_0 = \frac{\partial}{\partial x^1}. \end{aligned}$$

Passing to the coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ dual to the frame $(\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$, we obtain the following lemma.

Lemma 2.7. *The most general marked contact Engel structure (M, C, γ, σ) can be locally represented in terms of the following adapted coframe.*

$$\begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = \begin{pmatrix} dx^0 + x^1 dx^4 - 3x^2 dx^3 \\ dx^1 + 3t dx^2 + 3t^2 dx^3 + t^3 dx^4 \\ dx^2 + 2t dx^3 + t^2 dx^4 \\ dx^3 + t dx^4 \\ dx^4 \end{pmatrix}, \quad (2.8)$$

where $t = t(x^0, x^1, x^2, x^3, x^4)$ a smooth function on U .

The filtration (2.6) associated the a marked contact Engel structure is given in terms of this

coframe as

$$l^\sigma = Vect(\omega^0, \omega^1, \omega^2, \omega^3) \subset D^\sigma = Vect(\omega^0, \omega^1, \omega^2) \subset H^\sigma = Vect(\omega^0, \omega^1) \subset C^\sigma = Vect(\omega^0). \quad (2.9)$$

Definition 2.10. A marked contact Engel structure is flat if it can be locally represented in terms of an adapted coframe $(\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4)$ as in (2.3).
In other word, $t = 0$ for the marked contact Engel structure.

Then we obtain the following lemma.

Lemma 2.11. Any marked contact Engel structure admits an adapted coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ satisfying

$$\begin{aligned} d\omega^0 &= \omega^1 \wedge \omega^4 - 3\omega^2 \wedge \omega^3, \\ d\omega^1 &= \frac{3}{4}(b^2 - 4ac + M - P)\omega^0 \wedge \omega^2 + 3c\omega^1 \wedge \omega^2 - 3a\omega^2 \wedge \omega^3 + 3J\omega^2 \wedge \omega^4, \\ d\omega^2 &= \frac{1}{2}(b^2 - 4ac + M - P)\omega^0 \wedge \omega^3 + 2c\omega^1 \wedge \omega^3 - 2b\omega^2 \wedge \omega^3 + 2J\omega^3 \wedge \omega^4, \quad (2.12) \\ d\omega^3 &= \frac{1}{4}(b^2 - 4ac + M - P)\omega^0 \wedge \omega^4 + c\omega^1 \wedge \omega^4 - b\omega^2 \wedge \omega^4 + a\omega^3 \wedge \omega^4, \\ d\omega^4 &= 0. \end{aligned}$$

for functions a, b, c, J, M, P .

Proof. We differentiate the adapted coframe in the previous lemma.

$$\begin{aligned} d\omega^0 &= \omega^1 \wedge \omega^4 - 3\omega^2 \wedge \omega^3, \\ d\omega^1 &= 3dt \wedge \omega^2, \\ d\omega^2 &= 2dt \wedge \omega^3, \\ d\omega^3 &= dt \wedge \omega^4, \\ d\omega^4 &= 0. \end{aligned}$$

Let $dt = \sum_{i=0}^4 t_{\omega^i} \omega^i$. There is a unique solution for a, b, c, J, M, P in terms of the function t and its derivatives.

□

Applying again the exterior derivative on (2.12), we obtain the following result.

Lemma 2.13. *The functions a, b, c, J from the previous lemma satisfy*

$$\begin{aligned}
 dJ &= J_{\omega^0}\omega^0 + J_{\omega^1}\omega^1 + J_{\omega^2}\omega^2 + J_{\omega^3}\omega^3 + J_{\omega^4}\omega^4, \\
 da &= a_{\omega^0}\omega^0 + a_{\omega^1}\omega^1 + \frac{1}{4}(-3b^2 + M + 3P)\omega^2 + L\omega^3 + (a^3 - 2bJ - J_{\omega^3})\omega^4, \\
 db &= \frac{1}{4}(-4a_{\omega^1}b + 6b^2c - 8ac^2 + 4cM - M_{\omega^2} + P_{\omega^2} + 2bQ - 4aR)\omega^0 + (2c^2 + R)\omega^1 \\
 &\quad + (2a_{\omega^1} - 3bc - Q)\omega^2 + \frac{1}{2}(-b^2 + M - 3P)\omega^3 + (ab - 3cJ + J_{\omega^2})\omega^4, \\
 dc &= c_{\omega^0}\omega^0 + S\omega^1 + (c^2 - R)\omega^2 + (a_{\omega^1} - 2bc)\omega^3 + \frac{1}{4}(b^2 - 4J_{\omega^2} + M - P)\omega^4,
 \end{aligned} \tag{2.14}$$

for function L, Q, R, S .

Lemma 2.15. *The functions $a, b, c, J, L, M, P, Q, R, S$ are uniquely determined by (2.12) and (2.14). Explicitly,*

$$\begin{aligned}
 a &= t_{\omega^3}, \quad b = -t_{\omega^2}, \quad c = t_{\omega^1}, \quad J = -t_{\omega^4}, \quad M = 6t_{\omega^0} - 2(t_{\omega^2})^2 + 6t_{\omega^3}t_{\omega^1} + t_{\omega^2}\omega^3, \\
 P &= 2t_{\omega^0} - (t_{\omega^2})^2 + 2t_{\omega^3}t_{\omega^1} + t_{\omega^2}\omega^3, \quad Q = 2t_{\omega^3}\omega^1 + t_{\omega^2}\omega^3 + 3t_{\omega^2}\omega^1, \quad R = -t_{\omega^2}\omega^1 - 2(t_{\omega^1})^2, \\
 S &= t_{(\omega^1)^2}.
 \end{aligned}$$

3. The main invariants and a characterization of the flat model

In this section we will prove the following theorem.

Before we introduce the group $\mathbf{S}$ defined by

$$\mathbf{S} = \{(\mathbf{S}_{\nu}^{\mu})_{0 \leq \nu, \mu \leq 4} = \begin{pmatrix} s_0 & 0 & 0 & 0 & 0 \\ s_1 & s_5^3 & 0 & 0 & 0 \\ s_2 & s_5^2 s_7 & s_5^2 s_8 & 0 & 0 \\ s_3 & s_5 s_7^2 & 2s_7 s_5 s_8 & s_5 s_8^2 & 0 \\ s_4 & s_7^3 & 3s_7^2 s_8 & 3s_7 s_8^2 & s_8^3 \end{pmatrix} \mid \det(\mathbf{S}_{\nu}^{\mu}) = s_0 s_5^6 s_8^6 \neq 0\}.$$

Then

Theorem 3.1. *Given the most general marked contact Engel structure on U , consider an adapted coframe $\omega = (\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ that satisfies structure equations (2.12), and let J, L, M, P, Q, R, S be the functions defined via (2.12) and (2.14).*

- (1) *Let $\hat{\omega} = (\hat{\omega}^0, \hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3, \hat{\omega}^4)$ be another coframe related to ω via $\hat{\omega} = A \cdot f^*(\omega)$, with $f : U \rightarrow U$ a diffeomorphism and $A : U \rightarrow \mathbf{S}$ a function with values in the structure group $\mathbf{S}$.*

Suppose that $\hat{\omega}$ satisfies the structure equations (2.12) for some functions

$\widehat{a}, \widehat{b}, \widehat{c}, \widehat{J}, \widehat{M}, \widehat{P}$, and let $\widehat{Q}, \widehat{R}, \widehat{S}$ be the derived functions as in (2.14). Then

- (a) $J = 0$ iff $\widehat{J} = 0$
- (b) $J = L = 0$ iff $\widehat{J} = \widehat{L} = 0$
- (c) $J = L = M = 0$ iff $\widehat{J} = \widehat{L} = \widehat{M} = 0$
- (d) $J = L = M = P = 0$ iff $\widehat{J} = \widehat{L} = \widehat{M} = \widehat{P} = 0$
- (e) $J = L = M = P = Q = 0$ iff $\widehat{J} = \widehat{L} = \widehat{M} = \widehat{P} = \widehat{Q} = 0$
- (f) $J = L = M = P = Q = R = 0$ iff $\widehat{J} = \widehat{L} = \widehat{M} = \widehat{P} = \widehat{Q} = \widehat{R} = 0$
- (f) $J = L = M = P = Q = R = S = 0$ iff $\widehat{J} = \widehat{L} = \widehat{M} = \widehat{P} = \widehat{Q} = \widehat{R} = \widehat{S} = 0$.

