La puissance des flèches : une invitation à la théorie des représentations des carquois

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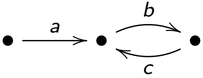
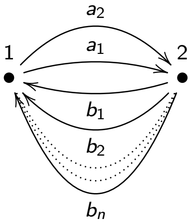
Zénon ! Cruel Zénon ! Zénon d'Élée !
M'as tu percé de cette flèche ailée
Qui vibre, vole, et qui ne vole pas !
Le son m'enfante et la flèche me tue !
[...]
P. Valéry, *Le cimetière marin*

Quivers

A quiver [carquois] $\mathbf{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a finite oriented graph :
$\mathcal{Q}_0$ finite set of vertices, $\mathcal{Q}_1$ finite set of arrows.
We denote by Q the *non-oriented* graph associated to $\mathbf{Q}$.





There are maps *tail* $t : \mathcal{Q}_1 \to \mathcal{Q}_0$ and *head* $h : \mathcal{Q}_1 \to \mathcal{Q}_0$ that do what their name says.
Whenever $h(a_1) = t(a_2)$, the arrows a_1 and a_2 can be concatenated and we can form a *path* $a_2 a_1$. By iteration one defines paths $a_m a_{m-1} \cdots a_2 a_1$ and an obvious operation of concatenation between paths.
For each vertex $x \in \mathcal{Q}_1$, we denote by e_x the trivial ("length zero") path starting and ending at x ; if p, q are paths in **Q** s.t. $t(p) = x = h(q)$, then we set $pe_x = p$, $e_x q = q$. In particular,

$$e_x e_x = e_x \,, \quad e_x e_y = 0 \quad \text{if } x \neq y \,.$$

So, the free abelian group B generated by $\{e_x\}_{x \in \mathcal{Q}_0}$ has a natural ring structure and $\sum_{x \in \mathcal{Q}_0} e_x = 1_B$.

The *path algebra* $\mathbb{Z}\mathbf{Q}$ is the $\mathbb{Z}$-algebra generated by paths in $\mathbf{Q}$ (including trivial paths) and endowed with the product given by concatenation of paths.

If R is any commutative ring, the path algebra of $\mathbf{Q}$ over R is the R-algebra $R\mathbf{Q} = \mathbb{Z}\mathbf{Q} \otimes_{\mathbb{Z}} R$. In particular, for any field $\mathbb{K}$, we have the path algebra $\mathbb{K}\mathbf{Q}$ over $\mathbb{K}$.

$R\mathbf{Q}$ is an associative algebra with unity $1 = \sum_{x \in \mathcal{Q}_0} e_x$; it is naturally graded $\mathbb{Z}$-graded by path length and $(R\mathbf{Q})_0 = \sum Re_x$ is semisimple.

Examples.

1) Let $\mathbf{L} =$  be the Jordan 1-loop quiver : $\mathbb{K}\mathbf{L} \simeq \mathbb{K}[X]$.

2) Let $\mathbf{L}_n$ the n-loop quiver (the quiver with one vertex and $n \geq 2$ loops) : $\mathbb{K}\mathbf{L}_n \simeq$ free associative (non commutative) algebra over $\mathbb{K}$ in n generators.

3) Let $\mathbf{A}_n = \overset{1}{\bullet} \xleftarrow{a_2} \overset{2}{\bullet} \xleftarrow{a_3} \overset{3}{\bullet} \longleftarrow \cdots \longleftarrow \overset{n-1}{\bullet} \xleftarrow{a_n} \overset{n}{\bullet}$ be the *straight quiver* whose underlying graph is the Dynkin graph A_n ($n \geq 1$). There is an isomorphism

$$\psi \colon \mathbb{K}\mathbf{A}_n \simeq T_n(\mathbb{K}),$$

where $T_n(\mathbb{K})$ is the algebra of upper triangular $n \times n$ matrices : $\psi(e_i) = E_{i,i}$ for $i = 1, \ldots, n$, $\psi(a_k) = E_{k-1,k}$ for $k = 2, \ldots, n$ ($E_{i,j}$ is the matrix with 1 in the j-th column of the i-th row, and 0s everywhere else).

The path-algebra $\mathbb{K}\mathbf{Q}$ is finite-dimensional if and only if $\mathbf{Q}$ is a (finite) quiver without oriented cycles.

In categorical terms, a quiver can be equivalently defined as a functor as a functor from the Kronecker category $\bullet \rightrightarrows \bullet$ having 2 objects and 2 morphisms (plus identies) to the category of (finite) sets.

A quiver $\mathbf{Q}$ generates a category $\mathfrak{Q}$ whose objects are the vertices of $\mathbf{Q}$ and whose morphisms are the paths in $\mathbf{Q}$ (including the trivial paths) : the composition rule for morphisms is induced by the product rule (i.e. concatenation) for paths.

Given any two objects x, y in $\mathfrak{Q}$, the cardinality of $\mathrm{Hom}_{\mathfrak{Q}}(x, y)$ is equal to the cardinality of the set of all paths from x to y in $\mathbf{Q}$. Hence, $\mathrm{Hom}_{\mathfrak{Q}}(x, y)$ is finite for all x, y if and only if $\mathbf{Q}$ contains no oriented cycles.

Quiver representations

Let $\mathfrak{A}$ be a category.
A representation of $\mathcal{Q}$ in $\mathfrak{A}$ is a functor $\mathfrak{Q} \to \mathfrak{A}$ which maps each vertex $x \in \mathcal{Q}_0$ to an object V_x of $\mathfrak{A}$ and each arrow $a \in \mathcal{Q}_1$ to a morphism

$$X_a \colon V_{t(a)} \to V_{h(a)} \,.$$

A morphism (or better, a natural transformation) $\psi \colon (V, X) \to (W, Y)$ between two representations of $\mathcal{Q}$ in $\mathfrak{A}$ is a family of morphisms $\psi_i \colon V_x \to W_x$ such that the diagrams

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\psi_{t(a)}"] \arrow[d, "X_a"'] & W_{t(a)} \arrow[d, "Y_a"] \\
V_{h(a)} \arrow[r, "\psi_{h(a)}"] & W_{h(a)}
\end{tikzcd}$$

commute for every arrow $a \in E$.

