About Grothendieck. R and D Duality as philosophical and mathematical concept bis

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Je souhaite que le lecteur puisse se rapporter au roman d'Aragon intitulé : "La mise à mort". Je remercie Hélène Esnaut de m'y renvoyer. Le héros du roman Monsieur Célèbre a perdu son image dans le miroir, il ne se reflète plus. Je pense que le travail sur la dualité est issu de manière diversifiée de l'expérience de cette perte

Je vais faire d'abord une présentation générale de la question de la dualité selon Grothendieck. J'utilise le travail de Brian Conrad. *Grothendieck duality and Base change Springer Lectures notes 1750*
La théorie de la dualité sur des schémas nœthériens en particulier la notion de faisceau dualisant joue un rôle fondamental dans divers contextes comme la téorie arithmétique des formes modulaires et l'étude des espaces de modules des courbes. Le but de la théorie est de produire une application trace en termes de la quelle on peut formuler des résultats de dualité pour la cohomologie des faisceaux cohérents.

Il s'agit d'exprimer les conditions le s plus générales pour que l'on puisse parler de dualité. Le point de vue Serre - Grothendieck provient d'abord de partir de l'application trace. Dans le cas classique de la dualité de Serre pour un schéma propre, lisse, géométriquement connexe n -dimensionnel sur un corps k, l'application trace revient à une application linéaire

$$t_X : H^n(X, \Omega^n_{X/k}) \to k$$

telle que (entre autre chose) pour tout faisceau libre cohérent $\mathcal{F}$ sur X avec un faisceau dual $\mathcal{F}^{\checkmark} = Hom_{O_X}(\mathcal{F}, \mathcal{O}_X)$

Le cup produit nous donne un appariement

$$H^i(X, \mathcal{F}) \otimes H^{n-1}(X, \mathcal{F}^{\checkmark} \otimes \Omega^n_{X/k}) \to H^n(X, \Omega^n_{X/k}) \xrightarrow{t_X} k$$

Celle-ci est un appariement parfait pour tout i. En particulier en usant du fait que $\mathcal{F} = \mathcal{O}_X$ and $i = 0$ nous voyons que $dim_k H^n(X, \Omega^n X/k) = 1$ et t_X est non nul et t_{X_l} doit être un isomorphisme

La dualité de Grothendieck s'étend à une situation relative, mais même le cas relatifoù la base est un annneau de valuation discrète est hautement non trivial. Les fondements de la théorie de la dualité selon Grothendieck, fondé sur des complexes résiduels ont été établis dans *Résidus et dualité* (RD) notes de Hartshorne. ces fondements rendent la théorie de la dualité tout à fait calculable en termes de formes différentielles et de résidus, et une telle calculabilité peut être très utile (c'est développé dans la thèse de Berthelot. ou le travail de Mazur sur l'idéal de Eisentstein).

Dans la construction de cette théorie il y a des compatibilités et des explications de résultats abstraits qui ne sont pas prouvés et qui sont difficiles à vérifier. L'un des plus difficiles est la la compatibilité du changement de base dans le cas des morphismes propres Cohen-Macaulay avec relation relative (e. g. familles plates de courbes semi stables). Les questions de compatibilité sont importantes car elles régissent la possibilité de s'élever de façon homogène à des niveaux d'abstraction.

Si l'on ignore la question du changement de base il existe des méthodes plus simples pour obtenir les théorèmes de dualité das le cas C M *projectif* qui ont aussi des résultats dans le cas projectif non CM

Il ne semble pas qu'il y ait eu une preuve publiée du théorème de dualité dans le cas général propre CM sur une base localement nœtherienne, ni seulement une analyse de son comportement par rapport au *changement de base*. Par exemple, le cas spécial tout à fait important de la compatibilité de l'application trace par rapport au changement de base pour une fibre géométrique n'est pas du tout évident, même si nous restreignons notre attention à la dualité pour les applications projectives lisses.
Ce fut là l'original la source de motivation dans ce sujet et même ce cas spécial ne semble pas avoir été accessible dans la littérature.

Il y a plusieurs buts dans le livre de Brian. En relation avec *Résidues and Duality* by Hartshorne de notes qu'il a tirées d'abord d'un séminaire qu'il a organisé à Harvard sur sa théorie de la dualité pour les faisceaux cohérents théorie qui a été annoncée au Séminaire Bourbaki en 1957, et dans son exposé au Congrés International des Mathématiciens en 1958 mais n'avait jamais été systématiquement développée. Grothendieck a été d'accord pour que Hartshorne fasse un e rédaction , disant qu'il fournirait un plan de présentation du matériel si je remplissais les détails et qu'il rédiigerait les notes du séminaire. durant l'été 1963, il écrivit une série de "prénotes" qui devait être la base du séminaire.

Je cite dit Hartshorne à partir de la préface des prénotes.

> *Les présentes notes donnent une esquisse assez détaillée d'une théorie cohomologique de la dualité des modules cohérents sur les préschémas. Les idées principales de la théorie m'étaient connues dès 1959, mais le manque de fondements adéquats d'Algèbre Homologique m'avait empêché d'aborder une rédaction d'ensemble. Cette lacune de fondements est sur le point d'être comblée par la thèse de Verdier, ce qui rend en principe possible un exposé satisfaisant. Il est d'ailleurs apparu depuis qu'il existe des théories cohomologiques de dualité analogues formellement très*

dans toutes sortes d'autres contextes : faisceaux cohérents sur les espaces analytiques, faisceaux abéliens sur les espaces topologiques(Verdier), modules galoisiens (Verdier, Tate) faisceaux de torsion sur les schémas munis de leur topologie étale, corps de classe en tous genres...Cela me semble une raison assez sérieuse pour se familiariser avec le yoga général de la dualité dans un cas type, comme la théorie cohomologique des résidus.

1-If I refer to the Introduction of the book *Duality in 19th and 20th Century Mathematical thinking*, the word "duality" is used in modern mathematics for many phenomena. Some of these have been recognized long ago in history, albeit under different names, such as duality of Platonic solids (book XV of the *Elements*) and in Kepler's *Harmonices Mundi* (II and V), or the polar triangle in spherical geometry." And then : " The idea became more prominent and explicit in the context of projective geometry" starting around 1810, and it was in the context that the term "duality" was coined (although "reciprocity" or "polarity" were also used)".

2-I would like to remark that the effect of the first highlighting of this first major conceptual step in the history of duality gives a first glimpse of the concept in question (duality). The projective duality will be a key element of this comprehension. It wlil give me the opportunity to discuss some propositions by the philosopher Lautman.

Retrospective, recurrency and history have been often discussed. I will insist on two points : history is the place where the ancient theories, are reconstructed, but history allows us to reconstruct their mathematical meaning, it is not the pure positive observation of a fact. Whatever its complexity it must take a meaning or a sense. Second point : from these analyses in an immanent manner, inside the theories, philosophy must rise.

3-Guiding question in the book (*Duality...*) -according the authors- adresses the **essence** of duality and would amount to the following : What is "duality" ? What are the common features (if any) of the different manifestations ? The usage of the term "duality" in mathematical discourse is most probably a case of Wittgenstein "family resemblance", and thus, no mathematical definition of the term duality is possible. In my viewpoint, it ought be possible to give a definition, but this supposes a philosophical deep work. The authors say that it is more a question of method, if we want to study the usage of the term "dual".

That's right but when we can give the reason why this term is used, we also can deep the philosophical meaning of duality.

4- In the literature, we find several answers on the question of what duality is. Sometimes, it is called a "principle", so by Atiyah,

> *Duality in mathematics is not a theorem, but a « principle ». It has a simple origin, it is very powerful and useful, and has a long history going back hundred of years. Over time it has been adapted and modified and so we can still use it in novel situations. It appears in many subjects in mathematics (geometry, algebra, analysis) and in physics. Fundamentally duality gives two different points of view of looking at the same object .*

It's a fact that duality is a way of attributing a "double" to an object, an attribution of an second object that is necessarily linked to the first, in which the former is reflected in an extension. In this way, the later object takes possession of the space, giving it a kind of substantiality, see Pontriagine duality, the intrinsicality commands extrinsicality.

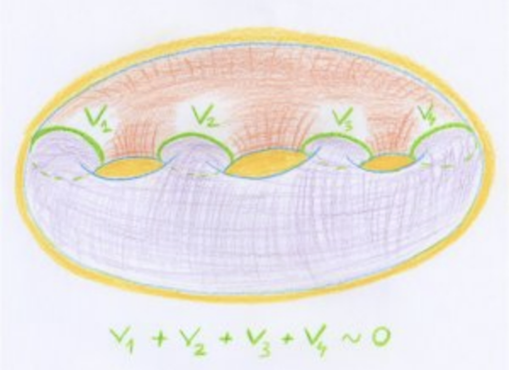
> *There are many things that have two different points of looking at the same object. There are many things that have two different points of view and in principle they are all dualities, says Atiyah*

5-My aim is to try to work the concept of duality sufficiently to produce a form of synthesis I claim that the most profound point of view is that of Grothendieck duality, and I claim that it is possible to construct a philosophical concept of this point of view. I chose some steps in its history. Why are we -so often- so driven to construct a double point of view on the same object ? More precisely to recognize the existence of such a double for so many objects ?

6- I will briefly explain the essential Poincaré's theorem - according to the Popescu's article (La dualité de Poincaré images des mathématiques)-. Popescu strongly emphasizes 19th century mathematicians were led to speak of a second space of dimension four. All the figures living in it and they were all spaces.

7- Let us consider a manifold V with p dimensions ; let be now a manifold W with q dimensions being part of V. Let us suppose that that the complete boundary is composed of continous submanifolds. with $q-1$ dimensions, $v_1, v_2, \cdots, v_\lambda$ It will be noticed $v_1 + v_2 + v_3 + \cdots + v_\lambda \simeq 0$ It is important to consider complete boundary.