(2) A marked contact Engel structure is flat if, and only if

$$J = L = M = P = Q = R = S = 0 \quad (3.2)$$

holds.

In this case the structure has a 9-dimensional algebra of infinitesimal symmetries isomorphic to the parabolic subalgebra $\mathfrak{p}_1$.

Proof. Let $\omega = (\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ an adapted coframe that satisfies (2.12).

We have a trivialization of the bundle of all adapted coframes, identified by $\pi : U \times \mathbf{S} \rightarrow U$. Then we lift ω to the lifted coframe

$$\theta^\mu = \sum_{\nu=0}^4 \mathbf{S}_\nu^\mu \omega^\nu, \quad (0 \leq \mu \leq 4)$$

on $U \times \mathbf{S}$.

With the equation (2.14) written in the following form

$$d\omega^\mu = -\frac{1}{2} \sum_{0 \leq \nu < \rho \leq 4} F_{\nu, \rho}^\mu \omega^\nu \wedge \omega^\rho, \quad (0 \leq \mu \leq 4)$$

we express the differentials $d\theta^0, \dots, d\theta^4$ as

$$d\theta^\mu = \sum_{\nu=0}^4 (d(\mathbf{S}_\nu^\mu) \wedge \omega^\nu + \mathbf{S}_\nu^\mu d\omega^\nu) = \sum_{\nu, \sigma=0}^4 d\mathbf{S}_\mu^\nu (\mathbf{S}^{-1})_\sigma^\rho \wedge \theta^\sigma - \sum_{\nu, \rho < \sigma, \alpha < \beta=0}^4 \mathbf{S}_\nu^\mu F_{\rho, \sigma}^\nu (\mathbf{S}^{-1})_\sigma^\rho (\mathbf{S}^{-1})_\beta^\alpha \theta^\alpha \wedge \theta^\beta.$$

We set

$$\delta = -s_5 s_8.$$

To prove this theorem, we will find the unknowns which are the group parameters $s_0, \dots, s_7, \delta$ and the 1-forms $\theta^5, \theta^6, \theta^8$ and θ^{12} .

We will find them in resolving the Maurer-Cartan equations of the parabolic subgroup $P_1 \subset \widetilde{G}_2$.

These equations are $e^i = 0$ for $i \in \{0, 1, \dots, 6, 8, 12\}$ where

$$\begin{aligned}
 e^0 &= d\theta^0 - (-6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3), \\
 e^1 &= d\theta^1 - (-3\theta^1 \wedge \theta^5 - 3\theta^1 \wedge \theta^8), \\
 e^2 &= d\theta^2 - (\theta^1 \wedge \theta^6 - 3\theta^2 \wedge \theta^5 - \theta^2 \wedge \theta^8), \\
 e^3 &= d\theta^3 - (2\theta^2 \wedge \theta^6 - 3\theta^3 \wedge \theta^5 + \theta^3 \wedge \theta^8), \\
 e^4 &= d\theta^4 - (3\theta^3 \wedge \theta^6 - 3\theta^4 \wedge \theta^5 + 3\theta^4 \wedge \theta^8), \\
 e^5 &= d\theta^5 - (-\theta^1 \wedge \theta^{12}), \\
 e^6 &= d\theta^6 - (6\theta^2 \wedge \theta^{12} + 2\theta^6 \wedge \theta^8), \\
 e^8 &= d\theta^8 - (-3\theta^1 \wedge \theta^{12}), \\
 e^{12} &= d\theta^{12} - (-3\theta^5 \wedge \theta^{12} - 3\theta^8 \wedge \theta^{12}).
 \end{aligned} \tag{3.3}$$

We describe the proof in several steps.

Step 1 : $e^0 = 0$.

Computing

$$d\theta^0 = \frac{1}{s_0} ds_0 \wedge \theta^0 - \frac{s_4}{\delta^3} \theta^0 \wedge \theta^1 + \frac{3s_3}{\delta^3} \theta^0 \wedge \theta^2 - 3\frac{s_2}{\delta^3} \theta^0 \wedge \theta^3 + \frac{s_1}{\delta^3} \theta^0 \wedge \theta^4 - \frac{s_0}{\delta^3} \theta^1 \wedge \theta^4 + 3\frac{s_0}{\delta^3} \theta^2 \wedge \theta^3,$$

and inserting it into $e^0 = 0$, it gives

$$(-1 - \frac{s_0}{\delta^3})\delta^1 \wedge \theta^4 \wedge \theta^0 + (3 + \frac{3s_0}{\delta^3})\theta^2 \wedge \theta^3 \wedge \theta^0 = 0,$$

whose the unique solution is

$$s_0 = -\delta^3. \tag{3.4}$$

Then the general solution for $e^0 = 0$ is

$$\theta^5 := \frac{1}{2\delta} d\delta + \frac{s_4}{6\delta^3} \theta^1 - \frac{s_3}{2\delta^3} \theta^2 + \frac{s_2}{2\delta^3} \theta^3 - \frac{s_1}{6\delta^3} \theta^4 - \frac{1}{6} u_0 \theta^0, \tag{3.5}$$

where u_0 is new variable.

Step 2 : $e^1 = 0$.

By $e^1 \wedge \theta^0 \wedge \theta^1 = 0$, we find

$$\frac{3(s_1\delta + as_5^3\delta - 3Js_5^4s_7)}{\delta^4} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 + \frac{3Js_5^5}{\delta^4} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = 0. \tag{3.6}$$

Since s_5 cannot be zero, it is equivalent to $J = 0$.

In other words, we have shown that under a transformation that maps one adapted coframe ω to another adapted coframe $\hat{\omega}$.

Then $J = 0$ if, and only if $\hat{J} = 0$.

Now, we assume $J = 0$.

By this assumption and (3.6) we have

$$s_1 = -as_5^3.$$

By $e^1 \wedge \theta^1 = 0$, we have

$$\frac{-s_5^3}{\delta^5}(M\delta + 2Ls_5s_7)\theta^0 \wedge \theta^1 \wedge \theta^2 + \frac{s_5^4}{\delta^5}L\theta^0 \wedge \theta^1 \wedge \theta^3 = 0. \quad (3.7)$$

By the same argument, we find $L = 0$ and that $L = 0$ if, and only if $\widehat{L} = 0$.

Now, we assume $J = L = 0$. Again by (3.7), we have $M = 0$ and that $M = 0$ if, and only if $\widehat{M} = 0$.

Then the simultaneous vanishing of J, L and M is an invariant property and necessary for flatness.

Now, we assume $J = L = M = 0$.

