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EFFECTIVE CARTAN-TANAKA CONNECTIONS ON $\mathscr{C}^6$ STRONGLY PSEUDOCONVEX HYPERSURFACES $M^3 \subset \mathbb{C}^2$

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1. SYNTHETIC STATEMENT OF THE RESULTS

Theorem 1.1. *Let $M^3 \subset \mathbb{C}^2$ be an arbitrary local Levi nondegenerate $\mathscr{C}^6$-smooth real 3-dimensional hypersurface of $\mathbb{C}^2$ which is represented in coordinates $(z,w) = (x+iy,\, u+iv)$ as a graph:*

$$v = \varphi(x,y,u) = x^2 + y^2 + \mathrm{O}(3),$$

and whose complex tangent bundle $T^cM = \mathrm{Re}\, T^{1,0}M$ is generated by the two explicit intrinsic vector fields:

$$H_1 := \frac{\partial}{\partial x} + \Big(\frac{\varphi_y - \varphi_x\,\varphi_u}{1+\varphi_u^2}\Big)\frac{\partial}{\partial u} \qquad \text{and} \qquad H_2 := \frac{\partial}{\partial y} + \Big(\frac{-\varphi_x - \varphi_y\,\varphi_u}{1+\varphi_u^2}\Big)\frac{\partial}{\partial u},$$

satisfying $H_1|_0 = \frac{\partial}{\partial x}\big|_0$ and $H_2|_0 = \frac{\partial}{\partial y}\big|_0$; introduce the three $\mathscr{C}^5$-smooth functions:

$$\Delta := 1 + \varphi_u^2, \qquad \Lambda_1 := \varphi_y - \varphi_x\,\varphi_u, \qquad \Lambda_2 := -\,\varphi_x - \varphi_y\,\varphi_u,$$

so that:

$$H_1 = \frac{\partial}{\partial x} + \frac{\Lambda_1}{\Delta}\,\frac{\partial}{\partial u} \qquad \text{and} \qquad H_2 = \frac{\partial}{\partial y} + \frac{\Lambda_2}{\Delta}\,\frac{\partial}{\partial u};$$

consider the third, Levi form-type Lie-bracket vector field:

$$\begin{aligned} T &:= \tfrac{1}{4}\,[H_1, H_2] \\ &= \Big(\tfrac{1}{4}\,\tfrac{1}{(1+\varphi_u^2)^2}\big\{ -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + \\ &\qquad + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy}\big\}\Big)\,\frac{\partial}{\partial u}, \end{aligned}$$

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satisfying $T\big|_0 = -\frac{\partial}{\partial u}\big|_0$ *which produces, jointly with* H_1 *and* H_2 *of which it is locally linearly independent, a frame for* TM *in a neighborhood of the origin; introduce the* $\mathscr{C}^4$*-smooth function:*

$$\begin{aligned}\Upsilon := -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + \\ + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy},\end{aligned}$$

which specifies the numerator of the Levi form-type Lie-bracket:

$$T = \tfrac{1}{4}\,[H_1, H_2] = \frac{1}{4}\,\frac{\Upsilon}{\Delta^2}\,\frac{\partial}{\partial u};$$

allow the two notational coincidences: $x_1 \equiv x$*,* $x_2 \equiv y$*; introduce the two length-three brackets:*

$$\big[H_i,\, T\big] = \tfrac{1}{4}\,\big[H_i, [H_1, H_2]\big] =: \Phi_i\,T \qquad (i=1,2),$$

which are both multiples of T *by means of two functions:*

$$\Phi_i := \frac{A_i}{\Delta^2\,\Upsilon} \qquad (i=1,2)$$

whose numerators are explicitly given by:

$$A_i := \Delta^2\,\Upsilon_{x_i} + \Delta\big(-2\,\Delta_{x_i}\,\Upsilon + \Lambda_i\,\Upsilon_u - \Upsilon\,\Lambda_{i,u}\big) - \Lambda_i\,\Upsilon\,\Delta_u \qquad (i=1,2);$$

introduce furthermore the following $4+8+16$ *iterated brackets for* $i, k_1, k_2, k_3 = 1, 2$*:*

$$\begin{aligned}\big[H_{k_1}, \big[H_i,\, T\big]\big] &= \tfrac{1}{4}\,\big[H_{k_1}, \big[H_i,\, [H_1, H_2]\big]\big] \\ \big[H_{k_2}, \big[H_{k_1}, \big[H_i,\, T\big]\big]\big] &= \tfrac{1}{4}\,\big[H_{k_2}, \big[H_{k_1}, \big[H_i,\, [H_1, H_2]\big]\big]\big] \\ \big[H_{k_3}, \big[H_{k_2}, \big[H_{k_1}, \big[H_i,\, T\big]\big]\big]\big] &= \tfrac{1}{4}\,\big[H_{k_3}, \big[H_{k_2}, \big[H_{k_1}, \big[H_i,\, [H_1, H_2]\big]\big]\big]\big],\end{aligned}$$

up to length 6 *that are all multiples of* T*:*

$$\begin{aligned}\big[H_{k_1}, [H_i,\, T]\big] &= \big(H_{k_1}(\Phi_i) + \Phi_i\,\Phi_{k_1}\big)\,T, \\ \big[H_{k_2}, \big[H_{k_1}, [H_i,\, T]\big]\big] &= \big(H_{k_2}(H_{k_1}(\Phi_i)) + \Phi_{k_1}\,H_{k_2}(\Phi_i) + \Phi_i\,H_{k_2}(\Phi_{k_1}) + \\ &\quad + \Phi_{k_2}\,H_{k_1}(\Phi_i) + \Phi_i\,\Phi_{k_1}\Phi_{k_2}\big)\,T, \\ \big[H_{k_3}, \big[H_{k_2}, \big[H_{k_1}, [H_i,\, T]\big]\big]\big] &= \big(H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) + \Phi_{k_1} H_{k_3}(H_{k_2}(\Phi_i)) + \Phi_i\,H_{k_3}(H_{k_2}(\Phi_{k_1})) + \\ &\quad + \Phi_{k_2}\,H_{k_3}(H_{k_1}(\Phi_i)) + \Phi_{k_3} H_{k_2}(H_{k_1}(\Phi_i)) + \\ &\quad + H_{k_3}(\Phi_{k_1})\,H_{k_2}(\Phi_i) + H_{k_3}(\Phi_i)\,H_{k_2}(\Phi_{k_1}) + H_{k_3}(\Phi_{k_2})\,H_{k_1}(\Phi_i) + \\ &\quad + \Phi_{k_1}\Phi_{k_2} H_{k_3}(\Phi_i) + \Phi_i\,\Phi_{k_2} H_{k_3}(\Phi_{k_1}) + \Phi_i\,\Phi_{k_1} H_{k_3}(\Phi_{k_2}) + \\ &\quad + \Phi_{k_1}\Phi_{k_3} H_{k_2}(\Phi_i) + \Phi_i\,\Phi_{k_3} H_{k_2}(\Phi_{k_1}) + \Phi_{k_2}\Phi_{k_3} H_{k_1}(\Phi_i) + \\ &\quad + \Phi_i\,\Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\big)\,T\end{aligned}$$

and in the expressions of which the H_k*-iterated derivatives of the functions* Φ_i *up to order* 3*:*

$$\begin{aligned}H_{k_1}(\Phi_i) &= \frac{A_{i,k_1}}{\Delta^4\,\Upsilon^2} && (i, k_1 = 1,2), \\ H_{k_2}(H_{k_1}(\Phi_i)) &= \frac{A_{i,k_1,k_2}}{\Delta^6\,\Upsilon^3} && (i, k_1, k_2 = 1,2), \\ H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) &= \frac{A_{i,k_1,k_2,k_3}}{\Delta^8\,\Upsilon^4} && (i, k_1, k_2, k_3 = 1,2),\end{aligned}$$

have numerators A_{i,k_1}, A_{i,k_1,k_2}, A_{i,k_1,k_2,k_3} *that are certain differential polynomials whose* completely explicit *expressions in terms of the jets* $J^4_{x,y,u}\varphi$, $J^5_{x,y,u}\varphi$, $J^6_{x,y,u}\varphi$ *of the graphing function* $\varphi(x,y,u)$ *of orders* 4, 5, 6 *(respectively) are provided through the induction formulas:*

$$\begin{aligned}
A_{i,k_1} &:= \Delta^2\big(\Upsilon\, A_{i,x_{k_1}} - \Upsilon_{x_{k_1}}\, A_i\big) + \Delta\big(-2\,\Delta_{x_{k_1}}\,\Upsilon\, A_i + \Upsilon\,\Lambda_{k_1}\, A_{i,u} - \Upsilon_u\,\Lambda_{k_1}\, A_i\big) - \\
&\quad - 2\,\Delta_u\,\Upsilon\,\Lambda_{k_1}\, A_i \qquad\qquad {\scriptstyle (i,\,k_1=1,2)},\\
A_{i,k_1,k_2} &:= \Delta^2\big(\Upsilon\, A_{i,k_1,x_{k_2}} - 2\,\Upsilon_{x_{k_2}}\, A_{i,k_1}\big) + \Delta\big(-3\,\Delta_{x_{k_2}}\,\Upsilon\, A_{i,k_1} + \Upsilon\,\Lambda_{k_2}\, A_{i,k_1,u} - \\
&\quad - 2\,\Upsilon_u\,\Lambda_{k_2}\, A_{i,k_1}\big) - 3\,\Delta_u\,\Upsilon\,\Lambda_{k_2}\, A_{i,k_1} \qquad\qquad {\scriptstyle (i,\,k_1,\,k_2=1,2)},\\
A_{i,k_1,k_2,k_3} &= \Delta^2\big(\Upsilon\, A_{i,k_1,k_2,x_{k_3}} - \Upsilon_{x_{k_3}}\, A_{i,k_1,k_2}\big) + \Delta\big(-6\,\Delta_{x_{k_3}}\,\Upsilon\, A_{i,k_1,k_2} + \Upsilon\,\Lambda_{k_3}\, A_{i,k_1,k_2,u} - \\
&\quad - 3\,\Upsilon_u\,\Lambda_{k_3}\, A_{i,k_1,k_2}\big) - 6\,\Delta_u\,\Upsilon\,\Lambda_{k_3}\, A_{i,k_1,k_2} \qquad\qquad {\scriptstyle (i,\,k_1,\,k_2,\,k_3=1,2)}.
\end{aligned}$$

Then (first, preliminary effective assertion): these iterated derivatives identically satisfy:

$$H_2(\Phi_1) \equiv H_1(\Phi_2),$$

together with the following four third-order relations:

$$\begin{aligned}
0 &\equiv -H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\
&\quad - \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)),
\end{aligned}$$

$$\begin{aligned}
0 &\equiv -H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \\
&\quad - \Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)),
\end{aligned}$$

$$\begin{aligned}
0 &\equiv -H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \\
&\quad + \Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)),
\end{aligned}$$

$$\begin{aligned}
0 &\equiv H_2(H_2(H_1(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_1(H_2(H_2(\Phi_2))) - \\
&\quad - \Phi_2\,H_2(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)).
\end{aligned}$$

Moreover (second, well known effective assertion), the model Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$ *whose graphing function simply has no* $\mathrm{O}(3)$ *remainder:*

$$v = x^2 + y^2$$

possesses an eight-dimensional graded Lie algebra:

$$\mathfrak{hol}(\mathbb{H}^3) = \mathfrak{h}_{-2} \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

of (local or global) holomorphic vector fields X *whose real parts* $\frac{1}{2}\big(\mathsf{X} + \overline{\mathsf{X}}\big)$ *are tangent to it, having components:*

$$\begin{aligned}
\mathfrak{h}_{-2} &= \mathbb{R}\,\mathsf{T}, \qquad \mathfrak{h}_{-1} = \mathbb{R}\,\mathsf{H}_1 \oplus \mathbb{R}\,\mathsf{H}_2\\
\mathfrak{h}_0 &= \mathbb{R}\,\mathsf{D} \oplus \mathbb{R}\,\mathsf{R},\\
\mathfrak{h}_1 &= \mathbb{R}\,\mathsf{I}_1 \oplus \mathbb{R}\,\mathsf{I}_2, \qquad \mathfrak{h}_2 = \mathbb{R}\,\mathsf{J}.
\end{aligned}$$

where:

$$\mathfrak{h}_{-2}\colon \big\{ \mathsf{T} := \partial_w \qquad \mathfrak{h}_{-1}\colon \begin{cases} \mathsf{H}_1 := \partial_z + 2iz\,\partial_w \\ \mathsf{H}_2 := i\,\partial_z + 2z\,\partial_w \end{cases}$$

$$\mathfrak{h}_0\colon \begin{cases} \mathsf{D} := z\,\partial_z + 2w\,\partial_w \\ \mathsf{R} := iz\,\partial_z \end{cases}$$

$$\mathfrak{h}_1\colon \begin{cases} \mathsf{I}_1 := (w + 2iz^2)\,\partial_z + 2izw\,\partial_w \\ \mathsf{I}_2 := (iw + 2z^2)\,\partial_z + 2zw\,\partial_w \end{cases} \qquad \mathfrak{h}_2\colon \big\{ \mathsf{J} := zw\,\partial_z + w^2\,\partial_w,$$

and these eight holomorphic fields enjoy the following commutator table with real (in fact, integer) structure constants:

	T	H_1	H_2	D	R	I_1	I_2	J
T	0	0	0	$2\,\mathsf{T}$	0	H_1	H_2	D
H_1	$*$	0	$4\,\mathsf{T}$	H_1	H_2	$6\,\mathsf{R}$	$2\,\mathsf{D}$	I_1
H_2	$*$	$*$	0	H_2	$-\mathsf{H}_1$	$-2\,\mathsf{D}$	$6\,\mathsf{R}$	I_2
D	$*$	$*$	$*$	0	0	I_1	I_2	$2\,\mathsf{J}$
R	$*$	$*$	$*$	$*$	0	$-\mathsf{I}_2$	I_1	0
I_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{J}$	0
I_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

Lastly (third, main effective assertion), to any $\mathscr{C}^6$ strongly pseudoconvex $M^3 \subset \mathbb{C}^2$ is uniquely associated an effective *local Cartan connection:*

$$\omega\colon \quad T\mathscr{P} \longrightarrow \mathfrak{g}$$

on the local principal bundle:

$$\mathscr{P} := M^3 \times H^5$$

which is the Cartesian product of M with the unique (connected and simply connected) local 5-dimensional Lie group H equipped with some 5 real coordinates:

$$(a, b, c, d, e) \in \mathbb{R}^5,$$

that is associated to the isotropy Lie subalgebra:

$$\mathfrak{hol}(\mathbb{H}^3, 0) = \mathbb{R}\,\mathsf{D} \oplus \mathbb{R}\,\mathsf{R} \oplus \mathbb{R}\,\mathsf{I}_1 \oplus \mathbb{R}\,\mathsf{I}_2 \oplus \mathbb{R}\,\mathsf{J}$$

of the origin $0 \in \mathbb{H}^3$; this Cartan connection $\omega\colon T\mathscr{P} \longrightarrow \mathfrak{g}$ is valued in the eight-dimensional abstract *real Lie algebra:*

$$\mathfrak{g} := \mathbb{R}\,\mathsf{t} \oplus \mathbb{R}\,\mathsf{h}_1 \oplus \mathbb{R}\,\mathsf{h}_2 \oplus \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r} \oplus \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2 \oplus \mathbb{R}\,\mathsf{j}$$

spanned by some eight abstract vectors enjoying the same commutator table:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0;

this Cartan connection $\omega\colon T\mathscr{P} \longrightarrow \mathfrak{g}$ is normal and regular in the sense of Tanaka ([8, 14]*, 71 below), and if one denotes the Lie algebra of H by:*

$$\mathfrak{h} := \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r} \oplus \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2 \oplus \mathbb{R}\,\mathsf{j},$$

with corresponding five left-invariant vector fields on H of the form:

$$\begin{aligned}
D &:= -a\,\frac{\partial}{\partial a} - b\,\frac{\partial}{\partial b} - c\,\frac{\partial}{\partial c} - d\,\frac{\partial}{\partial d} - 2e\,\frac{\partial}{\partial e} \\
R &:= -b\,\frac{\partial}{\partial a} + a\,\frac{\partial}{\partial b} + d\,\frac{\partial}{\partial c} - c\,\frac{\partial}{\partial d} \\
I_1 &:= \frac{\partial}{\partial a} - b\,\frac{\partial}{\partial e} \\
I_2 &:= \frac{\partial}{\partial b} + a\,\frac{\partial}{\partial e} \\
J &:= \frac{1}{2}\,\frac{\partial}{\partial e}
\end{aligned}$$

near the origin $(a_0, b_0, c_0, d_0, e_0) := (0, 0, 1, 1, 0)$, then the curvature function:

$$\kappa \in \mathscr{C}^0\big(\mathscr{P},\, \Lambda^2(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}\big)$$

of the Cartan connection $\omega\colon \mathscr{P} \longrightarrow \mathfrak{g}$, a function $\kappa(p)$ of the eight real variables:

$$\mathscr{P} \ni p := (x, y, u, a, b, c, d, e) \in M^3 \times H$$

has an algebraic expression which reduces to:

$$\begin{aligned}
\kappa(p) = \kappa_{i_1}^{h_1 t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1 t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2 t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \\
+ \kappa_{i_2}^{h_2 t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1 t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},
\end{aligned}$$

where the two main curvature coefficients are explicitly *given by:*

$$\begin{aligned}
\kappa_{i_1}^{h_1 t}(p) &= -\,\boldsymbol{\Delta_1}\,c^4 - 2\,\boldsymbol{\Delta_4}\,c^3 d - 2\,\boldsymbol{\Delta_4}\,c d^3 + \boldsymbol{\Delta_1}\,d^4, \\
\kappa_{i_2}^{h_1 t}(p) &= -\,\boldsymbol{\Delta_4}\,c^4 + 2\,\boldsymbol{\Delta_1}\,c^3 d + 2\,\boldsymbol{\Delta_1}\,c d^3 + \boldsymbol{\Delta_4}\,d^4,
\end{aligned}$$

in which the two functions $\mathbf{\Delta_1}$ *and* $\mathbf{\Delta_4}$ *of only the three variables* (x,y,u) *are* explicitly *given by the symmetric expressions:*

$$\begin{aligned}\mathbf{\Delta_1} = \tfrac{1}{384}\Big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\,H_1(H_2(H_1(\Phi_2))) - 11\,H_2(H_1(H_2(\Phi_1)))+\\ &+ 6\,\Phi_2\,H_2(H_1(\Phi_1)) - 6\,\Phi_1\,H_1(H_2(\Phi_2)) - 3\,\Phi_2\,H_1(H_1(\Phi_2)) + 3\,\Phi_1\,H_2(H_2(\Phi_1))-\\ &-\,3\,\Phi_1\,H_1(H_1(\Phi_1)) + 3\,\Phi_2\,H_2(H_2(\Phi_2)) - 2\,\Phi_1\,H_1(\Phi_1) + 2\,\Phi_2\,H_2(\Phi_2)-\\ &-\,2\,(\Phi_2)^2\,H_1(\Phi_1) + 2\,(\Phi_1)^2\,H_2(\Phi_2) - 2\,(\Phi_2)^2\,H_2(\Phi_2) + 2\,(\Phi_1)^2\,H_1(\Phi_1)\Big],\\ \mathbf{\Delta_4} = \tfrac{1}{384}\Big[&-\,3\,H_2(H_1(H_2(\Phi_2))) - 3\,H_1(H_2(H_1(\Phi_1))) + 5\,H_1(H_2(H_2(\Phi_2))) + 5\,H_2(H_1(H_1(\Phi_1)))+\\ &+ 4\,\Phi_1\,H_1(H_1(\Phi_2)) + 4\,\Phi_2\,H_2(H_1(\Phi_2)) - 3\,\Phi_2\,H_1(H_1(\Phi_1)) - 3\,\Phi_1\,H_2(H_2(\Phi_2))-\\ &-\,7\,\Phi_2\,H_1(H_2(\Phi_2)) - 7\,\Phi_1\,H_2(H_1(\Phi_1)) - 2\,H_1(\Phi_1)\,H_1(\Phi_2) - 2\,H_2(\Phi_2)\,H_2(\Phi_1)+\\ &+ 4\,\Phi_1\Phi_2\,H_1(\Phi_1) + 4\,\Phi_1\Phi_2\,H_2(\Phi_2)\Big],\end{aligned}$$

and where the remaining four secondary curvature coefficients are given by:

$$\begin{aligned}\kappa_{i_1}^{h_2t} &= \kappa_{i_2}^{h_1t},\\ \kappa_{i_2}^{h_2t} &= -\,\kappa_{i_1}^{h_1t},\\ \kappa_{j}^{h_1t} &= \widehat{H}_1\big(\kappa_{i_2}^{h_2t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1t}\big),\\ \kappa_{j}^{h_2t} &= -\widehat{H}_1\big(\kappa_{i_1}^{h_2t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1t}\big),\end{aligned}$$

if one denotes the eight constant vector fields on $\mathscr{P}$ *associated to the Cartan connection by:*

$$\begin{aligned}&\widehat{T} := \omega^{-1}(\mathsf{t}), \qquad \widehat{H}_1 := \omega^{-1}(\mathsf{h}_1), \qquad \widehat{H}_2 := \omega^{-1}(\mathsf{h}_2),\\ &\widehat{D} := \omega^{-1}(\mathsf{d}), \qquad \widehat{R} := \omega^{-1}(\mathsf{r}), \qquad \widehat{I}_1 := \omega^{-1}(\mathsf{i}_1), \qquad \widehat{I}_2 := \omega^{-1}(\mathsf{i}_2), \qquad \widehat{J} := \omega^{-1}(\mathsf{j});\end{aligned}$$

furthermore and for completeness, the 22 *coefficients* $\alpha_{\bullet\bullet}$ *of these eight constant fields with respect to the frame* $\{T, H_1, H_2, D, R, I_1, I_2, J\}$ *on* $\mathscr{P}$*:*

$$\begin{aligned}\widehat{T} &= \alpha_{tt}\,T + \alpha_{th_1}\,H_1 + \alpha_{th_2}\,H_2 + \alpha_{td}\,D + \alpha_{tr}\,R + \alpha_{ti_1}\,I_1 + \alpha_{ti_2}\,I_2 + \alpha_{tj}\,J\\ \widehat{H}_1 &= \alpha_{h_1h_1}\,H_1 + \alpha_{h_1h_2}\,H_2 + \alpha_{h_1d}\,D + \alpha_{h_1r}\,R + \alpha_{h_1i_1}\,I_1 + \alpha_{h_1i_2}\,I_2 + \alpha_{h_1j}\,J\\ \widehat{H}_2 &= \alpha_{h_2h_1}\,H_1 + \alpha_{h_2h_2}\,H_2 + \alpha_{h_2d}\,D + \alpha_{h_2r}\,R + \alpha_{h_2i_1}\,I_1 + \alpha_{h_2i_2}\,I_2 + \alpha_{h_2j}\,J\\ \widehat{D} &= D\\ \widehat{R} &= R\\ \widehat{I}_1 &= I_1\\ \widehat{I}_2 &= I_2\\ \widehat{J} &= J\end{aligned}$$

are given, in terms of the five variables (a,b,c,d,e) *of the structure group* H *of the principal bundle* $\mathscr{P}$ *and in terms of the* H_k*-derivatives (up to order* 3*) of the fundamental coefficient functions* Φ_i*, explicitly by:*

$$\begin{aligned}&\alpha_{tt} = c^2 + d^2, \quad \alpha_{th_1} = bd - ac, \quad \alpha_{th_2} = -ad - bc, \quad \alpha_{h_1h_1} = c,\\ &\alpha_{h_1h_2} = d, \quad \alpha_{h_2h_1} = -d, \quad \alpha_{h_2h_2} = c,\\ &\alpha_{h_1d} = -2b + \tfrac{1}{2}\Phi_1 c + \tfrac{1}{2}\Phi_2 d, \quad \alpha_{h_2d} = 2a + \tfrac{1}{2}\Phi_2 c - \tfrac{1}{2}\Phi_1 d,\\ &\alpha_{h_1r} = -6a - \tfrac{1}{2}\Phi_2 c + \tfrac{1}{2}\Phi_1 d, \quad \alpha_{h_2r} = -6b + \tfrac{1}{2}\Phi_1 c + \tfrac{1}{2}\Phi_2 d,\end{aligned}$$

$$\alpha_{td} = \tfrac{1}{2}(bd-ac)\Phi_1 - \tfrac{1}{2}\Phi_2(bc+ad)\Phi_2 - 2e,$$
$$\alpha_{tr} = \tfrac{1}{32}(H_1(\Phi_1)H_2(\Phi_2))c^2 + \tfrac{1}{32}(H_1(\Phi_1)+H_2(\Phi_2))d^2 - \tfrac{1}{2}(ad+bc)\Phi_1 + \tfrac{1}{2}(ac-bd)\Phi_2 + 3a^2 + 3b^2,$$
$$\alpha_{h_1i_1} = \tfrac{1}{2}(bd+ac)\Phi_1 - \tfrac{1}{2}(bc-ad)\Phi_2 - 4ab - 2e,$$
$$\alpha_{h_1i_2} = \tfrac{1}{32}(H_1(\Phi_1)+H_2(\Phi_2))c^2 + \tfrac{1}{32}(H_1(\Phi_1)+H_2(\Phi_2))d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + 3a^2 - b^2,$$
$$\alpha_{h_2i_1} = -\tfrac{1}{32}H_1((\Phi_1)+H_2(\Phi_2))c^2 - \tfrac{1}{32}(H_1(\Phi_1)+H_2(\Phi_2))d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + a^2 - 3b^2,$$
$$\alpha_{h_2i_2} = -\tfrac{1}{2}(ac+bd)\Phi_1 - \tfrac{1}{2}(ad-bc)\Phi_2 + 4ab - 2e,$$

$$\begin{aligned}\alpha_{ti_1} = {} & \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]d^3 + \\ & + \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]c^3 + \\ & + \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]cd^2 + \\ & + \tfrac{1}{16}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + \\ & + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^2d + \tfrac{1}{16}\big[H_2(\Phi_2) + H_1\Phi_1\big]bd^2 + \\ & + \tfrac{1}{2}\big[-\Phi_1a^2c + 4b^3 - \Phi_1b^2c + 4ba^2 - \Phi_2b^2d - \Phi_2a^2d\big],\end{aligned}$$

$$\begin{aligned}\alpha_{ti_2} = {} & \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^3 - \\ & - \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ac^2 - \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ad^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - \\ & - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]cd^2 + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - \\ & - H_2(\Phi_2)\Phi_1 + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]c^2d + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 + \\ & + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]d^3 - \tfrac{1}{2}\big[\Phi_2a^2c + \Phi_2b^2c - \Phi_1b^2d - \Phi_1a^2d - 4ab^2 + 4a^3\big],\end{aligned}$$

$$\begin{aligned}\alpha_{h_1j} = {} & \tfrac{1}{96}\big[-H_2(H_2(\Phi_2) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + 7H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]c^3 - \\ & + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^2d + \\ & + \tfrac{1}{96}\big[-H_2(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + \tfrac{7}{16}H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]cd^2 + \\ & + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]d^3 - \\ & - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ac^2 - \tfrac{1}{8}\big[\frac{1}{8}H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2 - \Phi_2a^2c - \Phi_2b^2c + 2\Phi_1ce - \\ & - 8be + 2\Phi_2de + \Phi_1b^2d + \Phi_1a^2d - 4ab^2 - 4a^3,\end{aligned}$$

$$\begin{aligned}\alpha_{h_2j} = {} & \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^3 + \\ & - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + 8H_1(H_1(\Phi_2)) - \\ & - 7H_2(H_1(\Phi_1))\big]d^3 - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - \\ & - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]cd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + \\ & + 8H_1(H_1(\Phi_2)) - 7H_2(H_1(\Phi_1))\big]c^2d + \Phi_1a^2c - 2\Phi_1de + \Phi_2b^2d + \Phi_2a^2d + 8ae4b^3 + \Phi_1b^2c - \\ & - 4a^2b + 2\Phi_2ce,\end{aligned}$$

$$\begin{aligned}\alpha_{tj} = {} & 3a^4 + 3b^4 - 4e^2 - \Phi_1a^2bc + ca\Phi_2b^2 - \Phi_1ab^2d - \Phi_2a^2bd - 2\Phi_2bce - 2\Phi_1ace - 2\Phi_2ade + 2\Phi_1bde - \\ & - \Phi_1a^3d + \Phi_2a^3c - \Phi_1b^3c - \Phi_2b^3d + 6a^2b^2 + \big[\tfrac{3}{16}H_1(\Phi_1) + \tfrac{3}{16}H_2(\Phi_2)\big]b^2d^2 + \\ & + \big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1) - \tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1 - \tfrac{11}{3072}H_2({\Phi_2}^2) + \tfrac{1}{384}{\Phi_2}^2H_2(\Phi_2) - \tfrac{11}{3072}H_1(\Phi_1^2) + \\ & + \tfrac{1}{384}\Phi_1^2H_1(\Phi_1) + \tfrac{1}{48}H_1(H_2(H_1(\Phi_2))) + \tfrac{1}{384}H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{384}H_1(H_1(H_1(\Phi_1))) + \tfrac{1}{384}{\Phi_2}^2H_1(\Phi_1) - \\ & - \tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2 + \tfrac{1}{48}H_2(H_1(H_1(\Phi_2))) + \tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2 - \tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2)) + \tfrac{1}{384}{\Phi_1}^2H_2(\Phi_2) - \\ & - \tfrac{7}{384}H_2(H_2(H_1(\Phi_1))) + \tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1 - \tfrac{7}{384}H_1(H_1(H_2(\Phi_2))) - \tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]d^4 + \\ & + \big[-\tfrac{11}{768}H_2(\Phi_2)H_1(\Phi_1) - \tfrac{7}{192}H_2(H_2(H_1(\Phi_1))) + \tfrac{1}{192}H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{192}H_1(H_1(H_1(\Phi_1))) +\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{24}H_1(H_2(H_1(\Phi_2)))-\tfrac{1}{96}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{32}H_1(H_2(\Phi_2))\Phi_1+\tfrac{1}{192}\Phi_2^2H_1(\Phi_1)-\tfrac{7}{192}H_1(H_1(H_2(\Phi_2)))+\\
&+\tfrac{1}{192}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{1536}H_1(\Phi_1^2)-\tfrac{1}{24}\Phi_2H_1(H_1(\Phi_2))-\tfrac{11}{1536}H_2(\Phi_2^2)+\tfrac{1}{32}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{96}H_1(H_1(\Phi_1))\Phi_1+\\
&+\tfrac{1}{192}\Phi_1^2H_2(\Phi_2)+\tfrac{1}{192}\Phi_1^2H_1(\Phi_1)-\tfrac{1}{24}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{24}H_2(H_1(H_1(\Phi_2)))\big]c^2d^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\\
&+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bcd^2+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\\
&-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]acd^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\\
&+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]ad^3+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]ac^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2d^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]bd^3+\\
&+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bc^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2c^2+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]b^2c^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-\\
&-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]dbc^2+\tfrac{1}{32}\big[-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))+H_1(\Phi_1)\Phi_1\big]ac^2d+\\
&+\big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1-\tfrac{11}{3072}H_2(\Phi_2^2)+\tfrac{1}{384}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{3072}H_1(\Phi_1^2)+\\
&+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))+\tfrac{1}{384}\Phi_2^2H_1(\Phi_1)-\\
&-\tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{384}\Phi_1^2H_2(\Phi_2)-\\
&-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]c^4.
\end{aligned}
$$

For a conceptional, motivational and historical introduction to the domain that it would be quite useless to reproduce here, the reader is referred to the excellent expository article ([14]) by Vladimir Ezhov, Ben McLaughlin and Gerd Schmalz which appeared recently in the *Notices of the American Mathematical Society*. By performing the above choice $\{H_1, H_2, T\}$ of an initial frame for TM which is explicit in terms of the graphing function $\varphi(x,y,u)$, we deviate from the initial normalization made in [14] (with a more geometric-minded approach), our computational objective being to provide a Cartan-Tanaka connection all elements of which are completely effective in terms of $\varphi(x,y,u)$ — assuming only $\mathscr{C}^6$-smoothness of M.

It took about fifteen years (between 1810 and 1826) to Gauss to derive what he considered to be a completely convincing proof that the (Gaussian) curvature $\kappa = \kappa(u,v)$ of a surface equipped with a metric:

$$ds^2 = E(u,v)\,du^2 + 2F(u,v)\,dudv + G(u,v)\,dv^2$$

is a completely *intrinsic* invariant through infinitesimal isometries *because* it expresses (*Theorema Egregium*, [15]) for *any* surface as the following explicit rational differential in the second-order jet of the three elements E, F, G, as Gauss showed:

$$
\begin{aligned}
\kappa = \frac{1}{4\,(EG-F^2)^2}\bigg\{ & E\bigg[\frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial v} + \frac{\partial G}{\partial u}\cdot\frac{\partial G}{\partial u}\bigg]+\\
&+F\bigg[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial v}\cdot\frac{\partial F}{\partial v} + 4\,\frac{\partial F}{\partial u}\cdot\frac{\partial F}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial u}\bigg]+\\
&+G\bigg[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial u}\cdot\frac{\partial F}{\partial v} + \frac{\partial E}{\partial v}\cdot\frac{\partial E}{\partial v}\bigg]+\\
&+2\,\big(EG-F^2\big)\bigg[-\frac{\partial^2 E}{\partial v^2} + 2\,\frac{\partial^2 F}{\partial u\partial v} - \frac{\partial^2 G}{\partial u^2}\bigg]\bigg\}.
\end{aligned}
$$

However, for what is commonly considered to constitute the *simplest* instance of Cauchy-Riemann geometry, namely for the case of (embedded) Levi nondegenerate real hypersurfaces $M^3 \subset \mathbb{C}^2$, what would correspond to the above *formula egregia* concerning CR curvature seems not to have yet ever been achieved, perhaps due to the fact that in modern differential geometry — and in Cartan's theory of the problem of equivalence as well —, the *causality* of 'intrinsic-ness' never relies upon some elimination computations (Gauss' proof), but it is set *ab initio* in theories. Nonetheless, a folklore yet unresolved question seems to remain: can one characterize the vanishing of curvature explicitly in terms of $\varphi(x, y, u)$?