Let $\mathfrak{Rep}_{\mathfrak{A}}(\mathbf{Q})$ be the category of representations of the quiver $\mathbf{Q}$ in the category $\mathfrak{A}$. Given a commutative ring R, the category of representations of $\mathbf{Q}$ in the category of R-modules will be denoted by $\mathfrak{Rep}_{R}(\mathbf{Q})$ (in particular, we may take $R = \mathbb{K}$).

> The category $\mathfrak{Rep}_{R}(\mathbf{Q})$ is equivalent to the category of left $R\mathbf{Q}$-modules.

Given a left $R\mathbf{Q}$-module M one define the representation (V, X) by setting : $V_x = e_x M$ and, for any arrow a such that $t(a) = x$, $h(a) = y$ and any $v \in V_x = e_x M$, $X_a(v) = av \in e_y M = V_y$. Conversely, given a representation (V, X), one easily shows that $M = \oplus_{x \in \mathcal{Q}_0} V_x$ has a natural left R-module structure.

If the category $\mathfrak{A}$ is abelian, then the category $\mathfrak{Rep}_{\mathfrak{A}}(\mathbf{Q})$ is abelian. [Mozgovoy 2020]

Quiver representations in categories that are *not* abelian have also been studied. Let $\mathfrak{T}$ be the category whose objects are finite pointed sets $(S, *_S)$ and whose morphisms are pointed maps $f\colon (S, *_S) \to (T, *_T)$ such that $f|_{S \smallsetminus f^{-1}(*_T)}$ is an injection. This category is usually thought of as the category of "vector spaces over $\mathbb{F}_1$" (as first observed by J. Tits in 1956, linear algebra over finite fields $\mathbb{F}_q$ "reduces" in the limit $q \to 1$ to combinatorics of finite sets ; this idea can be be made precise, for example, in the framework of Deitmar's"schemes over $\mathbb{F}_1$" [Deitmar 2006], or in the setting of other proposed "geometries over $\mathbb{F}_1$" [López Peña & Lorscheid 2009]).

The category $\mathfrak{T}$ is proto-exact. For every quiver $\mathbf{Q}$ the category $\mathfrak{Rep}_{\mathfrak{T}}(\mathbf{Q})$ is proto-exact and there is a fully faithful functor from this category to the category of finite left $M_{\mathbf{Q}}$-modules, where $M_{\mathbf{Q}}$ is the path monoid associated to $\mathbf{Q}$. However, this functor may fail to be an equivalence of categories. [Szczesny 2012], [Jun & Sistko 2023], [Mozgovoy 2025].

Representations in $\mathfrak{Rep}_{\mathfrak{T}}(\mathbb{Q})$ are often called $\mathbb{F}_1$-representations of $\mathbf{Q}$.

Given a $\mathbb{K}$-representation (V, X) of $\mathbf{Q}$, its dimension vector is $\mathbf{dim}\, V = \mathbf{v} = (\dim V_x)_{x \in \mathcal{Q}_0} \in \mathbb{N}^{\mathcal{Q}_0}$.

Examples

1) A $\mathbb{K}$-representation of the Jordan quiver $\overset{\circlearrowright}{\bullet}$ of dimension m is a pair (V, X), where V is an m-dim. $\mathbb{K}$-vector space and $X \in \mathrm{End}_{\mathbb{K}}(V)$. The problem of classifying representations of $\mathbf{L}$ is equivalent to the classical linear algebra problem of classifying $\mathbb{K}$-endomorphisms of a vector space up to a change of basis (if $\mathbb{K}$ is algebraically closed, the problem is solved by the Jordan-Chevalley decomposition theorem).

2) Classifying the $(m, n$ dim. $\mathbb{K}$-representations of the Kronecker quiver $\mathbf{\Theta}_2 \;\; \overset{x}{\bullet} \overset{a}{\underset{b}{\rightrightarrows}} \overset{x}{\bullet}$ amounts to classifying pairs of $\mathbb{K}$-homomorphisms $X_a, X_b \colon V_x \to V_y$, where $\dim V_x = m$, $\dim V_y = n$, up to changes of basis in V_x, V_y. This is a slightly harder problem, which is classically called the problem of "equivalence of pencils of matrices". This problem was solved by L. Kronecker in 1890 in relation to the problem of classification of bilinear forms. See [Kirillov 2016] for a quiver-theoretic approach.

3) The path algebra h-Kronecker quiver $\mathbf{\Theta}_h$ $\overset{x}{\bullet} \underset{a_h}{\overset{a_1}{\underset{\vdots}{\rightrightarrows}}} \overset{y}{\bullet}$ can be written as $\mathbb{K}\mathbf{\Theta}_h = \begin{pmatrix} \mathbb{K} & H \\ 0 & \mathbb{K} \end{pmatrix} = L \oplus H$, where $L = \mathbb{K}e_x \oplus \mathbb{K}e_y$ and $H = \langle a_1, \dots, a_h \rangle_{\mathbb{K}}$ is endowed with an L-bimodule structure characterised by the equations $a_i e_x = e_y a_i = a_i$, $e_x a_i = a_i e_y = 0$ (notice that $H \otimes_L H = 0$, so that $A = L \oplus H$ is the tensor algebra of H over L). A left $\mathbb{K}\mathbf{\Theta}_h$-module structure on M is thus a left L-module structure together with a left L-module map $M \otimes_L H \to M$. The former is the same as a direct sum decomposition $M = V \oplus W$, where $V = e_x M$ and $W = e_y M$, while the latter is completely determined by a $\mathbb{K}$-linear "multiplication" map $\alpha \colon V \otimes H \to W$ (for any $v \in V$ one has $a_i v = a_i e_x v = e_y a_i v \in W$ and for any $w \in W$ one has $a_i w = a_i e_y w = 0$). A triple $(V, W, \alpha \colon V \otimes H \to W)$ is often called an *H-Kronecker module*.

Kronecker modules play a prominent role in linear algebra and in the study of moduli spaces of sheaves over projective varieties. For example, [Barth 1977] showed that any stable rank 2 bundle F (with $c_1 = 0$) over the projective plane $\mathbb{P}^2_{\mathbb{C}}$ could be recovered from the Kronecker module $\alpha_F \colon H^1(F(-2)) \otimes H \to H^1(F(-1))$ where $H = H^0(\mathcal{O}(1))$ (Barth's pioneering paper was at the origin of the "monad" machinery; cf. [Okonek, Schneider & Spindler 1980]).