We have the general solution for $e^1 = 0$:

$$\begin{aligned} \theta^8 := & -\frac{1}{2\delta}d\delta + \frac{1}{s_5}ds_5 \\ & - \frac{-6cs_2\delta^2 + 2as_5\delta^3 + 2as_4s_5^4 - 6acs_5^2s_7\delta^2 - 6as_3s_5^3s_7 + 6as_2s_5^2s_7^2 + 2a^2s_5^4s_7^3 - s_5u_0\delta^6}{6s_5\delta^6}\theta^0 \\ & + \frac{2c\delta^2 + s_3s_5 - 2as_5^2s_7^2}{2s_5\delta^3}\theta^2 - \frac{s_2 - 2as_5^2s_7}{2\delta^3}\theta^3 - \frac{as_5^3}{2\delta^3}\theta^4 - \frac{1}{3}u_1\theta^4, \end{aligned}$$

where u_1 is a new variable.

Step 3 : $e^2 = 0$.

By $e^2 \wedge \theta^0 \wedge \theta^1 = 0$, we have

$$s_2 = \frac{1}{2}bs_5\delta - as_5^2s_7. \quad (3.8)$$

By $e^2 \wedge \theta^1 = 0$, we find the invariance property of the simultaneous vanishing of J, L, M, P and

$$u_0 = \frac{(4a - 6bc - 3Q)\delta^3 + 3b^2s_5s_7\delta^2 - 3(s_3 + 2as_7^2s_5)bs_5\delta + 2as_5^2(-s_4s_5 + 3s_3s_7 + 2as_5s_7^3)}{2\delta^6}. \quad (3.9)$$

Then the general solution for $e^2 = 0$ is

$$\begin{aligned}
\theta^6 := & \frac{s_7}{s_5\delta}d\delta - \frac{s_7}{s_5^2}ds_5 - \frac{1}{s_5}ds_7 \\
& - \left[\frac{2(2c^2 + R)\delta^4 - 2(4bc + Q)s_5s_7\delta^3 + (8cs_3 + 5b^2s_5s_7^2 + 8acs_5s_7^2)s_5\delta^2}{4s_5^2\delta^6} \right. \\
& + \frac{2(s_4s_5 - 4s_3s_7 - 6as_5s_7^3)bs_5^2\delta - 4(s_4s_5 - 3s_3s_7 - 2as_5s_7^3)as_5^3s_7}{4s_5^2\delta^6} \Big] \theta^0 \\
& + \frac{2s_5^2u_1\delta^3 + 6cs_7\delta^2 - 12bs_5s_7^2\delta - 3s_4s_5^2 + 18as_5^2s_7^3}{6s_5^2}\theta^2 \\
& - \frac{2c\delta^2 - 2bs_5s_7\delta + 3as_5^2s_7^2}{s_5\delta^3}\theta^3 \\
& - \frac{s_5(b\delta - 2as_5s_7)}{2\delta^3}\theta^4 \\
& + u_2\theta^1.
\end{aligned}$$

Step 4 : $e^3 = 0$.

By $e^3 \wedge \theta^0 = 0$, we have

$$s_3 = \frac{-c\delta^2 + bs_5s_7\delta - as_5^2s_7^2}{s_5}, \quad (3.10)$$

$$u_1 = \frac{-3(2cs_7\delta^2 - 2bs_5s_7^2\delta + s_4s_5^2 + 2as_5^2s_7^3)}{2s_5^2\delta^3}, \quad (3.11)$$

$$u_2 = \frac{-s_7^2(c\delta^2 - bs_5s_7\delta + as_5^2s_7^2)}{s_5^3\delta^3}. \quad (3.12)$$

By the expression of $e^3 = 0$, we find the invariance property of the simultaneous vanishing of J, L, M, P, Q .

Then assuming the simultaneous vanishing of J, L, M, P, Q , again by $e^3 = 0$, we find the simultaneous vanishing of J, L, M, P, Q and R .

Step 5 : $e^4 = 0$.

By $e^4 = 0$, we find the general solution for θ^{12} :

$$\begin{aligned} \theta^{12} = & -\frac{cs_7\delta^2bs_5s_7^2\delta + s_4s_5^2 + as_5^2s_7^3}{2s_5^2\delta^4}d\delta + \frac{1}{6\delta^3}ds_4 + \frac{cs_7\delta^2bs_5s_7^2\delta + s_4s_5^2 + as_5^2s_7^3}{2s_5^3\delta^3}ds_5 \\ & + \frac{c\delta^2bs_5s_7\delta + as_5^2s_7^2}{2s_5^2\delta^3}ds_7 + \left[\frac{3c^2s_7^2\delta 46bcs_5s_7^3\delta^3 + 3(cs_4s_5^2s_7 + b^2s_5^2s_7^4 + 2acs_5^2s_7^4)\delta^2}{6s_5^4\delta^6} \right. \\ & - \left. \frac{3(bs_4s_5^3s_7^2 + 2abs_5^3s_7^5)\delta + s_4^2s_5^4 + 3as_4s_5^4s_7^3 + 3a^2s_5^4s_7}{6s_5^4\delta^6} \right] \theta^1 \\ & + \frac{bcs_7^2\delta^3 + (cs_4s_5b^2s_5s_7^3 2acs_5s_7^3)\delta^2 + 3abs_5^2s_7^4as_4s_5^3s_7^2 2a^2s_5^3s_7^5}{2s_5^2\delta^6} \theta^2 \\ & + \frac{c^2\delta^4 2bcs_5s_7\delta^3 + (b^2s_5^2s_7^2 + 3acs_5^2s_7^2)\delta^2 3abs_5^3s_7^3\delta + as_4s_5^4s_7 + 2a^2s_5^4s_7^4}{2s_5^2\delta^6} \theta^3 \\ & + \frac{4a\delta^3(3b^2s_5s_7 + 12acs_5s - 7)\delta^2 + 12abs_5^2s_7^2\delta 4as_4s_5^3 8a^2s_5^3s_7^3}{24\delta^6} \theta^4 \\ & + \frac{1}{6}u_3\theta^0, \end{aligned}$$

where u_3 is a new variable.

Step 6 : $e^5 = 0$.

By $e^5 = 0$, we find

$$\begin{aligned} u_3 = & \frac{8c^3\delta^6 + 24bc^2s_5s_7\delta^5(21b^2cs_5^2s_7^2 + 36ac^2s_5^2s_7^2)\delta^4 + (6bcs_4s_5^3 + 5b^3s_5^3s_7^3 + 60abcs_5^3s_7^3)\delta^3}{4s_5^3\delta^9} \\ & + \frac{(3b^2s_4s_5^4s_7 + 24acs_4s_5^4s_7 + 21ab^2s_5^4s_7^4 + 36a^2cs_5^4s_7^4)\delta^2}{4s_5^3\delta^9} \\ & - \frac{(18abs_4s_5^5s_7^2 + 24a^2bs_5^5s_7^5)\delta + 4s_4^2s_5^6 + 12a^2s_4s_5^6s_7^3 + 8a^3s_5^6s_7^6}{4s_5^3\delta^9}. \end{aligned}$$

Step 7 :

We show that the 1-form θ^i for $i \in \{0, \dots, 6, 8, 12\}$ satisfy $e^6 = 0$, $e^8 = 0$, and $e^{12} = 0$.

In assuming $J = L = M = P = Q = R = S = 0$, we obtain the solution for the Maurer-Cartan equations for P_1 . Then the symmetry group of the coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ is P_1 (since the structure functions are constant).

Finally, in assuming $J = L = M = P = Q = R = S = 0$, we uniquely determined

- (1) a 9-dimensional subbundle $\mathcal{P}$ of the 13-dimensional bundle $\pi : U \times \mathbf{S} \rightarrow U$.
- (2) a well defined coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12})$ on $\mathcal{P}$ satisfying the Maurer-Cartan equations of P_1 whose the first five forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ when pulled back with respect to any section of $\mathcal{P} \rightarrow U$ are contained in the equivalence class $[(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)]$.