The notation $k_1 v_1 + k_2 v_2 \simeq k_3 v_3 + k_4 v_4$ k_l integers and v varieties with $q-1$ dimensions, means that there exist a manifold W with q dimensions being part of W with q dimensions whose complete boundary is composed of k_i $i = 1, 2$ varieties little different from v_i and of k_j opposite varieties little different from $v_j, j = 3, 4$. These relations will be called *homologies*.


V_1
V_2
V_3
V_4
$V_1 + V_2 + V_3 + V_4 \sim 0$

8-I recall the definition of the Betti numbers. If there exists $P_m - 1$ closed varieties with m dimensions and linearly independent, (that means no homology links them) and if there exists only $P_m - 1$ it will said that the connexion order of V with respect to varieties with m dimensions is equal to P_m. We defined concerning a variety with m dimensions, $m - 1$ numbers that are called $P_1, P_2, \cdots, P_{m-1}$ and which are the connexion order with respect to varieties of $1, 2, \cdots, m-1$ dimensions. They are called *Betti number*.

Poincaré noticed $P_m - 1$ which we called *Betti number*. Poincarés theorem is the following. As for a closed variety the Betti numbers equally distant from the extremes are equal. The duality theorem says that in a closed manifold (and orientable) of dimension n there are as many independent submanifolds from the point of view of homology relations of a given dimension p as of the complementary dimension $n - p$. It is a first form of this theorem, I will later explain its cohomological form that gives its conceptual meaning. Obviously these statements are abrupt, and it would be important to analyse the presence of the geometry even in these abstract forms.

I can't explain what the corrections by Heegard to Poincaré's concept of a Betti number consists of. The Krömer and Haffner book is very clear in this topic.

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I will add some remarks on Pontrjagine duality. In comparison with Alexander's original, Pontrjagine describes the major innovation contained in his own paper as follows (I recall Krömer 's presentation) as follows . In the present paper I want to show that a certain duality exists not only between the Betti numbers, but also between the homology bases of K^λ and $R^n - K^\lambda$

" [Veblen showed in 1923] that the r-th and the $(n-r)$-th Betti basis of n- dimensional closed manifold can be chosen in such a way that the matrix of of the intersection numbers of the elements of the two bases is the unit matrix, a fact which obviously contains the theorem of Poincaré and generalizes it essentially.

If one compares this formulation of the Poincaré Veblen duality theorem with a generalization of Alexander's duality theorem - which asserts that the r-dimensional Betti basis complex in R^n and the $(n-r-1)$ dilmensional Betti basis of the complement space $\mathbb{R}^n - K$ can always be chosen in such a way that the matrix of the looping coefficients of the elements of the two bases is the unit matrix - a certain analogy between theses two theorems is obvious."

Pondrjagine devoted himself to a generalization and unification of the theorems by Poincaré and Alexander. In this work he expressed the theorem by isomorphisms of homology groups. In order to re-prove the theorems in this group-theoretical form he introduced (Krömer) a new group-theoretical concept, namely the so-called group pairing. In order to generalize to general, space he introduced limit groups (extending Alexandroff's notion of projection spectrum to the group theoretical context). I continue my quote. "In the present work, this analogy is fully clarified by reducing the two dualities theorems[...] to the application of one and the same algebraic principle."

I refer the reader to the well known definitions. This algebraic principle was the introduction of the notion of group pairings. Pondrjagine describes how he proceeds for tranferring his results from complexes to arbitrary closed sets by a use of Alexandroff's notion of projection spectrum and corresponding grouptheoretical notions Pondrjagine invented for this purpose.

It will be about direct and inverse sequences of groups. And I will add for our exposition. "The most important consequence of this theory is undoubtedly the proof it contains the fact that the reduced Bett group of the complementary space to a closed set is a topological invariant of that set. Later the distinction is made between "the reduced N Betti group" and the "full Betti group".

I would like to insist on the role played by algebra in Pondrjagine's theory. As Ralph K said repeatedly Pondrjagine pointed out an analogy of the situations of Poincaré and the Alexander duality theorems, namely the fact , says R Krömer, that in both cases the corresponding Betti bases could be so chosen that the matrix composed of intersection number or link numbers, respectively is the unit matrix. And Pondrjagine has to dig somehat deeper to find a valid algebraic criterion for the group being isomorphic : it is a certain property so-called group pairing.

This criterion is interesting for what we are looking for. I give only one property. Let M be a cyclic group of order μ. Pondrjagine writes $\mu = 0$ in the case of infinite group and he explicitly makes the agreement that M is the additive group of the integers if $\mu = 0$, and the least (nonnegative) residue system modulo μ if $\mu > 0$. An element of M always is an integer.

Two groups U and V form a pairing group with respect to (the module)M, if to each ordered pair of elements x, y -where x is an element of U and y an element of V is asigned an elemnt k of M the product of the two elements x and y :

$$k = x \cdot y$$

with two laws of distributivity.

L'important ici est de voir la structure de groupes est explicitement et développée den :même temps que celle de couplage (pairing).Ce qui fait de la dualité une structure qui repose sur la structure de groupe, que nous devons retouver dans la plupart des définitions de la dualitÃ©. Comme l'indique la définition de Grothendieck que nous faisons surplomber.

9- I drop to the analysis by J P Marquis (*Duality in XIX th and XXth Century ...*). The interaction between duality and category theory clearly illustrates one of the facets of structuralism in contemporary mathematics in the spirit of Bourbaki. The classical Pontrjagin and Stone type dualities are expressed in terms of isomorphisms of structures. When we move to the categorical framework, these isomorphisms become part of an equivalence of categories with additional structure data.

One of the challenges we face is to come up with a theoretical analysis and a typology of the different kinds of dualities one finds. As I wrote, there is no such thing, but there are, as we will also indicate, partial attempts and syntheses available.

Be that as it may, one interesting aspect that comes up in our history has to do with the place duality occupies between *metamathematics and mathematics*. The place also showed up early in the context of the interaction between duality and category theory. Thus, Marquis adds, focussing on duality also allows us to show how the status of CT evolved in the 1950s and the 1960s, oscillating between a meta- and mathematical role. Genuine mathematical concepts Kan, Grothendieck, Lawvere wanted to put categories at the very foundations of mathematics. The meta-, mathematical, and foundational interacted with another.

50-It is this form of doubling that Grothendieck explored. I give very simple philosophical remarks. As first remark this duality (Grothendieck) is not only production of connection but also an operational production. And we will develop this remark thanks to CT and the link of this operation with CT. As second remark this production is a kind of internal production, a relation of concept with itself. My aim is to describe *this specific synthesis of one thing with itself*. We dispose a concept twice but the second concept is not the same as the first. I will use in a first step Hartshorne *Algebraic Geometry* where the issues of the theorem are not present, we will see these ones in Conrad's book.

51-To realize this machine or this set of machine devices we need a few conceptual tools, and these tools will be in turn realized through their operation. It is a fact *relationships* become objects for higher-ranking relationships in various ways. . This what one calls *thematization* : in these constructions one can observe numerous forms of thematization where for example one morphism becomes the object for another morphism and this in repetition.

52- **Ext Groups** and $\mathcal{S}heaves$. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$ modules we denote by $Hom(\mathcal{F}, \mathcal{G})$ the group of $\mathcal{O}_X$-module homomorphisms and by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the sheaf *Hom*. Like Hartshorne we put a subscript X to indicate which space we are on : $Hom_X(\mathcal{F}, \mathcal{G})$. Recall that for fixed $\mathcal{F}$, $Hom(\mathcal{F}, \cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ and $\mathcal{H}om(\mathcal{F}, .)$ is a left exact covariant functor $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Since $\mathfrak{Mod}(X)$ has enough injectives we can make the following definition.

53-Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ an $\mathcal{O}_X$-module.We define the functors $Ext^i(\mathcal{F}, \cdot)$ as the right derived functors of $Hom(\mathcal{F}, \cdot)$ and $\mathcal{E}xt^i(\mathcal{F}, \cdot)$ the right derived functor of $\mathcal{H}om(\mathcal{F}, \cdot)$

54-If $\mathcal{I}$ is an injective object of $\mathfrak{Mod}(X)$, then for any open subset $U \subseteq X, \mathcal{I}_U$ is an injective object of $\mathfrak{Mod}(U)$. Consequently, according to the general properties of derived functors, we have $Ext^0 = Hom$. A long exact sequence for a short exact sequence in the second variable, $Ext^i((\mathcal{F}, \mathcal{G}) = 0$ for $i > 0$ $\mathcal{G}$ injective in $\mathfrak{Mod}(X)$ and ditto for the $\mathcal{E}xt$ sheaves.

55-These definitions are well known. Remind that I recall them as ingredient of a new theory of duality. For example, the concept of derived functor is a construction at the double level : we are installed in the word of abstract reflexivity where duality has to live. Let be the following proposition. For any subset $U \subset X$ we have $\mathcal{E}xt(\mathcal{F}, \mathcal{G})|_U \simeq \mathcal{E}xt(\mathcal{F}|_U, \mathcal{G}|_U)$

56-For any $\mathcal{G} \in \mathfrak{Mod}(X)$ we have

$$a)\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G} \quad (1)$$

$$b)\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = 0 \text{ for } i > 0 \quad (2)$$

$$c)Ext^i(O_X, \mathcal{G}) = H^i(X, \mathcal{G}) \text{for all } i \geq 0 \quad (3)$$

57- If $0 \to Hom(\mathcal{F}'', \mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G}) \to Hom(\mathcal{F}', \mathcal{G})$
$\to Ext^i(\mathcal{F}'', \mathcal{G}) \to Ext^1(\mathcal{F}'', \mathcal{G}) \to \cdots$ and similarly for the $\mathcal{E}xt$ sheaves.