Corollary 1.1. *The local biholomorphic equivalence to the Heisenberg sphere:* $v' = x'^2 + y'^2$ *of an arbitrary real analytic Levi nondegenerate hypersurface* $M^3 \subset \mathbb{C}^2$ *represented as a graph of the form:*

$$v = \varphi(x, y, u)$$

with $\varphi_{xx}(0) + \varphi_{yy}(0) \neq 0$ *is explicitly characterized by the identical vanishing:*

$$0 \equiv \mathbf{\Delta_1} \equiv \mathbf{\Delta_4},$$

of the two main functions of (x, y, u) *appearing in the curvature function of the above effective Cartan-Tanaka connection.* □

Of course, thanks to our extensive theorem stated in length, expansions of these two principal functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta}_4$ can straightforwardly be achieved on a computer by just applying the induction formulas to which the numerators A_i, A_{i,k_1}, A_{i,k_1,k_2}, A_{i,k_1,k_2,k_3} of the above fundamental functions Φ_i, $H_{k_1}(\Phi_i)$, $H_{k_2}(H_{k_1}(\Phi_i))$, $H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i)))$ are subjected. As a mathematically satisfactory fact, the numerators of both expressions of $\mathbf{\Delta_1}$ and $\mathbf{\Delta}_4$ then become a completely explicit differential polynomial in the sixth-order jet $J^6_{x,y,u}\varphi$ of the graphing function (the same can be done of course for the functions $\alpha_{\bullet\bullet}$ too). But their prohibitive lengths — nearly one thousand pages long on a computer, not copied in this LaTeX file — presumably explain why no reference in the literature (*cf. e.g.* [9, 10, 11, 14, 19, 20, 24, 29]) succeeded to be fully effective on the topic, whence unexpectedly and a bit paradoxically also, DIE GAUSSCHE STRENGE (the Gaussian requirement) for total computational effectiveness in mathematics happens to be unsatisfiable at human scale even in the case of the simplest $M^3 \subset \mathbb{C}^2$.

Acknowledgments. One year ago, Gerd Schmalz kindly provided us with a pdf copy of the accepted version of [14], and this was of great help during the (painful) preparation of the present paper. Another article [6] of Valerii Beloshapka, Vladimir Ezhov and Gerd Schmalz was also used to enter the theory in the right way. The authors would also like to thank Gerd Schmalz and Ben McLaughlin for very helpful explanations through e-mail exchanges.

2. INFINITESIMAL CR AUTOMORPHISMS OF REAL ANALYTIC GENERIC SUBMANIFOLDS OF $\mathbb{C}^{n+d}$

2.1. **Real and complex local equations for generic submanifolds.** Consider a local generic CR submanifold $M \subset \mathbb{C}^{n+d}$ of positive codimension $d \geqslant 1$ and of positive CR dimension $n \geqslant 1$ and let p be a point of M. In any system of local holomorphic coordinates $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_d)$ decomposed in real and imaginary parts as $(z, w) = (x + iy, u + iv)$ which vanish at p and for which $T_pM = \{\mathrm{Re}\, w_j = 0, j = 1, \dots, d\}$, the generic submanifold $M \subset \mathbb{C}^{n+d}$ is locally represented by d real equations:

(1) $$u_j = \varphi_j(x, y, v) \qquad (j = 1 \cdots d)$$

as a graph over the d-codimensional plane T_pM with of course the property that the first order jet of each graphing function φ_j is zero at the origin:

$$0 = \varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_k}\varphi_j(0) = \partial_{v_{j'}}\varphi_j(0) \qquad (k = 1 \cdots n\,;\; j,\, j' = 1 \cdots d).$$

We shall assume in this section that M is *real analytic*, so that the functions $\varphi_1, \dots, \varphi_d$ are all expandable in Taylor series converging in a certain neighborhood of the origin in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$.

In fact, the adequate invariants of (CR mappings between) CR manifolds can be viewed mostly when M is represented by d so-called *complex defining equations*. Such equations may be obtained by rewriting the above real equations just as:

$$\frac{w_j + \overline{w}_j}{2} = \varphi_j\Big(\frac{z+\overline{z}}{2},\, \frac{z-\overline{z}}{2\sqrt{-1}},\, \frac{w-\overline{w}}{2\sqrt{-1}}\Big) \qquad (j = 1 \cdots d),$$

and then by solving the so written equations with respect to the variables w_j by means of the *analytic* implicit function theorem; in this way, one obtains a collection of d equations of the shape[1] (written in vectorial notation):

$$w = \Theta\big(z, \overline{z}, \overline{w}\big) = \sum_{\substack{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^n,\, \gamma\in\mathbb{N}^d \\ |\alpha|+|\beta|+|\gamma|\geqslant 1}} \Theta_{\alpha,\beta,\gamma}\, z^\alpha\, \overline{z}^\beta\, \overline{w}^\gamma \in \mathbb{C}\big\{z, \overline{z}, \overline{w}\big\}^d,$$

whose right-hand side converges of course near the origin $(0, 0, 0) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$ and whose (vector) coefficients $\Theta_{\alpha,\beta,\gamma} \in \mathbb{C}^d$ are *complex*. Since $d\varphi(0) = 0$, one has $\Theta = -\overline{w} + \mathsf{order}\, 2\, \mathsf{terms}$.

The paradox that any such d *complex* equations provide in fact $2d$ real defining equations for the *real* generic submanifold $M \subset \mathbb{C}^{n+d}$ which is d-codimensional, and also in addition the fact that one could as well have chosen to solve the above equations with respect to the $\overline{w}_j$, instead of the w_j, these two apparent "contradictions" are corrected by means of a fundamental, elementary statement that transfers to Θ (in a natural way) the condition of reality enjoyed by the initial defining $\mathbb{R}^d$-valued map φ:

$$\overline{\varphi(x, y, v)} = \sum_{|\alpha|+|\beta|+|\gamma|\geqslant 1} \overline{\varphi_{\alpha,\beta,\gamma}}\, \overline{x}^\alpha \overline{y}^\beta \overline{v}^\gamma = \sum_{|\alpha|+|\beta|+|\gamma|\geqslant 1} \varphi_{\alpha,\beta,\gamma}\, x^\alpha y^\beta v^\gamma = \varphi(x, y, v).$$

In the sequel, we shall work exclusively with the complex graphing functions Θ_j, so we recall a basic result[2]. The complex analytic $\mathbb{C}^d$-valued map $\Theta = \Theta(z, \overline{z}, \overline{w})$

---

[1] Recall that $\mathbb{C}\{\mathsf{x}_1, \dots, \mathsf{x}_\nu\}^d$ denotes the ring of power series $\sum_{\alpha_1,\dots,\alpha_n\in\mathbb{N}} C_{\alpha_1,\dots,\alpha_n} \cdot \mathsf{x}_1^{\alpha_1} \cdots \mathsf{x}_n^{\alpha_n}$ with $\mathbb{C}^d$-valued complex coefficients $C_{\alpha_1,\dots,\alpha_n}$ which converge in some neighborhood of the origin.

[2] According to a general, common convention, given a power series $\Phi(Z) = \sum_{\delta\in\mathbb{N}^N} \Phi_\delta\, Z^\delta$, $Z \in \mathbb{C}^N$, $\Phi_\delta \in \mathbb{C}$, one defines the series $\overline{\Phi}(Z) := \sum_{\delta\in\mathbb{N}^N} \overline{\Phi}_\delta\, Z^\delta$ by conjugating only its complex

with $\Theta = -\overline{w} + \mathrm{O}(2)$ together with its complex conjugate:

$$\overline{\Theta} = \overline{\Theta}\big(\overline{z}, z, w\big) = \sum_{\alpha\in\mathbb{N}^n,\,\beta\in\mathbb{N}^n,\,\gamma\in\mathbb{N}^d} \overline{\Theta}_{\alpha,\beta,\gamma}\,\overline{z}^\alpha\,z^\beta\,w^\gamma \in \mathbb{C}\big\{\overline{z},\,z,\,w\big\}^d$$

satisfy the two (equivalent by conjugation) collections of d functional equations:

$$(2)\qquad \begin{aligned} \overline{w}_j &\equiv \overline{\Theta}_j\big(\overline{z}, z, \Theta(z,\overline{z},\overline{w})\big) && (j=1\cdots d),\\ w_j &\equiv \Theta_j\big(z,\overline{z},\overline{\Theta}(\overline{z},z,w)\big) && (j=1\cdots d); \end{aligned}$$

conversely, given a local holomorphic $\mathbb{C}^d$-valued map $\Theta(z,\overline{z},\overline{w}) \in \mathbb{C}\{z,\overline{z},\overline{w}\}^d$, $\Theta = -\overline{w} + \mathrm{O}(2)$ which, in conjunction with its complex conjugate $\overline{\Theta}(\overline{z},z,w)$, satisfies this pair of equivalent identities, then the two zero-sets:

$$\big\{0 = -w + \Theta\big(z,\,\overline{z},\,\overline{w}\big)\big\} \qquad \text{and} \qquad \big\{0 = -\overline{w} + \overline{\Theta}\big(\overline{z},\,z,\,w\big)\big\}$$

coincide and define a local generic d-codimensional real analytic submanifold passing through the origin in $\mathbb{C}^{n+d}$. In fact more precisely, one may show ([26, 27]) that there is an invertible $d\times d$ matrix $a(z,w,\overline{z},\overline{w})$ of analytic functions defined near the origin such that one has:

$$w - \Theta(z,\overline{z},\overline{w}) \equiv a(z,w,\overline{z},\overline{w})\,\big[\overline{w} - \overline{\Theta}(\overline{z},z,w)\big],$$

identically in $\mathbb{C}\{z,w,\overline{z},\overline{w}\}$, whence the coincidence of the two zero-sets immediately follows.

2.2. **Extrinsic complexification.** As is known in local analytic CR geometry, it is natural to introduce new independent complex variables $(\underline{z},\underline{w}) \in \mathbb{C}^n\times\mathbb{C}^d$ — underlining should *not* be confused here with conjugating — and to define the so-called *extrinsic complexification* M^{e_c} of M as being the complex analytic d-codimensional submanifold of $\mathbb{C}^{n+d}\times\mathbb{C}^{n+d}$ equipped with the $2n+2d$ coordinates $(z,w,\underline{z},\underline{w})$ which is defined by the d equations:

$$w_j = \Theta_j(z,\underline{z},\underline{w}) \qquad (j=1\cdots d).$$

Notice that the replacement $(\overline{z},\overline{w})$ by $(\underline{z},\underline{w})$ in the Taylor series of Θ is really meaningful:

$$\Theta_j(z,\underline{z},\underline{w}) := \sum_{\alpha\in\mathbb{N}^n,\,\beta\in\mathbb{N}^n,\,\gamma\in\mathbb{N}^d} \Theta_{j,\alpha,\beta,\gamma}\,z^\alpha\,\underline{z}^\beta\,\underline{w}^\gamma,$$

thanks to the fact that the series converges locally. Equivalently, M^{e_c} is defined by the d equations $\underline{w}_j = \overline{\Theta}_j(\underline{z},z,w)$. Then M is recovered from M^{e_c} by just replacing these independent variables $(\underline{z},\underline{w})$ by the original conjugates $(\overline{z},\overline{w})$. The following standard uniqueness principle is useful.

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coefficients. Then the complex conjugation operator distributes oneself simultaneously on functions and on variables: $\overline{\Phi(Z)} \equiv \overline{\Phi}(\overline{Z})$, a trivial property which is nonetheless frequently used in the formal CR reflection principle ([26, 27, 32]).

Lemma 2.1. *Consider a complex-valued converging power series:*

$$\Phi = \Phi(z, w, \overline{z}, \overline{w}) = \sum_{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^d,\, \gamma\in\mathbb{N}^n,\, \delta\in\mathbb{N}^d} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \overline{z}^\gamma\, \overline{w}^\delta$$

in $\mathbb{C}\{z, w, \overline{z}, \overline{w}\}$ *having complex coefficients* $\Phi_{\alpha,\beta,\gamma,\delta} \in \mathbb{C}$. *Then the following four properties are equivalent:*

- **(i)** Φ *takes only the value zero when the point* (z, w) *varies (without restriction) on* $M \subset \mathbb{C}^n$*;*
- **(ii)** *the extrinsic complexification of* Φ*:*

$$\Phi^{e_c} = \Phi^{e_c}(z, w, \underline{z}, \underline{w}) := \sum_{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^d,\, \gamma\in\mathbb{N}^n,\, \delta\in\mathbb{N}^d} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \underline{z}^\gamma\, \underline{w}^\delta$$

  takes only the value zero when the point $(z, w, \underline{z}, \underline{w})$ *varies (without restriction) on the complexification* $M^{e_c} \subset \mathbb{C}^{2n+2d}$*;*
- **(iii)** *after replacing* $\underline{w}$ *by* $\overline{\Theta}(\underline{z}, z, w)$ *in the extrinsic complexification* Φ^{e_c} *of* Φ*, the result is an identically zero series in* $\mathbb{C}\{\underline{z}, z, w\}^d$*, namely:*

$$0 \equiv \sum_{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^d,\, \gamma\in\mathbb{N}^n,\, \delta\in\mathbb{N}^d} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \underline{z}^\gamma\, [\overline{\Theta}(\underline{z}, z, w)]^\delta;$$

- **(iv)** *after replacing* w *by* $\Theta(z, \underline{z}, \underline{w})$ *in the extrinsic complexification* Φ^{e_c}*, the result is an identically zero power series in* $\mathbb{C}\{z, \underline{z}, \underline{w}\}^d$*, namely:*

$$0 \equiv \sum_{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^d,\, \gamma\in\mathbb{N}^n,\, \delta\in\mathbb{N}^d} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, [\Theta(z, \underline{z}, \underline{w})]^\beta\, \underline{z}^\gamma\, \underline{w}^\delta. \quad \square$$

Let $Z \in \mathbb{C}^N$. A converging power series $\Phi(Z) \in \mathbb{C}\{Z\}$ will be called a *holomorphic* function. A converging power series $\Pi(\overline{Z}) \in \mathbb{C}\{\overline{Z}\}$ will be called an *antiholomorphic* function. But in general, in local analytic Cauchy-Riemann geometry, some variables Z and $\overline{Z}$ are mixed or considered together. Because any converging power series $\Psi(Z, \overline{Z}) \in \mathbb{C}\{Z, \overline{Z}\}$ may also be considered as the series:

$$\Psi^\sim(\operatorname{Re} Z,\, \operatorname{Im} Z) := \Psi(\operatorname{Re} Z + i \operatorname{Im} Z,\ \operatorname{Re} Z - i \operatorname{Im} Z),$$

belonging to $\mathbb{C}\{\operatorname{Re} Z,\, \operatorname{Im} Z\}$ (in terms of the basic *real* $2N$ variables $\operatorname{Re} Z$ and $\operatorname{Im} Z$), such a series Ψ will be called a *real analytic* function, not only when it has purely real values, namely when $\Psi^\sim \in \mathbb{R}\{\operatorname{Re} Z,\, \operatorname{Im} Z\}$, but also when it has complex values, namely when $\Psi^\sim \in \mathbb{C}\{\operatorname{Re} Z,\, \operatorname{Im} Z\}$. Thus, the terminology "*real analytic*" is used for $(Z, \overline{Z})$-dependence.

2.3. **Holomorphic and antiholomorphic tangent vector fields.** In such coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, we claim that the bundle $T^{1,0}M$ and its conjugate $T^{0,1}M = \overline{T^{1,0}M}$ are generated, respectively, by the two collections of mutually independent $(1,0)$ and $(0,1)$ vector fields:

$$\mathscr{L}_k = \frac{\partial}{\partial z_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial z_k}(z, \overline{z}, \overline{w}) \frac{\partial}{\partial w_j} \qquad (k = 1 \cdots n),$$

$$\overline{\mathscr{L}}_k = \frac{\partial}{\partial \overline{z}_k} + \sum_{j=1}^{d} \frac{\partial \overline{\Theta}_j}{\partial \overline{z}_k}(\overline{z}, z, w) \frac{\partial}{\partial \overline{w}_j} \qquad (k = 1 \cdots n)$$

having real analytic coefficients. Indeed, one checks immediately that $0 \equiv \mathscr{L}_k\big(w_j - \Theta_j(z, \overline{z}, \overline{w})\big)$ and that $0 \equiv \overline{\mathscr{L}}_k\big(\overline{w_j} - \overline{\Theta}(\overline{z}, z, w)\big)$, and since the two complex vector bundles $T^{1,0}M$ and $T^{0,1}M$ are known ([7, 3, 32]) to be of rank $n = \mathrm{CRdim}\, M$ (which truly means that M is generic), the claim is clear. Of course, these two collections of vector fields have extrinsic complexifications:

$$\mathscr{L}_k^{e_c} = \frac{\partial}{\partial z_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial z_k}(z, \underline{z}, \underline{w}) \frac{\partial}{\partial w_j} \qquad (k = 1 \cdots n),$$

$$\underline{\mathscr{L}}_k^{e_c} = \frac{\partial}{\partial \underline{z}_k} + \sum_{j=1}^{d} \frac{\partial \overline{\Theta}_j}{\partial \underline{z}_k}(\underline{z}, z, w) \frac{\partial}{\partial \underline{w}_j} \qquad (k = 1 \cdots n).$$

2.4. **Intrinsic generators of T^cM.** It is also useful to write the $(1,0)$ and $(0,1)$ vector fields tangent to M in terms of the *real* graphed defining equations $u_j = \varphi_j(x, y, v)$ of M. For any $k = 1, \dots, n$, a $(1,0)$ vector field of the general form:

$$\mathscr{L}_k = \frac{\partial}{\partial z_k} + \sum_{l=1}^{d} \mathtt{A}_{k,l} \frac{\partial}{\partial w_l}$$

is tangent to the d real equations of M:

$$0 = -u_j + \varphi_j(x, y, u) \qquad (j = 1 \cdots d)$$

if and only if its d complex coefficients $\mathtt{A}_{k,l}$ satisfy, on restriction to M, the following $n\,d$ scalar equations:

$$0 = -\frac{1}{2}\mathtt{A}_{k,j} - \frac{i}{2}\sum_{l=1}^{d} \mathtt{A}_{k,l}\, \varphi_{j,v_l} + \varphi_{j,z_k} \qquad (k = 1 \cdots n;\ j = 1 \cdots d).$$

Fixing k, if one introduces the column matrix $\mathtt{A}_k := \big(\mathtt{A}_{k,1}, \dots, \mathtt{A}_{k,d}\big)^{\mathtt{t}}$, the column matrix $\varphi_{z_k} := \big(\varphi_{1,z_k}, \dots, \varphi_{d,z_k}\big)^{\mathtt{t}}$ and the $d \times d$ matrix $\varphi_v := \big(\varphi_{j,v_l}\big)_{1\leqslant j\leqslant d}^{1\leqslant l\leqslant d}$ in which j is the index of lines, the corresponding d equations, when rewritten as:

$$2\,\varphi_{z_k} = \Big(\big(\delta_{j,l} + i\,\varphi_{j,v_l}\big)_{1\leqslant j\leqslant d}^{1\leqslant l\leqslant d}\Big) \cdot \mathtt{A}_k \qquad (j = 1 \cdots d)$$

constitute a linear system of d equations in the d unknowns $\mathtt{A}_{k,1}, \dots, \mathtt{A}_{k,d}$ which may be solved by means of a matrix inversion:

$$\mathtt{A}_k = 2\,\big(I + i\,\varphi_v\big)^{-1} \cdot \varphi_{z_k}.$$

Then the decomposition in real and imaginary parts:

$$\mathtt{A}_k = \mathtt{A}_k' + i\,\mathtt{A}_k'' \qquad\qquad {\scriptstyle (k=1\cdots n)}$$

of these coefficients writes:

$$\begin{aligned}\mathtt{A}_k' &= \big(I + i\,\varphi_v\big)^{-1}\cdot\varphi_{z_k} + \big(I - i\,\varphi_v\big)^{-1}\cdot\varphi_{\overline{z}_k},\\ \mathtt{A}_k'' &= -\,i\,\big(I + i\,\varphi_v\big)^{-1}\cdot\varphi_{z_k} + i\,\big(I - i\,\varphi_v\big)^{-1}\cdot\varphi_{\overline{z}_k}.\end{aligned}$$

In this way, transposing the column matrix of basic $\frac{\partial}{\partial w_l}$ derivations, we obtain precisely the right number $n = \mathrm{CRdim}\,M$ linearly independent generators of $T^{1,0}M$:

$$\begin{aligned}\mathscr{L}_k &= \frac{1}{2}\,\frac{\partial}{\partial x_k} - \frac{i}{2}\,\frac{\partial}{\partial y_k} + \bigg(\frac{\partial}{\partial w}\bigg)^{\mathtt{t}}\cdot\mathtt{A}_k\\ &= \frac{1}{2}\,\frac{\partial}{\partial x_k} - \frac{i}{2}\,\frac{\partial}{\partial y_k} + \bigg(\frac{1}{2}\,\frac{\partial}{\partial u} - \frac{i}{2}\,\frac{\partial}{\partial v}\bigg)^{\mathtt{t}}\cdot\big(\mathtt{A}_k' + i\,\mathtt{A}_k''\big).\end{aligned}$$

However, such n generators $\mathscr{L}_k$ of $T^{1,0}M$ are still *extrinsic*, namely they involve the coordinates u_j, and if we want to pull-back them to the generic submanifold M that needs only its intrinsic coordinates (x,y,v):

$$\begin{aligned}H_k^1 &:= 2\,\mathrm{Re}\,\big(\mathscr{L}_k\big|_M\big),\\ H_k^2 &:= -\,2\,\mathrm{Im}\,\big(\mathscr{L}_k\big|_M\big),\end{aligned}$$

we just have to drop the $\frac{\partial}{\partial u}$ above, and we receive in this way $2n$ generators for the intrinsic complex tangent bundle $T^cM = \mathrm{Re}\,T^{1,0}M$:

$$\begin{aligned}H_k^1 &= \frac{\partial}{\partial x_k} + \bigg(\frac{\partial}{\partial v}\bigg)^{\mathtt{t}}\cdot\mathtt{A}_k'' \qquad\qquad {\scriptstyle (k=1\cdots n)},\\ H_k^2 &= \frac{\partial}{\partial y_k} + \bigg(\frac{\partial}{\partial v}\bigg)^{\mathtt{t}}\cdot\mathtt{A}_k' \qquad\qquad {\scriptstyle (k=1\cdots n)}.\end{aligned}$$

2.5. **Infinitesimal CR automorphisms.** According to [38, 4, 6], a *(local) infinitesimal CR-automorphism of* M, when understood extrinsically, is a holomorphic vector field:

$$\mathsf{X} = \sum_{k=1}^{n} Z^k(z,w)\frac{\partial}{\partial z_k} + \sum_{j=1}^{d} W^j(z,w)\frac{\partial}{\partial w_j}$$

whose real part $\mathrm{Re}\,X = \frac{1}{2}(\mathsf{X} + \overline{\mathsf{X}})$ is tangent to M. (One should mind that, contrary to the above $(1,0)$ generators $\mathscr{L}_k$ of $T^{1,0}M$, such an X is supposed to have purely holomorphic coefficients, whereas the $\frac{\partial\Theta_j}{\partial z_k}(z,\overline{z},\overline{w})$ are — most of the time — neither purely holomorphic, nor purely antiholomorphic, but only real analytic.) Determining all such X's is the same as knowing the *CR symmetries* of M, a question which lies at the heart of the problem of classifying all local analytic CR manifolds up to biholomorphisms.

By integration, the *real* flow:

$$(t,z,w) \longmapsto \exp(t\,\mathsf{X})(z,w) \qquad\qquad {\scriptstyle (t\in\mathbb{R}\ \text{small})}$$

constitutes a local one-parameter group of local biholomorphisms of $\mathbb{C}^n$; because X is tangent to M, this flow leaves M invariant (locally), that is to say: through this flow, points of M are transferred to points of M. We note *passim* that this real flow coincides with restricting the consideration of the complex (holomorphic) flow:

$$(\tau, z, w) \longmapsto \exp(\tau \, \mathsf{X})(z, w) \qquad\qquad (\tau \in \mathbb{C} \ \text{small})$$

to real time parameters $\tau := t \in \mathbb{R}$, a fact we will not use. Conversely, one may show:

Lemma 2.2. ([38, 3]) *If $M \subset \mathbb{C}^{n+d}$ is a generic submanifold and if $(z, w) \longmapsto \phi_t(z, w)$ is a local real one-parameter group of* holomorphic *self-transformations of $\mathbb{C}^{n+d}$ which stabilizes M locally, then the vector field:*

$$\tfrac{d}{dt}\big|_0\big(\phi_t(z, w)\big)$$

has holomorphic coefficients and its real part is tangent to M. □

Since holomorphy of coefficients and tangency to a submanifold is preserved under taking Lie brackets, the collection $\mathfrak{hol}(M)$ of all such X is obviously a Lie algebra. Also, when $\mathfrak{hol}(M)$ is finite-dimensional (which occurs except in degenerate situations, *see e.g.* [16]), the corresponding finite-dimensional local Lie group is real, whence $\mathfrak{hol}(M)$ constitutes a *real* Lie algebra. So according to one of Lie's fundamental theorems ([31], Chap. 9), if $\mathsf{X}_1, \dots, \mathsf{X}_r$ denote any basis of $\mathfrak{hol}(M)$ as a vector space, there are *real* structure constants $c_{jk}^s \in \mathbb{R}$ such that:

$$\big[\mathsf{X}_j, \mathsf{X}_k\big] = \sum_{s=1}^{r} c_{jk}^s \, \mathsf{X}_s. \tag{3}$$

For an explicitly given $M \subset \mathbb{C}^{n+d}$, determining a basis of the Lie algebra $\mathfrak{hol}(M)$ is a natural problem for which systematic computational procedures exists, as we will establish in a while. The groundbreaking works of Sophus Lie and his followers (Friedrich Engel, Georg Scheffers, Gerhard Kowalewski, Ugo Amaldi and others) showed that the most fundamental question in concern here is to draw up lists of possible Lie algebras $\mathfrak{hol}(M)$ which would classify all possible M's according to their CR symmetries.

Alternatively, if one prefers to view the CR manifold M in a purely intrinsic way, one may consider the local group $\mathrm{Aut}_{CR}(M)$ of automorphisms of the CR structure, namely of local $\mathscr{C}^\infty$ diffeomorphisms $g \colon M \to M$ (close to the identity mapping) which satisfy:

$$dg_p\big(T_p^c M\big) = T_{g(p)}^c M \quad \text{ and } \quad dg_p\big(J(v_p)\big) = J_{g(p)}\big(dg_p(v_p)\big)$$

at any point $p \in M$ and for any complex-tangent vector $v_p \in T_p^c M$. In other words, g belongs to $\mathrm{Aut}_{CR}(M)$ if and only if it is a (local) *CR-diffeomorphism* of M, namely a diffeomorphism which respects the (intrinsic) CR structure of M. As did Lie most of the time in his original theory ([31, 13]), we shall consider only a neighborhood of the identity mapping, hence all our groups will be *local Lie groups*; the reader is again referred to [31, 33, 16]) for fundamentals about local Lie groups in general, especially concerning the fact that it is essentially useless to point out open sets and domains in which mappings and transformations are defined.

Accordingly, let:

$$\mathfrak{aut}_{CR}(M)$$

denote the collection of all (real) vector fields Y on M the flow of which $(t,p) \mapsto \exp(t\mathsf{Y})(p)$ becomes a local CR diffeomorphism of M. When $\mathrm{Aut}_{CR}(M)$ is a finite-dimensional Lie group, $\mathfrak{aut}_{CR}(M)$ is just its Lie algebra.

Lemma 2.3. ([31], Chap. 8; [3]) *A local real analytic vector field* Y *on* M *belongs to* $\mathfrak{aut}_{CR}(M)$*, if and only if for every local section* L *of the complex tangent bundle* T^cM*, the Lie bracket* $[\mathsf{Y},\, L]$ *is again a section of* T^cM*.* □

In all cases which are of interest, namely when M is nondegenerate in some sense (the interested reader is referred to [3, 16, 26, 27, 32], for we prefer not to dwell on that topic here), such $\mathscr{C}^\infty$ flows $(t,p) \mapsto \exp(t\mathsf{Y})(p)$ happen to be in fact *real analytic*, whence, according to a classical theorem, they extend as local *biholomorphic* maps from a neighborhood of M in $\mathbb{C}^{n+d}$. If follows that any such intrinsic Y happens in fact to be a restriction $\mathsf{Y} = \mathsf{X}\big|_M$ to M of some extrinsic $\mathsf{X} \in \mathfrak{hol}(M)$. In all these circumstances which cover a broad universe of yet unstudied CR structures, one has the fundamental relation:

$$\boxed{\mathfrak{aut}_{CR}(M) = \mathrm{Re}\big(\mathfrak{hol}(M)\big)},$$

where both sides are finite-dimensional, spanned by vector fields whose coefficients are expandable in converging power series. Thus, one may work exclusively with the *holomorphic* vector fields generating $\mathfrak{hol}(M)$, as we will do from now on. And in any case, there will be no confusion to call an *infinitesimal CR automorphism* either the holomorphic vector field $\mathsf{X} \in \mathfrak{hol}(M)$ or its real part $\frac{1}{2}(\mathsf{X} + \overline{\mathsf{X}}) \in \mathfrak{aut}_{CR}(M)$.

Since holomorphic vector fields obviously commute with antiholomorphic vector fields, we deduce from (3) that when $\mathfrak{hol}(M) = \mathbb{R}\mathsf{X}_1 \oplus \cdots \oplus \mathbb{R}\mathsf{X}_r$ is r-dimensional, the real parts of the X_j which generate $\mathfrak{aut}_{CR}(M)$ simply have the same (real) structure constants:

$$\big[\mathsf{X}_j + \overline{\mathsf{X}}_j,\, \mathsf{X}_k + \overline{\mathsf{X}}_k\big] = \big[\mathsf{X}_j,\, \mathsf{X}_k\big] + \big[\overline{\mathsf{X}}_j,\, \overline{\mathsf{X}}_k\big] = \sum_{s=1}^{r} c_{jk}^s \,\big(\mathsf{X}_s + \overline{\mathsf{X}}_s\big). \tag{4}$$

To conclude these generalities, at any fixed point $p \in M$, one may also consider the Lie subalgebras $\mathfrak{hol}(M,p)$ of $\mathfrak{hol}(M)$ and $\mathfrak{aut}_{CR}(M,p)$ of $\mathfrak{aut}_{CR}(M)$ consisting of those vector fields whose values vanish at p. Then $\mathfrak{hol}(M,p)$ and $\mathfrak{aut}_{CR}(M,p)$ are the Lie algebra of the subgroups $\mathrm{Hol}(M,p)$ of $\mathrm{Hol}(M)$ and $\mathrm{Aut}_{CR}(M,p)$ of $\mathrm{Aut}_{CR}(M)$ consisting of only the maps that fix the point p. Of course, one has $\mathfrak{aut}_{CR}(M,p) = \mathrm{Re}\big(\mathsf{hol}(M,p)\big)$.