□

4. A rigid coframe for marked contact Engel structure

In the previous section, we have explicitly construe a coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12})$ on a 9-dimensional bundle over the flat marked contact Engel structure, called a **rigid coframe**.

In this section we will apply the Cartan's equivalence methods to show how to associate a rigid coframe on a 9-dimensional bundle to a general marked contact Engel structure.

Let a marked contact Engel structure with an adapted coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ satisfying (2.12).

Let $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ the lifted coframe

$$\theta^\mu = \sum_{\nu=0}^4 S_\nu^\mu \omega^\nu, \quad (0 \leq \mu \leq 4)$$

on $U \times S$.

With the equation (2.14) written in the following form

$$d\omega^\mu = -\frac{1}{2} \sum_{0 \leq \nu < \rho \leq 4} F_{\nu, \rho}^\mu \omega^\nu \wedge \omega^\rho, \quad (0 \leq \mu \leq 4)$$

we express the differentials $d\theta^0, \dots, d\theta^4$ as

$$d\theta^\mu = \sum_{\nu=0}^4 (d(S_\nu^\mu) \wedge \omega^\nu + S_\nu^\mu d\omega^\nu) = \sum_{\nu, \sigma=0}^4 dS_\mu^\nu (S^{-1})_\sigma^\rho \wedge \theta^\sigma - \sum_{\nu, \rho < \sigma, \alpha < \beta=0}^4 S_\nu^\mu F_{\rho, \sigma}^\nu (S^{-1})_\sigma^\rho (S^{-1})_\beta^\alpha \theta^\alpha \wedge \theta^\beta.$$

Remark 4.1. Next, we have to apply the **absorption** step.

However, a method to avoid the absorption step and its complicated expression, we will compute the product of $d\theta^\mu$'s with some θ^ν to determine equations allowing to go directly to the **normalization** step.

* Begin with $d\theta^0$, we have

$$d\theta^0 \wedge \theta^0 = -\frac{s_0}{\delta^3} \theta^1 \wedge \theta^4 \wedge \theta^0 + \frac{3s_0}{\delta^3} \theta^2 \wedge \theta^3 \wedge \theta^0.$$

Then we can also normalize the coefficient of the $\theta^1 \wedge \theta^4 \wedge \theta^0$ to 1 by setting

$$s_0 = -\delta^3 \quad (4.2)$$

We already have seen, in the previous theorem, that there exists a 1-form θ^5 , which is uniquely defined up to addition of multiples of θ^0 , satisfying the equation

$$d\theta^0 = -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3.$$

* Next, with $d\theta^1$, we compute

$$d\theta^1 \wedge \theta^0 \wedge \theta^1 \wedge \theta^4 = \frac{3(s_1\delta + as_5^3\delta - 3Js_5^4s_7)}{\delta^4} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$$

By normalization of the coefficient to 0, we have

$$s_1 = \frac{-a\delta s_5^3 + 3Js_5^4s_7}{\delta}. \quad (4.3)$$

Then there exists a 1-form θ^8 , uniquely defined up to addition of multiples of θ^0 and θ^1 , satisfying

$$d\theta^1 = -3\theta^1 \wedge \theta^5 - 3\theta^1 \wedge \theta^8.$$

* Now with $d\theta^2$, we compute

$$\begin{aligned} d\theta^2 \wedge \theta^0 \wedge \theta^1 &= \frac{2(2\delta s_2 - b\delta^2 s_5 + 2a\delta s_5^2 s_7 - 3Js_5^3 s_7^2)}{\delta^4} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 - 3\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^5 \\ &\quad - \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^8 + \frac{2Js_5^5}{\delta^4} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4, \end{aligned}$$

and we can normalize the $\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ -term to 0 by setting

$$s_2 = \frac{s_5}{2\delta} (b\delta^2 - 2a\delta s_5 s_7 + 3Js_5^2 s_7^2). \quad (4.4)$$

* Next with $d\theta^3$, we compute

$$d\theta^3 \wedge \theta^0 \wedge \theta^2 \wedge \theta^3 = -\frac{c\delta^3 + \delta s_3 s_5 - b\delta^2 s_5 s_7 + a\delta s_5^2 s_7^2 - Js_5^3 s_7^3}{\delta^4 s_5} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4$$

and we can normalize the $\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4$ -term to 0 by setting

$$s_3 = -\frac{1}{\delta s_5} (c\delta^3 - b\delta^2 s_5 s_7 + a\delta s_5^2 s_7^2 - Js_5^3 s_7^3). \quad (4.5)$$

With this normalization, we obtain, on the subbundle $\mathcal{G}^9 \subset U \times \mathbf{S}$ defined by (4.2), (4.3), (4.4) and (4.5), the following new lifted coframe :

$$\begin{aligned} \theta^0 &= -\delta^3 \omega^3, \\ \theta^1 &= \frac{s_5^3 (3Js_5 s_7 - a\delta)}{\delta} \omega^0 + s_5^3 \omega^1, \\ \theta^2 &= \frac{s_5 (b\delta^2 - 2a\delta s_5 s_7 + 3Js_5^2 s_7^2)}{2\delta} \omega^0 + s_5^2 s_7 \omega^1 - \delta s_5 \omega^2, \\ \theta^3 &= \frac{-c\delta^3 + b\delta^2 s_5 s_7 - a\delta s_5^2 s_7^2 + Js_5^3 s_7^3}{s_5} \omega^0 + s_5 s_7^2 \omega^1 - 2\delta s_7 \omega^2 + \frac{\delta^2}{s_5} \omega^3, \\ \theta^4 &= s_4 \omega^0 + s_7^3 \omega^1 - \frac{3\delta s_7^2}{s_5} \omega^2 + \frac{3\delta^2 s_7}{s_5^2} \omega^3 - \frac{\delta^3}{s_5^3} \omega^4. \end{aligned} \quad (4.6)$$

We can summarize this result in this proposition.

Proposition 4.7. *The five forms (4.6) on the 9-dimensional subbundle $\mathcal{G}^9 \subset U \times \mathbf{S}$ given by (4.2), (4.3), (4.4) and (4.5) can be supplemented to a rigid coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12})$ which is uniquely determined by the fact that it satisfies*

$$\begin{aligned} d\theta^0 &= -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\ d\theta^1 &= -3\theta^1 \wedge \theta^5 - 3\theta^1 \wedge \theta^8 + T_{0,2}^1 \theta^0 \wedge \theta^2 + T_{0,3}^1 \theta^0 \wedge \theta^3 + T_{0,4}^1 \theta^0 \wedge \theta^4 + T_{0,6}^1 \theta^0 \wedge \theta^6 + T_{2,4}^1 \theta^2 \wedge \theta^4, \\ d\theta^2 &= \theta^1 \wedge \theta^6 - 3\theta^2 \wedge \theta^5 - \theta^2 \wedge \theta^8 + T_{0,3}^2 \theta^0 \wedge \theta^3 - T_{0,4}^2 \theta^0 \wedge \theta^4 + T_{3,4}^2 \theta^3 \wedge \theta^4, \\ d\theta^3 &= 2\theta^2 \wedge \theta^6 - 3\theta^3 \wedge \theta^5 + \theta^3 \wedge \theta^8 + T_{0,1}^3 \theta^0 \wedge \theta^1 + T_{0,2}^3 \theta^0 \wedge \theta^2 - T_{0,3}^3 \theta^0 \wedge \theta^3 + T_{0,4}^3 \theta^0 \wedge \theta^4, \\ d\theta^4 &= 6\theta^0 \wedge \theta^{12} + 3\theta^3 \wedge \theta^6 - 3\theta^4 \wedge \theta^5 + 3\theta^4 \wedge \theta^8, \end{aligned} \quad (4.8)$$

for some functions $T_{j,k}^i$, and the additional normalization that $d\theta^5$, when written with respect to the basis of forms $\theta^i \wedge \theta^j$, has zero coefficient at the $\theta^0 \wedge \theta^1$ term.