58-Suppose that there is an exact sequence

$$\cdots \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

in $\mathfrak{Mod}(X)$, where the $\mathcal{L}_i$ are locally free sheaves of finite rank (in this case we say $\mathcal{L}.$ is a locally free resolution of $\mathcal{F}$).
Then for any $\mathcal{G} \in \mathfrak{Mod}(X)$ we have

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \simeq h^i(\mathcal{H}om(\mathcal{L}, \mathcal{G})).$$

59-First. A (covariant) δ functor from $\mathfrak{A}$ to $\mathfrak{B}$ (abelian categories) is a collection of of functors $T = (T^I)_{i\geq 0}$ together with a morphism $\delta^i : T^i(A") \to T^{i+1}(A)$ for each short exact sequence $0 \to A' \to A \to A" \to 0$ and each $i \geq 0$, such that,

$$0 \to T^0(A') \to T^0(A) \to T^0(A") \xrightarrow{\delta^0} T^i(A') \to \cdots$$

60-Second. for each morphism of one short exact sequence into another $0 \to B' \to B \to B" \to 0$ the δ's gives a commutative diagram

$$\begin{array}{ccc} T^i(A") & \longrightarrow & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B) & \longrightarrow & T^{i+1}(B') \end{array}$$

61- And we have the well known definitions. The δ-functor $T = (T^i) : \mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, given any other δ-functor $T' = (T'^i) : \mathfrak{A} \to \mathfrak{B}$, and given any morphism of functors $f^0 : T^0 \to T'^0$ there exists a unique sequence of morphisms $f^i : T^i \to T'^i$ for each $i \geq 0$, starting with the given f^0which commute with the δ^i for each short exact sequence.

I give these definitions as general remarks. It has to be noticed that when I rise to a higher level I must keep the same structures, here the notion of universality for the δ- functor. We can see here an operational feature of thematization, that Cavaillès uses to describe mathematical process. What is important is the level of the syntheses. Some comment

62-I need some additional definition and notions before entering the duality at the higher level. There are satellite functor, effaceable functor. I recall first of all definition of abelian category. It is a category $\mathfrak{A}$ such that for each $A, B \in Ob\mathfrak{A}$, $Hom(A, B)$ has a structure of an abelian group and the composition is linear ; finite sum exist ; every morphism has a kernel and a cokernel, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel ; and finally, every morphism can be factored into an epimorphism followed by a monomorhism. Let us remark the synthetic powerful of this well known definition. *Tohoku* The more we show that asome notion like sheaf or fibration can be

63-First he does the case of projective space itself, which follows easily from the explicit calculations.
Second. Then on an arbitrary projective scheme X, he shows that there is a coherent sheaf ω_X^0. Its role in duality theory is similar to the canonical sheaf of a non singular variety.
In particular, if X is Cohen-Macaulay (see below) it gives a duality theorem just like the one on projective space.

64-Finally, if X is a non-singular variety over an algebraically closed field, we show that the dualizing sheaf ω_X^0 coincides with the canonical sheaf ω_X.

Recall .We say that a local nœtherian ring A is Cohen Macaulay if depth $A = dim\, A$.

65-Remark. In these cases one can see the importance of the role played by the projective. For projectivity is maintained at all levels Why ? While this property keeps sense at all levels.
First many concepts of A G are developed in the frame of the projectivity.

67-Duality for $\mathbf{P}^k_k$. Let $X = \mathbf{P}^n_k$ over a field k. Then
(a) $H^n(X, \omega_X) \simeq k$. Fix one such isomorphism
(b) for any coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$Hom(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \simeq k$$

is perfect pairing of finite -dimensional vector spaces over k ;

(c) for every $i \geq 0$ there is a natural functorial isomorphism

$$Ext(\mathcal{F}, \omega) \simeq H^{n-1}(X, \mathcal{F})',$$

where ' denotes the dual vector space, which is for $i = 0$ the one induced by the pairing of (b). 30 mn

68-Let us consider the proofs. For a) We know that $\omega_X \simeq \mathcal{O}_X(-n-1)$. One use two exact sequences
Firstable- Let A be a ring , let $Y = SpecA$, let $X = \mathbf{P}^n_A$, then there is an exact sequence of sheaves

$$0 \to \Omega_{X/Y} \to \prime_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

The exponent $n+1$ in the middle means a direct sum of $n+1$ copies of $\mathcal{O}_X(-1)$.

69- Secondly. Let $X = \mathbf{P}^n_k$. Taking the dual of the sequence about gives us this exact sequence involving the tangent sheaf of $\mathbf{P}^n$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathcal{I}_X \to 0.$$

and a theorem says $H^r(X, \mathcal{O}_X(-r-1)) \simeq A$

69-bis Comment on these abstractions. Let us remember we established a level where duality keeps its validity and a meaning. In a general frame of sheaf, we use the canonical sheaf on X, $X = \mathbf{P}^n_k$. I will try to give some explanation of the role sheaf plays as a basis in the construction of duality.

The rise towards abstraction most often takes two forms : a) maintaining, keeping, the formal-intuitive structure (duality) within the new concepts, (here sheaves). b) the increase in meaning that this new synthesis allows to get. Roughly speaking the concept of a sheaf provides a systematic way of keeping track of local algebraic data on a topological space. For example, the regular functions on open subsets of a variety form a sheaf.

70- I could say that the introduction of duality in the frame of concept of a sheaf is a way to enlarge the duality by its operating on local algebraic data. This gives duality a much greater scope. In this case (of this theorem) we should be more precise. The concept which supports the duality is the concept of the *n*-dimensional projective space over k (field).

71- First step needs to consider projective duality. You all know the story of ancient duality. It expresses duality meaning. It's true : this duality gives basic formalism, f. e. line/ point, in a certain sense intuitive their intuitive meaning is extended. (Hilbert).

In this theorem we start from the cohomology of coherent sheaves on a projective scheme. Let us consider the proposition a) $H^n(X, \Omega_X \simeq k$. Fix one such isomorphism

Hom(F , Ω) $\times H^n(X, \mathcal{F}) \to H^n(X, \Omega) \simeq k$ is a perfect pairing of finite -dimensional vector space over k.
For every $i \geq 0$ there is a natural fonctorial isomorphism
$Ext^i(\mathcal{F}, \omega) \xrightarrow{\cong} H^{n-1}(X, \mathcal{F})'$

72- For each object A of $\mathfrak{A}$, choose once and for all an injective resolution $I^{\cdot}$ of A. Then we define $R^{\cdot}F(A) = h^i(F(I^{\cdot}))$.

I recall that the i th *cohomology object* $h^i(A)$ of the complex $A^{\cdot}$ is defined to be $kerd^i/imd^{i-1}$. If $f : A^{\cdot} \rightarrow B^{\cdot}$ is a morphism of complexes, then f induces a natural map $h^i(f) : h^i(A^{\cdot}) \rightarrow h^i(B^{\cdot})$.

73-Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to B$ a covariant left exact functor to anther abelian category $\mathfrak{B}$ then

- For each $i \geq 0$, $R^i F$ as defined above isan additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Furthermore, it is independant of the choices of injective resolution made.
- There is a natural isomorphism $F \simeq R^0 F$
- For each short exact sequence $0 \to A' \to A" \to 0$ and for each $i \geq 0$ there is a natural morphism $^i : R^{\cdot} F(A") \to R^{i+1} F(A')$ such that we obtain a long exact sequence

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$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A") \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \cdots$ Given a morphism of the exact sequence of the sequence above to another $0 \to B' \to B \to B" \to B"$ the δ^i give the commutative diagram

$$\begin{array}{ccc} R^iF(A") & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B") & \overset{\delta^i}{\dashrightarrow} & R^{i+1}F(B') \end{array}$$

Remark. For each injective object I of $\mathfrak{A}$,for $i \geq 0$ we have $R^i F(I) = 0$

75-Recall. We work on a ringed space $(X, \mathcal{O}_X)$ all sheaves will be sheaves of $\mathcal{O}_X$-modules. And as said above, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$ modules, we denote by $Hom(\mathcal{F}, \mathcal{G})$ the group of $\mathcal{O}_X$- modules homomorphisms, and by $\mathcal{H}om$, the sheaf Hom. For fixed $\mathcal{F}$ Hom $(\mathcal{F}, \cdot)$ is a left covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ and $\mathcal{H}om(\mathcal{F}, \cdot)$ is a left covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$.

What is a pairing ?In mathematics a pairing or pair over a field $\mathbb{K}$ is a triple (X, Y, b), which also be denoted $b(X, Y)$ consisting of two vector spaces X and Y over $\mathbb{K}$ and a bilinear map $b : X \times Y \to \mathbb{K}$ called the bilinear map associated with the pairing, or more simply called the pairing's map or its bilinear form.The mathematical theory is general.

76-While a pairing ? I will give a very simple presentation of Poincaré duality. If m is the dimension of M then there exists a natural pairing between p-forms and $m-p$-forms, which can be defined in the following way

$$\langle \omega_p, \eta_{m-p} \rangle = \int_M \omega_p \wedge \eta_{m-p}$$

where $\omega_p \in \Omega^p(M)$ and $\eta_{m-p} \in \Omega^{m-p}(M)$.

If both ω_p and η_{m-p} are closed then by using the integration by part we obtain

$$\langle \omega_p + d\alpha_{p-1}, \eta_{m-p} \rangle = \langle \omega_p, \eta_{m-p} \rangle$$

and

$$\langle \omega_p, \eta_{m-p} + d\beta_{m-p-1} \rangle = \langle \omega_p, \eta_{m-p} \rangle$$

This pairing passes into cohomology and we define a bilinear map :

$$H^p_{dR}(M) \times H^{m-p}_{dR}(M) \to \mathbb{R}.$$

77-The Poincaré theorem says that this pairing is not degenerated, that means that we get isomorphisms

$$(H^p_{dR}(M))' \simeq (H^{m-p}_{dR}(M)).$$

where $(H^p_{dR}(M))'$ is the dual of $(H^p_{dR}(M))'$. It is the **Poincaré duality**. This is the classical form of Poincaré duality.