[Álvarez-Cónsul & King 2007] used Kronecker modules to define a natural functor mapping "nice" coherent sheaves over a smooth projective variety X over an algebraically closed field $\mathbb{K}$ to representations of a Kronecker quiver.

Their idea (which combines Grothendieck's original construction of the Quot scheme, Mumford's GIT and a clever construction due to Simpson) is based on the fact that, after fixing a very ample line bundle $\mathcal{O}(1)$ on X, one can choose two sufficiently large integers n, m, $m \gg n$, such that for every (Gieseker) semistable sheaf $\mathcal{F}$ on X with Hilbert polynomial $P_{\mathcal{F}}(d) = \chi(X, \mathcal{F}(d))$ one has $P_{\mathcal{F}}(n) = h^0(\mathcal{F}(n))$, $P_{\mathcal{F}}(m) = h^0(\mathcal{F}(m))$, and *surjective* maps

$H^0\big(\mathcal{F}(n)\big) \otimes \mathcal{O}(-n) \to \mathcal{F}$ (the natural evaluation map)

$\alpha_{\mathcal{F}} \colon H^0\big(\mathcal{F}(n)\big) \otimes H^0\big(\mathcal{O}(m-n)\big) \to H^0\big(\mathcal{F}(m)\big)$ (obtained by the previous one by applying the functor $H^0\big(- \otimes \mathcal{O}(m)\big)$).

So, by setting $H = H^0\big(\mathcal{O}(m-n)\big)$, $V = H^0\big(\mathcal{F}(n)\big)$, $W = H^0\big(\mathcal{F}(m)\big)$, we get an H-Kronecker module $(V, W, \alpha_{\mathcal{F}})$ that uniquely determines a representation of the quiver $\mathbf{\Theta}_h$ with $h = \dim H$.

4) Let $\mathfrak{A}$ be the category of holomorphic vector bundles over a complex manifold X (or the category of algebraic vector bundles over an algebraic variety). A representation of a quiver $\mathbf{Q}$ in $\mathfrak{A}$ is usually called a quiver bundle ($\mathbf{Q}$-bundle).

- If $\mathcal{Q}$ is the quiver $\bullet \longrightarrow \bullet$, then a $\mathbf{Q}$-bundle is a *holomorphic triple* as defined in [García-Prada 1994], [Bradlow & García-Prada 1996].

- There is a generalization of the notion of $\mathbf{Q}$-bundle to that of *twisted* $\mathbf{Q}$*-bundle*, where each morphisms X_a is "twisted" by a vector bundle E_a [Gothen & King 2005]. An important example of this is provided by a *Higgs bundle* on a Riemann surface X, a notion introduced by N. Hitchin [1987] : a Higgs bundle is a pair (E, φ), where E is a vector bundle and $\varphi\colon E \to E \otimes K_X$ is a morphism of vector bundles (K_X is the canonical bundle of X). So a Higgs bundle is a twisted $\mathbf{Q}$-bundle, where $\mathbf{Q}$ is the Jordan quiver $\begin{tikzcd}\bullet \arrow[loop right]\end{tikzcd}$.

Quelques citations sur le problème du “simple”

- Immanuel Kant, *Die Kritik der reinen Vernunft* :

> **Die antinomie der reinen Vernunft. Zweiter Widerstreit der transzendentalen Ideen**
> *Thesis*. Eine jede zusammengesetzte Substanz in der Welt bestehet aus einfachen Teilen, und es existiert überall nichts als das Einfache, oder das, was aus diesen zusammengesetzt ist.
> *Antithesis*. Kein zusammengesetztes Ding in der Welt besteht aus einfachen Teilen, und es existiert überall nichts Einfaches in derselben.

[**Les antinomies de la raison pure. Deuxième conflit des idées transcendentales**
Thèse. Toute substance composée, dans le monde, se compose de parties simples, et il n'existe absolument rien que le simple ou ce qui en est composé.
Antithèse. Aucune chose composée, dans le monde, n'est formée de parties simples, et il n'existe rien de simple dans le monde.
Traduction par A. Tremesaygues & B. Pacaud]

- Alexander Grothendieck, *Récoltes et semailles*, Gallimard, Paris 2021 :

> L'idée même de schéma est d'une simplicité enfantine – si simple, si humble, que personne avant moi n'avait songé à se pencher si bas. Si « bébête » même, pour tout dire, que pendant des années encore et en dépit de l'évidence, pour beaucoup de mes savants collègues, ça faisait vraiment « pas sérieux » !
> R & S, I, pp. 55–56
>
> Et il n'y a mûrissement qui ne soit aussi retour tant soit peu – retour à l'enfant, et à la simplicité, à l'innocence de l'enfant.
> R & S, II, p. 863
>
> [...] le « paquet attracteur yang » [...] :
>
> le **simple**-le complexe
> l'**abstrait**-le concret (ou le réel)
> le **précis**-le vague
> **ordre**-chaos
> **structure**-substance.
>
> Parmi les cinq attracteurs yang qui figurent dans ce paquet, il en est deux qui me semblent jouer un rôle primordial
> **le simple** (ou la simplicité), et **l'ordre**.
> R & S, II, p. 1198

« Simplicité » et « ordre » sont des qualités étroitement liées, au point qu'on serait tenté de les déclarer identiques. Pourtant, l'ordre que nous décelons dans les choses peut lui-même être plus ou moins « simple », ou plus ou moins « complexe », suivant le degré de profondeur où il nous fait pénétrer dans notre appréhension de l'harmonie des choses. Mais si subtil et si complexe que soit l'ordre perçu et exprimé par la pensée, toujours il incarne, par sa nature même, une « simplicité », elle-même plus ou moins « simple » (voire « simpliste »), ou plus ou moins délicate ou « complexe ». Et inversement, reconnaître le simple dans le complexe, c'est bien y voir apparaître un ordre qui nous avait échappé jusque-là. [. . .] La simplicité parfaite est celle qui exprime et épouse de façon parfaite l'ordre caché inhérent aux choses elles-mêmes.
Aussi, on pourrait dire que « simplicité » et « ordre » sont comme l'âme et le corps d'une seule et même qualité.
R & S, II, p. 1198

[...] **le miracle de la simplicité** [...]
R & S, II, p. 1237

- Johann Wolfgang Goethe, lettre à Carl Friedrich Zelter (compositeur, maître de musique de F. Mendelssohn) du 29 mars 1827

> [. . .] *il faut croire à la simplicité* ! zu Deutsch : man muß an die Einfalt, an das Einfache, an das urständig Prouctive glauben, wenn man den rechten Weg gewinnen will. Dieses ist aber nicht jedem gegeben

([...] il faut croire à la simplicité ! [. . .] il faut croire en la simplicité, en ce qui est simple, en ce qui est véritablement productif, si l'on veut trouver le bon chemin. Mais cela n'est pas donné à tous [...])