Remark 4.9. The bundle $\mathcal{G}^9$ has as structure group the subgroup of $\mathbf{S}$ of matrix of the form

$$\begin{pmatrix} -\delta^3 & 0 & 0 & 0 & 0 \\ 0 & s_5^3 & 0 & 0 & 0 \\ 0 & s_5^2 s_7 & -\delta s_5 & 0 & 0 \\ 0 & s_5 s_7^2 & -2\delta s_7 & \frac{\delta^2}{s_5} & 0 \\ s_4 & s_7^3 & -\frac{3\delta s_7^2}{s_5} & \frac{3\delta^2 s_7}{s_5^2} & -\frac{\delta^3}{s_5^3} \end{pmatrix}.$$

Remark 4.10. By the proof of the proposition, we obtain some expressions of the $T_{j,k}^i$

$$\begin{aligned} T_{2,4}^1 &= -T_{0,6}^1 = \frac{3}{2}T_{3,4}^2 = \frac{3Js_5^5}{\delta^4}, \\ T_{0,2}^1 &= \frac{s_5^2(\delta^4 M - 6\delta Js_4 s_5^3 - 9c\delta^3 Js_5 s_7 - 3\delta^3 J_{\omega^2} s_5 s_7 + 2\delta^3 L s_5 s_7 - 9b\delta^2 Js_5^2 s_7^2)}{\delta^8} \\ &\quad + \frac{s_5^2(-9\delta^2 J_{\omega^3} s_5^2 s_7^2 + 21a\delta Js_5^3 s_7^3 - 9\delta J_{\omega^4} s_5^3 s_7^3 - 27J^2 s_5^4 s_7^4)}{\delta^8}. \end{aligned} \quad (4.11)$$

To obtain all expressions of $T_{j,k}^i$'s we have to differentiate the lifted coframe (4.6).

5. A tree of homogeneous models

In this last section, we reach the goal of the marked contact Engel structure : it is to find all locally non-equivalent homogeneous model of marked contact Engel structure with symmetry group of dimension greater than or equal to 6.

The structure of the classification is the following :

- (1) Beginning with the coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ and the rigid coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12})$ viewed in the previous proposition, we divide the marked contact Engel structure into classes of mutually non-equivalent structures.
- (2) $J \neq 0$; $J = 0, L \neq 0$; $J = L = 0, M = 0$; $J = L = M = 0, P = 0$; $J = L = M = P = 0, Q \neq 0$.
- (3) With this distinction, we will obtain different structure equations express in terms of the wedge product $\theta^i \wedge \theta^j$ of the coframe forms

$$d\theta^k = \sum_{\substack{i < j \\ i, j \in \{0, \dots, 6, 8, 12\}}} T_{i,j}^k \theta^i \wedge \theta^j,$$

where the $T_{i,j}^k$ are the structure functions.

Remark : We forget the Maurer-Cartan forms of the subgroup of $\mathbf{S}$.

- (4) Then we will take the constant structure functions i.e. equal to zero if there dependency of group and variable parameters.
Thus, we will have for each case Maurer-Cartan equations of a some Lie G consisting the symmetry group.

Finally we will have the homogeneous models given by G/G_e where G_e is the isotropy group of e .

We will often abus notation : we will denote various different sub bundles $\mathcal{G}^i \subset \mathcal{G}^9$ of dimension i by the same symbol. Moreover, we will frequently pullback the forms $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12}$ to these various sub bundles and always reuse the same names for the pulled back forms.

Classification :

* Step 1 : $J \neq 0$.

We assume $J \neq 0$.

The structure equation for $d\theta^0$ already has constant structure functions, so we look $d\theta^1$ in the previous proposition, we see that we can normalize the coefficient $T_{2,4}^1 = \frac{3s_5^5}{\delta^4}J$ by 3. Also we can normalize the coefficient $T_{0,2}^1$ to zero.

This means that we restrict to a subbundle $\mathcal{G}^7 \subset \mathcal{G}^9$ given by

$$\begin{aligned} s_5 &= \left(\frac{\delta^4}{J}\right)^{\frac{1}{5}}, \\ s_4 &= \frac{\delta^4 M - 9c\delta^3 J s_5 s_7 - 3\delta^3 J_{\omega^2} s_5 s_7 + 2\delta^3 L s_5 s_7 - 9b\delta^2 J s_5^2 s_7^2 - 9\delta^2 J_{\omega^3} s_5^2 s_7^2 + 21a\delta J s_5^3 s_7^3}{6\delta J s_5^3} \\ &\quad + \frac{-9\delta J_{\omega^4} s_5^3 s_7^3 - 27J^2 s_5^4 s_7^4}{6\delta J s_5^3}. \end{aligned}$$

Then we have a new coframe $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^8, \theta^{12}$. We compute the new structure equations and look $d\theta^1$. We normalize the coefficient of $d\theta^1$ at the $\theta^1 \wedge \theta^4$ -term to zero, which determines a 6-dimensional subbundle $\mathcal{G}^6 \subset \mathcal{G}^7$ given by

$$s_7 = \frac{\delta^{\frac{1}{5}}(3aJ - J_{\omega^4})}{14J^{\frac{9}{5}}}.$$

Then we have a new coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ satisfying