78- Now I 'll go to the work *Residues and duality.* I restrict my self to some remarks and comments. In this famous volume we dispose a specific building of a notion of duality. As the author said the main purpose of these notes is to prove a duality theorem for cohomology of quasi-coherent sheaves, with respect to a proper morphism of locally nœtherian preschemes.

79-Before going to the text (*R and D*) by Hartshorne I would like to describe duality theory on nœtherian schemes, particularly the notion of a dualizing sheaf that plays a fundamental role in diverse contexts. I will question the conceptual form of this profound theory. We can consider that the properties that it adds contribute to deepening the same conceptual target. The goal of the theory is to produce a trace map in terms of which one can formulate duality results for the cohomology of cohe rent sheaves.

I don't reproduce the theorem in detail and its demonstration for projective scheme. It is the duality for a projective scheme. Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω^0_X be a dualizing sheaf on X , and $\mathcal{O}(1)$ be a very ample sheaf on X. then, for all $i \geq 0$ and $\mathcal{F}$ coherent on X there are natural functorial map such that $\theta^i : Ext^i(\mathcal{F}, \omega_X^0)^{n-i}(X, \mathcal{F})'$, is the map given in the definition of dualizing sheaf : a dualizing sheaf for X a proper scheme of dimensnon n a over an algebraically closed field. is a coherent sheaf ω_X, on X with a trace morphism $t : H^n(X, \omega_X) \to K$, such that forall coherent sheaves $\mathcal{F}$ on X the natural pairing $Hom(\Omega_X) \times H^n(X, \mathcal{F}) \to (H^n(X, \omega_X)$ followed by t gives an isomorphism $Hom(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^n(X, \mathcal{F})$

80-In the case of Serre duality for a proper, smooth, geometrically connected n-dimensional scheme X over a field k the trace map amounts to a canonical k-linear map $t_X : H^n(X, \Omega^n_{X/k}) \to k$ such that (among other things)for any locally free coherent sheaf F on $X$with dual sheaf $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$ the cup product yields a pairing of finite-dimensional k-vector spaces

$$H^i(X, \mathcal{F}) \otimes H^{n-i}(X, \mathcal{F}^\vee \otimes \Omega^n_{X/k}) \to H^n(X, \Omega^n_{X/k}) \stackrel{t_X}{\to} k.$$

82-Which is a perfect pairing for all i. In particular using $\mathcal{F} = \mathcal{O}_X$ and $i = 0$ we see that $dim_k H^n(X, \Omega^n_{X/k}) = 1$ and t_X is non zero , so t_X must be an isomorphism. The foundations make the duality theory, based on on residual complexe, make the duality theory quite computable in terms of differential forms (see Serre above) and residues, and such computability can be very useful.

One of the most important, is the base change compatibilty of the trace map in the case of of proper Cohen -Macaulay morphism with pure relative dimension (e. g. flat families of semi stable curves Ignoring the the base change question, there are simpler methods for obtaining duality theorems in the *projective* CM case.)

83-However, there does not seem to be a published proof of the duality theorem in the general proper CM case over a locally nœtherian base, let alone analysis of its behavior with respect the base change. f. e. the rather important special case of the compatibility of the trace map with respect to base change to a geometric fiber is not all obvious, even if we restrict attention to duality for projective *smooth* maps. Source of motivation by Conrad

I'll use the book by Conrad. This book, its aim, is to prove the hard unproven in particular base change compatibilities of the trace map and some consequences. But Conrad says its book is a companion of Grothendieck's famous article.

That what is important for us. Conrad thus says that in the construction of this theory in *R and D* there are some essential compatibilities and explications of abstract results which are not proven and are quite difficult to verify.

83-The main interest of Grothendieck's work lies in an original position concerning the duality. I will give there a synthetic conceptual (and no complete !) abstract of this building and its aim. Let us consider the various duality theorems the author is quoting in the article. I repeat. First he is citing as typical the duality theorem for a non-singular complete curve X over an algebraically closed field k which says that

$$h^0(D) = h^1(K - D),$$

where D is a divisor, K is the canonical divisor, and

$h^i(D) = dim_K H^i(X, L(D))$ for any i and any divisor D.

84-Some comments need to be made there.
First we are at a certain level of abstraction. D is a divisor. And K the canonical divisor. That means that we present the theorem directly from the point of view of divisors. (example Element of the free abelian group generated by the set of points of a curve X). One consider the difference $K-D$. And one can see the theorem as the expression of cohomological groups in the case of D and $K-D$. It is in a certain sense the deep significance of the duality. (see above).

Recall. Let C be a non singular projective curve in $\mathbf{P}_k^2$, the projective plane over an algebraically closed field k.

For each line L in $\mathbf{P}^2$ we consider $L \cap C$ which is a finite set of points on C.If C is a curve of degree$d$, and if we coint the points with proper multiplicity, then $L \cap C$ will consist of exactly d points with proper multiplicity, then $l \cap C$will consist of exactly d points. We write $L \cap C = \Sigma \eta_i P_i$, where $P_i \in C$ are the points, and η_i the multiplicities, and we call this formal sum a divisor on C. (Hartshorne).

What could be a generalization of this theorem ? Various attempts were made to generalize this theorem to varieties of higher dimension. I return to the book (*R and D*). And as Zariski points out in his report, his generalization. O. Zariski, "Algebraic sheaf theory. Scientific Report on the second summer institute, Paris III, Bull. Amer.Math.Soc. 62, 1956 117-141, le lemme d'Enriques-Severi, Néron, Séminaire Bourbaki 1956)

A generalization of lemma of Enrique-Severi, is equivalent to the statement that for a normal variety X of dimension n over K

$$h^0(D) = h^n(K - D).$$

This is also equivalent to a theorem of Serre [1]

1. FAC §76 theorem 4

For m> 1 a satisfactory algebro - geometric characterisation of the cohomological dimension $h^q(D)$, $q > 0$ is still missing, except for $q = m$ in which case we have a fundamental duality relation similar

$$h^m(D) = h^0(K - D)$$

One knows that the divisor classes D and K_D play a dual role in the theorem of Riemann-Roch + 30 mn

85-Before my return to Grothendieck I will recall you an other way to consider the theorem in the frame of $A^{p,q}$ space. Serre in the paper I will partly present to you has the aim to precise and to enlarge this result and to extend it to the case of arbitrary fibered space V. He will prove that under sufficiently large hypothesis the vector spaces $H^q(X, S(V))$ and $\mathrm{H}_*^{n-q}(X, S(\tilde{V}))$ are in duality. X is an analytic complex variety with dimension n, V is a fibered analytic space with the base X whose the fiber is a vector space of dimension r over $\mathbb{C}$ The sheaf $S(V)$ of the germ of the holomorphic sections of V of$V$ is a analytic coherent sheaf over X.

$*, *$

86- I'll go directly to the Serre's theorem.
a)One needs to define the cohomology with coefficients in a sheaf
b) Sheaves of differential forms over X.
c) Differential forms with coefficients in a analytic fibered space with vector fiber. One needs to define a topology over the space $A^{p,q}$. $A^{p,q}$ differential forms of type p, q.

87- We define a family of semi normes over the space $A^{p,q}(V)$ of sections of $\mathbf{A}^{p,q}(V)$ Let us consider the systems (K, ϕ, ψ, k)
K is a compact of X
ϕ is an analytic homeomorphism from a neighborhood U of K to an open of $\mathbf{C}^n$
ψ is an isomorphism from $\pi^{-1}(U)$ on $Y \times \mathbf{C}^r$ π is the projection from V to K.
k is a sequence of $2n$ integer numbers ≥ 0 : $r_1, \cdots, r_n, s_1, \cdots, s_n$

88-If ω is a membership of $A^{p,q}(V)$, the restriction of ω to U can be identified (by ψ) with a system of r differential forms of type p, q system which can be identified (by ϕ) with a system of $r \begin{pmatrix} n \\ p \end{pmatrix}$. $\begin{pmatrix} n \\ q \end{pmatrix}$ = N differentiable functions on ϕU. one notes these functions $\omega_{i,\phi,\psi}$, $1 \geq i \geq N$

89-Let D^k be the differential operator $\frac{\partial^{r_1+\cdots+r_n+s_1+\cdots+s_n}}{\partial z_1^{r_1}\cdots\partial z_n^{r_n}\partial\overline{z}_1^{s_1}\cdots\overline{z}_n^{s_n}}$

We will pose : $\mathcal{P}_{K,\phi,\psi,k}(\omega) = Sup_{z\in\phi(K)}Sup_{1\geq i\geq N}|D^k\omega_{i,\phi,\psi}(z)|$

The functions $\mathcal{P}_{K,\phi,\psi,k}(\omega)$ are norms ; lwhen K, ϕ, ψ, k vary in every possible way, these semi norms define a topology on $A^{p,q}(V)$ which is separated.

90-Any sequel ω^n of elements of $A^{p,q}(V)$ tends toward zero in the sense of the previous topology if in the neighborhood of any point of X, the N functions which represent locally ω^n tend uniformly toward zero with each of their partial derivates. One can say that the topology of $A^{p,q}(V)$ is that of local uniform convergence. This space is analogous to the space **E** of Schwartz . One verifies that it complete like **E**, that means that it is a Fréchet space.

One gets easily the topological dual of $A^{p,q}(V)$. The topological dual of **E** can be identified with the space of the distributions of compact support. Let V^* be the fibered dual space of V. : if V is defined by means of the the principal fibered space P, one can define V^* as the space associated to P with fiber of type $\mathbf{C}^r$ on which $\mathbf{GL}_r$ **(C)** operates, by the contravariant representation of the usual representation.