2.6. **Effective tangency equations.** In order to compute $\mathfrak{hol}(M)$ for an explicitly given generic submanifold $M \subset \mathbb{C}^{n+d}$, it is most convenient, as already pointed out, to work with *complex defining equations* of the specific shape ([32, 26]):

$$\overline{w}_j + w_j = \overline{\Xi}_j(\overline{z}, z, w) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

so that, compared to the notations introduced a moment ago, one should consider that the following *notational coincidence* holds:

$$\overline{\Theta}_j(\overline{z}, z, w) \equiv -w_j + \overline{\Xi}_j(\overline{z}, z, w) \qquad\qquad (j = 1 \cdots d).$$

Concretely and precisely, the condition that a general holomorphic vector field $\mathsf{X} = \sum_{k=1}^n Z^k(z,w)\,\frac{\partial}{\partial z_k} + \sum_{l=1}^d W^l(z,w)\,\frac{\partial}{\partial w_l}$ belongs to $\mathfrak{hol}(M)$, namely that $\overline{\mathsf{X}} + \mathsf{X}$ is tangent to M, means that each one of the following d differentiated equation:

$$\begin{aligned} 0 = (\overline{\mathsf{X}} + \mathsf{X})\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] = \\ &= \overline{\mathsf{X}}\,\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] + \mathsf{X}\,\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] \\ &= \overline{W}^j(\overline{z}, \overline{w}) - \sum_{k=1}^n \overline{Z}^k(\overline{z}, \overline{w})\,\frac{\partial \overline{\Xi}_j}{\partial \overline{z}_k}(\overline{z}, z, w) + \\ &\quad + W^j(z,w) - \sum_{k=1}^n Z^k(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial z_k}(\overline{z}, z, w) - \sum_{l=1}^d W^l(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial w_l}(\overline{z}, z, w) \\ &\qquad (j = 1 \cdots d) \end{aligned}$$

should vanish for every $(z, w) \in M$. According to Lemma 2.1, this condition holds true if and only if, after extrinsic complexification and replacement of $\underline{w}$ by $-w + \overline{\Xi}(\underline{z}, z, w)$, the d power series obtained in $\mathbb{C}\{\underline{z}, z, w\}$ vanish identically, namely if and only if:

$$\begin{aligned} 0 \equiv \Bigg[&\overline{W}^j(\underline{z}, \underline{w}) - \sum_{k=1}^n \overline{Z}^k(\underline{z}, \underline{w})\,\frac{\partial \overline{\Xi}_j}{\partial \underline{z}_k}(\underline{z}, z, w) + \\ &+ W^j(z,w) - \sum_{k=1}^n Z^k(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial z_k}(\underline{z}, z, w) - \sum_{l=1}^d W^l(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial w_l}(\underline{z}, z, w)\Bigg]_{\underline{w} = -w + \overline{\Xi}(\underline{z}, z, w)} \\ &\qquad (j = 1 \cdots d), \end{aligned}$$

or else in greater details, when one really performs the said substitution:

$$\begin{aligned} 0 \equiv\; &\overline{W}^j\big(\underline{z},\, -w + \overline{\Xi}(\underline{z}, z, w)\big) - \sum_{k=1}^n \overline{Z}^k\big(\underline{z},\, -w + \overline{\Xi}(\underline{z}, z, w)\big)\,\frac{\partial \overline{\Xi}_j}{\partial \underline{z}_k}(\underline{z}, z, w) + \\ &+ W^j(z,w) - \sum_{k=1}^n Z^k(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial z_k}(\underline{z}, z, w) - \sum_{l=1}^d W^l(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial w}(\underline{z}, z, w) \\ &\qquad (j = 1 \cdots d). \end{aligned} \tag{5}$$

Interestingly enough, this condition may also be interpreted as saying that the complexified sum of vector fields:

$$\begin{aligned} (\overline{\mathsf{X}})^{e_c} + \mathsf{X}^{e_c} &:= \sum_{k=1}^n \overline{Z}_k(\underline{z}, \underline{w})\,\frac{\partial}{\partial \underline{z}_k} + \sum_{l=1}^d \overline{W}^l(\underline{z}, \underline{w})\,\frac{\partial}{\partial \underline{w}_l} + \sum_{k=1}^n Z^k(z,w)\,\frac{\partial}{\partial z_k} + \sum_{l=1}^d W^l(z,w)\,\frac{\partial}{\partial w_l} \\ &=: \underline{\mathsf{X}} + \mathsf{X} \end{aligned}$$

is tangent to M^{e_c}, *cf.* [28] for similar considerations in the broader context of completely integrable analytic systems of partial differential equations. But we must now analyze further what this condition really means.

To this aim, we may at first introduce the expansions of the coefficients of such a sought X with respect to the powers of z:

$$Z^k(z,w)=\sum_{\alpha\in\mathbb{N}^n} z^\alpha\, Z^{k,\alpha}(w) \quad \text{and} \quad W^l(z,w)=\sum_{\alpha\in\mathbb{N}^n} z^\alpha\, W^{l,\alpha}(w),$$

where the $Z^{k,\alpha}(w)$ and the $W^{l,\alpha}(w)$ are local holomorphic functions. We will show that the identical vanishing of the d equations (5) in $\mathbb{C}\{\underline{z},z,w\}$ is equivalent to a certain (in general complicated) linear system of partial differential equations involving the $\frac{\partial^{|\gamma|}Z^{k,\alpha}}{\partial w^\gamma}(w)$, the $\frac{\partial^{|\gamma'|}Z^{k',\alpha'}}{\partial w^{\gamma'}}(w)$, the $\frac{\partial^{|\gamma''|}W^{l,\alpha''}}{\partial w^{\gamma''}}(w)$ and the $\frac{\partial^{|\gamma'''|}W^{l',\alpha'''}}{\partial w^{\gamma'''}}(w)$.

Applying these expansions with respect to the powers of z, we get:

$$\begin{aligned}0\equiv &\sum_{\alpha\in\mathbb{N}^n}\underline{z}^\alpha\,\overline{W}^{j,\alpha}\big(-w+\overline{\Xi}\big)-\sum_{k=1}^n\sum_{\alpha\in\mathbb{N}^n}\underline{z}^\alpha\,\overline{Z}^{k,\alpha}\big(-w+\overline{\Xi}\big)\,\frac{\partial\overline{\Xi}_j}{\partial\underline{z}_k}(\underline{z},z,w)+\\ &+\sum_{\beta\in\mathbb{N}^n}z^\beta\,W^{j,\beta}(w)-\sum_{k=1}^n\sum_{\beta\in\mathbb{N}^n}z^\beta\,Z^{k,\beta}(w)\,\frac{\partial\overline{\Xi}_j}{\partial z_k}(\underline{z},z,w)-\sum_{l=1}^d\sum_{\beta\in\mathbb{N}^n}z^\beta\,W^{l,\beta}(w)\,\frac{\partial\overline{\Xi}_j}{\partial w_l}(\underline{z},z,w)\\ &\qquad (j=1\cdots d).\end{aligned}$$

Since in these equations, w is the argument both of all the $Z^{k,\beta}$ and of all the $W^{l,\beta}$ appearing in the second line, one should arrange that the same argument w takes place inside the functions $\overline{W}^{j,\alpha}$ and $\overline{Z}^{k,\alpha}$ appearing in the first line. Thus, one is led, for an arbitrary converging holomorphic power series $\overline{A}=\overline{A}(w)=\sum_{\gamma\in\mathbb{N}^d}\frac{\partial^{|\gamma|}\overline{A}}{\partial w^\gamma}(0)\,w^\gamma$, to apply the well known basic infinite Taylor series formula under the following slightly artificial form:

$$\begin{aligned}\overline{A}\big(-w+\overline{\Xi}\big)&=\overline{A}\big(w+(-2w+\overline{\Xi})\big)\\ &=\sum_{\gamma\in\mathbb{N}^d}\frac{\partial^{|\gamma|}\overline{A}}{\partial w^\gamma}(w)\,\frac{1}{\gamma!}\,\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma.\end{aligned}$$

When one does this, one transforms the first lines of the previous d equations as follows:

(6)

$$\begin{aligned}0\equiv &\sum_{\alpha\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d}\frac{1}{\gamma!}\,\underline{z}^\alpha\,\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma\,\frac{\partial^{|\gamma|}\overline{W}^{j,\alpha}}{\partial w^\gamma}(w)-\\ &-\sum_{k=1}^n\sum_{\alpha\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d}\frac{1}{\gamma!}\,\underline{z}^\alpha\,\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma\,\frac{\partial^{|\gamma|}\overline{Z}^{k,\alpha}}{\partial w^\gamma}(w)+\\ &+\sum_{\beta\in\mathbb{N}^n}z^\beta\,W^{j,\beta}(w)-\sum_{k=1}^n\sum_{\beta\in\mathbb{N}^n}z^\beta\,Z^{k,\beta}(w)\,\frac{\partial\overline{\Xi}_j}{\partial z_k}(\underline{z},z,w)-\sum_{l=1}^d\sum_{\beta\in\mathbb{N}^n}z^\beta\,W^{l,\beta}(w)\,\frac{\partial\overline{\Xi}_j}{\partial w_l}(\underline{z},z,w)\\ &\qquad (j=1\cdots d).\end{aligned}$$

But still, we must expand and reorganize everything in terms of the powers $\underline{z}^\alpha\,z^\beta$ of $(\underline{z},z)$. At first, we must do this for the multipowers $\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma$.

2.7. **Expansion, reorganization and associated linear** PDE **system.** To begin with, let us denote the $(\underline{z}, z)$-power series expansion of $-2w_j + \overline{\Xi}_j$ by:

$$-2w_j + \overline{\Xi}_j(\underline{z}, z, w) = \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\, \overline{\Xi}_{j,\alpha,\beta}^{\sim}(w) \qquad\qquad {\scriptstyle (j=1\cdots d)},$$

with the understanding that the coefficients of the expansion of $\overline{\Xi}_j$ would be denoted plainly $\overline{\Xi}_{j,\alpha,\beta}(w)$, without $\sim$ sign. Hence, as $\overline{\Xi}_j$ was assumed to be an $\mathrm{O}(2)$ at the beginning, we adopt the convention that in this right-hand side, the $\overline{\Xi}_{j,\alpha,\beta}^{\sim}(w)$ for $\alpha = \beta = 0$ comes not from $\overline{\Xi}_j$ itself, but from the supplementary first-order term $-2\,w_j$.

Thus, denoting:

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_d) \in \mathbb{N}^d,$$

we may expand explicitly the exponentiated product under consideration, and the intermediate, detailed computations read as follows:

$$\begin{aligned}
&\prod_{j=1}^d \big(-2\,w_j + \overline{\Xi}_j(\underline{z}, z, w)\big)^{\gamma_j} = \\
&= \prod_{j=1}^d \bigg(\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\, \overline{\Xi}_{j,\alpha,\beta}^{\sim}(w)\bigg)^{\gamma_j} \\
&= \prod_{j=1}^d \bigg[\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta \bigg(\sum_{\substack{\alpha_1+\cdots+\alpha_{\gamma_j}=\alpha \\ \beta_1+\cdots+\beta_{\gamma_j}=\beta}} \overline{\Xi}_{j,\alpha_1,\beta_1}^{\sim}(w)\cdots \overline{\Xi}_{j,\alpha_{\gamma_j},\beta_{\gamma_j}}^{\sim}(w)\bigg)\bigg] \\
&= \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta \bigg[\sum_{\substack{\alpha^1+\cdots+\alpha^d=\alpha \\ \beta_1+\cdots+\beta^d=\beta}} \sum_{\substack{\alpha_1^1+\cdots+\alpha_{\gamma_1}^1=\alpha^1 \\ \beta_1^1+\cdots+\beta_{\gamma_1}^1=\beta^1}} \cdots \sum_{\substack{\alpha_1^d+\cdots+\alpha_{\gamma_d}^d=\alpha^d \\ \beta_1^d+\cdots+\beta_{\gamma_d}^d=\beta_d}} \\
&\qquad\qquad \overline{\Xi}_{1,\alpha_1^1,\beta_1^1}^{\sim}(w)\cdots \overline{\Xi}_{1,\alpha_{\gamma_1}^1,\beta_{\gamma_1}^1}^{\sim}(w)\cdots\cdots \overline{\Xi}_{d,\alpha_1^d,\beta_1^d}^{\sim}(w)\cdots \overline{\Xi}_{d,\alpha_{\gamma_d}^d,\beta_{\gamma_d}^d}^{\sim}(w)\bigg] \\
&=: \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\, \mathscr{A}_{\alpha,\beta,\gamma}\Big(\big\{\overline{\Xi}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}^{\sim}(w)\big\}_{\widehat{j}\in\mathbb{N},\widehat{\alpha}\in\mathbb{N}^n,\widehat{\beta}\in\mathbb{N}^n}\Big),
\end{aligned}$$

where we introduce a collection of certain polynomial functions $\mathscr{A}_{\alpha,\beta,\gamma}$ of all the $\overline{\Xi}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}^{\sim}(w)$ that appear naturally in the large brackets of the penultimate equality, namely where we set:

$$\begin{aligned}
\mathscr{A}_{\alpha,\beta,\gamma}\Big(\big\{\overline{\Xi}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}^{\sim}(w)\big\}_{\widehat{j}\in\mathbb{N},\widehat{\alpha}\in\mathbb{N}^n,\widehat{\beta}\in\mathbb{N}^n}\Big) &:= \sum_{\substack{\alpha^1+\cdots+\alpha^d=\alpha \\ \beta_1+\cdots+\beta^d=\beta}} \sum_{\substack{\alpha_1^1+\cdots+\alpha_{\gamma_1}^1=\alpha^1 \\ \beta_1^1+\cdots+\beta_{\gamma_1}^1=\beta^1}} \cdots \sum_{\substack{\alpha_1^d+\cdots+\alpha_{\gamma_d}^d=\alpha^d \\ \beta_1^d+\cdots+\beta_{\gamma_d}^d=\beta_d}} \\
&\overline{\Xi}_{1,\alpha_1^1,\beta_1^1}^{\sim}(w)\cdots \overline{\Xi}_{1,\alpha_{\gamma_1}^1,\beta_{\gamma_1}^1}^{\sim}(w)\cdots\cdots \overline{\Xi}_{d,\alpha_1^d,\beta_1^d}^{\sim}(w)\cdots \overline{\Xi}_{d,\alpha_{\gamma_d}^d,\beta_{\gamma_d}^d}^{\sim}(w).
\end{aligned}$$

At present, coming back to the d equations (6) we left momentarily untouched, we see that in them, five sums are extant and we now want to expand and to reorganize properly each one of these terms as a $(\underline{z}, z)$-power series $\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\big(\cdots\big)$. For the sum in (6), we therefore compute, changing in advance the index α to α':

$$
\begin{aligned}
&\sum_{\alpha'\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d}\frac{1}{\gamma!}\,\underline{z}^{\alpha'}\,\frac{\partial^{|\gamma|}\overline{W}^{j,\alpha'}}{\partial w^\gamma}(w)\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma =\\
&=\sum_{\alpha'\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\frac{1}{\gamma!}\,\underline{z}^{\alpha'}\,\frac{\partial^{|\gamma|}\overline{W}^{j,\alpha'}}{\partial w^\gamma}(w)\sum_{\alpha''\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^{\alpha''}\,z^\beta\,\mathscr{A}_{\alpha'',\beta,\gamma}\Big(\big\{\overline{\Xi}^{\sim}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}(w)\big\}\Big)\\
&=\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\Big[\sum_{\gamma\in\mathbb{N}^d}\sum_{\alpha=\alpha'+\alpha''}\frac{1}{\gamma!}\,\mathscr{A}_{\alpha'',\beta,\gamma}\Big(\big\{\overline{\Xi}^{\sim}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}(w)\big\}\Big)\cdot\frac{\partial^{|\gamma|}\overline{W}^{j,\alpha'}}{\partial w^\gamma}(w)\Big].
\end{aligned}
\tag{7}
$$

The computations for the second sum in (6) are essentially exactly the same:

$$
\begin{aligned}
&-\sum_{k=1}^n\sum_{\alpha'\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d}\frac{1}{\gamma!}\,\underline{z}^{\alpha'}\,\frac{\partial^{|\gamma|}\overline{Z}^{k,\alpha'}}{\partial w^\gamma}(w)\big(-2w+\overline{\Xi}(\underline{z},z,w)\big)^\gamma\\
&=-\sum_{k=1}^n\sum_{\alpha'\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d}\frac{1}{\gamma!}\,\underline{z}^{\alpha'}\,\frac{\partial^{|\gamma|}\overline{Z}^{k,\alpha'}}{\partial w^\gamma}(w)\sum_{\alpha''\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^{\alpha''}\,z^\beta\,\mathscr{A}_{\alpha'',\beta,\gamma}\Big(\big\{\overline{\Xi}^{\sim}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}(w)\big\}\Big)\\
&=\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\Big[-\sum_{k=1}^n\sum_{\gamma\in\mathbb{N}^d}\sum_{\alpha'+\alpha''=\alpha}\frac{1}{\gamma!}\,\mathscr{A}_{\alpha'',\beta,\gamma}\Big(\big\{\overline{\Xi}^{\sim}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}(w)\big\}\Big)\cdot\frac{\partial^{|\gamma|}\overline{Z}^{k,\alpha'}}{\partial w^\gamma}(w)\Big].
\end{aligned}
\tag{8}
$$

The third sum in (6) is already almost well written, for we indeed have, if we denote by $\mathbf{0}=(0,\dots,0)\in\mathbb{N}^n$ the zero-multiindex:

$$
\sum_{\beta\in\mathbb{N}^n}z^\beta\,W^{j,\beta}(w)=\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\,\big[\delta_\alpha^{\mathbf{0}}\cdot W^{j,\beta}(w)\big],
\tag{9}
$$

where $\delta_{\mathsf{a}}^{\mathsf{b}}=0$ if $\mathsf{a}\neq\mathsf{b}$ and 1 if $\mathsf{a}=\mathsf{b}$. To transform the fourth sum in (6), we must at first compute, for each $k=1,\dots,n$ (and for each $j=1,\dots,d$), the first-order partial derivatives $\frac{\partial\overline{\Xi}_j}{\partial z_k}$, which gives, if we denote simply by $\mathbf{1}_k$ the multiindex $(0,\dots,1,\dots,0)$ of $\mathbb{N}^n$ with 1 at the k-th place and zero elsewhere:

$$
\begin{aligned}
\frac{\partial\overline{\Xi}_j}{\partial z_k}(\underline{z},z,w)&=\sum_{\alpha\in\mathbb{N}^n}\sum_{\substack{\beta\in\mathbb{N}^n\\ \beta_k\geqslant 1}}\underline{z}^\alpha\,\beta_k\,z^{\beta-\mathbf{1}_k}\,\overline{\Xi}^{\sim}_{j,\alpha,\beta}(w)\\
&=\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\,(\beta_k+1)\,\overline{\Xi}^{\sim}_{j,\alpha,\beta+\mathbf{1}_k}(w).
\end{aligned}
$$

Thanks to this, the fourth sum in (6) may be reorganized as wanted:

$$
\begin{aligned}
&-\sum_{k=1}^n\sum_{\beta'\in\mathbb{N}^n}z^{\beta'}\,Z^{k,\beta'}(w)\,\frac{\partial\overline{\Xi}_j}{\partial z_k}(\underline{z},z,w)=\\
&=-\sum_{k=1}^n\sum_{\beta'\in\mathbb{N}^n}z^{\beta'}\,Z^{k,\beta'}(w)\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta''\in\mathbb{N}^n}\underline{z}^\alpha\,z^{\beta''}\,(1+\beta''_k)\,\overline{\Xi}^{\sim}_{j,\alpha,\beta''+\mathbf{1}_k}(w)\\
&=\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\Big[-\sum_{k=1}^n\sum_{\beta'+\beta''=\beta}(\beta''_k+1)\,\overline{\Xi}^{\sim}_{j,\alpha,\beta''+\mathbf{1}_k}(w)\cdot Z^{k,\beta'}(w)\Big].
\end{aligned}
\tag{10}
$$

Lastly, in order to transform the fifth sum in (6), we must at first compute, for each $l=1,\dots,d$ (and for each $j=1,\dots,d$), the first-order partial derivatives $\frac{\partial\overline{\Xi}_j}{\partial w_l}$, and to this aim, we start by rewriting:

$$\overline{\Xi}_j(\underline{z},z,w) = 2w_j + \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\, \overline{\Xi}_{j,\alpha,\beta}^{\sim}(w),$$

whence it immediately follows:

$$\frac{\partial\overline{\Xi}_j}{\partial w_l}(\underline{z},z,w) = 2\,\delta_j^l + \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \underline{z}^\alpha\, z^\beta\,\big(\partial\overline{\Xi}_{j,\alpha,\beta}^{\sim}(w)/\partial w_l\big).$$

Thanks to this, the fifth sum in (6), too, may be reorganized appropriately:

(11)

$$\begin{aligned}
&-\sum_{l=1}^d\sum_{\beta'\in\mathbb{N}^n} z^{\beta'}\,W^{l,\beta'}(w)\,\frac{\partial\overline{\Xi}_j}{\partial w_l}(\underline{z},z,w) = \\
&= -\sum_{l=1}^d\sum_{\beta'\in\mathbb{N}^n} z^{\beta'}\,W^{l,\beta'}(w)\left[2\,\delta_j^l + \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta''\in\mathbb{N}^n}\underline{z}^\alpha\,z^{\beta''}\,\big(\partial\overline{\Xi}_{j,\alpha,\beta''}^{\sim}(w)/\partial w_l\big)\right] \\
&= \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\left[-2\,\delta_\alpha^{\mathbf{0}}\cdot W^{j,\beta}(w) - \sum_{l=1}^d\sum_{\beta'+\beta''=\beta}\big(\partial\overline{\Xi}_{j,\alpha,\beta''}^{\sim}(w)/\partial w_l\big)\cdot W^{l,\beta'}(w)\right].
\end{aligned}$$

Summing up these five reorganized sums appearing in (6) as a double sum $\sum_\alpha\sum_\beta \underline{z}^\alpha\, z^\beta\,\big(\mathsf{coeff}_{j,\alpha,\beta}\big)$, and equating to zero all the obtained coefficients (7), (8), (9), (10) and (11), we deduce the following fundamental statement.

Theorem 2.1. *Let M be a generic real analytic CR-submanifold of $\mathbb{C}^{n+d}$ having positive codimension $d\geqslant 1$ and positive CR dimension $n\geqslant 1$ which is represented, in local holomorphic coordinates $(z,w)=(z_1,\dots,z_n,w_1,\dots,w_d)$ by d complex defining equations of the shape:*

$$\overline{w}_j + w_j = \overline{\Xi}_j(\overline{z},z,w) \qquad (j=1\cdots d),$$

denote by $(\underline{z},\underline{w})$ the extrinsic complexifications of the antiholomorphic variables $(\overline{z},\overline{w})$ and introduce the power series expansion with respect to the variables $(\underline{z},z)$:

$$-2w_j + \overline{\Xi}_j(\underline{z},z,w) =: \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\underline{z}^\alpha\,z^\beta\,\overline{\Xi}_{j,\alpha,\beta}^{\sim}(w) \qquad (j=1\cdots d).$$

For every multiindex $\alpha\in\mathbb{N}^n$, every multiindex $\beta\in\mathbb{N}^n$ and every multiindex $\gamma\in\mathbb{N}^d$, introduce also the explicit universal polynomial:

$$\begin{aligned}
\mathscr{A}_{\alpha,\beta,\gamma}\Big(\big\{\overline{\Xi}_{\widehat{j},\widehat{\alpha},\widehat{\beta}}^{\sim}(w)\big\}_{\widehat{j}\in\mathbb{N},\widehat{\alpha}\in\mathbb{N}^n,\widehat{\beta}\in\mathbb{N}^n}\Big) := \sum_{\substack{\alpha^1+\cdots+\alpha^d=\alpha\\ \beta_1+\cdots+\beta^d=\beta}}\ \sum_{\substack{\alpha_1^1+\cdots+\alpha_{\gamma_1}^1=\alpha^1\\ \beta_1^1+\cdots+\beta_{\gamma_1}^1=\beta^1}}\cdots\sum_{\substack{\alpha_1^d+\cdots+\alpha_{\gamma_d}^d=\alpha^d\\ \beta_1^d+\cdots+\beta_{\gamma_d}^d=\beta_d}} \\
\overline{\Xi}_{1,\alpha_1^1,\beta_1^1}^{\sim}(w)\cdots\overline{\Xi}_{1,\alpha_{\gamma_1}^1,\beta_{\gamma_1}^1}^{\sim}(w)\cdots\cdots\overline{\Xi}_{d,\alpha_1^d,\beta_1^d}^{\sim}(w)\cdots\overline{\Xi}_{d,\alpha_{\gamma_d}^d,\beta_{\gamma_d}^d}^{\sim}(w).
\end{aligned}$$

Then a general holomorphic vector field:

$$\mathsf{X} = \sum_{k=1}^n Z^k(z,w)\,\frac{\partial}{\partial z_k} + \sum_{l=1}^d W^l(z,w)\,\frac{\partial}{\partial w_l}$$

is an infinitesimal CR-automorphism of M *belonging to* $\mathfrak{hol}(M)$*, namely it has the property that* $\overline{\mathsf{X}} + \mathsf{X}$ *is tangent to* M if and only if, *for every* $j = 1, \dots, d$*, for every* $\alpha \in \mathbb{N}^n$ *and for every* $\beta \in \mathbb{N}^n$*, the following linear holomorphic partial differential equation:*

$$\boxed{\begin{aligned}
0 \equiv & \sum_{\gamma \in \mathbb{N}^d} \sum_{\alpha = \alpha' + \alpha''} \frac{1}{\gamma!} \mathscr{A}_{\alpha'', \beta, \gamma} \Big(\big\{ \overline{\Xi}^{\sim}_{\widehat{j}, \widehat{\alpha}, \widehat{\beta}}(w) \big\}_{\widehat{j} \in \mathbb{N}, \widehat{\alpha} \in \mathbb{N}^n, \widehat{\beta} \in \mathbb{N}^n} \Big) \cdot \frac{\partial^{|\gamma|} \overline{W}^{j, \alpha'}}{\partial w^\gamma}(w) - \\
& - \sum_{k=1}^n \sum_{\gamma \in \mathbb{N}^d} \sum_{\alpha' + \alpha'' = \alpha} \frac{1}{\gamma!} \mathscr{A}_{\alpha'', \beta, \gamma} \Big(\big\{ \overline{\Xi}^{\sim}_{\widehat{j}, \widehat{\alpha}, \widehat{\beta}}(w) \big\} \Big) \cdot \frac{\partial^{|\gamma|} \overline{Z}^{k, \alpha'}}{\partial w^\gamma}(w) + \\
& + \delta_\alpha^{\mathbf{0}} \cdot W^{j, \beta}(w) - \\
& - \sum_{k=1}^n \sum_{\beta' + \beta'' = \beta} (\beta''_k + 1) \, \overline{\Xi}^{\sim}_{j, \alpha, \beta'' + \mathbf{1}_k}(w) \cdot Z^{k, \beta'}(w) - \\
& - 2 \, \delta_\alpha^{\mathbf{0}} \cdot W^{j, \beta}(w) - \sum_{l=1}^d \sum_{\beta' + \beta'' = \beta} \big(\partial \overline{\Xi}^{\sim}_{j, \alpha, \beta''}(w) / \partial w_l \big) \cdot W^{l, \beta'}(w)
\end{aligned}}$$

which is linear *with respect to the partial derivatives:*

$$\frac{\partial^{|\gamma|} Z^{k, \alpha}}{\partial w^\gamma}(w), \qquad \frac{\partial^{|\gamma'|} Z^{k', \alpha'}}{\partial w^{\gamma'}}(w), \qquad \frac{\partial^{|\gamma''|} W^{l, \alpha''}}{\partial w^{\gamma''}}(w), \qquad \frac{\partial^{|\gamma'''|} W^{l', \alpha'''}}{\partial w^{\gamma'''}}(w)$$

is satisfied identically in $\mathbb{C}\{w\}$ *by the four families of functions:*

$$Z^{k, \alpha}(w), \qquad Z^{k', \alpha'}(w), \qquad W^{l, \alpha'''}(w), \qquad W^{l', \alpha'''}(w).$$

depending only upon the d *holomorphic variables* $(w_1, \dots, w_d)$.

Then the resolution of this linear system of holomorphic partial differential equations (having nonconstant coefficients in general) is often delicate when dealing with concrete, specific functions $\overline{\Xi}_j(\underline{z}, z, w)$. Of course, starting from an M of equation $v = \varphi(x, y, u)$ with $T_0 M = \{\operatorname{Im} w = 0\}$, instead of $T_0 M = \{\operatorname{Re} w = 0\}$, the same process of extracting linear partial differential equations providing (after resolution) access to all $\mathsf{X} \in \mathfrak{hol}(M)$ may be conducted quite similarly, the only difference being that an i-factor comes regularly into play, for the complex defining equations of M must then be thought to be of the general form:

$$w_j = \overline{w}_j + i \, \Xi_j(z, \overline{z}, \overline{w}) \qquad \scriptstyle (j = 1 \cdots d),$$

because each $w_j - \overline{w}_j = v_j$ is purely real, or because a reality condition like (2) must hold true. The presence of the i-factor is especially visible in the case where the M under consideration is of the particular (and quite convenient) form, sometimes called *rigid* in the literature, where the right-hand side functions Ξ_j are *completely independent* of the variable $w \in \mathbb{C}^d$, since in this case if one writes:

$$w_j = \overline{w}_j + i \, \Xi_j(z, \overline{z}) \qquad \scriptstyle (j = 1 \cdots d),$$

it is clear that each right-hand side function $\Xi_j(z, \overline{z})$ must be purely real, namely must satisfy:

$$\Xi_j(z, \underline{z}) \equiv \overline{\Xi}_j(\underline{z}, z) \qquad \scriptstyle (j = 1 \cdots d)$$

identically in $\mathbb{C}\{z, \underline{z}\}$. A concrete example is on.

3. CR symmetries of the Heisenberg Sphere $\mathbb{H}^3 \subseteq \mathbb{C}^2$

3.1. **Infinitesimal CR automorphisms of $\mathbb{H}^3$.** We now consider the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$, equipped with coordinates (z, w), of equation:

$$0 = w - \overline{w} - 2i\, z\overline{z}.$$

A local $(1,0)$ vector field defined in a neighborhood of the origin:

$$\mathsf{X} = Z(z,w)\,\frac{\partial}{\partial z} + W(z,w)\,\frac{\partial}{\partial w}$$

having *holomorphic* coefficients $Z(z,w)$ and $W(z,w)$ is an *infinitesimal CR automorphism* of M if and only if $X + \underline{X}$ is tangent to the extrinsic complexification M^{e_c}, that is to say, if and only if the following equation:

$$0 \equiv \big[W - 2i\underline{z}\, Z - \overline{W} - 2iz\,\overline{Z}\big]_{w=\underline{w}+2i\,z\underline{z}}$$

holds identically in $\mathbb{C}\{z, \underline{z}, \underline{w}\}$, that is to say if again, and only if:

$$0 \equiv W(z,\, \underline{w} + 2i\,z\underline{z}) - 2i\underline{z}\,Z(z,\, \underline{w} + 2i\,z\underline{z}) - \overline{W}(\underline{z},\, \underline{w}) - 2iz\,\overline{Z}(\underline{z}, \underline{w}). \tag{12}$$

Since the two coefficients Z and W of L are analytic, we may expand them with respect to the powers of z:

$$Z(z,w) = \sum_{k\in\mathbb{N}} z^k\, Z_k(w) \qquad \text{and} \qquad W(z,w) = \sum_{k\in\mathbb{N}} z^k\, W_k(z,w),$$

and the fundamental equation (12) just written becomes:

$$\begin{aligned} 0 \equiv \sum_{k\in\mathbb{N}} z^k\, W_k(\underline{w} + 2i\,z\underline{z}) - 2i\underline{z} \sum_{k\in\mathbb{N}} z^k\, Z_k(\underline{w} + 2i\,z\underline{z}) - \\ - \sum_{k\in\mathbb{N}} \underline{z}^k\, \overline{W}^k(\underline{w}) - 2iz \sum_{k\in\mathbb{N}} \underline{z}^k\, \overline{Z}_k(\underline{w}). \end{aligned}$$

Furthermore, if $A = A(\underline{w}) = \sum_{l\in\mathbb{N}} A_{\underline{w}^l}(0)\, \frac{1}{l!}\, \underline{w}^l$ is a function holomorphic with respect to $\underline{w}$ near the origin, where $A_{\underline{w}}, A_{\underline{w}^2}, \dots, A_{\underline{w}^l}$ denote (partial) derivatives, we may yet expand:

$$A(\underline{w} + 2i\,z\underline{z}) = \sum_{l\in\mathbb{N}} A_{\underline{w}^l}(\underline{w})\, (2i\,z\underline{z})^l\, \frac{1}{l!}, \tag{13}$$

and here, this gives us:

$$\begin{aligned} 0 \equiv \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} \bigg(z^k\, (2i\,z\underline{z})^l\, \frac{1}{l!}\, W_{k,\underline{w}^l}(\underline{w}) - 2i\,\underline{z}\, z^k\, (2i\,z\underline{z})^l\, \frac{1}{l!}\, Z_{k,\underline{w}^l}(\underline{w}) \bigg) - \\ - \sum_{k\in\mathbb{N}} \underline{z}^k \Big(\overline{W}_k(\underline{w}) + 2i\,z\,\overline{Z}_k(\underline{w}) \Big). \end{aligned}$$

In this equation, the coefficients of the monomials $\underline{z}^k$ for all $k \geqslant 2$ and the coefficients of the monomials $z\,\underline{z}^{k'}$ for all $k' \geqslant 3$ must vanish, and this simply yields:

$$0 \equiv \overline{W}_k(\underline{w}) \quad \text{for all} \quad k \geqslant 2 \qquad \text{and} \qquad 0 \equiv \overline{Z}_{k'}(\underline{w}) \quad \text{for all} \quad k' \geqslant 3.$$

Consequently, the two coefficients Z and W of our infinitesimal CR automorphism greatly simplify, and they receive the (truncated) form:

$$\left[\begin{aligned} Z(z,w) &= Z_0(w) + z\,Z_1(w) + z^2\,Z_2(w) \\ W(z,w) &= W_0(w) + z\,W_1(w). \end{aligned}\right.$$

After this simplification, the fundamental equation (12) becomes:

$$\begin{aligned} 0 \equiv W_0(\underline{w} + 2i\,z\underline{z}) &+ z\,W_1(\underline{w} + 2i\,z\underline{z}) - \\ &- 2i\underline{z}\,Z_0(\underline{w} + 2i\,z\underline{z}) - 2iz\underline{z}\,Z_1(\underline{w} + 2i\,z\underline{z}) - 2iz^2\underline{z}\,Z_2(\underline{w} + 2i\,z\underline{z}) - \\ &- \overline{W}_0(\underline{w}) - \underline{z}\,\overline{W}_1(\underline{w}) - 2iz\,\overline{Z}_0(\underline{w}) - 2iz\underline{z}\,\overline{Z}(\underline{w}) - 2iz\underline{z}^2\,\overline{Z}_2(\underline{w}). \end{aligned}$$

We now expand all the series $A(\underline{w} + 2i\,z\underline{z})$ appearing in the first two lines, using (13):

$$\begin{aligned} 0 \equiv W_0(\underline{w}) &+ 2iz\underline{z}\,W_{0,\underline{w}}(\underline{w}) - 4z^2\underline{z}^2\,\tfrac{1}{2!}\,W_{0,\underline{w}^2}(\underline{w}) - 8iz^3\underline{z}^3\,\tfrac{1}{3!}\,W_{0,\underline{w}^3}(\underline{w}) + \cdots \\ &+ z\,W_1(\underline{w}) + 2iz^2\underline{z}\,W_{1,\underline{w}}(\underline{w}) - 4z^3\underline{z}^2\,\tfrac{1}{2!}\,W_{1,\underline{w}^2}(\underline{w}) - \cdots \\ &-\ 2i\underline{z}\,Z_0(\underline{w}) + 4\,z\underline{z}^2\,Z_{0,\underline{w}}(\underline{w}) + 8iz^2\underline{z}^3\,\tfrac{1}{2!}\,Z_{0,\underline{w}^2}(\underline{w}) + \cdots \\ &-\ 2i\underline{z}z\,Z_1(\underline{w}) + 4z^2\underline{z}^2\,Z_{1,\underline{w}}(\underline{w}) + 8iz^3\underline{z}^3\tfrac{1}{2!}\,Z_{1,\underline{w}^2}(\underline{w}) + \cdots \\ &-\ 2i\underline{z}z^2 Z_2(\underline{w}) + 4z^3\underline{z}^2\,Z_{2,\underline{w}} + \cdots \\ &-\ \overline{W}_0(\underline{w}) - \underline{z}\,\overline{W}_1(\underline{w}) - 2iz\,\overline{Z}_0(\underline{w}) - 2iz\underline{z}\,\overline{Z}_1(\underline{w}) - 2iz\underline{z}^2\,\overline{Z}_2(\underline{w}). \end{aligned}$$

Now, we extract the coefficients of the monomials $z^\mu\underline{z}^\nu$ for small values of μ and ν, and these coefficients must all vanish identically in $\mathbb{C}\{\underline{w}\}$. What is left out in the cdots will not be useful to us.