$$\begin{aligned}
d\theta^0 &= -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\
d\theta^1 &= \frac{\alpha_1}{\delta^3} \theta^0 \wedge \theta^1 + \frac{\alpha_2}{\delta^{\frac{12}{5}}} \theta^0 \wedge \theta^2 + \frac{\alpha_3}{\delta^{\frac{9}{5}}} \theta^0 \wedge \theta^3 + \frac{\alpha_4}{\delta^{\frac{6}{5}}} \theta^0 \wedge \theta^4 + \frac{\alpha_5}{\delta^{\frac{9}{5}}} \theta^1 \wedge \theta^2 + \frac{\alpha_6}{\delta^{\frac{6}{5}}} \theta^1 \wedge \theta^3 \\
&\quad - \frac{24}{5} \theta^1 \wedge \theta^5 + 3\theta^2 \wedge \theta^4, \\
d\theta^2 &= \frac{\alpha_7}{\delta^{\frac{18}{5}}} \theta^0 \wedge \theta^1 + \frac{\alpha_8}{\delta^3} \theta^0 \wedge \theta^2 + \frac{\alpha_9}{\delta^{\frac{12}{5}}} \theta^0 \wedge \theta^3 + \frac{5\alpha_5}{6\delta^{\frac{9}{5}}} \theta^0 \wedge \theta^4 + \frac{\alpha_{10}}{\delta^{\frac{12}{5}}} \theta^1 \wedge \theta^2 + \frac{\alpha_{11}}{\delta^{\frac{9}{5}}} \theta^1 \wedge \theta^3 \\
&\quad - \frac{3\alpha_4 + 5\alpha_6}{9\delta^{\frac{6}{5}}} \theta^1 \wedge \theta^4 + \frac{\alpha_6}{3\delta^{\frac{6}{5}}} \theta^2 \wedge \theta^3 - \frac{18}{5} \theta^2 \wedge \theta^5 + 2\theta^3 \wedge \theta^4, \\
d\theta^3 &= \frac{\alpha_{12}}{\delta^{\frac{21}{5}}} \theta^0 \wedge \theta^1 + \frac{\alpha_{13}}{\delta^{\frac{18}{5}}} \theta^0 \wedge \theta^2 + \frac{\alpha_{14}}{\delta^3} \theta^0 \wedge \theta^3 + \frac{6\alpha_9 + 75\alpha_{10} + 25\alpha_2}{15\delta^{\frac{12}{5}}} \theta^0 \wedge \theta^4 + \frac{2(\alpha_1 - 3\alpha_8)}{3\delta^3} \theta^1 \wedge \theta^2 \\
&\quad - \frac{3\alpha_{10} + \alpha_2}{3\delta^{\frac{12}{5}}} \theta^1 \wedge \theta^3 + \frac{\alpha_5 + 6\alpha_{11}}{3\delta^{\frac{9}{5}}} \theta^2 \wedge \theta^3 - \frac{6\alpha_4 + 10\alpha_6}{9\delta^{\frac{6}{5}}} \theta^2 \wedge \theta^4 - \frac{12}{5} \theta^3 \wedge \theta^5, \\
d\theta^4 &= \frac{\alpha_{15}}{\delta^{\frac{24}{5}}} \theta^0 \wedge \theta^1 + \frac{\alpha_{16}}{\delta^{\frac{21}{5}}} \theta^0 \wedge \theta^2 + \frac{\alpha_{17}}{\delta^{\frac{18}{5}}} \theta^0 \wedge \theta^3 + \frac{\alpha_{18}}{\delta^3} \theta^0 \wedge \theta^4 - \frac{\alpha_1 - 3\alpha_8}{\delta^3} \theta^1 \wedge \theta^3 - \frac{3\alpha_{10} + \alpha_2}{\delta^{\frac{12}{5}}} \theta^1 \wedge \theta^4 \\
&\quad + \frac{\alpha_2}{\delta^{\frac{12}{5}}} \theta^2 \wedge \theta^3 + \frac{\alpha_5}{\delta^{\frac{9}{5}}} \theta^2 \wedge \theta^4 - \frac{3\alpha_4 + 2\alpha_6}{3\delta^{\frac{6}{5}}} \theta^3 \wedge \theta^4 - \frac{6}{5} \theta^4 \wedge \theta^5, \\
d\theta^5 &= \frac{\alpha_{19}}{\delta^{\frac{27}{5}}} \theta^0 \wedge \theta^1 + \frac{\alpha_{20}}{\delta^{\frac{24}{5}}} \theta^0 \wedge \theta^2 + \frac{\alpha_{21}}{\delta^{\frac{21}{5}}} \theta^0 \wedge \theta^3 + \frac{\alpha_{22}}{\delta^{\frac{18}{5}}} \theta^0 \wedge \theta^4 - \frac{3\alpha_{12} + \alpha_{16}}{6\delta^{\frac{21}{5}}} \theta^1 \wedge \theta^2 - \frac{\alpha_{17} - 3\alpha_7}{6\delta^{\frac{18}{5}}} \theta^1 \wedge \theta^3 \\
&\quad - \frac{\alpha_1 + \alpha_{18}}{6\delta^3} \theta^1 \wedge \theta^4 + \frac{\alpha_8 + \alpha_{14}}{2\delta^3} \theta^2 \wedge \theta^3 - \frac{6\alpha_9 + 75\alpha_{10} + 20\alpha_2}{30\delta^{\frac{12}{5}}} \theta^2 \wedge \theta^4 - \frac{2\alpha_3 - 5\alpha_5}{12\delta^{\frac{9}{5}}} \theta^3 \wedge \theta^4,
\end{aligned} \tag{5.1}$$

where $\alpha_1, \dots, \alpha_{22}$ are the pullbacks of fonctions on U , that is, as functions on $\mathcal{G}^6$ they do not depend on δ .

Now we are looking for homogeneous model with 6-dimensional symmetry group. For a such structure all of the structure functions are constants. In particular, all those that depend on δ have to be identically zero.

Then we have :

$$\begin{aligned}
d\theta^0 &= -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\
d\theta^1 &= -\frac{24}{5} \theta^1 \wedge \theta^5 + 3\theta^2 \wedge \theta^4, \\
d\theta^2 &= -\frac{18}{5} \theta^2 \wedge \theta^5 + 2\theta^3 \wedge \theta^4, \\
d\theta^3 &= -\frac{12}{5} \theta^3 \wedge \theta^5, \\
d\theta^4 &= -\frac{6}{5} \theta^4 \wedge \theta^5, \\
d\theta^5 &= 0.
\end{aligned} \tag{5.2}$$

This means there is a unique local model with 6-dimensional symmetry group in the branch whose symmetry algebra has Maurer-Cartan equations (5.2).

* Step 2 : $J = 0, L \neq 0$.

We assume $J = 0, L \neq 0$.

We look $d\theta^1$ from the previous proposition. We normalize the coefficient $T_{0,2}^1 = 0$ and $T_{0,3}^1 = 1$ then we can express θ^6 and θ^8 in terms of the remaining forms, which constitute a coframe.

We recompute the structure equations.

We look $d\theta^3$. We normalize the coefficient at the $\theta^1 \wedge \theta^3$ -term to 0.

Then we have a 6-dimensional subbundle $\mathcal{G}^6 \subset \mathcal{G}^9$ defined by

$$\begin{aligned} s_7 &= \frac{M}{2L^{\frac{4}{5}}s_5^{\frac{1}{5}}}, \\ \delta &= -L^{\frac{1}{5}}s_5^{\frac{4}{5}}, \\ s_4 &= -\frac{8L^2L_{\omega^1} + 16cL^2M - 4LL_{\omega^2}M + 8bLM^2 + 2L_{\omega^3}M^2 + aM^3}{8L^{\frac{12}{5}}s_5^{\frac{3}{5}}}, \end{aligned}$$

which (the pullbacks of) the forms $\theta^0, \dots, \theta^5$ define a coframe.

If there is a homogeneous model with a 6-dimensional symmetry group in this branch, then for these structures all of the structure functions of the structure equations with respect to the coframe $(\theta^0, \dots, \theta^5)$ on $\mathcal{G}^6$ have to be constants.

However, this assumption leads to a contradiction. So there is no homogeneous model 6-dimensional symmetry group in this branch.

Then there are structures with 5-dimensional symmetry group in this branch ;

The structure equations leads up to distinguish two subclass of structures.

Case 1 : $M - P \neq 0$.

This allows us to normalize the coefficient at the $\theta^0 \wedge \theta^3$ -term of $d\theta^2$. This determines a 5-dimensional subbundle of $\mathcal{G}^6 \rightarrow U$, and thus a rigid coframe $(\theta^0, \dots, \theta^4)$ on U .

However, assuming that the structure equations with respect this coframe have only constant structure functions quickly leads to a contradiction.

Then there is no homogeneous model in this branch.

Case 2 : $M - P = 0$.

In this case, we obtain a new relative invariant, namely $5bL + 2L_{\omega^3}$.

Case 2.1 : $5bL + 2L_{\omega^3} \neq 0$.

We can normalize the coefficients of $d\theta^1$ at the $\theta^1 \wedge \theta^3$ -term to $\frac{3}{8}$.

This determines a 5-dimensional subbundle $\mathcal{G}^5 \subset \mathcal{G}^6$ given by

$$s_5 = \left(\frac{(5bL + 2L_{\omega^3})^5}{L^7} \right)^{\frac{1}{3}},$$

and a rigid coframe $\theta^0, \dots, \theta^4$ on U .