Gabriel's theorem

Let $\mathbb{K}$ be a field and $\mathbf{Q}$ a quiver. Since the category $\mathfrak{Rep}_{\mathbb{K}}(\mathbf{Q})$ is abelian, we have obvious notions of direct sums, subrepresentations, kernel and cokernels (and there are good quotients too).
A representation $(V, X) \in \mathfrak{Rep}_{\mathbb{K}}(\mathbf{Q})$ is called
- simple if it contains no nontrivial subrepresentations ;
- semisimple if is isomorphic to a direct sum of simple representations ;
- indecomposable if it cannot be written as a direct sum of nonzero representations.

Example. Let $\mathbf{Q} = \bullet \longrightarrow \bullet$. The representations $\overset{0}{\bullet} \longrightarrow \overset{\mathbb{K}}{\bullet}$, $\overset{\mathbb{K}}{\bullet} \longrightarrow \overset{0}{\bullet}$ are simple ; the representation $\overset{\mathbb{K}}{\bullet} \xrightarrow{1} \overset{\mathbb{K}}{\bullet}$ is indecomposable but not semisimple.
For any quiver $\mathbf{Q}$ denote by $S(x)$ the representation defined by setting ; $S(x)_x = \mathbb{K}$, $S(x)_y = 0$ for all $y \neq x$, $X_a = 0$ for all $a \in \mathcal{Q}_1$.
Each representation $S(x)$ is simple ; moreover, $S(x)$ is not isomorphic to $S(y)$ if $x \neq y$.

> Let $\mathbf{Q}$ be a quiver without oriented cycles. Then a representation of $\mathbf{Q}$ is simple if and only if it is of the form $S(x)$ for some $x \in \mathcal{Q}_0$.

Krull-Schmidt theorem. Any finite-dimensional $\mathbb{K}$-representation of a quiver $\mathbf{Q}$ can be uniquely (up to reordering) decomposed into a direct sum of indecomposable representations.

Remark : the Krull-Schmidt holds also true in the category of $\mathbb{F}_1$-representations [Szczesny 2012].

One of the main goals of the theory of quiver representations is to (try to) classify indecomposable representations of a given quiver.
One say that a quiver $\mathbf{Q}$ is
• of finite type if, for any dimension vector $\mathbf{v} \in \mathbb{N}^{Q_0}$, the set of isomorphism classes of indecomposable $\mathbb{K}$-representations of dimension $\mathbf{v}$ is finite ;
• tame if, for any dimension vector $\mathbf{v} \in \mathbb{N}^{Q_0}$, the set of isomorphism classes of indecomposable $\mathbb{K}$-representations of dimension $\mathbf{v}$ is a union of one-parameter families and a finite set of isolated points [Crawley-Boevey 1988] ;
• wild if it is neither of finite type nor tame.
Note that, by definition, every quiver of finite type is tame.

Let us assume from now on that the field $\mathbb{K}$ is algebraically closed.

Examples

1) The quiver $\bullet \longrightarrow \bullet$ is of finite type.

2) The Jordan 1-loop quiver $\mathbf{L} = \begin{tikzcd}\bullet \arrow[loop above]\end{tikzcd}$ is tame but not of finite type : for every $d \in \mathbb{N}$ there exists precisely one one-parameter family of indecomposable $\mathbb{K}$-representations of dimension d (this family is a one-to-one correspondence with the family of Jordan blocks $\{J_{\lambda,d}\}_{\lambda\in\mathbb{K}}$ of size d and eigenvalue λ.

2) For the Kronecker quiver $\mathbf{\Theta}_2 = \begin{tikzcd}\overset{x}{\bullet} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \overset{x}{\bullet}\end{tikzcd}$ isomorphism classes of indecomposable $\mathbb{K}$-representations of dimension $(1,1)$ are parameterized by $\mathbb{P}^1_{\mathbb{K}}$. In fact, one shows that $\mathbf{\Theta}_2$ is tame but not of finite type.

3) For the 2-loop quiver $\mathbf{L}_2 = \begin{tikzcd}\bullet \arrow[loop above, "a"] \arrow[loop below, "b"]\end{tikzcd}$ isomorphism classes of indecomposable $\mathbb{K}$-representations of dimension $(1,1)$ are parameterized by $\mathbb{K}^2$. In fact, $\mathbf{L}_2 =$ is wild.

The Euler form associated to a quiver $\mathbf{Q}$ is the bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{Q}} : \mathbb{Z}^{\mathcal{Q}_0} \times \mathbb{Z}^{\mathcal{Q}_0} \to \mathbb{Z}$ given by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{Q}} = \sum_{x \in \mathcal{Q}_0} v_x w_x - \sum_{a \in \mathcal{Q}_1} v_{t(a)} w_{h(a)}$$

> If V, W are representations of $\mathbf{Q}$ having dimension vectors $\mathbf{v}, \mathbf{w}$, resp., then $\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{Q}} = \dim \mathrm{Hom}_{\mathbb{K}}(V, W) - \dim \mathrm{Ext}^1(V, W)$.

Notice that $\langle \mathbf{dim}\, S(x), \mathbf{dim}\, S(y) \rangle_{\mathbf{Q}} = \delta_{xy} - \sharp\{\text{arrows } x \to y\}$
We can make the bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{Q}}$ independent of orientation by symmetrizing it :
$(\mathbf{v}, \mathbf{w})_{\mathbf{Q}} = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{Q}} + \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{Q}} = \sum_{x \in \mathcal{Q}_0} (2 - A_{xx}) v_x w_x - \sum_{x, y \in \mathcal{Q}_0, x \neq y} A_{xy} v_x w_y$
where $A_{xy} = \sharp\{\text{edges joining } x \text{ and } y\}$.
If the quiver $\mathbf{Q}$ has no loops, then A_{xy} is just the adjacency matrix of the non-oriented graph Q underlying $\mathbf{Q}$ and therefore the symmetrized Euler is the Cartan matrix C of Q : $C_{xy} = 2\delta_{xy} - A_{xy}$.
The quadratic form $q_{\mathbf{Q}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle_{\mathbf{Q}} = \frac{1}{2}(\mathbf{v}, \mathbf{v})_{\mathbf{Q}}$ is called the Tits form of $\mathbf{Q}$ (or, equivalently, of Q).