First of all, for (μ,ν) equal to $(0,0)$ and to $(1,0)$, we get two equations:

$$0 \equiv W_0(\underline{w}) - \overline{W}_0(\underline{w}) \tag{14}$$

$$0 \equiv W_1(\underline{w}) - 2i\,\overline{Z}_0(\underline{w}), \tag{15}$$

holding identically in $\mathbb{C}\{\underline{w}\}$, while for $(\mu,\nu) = (0,1)$, we get $0 \equiv -\overline{W}_1(\underline{w}) - 2i\,Z_0(\underline{w})$ which is fully equivalent to (15), after conjugation and replacement of the variable w by $\underline{w}$ (a power series $\varphi(\underline{w})$ is identically zero if and only if $\varphi(w)$ is identically zero). Next, for $(\mu,\nu) = (2,0)$, nothing comes, while for $(\mu,\nu) = (1,1)$, we obtain:

$$0 \equiv 2i\,W_{0,\underline{w}}(\underline{w}) - 2i\,Z_1(\underline{w}) - 2i\,\overline{Z}_1(\underline{w}). \tag{16}$$

Next, for $(2,1)$ and for $(1,2)$ we obtain:

$$0 \equiv 2i\,W_{1,\underline{w}}(\underline{w}) - 2i\,Z_2(\underline{w}) \tag{17}$$

and: $0 \equiv 4\,Z_{0,w}(\underline{w}) - 2i\,\overline{Z}_2(\underline{w})$, but this second equation visibly follows from the ones already obtained, hence will be disregarded. Next, for $(2,2)$, for $(2,3)$, for

$(3,2)$ and for $(3,3)$, we obtain:

$$0 \equiv -4\,\tfrac{1}{2!}\,W_{0,\underline{w}^2}(\underline{w}) + 4\,Z_{1,\underline{w}}(\underline{w}) \tag{18}$$

$$0 \equiv 8i\,\tfrac{1}{2!}\,Z_{0,\underline{w}^2}(\underline{w}) \tag{19}$$

$$0 \equiv -4\,\tfrac{1}{2!}\,W_{1,\underline{w}^2}(\underline{w}) + 4\,Z_{2,\underline{w}}(\underline{w}) \tag{20}$$

$$0 \equiv -8i\,\tfrac{1}{3!}\,W_{0,\underline{w}^3}(\underline{w}) - 8i\,\tfrac{1}{2!}\,Z_{1,\underline{w}^2}(\underline{w}). \tag{21}$$

Clearly, (19) yields that Z_0 is affine:

$$Z_0(w) = \mathsf{z}_{0,0} + \mathsf{z}_{0,1}\,w, \tag{22}$$

where $\mathsf{z}_{0,0} = \mathsf{x}_{0,0} + i\,\mathsf{y}_{0,0}$ and $\mathsf{z}_{0,1} = \mathsf{x}_{0,1} + i\,\mathsf{y}_{0,1}$ are *complex* constants in $\mathbb{C}$. From (15), it then follows immediately that:

$$W_1(w) = 2i\,\overline{\mathsf{z}}_{0,0} + 2i\,\overline{\mathsf{z}}_{0,1}\,w. \tag{23}$$

Next, differentiating (18) once with respect to $\underline{w}$ and comparing to (21), we get:

$$0 \equiv W_{0,\underline{w}^3}(\underline{w}) \qquad \text{and} \qquad 0 \equiv Z_{1,\underline{w}}(\underline{w}).$$

It follows firstly that W_0 is quadratic:

$$W_0(w) = \mathsf{u}_{0,0} + \mathsf{u}_{0,1}\,w + \mathsf{u}_{0,2}\,w^2, \tag{24}$$

but taking account of (14), we see that the three appearing coefficients $\mathsf{u}_{0,0}$, $\mathsf{u}_{0,1}$, $\mathsf{u}_{0,2}$ must even all be *real*. Secondly, it follows that $Z_1(w) = \mathsf{z}_{1,0} + \mathsf{z}_{1,1}\,w$ is affine, but moreover, taking in addition account of (16) and of (18), we see that:

$$Z_1(w) = \tfrac{1}{2}\,\mathsf{u}_{0,1} + i\,\mathsf{y}_{1,0} + \mathsf{u}_{0,2}\,w. \tag{25}$$

Finally, (17) and (23) give that $Z_2(w)$ is constant:

$$Z_2(w) = 2\,\mathsf{y}_{0,1} + 2i\,\mathsf{x}_{0,1}.$$

3.2. **Solution.** The eight real constants found in this way:

$$\mathsf{x}_{0,0},\quad \mathsf{y}_{0,0},\quad \mathsf{x}_{0,1},\quad \mathsf{y}_{0,1},\quad \mathsf{u}_{0,0},\quad \mathsf{u}_{0,1},\quad \mathsf{u}_{0,2},\quad \mathsf{y}_{1,0}$$

give us eight $\mathbb{R}$-linearly independent infinitesimal automorphisms of the Heisenberg sphere, when one sets one of these constants equal to 1, while the 7 remaining constants are set equal to 0:

$$\begin{aligned}
&\partial_z + 2iz\,\partial_w\\
&i\,\partial_z + 2z\,\partial_w\\
&(w + 2iz^2)\,\partial_z + 2izw\,\partial_w\\
&(iw + 2z^2)\,\partial_z + 2zw\,\partial_w\\
&\partial_w\\
&\tfrac{1}{2}z\,\partial_z + w\,\partial_w\\
&zw\,\partial_z + w^2\,\partial_w\\
&iz\,\partial_z.
\end{aligned}$$

By straightforward computations, one verifies that indeed the real part of $\mathsf{X} + \underline{\mathsf{X}}$, where X is any of these eight holomorphic vector fields, is tangent to M^{e_c}; for instance, for the third vector field, we get:

$$\begin{aligned} & 4z^2\underline{z} - 4z\underline{z}^2 - 2i\underline{z}w - 2iz\underline{w} + 2izw + 2i\underline{z}\underline{w} \\ & = (w - \underline{w} - 2iz\underline{z})[2iz - 2i\underline{z}], \end{aligned}$$

which identifies to the equation of M^{e_c} multiplied by a factor, hence vanishes on M^{e_c}.

3.3. **Homogeneities and graded structure.** The anisotropic real dilation $(z, w) \longmapsto (c\,z,\, c^2\,w)$ with $c \in \mathbb{R}$ visibly stabilizes $\mathbb{H}^3$, hence it is natural to ascribe homogeneity 1 to the variable z and homogeneity 2 to the variable w. Accordingly, ∂_z and ∂_w have homogeneity -1 and -2, respectively, and a holomorphic field like $zw\,\partial_w$, for instance, has homogeneity $1 + 2 - 2 = 1$. One may thus list the eight generators found above according to their homogeneities, which take the values -2, -1, 0, 1 and 2 and this conducts us to represent:

$$\mathfrak{hol}(\mathbb{H}^3) = \mathfrak{h}_{-2} \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

as a direct sum of five components of dimensions 1, 2, 2, 2, 1 defined by:

$$\begin{aligned} \mathfrak{h}_{-2} &= \mathbb{R}\,\mathsf{T}, \qquad \mathfrak{h}_{-1} = \mathbb{R}\,\mathsf{H}_1 \oplus \mathbb{R}\,\mathsf{H}_2 \\ \mathfrak{h}_0 &= \mathbb{R}\,\mathsf{D} \oplus \mathbb{R}\,\mathsf{R}, \\ \mathfrak{h}_1 &= \mathbb{R}\,\mathsf{I}_1 \oplus \mathbb{R}\,\mathsf{I}_2, \qquad \mathfrak{h}_2 = \mathbb{R}\,\mathsf{J}. \end{aligned}$$

where:

$$\mathfrak{h}_{-2}\colon\ \big\{\mathsf{T} := \partial_w \qquad\qquad \mathfrak{h}_{-1}\colon\ \begin{cases} \mathsf{H}_1 := \partial_z + 2iz\,\partial_w \\ \mathsf{H}_2 := i\,\partial_z + 2z\,\partial_w \end{cases}$$

$$\mathfrak{h}_0\colon\ \begin{cases} \mathsf{D} := z\,\partial_z + 2w\,\partial_w \\ \mathsf{R} := iz\,\partial_z \end{cases}$$

$$\mathfrak{h}_1\colon\ \begin{cases} \mathsf{I}_1 := (w + 2iz^2)\,\partial_z + 2izw\,\partial_w \\ \mathsf{I}_2 := (iw + 2z^2)\,\partial_z + 2zw\,\partial_w \end{cases} \qquad\qquad \mathfrak{h}_2\colon\ \big\{\mathsf{J} := zw\,\partial_z + w^2\,\partial_w.$$

Here with $t \in \mathbb{R}$, the flow $(z, w) \mapsto (z,\, w + t)$ of T is $\underline{\text{t}}$ransversal to the complex tangent bundle T^cM, spanned by $\mathrm{Re}\,\mathscr{L}$ and by $\mathrm{Im}\,\mathscr{L}$, where:

$$\mathscr{L} := \partial_z + 2i\overline{z}\,\partial_w;$$

the flows of H_1 and H_2, namely $(z+t,\ w+2izt+it^2)$ and $(z+it,\ w+2zt+it^2)$ are somewhat $\underline{\text{h}}$orizontal; the flow of D is just the $\underline{\text{d}}$ilation $(e^t z,\ e^{2t}\,w)$; the flow of R is just the imaginary $\underline{\text{r}}$otation of the z-coordinate $(e^{it}z,\ w)$. On the other hand, it is known since Poincaré [35] that any holomorphic automorphism of the Heisenberg sphere fixing the origin is a fractional linear transformation of the general form:

$$(z,\ w) \longmapsto \Big(\tfrac{c(z+aw)}{1-2i\overline{a}z-(r+ia\overline{a})w},\ \ \tfrac{\rho w}{1-2i\overline{a}z-(r+ia\overline{a})w}\Big),$$

where $c \in \mathbb{C}$, $a \in \mathbb{C}$, $r \in \mathbb{R}$ and $\rho \in \mathbb{R}$. Such a general expression may be recovered by concatenating the eight flows, after a change of parameters. However, as understood originally by Lie himself ([13, 31]), except in some specific situations, it is essentially useless to explicit the finite equations of a local transformation group, because the infinitesimal description shows better the structures.

With the convention that $\mathfrak{h}_k = \{0\}$ for either $k \leqslant -3$ or $k \geqslant 3$, one may then verify the property that:

$$\big[\mathfrak{h}_{k_1}, \mathfrak{h}_{k_2}\big] \subset \mathfrak{h}_{k_1+k_2}$$

for any two $k_1 \leqslant k_2 \in \mathbb{Z}$, and more precisely, this fact follows by inspecting the full commutator table between these eight generators of $\mathfrak{hol}(\mathbb{H}^3)$:

	T	H_1	H_2	D	R	I_1	I_2	J
T	0	0	0	$2\,\mathsf{T}$	0	H_1	H_2	D
H_1	$*$	0	$4\,\mathsf{T}$	H_1	H_2	$6\,\mathsf{R}$	$2\,\mathsf{D}$	I_1
H_2	$*$	$*$	0	H_2	$-\mathsf{H}_1$	$-2\,\mathsf{D}$	$6\,\mathsf{R}$	I_2
D	$*$	$*$	$*$	0	0	I_1	I_2	$2\,\mathsf{J}$
R	$*$	$*$	$*$	$*$	0	$-\mathsf{I}_2$	I_1	0
I_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{J}$	0
I_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

Clearly also, the isotropy algebra of the origin is just the nonnegative part of the sum:

$$\mathfrak{hol}(\mathbb{H}^3, 0) = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

4. Tanaka Prolongation

4.1. **The prolongation procedure in the CR context.** Consider a finite-dimensional graded real Lie algebra indexed by negative integers:

$$\mathfrak{g}_- = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

satisfying $[\mathfrak{g}_{-l_1}, \mathfrak{g}_{-l_2}] \subset \mathfrak{g}_{-l_1-l_2}$ with the convention that $\mathfrak{g}_k = 0$ for $k \leqslant -\mu - 1$. Following [39], $\mathfrak{g}_-$ will be said to be *of μ-th kind*. Assume that there is a complex structure $J \colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ such that $J^2 = -\mathrm{Id}$, whence $\mathfrak{g}_{-1}$ is even-dimensional and bears a natural structure of a complex vector space. Tanaka's prolongation of $\mathfrak{g}_-$ is an algebraic procedure which generates a certain larger graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots$$

in the following way.

By definition, the order-zero component $\mathfrak{g}_0$ consists of all linear endomorphisms $\mathsf{d} \colon \mathfrak{g}_- \to \mathfrak{g}_-$ which preserve gradation: $\mathsf{d}(\mathfrak{g}_k) \subset \mathfrak{g}_k$, which respect the complex structure: $\mathsf{d}(J\,\mathsf{x}) = J\mathsf{d}(\mathsf{x})$ for all $\mathsf{x} \in \mathfrak{g}_{-1}$ and which are *derivations*, namely satisfy $\mathsf{d}([\mathsf{y}, \mathsf{z}]) = [\mathsf{d}(\mathsf{x}), \mathsf{y}] + [\mathsf{x}, \mathsf{d}(\mathsf{y})]$ for every $\mathsf{y}, \mathsf{z} \in \mathfrak{g}_-$. Then the bracket between a $\mathsf{d} \in \mathfrak{g}_0$ and an $\mathsf{x} \in \mathfrak{g}_-$ is simply defined by $[\mathsf{d}, \mathsf{x}] := \mathsf{d}(\mathsf{x})$, while the bracket between *two* elements $\mathsf{d}', \mathsf{d}'' \in \mathfrak{g}_0$ is defined to be the commutator $\mathsf{d}' \circ \mathsf{d}'' - \mathsf{d}'' \circ \mathsf{d}'$ between endomorphisms. One checks at once that Jacobi relations hold, hence $\mathfrak{g}_- \oplus \mathfrak{g}_0$ becomes a true Lie algebra.

By contrast, for any $l \geqslant 1$, no constraint with respect to J is required. Assuming that the components $\mathfrak{g}_{l'}$ are already constructed for any $l' \leqslant l - 1$, the l-th component $\mathfrak{g}_l$ of the prolongation consists of l-shifted graded linear morphisms $\mathfrak{g}_- \to \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ that are derivations, namely:

(26)
$$\mathfrak{g}_l = \Big\{ \mathsf{d} \in \bigoplus_{k \leqslant -1} \mathrm{Lin}(\mathfrak{g}_k,\, \mathfrak{g}_{k+l}) \colon \mathsf{d}([\mathsf{y},\, \mathsf{z}]) = [\mathsf{d}(\mathsf{y}),\, \mathsf{z}] + [\mathsf{y},\, \mathsf{d}(\mathsf{z})], \quad \forall\, \mathsf{y},\, \mathsf{z} \in \mathfrak{g}_- \Big\}.$$

Now, for $\mathsf{d} \in \mathfrak{g}_k$ and $\mathsf{e} \in \mathfrak{g}_l$, by induction on the integer $k + l \geqslant 0$, one defines the bracket $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_{k+l} \otimes \mathfrak{g}_-^*$ by:

(27)
$$[\mathsf{d},\, \mathsf{e}](\mathsf{x}) = \big[[\mathsf{d},\, \mathsf{x}],\, \mathsf{e}\big] + \big[\mathsf{d},\, [\mathsf{e},\, \mathsf{x}]\big] \qquad \text{for } \mathsf{x} \in \mathfrak{g}_-.$$

One notes that, for $k = l = 0$, this definition coincides with the above one for $[\mathfrak{g}_0,\, \mathfrak{g}_0]$. It follows by induction ([39, 41]) that $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_{k+l}$ and that with this bracket, the sum $\mathfrak{g}_- \bigoplus_{k \geqslant 1} \mathfrak{g}_k$ becomes a graded Lie algebra, because the general Jacobi identity:

$$0 = \big[[\mathsf{d},\, \mathsf{e}],\, \mathsf{f}\big] + \big[[\mathsf{f},\, \mathsf{d}],\, \mathsf{e}\big] + \big[[\mathsf{e},\, \mathsf{f}],\, \mathsf{d}\big]$$

for $\mathsf{d} \in \mathfrak{g}_k$, $\mathsf{e} \in \mathfrak{g}_l$ and $\mathsf{f} \in \mathfrak{g}_m$ follows by definition when one of k, l, m is negative, and can be shown by induction on the integer $k + l + m \geqslant 0$ when all of k, l, m are nonnegative.

4.2. **The Heisenberg algebra.** The symbol Lie algebra $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ associated to any Levi nondegenerate CR manifold $M^3 \subset \mathbb{C}^2$ equipped with the distribution T^cM is three-dimensional, with $\mathfrak{g}_{-2} = \mathbb{R}\,\mathsf{x}_1$, with $\mathfrak{g}_{-1} = \mathbb{R}\,\mathsf{x}_2 \oplus \mathbb{R}\mathsf{x}_3$ with $\mathsf{x}_3 = J(\mathsf{x}_2)$ and with only nonzero Lie bracket $[\mathsf{x}_2,\, \mathsf{x}_3] = 4\,\mathsf{x}_1$. If one disregards J, such a $\mathfrak{g}_-$ is the unique irreducible three-dimensional nilpotent Lie algebra, denoted $\mathfrak{n}_3^1$ in [17]. Now, what is the Tanaka prolongation of $\mathfrak{g}_-$?

By definition, an element of $\mathfrak{g}_0$ is a derivation $\mathsf{d} \in (\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-2}^*) \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}^*)$, hence it writes:

$$\mathsf{d}(\mathsf{x}_1) = k\mathsf{x}_1, \qquad \mathsf{d}(\mathsf{x}_2) = r_1\mathsf{x}_2 + r_2\mathsf{x}_3, \qquad \mathsf{d}(\mathsf{x}_3) = r_3\mathsf{x}_2 + r_4\mathsf{x}_3,$$

for some five real, unknown constants. But because d preserves the complex structure J on $\mathfrak{g}_{-1}$, one also has $\mathsf{d}(J\mathsf{x}_2) = J\mathsf{d}(\mathsf{x}_2)$, i.e. $r_3\mathsf{x}_2 + r_4\mathsf{x}_3 = r_1\mathsf{x}_3 - r_2\mathsf{x}_2$, that is to say: $r_1 = r_4$ and $r_2 = -r_3$. Moreover, applying the derivation property of d, one must have:

$$\begin{aligned} 4k\,\mathsf{x}_1 = 4\mathsf{d}(\mathsf{x}_1) = \mathsf{d}([\mathsf{x}_2, \mathsf{x}_3]) &= [\mathsf{d}(\mathsf{x}_2), \mathsf{x}_3] + [\mathsf{x}_2, \mathsf{d}(\mathsf{x}_3)] \\ &= [r_1\mathsf{x}_2 + r_2\mathsf{x}_3, \mathsf{x}_3] + [\mathsf{x}_2, r_3\mathsf{x}_2 + r_4\mathsf{x}_3] \\ &= 4r_1\mathsf{x}_1 + 4r_4\mathsf{x}_1, \end{aligned}$$

which yields that $k = r_1 + r_4$. These three linear equations solve as $r_3 = -r_2$, $r_4 = r_1$ and $k = 2r_1$ with free r_1 and r_2. It follows that $\mathfrak{g}_0$ is two-dimensional and generated over $\mathbb{R}$ by two derivations (corresponding to the two choices: $r_1 = -1$, $r_2 0$ and $r_1 = 0$, $r_2 = -1$) that we will denote x_4 and x_5 and which are defined by:

$$\begin{aligned} &\mathsf{x}_4\colon \quad \mathsf{x}_1 \mapsto -2\,\mathsf{x}_1, \qquad \mathsf{x}_2 \mapsto -\mathsf{x}_2, \qquad \mathsf{x}_3 \mapsto -\mathsf{x}_3, \\ &\mathsf{x}_5\colon \quad \mathsf{x}_1 \mapsto 0, \qquad\quad\;\; \mathsf{x}_2 \mapsto -\mathsf{x}_3, \qquad \mathsf{x}_3 \mapsto \mathsf{x}_2. \end{aligned}$$

Then the commutator $\mathsf{x}_4 \circ \mathsf{x}_5 - \mathsf{x}_5 \circ \mathsf{x}_4 = 0$ vanishes, and at this stage, the Lie brackets between the obtained x_k read as follows, if listed by increasing homogeneity:

$$\begin{array}{ll} \boxed{-3}: & \big\{ \ [\mathsf{x}_1,\, \mathsf{x}_2] = 0, \qquad [\mathsf{x}_1,\, \mathsf{x}_3] = 0, \\ \boxed{-2}: & \big\{ \ [\mathsf{x}_2,\, \mathsf{x}_3] = 4\,\mathsf{x}_1, \quad [\mathsf{x}_1,\, \mathsf{x}_4] = 2\,\mathsf{x}_1, \qquad [\mathsf{x}_1,\, \mathsf{x}_5] = 0, \\ \boxed{-1}: & \big\{ \ [\mathsf{x}_2,\, \mathsf{x}_4] = \mathsf{x}_2, \qquad [\mathsf{x}_3,\, \mathsf{x}_4] = \mathsf{x}_3, \qquad [\mathsf{x}_2,\, \mathsf{x}_5] = \mathsf{x}_3, \qquad [\mathsf{x}_3,\, \mathsf{x}_5] = -\mathsf{x}_2 \\ \boxed{0}: & \big\{ \ [\mathsf{x}_4,\, \mathsf{x}_5] = 0. \end{array}$$

Here, we see that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is a Lie algebra in itself. Moreover, we observe that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is isomorphic to the isotropy subalgebra $\mathfrak{h}_{-2} \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_0$ of the Heisenberg sphere through the simple map:

$$\mathsf{T} \mapsto \mathsf{x}_1, \qquad \mathsf{H}_1 \mapsto \mathsf{x}_2, \qquad \mathsf{H}_2 \mapsto \mathsf{x}_3, \qquad \mathsf{D} \mapsto \mathsf{x}_4, \qquad \mathsf{R} \mapsto \mathsf{x}_5.$$

Next, we compute $\mathfrak{g}_1$. An element d of $\mathfrak{g}_1$ belongs to $(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}^*) \oplus (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*)$, hence it writes:

$$\mathsf{d}(\mathsf{x}_1) = k\mathsf{x}_2 + l\mathsf{x}_3, \qquad \mathsf{d}(\mathsf{x}_2) = m\mathsf{x}_4 + n\mathsf{x}_5, \qquad \mathsf{d}(\mathsf{x}_3) = p\mathsf{x}_4 + q\mathsf{x}_5,$$

for some six real, unknown constants. But the condition that d be a derivation gives us exactly three constraints, firstly:

$$0 = \mathsf{d}([\mathsf{x}_1,\, \mathsf{x}_2]) = [k\mathsf{x}_2 + l\mathsf{x}_3,\, \mathsf{x}_2] + [\mathsf{x}_2,\, m\mathsf{x}_4 + n\mathsf{x}_5] = -4l\mathsf{x}_1 + 2m\mathsf{x}_1$$

that is to say: $0 = -2l + m$; secondly:

$$0 = \mathsf{d}([\mathsf{x}_1,\, \mathsf{x}_3]) = [k\mathsf{x}_2 + l\mathsf{x}_3,\, \mathsf{x}_3] + [\mathsf{x}_1,\, p\mathsf{x}_4 + q\mathsf{x}_5] = 4k\mathsf{x}_1 + 2p\mathsf{x}_1,$$

that is to say: $0 = 2k + p$; thirdly and lastly:

$$\begin{aligned} 4k\mathsf{x}_2 + 4l\mathsf{x}_3 = 4\mathsf{d}(\mathsf{x}_1) = \mathsf{d}([\mathsf{x}_1,\, \mathsf{x}_2]) = [m\mathsf{x}_4 + n\mathsf{x}_5,\, \mathsf{x}_3] + [\mathsf{x}_2,\, p\mathsf{x}_4 + q\mathsf{x}_5] = \\ = -m\mathsf{x}_3 + n\mathsf{x}_2 + p\mathsf{x}_2 + q\mathsf{x}_3, \end{aligned}$$

that is to say: $4k = n + p$ and $4l = -m + q$. These four linear equations solve as $m = 2l$, $p - 2k$, $n = 6k$, $q = 6l$ with free k and l. It follows that $\mathfrak{g}_1$ is two-dimensional and generated over $\mathbb{R}$ by two derivations (corresponding to the two choices: $k = -1$, $l = 0$ and $k = 0$, $l = -1$):

$$\begin{array}{llll} \mathsf{x}_6: & \mathsf{x}_1 \mapsto -\mathsf{x}_2, & \mathsf{x}_2 \mapsto -6\,\mathsf{x}_5, & \mathsf{x}_3 \mapsto 2\,\mathsf{x}_4, \\ \mathsf{x}_7: & \mathsf{x}_1 \mapsto -\mathsf{x}_3, & \mathsf{x}_2 \mapsto -2\,\mathsf{x}_4, & \mathsf{x}_3 \mapsto -6\,\mathsf{x}_5. \end{array}$$

We still need to know the brackets structures $[\mathfrak{g}_1,\, \mathfrak{g}_0]$ and $[\mathfrak{g}_1,\, \mathfrak{g}_1]$. At this stage in fact, we can only determine $[\mathfrak{g}_1,\, \mathfrak{g}_0]$. By definition, with $\mathsf{d} \in \mathfrak{g}_1$ and $\mathsf{e} \in \mathfrak{g}_0$, the bracket $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_1 \subset (\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}^*) \oplus (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*)$ is determined by his action on the three vectors $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3$ generating $\mathfrak{g}_-$ through the formula (27), hence we compute three times at once:

$$[\mathsf{x}_4,\, \mathsf{x}_6]\begin{pmatrix} \mathsf{x}_1 \\ \mathsf{x}_2 \\ \mathsf{x}_3 \end{pmatrix} := \Bigg[\bigg[\mathsf{x}_4,\, \begin{matrix} \mathsf{x}_1 \\ \mathsf{x}_2 \\ \mathsf{x}_3 \end{matrix}\bigg],\, \mathsf{x}_6\Bigg] + \Bigg[\mathsf{x}_4,\, \bigg[\mathsf{x}_6,\, \begin{matrix} \mathsf{x}_1 \\ \mathsf{x}_2 \\ \mathsf{x}_3 \end{matrix}\bigg]\Bigg] = \Bigg[\begin{matrix} -2\,\mathsf{x}_1 \\ -\mathsf{x}_2 \\ -\mathsf{x}_3 \end{matrix},\, \mathsf{x}_6\Bigg] + \Bigg[\mathsf{x}_4,\, \begin{matrix} -\mathsf{x}_2 \\ -6\,\mathsf{x}_5 \\ -2\,\mathsf{x}_4 \end{matrix}\Bigg]$$

$$= \begin{pmatrix} -2\,\mathsf{x}_2 \\ -6\,\mathsf{x}_5 \\ 2\,\mathsf{x}_4 \end{pmatrix} + \begin{pmatrix} \mathsf{x}_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathsf{x}_2 \\ -6\,\mathsf{x}_5 \\ 2\,\mathsf{x}_4 \end{pmatrix} = \mathrm{map}\,\mathsf{x}_6\begin{pmatrix} \mathsf{x}_1 \\ \mathsf{x}_2 \\ \mathsf{x}_3 \end{pmatrix}.$$

and we recognize x_6, as a linear map, whence $[\mathsf{x}_4, \mathsf{x}_6] = \mathsf{x}_6$. In a similar way, one may compute the three remaining brackets. In summary, one obtains the following supplementary brackets, listed by increasing homogeneity:

$$\boxed{-1}\colon \ \big\{ \ [\mathsf{x}_1, \mathsf{x}_6] = \mathsf{x}_2, \qquad [\mathsf{x}_1, \mathsf{x}_7] = \mathsf{x}_3,$$
$$\boxed{0}\colon \ \big\{ \ [\mathsf{x}_2, \mathsf{x}_6] = 6\,\mathsf{x}_5, \quad [\mathsf{x}_3, \mathsf{x}_6] = -2\,\mathsf{x}_4, \quad [\mathsf{x}_2, \mathsf{x}_7] = 2\,\mathsf{x}_4, \quad [\mathsf{x}_3, \mathsf{x}_7] = 6\,\mathsf{x}_5,$$
$$\boxed{1}\colon \ \big\{ \ [\mathsf{x}_4, \mathsf{x}_6] = \mathsf{x}_6, \qquad [\mathsf{x}_4, \mathsf{x}_7] = \mathsf{x}_7, \qquad [\mathsf{x}_5, \mathsf{x}_6] = -\mathsf{x}_7, \quad [\mathsf{x}_5, \mathsf{x}_7] = \mathsf{x}_6.$$

Now, we are in a position to compute $\mathfrak{g}_2$. An element d of $\mathfrak{g}_2$ belongs to $(\mathfrak{g}_0 \otimes \mathfrak{g}_{-2}^*) \oplus (\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1}^*)$, hence it writes:

$$\mathsf{d}(\mathsf{x}_1) = k\mathsf{x}_4 + l\mathsf{x}_5, \qquad \mathsf{d}(\mathsf{x}_2) = m\mathsf{x}_6 + n\mathsf{x}_7, \qquad \mathsf{d}(\mathsf{x}_3) = p\mathsf{x}_6 + q\mathsf{x}_7,$$

for some six real, unknown constants. Again, the condition that d be a derivation gives exactly three constraints, firstly:

$$0 = \mathsf{d}([\mathsf{x}_1, \mathsf{x}_2]) = [k\mathsf{x}_4 + l\mathsf{x}_5, \mathsf{x}_2] + [\mathsf{x}_1, m\mathsf{x}_6 + n\mathsf{x}_7] = -k\mathsf{x}_2 - l\mathsf{x}_3 + m\mathsf{x}_2 + n\mathsf{x}_3,$$

that is to say: $0 = -k + m$ and $0 = -l + n$; secondly:

$$0 = \mathsf{d}([\mathsf{x}_1, \mathsf{x}_3]) = [k\mathsf{x}_4 + l\mathsf{x}_5, \mathsf{x}_3] + [\mathsf{x}_1, p\mathsf{x}_6 + q\mathsf{x}_7] = -k\mathsf{x}_3 + l\mathsf{x}_2 + p\mathsf{x}_2 + q\mathsf{x}_3,$$

that is to say: $0 = l + p$ and $0 = -k + q$; thirdly and lastly:

$$\begin{aligned} 4k\mathsf{x}_4 + 4l\mathsf{x}_5 = 4\mathsf{d}(\mathsf{x}_1) = \mathsf{d}([\mathsf{x}_2, \mathsf{x}_3]) &= [m\mathsf{x}_6 + n\mathsf{x}_7, \mathsf{x}_3] + [\mathsf{x}_2, p\mathsf{x}_6 + q\mathsf{x}_7] \\ &= 2m\mathsf{x}_4 - 6n\mathsf{x}_5 + 6p\mathsf{x}_5 + 2q\mathsf{x}_4, \end{aligned}$$

that is to say: $4k = 2m + 2q$ and $4l = -n + p$. These four linear equations solve, up to a dilation factor, as $0 = l = n = p$ and $m = q = k = -1$, whence it follows that $\mathfrak{g}_2$ is one-dimensional and generated over $\mathbb{R}$ by the single derivation:

$$\mathsf{x}_8\colon \quad \mathsf{x}_1 \mapsto -\mathsf{x}_4, \qquad \mathsf{x}_2 \mapsto -\mathsf{x}_6, \qquad \mathsf{x}_3 \mapsto -\mathsf{x}_7.$$

We still need to know the bracket structures $[\mathfrak{g}_0, \mathfrak{g}_2]$ and $[\mathfrak{g}_1, \mathfrak{g}_1]$. After a few computations using the already known brackets, one obtains the supplementary brackets:

$$\boxed{0}\colon \ \big\{ \ [\mathsf{x}_1, \mathsf{x}_8] = \mathsf{x}_4,$$
$$\boxed{1}\colon \ \big\{ \ [\mathsf{x}_2, \mathsf{x}_8] = \mathsf{x}_6, \qquad [\mathsf{x}_3, \mathsf{x}_8] = \mathsf{x}_7,$$
$$\boxed{2}\colon \ \big\{ \ [\mathsf{x}_4, \mathsf{x}_8] = 2\,\mathsf{x}_8, \qquad [\mathsf{x}_5, \mathsf{x}_8] = 0, \qquad [\mathsf{x}_6, \mathsf{x}_7] = 4\,\mathsf{x}_8,$$

Finally, the prolongation stops, for one may easily verify that $\mathfrak{g}_3 = \{0\}$, while it is known that $\mathfrak{g}_k\{0\}$ for some $k \geqslant 0$ implies $\mathfrak{g}_l = \{0\}$ for every $l \geqslant k$ ([41], p. 433). Also, $0 = [\mathfrak{g}_1, \mathfrak{g}_2] = [\mathfrak{g}_2, \mathfrak{g}_2]$, which ends up the process, all brackets being computed and:

$$\mathfrak{g} := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

being a graded $1 + 2 + 2 + 2 + 1 =$ eight-dimensional Lie algebra.