Assuming that all of the structure functions with respect the coframe are constant and using $d^2 = 0$, we find that there is a locally unique homogeneous model with a 5-dimensional

symmetry group in this branch. It has Maurer-Cartan equations :

$$\begin{aligned}
 d\theta^0 &= -\frac{5}{6}\theta^0 \wedge \theta^3 + 24\theta^0 \wedge \theta^4 + \theta^1 \wedge \theta^3 - 3\theta^2 \wedge \theta^3, \\
 d\theta^1 &= \theta^0 \wedge \theta^3 - \frac{2}{3}\theta^1 \wedge \theta^3 - 30\theta^1 \wedge \theta^4, \\
 d\theta^2 &= -\frac{1}{2}\theta^2 \wedge \theta^3 + 18\theta^2 \wedge \theta^4, \\
 d\theta^3 &= -6\theta^3 \wedge \theta^4, \\
 d\theta^4 &= -\frac{1}{6}\theta^3 \wedge \theta^4.
 \end{aligned} \tag{5.3}$$

Case 2.2 : $5bL + 2L_{\omega^3} = 0$.

In this subclass, we obtain a contradiction. Then there is no homogeneous model in this branch.

* Step 3 : $J = L = 0, M \neq 0$.

We assume $J = L = 0, M \neq 0$.

We look $d\theta^1$ and we normalize the coefficient $T_{0,2}^1$ by $\epsilon = \text{sign}(M)$ to obtain a 8-dimensional subbundle $\mathcal{G}^8 \subset \mathcal{G}^9$ defined by

$$s_5 = \frac{\delta^2}{\sqrt{\epsilon M}}.$$

Then we can find with the new 1-forms $\theta^0, \dots, \theta^6, \theta^8, \theta^{12}$ on $\mathcal{G}^8$ that θ^8 is linearly dependent on the other 1-forms.

We recompute the $d\theta^i$ for $i \in \{0, \dots, 6, 12\}$ on $\mathcal{G}^8$.

The coefficient of $d\theta^5$ at the $\theta^2 \wedge \theta^3$ -term is

$$\frac{\sqrt{\epsilon M}Q + 6\delta P s_7}{2\delta^3 \sqrt{\epsilon M}}.$$

Then we distinguish :

Case 1 : $P \neq 0$.

This coefficient normalized to 0 gives a 7-dimensional subbundle $\mathcal{G}^7 \subset \mathcal{G}^8$ given by

$$s_7 = -\frac{\sqrt{\epsilon M}Q}{6\delta P}.$$

Again, we compute the structure equations and we normalize the coefficient of $d\theta^4$ at the $\theta^2 \wedge \theta^3$ -term to 0. Then we obtain a 6-dimensional subbundle $\mathcal{G}^6 \subset \mathcal{G}^7$ given by

$$\begin{aligned}
 s_4 &= (\epsilon M)^{\frac{3}{2}} \left(\frac{60cMPQ - 30M_{\omega^2}PQ + 180cP^2Q + 126PP_{\omega^2}Q + 84a_{\omega^0}Q^2 + 84aa_{\omega^1}Q^2}{432(\delta P)^3} \right. \\
 &\quad \left. + \frac{-21b^3Q^2 - 21bMQ^2 + 81bPQ^2 + 2aQ^3 - 72P^2Q_{\omega^2}}{432(\delta P)^3} \right),
 \end{aligned}$$

on which we express θ^{12} in terms of the remaining forms.

Moreover, since $P \neq 0$, we have $2cM - M_{\omega^2} \neq 0$, this allows us to normalize the coefficient

of $d\theta^1$ at the $\theta^1 \wedge \theta^2$ -term with respect to the coframe $\mathcal{G}^6$ to 12. This determines a 5-dimensional subbundle $\mathcal{G}^5 \subset \mathcal{G}^6$ given by

$$\delta = \left(\frac{2cM - M_{\omega^2}}{8\sqrt{\epsilon M}} \right)^{\frac{1}{3}}.$$

We have obtained a unique coframe on the 5-dimensional manifold. Then there are two locally non-equivalent homogeneous models with 5-dimensional symmetry group in this branch, whose Maurer-Cartan equations are

$$\begin{aligned} d\theta^0 &= -\frac{15}{2}\theta^0 \wedge \theta^2 - \frac{1}{6}\epsilon\theta^0 \wedge \theta^4 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\ d\theta^1 &= \epsilon\theta^0 \wedge \theta^2 - 3\theta^1 \wedge \theta^2 - 3\theta^1 \wedge \theta^3 - \frac{1}{3}\epsilon\theta^1 \wedge \theta^4, \\ d\theta^2 &= \frac{1}{4}\theta^0 \wedge \theta^1 - \frac{1}{12}\epsilon\theta^0 \wedge \theta^3 - \frac{1}{2}\theta^1 \wedge \theta^3 - \frac{1}{6}\epsilon\theta^2 \wedge \theta^4, \\ d\theta^3 &= \frac{9}{2}\theta^0 \wedge \theta^2 + \frac{1}{6}\epsilon\theta^0 \wedge \theta^4 + 9\epsilon\theta^1 \wedge \theta^2 + 3\theta^2 \wedge \theta^3, \\ d\theta^4 &= -\frac{27}{4}\epsilon\theta^0 \wedge \theta^1 + \frac{9}{4}\theta^0 \wedge \theta^3 + \frac{27}{2}\epsilon\theta^1 \wedge \theta^3 + \frac{9}{2}\theta^2 \wedge \theta^4, \end{aligned} \quad (5.4)$$

where $\epsilon = \pm 1$.

Case 2 : $P = 0$.

We obtain

$$d\theta^1 = -\frac{9Q}{2\delta^3}\theta^0 \wedge \theta^1 + \epsilon\theta^0 \wedge \theta^2 + \frac{3(2cM - M_{\omega^2})}{2\delta^3\sqrt{\epsilon M}}\theta^1 \wedge \theta^2 - 12\theta^1 \wedge \theta^5.$$

Case 2.1 : $Q \neq 0$.

This allows to perform further normalization, which determines a unique coframe on the 5-dimensional manifold. But further analyst shows that there is no homogeneous model in this branch.

Case 2.2 : $Q = 0$.

In this case, the differential consequence allows us to obtain that $R = S = 0$.

The only structure satisfying this assumption are defined by

$$\begin{aligned} d\theta^0 &= -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\ d\theta^1 &= \epsilon\theta^0 \wedge \theta^2 - 12\epsilon\theta^1 \wedge \theta^4, \\ d\theta^2 &= \frac{3}{4}\epsilon\theta^0 \wedge \theta^3 + \theta^1 \wedge \theta^6 - 6\theta^2 \wedge \theta^5, \\ d\theta^3 &= \frac{1}{2}\epsilon\theta^0 \wedge \theta^4 + 2\theta^2 \wedge \theta^6, \\ d\theta^4 &= 6\theta^0 \wedge \theta^{12} + 3\theta^3 \wedge \theta^6 + 6\theta^4 \wedge \theta^5, \\ d\theta^5 &= -\frac{1}{12}\epsilon\theta^0 \wedge \theta^6 - \theta^1 \wedge \theta^{12} + \frac{1}{12}\epsilon\theta^2 \wedge \theta^4, \\ d\theta^6 &= 6\theta^2 \wedge \theta^{12} - \frac{3}{4}\epsilon\theta^3 \wedge \theta^4 - 6\theta^5 \wedge \theta^6, \\ d\theta^{12} &= \frac{1}{6}\epsilon\theta^4 \wedge \theta^6 - 12\theta^5 \wedge \theta^{12}. \end{aligned} \quad (5.5)$$

These are Maurer-Cartan equations for $\mathfrak{sl}(3, \mathbb{R})$ if $\epsilon < 0$ and for $\mathfrak{su}(2, 1)$ if $\epsilon > 0$.

Remark 5.6. $\mathfrak{su}(2, 1)$ is the Lie algebra of $SU(2, 1)$ the special unitary group of $O(2, 1)$. Recall, $O(p, q)$ is the subgroup of $GL(p + q, \mathbb{R})$ constituted of isometries for the standard quadratic form of signature (p, q) .