The space (affine variety) of $\mathbf{v}$-dimensional $\mathbb{K}$-representations of a quiver $\mathbf{Q}$ will be denoted by $\mathrm{Rep}(\mathbf{Q},\mathbf{v})$. Since there is an isomorphism

$$\mathrm{Rep}(\mathbf{Q},\mathbf{v}) \simeq \bigoplus_{a\in\mathcal{Q}_1} \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^{v_{t(a)}},\mathbb{K}^{v_{h(a)}}\right),$$

there is an induced action of $GL_{\mathbf{v}} = \prod_{x\in\mathcal{Q}_0} GL(v_x;\mathbb{K})$ on $\mathrm{Rep}(\mathbf{Q},\mathbf{v})$.

> The isomorphism classes of representations of dim. $\mathbf{v}$ are in a one-to-one correspondence with the orbits of the $GL_{\mathbf{v}}$-action on $\mathrm{Rep}(\mathbf{Q},\mathbf{v})$.

The subgroup $GL_1 = \{(\lambda I_{v_x})_{x\in\mathcal{Q}_0} \mid \lambda\in\mathbb{K}^\times\}$ acts trivially on $\mathrm{Rep}(\mathbf{Q},\mathbf{v})$. Hence, the action of $GL_{\mathbf{v}}$ factorizes through the action of the projective linear group $\mathbb{P}GL_{\mathbf{v}} = GL_{\mathbf{v}}/GL_1$.
One has

$$\dim\mathbb{P}GL_{\mathbf{v}} - \dim\mathrm{Rep}(\mathbf{Q},\mathbf{v}) = \Big(\sum_{x\in\mathcal{Q}_0} v_x^2\Big) - 1 - \sum_{a\in\mathcal{Q}_a} v_{t(a)}v_{h(a)} = q_{\mathbf{Q}} - 1\,.$$

From this very elementary fact follows a key result.

If $\mathbf{Q}$ is of finite type, then its Tits form $q_{\mathbf{Q}} - 1$ is definite positive.

(If $\mathbf{Q}$ is of finite type, then, for each dimension vector $\mathbf{v}$, there only finitely many isomorphism classes of $\mathbf{v}$-dimensional representations of $\mathbf{Q}$ or, equivalently, the set of $GL_{\mathbf{v}}$-orbits in $\mathrm{Rep}(\mathbf{Q}, \mathbf{v})$ is finite. The result follows from standard results about orbits of an algebraic group acting on an affine variety).

Theorem. Let Γ be a connected graph. The Tits form q_Γ is definite positive definite if and only if Γ is Dynkin (ADE).

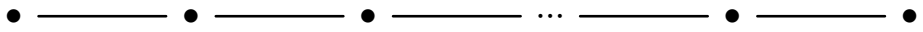
The proof is quite elementary. From the previous two result we deduce the following corollary.

If $\mathbf{Q}$ is a connected quiver of finite type, then its underlying graph Q is Dynkin (ADE).

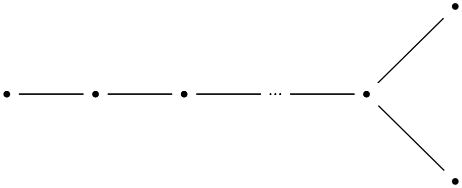
This is the first half of Gabriel's theorem

The Dynkin graphs (ADE)

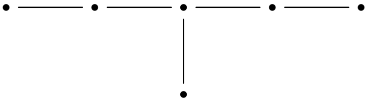
A_n, $n \geq 1$ ($n-1$ edges) :



D_n, $n \geq 4$:



E_6 :



E_7, E_8

The second – non trivial – half of Gabriel's theorem [1972] is as follows.

> Let **Q** be a quiver. If its underlying graph is Dynkin (ADE), then **Q** is of finite type.

In 1973 Bernštein, Gelfand & Ponomarev devised an extremely beautiful proof of Gabriel's theorem based on the so-called "reflections functors" (sort of categorical versions of Coxeter's reflections operators).

Actually, Gabriel's theorem can be stated in a much precise way.
Let **Q** be a quiver without loops and let $\alpha_x = \mathbf{dim}\, S(x)$ for every $x \in \mathcal{Q}_0$.
The root lattice L is defined as $L = \bigoplus_{x\in\mathcal{Q}_0} \mathbb{Z}\alpha_x \simeq \mathbb{Z}^{\mathcal{Q}_0}$ and the α_x are called simple roots. On L we have an integral symmetric bilinear form $(\cdot,\cdot)_Q$, namely, the symmetrized Euler form of **Q**, which coincides with the bilinear form defined by the Cartan matrix of Q.
Assume that the underlying graph Q is Dynkin (ADE) (iff its Tits form is positive definite) or Euclidean (ADE) (iff its Tits form is positive semidefinite) . The set $R = \{\alpha \in L \smallsetminus \{0\} \mid (\alpha,\alpha)_Q \leq 2\}$ is called the root system associated to **Q** and it can be shown that there is a decomposition $R = R_+ \sqcup R_-$ into the sets of positive and negative roots.

Finally, we recall that a root α is called real if there is some $x \in \mathcal{Q}_0$ and some element w of the Weyl group W_Q such that $\alpha = w\alpha_x$ (the Weyl group is the subgroup of $GL(L \otimes_{\mathbb{Z}} \mathbb{R})$ generated by simple reflections $\alpha \mapsto \alpha - (\alpha, \alpha_x)_Q \alpha_x$). A root is imaginary if and only if it is not real.