4.3. **Fundamental isomorphism.** Now by inspecting all brackets just obtained, we observe that this J-compatible Tanaka prolongation $\mathfrak{g}$ (of the above three-dimensional Heisenberg nilpotent Lie algebra $\mathfrak{n}_3^1$) *isomorphically coincides* with the Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ of CR-automorphisms of the Heisenberg sphere through the plain identifications:

$$\mathsf{x}_1 \leftrightarrow \mathsf{T},\ \ \mathsf{x}_2 \leftrightarrow \mathsf{H}_1,\ \ \mathsf{x}_3 \leftrightarrow \mathsf{H}_2,\ \ \mathsf{x}_4 \leftrightarrow \mathsf{D},\ \ \mathsf{x}_5 \leftrightarrow \mathsf{R},\ \ \mathsf{x}_6 \leftrightarrow \mathsf{I}_1,\ \ \mathsf{x}_7 \leftrightarrow \mathsf{I}_2,\ \ \mathsf{x}_8 \leftrightarrow \mathsf{J}.$$

In fact, this coincidence comes from Tanaka's general ([41]) theorem that the prolongation $\mathfrak{g}$ can naturally be identified with the Lie algebra of all J-compatible infinitesimal automorphisms of the unique connected simply connected three-dimensional Lie group with the Lie algebra $[\mathsf{x}_2, \mathsf{x}_3] = 4\,\mathsf{x}_1$, $\mathsf{x}_3 = J(\mathsf{x}_2)$. For more clarity, we will employ from now on the letters t, h_1, h_2, d, r, i_1, i_2, j instead of $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4, \mathsf{x}_5, \mathsf{x}_6, \mathsf{x}_7, \mathsf{x}_8$ as generators of $\mathfrak{g}$. For later use, we draw up the full commutator table between these eight generators of the abstract Lie algebra $\mathfrak{g}$:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

5. Second Cohomology of the Heisenberg Lie Algebra

Specifically for a Cartan geometry modeled on a pair $(\mathfrak{g}_-, \mathfrak{g})$ of Lie algebras $\mathfrak{g}_- \subset \mathfrak{g}$, the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ *a priori* provides some useful algebraic information about the number of functionally independent Cartan curvatures. Similarly as for Tanaka prolongations, the computations for $H^2(\mathfrak{g}_-, \mathfrak{g})$ are of purely algebraic nature, without any differentialo-geometric invariant coming into the picture hence more elementary. In Section 10, general considerations and formulas about (second) cohomologies of (graded) Lie algebra are set up.

Thus, let $\mathfrak{g}$ be an r-dimensional Lie algebra and let $\mathfrak{g}_-$ be an n-dimensional ($1 \leqslant n \leqslant r-1$) subalgebra of $\mathfrak{g}$. For any $k \geqslant 1$, the space $\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$ of *k-cochains* consists by definition ([17], Chap. 3) of the space of linear maps from $\Lambda^k \mathfrak{g}_-$ to $\mathfrak{g}$, that is to say:

$$\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g}) = \mathrm{Lin}\big(\Lambda^k \mathfrak{g}_-,\, \mathfrak{g}\big).$$

When $\mathfrak{g}$ is equipped with the structure of a graded Lie algebra:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu,$$

with $\big[\mathfrak{g}_{k_1},\, \mathfrak{g}_{k_2}\big] \subset \mathfrak{g}_{k_1+k_2}$ for any $k_1, k_2 \in \mathbb{Z}$ (with the convention that $\mathfrak{g}_k = \{0\}$ whenever $k \leqslant -\mu - 1$ or $k \geqslant \nu + 1$), each vector space $\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$ naturally splits into a direct sum of so-called homogeneous cochains as follows: a k-cochain $\Phi \in \mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$ is said to be *of homogeneity* $h \in \mathbb{Z}$ whenever for arbitrary elements:

$$\mathsf{z}_{i_1} \in \mathfrak{g}_{i_1}, \ldots\ldots, \mathsf{z}_{i_k} \in \mathfrak{g}_{i_k}$$

belonging to certain arbitrary determined $\mathfrak{g}$-component, its value:

$$\Phi(\mathsf{z}_{i_1}, \ldots \mathsf{z}_{i_k}) \in \mathfrak{g}_{i_1+\cdots+i_k+h}$$

belongs to the $(i_1 + \ldots + i_k + i)$-th component of $\mathfrak{g}$. In fact, one easily convinces oneself that any k-cochain Φ splits as a direct sum of k-cochains of fixed homogeneity:

$$\Phi = \cdots + \Phi^{[h-1]} + \Phi^{[h]} + \Phi^{[h+1]} + \cdots$$

where we denote the h-th component of Φ just by $\Phi^{[h]}$.

For each $k = 0, 1, \ldots, n$, the *differential operator* ([17, 8, 14]):

$$\partial^k : \quad \mathscr{C}^k\big(\mathfrak{g}_-,\, \mathfrak{g}\big) \longrightarrow \mathscr{C}^{k+1}\big(\mathfrak{g}_-,\, \mathfrak{g}\big)$$

assigns to a k-cochain $\Phi \in \mathscr{C}^k\big(\mathfrak{g}_-, \mathfrak{g}\big)$ the $k+1$-cochain $\partial^k\Phi$ whose value on any collection of $k+1$ vectors $\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_k$ is defined through the formula:

$$\begin{aligned}(\partial^k\Phi)\big(\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_k\big) := \sum_{i=0}^{k} (-1)^i \big[\mathsf{z}_i,\, \Phi(\mathsf{z}_0, \ldots, \widehat{\mathsf{z}}_i, \ldots, \mathsf{z}_k)\big]_{\mathfrak{g}} + \\ + \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j}\, \Phi\big([\mathsf{z}_i, \mathsf{z}_j]_{\mathfrak{g}}, \mathsf{z}_0, \ldots, \widehat{\mathsf{z}}_i, \ldots, \widehat{\mathsf{z}}_j, \ldots, \mathsf{z}_k\big),\end{aligned} \tag{28}$$

where $\widehat{\mathsf{z}}_l$ means removal of the term z_l. This action is linear with respect to each argument z_i, $i = 0, 1, \ldots, k$. One can check that the composition $\partial^{k+1} \circ \partial^k$ vanishes for any $k \in \mathbb{N}$ and we have the following *cochain complex*:

$$0 \xrightarrow{\partial^0} \mathscr{C}^1 \xrightarrow{\partial^1} \mathscr{C}^2 \xrightarrow{\partial^2} \cdots \xrightarrow{\partial^{n-2}} \mathscr{C}^{n-1} \xrightarrow{\partial^{n-1}} \mathscr{C}^n \xrightarrow{\partial^n} 0.$$

The k-th cohomological space $H^k(\mathfrak{g}_-, \mathfrak{g})$ is then defined as being the following quotient:

$$H^k\big(\mathfrak{g}_-, \mathfrak{g}\big) = \frac{\ker(\partial^k)}{\mathrm{im}(\partial^{k-1})}.$$

Before entering specific computations, let us briefly motivate and anticipate. Only the second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ will be of interest to us. In Section 8 (*cf.* also [8, 14]), we will indeed see that the the curvature function (Definition 8.4) associated to a Cartan connection takes its image in the set of 2-cochains $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$. Also, the curvature function naturally splits in homogeneous components. As explained in [14], the so-called *Bianchi-Tanaka identities* stated by Proposition 8.10 below entail in particular that the lowest order nonvanishing curvature must be ∂-closed, and more generally, any homogeneous curvature component is determined by the lower components up to a ∂-closed component (*see* also Proposition 8.13 below). Hence, some of the linear-like properties of Cartan curvatures rely on probing the corresponding second cohomological spaces.

5.1. **Cochains $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$ for the prolonged Heisenberg Lie algebra.** Now, let $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the eight-dimensional (abstract) graded Lie algebra under study with:

$$\mathfrak{g}_{-2} = \mathbb{R}\,\mathsf{t}, \quad \mathfrak{g}_{-1} = \mathbb{R}\,\mathsf{h}_1 \oplus \mathbb{R}\,\mathsf{h}_2, \quad \mathfrak{g}_0 = \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r}, \quad \mathfrak{g}_1 = \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2, \quad \mathfrak{g}_2 = \mathbb{R}\,\mathsf{j},$$

its commutator table being shown above. Let $\{\mathsf{t}^*, \mathsf{h}_1^*, \mathsf{h}_2^*\}$ be the dual basis of $\mathfrak{g}_- = \mathbb{R}\,\mathsf{t} \oplus \mathbb{R}\,\mathsf{h}_1 \oplus \mathbb{R}\,\mathsf{h}_2$. Then with such bases, a general 2-cochain $\Phi \in \Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ writes under the explicit expanded form:

$$\begin{aligned}\Phi = \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \Big(&\phi_t^{th_1}\,\mathsf{t} + \phi_{h_1}^{th_1}\,\mathsf{h}_1 + \phi_{h_2}^{th_1}\,\mathsf{h}_2 + \phi_d^{th_1}\,\mathsf{d} + \phi_r^{th_1}\,\mathsf{r} + \phi_{i_1}^{th_1}\,\mathsf{i}_1 + \phi_{i_2}^{th_1}\,\mathsf{i}_2 + \phi_j^{th_1}\,\mathsf{j}\Big) + \\ + \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \Big(&\phi_t^{th_2}\,\mathsf{t} + \phi_{h_1}^{th_2}\,\mathsf{h}_1 + \phi_{h_2}^{th_2}\,\mathsf{h}_2 + \phi_d^{th_2}\,\mathsf{d} + \phi_r^{th_2}\,\mathsf{r} + \phi_{i_1}^{th_2}\,\mathsf{i}_1 + \phi_{i_2}^{th_2}\,\mathsf{i}_2 + \phi_j^{th_2}\,\mathsf{j}\Big) + \\ + \mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \Big(&\phi_t^{h_1h_2}\,\mathsf{t} + \phi_{h_1}^{h_1h_2}\,\mathsf{h}_1 + \phi_{h_2}^{h_1h_2}\,\mathsf{h}_2 + \phi_d^{h_1h_2}\,\mathsf{d} + \phi_r^{h_1h_2}\,\mathsf{r} + \phi_{i_1}^{h_1h_2}\,\mathsf{i}_1 + \phi_{i_2}^{h_1h_2}\,\mathsf{i}_2 + \phi_j^{h_1h_2}\,\mathsf{j}\Big),\end{aligned}$$

where the 24 real coefficients $\phi_t^{th_1}, \ldots, \phi_j^{h_1h_2}$ are arbitrary. This 2-cochain could also be written (*cf.* [6]) under a condensed symbolic form as:

$$\Phi = \sum_{x<y} \sum_v \phi_v^{xy}\,\mathsf{x}^* \wedge \mathsf{y}^* \otimes \mathsf{v},$$

for $\mathsf{x}, \mathsf{y} \in \{\mathsf{h}_1^*, \mathsf{h}_2^*, \mathsf{t}^*\}$ and for $\mathsf{v} \in \{\mathsf{t}, \mathsf{h}_1, \mathsf{h}_2, \mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}\}$, but the first, complete writing is much more suited to effective calculations. Also, it is useful to reorganize the 24 components of Φ by collecting, in one and a single line, all those which have the same homogeneity:

$$\begin{aligned}
&\Phi = \phi_t^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \qquad \boxed{0}\\
&\boxed{1} \quad + \phi_t^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{t} + \phi_t^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \phi_{h_1}^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{t} + \phi_{h_2}^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_2 + \\
&\boxed{2} \quad + \phi_{h_1}^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{h}_1 + \phi_{h_2}^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{h}_2 + \phi_{h_1}^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_1 + \phi_{h_2}^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{h}_2 + \\
&\qquad\qquad + \phi_d^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{d} + \phi_r^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{r} + \\
&\boxed{3} \quad + \phi_d^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{d} + \phi_r^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{r} + \phi_d^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{d} + \phi_r^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{r} \\
&\qquad\qquad + \phi_{i_1}^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2{}^* \otimes \mathsf{i}_1 + \phi_{i_2}^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 + \\
&\boxed{4} \quad + \phi_{i_1}^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_1 + \phi_{i_2}^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_2 + \phi_{i_1}^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_1 + \phi_{i_2}^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 \\
&\qquad\qquad + \phi_j^{h_1h_2}\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{j} + \\
&\boxed{5} \quad + \phi_j^{th_1}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{j} + \phi_j^{th_2}\,\mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{j},
\end{aligned}$$

starting from homogeneity 0 (first line) up to homogeneity 5 (last line). Thus, the graded dimensions of $\mathscr{C}_0^2$ $\mathscr{C}_1^2$, $\mathscr{C}_2^2$, $\mathscr{C}_3^2$, $\mathscr{C}_4^2$, $\mathscr{C}_5^2$ are equal, respectively, to: 1, 4, 6, 6, 5, 2, *cf.* also the summarizing table at the end of this section.

5.2. **Computations of $\mathscr{Z}^2(\mathfrak{g}_-, \mathfrak{g})$.** Now, such a general 2-cochain Φ belongs to $\mathscr{Z}^2$ if and only if the value of $\partial\Phi$ on each antisymmetric 3-vector of $\Lambda^3\mathfrak{g}_-$ vanishes. But here, $\Lambda^3\mathfrak{g}_-$ is one-dimensional, generated by just $\mathsf{t} \wedge \mathsf{h}_1 \wedge \mathsf{h}_2$. Consequently, applying the definition (28), the cocycle condition amounts to the single equation:

$$\begin{aligned}0 &= \partial\Phi(\mathsf{t}, \mathsf{h}_1, \mathsf{h}_2) \\ &= \big[\mathsf{t},\, \Phi(\mathsf{h}_1, \mathsf{h}_2)\big]_{\mathfrak{g}} - \big[\mathsf{h}_1,\, \Phi(\mathsf{t}, \mathsf{h}_2)\big]_{\mathfrak{g}} + \big[\mathsf{h}_2,\, \Phi(\mathsf{t}, \mathsf{h}_1)\big]_{\mathfrak{g}} - \\ &\quad - \Phi\big([\mathsf{t}, \mathsf{h}_1]_{\mathfrak{g}},\, \mathsf{h}_2\big) + \Phi\big([\mathsf{t}, \mathsf{h}_2]_{\mathfrak{g}},\, \mathsf{h}_1\big) - \Phi\big([\mathsf{h}_1, \mathsf{h}_2]_{\mathfrak{g}},\, \mathsf{t}\big),\end{aligned}$$

and then, after substituting all corresponding values of $\Phi(\cdot,\cdot)$, we get:

$$
\begin{aligned}
0 = \big[\mathsf{t},\, &\phi_t^{h_1h_2}\,\mathsf{t}+\phi_{h_1}^{h_1h_2}\,\mathsf{h}_1+\phi_{h_2}^{h_1h_2}\,\mathsf{h}_2+\phi_d^{h_1h_2}\,\mathsf{d}+\phi_r^{h_1h_2}\,\mathsf{r}+\phi_{i_1}^{h_1h_2}\,\mathsf{i}_1+\phi_{i_2}^{h_1h_2}\,\mathsf{i}_2+\phi_j^{h_1h_2}\,\mathsf{j}\big]-\\
&-\big[\mathsf{h}_1,\,\phi_t^{th_2}\,\mathsf{t}+\phi_{h_1}^{th_2}\,\mathsf{h}_1+\phi_{h_2}^{th_2}\,\mathsf{h}_2+\phi_d^{th_2}\,\mathsf{d}+\phi_r^{th_2}\,\mathsf{r}+\phi_{i_1}^{th_2}\,\mathsf{i}_1+\phi_{i_2}^{th_2}\,\mathsf{i}_2+\phi_j^{th_2}\,\mathsf{j}\big]+\\
&+\big[\mathsf{h}_2,\,\phi_t^{th_1}\,\mathsf{t}+\phi_{h_1}^{th_1}\,\mathsf{h}_1+\phi_{h_2}^{th_1}\,\mathsf{h}_2+\phi_d^{th_1}\,\mathsf{d}+\phi_r^{th_1}\,\mathsf{r}+\phi_{i_1}^{th_1}\,\mathsf{i}_1+\phi_{i_2}^{th_1}\,\mathsf{i}_2+\phi_j^{th_1}\,\mathsf{j}\big]-\\
&-\underline{\Phi(0,\mathsf{h}_2)}_\circ+\underline{\Phi(0,\mathsf{h}_1)}_\circ-\underline{\Phi(4\mathsf{t},\mathsf{t})}_\circ.
\end{aligned}
$$

Using the commutator table, we may then replace each appearing Lie bracket:

$$
\begin{aligned}
0 = 2&\phi_d^{h_1h_2}\,\mathsf{t}+\phi_{i_1}^{h_1h_2}\,\mathsf{h}_1+\phi_{i_2}^{h_1h_2}\,\mathsf{h}_2+\phi_j^{h_1h_2}\,\mathsf{d}-\\
&-4\phi_{h_2}^{th_2}\,\mathsf{t}-\phi_d^{th_2}\,\mathsf{h}_1-\phi_r^{th_2}\,\mathsf{h}_2-6\phi_{i_1}^{th_2}\,\mathsf{r}-2\phi_{i_2}^{th_2}\,\mathsf{d}-\phi_j^{th_2}\,\mathsf{i}_1-\\
&-4\phi_{h_1}^{th_1}\,\mathsf{t}+\phi_d^{th_1}\,\mathsf{h}_2-\phi_r^{th_1}\,\mathsf{h}_1-2\phi_{i_1}^{th_1}\,\mathsf{d}+6\phi_{i_2}^{th_1}\,\mathsf{r}+\phi_j^{th_1}\,\mathsf{i}_2,
\end{aligned}
$$

and lastly, gather the coefficients of the appearing vectors $\mathsf{t},\dots,\mathsf{i}_2$:

$$
\begin{aligned}
0 = (2\phi_d^{h_1h_2}-4\phi_{h_2}^{th_2}-4\phi_{h_1}^{th_1})\,\mathsf{t}+(\phi_{i_1}^{h_1h_2}-\phi_d^{th_2}-\phi_r^{th_1})\,\mathsf{h}_1+(\phi_{i_2}^{h_1h_2}-\phi_r^{th_2}+\phi_d^{th_1})\,\mathsf{h}_2+\\
+(\phi_j^{h_1h_2}-2\phi_{i_2}^{th_2}-2\phi_{i_1}^{th_1})\,\mathsf{d}+(-6\phi_{i_1}^{th_2}+6\phi_{i_2}^{th_1})\,\mathsf{r}+(-\phi_j^{th_2})\,\mathsf{i}_1+(\phi_j^{th_1})\,\mathsf{i}_2.
\end{aligned}
$$

Thus, a 2-cochain Φ is a 2-cocycle if and only if its 24 coefficients satisfy the following seven linear equations, ordered by increasing homogeneity:

$$
\begin{array}{lll}
\boxed{2} & 0 = 2\phi_d^{h_1h_2}-4\phi_{h_2}^{th_2}-4\phi_{h_1}^{th_1}, & \\
\boxed{3} & 0 = \phi_{i_1}^{h_1h_2}-\phi_d^{th_2}-\phi_r^{th_1}, & 0 = \phi_{i_2}^{h_1h_2}-\phi_r^{th_2}+\phi_d^{th_1},\\
\boxed{4} & 0 = \phi_j^{h_1h_2}-2\phi_{i_2}^{th_2}-2\phi_{i_1}^{th_1}, & 0 = -6\phi_{i_1}^{th_2}+6\phi_{i_2}^{th_1},\\
\boxed{5} & 0 = -\phi_j^{th_2}, \qquad 0 = \phi_j^{th_1}. &
\end{array}
$$

All these equations are visibly linearly independent, and we deduce that the homogeneous components $\mathscr{Z}^2_{[h]}$ of:

$$
\mathscr{Z}^2 = \mathscr{Z}^2_{[0]}\oplus\mathscr{Z}^2_{[1]}\oplus\mathscr{Z}^2_{[2]}\oplus\mathscr{Z}^2_{[3]}\oplus\mathscr{Z}^2_{[4]}\oplus\mathscr{Z}^2_{[5]}
$$

have codimensions within $\mathscr{C}^2_{[h]}$ equal to 0, 0, 1, 2, 2, 2, hence are of dimensions equal to 1, 4, 5, 4, 3, 0 respectively,

5.3. **Determination of $\mathscr{B}^2(\mathfrak{g}_-,\mathfrak{g})$.** Now, let $\Psi\in\Lambda^1\mathfrak{g}_-^*\otimes\mathfrak{g}$ be a general 1-cochain. In terms of the bases $\{\mathsf{t}^*,\,\mathsf{h}_1^*,\,\mathsf{h}_2^*\}$ of $\mathfrak{g}_-^*$ and $\{\mathsf{t},\,\mathsf{h}_1,\,\mathsf{h}_2,\,\mathsf{d},\,\mathsf{r},\,\mathsf{i}_1,\,\mathsf{i}_2,\,\mathsf{j}\}$ of $\mathfrak{g}$, it writes under the explicit expanded form:

$$
\begin{aligned}
\Psi = \mathsf{t}^*\otimes\Big(&\psi_t^t\,\mathsf{t}+\psi_{h_1}^t\,\mathsf{h}_1+\psi_{h_2}^t\,\mathsf{h}_2+\psi_d^t\,\mathsf{d}+\psi_r^t\,\mathsf{r}+\psi_{i_1}^t\,\mathsf{i}_1+\psi_{i_2}^t\,\mathsf{i}_2+\psi_j^t\,\mathsf{j}\Big)+\\
&+\mathsf{h}_1^*\otimes\Big(\psi_t^{h_1}\,\mathsf{t}+\psi_{h_1}^{h_1}\,\mathsf{h}_1+\psi_{h_2}^{h_1}\,\mathsf{h}_2+\psi_d^{h_1}\,\mathsf{d}+\psi_r^{h_1}\,\mathsf{r}+\psi_{i_1}^{h_1}\,\mathsf{i}_1+\psi_{i_2}^{h_1}\,\mathsf{i}_2+\psi_j^{h_1}\,\mathsf{j}\Big)+\\
&+\mathsf{h}_1^*\otimes\Big(\psi_t^{h_2}\,\mathsf{t}+\psi_{h_1}^{h_2}\,\mathsf{h}_1+\psi_{h_2}^{h_2}\,\mathsf{h}_2+\psi_d^{h_2}\,\mathsf{d}+\psi_r^{h_2}\,\mathsf{r}+\psi_{i_1}^{h_2}\,\mathsf{i}_1+\psi_{i_2}^{h_2}\,\mathsf{i}_2+\psi_j^{h_2}\,\mathsf{j}\Big),
\end{aligned}
$$

where the 24 real coefficients $\psi_t^t,\dots,\psi_j^{j_2}$ are arbitrary. Equivalently, by collecting in one and a single line all components having equal homogeneity, such a general

1-cochains writes:

$$
\begin{aligned}
&\Psi = \psi_t^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{t} + \psi_t^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{t} + && \boxed{\text{-1}}\\
\boxed{0}\quad &+ \psi_t^t\,\mathsf{t}^* \otimes \mathsf{t} + \psi_{h_1}^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{h}_1 + \psi_{h_2}^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{h}_2 + \psi_{h_1}^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{h}_1 + \psi_{h_2}^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{h}_2 +\\
\boxed{1}\quad &+ \psi_{h_1}^t\,\mathsf{t}^* \otimes \mathsf{h}_1 + \psi_{h_2}^t\,\mathsf{t}^* \otimes \mathsf{h}_2 + \psi_d^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{d} + \psi_r^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{r} + \psi_d^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{d} + \psi_r^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{r} +\\
\boxed{2}\quad &+ \psi_d^t\,\mathsf{t}^* \otimes \mathsf{d} + \psi_r^t\,\mathsf{t}^* \otimes \mathsf{r} + \psi_{i_1}^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{i}_1 + \psi_{i_2}^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{i}_2 + \psi_{i_1}^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{i}_1 + \psi_{i_2}^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{i}_2 +\\
\boxed{3}\quad &+ \psi_{i_1}^t\,\mathsf{t}^* \otimes \mathsf{i}_1 + \psi_{i_2}^t\,\mathsf{t}^* \otimes \mathsf{i}_2 + \psi_j^{h_1}\,\mathsf{h}_1^* \otimes \mathsf{j} + \psi_j^{h_2}\,\mathsf{h}_2^* \otimes \mathsf{j} +\\
\boxed{4}\quad &+ \psi_j^t\,\mathsf{t}^* \otimes \mathsf{j}.
\end{aligned}
$$

In order to characterize when a 2-cochain Φ is of the form $\partial\Psi$ namely is a coboundary, applying the definition (28), we at first compute the values of $\partial\Psi$ on each of the three antisymmetric 2-vectors $\mathsf{t} \wedge \mathsf{h}_1$, $\mathsf{t} \wedge \mathsf{h}_2$, $\mathsf{h}_1 \wedge \mathsf{h}_2$ which make up a natural basis for $\Lambda^2\mathfrak{g}_-$, and with intermediate details, we obtain:

$$
\begin{aligned}
(\partial\Psi)(\mathsf{t},\mathsf{h}_1) &= \big[\mathsf{t},\Psi(\mathsf{h}_1)\big] - \big[\mathsf{h}_1,\Psi(\mathsf{t})\big] - \Psi\big([\mathsf{t},\mathsf{h}_1]\big)\\
&= \big[\mathsf{t},\,\psi_t^{h_1}\,\mathsf{t} + \psi_{h_1}^{h_1}\,\mathsf{h}_1 + \psi_{h_2}^{h_1}\,\mathsf{h}_2 + \psi_d^{h_1}\,\mathsf{d} + \psi_r^{h_1}\,\mathsf{r} + \psi_{i_1}^{h_1}\,\mathsf{i}_1 + \psi_{i_2}^{h_1}\,\mathsf{i}_2 + \psi_j^{h_1}\,\mathsf{j}\big] -\\
&\quad - \big[\mathsf{h}_1,\,\psi_t^t\,\mathsf{t} + \psi_{h_1}^t\,\mathsf{h}_1 + \psi_{h_2}^t\,\mathsf{h}_2 + \psi_d^t\,\mathsf{d} + \psi_r^t\,\mathsf{r} + \psi_{i_1}^t\,\mathsf{i}_1 + \psi_{i_2}^t\,\mathsf{i}_2 + \psi_j^t\,\mathsf{j}\big] -\\
&\quad - \underline{\Psi(0)}_\circ\\
&= 2\psi_d^{h_1}\,\mathsf{t} + \psi_{i_1}^{h_1}\,\mathsf{h}_1 + \psi_{i_2}^{h_1}\,\mathsf{h}_2 + \psi_j^{h_1}\,\mathsf{d} -\\
&\quad - 4\psi_{h_2}^t\,\mathsf{t} - \psi_d^t\,\mathsf{h}_1 - \psi_r^t\,\mathsf{h}_2 - 6\psi_{i_1}^t\,\mathsf{r} - 2\psi_{i_2}^t\,\mathsf{d} - \psi_j^t\,\mathsf{i}_1\\
&= (2\psi_d^{h_1} - 4\psi_{h_2}^t)\,\mathsf{t} + (\psi_{i_1}^{h_1} - \psi_d^t)\,\mathsf{h}_1 + (\psi_{i_2}^{h_1} - \psi_r^t)\,\mathsf{h}_2 + (\psi_j^{h_1} - 2\psi_{i_2}^t)\,\mathsf{d} +\\
&\quad + (-6\psi_{i_1}^t)\,\mathsf{r} + (-\psi_j^t)\,\mathsf{i}_1 + 0\,\mathsf{i}_2 + 0\,\mathsf{j},
\end{aligned}
$$

$$
\begin{aligned}
(\partial\Psi)(\mathsf{t},\mathsf{h}_2) &= \big[\mathsf{t},\Psi(\mathsf{h}_2)\big] - \big[\mathsf{h}_2,\Psi(\mathsf{t})\big] - \Psi\big([\mathsf{t},\mathsf{h}_2]\big)\\
&= \big[\mathsf{t},\,\psi_t^{h_2}\,\mathsf{t} + \psi_{h_1}^{h_2}\,\mathsf{h}_1 + \psi_{h_2}^{h_2}\,\mathsf{h}_2 + \psi_d^{h_2}\,\mathsf{d} + \psi_r^{h_2}\,\mathsf{r} + \psi_{i_1}^{h_2}\,\mathsf{i}_1 + \psi_{i_2}^{h_2}\,\mathsf{i}_2 + \psi_j^{h_2}\,\mathsf{j}\big] -\\
&\quad - \big[\mathsf{h}_2,\,\psi_t^t\,\mathsf{t} + \psi_{h_1}^t\,\mathsf{h}_1 + \psi_{h_2}^t\,\mathsf{h}_2 + \psi_d^t\,\mathsf{d} + \psi_r^t\,\mathsf{r} + \psi_{i_1}^t\,\mathsf{i}_1 + \psi_{i_2}^t\,\mathsf{i}_2 + \psi_j^t\,\mathsf{j}\big] -\\
&\quad - \underline{\Psi(0)}_\circ\\
&= 2\psi_d^{h_1}\,\mathsf{t} + \psi_{i_1}^{h_2}\,\mathsf{h}_1 + \psi_{i_2}^{h_2}\,\mathsf{h}_2 + \psi_j^{h_2}\,\mathsf{d} +\\
&\quad + 4\psi_{h_1}^t\,\mathsf{t} - \psi_d^t\,\mathsf{h}_2 + \psi_r^t\,\mathsf{h}_1 + 2\psi_{i_1}^t\,\mathsf{d} - 6\psi_{i_2}^r\,\mathsf{r} - \psi_j^t\,\mathsf{i}_2\\
&= (2\psi_d^{h_2} + 4\psi_{h_1}^t)\,\mathsf{t} + (\psi_{i_1}^{h_2} + \psi_r^t)\,\mathsf{h}_1 + (\psi_{i_2}^{h_2} - \psi_d^t)\,\mathsf{h}_2 + (\psi_j^{h_2} + 2\psi_{i_1}^t)\,\mathsf{d} +\\
&\quad + (-6\psi_{i_2}^t)\,\mathsf{r} + 0\,\mathsf{i}_1 + (-\psi_j^t)\,\mathsf{i}_2 + 0\,\mathsf{j},
\end{aligned}
$$

$$
\begin{aligned}
(\partial\Psi)(\mathsf{h}_1,\mathsf{h}_2) &= \big[\mathsf{h}_1,\Psi(\mathsf{h}_2)\big] - \big[\mathsf{h}_2,\Psi(\mathsf{h}_1)\big] - \Psi\big([\mathsf{h}_1,\mathsf{h}_2]\big)\\
&= \big[\mathsf{h}_1,\,\psi_t^{h_2}\,\mathsf{t} + \psi_{h_1}^{h_2}\,\mathsf{h}_1 + \psi_{h_2}^{h_2}\,\mathsf{h}_2 + \psi_d^{h_2}\,\mathsf{d} + \psi_r^{h_2}\,\mathsf{r} + \psi_{i_1}^{h_2}\,\mathsf{i}_1 + \psi_{i_2}^{h_2}\,\mathsf{i}_2 + \psi_j^{h_2}\,\mathsf{j}\big] -\\
&\quad - \big[\mathsf{h}_2,\,\psi_t^{h_1}\,\mathsf{t} + \psi_{h_1}^{h_1}\,\mathsf{h}_1 + \psi_{h_2}^{h_1}\,\mathsf{h}_2 + \psi_d^{h_1}\,\mathsf{d} + \psi_r^{h_1}\,\mathsf{r} + \psi_{i_1}^{h_1}\,\mathsf{i}_1 + \psi_{i_2}^{h_1}\,\mathsf{i}_2 + \psi_j^{h_1}\,\mathsf{j}\big] -\\
&\quad - 4\psi_t^t\,\mathsf{t} - 4\psi_{h_1}^t\,\mathsf{h}_1 - 4\psi_{h_2}^t\,\mathsf{h}_2 - 4\psi_d^t\,\mathsf{d} - 4\psi_r^t\,\mathsf{r} - 4\psi_{i_1}^t\,\mathsf{i}_1 - 4\psi_{i_2}^t\,\mathsf{i}_2 - 4\psi_j^t\,\mathsf{j}\\
&= 4\psi_{h_2}^{h_2}\,\mathsf{t} + \psi_d^{h_2}\,\mathsf{h}_1 + \psi_r^{h_2}\,\mathsf{h}_2 + 6\psi_{i_1}^{h_2}\,\mathsf{r} + 2\psi_{i_2}^{h_2}\,\mathsf{d} + \psi_j^{h_2}\,\mathsf{i}_1 +\\
&\quad + 4\psi_{h_1}^{h_1}\,\mathsf{t} - \psi_d^{h_1}\,\mathsf{h}_2 + \psi_r^{h_1}\,\mathsf{h}_1 + 2\psi_{i_1}^{h_1}\,\mathsf{d} - 6\psi_{i_2}^{h_1}\,\mathsf{r} - \psi_j^{h_1}\,\mathsf{i}_2 -\\
&\quad - 4\psi_t^t\,\mathsf{t} - 4\psi_{h_1}^t\,\mathsf{h}_1 - 4\psi_{h_2}^t\,\mathsf{h}_2 - 4\psi_d^t\,\mathsf{d} - 4\psi_r^t\,\mathsf{r} - 4\psi_{i_1}^t\,\mathsf{i}_1 - 4\psi_{i_2}^t\,\mathsf{i}_2 - 4\psi_j^t\,\mathsf{j}\\
&= (4\psi_{h_2}^{h_2} + 4\psi_{h_1}^{h_1} - 4\psi_t^t)\,\mathsf{t} + (\psi_d^{h_2} + \psi_r^{h_1} - 4\psi_{h_1}^t)\,\mathsf{h}_1 + (\psi_r^{h_2} - \psi_d^{h_1} - 4\psi_{h_2}^t)\,\mathsf{h}_2 +\\
&\quad + (2\psi_{i_2}^{h_2} + 2\psi_{i_1}^{h_1} - 4\psi_d^t)\,\mathsf{d} + (6\psi_{i_1}^{h_2} - 6\psi_{i_2}^{h_1} - 4\psi_r^t)\,\mathsf{r} + (\psi_j^{h_2} - 4\psi_{i_1}^t)\,\mathsf{i}_1 +\\
&\quad + (-\psi_j^{h_1} - 4\psi_{i_2}^t)\,\mathsf{i}_2 + (-4\psi_j^t)\,\mathsf{j}.
\end{aligned}
$$