For any $x \in X$, there exists a canonical bilinear form on $V_x \times V_x^*$ which places these two spaces in duality. It defines a **O**-linear homomorphism from $\mathbf{S}(V) \otimes_0 \mathbf{S}(V^*)$ in **O** ;

91-From the other side, the operation of exterior product defines an **O**-linear homomorphism from $\mathbf{A}^{p,q} \otimes_0 \mathbf{K}^{p',q'} in \mathbf{K}^{p+q',q+q'}$ q and q' any integers ≥ 0. If one passes to the tensor product one gets an **O**-linear homomorphism

$$\epsilon : \mathbf{A}^{p,q}(V) \otimes_0 \mathbf{K}^{p',q'}(V^*) \to \mathbf{K}^{p+p',q+q'}.$$

92-It $\omega \in A^{p,q}(V)$ and $T \in K^{p',q'}(V^*)$ the image of $\omega \otimes T$ by ϵ will be noticed by $\omega \wedge T$; it an element of $\mathsf{K}^{p+p',q+q'}$, i. e. a compactly current of type $(p+p', q+q')$. If one takes a local cart of V and the corresponding cart to V^* the form ω is identified with r forms ω_i and the current $> T$ with r currents T_i. and $\omega \wedge T$ is equal to $\Sigma_{i=1}^{i=r} \omega_i \wedge T_i$.

If we take $p' = n - p$ and $q' = n - q$ then $\omega \wedge T$ is a compact support current of type (n, n) that we can integrate on X, we set

$$< \omega, T >= \int_X \omega \wedge T.$$

For T fixed, the map $\omega \to < \omega, T >$ is a linear form on $A^{p,q}(V)$ that we call L_T.

93-The map $T \to L_T$ is an isomorphism from $K^{n-p,n-q}$ to the topological dual of $A^{p,q}(V)$. It is easy to see that $L_T = 0$ implies $T = 0$. One has to prove a) that L_T is continuous, b) that every linear form L continue on $A^{p,q}(V)$ is equal to some form L_T.

I'll skip the demonstration. And then we have the proposition. The linear map $d" : A^{p,q}(V) \to A^{p,q+1}(V)$ is continue and its transpose is $(-1)^{p+q+1}d" : (K_*^{n-p,n-q-1}(V^*) \to K_*^{n-p,n-q}(V^*)$ The proof is brief ? Let $\omega \in A^{p,q}(V)$ and $T \in K_*^{n-p,n-q-1}(V^*)$ one has

$$d(\omega \wedge T) = d"(\omega \wedge T) = d"(\omega) + (-1)^{p+q}\omega \wedge d"(T)$$

and while $\int_X d(\omega \wedge T) = 0$, one deduces from this

$$< d"(\omega), T > +(-1^{p+q}) < \omega, d"(T) >= 0,$$

which proves the proposition.

94-One needs one lemma. Let L, M, N be three Fréchet spaces, and $u : L \to M$, $v : M \to N$ two linear homomorphisms such that $v` = 0$. Let L^*, M^*, N^* be the topological duals of L, M, N, and let ${}^t u, {}^t v$ be the transpose maps of u, v. One poses $C = v^{-1}(0), B = u(L), H = C/B$ and let C' $=$ ${}^t u^{-1}(0), B' = {}^t v(N*), H'C'/B'$. Then H is a Fréchet space whose topological dual is isomorphic to H'.

95-Serre applies the above lemma with $L = A^{p,q-1}(V)$, $M = A^{p,q}(V)$, $N = A^{p,q+1}(V)$, and $u = d''$, $v = d''$. After the proposition ** $L^* = K_*^{n-p,n-q+1}(V^*)$, $M^* = K_*^{n-q,n-q}(V^*)$, $N^* = K_*^{n-p,n-q-1}(V^*)$.

Moreover, we have ${}^t u = -(1)^{p+q} d"$, ${}^t v = (-1)^{p+q+1} d"$. The above theorem says that the group $H^q_\phi(X, \Omega(V))$ is isomorphic to $H^{p,q}(A_\phi(V)$ and to $H^{p,q}(K_\phi(V)$.

$*, *$ 96-After some work on differential forms and the complex $K_*(V^*)$, we pass to cohomological groups thanks to a lemma (see 95) and we obtain the following theorem. Let X be an analytic complex manifold infinitely countable, with complex dimension n , and let V be an analytic fiber bundle with vectorial fiber with base X. Suppose both linear maps

$$A^{p,q-1}(V) \xrightarrow{d''} A^{p,q}(V) \xrightarrow{d''} A^{p,q+1}(V)$$

are homeomorphisms ? Then the dual of the Frechet space $H^q(X, \Omega^p(V)$ is canonically isomorphic to $H^{n-q}_*(X, \Omega^{n-q}(V^*))$

We presented here a purely topological duality theorem. It there fore has a purely topological meaning. What does it mean to have a purely topological meaning ? The choice to present the topological point of view as essential must be philosophically justified. It is spatial spread which gives it a spatial meaning. The duality is first for all spatial. Here we have not the same frame as the Grothendieck's one which is not geometric.

And since the duality expresses an important feature of mathematics (see the beginning), the spatial duality expresses its spatial form. I would quote but in a more general sense (not only algebraic Erwein , Voelke, Volkert, focused on the book *it Einführung in die analytische Geometrie and Algebra by Otto Schreier and Emmanuel Sperner* "dual statements are nothing else than different interpretation of one and the same algebraic results.

The fact of this twofold interpretability is called "the principle of duality". This principle in my view is much more general, this principle spreads in all mathematics whatever the discipline can be, and in a meaning that is for me interesting : the philosophical one. ("my" philosophy).)

97-For this purpose I will only recall the example above : the first one is Poincaré's theorem in his second version. The second one would be Tanaka Tadao's paper on duality of non-Abelian groups. First let us briefly recall Poincaré 's result concerning the so-called Betti numbers .

98-Grothendieck Hartshorne's duality. It is the one that is built basically in algebraic geometry. I will explain the reasons of this fact starting from Hartshorne. I recall that the duality theorem for a non-singular complete curve X over an algebraically closed field k, which says that

$$h^0(D) = h^1(K - D)$$

where D is a divisor, K, is the canonical divisor, and

$$h^i(D) = dim_K H^i(X, L(D))$$

for any i, and any divisor D. 30 mn

In terms of sheaves, this result corresponds to the fact that the k-vector spaces $H^i(X, F)$ and $H^{n-i}(X, F^{\vee} \otimes \omega)$ are dual to each other, where F is a locally free sheaf, $F^{\vee}$ is the dual sheaf $\underline{Hom}(F, O_X)$, and $\omega = \Omega^n_{X/k}$ is the sheaf of n- differential forms on X. I recall that Serre gaves a proof of this same theorem by analytic method for a compact complex analytic manifold X. See above the exposition of Serre's duality where K is a canonical divisor on V, i.e., the divisor of an m-fold differential on V This equality follows readily from the so-called« lemma of Enriques-Severi- Zariski »proved by Zariski which states that *« if D is any divisor on V and $C_n$is a general section of V by a hypersurface of order n, then for n sufficiently large the divisor system $Trc_n|D|$, cut out on C_n by the complete linear system $|D|$ is it self complete »*

Let us consider - one more time- the case of projective space $X = \mathbb{P}_k^n$ over an algebraically closed field. Then there is canonical isomorphism.

$$H^n(X, \omega) \simeq k$$

where $\omega = \Omega^n_{X/k}$.

102- I 'll give some complements. 1) Let A be a ring. Let $Y = SpecA$ and let $X = \mathbb{P}^n_A$. then there is an exact sequence of sheaves on X

$$0 \to \Omega_{X/Y} \to \mathcal{O}_{X/Y}(-1)^{n+1} \to \mathcal{O}_X \to 0$$

(The exposant $n+1$ in the middle means a direct sum of $n+1$ copies of $\mathcal{O}_X(-1)$.)

And then we use the above argument to finish. Let $X = \mathbb{P}^n_k$. Taking the dual of the exact sequence of above gives the exact sequence involving the tangent sheaf of $\mathbb{P}^n$.

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathcal{I}_X \to 0.$$

To obtain the canonical sheaf of $\mathbb{P}^n$, we take the highest exterior power of the exact sequence above and we find $\omega_X \simeq \mathcal{O}_X(-n-1)$

103-A result in Hartshorne says that
Let A be a nœtherian ring an d let $X = \mathbb{P}^r_A$ with $r \geq 1$ Then
$H^r(X, \mathcal{O}_X(-r-1)) \simeq A$
Combining this with the Yoneda pairing

$$H^i(X, F) \times Ext_X^{n-1}(F, \omega) \to (H^n(X, \omega)$$

we obtain a pairing

$$H^i(X, F) \times Ext_X^{n-1}(F, \omega) \to k$$

103 To motivate the statement of this main theorem, let us consider the case of projective space $X = \mathbb{P}^n_k$ over an algebraically closed field k. Then there is a canonical isomorphism

$$H^n(X, \omega) \simeq k$$

where $\omega = \Omega^n_{X/k}$ is the sheaf of n-differentials. See above,Combining with the Yoneda pairing

$$H^i(X, F) \times Ext^{n-1}_X(F, \omega)$$

105-In order to go up at the level of Grothendieck 's abstraction last formulation, we need some notions : as first a general right derived functor. Under reasonable conditions there is a right derived functor $RF : D(A) \to D(B)$ with the property that for any $X \in ObA$, if X denotes also the complex which is X in degree zero, and zero elsewhere, then $H^i(RF(X)) = R^iF(X)$, where R^iF is the ordinary i^{th} right derived functor of F. Finally , if $F : A \to B$ and $G : B \to C$ are two functors then $\underline{\underline{R}}(G \circ F) = \underline{\underline{R}}G \circ \underline{\underline{R}}F$

106-And now it is possible to jazz up the duality for projective space as follows. One replaces k by a prescheme Y, so that $X = \mathbb{P}^n_Y$. We consider the derived categories $D(X)$ and $D(Y)$ of the categories of O_X-modules and O_Y-modules, respectively. Then cohomology H^i *becomes*$\underline{\underline{R}}$ f$_*$ where $f : X \to Y$ is the projection. Ext becomes the derived functor $\underline{\underline{R}}f_*$ where $f : X \to Y$ is the projection. Ext becomes the derived functor $\underline{\underline{R}}Hom$ of Hom. I use exactly Hartshorne's terminology.