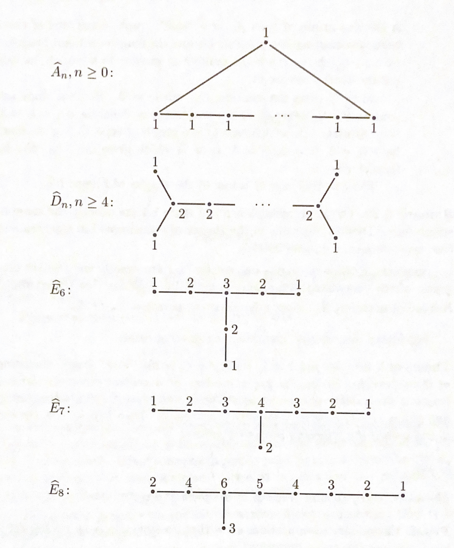
If Q is Dynkin (ADE), then there are no imaginary roots and $R = \{\alpha \in L \mid (\alpha, \alpha)_Q = 2\}$.
If Q is Euclidean (ADE), then the real roots are $R^{\mathrm{re}} = \{\alpha \in L \mid (\alpha, \alpha)_Q = 2\}$ and the imaginary roots are $R^{\mathrm{im}} = \{p\delta \mid p \in \mathbb{Z} \smallsetminus \{0\}\}$, where $\delta = \sum \delta_x \alpha_x$ is a generator of the (one-dimensional) radical of the symmetrized Euler form such that $\delta_x > 0$ for all $x \in \mathcal{Q}_0$ and $\min \delta_x = 1$.

Since $\mathfrak{Rep}_{\mathbb{K}}(\mathbf{Q})$ is an abelian category, we can form its Grothendieck group $K_{\mathbb{K}}(\mathbf{Q})$. Note that the classes $[S(x)]$ generate $K_{\mathbb{K}}(\mathbf{Q})$; moreover, if $\mathbf{Q}$ is a quiver without loops, then the map $\mathbf{dim} : K_{\mathbb{K}}(\mathbf{Q}) \to \mathbb{Z}^{\mathcal{Q}_0} \simeq L$ is an isomorphism.

Theorem. Let $\mathbf{Q}$ a quiver whose underlying graph is Dynkin (ADE). Then the map $\mathbf{dim}$ is a bijection between the set of (nonzero) isomorphism classes of indecomposable $\mathbb{K}$-representations of $\mathbf{Q}$ and the set $R_+ \subset L$ of positive roots..

Example. The previous theorem can be exploited to describe isomorphism classes of indecomposable representations of a quiver $\mathbf{Q}$ whose underlying graph is A_n. It can be shown that in this case for each positive root $(\alpha_i)_{1\leq i\leq n}$ of the Tits form $q_{\mathbf{Q}}$ the exist integers $1 \leq k \leq l \leq n$ such that $\alpha_i = 1$ for all $i \in [k, l]$ and $\alpha_i = 0$ otherwise. The representation association with the positive root determined by the interval $[k, l] \subset [1, n]$ has $V_i = \mathbb{K}$ for all $i \in [k, l]$ and $V_i = 0$ otherwise ; every map X_a such that $t(a) \notin [k, l)$ is zero, all other maps are the identity. The description of the indecomposable "interval representations" of A_n represents the main bridge from quiver representations to the theory of "algebraic persistence" (persistence modules, interval decompositions, barcodes) [Oudot 2015].

Euclidean graphs (ADE)



[Kirillov 2016]

Note that $\widetilde{A}_0$ is the Jordan quiver.

It is possible to completely classify the indecomposable representations of a Euclidean (ADE) quiver. The following theorem is stated and proved in [Kirillov 2016].

Let $\mathbf{Q}$ be a connected Euclidean quiver different form the Jordan quiver. Let R be the corresponding root system. Then :
(1) An indecomposable representation of $\mathbf{Q}$ of dimension $\mathbf{v}$ exists if and only if $\mathbf{v} \in R_+$.
(2) For any real positive root α the indecomposable representation of $\mathbf{Q}$ of dimension α is unique up to an isomorphism
(3) There exists a finite subset $D = \{p_1, \dots, p_k\} \subset \mathbb{P}^1$ and a set of positive integers $N_p > 1$ with $p \in D$, such that for any positive imaginary root $\alpha = n\delta$, the set of isomorphism classes of indecomposable representations of $\mathbf{Q}$ of dimension α is in bijection with the set $\left(\mathbb{P}^1 \setminus D\right) \cup \bigcup_{p \in D} \mathbb{Z}_{N_p}$.

It follows that every Euclidean (ADE) quiver is tame.
See [Kac 1980, 1983] for further generalizations.

Quotient spaces

Let $\mathbb{K} = \mathbb{C}$ and consider the action of $G_{\mathbf{v}} := \prod_{x \in \mathcal{Q}_0} \mathrm{GL}(v_x; \mathbb{C})$ on $\mathrm{Rep}(\mathbf{Q}, \mathbf{v})$.

The "rough orbit space" $\mathrm{Rep}(\mathbf{Q}, \mathbf{v})/G_{\mathbf{v}}$ is extremely badly behaved in most cases (not even Hausdorff).

To overcome these difficulties an appropriate notion of (semi)stability is introduced [King 1994 ; Rudakov 1997] :

- after choosing a parameter $\vartheta \in \mathbb{R}^{\mathcal{Q}_0}$, the slope of a $\mathbf{v}$-dimensional representation V of $\mathbf{Q}$ is

$$\mu_\vartheta(V) := \frac{\sum_{x \in \mathcal{Q}_0} \vartheta_x v_x}{\sum_{x \in \mathcal{Q}_0} v_x} ;$$

- a representation V of $\mathbf{Q}$ is said to be ϑ-semistable if, for any proper subrepresentation $U \subset V$, one has $\mu_\vartheta(U) \leq \mu_\vartheta(V)$.

According to GIT's general prescriptions :

- consider the subset $\mathrm{Rep}^{ss}_{\vartheta}(\mathbf{Q},\mathbf{v})$ of $\mathrm{Rep}(\mathbf{Q},\mathbf{v})$ consisting of semistable representations ;
- define the the coarse moduli space of $\mathbf{v}$-dimensional ϑ-semistable representations of $\mathbf{Q}$

$$\mathscr{M}(\mathbf{Q},\mathbf{v})_{\vartheta} = \mathrm{Rep}^{ss}_{\vartheta}(\mathbf{Q},\mathbf{v})/\!/_{\vartheta} G_{\mathbf{v}} .$$

If $\mathbf{v}$ *is a primitive vector, then the open subset*

$$\mathscr{M}^{s}(\mathbf{Q},\mathbf{v})_{\vartheta} \subset \mathscr{M}(\mathbf{Q},\mathbf{v})_{\vartheta}$$

consisting of stable representations makes up a fine moduli space [King 1994].

Geometry of the quotient space depends in an essential way on the choice of the parameter ϑ.