As a result, a 2-cochain Φ of the general form written above equals the coboundary $\partial\Psi$ of a 1-cochain if and only if there exist 24 quantities $\psi_\cdot^\cdot$ such that Ψ's three families of eight coefficients $\phi_\cdot^{th_1}$, $\phi_\cdot^{th_2}$, $\phi_\cdot^{h_1h_2}$ are equal, respectively, to the three collections of eight coefficients just found:

$\boxed{1}$	$\phi_t^{th_1} = 2\psi_d^{h_1} - 4\psi_{h_2}^t$	$\boxed{1}$	$\phi_t^{th_2} = 2\psi_d^{h_2} + 4\psi_{h_1}^t$	$\boxed{0}$	$\phi_t^{h_1h_2} = 4\psi_{h_2}^{h_2} + 4\psi_{h_1}^{h_1} - 4\psi_t^t$
$\boxed{2}$	$\phi_{h_1}^{th_1} = \psi_{i_1}^{h_1} - \psi_d^t$	$\boxed{2}$	$\phi_{h_1}^{th_2} = \psi_{i_1}^{h_2} + \psi_r^t$	$\boxed{1}$	$\phi_{h_1}^{h_1h_2} = \psi_d^{h_2} + \psi_r^{h_1} - 4\psi_{h_1}^t$
$\boxed{2}$	$\phi_{h_2}^{th_1} = \psi_{i_2}^{h_1} - \psi_r^t$	$\boxed{2}$	$\phi_{h_2}^{th_2} = \psi_{i_2}^{h_2} - \psi_d^t$	$\boxed{1}$	$\phi_{h_2}^{h_1h_2} = \psi_r^{h_2} - \psi_d^{h_1} + 4\psi_{h_2}^t$
$\boxed{3}$	$\phi_d^{th_1} = \psi_j^{h_1} - 2\psi_{i_2}^t$	$\boxed{3}$	$\phi_d^{th_2} = \psi_j^{h_2} + 2\psi_{i_1}^t$	$\boxed{2}$	$\phi_d^{h_1h_2} = 2\psi_{i_2}^{h_2} + 2\psi_{i_1}^{h_1} - 4\psi_d^t$
$\boxed{3}$	$\phi_r^{th_1} = -6\psi_{i_1}^t$	$\boxed{3}$	$\phi_r^{th_2} = -6\psi_{i_2}^t$	$\boxed{2}$	$\phi_r^{h_1h_2} = 6\psi_{i_1}^{h_2} - 6\psi_{i_2}^{h_1} - 4\psi_r^t$
$\boxed{4}$	$\phi_{i_1}^{th_1} = -\psi_j^t$	$\boxed{4}$	$\phi_{i_1}^{th_2} = 0$	$\boxed{3}$	$\phi_{i_1}^{h_1h_2} = \psi_j^{h_2} - 4\psi_{i_1}^t$
$\boxed{4}$	$\phi_{i_2}^{th_1} = 0$	$\boxed{4}$	$\phi_{i_2}^{th_2} = -\psi_j^t$	$\boxed{3}$	$\phi_{i_2}^{h_1h_2} = -\psi_j^{h_1} - 4\psi_{i_2}^t$
$\boxed{5}$	$\phi_j^{th_1} = 0$	$\boxed{5}$	$\phi_j^{th_2} = 0$	$\boxed{4}$	$\phi_j^{h_1h_2} = -4\psi_j^t.$

5.4. **Graded computation of** $H^2(\mathfrak{g}_-,\mathfrak{g})$**.** Now, the map $\Psi \mapsto \partial\Psi =: \Phi$ so obtained explicitly is visibly linear $(\psi_\cdot^\cdot) \longmapsto (\phi_\cdot^{\cdot\cdot})$, and furthermore, because $\partial\Psi \in \mathscr{Z}^2$ and because cochains naturally split in homogeneous components, this map happens to be a direct sum of six linear maps:

$$\mathscr{C}^1_{[0]} \to \mathscr{Z}^2_{[0]},\ \ \mathscr{C}^1_{[1]} \to \mathscr{Z}^2_{[1]},\ \ \mathscr{C}^1_{[2]} \to \mathscr{Z}^2_{[2]},\ \ \mathscr{C}^1_{[3]} \to \mathscr{Z}^2_{[3]},\ \ \mathscr{C}^1_{[4]} \to \mathscr{Z}^2_{[4]},\ \ \mathscr{C}^1_{[5]} \to \mathscr{Z}^2_{[5]},$$

the last one being just $\{0\} \to \{0\}$, that is to say a direct sum of the following five explicit nonzero linear maps:

$$\begin{aligned}
\partial_{[0]}:\ \ &\big(\psi_t^t,\psi_{h_1}^{h_1},\psi_{h_2}^{h_1},\psi_{h_1}^{h_2},\psi_{h_2}^{h_2}\big) \longmapsto \big(4\psi_{h_2}^{h_2} + 4\psi_{h_1}^{h_1} - 4\psi_t^t\big) \\
&\qquad = \big(\phi_t^{h_1h_2}\big) \\
\partial_{[1]}:\ \ &\big(\psi_{h_1}^t,\psi_{h_2}^t,\psi_d^{h_1},\psi_r^{h_1},\psi_d^{h_2},\psi_r^{h_2}\big) \longmapsto \big(2\psi_d^{h_1} - 4\psi_{h_2}^t, 2\psi_d^{h_2} + 4\psi_{h_1}^t, \psi_d^{h_2} + \psi_r^{h_1} - 4\psi_{h_1}^t, \\
&\qquad \psi_r^{h_2} - \psi_d^{h_1} + 4\psi_{h_2}^t\big) \\
&\qquad = \big(\phi_t^{th_1},\phi_t^{th_2},\phi_{h_1}^{h_1h_2},\phi_{h_2}^{h_1h_2}\big) \\
\partial_{[2]}:\ \ &\big(\psi_d^t,\psi_r^t,\psi_{i_1}^{h_1},\psi_{i_2}^{h_1},\psi_{i_1}^{h_2},\psi_{i_2}^{h_2}\big) \longmapsto \big(\psi_{i_1}^{h_1} - \psi_d^t, \psi_{i_2}^{h_1} - \psi_r^t, \psi_{i_1}^{h_2} + \psi_r^t, \psi_{i_2}^{h_2} - \psi_d^t, \\
&\qquad 2\psi_{i_2}^{h_2} + 2\psi_{i_1}^{h_1} - 4\psi_d^t, 6\psi_{i_1}^{h_2} - 6\psi_{i_2}^{h_1} - 4\psi_r^t\big) \\
&\qquad = \big(\phi_{h_1}^{th_1},\phi_{h_2}^{th_1},\phi_{h_1}^{th_2},\phi_{h_2}^{th_2},\phi_d^{h_1h_2},\phi_r^{h_1h_2}\big) \\
\partial_{[3]}:\ \ &\big(\psi_{i_1}^t,\psi_{i_2}^t,\psi_j^{h_1},\psi_j^{h_2}\big) \longmapsto \big(\psi_j^{h_1} - 2\psi_{i_2}^t, -6\psi_{i_1}^t, \psi_j^{h_2} + 2\psi_{i_1}^t, -6\psi_{i_2}^t, \psi_j^{h_2} - 4\psi_{i_1}^t, \\
&\qquad -\psi_j^{h_1} - 4\psi_{i_2}^t\big) \\
&\qquad = \big(\phi_d^{th_1},\phi_r^{th_1},\phi_d^{th_2},\phi_r^{th_2},\phi_{i_1}^{h_1h_2},\phi_{i_2}^{h_1h_2}\big) \\
\partial_{[4]}:\ \ &\big(\psi_j^t\big) \longmapsto \big(-\psi_j^t, 0, 0, -\psi_j^t, -4\psi_j^t\big) \\
&\qquad = \big(\phi_{i_1}^{th_1},\phi_{i_2}^{th_1},\phi_{i_1}^{th_2},\phi_{i_2}^{th_2},\phi_j^{h_1h_2}\big).
\end{aligned}$$

One checks easily that the images of $\partial_{[2]}$, $\partial_{[3]}$, $\partial_{[4]}$ satisfy the equations of $\mathscr{Z}^2_{[2]}$, $\mathscr{Z}^2_{[3]}$, $\mathscr{Z}^2_{[4]}$ found above. Now, it is easy to view the dimensions of the homogeneous components of the second cohomology space:

$$H^2(\mathfrak{g}_-,\mathfrak{g}) = \bigoplus_{h\in\mathbb{Z}} H^2_{[h]}(\mathfrak{g}_-,\mathfrak{g}) \quad \text{with} \quad H^2_{[h]}(\mathfrak{g}_-,\mathfrak{g}) := \frac{\mathscr{Z}^2_{[h]}(\mathfrak{g}_-,\mathfrak{g})}{\mathscr{B}^2_{[h]}(\mathfrak{g}_-,\mathfrak{g})}.$$

Remind that $\mathscr{Z}^2_{[0]} \simeq \mathbb{R}^1$, $\mathscr{Z}^2_{[1]} \simeq \mathbb{R}^4$, $\mathscr{Z}^2_{[2]} \simeq \mathbb{R}^5$, $\mathscr{Z}^2_{[3]} \simeq \mathbb{R}^4$, $\mathscr{Z}^2_{[4]} \simeq \mathbb{R}^3$. Clearly, $\partial_{[0]} \colon \mathbb{R}^5 \to \mathscr{Z}^2_{[0]} \simeq \mathbb{R}$ is onto, whence $H^2_{[0]} = \{0\}$. Similarly, one easily convinces oneself with almost no computations that $\partial_{[1]}$ is of rank 4, that $\partial_{[2]}$ is of rank 5, that $\partial_{[3]}$ is of rank 4 and that $\partial_{[5]}$ is of rank 1. It follows that $H^2_{[1]} = \{0\}$, that $H^2_2 = \{0\}$, that $H^2_{[3]} = \{0\}$, the only nonzero cohomology space being $H^2_{[4]} \simeq \mathbb{R}^2$ which is 2-dimensional. Finally, one also sees that $H^2(\mathfrak{g}_-, \mathfrak{g}) = H^2_{[4]}$ is generated by the following two independent 2-cochains:

$$\boxed{\begin{aligned} &\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 - 2\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{j} \\ \text{and:}\quad &\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_1 - \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_2. \end{aligned}}$$

In conclusion, let us summarize the results obtained by means of a dimensional table obtained in [14] that we recover here:

Homogeneity	$\dim \mathscr{C}^2$	$\dim \mathscr{Z}^2$	$\dim \mathscr{B}^2$	$\dim H^2$
0	1	1	1	0
1	4	4	4	0
2	6	5	5	0
3	6	4	4	0
4	5	3	1	2
5	2	0	0	0

5.5. **Codifferential.** When the Lie algebra $\mathfrak{g}$ is semi-simple, there exists another, degree-decreasing linear operator on the space of cochains:

$$\partial^{*k} \colon \quad \mathscr{C}^{k+1}\big(\mathfrak{g}_-,\, \mathfrak{g}\big) \longrightarrow \mathscr{C}^{k}\big(\mathfrak{g}_-,\, \mathfrak{g}\big),$$

called the *codifferential operator*, which is defined as follows. For a Lie algebra $\mathfrak{g}$ defined over a commutative field $\mathbb{K}$, recall that $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{K}} \mathfrak{g}$ denotes the adjoint action of $\mathfrak{g}$ on its space of endomorphisms:

$$(\mathrm{ad}(\mathsf{x}))(\mathsf{y}) := [\mathsf{x}, \mathsf{y}]_{\mathfrak{g}} \qquad\qquad (\mathsf{x}, \mathsf{y} \in \mathfrak{g}).$$

Recall also ([21]) that the Killing form $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is defined as being the symmetric bilinear form:

$$B(\mathsf{x}, \mathsf{y}) := \mathrm{Tr}\big(\mathrm{ad}(\mathsf{x}) \circ \mathrm{ad}(\mathsf{y})\big),$$

and that its nondegeneracy is equivalent to the semi-simplicity of $\mathfrak{g}$. Furthermore ([8]), if $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{\mu}$ is a graded semi-simple Lie algebra, then B induces an isomorphism $\mathfrak{g}_i^* \cong \mathfrak{g}_{-i}$ of $\mathfrak{g}_0$-modules for $i = 1, \ldots, \mu$. If we denote $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\mu$ by $\mathfrak{g}_+$, then each space $\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g}) \cong \wedge^k \mathfrak{g}_-^* \otimes \mathfrak{g}$ can be identified with the dual space of the space $\wedge^k \mathfrak{g}_+^* \otimes \mathfrak{g} \cong \mathscr{C}^k(\mathfrak{g}_+, \mathfrak{g})$. In particular, the negative of the dual map of $\partial^k \colon \quad \mathscr{C}^k\big(\mathfrak{g}_+,\, \mathfrak{g}\big) \longrightarrow \mathscr{C}^{k+1}\big(\mathfrak{g}_+,\, \mathfrak{g}\big)$ can be viewed as a linear map which is exactly the codifferential operator $\partial^{*k} \colon \quad \mathscr{C}^{k+1}\big(\mathfrak{g}_-,\, \mathfrak{g}\big) \longrightarrow \mathscr{C}^{k}\big(\mathfrak{g}_-,\, \mathfrak{g}\big)$. From this definition, it immediately follows that $\partial^{*(k-1)} \circ \partial^{*k} = 0$, whence one has a second *cochain complex*:

$$0 \xrightarrow{\partial^{*n}} \mathscr{C}^n \xrightarrow{\partial^{*(n-1)}} \mathscr{C}^{n-1} \xrightarrow{\partial^{*(n-2)}} \cdots \xrightarrow{\partial^{*2}} \mathscr{C}^2 \xrightarrow{\partial^{*1}} \mathscr{C}^1 \xrightarrow{\partial^{*0}} 0.$$

Lastly ([36]), for any $k+1$ elements $\mathsf{z}_1, \dots, \mathsf{z}_k$ of $\mathfrak{g}_-$ and for any $(k+1)$-cochain $\Psi \in \mathscr{C}^{k+1}(\mathfrak{g}_-, \mathfrak{g})$, the expression of $\partial^{*k}\Psi$ realizes as follows:

$$(\partial^{*k}\Psi)\big(\mathsf{z}_1, \dots, \mathsf{z}_k\big) := \sum_{i=1}^{n} \big[\mathsf{v}_i^*, \Psi\big(\mathsf{v}_i, \mathsf{z}_1, \dots, \mathsf{z}_k\big)\big]_{\mathfrak{g}} +$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{k} (-1)^{j+1}\, \Psi\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\big[\mathsf{v}_i^*, \mathsf{z}_j\big]_{\mathfrak{g}}\big),\ \mathsf{v}_i,\ \mathsf{z}_1, \dots, \widehat{\mathsf{z}}_j, \dots, \mathsf{z}_k\Big),$$

where $\mathsf{v}_1, \dots, \mathsf{v}_n$ are independent basis elements of $\mathfrak{g}_-$, where v_i^* $(i = 1, \dots, n)$ is the dual of v_i with respect to the Killing form, where $\mathsf{proj}_{\mathfrak{g}_-}\big(\big[\mathsf{v}_i^*, \mathsf{z}_j\big]_{\mathfrak{g}}\big)$ denotes the $\mathfrak{g}_-$-component of $\big[\mathsf{v}_i^*, \mathsf{z}_j\big]_{\mathfrak{g}}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$, and where $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

6. Initial Frame on a Strongly Pseudoconvex $M^3 \subset \mathbb{C}^2$

6.1. **Explicit CR structure.** Let M be a real $\mathscr{C}^1$-smooth hypersurface of $\mathbb{C}^2$, represented by:

$$v = \varphi(x, y, u)$$

in coordinates $(z, w) = (x + iy, u + iv)$. After a linear straightening, we may assume $0 \in M$ and $T_0 M = \{\operatorname{Im} w = 0\}$, so that $\varphi(0) = \varphi_x(0) = \varphi_y(0) = \varphi_u(0) = 0$. A $(0,1)$ vector field of the form:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \mathtt{A}\, \frac{\partial}{\partial \overline{w}}$$

is tangent to M if and only if its coefficient $\mathtt{A}$ satisfies:

$$0 = \frac{\mathtt{A}}{2i} + \varphi_{\overline{z}} + \frac{\mathtt{A}}{2}\, \varphi_u,$$

or equivalently:

$$\boxed{\mathtt{A} = \frac{2\,\varphi_{\overline{z}}}{i - \varphi_u}}.$$

Consequently the vector field:

$$\overline{\mathscr{L}} = \frac{1}{2}\,\frac{\partial}{\partial x} + \frac{i}{2}\,\frac{\partial}{\partial y} + \left(\frac{2\,\varphi_{\overline{z}}}{i - \varphi_u}\right)\left(\frac{1}{2}\,\frac{\partial}{\partial u} + \frac{i}{2}\,\frac{\partial}{\partial v}\right)$$

generates $T^{0,1}M$ in a neighborhood of the origin, since $T^{0,1}M$ is obviously of rank $\dim \mathbb{C}^2 - \mathrm{CRdim}\, M = 2 - 1 = 1$.

We notice that this $\overline{\mathscr{L}}$ is written here *extrinsically*, namely it involves the extra coordinate v and it lives in a neighborhood of M, in $\mathbb{C}^2$, while M itself, which is three-dimensional, is naturally equipped with the three real coordinates (x, y, u). Since we want two intrinsic sections of:

$$T^c M = \operatorname{Re}\big(T^{0,1}M\big),$$

we need at first to pullback this $\overline{\mathscr{L}}$ to M, which simply means dropping the basic vector field $\frac{\partial}{\partial v}$ and replacing v by $\varphi(x, y, u)$ in the coefficient functions (in fact here, no v appears), and we get the following section:

$$\overline{\mathscr{L}}\big|_M = \frac{1}{2}\,\frac{\partial}{\partial x} + \frac{i}{2}\,\frac{\partial}{\partial y} + \bigg(\frac{2\,\varphi_{\overline{z}}}{i-\varphi_u}\bigg)\bigg(\frac{1}{2}\,\frac{\partial}{\partial u}\bigg)$$

which generates $T^{0,1}M$, *intrinsically* (*see* also the basic first chapters of [3, 7, 20]).

So it remains only to decompose $\overline{\mathscr{L}}\big|_M$ in real and imaginary parts, and at first, we do this for the coefficient:

$$\mathtt{A} = \frac{\varphi_x + i\,\varphi_y}{i-\varphi_u} = \frac{\varphi_y - \varphi_x\,\varphi_u}{1+\varphi_u^2} + i\,\frac{-\varphi_x - \varphi_y\,\varphi_u}{1+\varphi_u^2}.$$

Hence we can provide an explicit representation of two independent real vector fields that are generators for T^cM near the origin, namely $2\,\mathrm{Re}\,\big(\overline{\mathscr{L}}\big|_M\big)$ and $2\,\mathrm{Im}\,\big(\overline{\mathscr{L}}\big|_M\big)$, multiplying by a factor 2 to simplify a bit.

Lemma 6.1. *For any local $\mathscr{C}^1$-smooth real hypersurface M^3 of $\mathbb{C}^2$ which is represented as a graph:*

$$v = \varphi(x,y,u)$$

in coordinates $(z,w) = (x+iy,\, u+iv)$, the complex tangent bundle $T^cM = \mathrm{Re}\,T^{0,1}M$ is generated by the two explicit vector fields:

$$\begin{cases} H_1 := \dfrac{\partial}{\partial x} + \bigg(\dfrac{\varphi_y - \varphi_x\,\varphi_u}{1+\varphi_u^2}\bigg)\dfrac{\partial}{\partial u} \\[2ex] H_2 := \dfrac{\partial}{\partial y} + \bigg(\dfrac{-\varphi_x - \varphi_y\,\varphi_u}{1+\varphi_u^2}\bigg)\dfrac{\partial}{\partial u}. \end{cases}$$

In fact, as one easily verifies, one does not need that $\varphi(0) = \varphi_x(0) = \varphi_y(0) = \varphi_u(0) = 0$ for the lemma to hold true (but we will always assume that such an affine normalization is done in advance, since it is free).

Some further notation will be useful. If we set:

$$\Delta := 1 + \varphi_u^2, \qquad \Lambda_1 := \varphi_y - \varphi_x\,\varphi_u, \qquad \Lambda_2 := -\,\varphi_x - \varphi_y\,\varphi_u,$$

our two intrinsic T^cM-tangent vector fields become:

$$H_1 = \frac{\partial}{\partial x} + \frac{\Lambda_1}{\Delta}\,\frac{\partial}{\partial u} \qquad \text{and} \qquad H_2 = \frac{\partial}{\partial y} + \frac{\Lambda_2}{\Delta}\,\frac{\partial}{\partial u}.$$

6.2. **Levi nondegeneracy assumption.** Now, we assume that M is Levi nondegenerate at the origin, so that second order terms can be assumed to be normalized as:

$$v = \varphi(x,y,u) = x^2 + y^2 + \mathrm{O}(3).$$

We may therefore compute the bracket $[H_1,\, H_2]$ using these notations, and realize that two terms underlined cancel:

$$\begin{aligned}
[H_1,\, H_2] &= \Big[\tfrac{\partial}{\partial x} + \big(\tfrac{\Lambda_1}{\Delta}\big)\tfrac{\partial}{\partial u},\;\; \tfrac{\partial}{\partial y} + \big(\tfrac{\Lambda_2}{\Delta}\big)\tfrac{\partial}{\partial u}\Big] \\
&= \Big(\tfrac{\Lambda_{2,x}}{\Delta} - \Lambda_2\,\tfrac{\Delta_x}{\Delta^2} + \tfrac{\Lambda_1}{\Delta}\,\tfrac{\Lambda_{2,u}}{\Delta} - \underline{\tfrac{\Lambda_1}{\Delta}\Lambda_2\tfrac{\Delta_u}{\Delta^2}}_{\circ} - \\
&\qquad - \tfrac{\Lambda_{1,y}}{\Delta} + \Lambda_1\,\tfrac{\Delta_y}{\Delta^2} - \tfrac{\Lambda_2}{\Delta}\,\tfrac{\Lambda_{1,u}}{\Delta} + \underline{\tfrac{\Lambda_2}{\Delta}\Lambda_1\tfrac{\Delta_u}{\Delta^2}}_{\circ}\Big)\,\tfrac{\partial}{\partial u},
\end{aligned}$$

so that the common denominator is *not* equal to Δ^3 as one would have expected, but is equal to Δ^2. Expanding the partial derivatives and simplifying either by hand or with a computer ([2]), we therefore get:

$$\boxed{\begin{aligned}\big[H_1,\,H_2\big] &= \Big[\tfrac{\partial}{\partial x}+\tfrac{\varphi_y-\varphi_x\,\varphi_u}{1+\varphi_u^2}\,\tfrac{\partial}{\partial u},\ \ \tfrac{\partial}{\partial y}+\tfrac{-\varphi_x+\varphi_y\,\varphi_u}{1+\varphi_u^2}\,\tfrac{\partial}{\partial u}\Big]\\ &= \Big(\tfrac{1}{(1+\varphi_u^2)^2}\big\{-\varphi_{xx}-\varphi_{yy}-2\,\varphi_y\,\varphi_{xu}-\varphi_x^2\,\varphi_{uu}+2\,\varphi_x\,\varphi_{yu}-\varphi_y^2\,\varphi_{uu}+\\ &\qquad\qquad +2\,\varphi_y\,\varphi_u\,\varphi_{yu}+2\,\varphi_x\,\varphi_u\,\varphi_{xu}-\varphi_u^2\,\varphi_{xx}-\varphi_u^2\,\varphi_{yy}\big\}\Big)\,\frac{\partial}{\partial u}.\end{aligned}}$$

Equivalently, as we want for later use to specify the numerator, if we set:

$$\begin{aligned}\Upsilon := -\varphi_{xx}-\varphi_{yy}-2\,\varphi_y\,\varphi_{xu}-\varphi_x^2\,\varphi_{uu}+2\,\varphi_x\,\varphi_{yu}-\varphi_y^2\,\varphi_{uu}+\\ +2\,\varphi_y\,\varphi_u\,\varphi_{yu}+2\,\varphi_x\,\varphi_u\,\varphi_{xu}-\varphi_u^2\,\varphi_{xx}-\varphi_u^2\,\varphi_{yy},\end{aligned}$$

we can write shortly:

$$\big[H_1,\,H_2\big]=\frac{\Upsilon}{\Delta^2}\,\frac{\partial}{\partial u}.$$

Now, thanks to the Levi-nondegeneracy assumption and because of the normalizations $0=\varphi(0)=\varphi_x(0)=\varphi_y(0)=\varphi_u(0)$, we have $\Upsilon(0)=-4$ (notice the minus sign), that is to say:

$$[H_1,H_2]\big|_0=-4\,\tfrac{\partial}{\partial u}\big|_0.$$

So, if we introduce the vector field (we choose a plus sign in the factor $\frac{1}{4}$):

$$T:=\frac{1}{4}\,\frac{\Upsilon}{\Delta^2}\,\frac{\partial}{\partial u},$$

we may rewrite:

$$\boxed{[H_1,\,H_2]=4\,T}.$$

6.3. **Length-three brackets.** At the next step, we must compute the two brackets $4\,[H_1,T]$ and $4\,[H_2,T]$, for instance the first one, in which we see how the denominator grows and of which we extract the numerator:

$$\begin{aligned}4\,\big[H_1,\,T\big] &= \Big[\tfrac{\partial}{\partial x}+\big(\tfrac{\Lambda_1}{\Delta}\big)\tfrac{\partial}{\partial u},\ \ \big(\tfrac{\Upsilon}{\Delta^2}\big)\tfrac{\partial}{\partial u}\Big]\\ &= \Big(\tfrac{\Upsilon_x}{\Delta^2}-2\,\Upsilon\,\tfrac{\Delta_x}{\Delta^3}+\tfrac{\Lambda_1}{\Delta}\,\tfrac{\Upsilon_u}{\Delta^2}-2\,\tfrac{\Lambda_1}{\Delta}\,\Upsilon\,\tfrac{\Delta_u}{\Delta^3}-\tfrac{\Upsilon}{\Delta^2}\,\tfrac{\Lambda_{1,u}}{\Delta}+\tfrac{\Upsilon}{\Delta^2}\,\Lambda_1\,\tfrac{\Delta_u}{\Delta^2}\Big)\frac{\partial}{\partial u}\\ &= \left(\frac{\Delta^2\big[\Upsilon_x\big]+\Delta\big[-2\,\Upsilon\,\Delta_x+\Lambda_1\,\Upsilon_u-\Upsilon\,\Lambda_{1,u}\big]-\Lambda_1\,\Upsilon\,\Delta_u}{\Delta^4}\right)\frac{\partial}{\partial u}.\end{aligned}$$

Exchanging H_1 with H_2, which means replacing Λ_1 by Λ_2 and $\frac{\partial}{\partial x}$ by $\frac{\partial}{\partial y}$, we get similarly and without any computation:

$$4\,\big[H_1,\,T\big]=\left(\frac{\Delta^2\big[\Upsilon_y\big]+\Delta\big[-2\,\Upsilon\,\Delta_y+\Lambda_2\,\Upsilon_u-\Upsilon\,\Lambda_{2,u}\big]-\Lambda_2\,\Upsilon\,\Delta_u}{\Delta^4}\right)\frac{\partial}{\partial u}.$$

Let us therefore introduce two new summarizing names:

$$\begin{aligned}A_1 &:= \Delta^2\big[\Upsilon_x\big]+\Delta\big[-2\,\Upsilon\,\Delta_x+\Lambda_1\,\Upsilon_u-\Upsilon\,\Lambda_{1,u}\big]-\Lambda_1\,\Upsilon\,\Delta_u,\\ A_2 &:= \Delta^2\big[\Upsilon_y\big]+\Delta\big[-2\,\Upsilon\,\Delta_y+\Lambda_2\,\Upsilon_u-\Upsilon\,\Lambda_{2,u}\big]-\Lambda_2\,\Upsilon\,\Delta_u,\end{aligned}$$

for the two appearing numerators. Now, for later use, we want to re-express these two brackets $[H_1, T]$ and $[H_2, T]$ in terms of the third field T transverse to T^cM, and for this, it suffices to simply replace the basic field:

$$\frac{\partial}{\partial u} = \frac{4\,\Delta^2}{\Upsilon}\,T,$$

so that doing this just yields expressions of the two supplementary brackets

$$\big[H_1,\,T\big] = \frac{1}{4}\,\frac{A_1}{\Delta^4}\,\frac{4\,\Delta^2}{\Upsilon}\,T = \frac{A_1}{\Delta^2\,\Upsilon}\,T,$$
$$\big[H_2,\,T\big] = \frac{1}{4}\,\frac{A_2}{\Delta^4}\,\frac{4\,\Delta^2}{\Upsilon}\,T = \frac{A_2}{\Delta^2\,\Upsilon}\,T.$$

However, these two numerators A_1 and A_2 are not yet expanded as explicit polynomials in the third-order jet $J^3_{x,y,u}\varphi$ of the graphing function $\varphi(x,y,u)$ for M. This can be done either by hand or using a computer ([2]), hence we directly summarize the fundamental result fully describing a useful initial frame for TM which is naturally produced by T^cM.