* Step 4 : $J = L = M = 0, P \neq 0$.

We look $d\theta^3$ and we normalize the coefficient $T_{0,3}^3$ to 0. Then we obtain a subbundle and we can express θ^6 in terms of the others forms.

After recomputing of the structures equations, we normalize the coefficient of $d\theta^4$ at the $\theta^2 \wedge \theta^3$ -term to 0, and the coefficient of $d\theta^2$ at the $\theta^0 \wedge \theta^3$ -term to $\frac{-5\epsilon}{4}$, where $\epsilon = \text{sign}(P)$. These normalization determines a 6-dimensional subbundle $\mathcal{G}^6 \subset \mathcal{G}^9$ on which $\theta^6, \theta^8, \theta^{12}$ are expressible in terms of the forms $\theta^0, \dots, \theta^5$ which form a coframe.

Assuming that the structure equations have only constant coefficients yields a contradiction, and we conclude that there is no homogeneous model with 6-dimensional symmetry group in this branch.

* Step 5 : $J = L = M = P = 0, Q \neq 0$.

With this assumption, we normalize the coefficient of $d\theta^3$ at the $\theta^0 \wedge \theta^3$ -term to 1 and the coefficient of $d\theta^6$ at the $\theta^0 \wedge \theta^3$ -term to 0.

This determines a 6-dimensional subbundle $\mathcal{G}^6 \subset \mathcal{G}^9$ given by

$$\begin{aligned} s_7 &= -\frac{R}{Q^{\frac{2}{3}}s_5}, \\ s_5 &= -Q^{\frac{1}{3}}, \\ s_4 &= \frac{2a_{(\omega^1)^2}Q^2 - 4c^3Q^2 + 8cQ^2R + 2QQ_{\omega^2}R - 3bQR^2 + 2aR^3 - 2Q^2R_{\omega^2} - 3bQ^2S}{2Q^2s_5^3}. \end{aligned}$$

We express the pullbacks of the forms θ^5, θ^6 , and θ^{12} to $\mathcal{G}^6$ in terms of $\theta^0, \dots, \theta^4, \theta^8$.

Assuming that the structure equations have only constant coefficients then quickly implies that they are of the form

$$\begin{aligned} d\theta^0 &= \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3, \\ d\theta^1 &= \frac{1}{2}\epsilon\theta^0 \wedge \theta^1 - 3\epsilon\theta^1 \wedge \theta^8, \\ d\theta^2 &= \frac{1}{2}\epsilon\theta^0 \wedge \theta^2 - \theta^2 \wedge \theta^8, \\ d\theta^3 &= -\frac{1}{2}\epsilon\theta^0 \wedge \theta^3 + \theta^3 \wedge \theta^8, \\ d\theta^4 &= -\frac{1}{2}\theta^0 \wedge \theta^4 + 3\theta^4 \wedge \theta^8, \\ d\theta^8 &= -\frac{1}{2}\epsilon\theta^1 \wedge \theta^4 + 3\theta^2 \wedge \theta^3. \end{aligned} \tag{5.7}$$

These are the Maurer-Cartan equations of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Then there is a locally unique maximally symmetric homogeneous model with 6-dimensional symmetry group.

Summary :

We summarize the result in the following theorem :

Theorem 5.8 (Manno-Nurowski-Sagerschnig 2018). (1) *Up to local equivalence, there is a unique homogeneous model of marked contact Engel structure with 9-dimensional symmetry group. The symmetry algebra is isomorphic to $\mathfrak{p}_1$. The structure is characterized by*

$$J = L = M = P = Q = R = S = 0.$$

(2) *Up to local equivalence, there are precisely two homogeneous models of marked contact Engel structure with a 8-dimensional symmetry groups. The symmetry algebras are isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{su}(1, 2)$. The structures are characterized by*

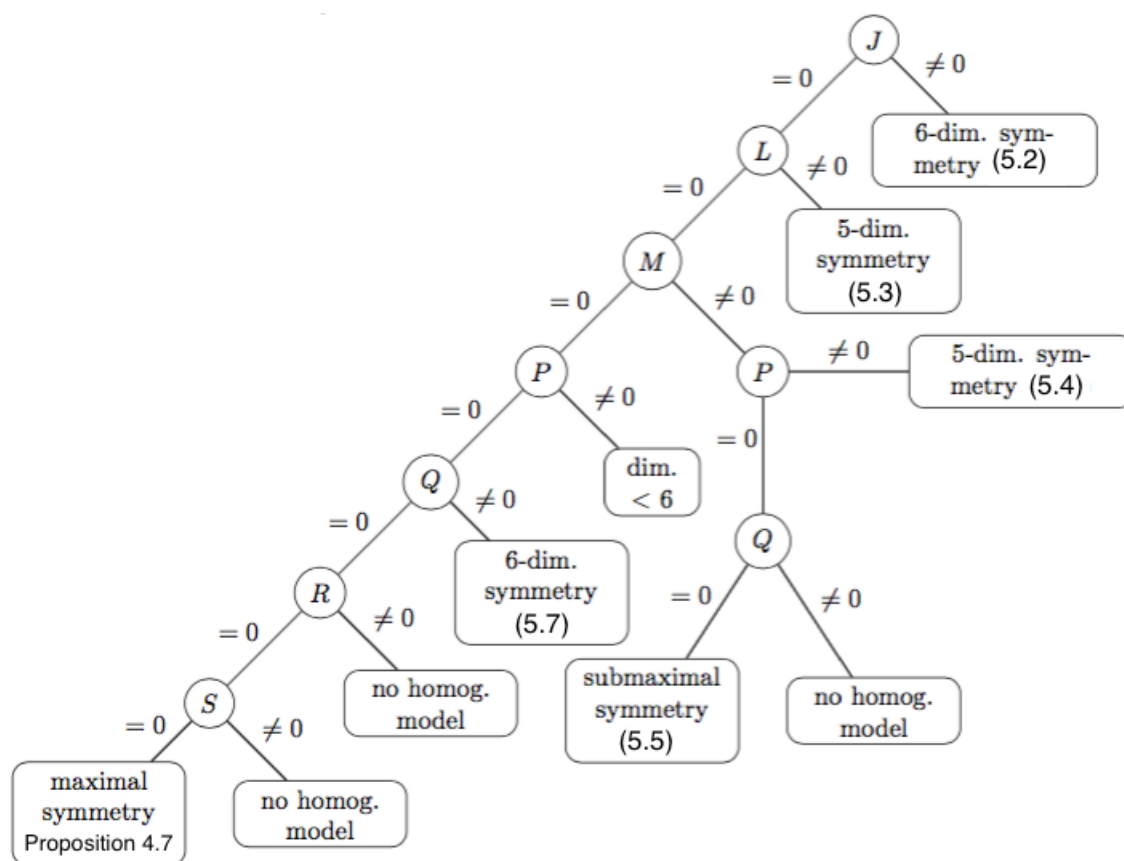
$$J = L = P = Q = 0 \text{ and } M \neq 0.$$

(3) *There is no homogeneous model of marked contact Engel structure with 7-dimension symmetry group.*

(4) *Up to local equivalence, there are precisely two homogeneous models marked contact Engel structures with 6-dimensional symmetry groups. The respective Maurer-Cartan equations are given by (XX) and (XX) ; the second symmetry algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.*

(5) *There are examples of homogeneous marked contact Engel structures with 5-dimensional symmetry groups, whose Maurer-Cartan equations are given by (XX) and (XX) .*

Remark 5.9. The case of classification of homogeneous models of marked contact Engel structures with 5-dimensional symmetry group is a problem still open.



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