Basic example

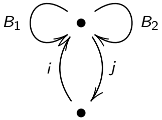
On the complex projective plane $\mathbb{P}^2$ we fix a "line at infinity" $\ell_\infty \simeq \mathbb{P}^1$ and a trivial rank r vector bundle $\mathcal{O}_{\ell_\infty}^{\oplus r}$.
A framed sheaf on $\mathbb{P}^2$ is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a torsionless sheaf on $\mathbb{P}^2$ and $\theta \colon \mathcal{E}|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism. This implies $\operatorname{rk} \mathcal{E} = r$, $c_1(\mathcal{E}) = 0$; hence, $\operatorname{ch}(\mathcal{E}) = (r, 0, c)$, with $c > 0$.
The moduli space $\mathcal{M}(r, 0, c)$ of framed sheaves $(\mathcal{E}, \theta)$ having Chern character $(r, 0, c)$ is a fine moduli space and admits an explicit description. Let

$$\mathcal{N}(r,c) = \Big\{ (B_1, B_2, i, j) \in \operatorname{End}(\mathbb{C}^c)^{\oplus 2} \oplus \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^c) \oplus \operatorname{Hom}(C^c, \mathbb{C}^r) \mid$$

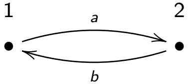
$$[B_1, B_2] + ij = 0 \text{ and a certain stability condition is fulfilled} \Big\}$$

The group $GL(c; \mathbb{C})$ acts on $\mathcal{N}(r, c)$ by conjugation. In 1984 Donaldson proved that there exists an isomorphism $\mathcal{M}(r, 0, c) \simeq \mathcal{N}(r, c)/GL(c; \mathbb{C})$.
• The "linear data" (B_1, B_2, i, j) are akin to the "linear data" of the ADHM description of instantons over $\mathbb{R}^4$.
• For $r = 1$ we get a moduli space isomorphic to the Hilbert scheme of points $\operatorname{Hilb}^c(\mathbb{C}^2)$.
• In 1990 Kronheimer & Nakajima gave a quiver description of instantons on ALE spaces.

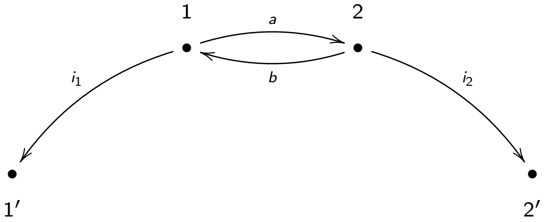
The linear data (B_1, B_2, i, j) are a quiver representation :



Given a quiver $\mathbf{Q}$ its framed quiver $\mathbf{Q}^{\mathrm{fr}}$ is defined as the quiver whose vertex set is $\mathcal{Q}_0 \sqcup \mathcal{Q}_0'$, where $\mathcal{Q}_0'$ is a copy of $\mathcal{Q}_0$ with a fixed bijection $x \to x'$ and whose arrow set $\mathcal{Q}_1^{\mathrm{fr}}$ is obtained by adding to $\mathcal{Q}_1$ new arrows $x \xrightarrow{d_x} x'$ for every $x \in \mathcal{Q}_0$. For instance, if $\mathbf{Q}$ is the quiver



then $\mathbf{Q}^{\mathrm{fr}}$ is the quiver



(1)

Nakajima quiver varieties

Given a quiver $\mathbf{Q}$ with vertex set $\mathcal{Q}_0$, for any $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{\mathcal{Q}_0}$, $\lambda \in \mathbb{C}^{\mathcal{Q}_0}$ and $\vartheta \in \mathbb{R}^{\mathcal{Q}_0}$, the associated Nakajima quiver variety $\mathcal{N}_{\lambda,\vartheta}(\mathbf{Q}, \mathbf{v}, \mathbf{w})$ is defined as follows [Nakajima 1994 ; Ginzburg 2012].

(i) Consider the framed quiver $\mathbf{Q}^{\mathrm{fr}}$ and the associated double $\overline{\mathbf{Q}^{\mathrm{fr}}}$ (the latter has the same vertex set as $\mathbf{Q}^{\mathrm{fr}}$, and for any arrow $x \xrightarrow{a} y$ in $\mathcal{Q}_1^{\mathrm{fr}}$, with $x, y \in \mathcal{Q}_0 \sqcup \mathcal{Q}_0'$, an *opposite* arrow $y \xrightarrow{a^*} x$ is added).
For all dimension vectors $(\mathbf{v}, \mathbf{w})$, there is an isomorphism

$$\mathrm{Rep}(\overline{\mathbf{Q}^{\mathrm{fr}}}, \mathbf{v}, \mathbf{w}) \simeq T^\vee \, \mathrm{Rep}(\mathbf{Q}^{\mathrm{fr}}, \mathbf{v}, \mathbf{w}) \, .$$

As a consequence of that, $\mathrm{Rep}(\overline{\mathbf{Q}^{\mathrm{fr}}}, \mathbf{v}, \mathbf{w})$ carries a canonical holomorphic symplectic form $\tilde{\omega} = \mathrm{tr}\left(\sum_{a \in \mathcal{Q}_1} \mathrm{d}X_a \wedge \mathrm{d}X_{a^*} + \sum_{x \in \mathcal{Q}_0} \mathrm{d}X_{d_x} \wedge \mathrm{d}X_{d_x^*}\right)$.
(ii) The group $G_{\mathbf{v}} := \prod_{x \in \mathcal{Q}_0} \mathrm{GL}(v_x)$ acts naturally on $\mathrm{Rep}(\overline{\mathbf{Q}^{\mathrm{fr}}}, \mathbf{v}, \mathbf{w})$ and the action is symplectic. So we can introduce a moment map

$$\mu \colon \mathrm{Rep}(\overline{\mathbf{Q}^{\mathrm{fr}}}, \mathbf{v}, \mathbf{w}) \to \mathfrak{g}_{\mathbf{v}}^* \simeq \mathfrak{g}_{\mathbf{v}} \, ,$$

$$(V \oplus W, X) \mapsto \sum_{a \in \mathcal{Q}_1} (X_a \circ X_{a^*} - X_{a^*} \circ X_a) + \sum_{x \in \mathcal{Q}_0} X_{d_x^*} \circ X_{d_x} \, . \qquad (2)$$

(iii) The fibre

$$\mu^{-1}\left(\sum_{i\in I}\lambda_i \mathbf{1}_{v_i}\right) \subset \mathrm{Rep}(\overline{\mathbf{Q}^{\mathrm{fr}}}, \mathbf{v}, \mathbf{w})$$

is the space of $(\mathbf{v}, \mathbf{w})$-dimensional representations of a certain quotient algebra $\Pi_\lambda(\mathbf{Q})$ of $\mathbb{C}\overline{\mathbf{Q}^{\mathrm{fr}}}$ (called the *framed preprojective algebra* of $\mathbf{Q}$).