Proposition 6.2. *If M^3 is an arbitrary local $\mathscr{C}^3$-smooth Levi nondegenerate real hypersurface of $\mathbb{C}^2$ represented in coordinates $(z,w) = (x+iy,\,u+iv)$ as a graph:*

$$v = \varphi(x,y,u) = x^2 + y^2 + \mathrm{O}(3),$$

and whose complex tangent bundle T^cM is generated by the two explicit vector fields:

$$H_1 := \tfrac{\partial}{\partial x} + \big(\tfrac{\varphi_y - \varphi_x\,\varphi_u}{1+\varphi_u^2}\big)\tfrac{\partial}{\partial u} \qquad \text{and} \qquad H_2 := \tfrac{\partial}{\partial y} + \big(\tfrac{-\varphi_x - \varphi_y\,\varphi_u}{1+\varphi_u^2}\big)\tfrac{\partial}{\partial u},$$

satisfying $H_1|_0 = \frac{\partial}{\partial x}\big|_0$ and $H_2|_0 = \frac{\partial}{\partial y}\big|_0$, then the third, bracketed vector field:

$$\begin{aligned}
T &:= \tfrac{1}{4}\,[H_1, H_2] \\
&= \Big(\tfrac{1}{4}\,\tfrac{1}{(1+\varphi_u^2)^2}\big\{ -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + \\
&\qquad\qquad + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy}\big\}\Big)\frac{\partial}{\partial u} \\
&=: \Big(\tfrac{1}{4}\,\tfrac{1}{(\Delta)^2}\,\big\{\Upsilon\big\}\Big)\frac{\partial}{\partial u}
\end{aligned}$$

satisfying $T\big|_0 = -\frac{\partial}{\partial u}\big|_0$ produces, jointly with H_1 and H_2 of which it is locally linearly independent, a frame for TM in a neighborhood of the origin. Furthermore, the remaining Lie bracket structure of this frame:

$$\boxed{[H_1, T] = \Phi_1\,T \qquad \text{and} \qquad [H_2, T] = \Phi_2\,T},$$

involves two further rational functions:

$$\Phi_1 = \frac{A_1}{\Delta^2\,\Upsilon} \qquad \text{and} \qquad \Phi_2 = \frac{A_2}{\Delta^2\,\Upsilon}$$

having common denominator equal to $\Delta^2\,\Upsilon$ *and whose two numerators* A_1 *and* A_2*, depending both upon the third-order jet* $J^3_{x,y,u}\varphi$*, read explicitly as follows:*

$$
\begin{aligned}
A_1 = & -\varphi_{xxx} - \varphi_{xyy} + 2\,\varphi_x\,\varphi_{xyu} - 3\,\varphi_y\,\varphi_{xxu} - \varphi_y\,\varphi_{yyu} - 3\,\varphi_y^2\,\varphi_{xuu} + \\
& + 2\,\varphi_x\,\varphi_y\,\varphi_{yuu} - \varphi_x^2\,\varphi_{xuu} - \varphi_x^2\,\varphi_y\,\varphi_{uuu} - \varphi_y^3\,\varphi_{uuu} - 2\,\varphi_x\,\varphi_y\,\varphi_{xu}\,\varphi_{uu} - \\
& - 3\,\varphi_x\varphi_{xx}\,\varphi_{uu} + \varphi_y^2\,\varphi_{uu}\,\varphi_{yu} - 2\,\varphi_y\,\varphi_{uu}\,\varphi_{xy} + 3\,\varphi_x^2\,\varphi_{yu}\,\varphi_{uu} - \\
& - \varphi_x\,\varphi_{yy}\,\varphi_{uu} - \varphi_x\,\varphi_y^2\,\varphi_{uu}^2 + 4\,\varphi_y\,\varphi_{xu}\,\varphi_{yu} - \varphi_x^3\,\varphi_{uu}^2 + \varphi_{yu}\,\varphi_{yy} + \\
& + 3\,\varphi_{xx}\,\varphi_{yu} - 2\,\varphi_{xu}\,\varphi_{xy} + 2\,\varphi_x\,\varphi_{xu}^2 - 2\,\varphi_x\,\varphi_{yu}^2 + \\
& + \varphi_u\Big(3\,\varphi_x\,\varphi_{xxu} + 2\,\varphi_y\,\varphi_{xyu} + \varphi_x\,\varphi_{yyu} + 4\,\varphi_x\,\varphi_y\,\varphi_{xuu} + 2\,\varphi_y^2\,\varphi_{yuu} - \\
& \quad - 2\,\varphi_x^2\,\varphi_{yuu} + \varphi_x\,\varphi_y^2\,\varphi_{uuu} + \varphi_x^2\,\varphi_{uuu} + 2\,\varphi_x^2\,\varphi_{uu}^2\,\varphi_y + 5\,\varphi_{uu}\,\varphi_{xu}\,\varphi_x^2 - \\
& \quad - 8\,\varphi_x\,\varphi_{xu}\,\varphi_{yu} + 7\,\varphi_y^2\,\varphi_{xu}\,\varphi_{uu} + \varphi_{yy}\,\varphi_{xu} + 2\,\varphi_y^3\,\varphi_{uu}^2 + 3\,\varphi_{xx}\,\varphi_{xu} + \\
& \quad + 8\,\varphi_y\,\varphi_{xu}^2 + 2\,\varphi_{xy}\,\varphi_{yu} - 2\,\varphi_x\,\varphi_y\,\varphi_{yu}\,\varphi_{uu}\Big) + \\
& + \varphi_u^2\Big(-3\,\varphi_{xxx} - 3\,\varphi_{xyy} - 6\,\varphi_y\,\varphi_{xxu} - 2\,\varphi_y\,\varphi_{yyu} + 4\,\varphi_x\,\varphi_{xyu} - 4\,\varphi_y^2\,\varphi_{xuu} - \\
& \quad - 4\,\varphi_x^2\,\varphi_{xuu} - \varphi_y^3\,\varphi_{uuu} - \varphi_y\,\varphi_x^2\,\varphi_{uuu} - 2\,\varphi_x\,\varphi_{uu}\,\varphi_{yy} + 7\,\varphi_x^2\,\varphi_{yu}\,\varphi_{uu} - \\
& \quad - 6\,\varphi_x\,\varphi_{uu}\,\varphi_{xx} - 4\,\varphi_y\,\varphi_{uu}\,\varphi_{xy} - 3\,\varphi_y^2\,\varphi_{uu}^2\,\varphi_x - 3\,\varphi_y^2\,\varphi_{uu}\,\varphi_{yu} - 4\,\varphi_{xu}\,\varphi_{xy} - \\
& \quad - 3\,\varphi_x^3\,\varphi_{uu}^2 + 6\,\varphi_{xx}\,\varphi_{yu} - 4\,\varphi_x\,\varphi_{yu}^2 - 4\,\varphi_x\,\varphi_{xu}^2 + 2\,\varphi_{yu}\,\varphi_{yy} - 10\,\varphi_x\,\varphi_y\,\varphi_{xu}\,\varphi_{uu}\Big) + \\
& + \varphi_u^3\Big(6\,\varphi_x\,\varphi_{xxu} + 4\,\varphi_y\,\varphi_{xyu} + 2\,\varphi_x\,\varphi_{yyu} + 4\,\varphi_x\,\varphi_y\,\varphi_{xuu} - 2\,\varphi_x^2\,\varphi_{yuu} + 2\,\varphi_y^2\,\varphi_{yuu} + \\
& \quad + \varphi_x^3\,\varphi_{uuu} + \varphi_x\,\varphi_y^2\,\varphi_{uuu} + 3\,\varphi_y^2\,\varphi_{xu}\,\varphi_{uu} - 8\,\varphi_{xu}\,\varphi_{yu}\,\varphi_x + 9\,\varphi_{uu}\,\varphi_{xu}\,\varphi_x^2 + \\
& \quad + 4\,\varphi_{xy}\,\varphi_{yu} + 8\,\varphi_y\,\varphi_{xy}^2 + 2\,\varphi_{yy}\,\varphi_{xu} + 6\,\varphi_{xx}\,\varphi_{xu} + 6\,\varphi_x\,\varphi_y\,\varphi_{yu}\,\varphi_{uu}\Big) + \\
& + \varphi_u^4\Big(-3\,\varphi_{xxx} - 3\,\varphi_{xyy} + 2\,\varphi_x\,\varphi_{xyu} - \varphi_y\,\varphi_{yyu} - 3\,\varphi_y\,\varphi_{xxu} - 3\,\varphi_x^2\,\varphi_{xuuu} - \\
& \quad - 2\,\varphi_x\,\varphi_y\,\varphi_{yuu} - \varphi_y^2\,\varphi_{xuu} - 3\,\varphi_x\,\varphi_{uu}\,\varphi_{xx} - \varphi_x\,\varphi_{uu}\,\varphi_{yy} - 6\,\varphi_x\,\varphi_{xu}^2 - \\
& \quad - 2\,\varphi_x\,\varphi_{yu}^2 - 2\,\varphi_y\,\varphi_{uu}\,\varphi_{xy} - 4\,\varphi_y\,\varphi_{xu}\,\varphi_{yu} - 2\,\varphi_{xu}\,\varphi_{xy} + 3\,\varphi_{xx}\,\varphi_{yu} + \varphi_{yu}\,\varphi_{yy}\Big) + \\
& + \varphi_u^5\Big(\varphi_x\,\varphi_{yyu} + 2\,\varphi_y\,\varphi_{xyu} + 3\,\varphi_x\,\varphi_{xxu} + \varphi_{yy}\,\varphi_{xu} + 3\,\varphi_{xx}\,\varphi_{xu} + 2\,\varphi_{xy}\,\varphi_{yu}\Big) + \\
& + \varphi_u^6\Big(-\varphi_{xxx} - \varphi_{xyy}\Big).
\end{aligned}
$$

while A_2 *is obtained from* A_1 *by just exchanging* x *and* y*.*

6.4. **Abstract shape of the initial frame on** $M^3 \subset \mathbb{C}^2$**.** From now on, we shall restart from the beginning by assuming that we are given an initial frame (H_1, H_2, T) for TM made of certain two linearly independent vector fields which generate T^cM locally:

$$
H_1 \in \Gamma(T^cM) \qquad \text{and} \qquad H_2 \in \Gamma(T^cM),
$$

together with their bracket:

$$
T := \tfrac{1}{4}\,[H_1, H_2] \in \Gamma(TM)
$$

enjoying the following commutator relations:

$$
[H_1, T] = \Phi_1\,T \qquad \text{and} \qquad [H_2, T] = \Phi_2\,T,
$$

were we now consider the two $\mathscr{C}^\infty$ functions $\Phi_1 \colon M \to \mathbb{R}$ and $\Phi_2 \colon M \to \mathbb{R}$ as basic data, without it to be necessary to know that they both depend explicitly on some local graphing function φ for M, as was stated by the preceding proposition. In the subsequent section, we shall construct a Cartan connection just in terms of Φ_1 and Φ_2, not trying to express the newly constructed functions and curvatures explicitly in terms of the graphing function φ, for the sizes of such expressions might explode dramatically. Only at the very end, after all the computations in terms of just Φ_1 and Φ_2 are finalized, will we give it to a computer to expand the gained curvatures in terms of the sixth-order jet $J^6_{x,y,u}\varphi$.

At least at the moment, it is useful two explore in advance what relations come out when one takes iterated brackets:

$$\begin{aligned}
\big[H_i,\,T\big] &= \tfrac{1}{4}\,\big[H_i,[H_1,H_2]\big] \\
\big[H_i,\big[H_j,\,T\big]\big] &= \tfrac{1}{4}\,\big[H_i,\big[H_j,\,[H_1,H_2]\big]\big] \\
\big[H_i,\big[H_j,\big[H_k,\,T\big]\big]\big] &= \tfrac{1}{4}\,\big[H_i,\big[H_j,\big[H_k,\,[H_1,H_2]\big]\big]\big] \\
\big[H_i,\big[H_j,\big[H_k,\big[H_l,\,T\big]\big]\big]\big] &= \tfrac{1}{4}\,\big[H_i,\big[H_j,\big[H_k,\big[H_l,\,[H_1,H_2]\big]\big]\big]\big]
\end{aligned}$$

up to length $\leqslant 6$, where $i,j,k,l = 1,2$. A first observation is as follows, but a more systematic exploration of higher order relations will be achieved in the next section.

Lemma 6.3. *The two functions $H_1(\Phi_2)$ and $H_2(\Phi_1)$ are equal.*

Proof. By what has been seen at the moment, we have by definition:

$$[H_1,T] = \Phi_1\,T, \qquad\qquad [H_2,T] = \Phi_2\,T,$$

whence, by bracketing the second (resp. first) equation with $[H_1,\cdot]$ (resp. $[H_2,\cdot]$):

$$\begin{aligned}
\big[H_1,[H_2,T]\big] &= \big[H_1,\Phi_2\,T\big] = H_1(\Phi_2)\,T + \Phi_2\Phi_1\,T, \\
\big[H_2,[H_1,T]\big] &= \big[H_2,\Phi_1\,T\big] = H_2(\Phi_1)\,T + \Phi_1\Phi_2\,T.
\end{aligned}$$

On the other hand, the Jacobi identity enables us to realize that these two iterated brackets of length 3 are in fact equal:

$$\big[H_1,[H_2,T]\big] - \big[H_2,[H_1,T]\big] = -\big[T,[H_1,H_2]\big] = \big[T,\,4\,T\big] = 0,$$

so that we deduce at once:

$$H_1(\Phi_2) = H_2(\Phi_1),$$

as was claimed. □

7. Free Lie Algebras of Rank Two and Relations Between Brackets of Length $\leqslant 6$

7.1. **Free Lie algebras of rank two.** To reach higher order relations, one must at first count the maximal number of iterated Lie brackets between H_1 and H_2 which are linearly independent modulo skew-symmetry and Jacobi identity, just abstractly, without using $[H_i,\,T] = \Phi_i\,T$. For this, one calls to the concept of free Lie algebra of rank 2, *cf.* the reference [32], pp. 9–11 of which we borrow the notations.

Let h_1, h_2 be two linearly independent elements of a certain vector space over $\mathbb{R}$. The *free Lie algebra* $\mathscr{F}$ of rank 2 is the smallest (non-commutative, non-associative) $\mathbb{R}$-algebra having h_1, h_2 as elements, with bilinear multiplication:

$$(h, h') \longmapsto [h, h'] \in \mathscr{F} \qquad\qquad {\scriptstyle (h,\, h' \in \mathscr{F})}$$

satisfying skew-symmetry:

$$0 = [h,\, h'] + [h',\, h] \qquad\qquad {\scriptstyle (h,\, h' \in \mathscr{F})}$$

and a general Jacobi-like identity:

$$0 = \big[h,\, [h',\, h'']\big] + \big[h'',\, [h,\, h']\big] + \big[h',\, [h'',\, h]\big] \qquad\qquad {\scriptstyle (h,\, h'\, h'' \in \mathscr{F})}.$$

Such an algebra $\mathscr{F}$ is unique up to isomorphism. Thus, the multiplication in $\mathscr{F}$ plays the role of the concrete Lie bracket between vector fields. But importantly, *no linear relation exists between iterated multiplications*, *i.e.* between iterated Lie brackets, *except those generated just by antisymmetry and by Jacobi identity*: this is *freeness* of the algebra.

Since the bracket multiplication is not associative, one must carefully write down the occurring brackets, for instance:

$$\big[[h_1,\, h_2],\, h_2\big], \qquad \big[h_1,\, [h_2,\, [h_1,\, h_2]]\big], \qquad \big[[h_1, h_2],\, [h_1,\, [h_1,\, h_2]]\big].$$

Writing all such words only with the alphabet $\{h_1, h_2\}$, we define the *length* of a word h to be the number of elements h_{i_α} in it, $i_\alpha = 1, 2$. For $\ell \in \mathbb{N}$ with $\ell \geqslant 1$, let $\mathscr{W}^\ell$ denote the set of words of length equal to ℓ and let $\mathscr{W} = \bigcup_{\ell \geqslant 1} \mathscr{W}^\ell$ be the set of all words.

Define $\mathscr{F}_1$ to be the $\mathbb{R}$-vector space generated by h_1, h_2 and for $\ell \geqslant 2$, define $\mathscr{F}_\ell$ to be the $\mathbb{R}$-vector space generated by all words of length $\leqslant \ell$. In this way, $\mathscr{F} = \bigcup_{\ell \geqslant 1} \mathscr{F}_\ell$ naturally becomes a graded Lie algebra, because by applying inductively the Jacobi identity, one may rather easily establish by induction that (but see also explicit examples below):

$$\big[\mathscr{F}_{\ell_1},\, \mathscr{F}_{\ell_2}\big] \subset \mathscr{F}_{\ell_1 + \ell_2}.$$

Again by an induction based on the Jacobi identity, it also follows that $\mathscr{F}_\ell$ is generated, as an $\mathbb{R}$-vector space, by only those words that of the form:

$$\big[h_{i_1},\, \big[h_{i_2},\, \big[\dots \big[h_{i_{\ell'-1}},\, h_{i_{\ell'}}\big] \dots\big]\big],$$

and which are called *simple*, where $\ell' \leqslant \ell$ and where $1 \leqslant i_1, i_2, \dots, i_{\ell'-1}, i_{\ell'} \leqslant 2$. For instance, the non-simple word $\big[[h_1,\, h_2],\, \big[h_1,\, [h_1,\, h_2]\big]\big]$ may be written as a certain linear combination of simple words of this kind having length 5, as we will see quite explicitly in a while. Let us denote by:

$$\mathscr{SW} = \bigcup_{\ell \geqslant 1} \mathscr{SW}^\ell$$

the set of all the simple words, where $\mathscr{SW}^\ell$ denotes the set of simple words of length ℓ. Thus, a rough induction argument based on Jacobi shows that $\mathscr{SW}$ generates $\mathscr{F}$ as a vector space over $\mathbb{R}$, but there are further linear dependence relations between simple words, as is known and as will be visible in examples.

7.2. **All relations up to length 5.** Thus, we are interested in words, namely in abstract-free Lie brackets, just up to length 6, and this, unexpectedly, will happen to already be a bit not straightforward. According to a known theorem (*see e.g.* [32], p. 11), the dimensions $n_\ell - n_{\ell-1}$ of $\mathscr{F}_\ell / \mathscr{F}_{\ell-1}$ satisfy the induction relations:

$$n_\ell - n_{\ell-1} = \frac{1}{\ell} \sum_{d \text{ divides } \ell} \mu(d)\, 2^{\frac{\ell}{d}},$$

where μ is the Möbius function:

$$\mu(d) = \begin{cases} 1, \text{ if } d = 1; \\ 0, \text{ if } d \text{ contains square integer factors}; \\ (-1)^\nu, \text{ if } d = p_1 \cdots p_\nu \text{ is the product of } \nu \text{ distinct prime numbers}. \end{cases}$$

Thus, a direct application of this general formula yields:

$$\begin{aligned} n_2 - n_1 &= \tfrac{1}{2}\big(\mu(1)\, 2^{\frac{2}{1}} + \mu(2)\, 2^{\frac{2}{2}}\big) = \tfrac{1}{2}\big(2^2 - 2\big) = 1, \\ n_3 - n_2 &= \tfrac{1}{3}\big(\mu(1)\, 2^{\frac{3}{1}} + \mu(3)\, 2^{\frac{3}{3}}\big) = \tfrac{1}{3}\big(8 - 2\big) = 2, \\ n_4 - n_3 &= \tfrac{1}{4}\big(\mu(1)\, 2^{\frac{4}{1}} + \mu(2)\, 2^{\frac{4}{2}} + \mu(4)\, 2^{\frac{4}{4}}\big) = \tfrac{1}{4}\big(16 - 4 + 0\big) = 3, \\ n_5 - n_4 &= \tfrac{1}{5}\big(\mu(1)\, 2^{\frac{5}{1}} + \mu(5)\, 2^{\frac{5}{5}}\big) = \tfrac{1}{5}\big(32 - 2\big) = 6, \\ n_6 - n_5 &= \tfrac{1}{6}\big(\mu(1)\, 2^{\frac{6}{1}} + \mu(2)\, 2^{\frac{6}{2}} + \mu(3)\, 2^{\frac{6}{3}} + \mu(6)\, 2^{\frac{6}{6}}\big) = \tfrac{1}{6}\big(64 - 8 - 4 + 2\big) = 9. \end{aligned}$$

Now, in length $\ell = 2$ it is clear that there is, up to skew-symmetry, only *one* simple word:

$$[h_1,\, h_2],$$

confirming $n_2 - n_1 = 1$ while $n_1 = 2$ of course, because h_1 and h_2 are two independent simple words of length 1.

Next, in length $\ell = 3$, it is again clear that up to skew-symmetry, there are only two simple words:

$$\big[h_1,\, [h_1,\, h_2]\big] \qquad \text{and} \qquad \big[h_2,\, [h_1,\, h_2]\big],$$

while no word is not simple.

It is only in length $\ell = 4$ that nontrivial relations come out. Indeed, again up to the skew-symmetry inside the 'core' $[h_1,\, h_2]$, there are *a priori* 4 distinct simple words generating $\mathscr{SW}^4$, namely:

$$\big[h_1,\, \big[h_1,\, [h_1,\, h_2]\big]\big], \quad \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big], \quad \big[h_2,\, \big[h_1,\, [h_1,\, h_2]\big]\big], \quad \big[h_2,\, \big[h_2,\, [h_1,\, h_2]\big]\big],$$

but an obvious Jacobi identity provides one linear relation between simple words[3]:

$$0 = \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big] + \underline{\big[[h_1,\, h_2],\, \big[h_1,\, h_2\big]\big]}_{\circ} + \big[[h_1,\, h_2],\, \big[h_1,\, h_2\big]\big],$$

[3] For arbitrary words h, h', h'' of length $\geqslant 1$, our convention for writing out any Jacobi identity under either one or the other form:

$$\begin{aligned} 0 &= \big[h,\, [h',\, h'']\big] + \big[h'',\, [h,\, h']\big] + \big[h',\, [h'',\, h]\big] \\ 0 &= \big[[h,\, h'],\, h''\big] + \big[[h'',\, h],\, h'\big] + \big[[h',\, h''],\, h\big] \end{aligned}$$

consists in subjecting the terms to a circular permutation, the last term being brought back to the first position while other terms are simultaneously shifted (pushed) from left to right.

where the central term trivially vanishes, and this is coherent with $n_4 - n_3 = 3$. Furthermore, one easily convinces oneself that, up to skew-symmetry, the Jacobi identity cannot produce any other nontrivial relation, for instance:

$$0 = \big[h_1,\, \big[h_1,\, [h_1,\, h_2]\big]\big] + \big[[h_1,\, h_2],\, \underline{[h_1,\, h_1]}_\circ\big] + \big[h_1,\, \big[[h_1,\, h_2],\, h_1,\big]\big],$$

is a trivial relation, it gives nothing. In fact, one realizes that *all* brackets between two words of length $\ell = 2$ vanish. As a basis for $\mathscr{SW}^4$, let us therefore choose the three simple words:

$$\big[h_1,\, \big[h_1,\, [h_1,\, h_2]\big]\big], \qquad \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big], \qquad \big[h_2,\, \big[h_2,\, [h_1,\, h_2]\big]\big],$$

remembering that the fourth simple word is simply given by:

(29) $$\big[h_2,\, \big[h_1,\, [h_1,\, h_2]\big]\big] = \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big].$$

Next, in length $\ell = 5$, applying $[h_i, \cdot]$, $i = 1, 2$, to these three simple words, we deduce that $\mathscr{SW}^5$ is generated by the following six simple words:

$$\big[h_1,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big], \qquad \big[h_1,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big], \qquad \big[h_1,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big],$$
$$\big[h_2,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big], \qquad \big[h_2,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big], \qquad \big[h_2,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big].$$

Are there other linear dependence relations between these six simple words? Certainly not, because of $n_5 - n_4 = 6$; alternatively, one could also realize this by trying to apply Jacobi to all possible triples of words, the sum-length of which equals 5. In addition, it is also important, for later use, to explicitly represent all length-5 multiple iterated brackets as specific linear combinations between simple brackets. For instance, there are exactly two brackets between two basic words of lengths 2 and 3, and the Jacobi identity gives[4]:

$$0 = \big[[\underline{h_1},\, \underline{h_2}],\, \underline{[h_1,[h_1,h_2]]}\big] + \big[\big[[h_1,\, [h_1,\, h_2]],\, h_1\big],\, h_2\big] + \big[\big[h_2,\, [h_1,[h_1,h_2]]\big],\, h_1\big],$$
$$0 = \big[[\underline{h_1},\, \underline{h_2}],\, \underline{[h_2,[h_1,h_2]]}\big] + \big[\big[[h_2,\, [h_1,\, h_2]],\, h_1\big],\, h_2\big] + \big[\big[h_2,\, [h_2,[h_1,h_2]]\big],\, h_1\big].$$

Here, in each one of the two lines, the last two words happen, thanks to skew-symmetry, to all be simple, whence (using (29) for the last term of the first line):

(30) $$\begin{aligned}\big[[h_1,\, h_2],\, [h_1,[h_1,h_2]]\big] &= -\big[h_2,\, \big[h_1,\, \big[h_1,\, [h_1,\, h_2]\big]\big]\big] + \big[h_1,\, \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big]\big],\\ \big[[h_1,\, h_2],\, [h_2,[h_1,h_2]]\big] &= -\big[h_2,\, \big[h_1,\, \big[h_2,\, [h_1,\, h_2]\big]\big]\big] + \big[h_1,\, \big[h_2,\, \big[h_2,\, [h_1,\, h_2]\big]\big]\big].\end{aligned}$$

7.3. **All relations in length 6.** In the next length $\ell = 6$, more complexity occurs. By applying $[h_i, \cdot]$, $i = 1, 2$, to the above six linearly independent simple words of length 5, we at first get the following twelve simple words:

$$\big[h_1,\big[h_1,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_1,\big[h_1,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_1,\big[h_1,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big]\big],$$
$$\big[h_1,\big[h_2,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_1,\big[h_2,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_1,\big[h_2,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big]\big],$$
$$\big[h_2,\big[h_1,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_2,\big[h_1,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_2,\big[h_1,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big]\big],$$
$$\big[h_2,\big[h_2,\big[h_1,\big[h_1,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_2,\big[h_2,\big[h_1,\big[h_2,[h_1,h_2]\big]\big]\big]\big], \quad \big[h_2,\big[h_2,\big[h_2,\big[h_2,[h_1,h_2]\big]\big]\big]\big].$$

However, according to the dimensional count made above, $n_6 - n_5 = 9$, so there must exist three independent linear relations between these twelve simple words.

We begin by exploring Lie brackets between two words of length 3. Such words are automatically simple. Since there are only two words of length 3, only one

[4] For clarity, we underline the three terms that are subjected to a circular permutation.

bracket exists, and the Jacobi identity can give only two different relations. The first relation is:

$$\begin{aligned}
0 &= \big[\underline{[h_1,[h_1,h_2]]},[\underline{h_2},\underline{[h_1,h_2]}]\big] + \big[[h_1,h_2],\big[[h_1,[h_1,h_2]],h_2\big]\big] + \big[h_2,\big[[h_1,h_2],[h_1,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[[h_1,h_2],\big[h_2,[h_1,[h_1,h_2]]\big]\big] + \big[h_2,\big[[h_1,h_2],[h_1,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] - \big[h_2,\big[h_2,\big[h_1,[h_1,[h_1,h_2]]\big]\big]\big]+ \\
&\qquad + \big[h_2,\big[h_1,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big].
\end{aligned}$$

where we leave untouched the first term, where we apply (29) to normalize the second term, and where the third term expresses as a linear combination of two simple words thanks to the first relation (30). The second relation, just with a different underlining for applying Jacobi, is:

$$\begin{aligned}
0 &= \big[[\underline{h_1},\underline{[h_1,h_2]}],\underline{[h_2,[h_1,h_2]]}\big] + \big[\big[[h_2,[h_1,h_2]],h_1\big],[h_1,h_2]\big] + \big[\big[[h_1,h_2],[h_2,[h_1,h_2]]\big],h_1\big] \\
&= \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] - \big[h_1,\big[[h_1,h_2],[h_2,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] + \big[h_1,\big[h_2,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big]- \\
&\qquad - \big[h_1,\big[h_1,\big[h_2,[h_2,[h_1,h_2]]\big]\big]\big].
\end{aligned}$$

Next, there are exactly six Jacobi identities for Lie brackets between two words having lengths 2 and 4. The first pair is:

$$\begin{aligned}
0 &= \big[\underline{[h_1,h_2]},[\underline{h_1},\underline{[h_1,[h_1,h_2]]}]\big] + \underline{\big[[h_1,[h_1,h_2]],[[h_1,h_2],h_1]\big]}_{\circ} + \big[h_1,\big[[h_1,[h_1,h_2]],[h_1,h_2]\big]\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_1,[h_1,h_2]]\big]\big] - \big[h_1,\big[[h_1,h_2],[h_1,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_1,[h_1,h_2]]\big]\big] + \big[h_1,\big[h_2,\big[h_1,[h_1,[h_1,h_2]]\big]\big]\big] - \big[h_1,\big[h_1,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big];
\end{aligned}$$

$$\begin{aligned}
0 &= \big[[\underline{h_1},\underline{h_2}],\underline{[h_1,[h_1,[h_1,h_2]]]}\big] + \big[\big[[h_1,[h_1,[h_1,h_2]]],h_1\big],h_2\big] + \big[\big[h_2,[h_1,[h_1,[h_1,h_2]]]\big],h_1\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_1,[h_1,h_2]]\big]\big] + \big[h_2,\big[h_1,\big[h_1,[h_1,[h_1,h_2]]\big]\big]\big] - \big[h_1,\big[h_2,\big[h_1,[h_1,[h_1,h_2]]\big]\big]\big].
\end{aligned}$$

The second pair is:

$$\begin{aligned}
0 &= \big[\underline{[h_1,h_2]},[\underline{h_1},\underline{[h_2,[h_1,h_2]]}]\big] + \big[[h_2,[h_1,h_2]],[[h_1,h_2],h_1]\big] + \big[h_1,\big[[h_2,[h_1,h_2]],[h_1,h_2]\big]\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] + \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[h_1,\big[[h_1,h_2],[h_2,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] + \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[h_1,\big[h_2,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big]- \\
&\qquad - \big[h_1,\big[h_1,\big[h_2,[h_2,[h_1,h_2]]\big]\big]\big];
\end{aligned}$$

$$\begin{aligned}
0 &= \big[[\underline{h_1},\underline{h_2}],\underline{[h_1,[h_2,[h_1,h_2]]]}\big] + \big[\big[[h_1,[h_2,[h_1,h_2]]],h_1\big],h_2\big] + \big[\big[h_2,[h_1,[h_2,[h_1,h_2]]]\big],h_1\big] \\
&= \big[[h_1,h_2],\big[h_1,[h_2,[h_1,h_2]]\big]\big] + \big[h_2,\big[h_1,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big] - \big[h_1,\big[h_2,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big].
\end{aligned}$$

The third pair is:

$$\begin{aligned}
0 &= \big[\underline{[h_1,h_2]},[\underline{h_2},\underline{[h_2,[h_1,h_2]]}]\big] + \underline{\big[[h_2,[h_1,h_2]],[[h_1,h_2],h_2]\big]}_{\circ} + \big[h_2,\big[[h_2,[h_1,h_2]],[h_1,h_2]\big]\big] \\
&= \big[[h_1,h_2],\big[h_2,[h_2,[h_1,h_2]]\big]\big] - \big[h_2,\big[[h_1,h_2],[h_2,[h_1,h_2]]\big]\big] \\
&= \big[[h_1,h_2],\big[h_2,[h_2,[h_1,h_2]]\big]\big] + \big[h_2,\big[h_2,\big[h_1,[h_2,[h_1,h_2]]\big]\big]\big] - \big[h_2,\big[h_1,\big[h_2,[h_2,[h_1,h_2]]\big]\big]\big];
\end{aligned}$$

$$\begin{aligned}
0 &= \big[[\underline{h_1},\underline{h_2}],\underline{[h_2,[h_2,[h_1,h_2]]]}\big] + \big[\big[[h_2,[h_2,[h_1,h_2]]],h_1\big],h_2\big] + \big[\big[h_2,[h_2,[h_2,[h_1,h_2]]]\big],h_1\big] \\
&= \big[[h_1,h_2],\big[h_2,[h_2,[h_1,h_2]]\big]\big] + \big[h_2,\big[h_1,\big[h_2,[h_2,[h_1,h_2]]\big]\big]\big] - \big[h_1,\big[h_2,\big[h_2,[h_2,[h_1,h_2]]\big]\big]\big].
\end{aligned}$$

We thus have got eight relations involving simple words: the single bracket between two words of length 3 and the six brackets between a word of length 2 and a word of length 4. We number these eight equations and, for easier readability, we underbrace the four non-simple words t, u, v, w:

$$0 \overset{1}{=} \underbrace{\big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big]}_{\mathtt{t}} - \underbrace{\big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big]}_{\mathtt{v}} - [h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]] + \\ + [h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]]$$

$$0 \overset{2}{=} \underbrace{\big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big]}_{\mathtt{t}} + \underbrace{\big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big]}_{\mathtt{v}} + [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - \\ - [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]],$$

$$0 \overset{3}{=} \underbrace{[[h_1,h_2],[h_1,[h_1,[h_1,h_2]]]]}_{\mathtt{u}} + [h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]] - [h_1,[h_1,[h_1,[h_2,[h_1,h_2]]]]],$$

$$0 \overset{4}{=} \underbrace{[[h_1,h_2],[h_1,[h_1,[h_1,h_2]]]]}_{\mathtt{u}} + [h_2,[h_1,[h_1,[h_1,[h_1,h_2]]]]] - [h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]],$$

$$0 \overset{5}{=} \underbrace{[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]]}_{\mathtt{v}} + \underbrace{\big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big]}_{\mathtt{t}} + [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - \\ - [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]],$$

$$0 \overset{6}{=} \underbrace{[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]]}_{\mathtt{v}} + [h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]].$$

$$0 \overset{7}{=} \underbrace{[[h_1,h_2],[h_2,[h_2,[h_1,h_2]]]]}_{\mathtt{w}} + [h_2,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - [h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]],$$

$$0 \overset{8}{=} \underbrace{[[h_1,h_2],[h_2,[h_2,[h_1,h_2]]]]}_{\mathtt{w}} + [h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]] - [h_1,[h_2,[h_2,[h_2,[h_1,h_2]]]]].$$

Visibly, the fifth equation coincides with the second one. There remain seven equations. Since four non-simple words are involved, one may expect to see here the three linearly independent relations between simple words that we are looking for. Firstly, subtracting the third equation to the fourth, we get a first relation of this kind:

$$0 \overset{9}{=} [h_1,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - 2\times[h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]] + \\ + [h_2,[h_1,[h_1,[h_1,[h_1,h_2]]]]].$$

Secondly, subtracting the eighth equation to the seventh, we get a second, visibly independent relation:

$$0 \overset{10}{=} [h_2,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - 2\times[h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]] + \\ + [h_1,[h_2,[h_2,[h_2,[h_1,h_2]]]]].$$

Thirdly and lastly, adding the sixth equation multiplied by 2 to the first one and subtracting the second one, we get a third independent relation between simple words:

$$0 \overset{11}{=} [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]] - 3\times[h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]] + \\ + 3\times[h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - [h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]].$$

7.4. **Iterated brackets of H_1 and H_2 on M.** Now, we come back to our two vector fields H_1 and H_2 on M satisfying the two specific relations:

$$\big[H_1,[H_1,H_2]\big]=\Phi_1[H_1,H_2] \qquad \text{and} \qquad \big[H_2,[H_1,H_2]\big]=\Phi_2[H_1,H_2],$$

for certain two functions Φ_1 and Φ_2 on M whose explicit expressions (not needed here), in terms of the third-order jet $J^3_{x,y,u}\varphi$ of the graphing function for M, have already been shown in Proposition 6.2. These two relations show well that H_1 and H_2 do *not* behave as the two abstract *totally free* elements h_1 and h_2 considered above. In fact, a straightforward induction argument shows that every iterated simple-word bracket:

$$\big[H_{i_1},\big[H_{i_2},\big[\cdots,\big[H_{i_{\ell-1}},H_{i_\ell}\big],\cdots\big]\big]\big]=\Phi_{i_1,i_2,\dots,i_{\ell-1},i_\ell}[H_1,H_2]$$

of arbitrary length $\ell\geqslant 2$, where $i_1,\dots,i_\ell=1,2$, must always be a multiple of $[H_1,\,H_2]$ by means of a certain function $\Phi_{i_1,\dots,i_\ell}$ which depends on Φ_1 and Φ_2, but whose explicit expression in terms of Φ_1 and Φ_2 is not immediate. For $\ell=4,5,6$, we must now compute all these $\Phi_{i_1,\dots,i_\ell}$ so that the abstract relations between iterated brackets of the free h_1 and h_2 computed above provide us with interesting relations that will be useful later.