107- We define $f^!(G) = f^*(G) \otimes \omega$ for $G \in D(Y)$ and we replace F by a complex of sheaves $F \in D(X)$. the the isomorphism gives us an isomorphism

$$\underline{\underline{R}} f_* f^! G \xrightarrow{\sim} G$$

which we call trace <u>*tracemap*</u>. The Yoneda pairing reappears as a natural map

$$\underline{\underline{R}} Hom_X(F, f^! G) \to \underline{\underline{R}}(Hom_Y(\underline{\underline{R}} Hom_Y(\underline{\underline{R}} f_* F, \underline{\underline{R}} f_* f^! G)))$$

which composed with the trace map above gives us the duality morphism see above

$$\underline{\underline{R}}Hom_X(F, f^! G) \to \underline{\underline{R}}Hom_Y(\underline{\underline{R}}f_* F, G)$$

obtained by composing the natural map above with Tr_f is an isomorphism for $F \in D(X)$ and $G \in D(G)$

108-I will comment this formula. This is generalization of the isomorphism given above. I repeat it.

$$\underline{\underline{R}}Hom_X(F, f^! G) \to \underline{\underline{R}}Hom_Y(\underline{\underline{R}}f_* F, G)$$

The result is the *highest level of abstraction* which expresses the duality theorem. I insist on the fact that the basic construction on which it rests is the construction of functor $f^!$.
First I will add some definitional details, after the volume (*R and D*) that allows us to deepen the conceptual meaning of the concepts that we have obtained. See Conrad below.

In triangulated category that are given in (*R and D)* the exact triangles generalize the short exact sequences in an abelian category, as well as fiber sequences and cofiber sequences in topology.

A shift or translation functor on a category D is an additive automorphism (or for some authors, an auto-equivalence) Σ from D to D. It is common to write $X[n] = \Sigma^n X$ for integers n.
A triangle (X, Y, Z, u, v, w) consists of three objects X, Y, and Z, together with morphisms $u : X \to Y$ $v : Y \to Z$ $v : Y$

and w: $Z \to X[1]$. Triangles are generally written in the unravelled form :

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$,

A triangulated category is an additive category D with a translation functor and a class of triangles, called *exact triangles* (or *distinguished triangles*), satisfying properties (TR 1), (TR 2), (TR 3) and (TR 4) I don't give here.. (These axioms are not entirely independent, since (TR 3) can be derived from the others.[3])

The last axiom is called the "octahedral axiom" because drawing all the objects and morphisms gives the skeleton of an octahedron, four of whose faces are exact triangles. The presentation is Verdier's own, and appears, complete with octahedral diagram, in (Hartshorne 1966). I don't give more details about these complementary notions.

Let $\mathfrak{A}$ be an abelian category and A a fixed object. An object I of $\mathfrak{A}$ is injective if the functor $Hom(\cdot, I)$ is exact. An injective resolution of an object A of $\mathfrak{A}$ is a complex $I^\cdot$, defined in degrees $i \geq 0$ together with a morphism $\epsilon : A \to I^0$ such that I^i is an injective object of $\mathfrak{A}$ for each $i \geq 0$ and such that the sequence

$$0 \to \xrightarrow{\epsilon} I^0 \to I^1 \cdots$$

is exact.

We defined for $\mathfrak{A}$ to have enough injectives. Recall and $\mathfrak{A}$ be an abelian category with enough injectives and let F :A$\rightarrow$ $\mathfrak{B}$ be a covariant left exact functor. Then we construct the right derived functor $R^iF, i \geq 0$ of $I^\cdot$ as follows. For each object A of $\mathfrak{A}$, choose one and for all an injective resolution $I^\cdot$ of A. Then we define $R^iF(A) = h^i(F(I^\cdot))$.

109- We have introduced a notion of derivation at the functor level. Here is the powerful of this conceptual building, raising the derivation to the next level. In this sense we can say that a certain analogy to the first level plays a driving role.

This new meaning allows us to develop a philosophical analysis somewhat different from Lautman's. From our point of view the construction extends the operational scope of this notion.

What are our differences from Lautman's analysis ? But, and here is the difference, we keep the same extended notion. We want to extend his point of view. Our main theoretical proposition lies in the fact that the philosophical base is for us in the analogical construction and int the expansive analogy we can detect in all Grothendieck construction. These analogies that we can highlight is the ones that lead to the derived functor. We arrive at a platform where relationship are established between objects and derived applications.Cavaillès's notion of thematization is multiplied and redoubled. Comment on Cavaillès and Lautman.

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. derived functors of $Hom(\mathcal{F}$ We define the functors $Ext^i(\mathcal{F}, \cdot)$ as the right derived functors of $Hom(\mathcal{F}, \cdot)$ and $\mathcal{E}xt^i(\mathcal{F}, \cdot)$ as the right derived functor of $\mathcal{H}\mathrm{om}(\mathcal{F}, \cdot)$.

According to the general properties of derived functors we have $Ext^0 = Hom$ a long exact sequence for a short exact sequence in the second variable, $Ext^i(\mathcal{F}, \mathcal{G}) = 0$ for $i \geq 0$, $\mathcal{G}$ injective in $\mathfrak{Mod}(X)$ and ditto for the $\mathcal{E}xt$ sheaves.

110-I will give some propositions about the functors $Ext^{\cdot}$ and $\mathcal{E}xt^{\cdot}$.
For any $\mathcal{G} \in \mathfrak{Mod}(X)$ we have
i)$\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$
ii) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ *for* i> 0
iii) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G} \simeq \mathrm{H}^i(X, \mathcal{G})$ for all $i \geq 0$.

ii) Recall. For any coherent sheaf $\mathcal{F}$ on X, the natural pairing $Hom(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \simeq k$. is a perfect pairing of finite dimensional vector spaces over k

iii) for every $i \geq 0$ there is a natural functorial isomorphism

$$Ext^i(\mathcal{F}, \omega) \simeq H^{n-1}(X, \mathcal{F})'$$

where ' denotes the dual vector space, which for $i = 0$ is the one induced by the pairing above.

It is an specific presentation. 116-SGA 60-61 , EGA §16. Definition 1.

If $A \to B$ is a morphism of rings, and M a B-module we define $Der_A(A, B)$ to be the A- module of derivations of B into M over A. We define $\Omega^1_{B/A}$ the module of relative one-differential of B over A, to be the module representing the functor

$$M \to Der_A(B, M)$$

.

In other words there is a derivation $d : B \to \Omega^1_{B/A}$ given , such that for any B- module M , the natural map

$$Hom_B(\Omega^1_{B/A}, M) \to Der_A(B, M)$$

is an isomorphism.

If $f : X \to Y$ is a morphism of preschemes, we define $\Omega^1_{X/Y}$, the sheaf of relative one-differentials of X over Y by considering open affines in X and Y, and glueing the corresponding modules $\Omega^1_{B/A}$

117-Definition 2

EGA (IV 6.8. 1) A morphism $f : X \to Y$ of preschemes is smooth if it flat, locally of finite presentation, and for every $y \in Y$, the fibre $f^{-1}(y)$ is locally nœtherian, and geometrically regular (i.e. « absolutely non-singular »). I give the two examples of these ones given by H. An open immersion is smooth. A prescheme X over a field k is smooth $\Leftrightarrow$ it is locally nœtherian and geometrically regular.

I will quote two propositions. *R and D* chaptre III. (SGA 60-61, II 4 3). Let $f : X \to Y$ be a smooth morphism of preschemes over another prescheme S. Then $\Omega^1_{X/Y}$ is locally free (or rank n, relative dimension of X over Y) and there is a an exact sequence

$$0 \to f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

We have a little later a similar proposition [SGA 60-61, II. 4. 3]

Definition 3 118-A closed subscheme Y of a prescheme X is locally a complete intersection if every point $y \in Y$ has a neighborhood U such that in U the ideal J_Y of Y is generated by an $\mathcal{O}_{\mathcal{X}}$ sequence, i. e. a collection of sections $s_1, \cdots, s_r$ such that s_1 is a non-zero divisor in $\mathcal{O}_X$, and for each $i = 2, \cdots, r$, s_i is a non-zero divisor in $\mathcal{O}_X/(s_1, \cdots, s_{i-1})$.

After these definitions we define $\Omega_{X/Y} = \wedge^n \Omega^1_{X/Y}$ and we define $\omega_{X/Y} = (\wedge^n (J/J^2))^{\checkmark}$ where J is the sheaf of ideals of X, $\checkmark$ denote the dual. We define also an isomorphism. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphism of locally nœtherian preschemes, f, g,fg smooth or a local complete intersection then

$$\zeta_{f,g} : \omega_{X/Z} \xrightarrow{\sim} f^* \Omega_{X/Y} \otimes \Omega_{X/Y}$$

Finally we have three occurrencies of theorem of duality.

Remark- and then we have here Gorenstein morphisms (complicated to give a definition : a Gorenstein local ring is a commutative local ring with finite injective dimension. G morphism is a little more complicated. This chapter achieves his goal : to define the functor $f^\sharp$. And in this way it will define the most important functor $f!$. I prefer - in order to save time - to directly go to an outline of the proof.