(iv) The Nakajima quiver variety $\mathcal{N}_{\lambda,\vartheta}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ is defined as the quotient

$$\mathcal{N}_{\lambda,\vartheta}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) = \mathrm{Rep}(\Pi_\lambda(\mathcal{Q}), \mathbf{v}, \mathbf{w})^{\mathrm{ss}}_{\vartheta} /\!/_{\vartheta} G_{\mathbf{v}}$$

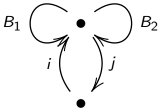
The variety $\mathcal{N}_{\lambda,\vartheta}(\mathbf{Q}, \mathbf{v}, \mathbf{w})$ may fail to be smooth but it has a natural Poisson structure $\{-,-\}$ induced by the Hamiltonian reduction.

(v) Let $C_{\mathbf{Q}}$ be the Cartan matrix of the quiver $\mathbf{Q}$.

If the triple $(\mathbf{v}, \lambda, \vartheta)$ *satisfies a certain regularity condition and* $\mathbf{w} \neq 0$, *then all* ϑ*-semistable representations of* Π_λ *are* ϑ*-stable, the variety* $\mathcal{N}_{\lambda,\vartheta}(\mathbf{Q}, \mathbf{v}, \mathbf{w})$ *is smooth and connected of dimension* $2\mathbf{w}\cdot\mathbf{v} - (C_{\mathbf{Q}}\mathbf{v})\cdot\mathbf{v}$, *and the Poisson structure* $\{-,-\}$ *is nondegenerate* [Ginzburg 2012].
In this case, the Nakajima variety $\mathcal{N}_{\lambda,\vartheta}(\mathbf{Q}, \mathbf{v}, \mathbf{w})$ can be described as a Kähler quotient.

Examples of Nakajima quiver varieties

1) **Moduli spaces of framed sheaves on $\mathbb{P}^2$.** If $\mathbf{L}$ is the Jordan (one-loop) quiver, then the quiver $\overline{\mathbf{L}^{\mathrm{fr}}}$



There is an isomorphism of algebraic varietes

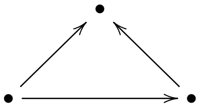
$$\mathcal{N}_{0,-1}(\mathfrak{J}_1, c, r) \simeq \mathcal{M}_{\mathbb{P}^2}(r, c)\,,$$

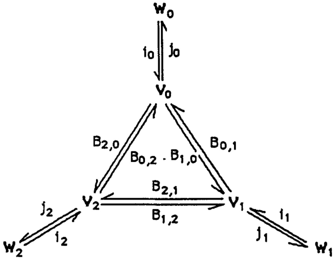
where $\mathcal{M}_{\mathbb{P}^2}(r,c)$ *is the moduli space of framed sheaves on* $\mathbb{P}^2$ *with Chern character* $(r, 0, c)$ [Donaldson 1984 ; Nakajima 1999].

- In particular, we get an isomorphism $\mathcal{M}_{\mathbb{P}^2}(1,c) = \mathrm{Hilb}^c(\mathbb{C}^2)$.

2) Let $\Gamma \subset SU(2)$ be a finite subgroup acting on $\mathbb{C}^2$. As is well known, 1) there is classification of all possible Γ's and each Γ can be uniquely associated to a Dynkin (ADE) graph ; 2) $\mathbb{C}^2/\Gamma$ is an algebraic variety with a Kleinian singularity at the origin ; 3) there is a crepant resolution $\widehat{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$.

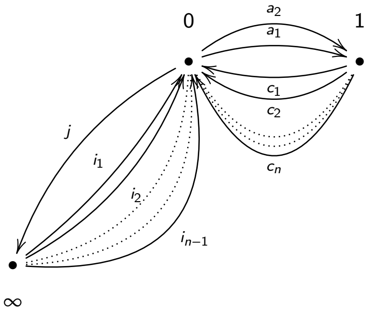
Kronheimer & Nakajima proved in 1990 that, for any Γ, $\widehat{\mathbb{C}^2/\Gamma}$ and $\mathbb{C}^2/\Gamma$ are Nakajima quiver varieties associated to the quiver whose underlying graph is the Euclidean (ADE) graph determined by Γ. For example, when $\Gamma = \mathbb{Z}_3$, we can take **Q** of the form . Hence, $\overline{\mathbf{Q}}^{\mathrm{fr}}$ is the quiver





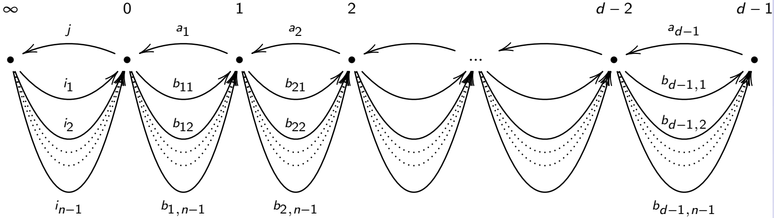
Other examples of quiver varieties

The quiver $\mathcal{Q}_n$, $n \geq 2$,



can be used to construct a quiver variety isomorphic to the Hilbert scheme of points $\mathrm{Hilb}^c(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$ [B, Bruzzo, Lanza & Rava 2017, 2020]. • The total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$ is isomorphic to the ALE space $\widetilde{\mathbb{C}^2/Z_2}$, hence it carries a holomorphic symplectic structure. The same is true for the Nakajima quiver variety $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2))$ [Kuznetsov 2007].

• For $n \neq 2$, the Hilbert scheme $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$ is not a holomorphic symplectic variety (hence not a variety of Nakajima type). However, it carries a natural Poisson structure induced by the Poisson bivector defined on the Hirzebruch surface Σ_n (as it follows from general results about moduli spaces of sheaves on Poisson surfaces [Bottacin 1998]).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$b_{d-1,1}$
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$b_{d-1,n-1}$

… a long story …

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