I would like to remind a very simple categorical abstract of the Grothendieck's project in this famous paper. Grothendieck duality theory on nœtherian schemes, particularly the notion of a dualizing sheaf, play a fundamental role in contexts as diverse as the arithmetic theory of modular forms, and the study of moduli spaces of curves. The goal of the theory is to produce a trace map in terms of which one can formulate duality results for the cohomology of coherent sheaves.

In the 'classical case' of Serre duality for a proper, smooth, geometrically connected n-dimensional scheme X over a field k, the trace map amounts to a canonical k linear map

$$t_X : H^n(X, \Omega^n_{X/k}) \to k$$

This means that G's formulation is the most general

for any locally free coherent $\mathcal{F}$ on. $X$with dual sheaf $\mathcal{F}^{\checkmark} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ the cup product yields a pairing of finite-dimensional k- vector spaces

$$H^i(X\mathcal{F}) \otimes H^{n-1}(X, \mathcal{F}^{\checkmark} \otimes \Omega^n_{X/k}) \to H^n(X, \Omega^n_{X/k}) \xrightarrow{t_X} k$$

which is a perfect pairing for all i. In particular, says Conrad, using $\mathcal{F} = \mathcal{O}_X$ and $i = 0$ we see that $dim_k H^n(X, \Omega^n_{X/k}) = 1$ and t_X is non-zero, so t_X must be an isomorphism. Grothendieck duality extends this to a relative situation., and Contrad adds that the relative case where the base is a discrete valuation ring is highly non-trivial.

I would like to add the important remark by Conrad : the foundations of Grothendieck duality - we saw it partially- based on residual complexe (I will explain a little still more precise) are worked out in Hartshorne's Residues and duality, make the duality theory quite computable in terms of differential forms and residues, and such computability can be very useful and Conrad refers to Berthelot's thesis or Mazur's pioneering work on the Eisentein ideal.

,

119- Ouline of the proof. I will sequel highligthing of Belman's paper which is himself inspired by Conrad.The geometric approach to Grothendieck duality can be summarized by the following slogan define $f^!$ by looking for a dualizing complex and *defining* the functor in terms of this complex.

And the author adds the whole setup of the book should be considered in this point of view

Dualizing complexes. I make a review of some facts from RD concerning Cousin complexes. Let X be a locally noetherian scheme and let $Z^\bullet = \{Z^p\}$ be a filtration of X of X by the subsets Z^p such that

- Each Z^p is stable under specialisation
- $Z^p \subseteq Z^{p-1}$ for all p
- $X = Z^p$ for sufficiently negative p and $\cap Z^p = \emptyset$, so X is disjoint union of $Z^p - Z^{p+1}$ over $p \in \mathbf{Z}$
- each $x \in Z^p - Z^{p+1}$ is not a specialisation of any other point of Z^p

I give some abstract of Hartshorne's RD.
120- Chapter 1 and 2 . As derived categories were still in their infancy and there was not a published texte available about them , they are first introduced and then applied to the situation of schemes.
Chapter 3 The proof of Grothendieck duality for projective morphisms. In this "easy" case we can do more explicit calculation, and control the dualizing object. Comment.
The idea is to factor sufficiently nice morphisms into

1. smooth morphisms
2. finite morphisms

121-and introduce the functor $f^!$ for each of these. The functor $f^!$ for a finite morphism is denoted $f^\flat$, the one for smooth morphism is $f^\sharp$. By checking compatibility of these two definitions (which are suggested by Ideal theorem, one obtains a theory of $f^!$ for these nice morphisms (but not all the required properties for $f^!$). Thus is then used to obtain Grothendieck duality for *projective morphisms* .

122- A similar idea of factoring morphisms into tractable ones occurs in section2 . Comment.
Chapter 4. As discussed in the section on the application of Grothendieck duality there is an interesting notion of local duality, related to local cohomology. B proceeds first to indicate we have to understant what happens in the
in the local case, as this is what is used to characterize the objects defined in the next chapter. Comment local / global more than by Lautman

Chapter 5. Recall that this approach to Grothendieck duality can be summarised by the following slogan
We define $f^!$ by looking for a dualizing complex and ıdefine this functor in terms of this complex.
It does give an explicit flavor to the machinery of complexes.
In this chapter the machinery ands properties of dualizing complexes are discussed.

124-The goal is to understand how dualizing complexes relate to local duality, how this behaves with respect to singularities and how we can interpret the dualizing complexes. Some of these properties are discussed in section 1. 4.

Unfortunately, dualizing complexes live in derived category and this is a non-local object. To solve this problem *residual problem* are inroduced. This is a manifestation of dualizing complexes in the non-derived category in the non derived category of chain complexes. A nice motivation for having a theory for both dualizing and residual complexe is given on 3 p. 106-107.

127-chapter 5
sequel. The complex above with its augmentation from$\Omega^1_{X/k}[1]$ plays a fundamental role in the classical construction of an isomorphism

$$H^1(X, \Omega^1_{X/k}) \cong k$$

via residues when X is *proper* over k. This isomorphism also determines Serre duality on such k schemes. Grothendieck's theory vastly generalizes this construction.

there are two complexes naturally lurking here. $\Omega^1_{X/k}[1]$ and the complex above. These are canonically quasi-isomorphic but are of quite different nature.
The first one is a bounded below complex with coherent cohomology and its terms involve the quasi-coherent injective at all $x \in X$ each appearing exactly 'once'

128-Meanwhile $\Omega^1_{X/k}[1]$ has coherent cohomology and finite injective dimension with the natural map

$$O_X \to \mathbf{R}Hom^\bullet_X(\mathbf{R}Hom^\bullet_X(O_X, \Omega^1_{X/k}[1], \Omega^1_{X/k}[1])$$

an isomorphism in $\mathbf{D}(X)$ since $\Omega^1_{K/k}$ is invertible.

recall that this approach to Grothendieck duality can be summarised by the following slogan
we define $f^!$ by looking a dualizing complex and *define* the functor in terms of this complex.
The statement of Grothendieck, I recall, doesn't mention dualizing complexes explicitly, Biemans says, it goes an explicit flavour to it.

In the chapter 6-the machinery and properties of dualizing complexe are discussed. The goal is to understand how dualizing complexes relate to local duality. How this behaves with respect to singularities and how we can interpret the dualizing complexes.

What is important is the fact that these complexes are quasi-isomorphic to each other, but the residual complex is a bunch of injective hulls taken together, which can be taken to live in a non-derived category. and still allow for computations. I can stop here.

Philosophically G'approach is profound because it is essentially intrinsic. This can explain its strength.

END

Now a theory for embeddable morphisms and residual complexes is developed, using functors f^y and $f^x$for finite and smooth morphisms. Their definitions depend on (pointwise) dualizing complexes, but (it is the point) they are truly functors on the non-derived level. This chapter is the technical part of the book.

Chapter 7. 130-In this chapter Grothendieck duality in its general form is finally proved. We have obtained many ordinary preliminary results, and this allows us to summarize the final proof.
1. Grothendieck duality for R*Hom* is local (and it implies the other statement) hence we reduce Y to the spectrum of a local ring (so the base is affine).
2. By some machinery of derived categories we can replace the complex by a single quasicoherent sheaf on $X*$.
3. We can replace the quasi-coherent sheaf on X by a coherent one using a direct limit argument.

4. As Y is affine the local statement for $\mathrm{R}Hom$ becomes a global statement for $\mathrm{R}\mathcal{H}om$.
5. We check compatibility of the global statement with composition of two morphisms (this is not a part of the conceptual flow of the proof in the opinion d f B). This where the residual complexes are required. One of the compatibilies requires a coherence condition, which explains the reduction in the third step.

6. Using nœtherian induction on X we can assume that the theorem is proven for every $g : Z \to Y$ where $i : Z \to X$ is a closed immersion with $Z \neq X$ and $g = f \circ i$

7.131- As X is proper over Y we apply Chow's lemma to find an X' which is projective over Y and a morphism $g : X \to X'$ which is an isomorphism over some non-empty open subset U. By the nœtherian induction we can assume the theorem proven for the complement, which allows us to reduce the statement to the projective morphism in the factorisation of Chow 'lemma.
8. Now we can apply the result we had for projective morphisms, and conclude.

END.

132-The passage from 1 to 4, and then 4 to 8

To proof Grothendieck duality one first prove it for embeddable morphisms.

Recall. Let S be a fixed prescheme. We say a morphism $f : X \to Y$ in the category of preschemes over S is *embeddable* (or S-embeddable), if there exists a smooth scheme P over S and a finite morphism $i : X \to P_X = P \times_S Y$ such that $f = P_2 \circ i$. Unless otherwise specified, embeddable will usually mean over $SpecZ$.

Example. A projective morphism $f : X \to Y$ where Y is quasi-compact and admits an ample sheaf is embeddable. Any morphism of finite type of affine schemes is embeddable in some affine space.

The main issue is that morphism of finite type (a very general class of morphism) are locally embeddable, but not globally embeddable. Hence tis approach does not yield a theory of $f^!$ in general. To overcome this issue we need the notion of dualizing and especially dualizing complexes.

Remark. We use the notion of dualizing and especially residual complexes

.

133-I recall once for all the definition of the functor $f^!$. We consider schemes which are nœtherian and admit a dualizing complex(or equivalently, admitting a residual complex which is bounded as a complex. All schemes to be considered, all complexes to be considered automatically have finite Krull dimension. Any scheme of finite type over a regular ring with finite Krull dimension admits a dualizing complex. This includes finite type schemes over $\mathbf{Z}$, a field or a complete local nœtherian ring. Recall what is needed for the construction of dualizing sheaves and duality theorem.

Let $f : X \to Y$ be a finite type morphism between nœtherian schemes which admit a dualizing complex.There is a map of triangulated categories

$$f^! : \mathbf{D}_c^+(Y) \to \mathbf{D}_c^+(X)$$

whose basic properties we have then review in the beginning of the chapter III of *R and D*.

134-The duality theorem for f requires f to be proper and uses a trace map of δ functors

$$Tr_f : \mathbf{R}f_* \circ f^! \to 1.$$

The idea is that Tr_f should make $f^!$ a right adjoint to $\mathbf{R}f_*$.

We will need to use duality 'functors' which are defined in terms of residual complexes. Let $K^\bullet$ be a residual complex on Y, so $K^\bullet$ is a bounded complex of quasi coherent injective $\mathcal{O}_Y$-modules and $K^\bullet$ has *coherent* cohomology sheaves. The contravariant 'duality' δ -functor

$$D_{K^\bullet} = D_{Y,K^\bullet} : \mathbf{D}_c(Y) \to \mathbf{D}_c(Y)$$

is defined to be $\mathbf{R}\mathcal{H}om_{Y^\bullet})$. For $K^\bullet$ fixed it is denoted D_Y.

This $\delta-$functior interchanges $\mathbf{D}_c^+$ and $\mathbf{D}_c^-$. Since residual complexes on Y are dualizing complexes in the derived category there is a canonical isomorphisms of δ-functors

$$\eta = \eta k^{\bullet} : 1 \cong DD$$

. We need to define what a dualizing complex consists of and ditto for residual complex.

135-We have to start with a "bounded complex of abelian groups with finitely generated cohomology". See B. By considering this particular complex we obtained a duality functor

$$D : M^* \to \mathbf{R}Hom^\bullet(M^\bullet, \mathbb{Z})$$

The derived category of the category of $\mathcal{O}_X$- modules is denoted $\mathbf{D}(X)$ and we have $\mathbf{D}^+_{qc}$ the full subcategoy in $\mathbf{D}$ consisting of complexes whose cohomologies are all quasi-coherent.

Let X be locally nœtherian. A dualizing complex is complex $\mathcal{R}^\bullet \in \mathbf{D}^+_{coh}(X)_{fid}$ such that for each $\mathcal{F}^\bullet \in \mathbf{D}(X)$ the morphism

$$\eta : \mathcal{F}^\bullet \to \underline{D} \circ \underline{D}(\mathcal{F}^\bullet) = \mathbf{R}Hom^\bullet(\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{R}^\bullet), R^\bullet)$$

136-The last step is very subtle and explains how to get the functor $f^!$. We cannot glue together $\mathbf{D}^+$dualizing complexes, as the derived category is not a local object. Hence (B) we need to do some work to use our dualizing complexes. The idea is to take a dualizing complex $\mathcal{R}^\bullet \in \mathbf{D}^+_{coh}(X)$ and turn it into an actual complex (i.e.in some $Ch(X)$ which will be called *residual complex*. and obtain $f^!$. As we can glue actual complexes together we can obtain a $f^!$ by gluing residual complexes together. And as B adds, we need to know that doesn't matter whether we use dualizing complexes or residual complexes.

We will look for a functor $E : \mathbf{D}^+_{coh}(X) \to Ch^+_{coh}(Qcoh_{inj}/X)$ such that $Q(\mathcal{R}^\bullet \mathcal{R}^\bullet$ where Q is the quotient functor in the construction of the derived category and. $Ch^+_{coh}(Qcoh_{inj}/X)$ is a model for $\mathbf{D}^+_{coh}(X)$.

The definition is the following. Let X be a locally nœtherian prescheme. A *residual complex* $K^\bullet$ on X is a bounded below complex of quasicoherent injective O_X- modules with coherent cohomology, together with an isomorphism

$$\bigoplus_{p \in \mathbb{Z}} \mathcal{K}^p \cong \bigoplus_{x \in X} J(x)$$

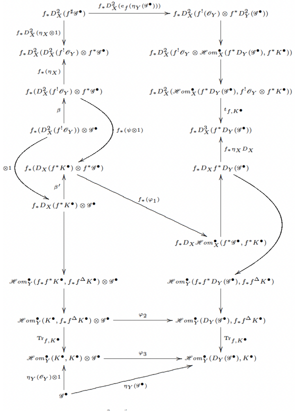
where $J(x)$ is the quasicoherent injective $\mathcal{O}_X$ module given by the constant sheaf with values in an injective hull of $k(x)$ over $\mathcal{O}_{X,x}$ on $(cl(xx)$, an zero elsewhere. from [*R and D* proposition V.34]
The functor E uses the theory of *Cousin complexes* [*R and D* chapter IV]

The conclusion will be the following.

I can give only some elements of the conclusion of the proof of duality theorem.

Let us consider a step of proof of duality theorem by Conrad

$$\begin{array}{ccc} \mathbf{R}f_* f^{\sharp}\mathscr{G}^{\bullet} & \xrightarrow{e_f} & \mathbf{R}f_* f^{!}\mathscr{G}^{\bullet} \\ \uparrow \simeq & & \downarrow \mathrm{Tr}_f(\mathscr{G}^{\bullet}) \\ \mathbf{R}f_*(f^{!}\mathscr{O}_Y) \overset{\mathbf{L}}{\otimes} \mathscr{G}^{\bullet} & \xrightarrow[\mathrm{Tr}_f(\mathscr{O}_Y)\overset{\mathbf{L}}{\otimes} 1]{} & \mathscr{G}^{\bullet} \end{array}$$



I will make some comments on this diagram, only in order to make clearer important steps. The terms $f_* D^2_X(f^\sharp \mathcal{G}^\bullet)$ and $f_* D_X(f^* K^\bullet) \otimes \mathcal{G}^\bullet$ in the left column are equal $Rf_* f^\sharp \mathcal{G}^\bullet$ and $\mathbf{R}f_*(f^! \mathcal{O}_Y) \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ respectively.

I allow myself to repeat only som remarks in order to remind that the goal is to make sense to consider the outside edge as a (large) diagram. in $\mathbf{D}(Y)$ Ditto $f_*(\eta_X)$ and β are quasi-isomorphism. They are some properties of this diagram that ensure its commutativity. It is the case for exemple for *derived category* inverse of the quasi isomorphism $f_*\eta_X D_X$ in the right column, which is $f_* D_X(\eta_X)$

We see also that ϕ_1, ϕ_2, ϕ_3 are special cases of the map

$$\mathcal{H}om^\bullet_Y(K^\bullet_1, K^\bullet_2) \otimes \mathcal{F}^\bullet \to \mathcal{H}om^\bullet_Y(\mathcal{F}^\bullet, K^\bullet_1)K,^\bullet_2)$$

We get some other properties from this diagram.

$$D^2_X(f^!\mathcal{O}_Y \longrightarrow D^2_X(f^!\mathcal{O}_Y \otimes \mathcal{H}om^\bullet_X(f^*K^\bullet, f^*K^\bullet))$$

$$\downarrow$$

$$D^2(\mathcal{H}om^\bullet_X(f^*K^*, f^!\mathcal{O}_Y \otimes f^*K^*))$$

$$\downarrow$$

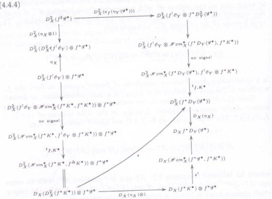
$$D^2(\mathcal{H}om^\bullet_X(f^*K^\bullet, f^\Delta K^\bullet$$

$$\downarrow$$

$$D_X(f^\bullet K^\bullet) \longleftarrow D_X \circ D^2_X(f^*K^*D)$$

I can add some remarks concerning this diagram. All subdiagram in the previous diagram are commutative on the level of complexes the commutativity of the outside edge of the diagram

(4.4.4)



In this diagram, the lower right map s' is defined by using the canonical map of complexes

(4.4.5) $$\mathscr{H}om^\bullet(\mathscr{F}^\bullet, \mathscr{E}^\bullet) \otimes \mathscr{H}^\bullet \to \mathscr{H}om^\bullet(\mathscr{H}om^\bullet(\mathscr{H}^\bullet, \mathscr{F}^\bullet), \mathscr{E}^\bullet)$$

for bounded below $\mathscr{E}^\bullet$, bounded $\mathscr{F}^\bullet$, and bounded above $\mathscr{H}^\bullet$, with an intervention of the sign $(-1)^{r(p+1)}$ on the term $\mathscr{H}om(\mathscr{F}^p, \mathscr{E}^q) \otimes \mathscr{H}^r$. Meanwhile, the 'diagonal' map s in (4.4.4) is the composite

$$D^3_X(f^*K^\bullet) \otimes f^*\mathscr{G}^\bullet \xrightarrow{t} D^3_X(\mathscr{H}om^\bullet_X(f^*\mathscr{G}^\bullet, f^*K^\bullet)) \to D^3_X(f^*D_Y(\mathscr{G}^\bullet)),$$

with t defined to be the composite

(4.4.6)
$$\begin{array}{ccc} D^3_X(f^*K^\bullet) \otimes f^*\mathscr{G}^\bullet & \longrightarrow & D_X(\mathscr{H}om^\bullet_X(f^*\mathscr{G}^\bullet, D^2_X(f^*K^\bullet))) \\ & & \downarrow \\ & & D_X(\mathscr{H}om^\bullet_X(D_X(f^*K^\bullet) \otimes f^*\mathscr{G}^\bullet, f^\Delta K^\bullet)) \\ & & \| \\ D^3_X(\mathscr{H}om^\bullet_X(f^*\mathscr{G}^\bullet, f^*K^\bullet)) & \longleftarrow & D^2_X(D_X(f^*K^\bullet) \otimes f^*\mathscr{G}^\bullet) \end{array}$$

defined by using (4.4.5) and two applications of the canonical map of complex