At first, applying the basic formula $[fX,\,gY]\;=\;fX(g)\,Y-g\,Y(f)\,X+fg\,[X,Y]$, we have in length $\ell=4$:

$$\begin{aligned}
\big[H_1,\,\big[H_1,\,[H_1,\,H_2]\big]\big]&=H_1(\Phi_1)\,[H_1,\,H_2]+\Phi_1\,\big[H_1,\,[H_1,\,H_2]\big]\\
&\overset{1}{=}\big(H_1(\Phi_1)+(\Phi_1)^2\big)\,[H_1,\,H_2],\\
\big[H_1,\,\big[H_2,\,[H_1,\,H_2]\big]\big]&=H_1(\Phi_2)\,[H_1,\,H_2]+\Phi_2\,\big[H_2,\,[H_1,\,H_2]\big]\\
&\overset{2}{=}\big(H_1(\Phi_2)+\Phi_2\Phi_1\big)\,[H_1,\,H_2],\\
\big[H_2,\,\big[H_1,\,[H_1,\,H_2]\big]\big]&=H_2(\Phi_1)\,[H_1,\,H_2]+\Phi_1\,\big[H_2,\,[H_1,\,H_2]\big]\\
&\overset{3}{=}\big(H_2(\Phi_1)+\Phi_1\Phi_2\big)\,[H_1,\,H_2],\\
\big[H_2,\,\big[H_2,\,[H_1,\,H_2]\big]\big]&=H_2(\Phi_2)\,[H_1,\,H_2]+\Phi_2\,\big[H_2,\,[H_1,\,H_2]\big]\\
&\overset{4}{=}\big(H_2(\Phi_2)+(\Phi_2)^2\big)\,[H_1,\,H_2].
\end{aligned}$$

But the known relation $\big[h_2,\,\big[h_1,\,[h_1,\,h_2]\big]\big]=\big[h_1,\,\big[h_2,\,[h_1,\,h_2]\big]\big]$ between free elements imposes here:

$$H_2(\Phi_1)+\Phi_1\Phi_2=H_1(\Phi_2)+\Phi_2\Phi_1,$$

a relation already seen in Lemma 6.3.

Next, setting aside the consideration of $\big[H_2,\,\big[H_1,\,[H_1,\,H_2]\big]\big]$, we compute as follows the six simple iterated brackets of length $\ell=5$ (we replace $H_2(\Phi_1)$ by $H_1(\Phi_2)$ wherever it occurs):

$$\begin{aligned}
\big[H_1,\big[H_1,\big[H_1,[H_1,H_2]\big]\big]\big]&=\big(H_1(H_1(\Phi_1))+2\Phi_1H_1(\Phi_1)+\Phi_1H_1(\Phi_1)+(\Phi_1)^3\big)[H_1,H_2]\\
&\overset{1}{=}\big(H_1(H_1(\Phi_1))+3\Phi_1H_1(\Phi_1)+(\Phi_1)^3\big)[H_1,H_2],\\
\big[H_1,\big[H_1,\big[H_2,[H_1,H_2]\big]\big]\big]&=\big(H_1(H_1(\Phi_2))+\Phi_1H_1(\Phi_2)+\Phi_2H_1(\Phi_1)+\Phi_1H_1(\Phi_2)+(\Phi_1)^2\Phi_2\big)[H_1,H_2]\\
&\overset{2}{=}\big(H_1(H_1(\Phi_2))+2\Phi_1H_1(\Phi_2)+\Phi_2H_1(\Phi_1)+(\Phi_1)^2\Phi_2\big)[H_1,H_2],\\
\big[H_1,\big[H_2,\big[H_2,[H_1,H_2]\big]\big]\big]&\overset{3}{=}\big(H_1(H_2(\Phi_2))+2\Phi_2H_1(\Phi_2)+\Phi_1H_2(\Phi_2)+\Phi_1(\Phi_2)^2\big)[H_1,H_2],
\end{aligned}$$

$$\begin{aligned}\big[H_2,\big[H_1,\big[H_1,[H_1,H_2]\big]\big]\big] &= \big(H_2(H_1(\Phi_1)) + 2\Phi_1 H_2(\Phi_1) + \Phi_2 H_1(\Phi_1) + (\Phi_1)^2\Phi_2\big)[H_1,H_2]\\ &\overset{4}{=} \big(H_2(H_1(\Phi_1)) + 2\Phi_1 H_1(\Phi_2) + \Phi_2 H_1(\Phi_1) + (\Phi_1)^2\Phi_2\big)[H_1,H_2],\\ \big[H_2,\big[H_1,\big[H_2,[H_1,H_2]\big]\big]\big] &= \big(H_2(H_1(\Phi_2)) + \Phi_1 H_2(\Phi_2) + \Phi_2 H_2(\Phi_1) + \Phi_2 H_1(\Phi_2) + \Phi_1(\Phi_2)^2\big)[H_1,H_2]\\ &\overset{5}{=} \big(H_2(H_1(\Phi_2)) + \Phi_1 H_2(\Phi_2) + 2\Phi_2 H_1(\Phi_2) + \Phi_1(\Phi_2)^2\big)[H_1,H_2],\\ \big[H_2,\big[H_2,\big[H_2,[H_1,H_2]\big]\big]\big] &\overset{6}{=} \big(H_2(H_2(\Phi_2)) + 3\Phi_2 H_2(\Phi_2) + (\Phi_2)^3\big)[H_1,H_2].\end{aligned}$$

Also, one may compute the two brackets between the unique simple word of length 2 and the two simple words of length 3, expanding $[H_1,H_2](\Psi)$ just as $H_1(H_2(\Psi)) - H_2(H_1(\Psi))$:

$$\begin{aligned}\big[[H_1,H_2],\big[H_1,[H_1,H_2]\big]\big] &= \big(H_1(H_2(\Phi_1)) - H_2(H_1(\Phi_1))\big)[H_1,H_2],\\ \big[[H_1,H_2],\big[H_2,[H_1,H_2]\big]\big] &= \big(H_1(H_2(\Phi_2)) - H_2(H_1(\Phi_2))\big)[H_1,H_2].\end{aligned}$$

Unexpectedly, if one looks at the two relations (30), one only gets twice the trivial relation $0 = 0$. Only in length $\ell = 6$ will one find new nontrivial relations.

Now, here are the twelve (not linearly independent) simple iterated brackets of length $\ell = 6$:

$$\begin{aligned}\big[H_1,\big[H_1,\big[H_1,\big[H_1,[H_1,H_2]\big]\big]\big]\big] &\overset{1}{=} \big(H_1(H_1(H_1(\Phi_1))) + 4\Phi_1 H_1(H_1(\Phi_1)) +\\ &\quad + 3H_1(\Phi_1)H_1(\Phi_1) + 6(\Phi_1)^2 H_1(\Phi_1) + (\Phi_1)^4\big)[H_1,H_2],\\ \big[H_1,\big[H_1,\big[H_1,\big[H_2,[H_1,H_2]\big]\big]\big]\big] &\overset{2}{=} \big(H_1(H_1(H_1(\Phi_2))) + 3\Phi_1 H_1(H_1(\Phi_2)) +\\ &\quad + \Phi_2 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) +\\ &\quad + 3\Phi_1\Phi_2 H_1(\Phi_1) + 3(\Phi_1)^2 H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],\\ \big[H_1,\big[H_1,\big[H_2,\big[H_2,[H_1,H_2]\big]\big]\big]\big] &\overset{3}{=} \big(H_1(H_1(H_2(\Phi_2))) + 2\Phi_1 H_1(H_2(\Phi_2)) +\\ &\quad + 2\Phi_2 H_1(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) +\\ &\quad + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],\\ \big[H_1,\big[H_2,\big[H_1,\big[H_1,[H_1,H_2]\big]\big]\big]\big] &\overset{4}{=} \big(H_1(H_2(H_1(\Phi_1))) + 2\Phi_1 H_1(H_1(\Phi_2)) +\\ &\quad + \Phi_2 H_1(H_1(\Phi_1)) + \Phi_1 H_2(H_1(\Phi_1)) + 3H_1(\Phi_1)H_2(\Phi_2) +\\ &\quad + 3\Phi_1\Phi_2 H_1(\Phi_1) + 3(\Phi_1)^2 H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],\\ \big[H_1,\big[H_2,\big[H_1,\big[H_2,[H_1,H_2]\big]\big]\big]\big] &\overset{5}{=} \big(H_1(H_2(H_1(\Phi_2))) + \Phi_1 H_1(H_2(\Phi_2)) + 2\Phi_2 H_1(H_1(\Phi_2)) +\\ &\quad + \Phi_1 H_2(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) +\\ &\quad + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],\\ \big[H_1,\big[H_2,\big[H_2,\big[H_2,[H_1,H_2]\big]\big]\big]\big] &\overset{6}{=} \big(H_1(H_2(H_2(\Phi_2))) + 3\Phi_2 H_1(H_2(\Phi_2)) +\\ &\quad + \Phi_1 H_2(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) +\\ &\quad + 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],\\ \big[H_2,\big[H_1,\big[H_1,\big[H_1,[H_1,H_2]\big]\big]\big]\big] &\overset{7}{=} \big(H_2(H_1(H_1(\Phi_1))) + 3\Phi_1 H_2(H_1(\Phi_1)) +\\ &\quad + \Phi_2 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) +\\ &\quad + 3(\Phi_1)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_1(\Phi_1) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],\\ \big[H_2,\big[H_1,\big[H_1,\big[H_2,[H_1,H_2]\big]\big]\big]\big] &\overset{8}{=} \big(H_2(H_1(H_1(\Phi_2))) + 2\Phi_1 H_2(H_1(\Phi_2)) + \Phi_2 H_2(H_1(\Phi_1)) +\\ &\quad + \Phi_2 H_1(H_1(\Phi_2)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_2(\Phi_2)H_1(\Phi_1) +\\ &\quad + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][H_2,[H_1,[H_2,[H_2,[H_1,H_2]]]]] \overset{9}{=} \big(&H_2(H_1(H_2(\Phi_2))) + 2\Phi_2 H_2(H_1(\Phi_2)) + \\ &+ \Phi_1 H_2(H_2(\Phi_2)) + \Phi_2 H_1(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + \\ &+ 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][H_2,[H_2,[H_1,[H_1,[H_1,H_2]]]]] \overset{10}{=} \big(&H_2(H_2(H_1(\Phi_1))) + 2\Phi_1 H_2(H_1(\Phi_2)) + \\ &+ 2\Phi_2 H_2(H_1(\Phi_1)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_1(\Phi_1)H_2(\Phi_2) + \\ &+ 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][H_2,[H_2,[H_1,[H_2,[H_1,H_2]]]]] \overset{11}{=} \big(&H_2(H_2(H_1(\Phi_2))) + \Phi_1 H_2(H_2(\Phi_2)) + \\ &+ 3\Phi_2 H_2(H_1(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + \\ &+ 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][H_2,[H_2,[H_2,[H_2,[H_1,H_2]]]]] \overset{12}{=} \big(&H_2(H_2(H_2(\Phi_2))) + 4\Phi_2 H_2(H_2(\Phi_2)) + \\ &+ 3H_2(\Phi_2)H_2(\Phi_2) + 6(\Phi_2)^2 H_2(\Phi_2) + (\Phi_2)^4\big)[H_1,H_2].\end{aligned}$$

Also, one may compute the single Lie bracket between two simple words of length 3 and the three Lie brackets between the single simple word of length 2 and the three simple words of length 3:

$$\begin{aligned}[][[H_1,[H_1,H_2]],[H_2,[H_1,H_2]]] \overset{13}{=} \big(&\Phi_1 H_1(H_2(\Phi_2)) - \Phi_1 H_2(H_1(\Phi_2)) - \\ &- \Phi_2 H_1(H_1(\Phi_2)) + \Phi_2 H_2(H_1(\Phi_1))\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][[H_1,H_2],[H_1,[H_1,H_2]]] \overset{14}{=} \big(&H_1(H_2(H_1(\Phi_2))) - H_2(H_1(H_1(\Phi_1))) + \\ &+ 2\Phi_1 H_1(H_1(\Phi_2)) - 2\Phi_1 H_2(H_1(\Phi_1))\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][[H_1,H_2],[H_1,[H_2,H_2]]] \overset{15}{=} \big(&H_1(H_2(H_1(\Phi_2))) - H_2(H_1(H_1(\Phi_2))) + \\ &+ \Phi_2 H_1(H_2(\Phi_1)) - \Phi_2 H_2(H_1(\Phi_1)) + \\ &+ \Phi_1 H_1(H_2(\Phi_2)) - \Phi_1 H_2(H_1(\Phi_2))\big)[H_1,H_2],\end{aligned}$$

$$\begin{aligned}[][[H_1,H_2],[H_2,[H_2,H_2]]] \overset{16}{=} \big(&H_1(H_2(H_2(\Phi_2))) - H_2(H_1(H_2(\Phi_2))) + \\ &+ 2\Phi_2 H_1(H_2(\Phi_2)) - 2\Phi_2 H_2(H_1(\Phi_2))\big)[H_1,H_2].\end{aligned}$$

Proposition 7.1. *The two functions Φ_1 and Φ_2 identically satisfy:*

$$H_2(\Phi_1) \equiv H_1(\Phi_2)$$

together with the following five third-order relations:

$$\begin{aligned}0 \overset{\text{I}}{\equiv} &- H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\ &- \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)),\end{aligned}$$

$$\begin{aligned}0 \overset{\text{II}}{\equiv} &- H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \\ &- \Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)),\end{aligned}$$

$$\begin{aligned}0 \overset{\text{III}}{\equiv} &- H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \\ &+ \Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)),\end{aligned}$$

$$\begin{aligned}0 \overset{\text{IV}}{\equiv} &\ H_2(H_2(H_1(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_1(H_2(H_2(\Phi_2))) - \\ &- \Phi_2\,H_2(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)),\end{aligned}$$

$$\begin{aligned}0 \overset{\text{V}}{\equiv} &\ H_1(H_1(H_2(\Phi_2))) - 3\,H_1(H_2(H_1(\Phi_2))) + 3\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\ &- \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)) - \Phi_1\,H_1(H_2(\Phi_2)) + \Phi_1\,H_2(H_1(\Phi_2)),\end{aligned}$$

the first four being linearly independent, while the fifth coincides with $\mathrm{I} - \mathrm{II}$.

Proof. Using the representations $\overset{1}{=}, \dots, \overset{16}{=}$ of the iterated brackets between H_1 and H_2 of lengths $\ell = 6$, one may substitute them in the eleven free-Lie relations $\overset{1}{=}$, $\dots, \overset{11}{=}$ Some of the obtained equations are redundant, and some reduce to the trivial identity $0 = 0$. □

Corollary 7.2. *The following two quantities are identically zero:*

$$\begin{aligned} 0 \overset{a}{\equiv} & - H_2(H_2(H_1(\Phi_1))) + H_2(H_1(H_1(\Phi_2))) + H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) + \\ & + \Phi_1\, H_1(H_2(\Phi_2)) - \Phi_1\, H_2(H_1(\Phi_2)) - \Phi_2\, H_1(H_1(\Phi_2)) + \Phi_2\, H_2(H_1(\Phi_1)), \end{aligned}$$

$$\begin{aligned} 0 \overset{b}{\equiv} & \, H_1(H_2(H_2(\Phi_2))) - 2\, H_2(H_1(H_2(\Phi_2))) + H_2(H_2(H_1(\Phi_2))) - \\ & - H_2(H_1(H_1(\Phi_1))) + 2\, H_1(H_2(H_1(\Phi_1))) - H_1(H_1(H_1(\Phi_2))) + \\ & \Phi_1\, H_1(H_1(\Phi_2)) + \Phi_2\, H_1(H_2(\Phi_2)) - \Phi_2\, H_2(H_1(\Phi_2)) - \Phi_1\, H_2(H_1(\Phi_1)). \end{aligned}$$

Proof. Indeed, using the proposition, the first identity is just $\mathrm{I}+\mathrm{II}$, while the second is just $\mathrm{III} + \mathrm{IV}$. □

8. Cartan Connections in Terms of Coordinates and Bases

8.1. **Definition of Cartan connections "à la Ehresmann":** Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let G be a local Lie group and let H be a local Lie subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras which are $\mathbb{K}$-vector spaces, with $\big[\mathfrak{h}, \mathfrak{h}\big]_{\mathfrak{g}} \subset \mathfrak{h}$. In order to set up a clear notational distinction, we shall write $[X, Y]$ for (Lie) brackets between vector fields, and $[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}$ for (Lie) brackets between vectors of an abstract Lie algebra $\mathfrak{g}$. Following [22, 37, 24], we start with a definition formulated independently of any coordinate system or of any basis for $\mathfrak{g}$.

Definition 8.1. A *Cartan geometry* on a $\mathscr{C}^\infty$ manifold M (over $\mathbb{K}$) modeled on $(\mathfrak{g}, \mathfrak{h})$ consists of the following data:

• a principal H bundle $\mathscr{P} \to M$ over M, the right action of H on $\mathscr{P}$ being on the right:

$$R_h\colon\ p \mapsto ph \qquad\qquad (p \in \mathscr{P});$$

• a $\mathfrak{g}$-valued 1-form ω on $\mathscr{P}$ which enjoys the following three properties:

(i) for every point $p \in \mathscr{P}$, the linear map:

$$\omega_p \colon T_p\mathscr{P} \longrightarrow \mathfrak{g}$$

is an isomorphism;

(ii) if, for every element $\mathsf{y} \in \mathfrak{h}$, one defines the fundamental vector field $Y^\dagger$ on $\mathscr{P}$ by differentiating the action of $\exp(t\mathsf{y}) \in H$ on $\mathscr{P}$:

$$Y^\dagger|_p := \frac{d}{dt}\Big|_0 \Big(p \exp\left(t\mathsf{y}\right)\Big) \qquad\qquad (p \in \mathscr{P}),$$

then ω satisfies:

$$\omega(Y^\dagger) = \mathsf{y};$$

(iii) again with the right translation $R_h \colon p \mapsto p\,h$ on $\mathscr{P}$ by means of an element $h \in H$, the $\mathfrak{g}$-valued 1-form ω satisfies:

$$\omega_{ph}\big(R_{h*}(v_p)\big) = \mathrm{Ad}(h^{-1})\big[\omega_p(v_p)\big],$$

for every tangent vector $v_p \in T_p\mathscr{P}$ at every point $p \in \mathscr{P}$.

Assuming that a Cartan connection 1-form $\omega \colon T\mathscr{P} \to \mathfrak{g}$ is given, our main aim in the next paragraphs will be to express more concretely its properties in terms of a certain local coordinate systems on M, H, $\mathscr{P}$, and in terms of a fixed basis for $\mathfrak{g}$.

8.2. **First consequences.** The way ω behaves through right translations **(iii)** may also be abbreviated without arguments as:

$$R_h^*(\omega) = \mathrm{Ad}(h^{-1}) \circ \omega,$$

where the composition is:

$$T\mathscr{P} \overset{\omega}{\longrightarrow} \mathfrak{g} \overset{\mathrm{Ad}(h^{-1})}{\longrightarrow} \mathfrak{g}.$$

Of course, the principal bundle $\mathscr{P}$ is foliated by copies of H. It will be convenient to denote by:

$$\mathscr{H}_p := \{ph \colon h \in H\}$$

the H-orbit ($\simeq H$) of an arbitrary point p in the H-principal bundle $\mathscr{P}$, which is a $\mathscr{C}^\infty$ submanifold of $\mathscr{P}$. Also, we shall denote by $\mathscr{H}$ the whole foliation of $\mathscr{P}$ by these copies of H. Then property **(ii)** means that each:

$$\omega_p \colon T_p\mathscr{H}_p \overset{\simeq}{\longrightarrow} \mathfrak{h} \qquad\qquad {\scriptstyle (p \in \mathscr{P})}$$

is the *identity* isomorphism, if one interprets $T_p\mathscr{H}_p$ as the tangent space to the Lie group copy $\mathscr{H}_p \simeq H$.

8.3. **Curvature 2-form and curvature function.** For reasons of clear notational distinction of objects that live in different spaces, we shall always denote by $\widehat{X}, \widehat{Y}, \widehat{Z}$ or by $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ vector fields on the bundle $\mathscr{P}$, while vector fields X, Y, Z on M will be systematically denoted without any hat or tilde.

Notably, for any vector $\mathsf{x} \in \mathfrak{g}$, the inverse image of x through ω at any point $p \in \mathscr{P}$, namely:

$$\widehat{X}_p := \omega_p^{-1}(\mathsf{x}) \qquad\qquad {\scriptstyle (p \in \mathscr{P})}$$

provides, as the point p varies all over $\mathscr{P}$, a well-defined $\mathscr{C}^\infty$ vector field which is sometimes called the *constant vector field*:

$$\widehat{X} := \omega^{-1}(\mathsf{x})$$

associated to x, the slight abuse of notation "$\omega^{-1}(\mathsf{x})$" being admissible here. Also, because all the $\omega_p \colon T_p\mathscr{P} \to \mathfrak{g}$ are isomorphisms, for any choice of a vector space basis $(\mathsf{x}_k)_{1 \leqslant k \leqslant \dim_{\mathbb{K}} \mathfrak{g}}$ for $\mathfrak{g}$, the collection of the r vector fields:

$$\widehat{X}_k := \omega_p^{-1}(\mathsf{x}_k) \qquad\qquad {\scriptstyle (k = 1 \cdots r)}$$

visibly makes a global frame on $\mathscr{P}$, that is to say, at every point $p \in \mathscr{P}$, the vectors $\widehat{X}_1|_p, \dots, \widehat{X}_r|_p$ make a basis of $T_p\mathscr{P}$. Furthermore, property **(ii)** that the $\mathfrak{g}$-valued

Cartan-connection 1-form ω should satisfy implies that for every element $\mathsf{y} \in \mathfrak{h}$, one has at every point $p \in \mathscr{P}$:

$$Y_p^\dagger = \omega_p^{-1}(\mathsf{y}) = \widehat{Y}_p,$$

so that (only) for these elements $\mathsf{y} \in \mathfrak{h}$, one has the coincidence of vector fields on $\mathscr{P}$:

$$Y^\dagger = \widehat{Y} \qquad\qquad (\mathsf{y} \in \mathfrak{h}).$$

Definition 8.2. The *curvature form* of the Cartan connection is the 2-form on $\mathscr{P}$ which acts on pairs of vectors $(\widetilde{X}_p, \widetilde{Y}_p)$ based at an arbitrary point $p \in \mathscr{P}$ through the formula:

$$\Omega_p\big(\widetilde{X}_p,\, \widetilde{Y}_p\big) := d\omega_p\big(\widetilde{X},\, \widetilde{Y}\big) + \big[\omega_p\big(\widetilde{X}\big),\, \omega_p\big(\widetilde{Y}\big)\big]_{\mathfrak{g}},$$

where $\widetilde{X}$ and $\widetilde{Y}$ denote any two local $\mathscr{C}^\infty$ extensions near p satisfying $\widetilde{X}\big|_p = \widetilde{X}_p$ and $\widetilde{Y}\big|_p = \widetilde{Y}_p$; the obtained value $\Omega_p\big(\widetilde{X}_p,\, \widetilde{Y}_p\big)$ is easily seen to be independent of these extensions, and also to be skew-symmetric.

Lemma 8.3. *The curvature vanishes as soon as at least one of its two arguments is tangent to the H-principal bundle foliation $\mathscr{H}$, namely:*

$$0 = \Omega_p\big(\widetilde{X}_p,\, \widetilde{Y}_p\big)$$

whenever either $\widetilde{X}_p \in T_p\mathscr{H}_p$ or $\widetilde{Y}_p \in T_p\mathscr{H}_p$.

Proof. Since the $\omega_p^{-1}(\mathsf{y})$ span $T_p\mathscr{H}_p$ when y varies in $\mathfrak{h}$, and since Ω is skew-symmetric, it suffices in fact to establish this vanishing curvature property:

$$0 = \Omega\big(\widehat{X}, \widehat{Y}\big),$$

for any constant vector field $\widehat{X} = \omega^{-1}(\mathsf{x})$ with $\mathsf{x} \in \mathfrak{g}$ arbitrary and any vertical constant vector field $\widehat{Y} = \omega^{-1}(\mathsf{y})$ with $\mathsf{y} \in \mathfrak{h}$ arbitrary. But then, this is a known consequence of **(iii)**. □

Now, at each point $p \in \mathscr{P}$, for every $\mathsf{x} \in \mathfrak{g}$, the inverse image through ω_p of the element $\mathsf{x} + \mathfrak{h}$ of the quotient $\mathfrak{g}/\mathfrak{h}$:

$$\omega_p^{-1}\big(\mathsf{x} + \mathfrak{h}\big) = \omega_p^{-1}(\mathsf{x}) + \omega_p^{-1}(\mathfrak{h}) = \omega_p^{-1}(\mathsf{x}) + T_p\mathscr{H}_p$$

is defined exactly up to the H-principal bundle tangent space. By bilinearity, it then follows immediately from the above proposition that

$$\Omega_p\big(\omega_p^{-1}(\mathsf{x}' + \mathfrak{h}),\, \omega_p^{-1}(\mathsf{x}'' + \mathfrak{h})\big) = \Omega_p\big(\omega_p^{-1}(\mathsf{x}'),\, \omega_p^{-1}(\mathsf{x}'')\big).$$

In other words, the curvature 2-form acts in fact on the quotient $T\mathscr{P}\big/T\mathscr{H}$ by the vertical H-foliation. This observation yields a path to a more concrete access to Cartan curvature which is quite useful for effective computations.

Definition 8.4. The *curvature function* of a Cartan connection $\omega\colon T\mathscr{P} \to \mathfrak{g}$ is the map:

$$\kappa \in \mathscr{C}^\infty\big(\mathscr{P},\, \mathrm{End}\big(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\, \mathfrak{g}\big)\big)$$

which sends every point $p \in \mathscr{P}$ to the following $\mathfrak{g}$-valued alternating bilinear map:

$$\kappa(p): \quad \begin{aligned} (\mathfrak{g}/\mathfrak{h}) \wedge (\mathfrak{g}/\mathfrak{h}) &\longrightarrow \mathfrak{g} \\ \big(\mathsf{x}' \bmod \mathfrak{h}\big) \wedge \big(\mathsf{x}'' \bmod \mathfrak{h}\big) &\longmapsto \Omega_p\big(\omega_p^{-1}(\mathsf{x}'),\, \omega_p^{-1}(\mathsf{x}'')\big) \\ &\quad = \Omega_p\big(\widehat{X}'_p, \widehat{X}''_p\big). \end{aligned}$$

Conversely, one easily convinces oneself that the curvature function determines the curvature form in a unique way.

Lemma 8.5. *For any two representatives* $\mathsf{x}', \mathsf{x}'' \in \mathfrak{g}$ *of two elements of* $\mathfrak{g}/\mathfrak{h}$*, one has:*

$$\kappa(p)(\mathsf{x}', \mathsf{x}'') = [\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}} - \omega_p\big(\big[\omega^{-1}(\mathsf{x}'),\, \omega^{-1}(\mathsf{x}'')\big]\big).$$

Proof. Starting from the definition just stated and applying the so-called *Cartan formula*[5] to expand the $d\omega_p$-term, we easily get:

$$\begin{aligned} \kappa(p)(\mathsf{x}', \mathsf{x}'') &= \Omega_p(\widehat{X}', \widehat{X}'') \\ &= d\omega_p\big(\widehat{X}', \widehat{X}''\big) + \big[\omega_p(\widehat{X}'),\, \omega_p(\widehat{X}'')\big]_{\mathfrak{g}} \\ &= \widehat{X}'\big(\omega_p(\widehat{X}'')\big) - \widehat{X}''\big(\omega_p(\widehat{X}')\big) - \omega_p\big([\widehat{X}',\, \widehat{X}'']\big) + [\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}} \\ &= \underline{\widehat{X}'(\mathsf{x}'')}_{\circ} - \underline{\widehat{X}''(\mathsf{x}')}_{\circ} - \omega_p\big([\omega^{-1}(\mathsf{x}'),\, \omega^{-1}(\mathsf{x}'')]\big) + [\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}, \end{aligned}$$

where underlined terms vanish as do all differentiated constants. □

Proposition 8.6. *For any* $h \in H$*, the curvature function enjoys the* Ad*-equivariancy property:*

$$\kappa\big(ph\big)\big(\mathsf{x}' \bmod \mathfrak{h},\, \mathsf{x}'' \bmod \mathfrak{h}\big) = \mathrm{Ad}(h^{-1})\big[\kappa(p)\big(\mathrm{Ad}(h)\,\mathsf{x}',\, \mathrm{Ad}(h)\,\mathsf{x}''\big)\big],$$

where $\mathsf{x}' \bmod \mathfrak{h}$ *and* $\mathsf{x}' \bmod \mathfrak{h}$ *are any two elements of* $\mathfrak{g}/\mathfrak{h}$*. Furthermore, for any fundamental field* $Y^\dagger = \frac{d}{dt}\big|_0 R_{\exp(ty)}$ *on* $\mathscr{P}$ *associated to an arbitrary* $\mathsf{y} \in \mathfrak{h}$*, one has:*

$$(Y^\dagger \kappa)(p)(\mathsf{x}', \mathsf{x}'') = -\big[\mathsf{y},\, \kappa(p)(\mathsf{x}', \mathsf{x}'')\big]_{\mathfrak{g}} + \kappa(p)\big([\mathsf{y}, \mathsf{x}']_{\mathfrak{g}},\, \mathsf{x}''\big) + \kappa(p)\big(\mathsf{x}',\, [\mathsf{y}, \mathsf{x}'']_{\mathfrak{g}}\big),$$

where the two arguments x' *and* x'' *of the curvature function* $\kappa(p)$ *are (implicitly) taken* $\bmod \mathfrak{h}$*.*

Proof. First of all, the right-equivariancy of the connection form and of the curvature may be read as two equations:

$$\begin{aligned} \omega_{ph}\big((R_h)_{*p}(\widetilde{X}_p)\big) &= \mathrm{Ad}(h^{-1})\big[\omega_p\big(\widetilde{X}_p\big)\big], \\ \Omega_{ph}\big((R_h)_{*p}(\widetilde{X}'_p),\ (R_h)_{*p}(\widetilde{X}''_p)\big) &= \mathrm{Ad}(h^{-1})\big[\Omega_p\big(\widetilde{X}'_p,\, \widetilde{X}''_p\big)\big], \end{aligned} \tag{31}$$

valid for any $h \in H$ and for any three (vector) fields $\widetilde{X}_p, \widetilde{X}'_p, \widetilde{X}''_p \in T_p\mathscr{P}$ based at an arbitrary point $p \in \mathscr{P}$. So, let us apply the first equivariancy in which we

[5] *See* [37], p. 58: if ω is an arbitrary 1-form on a smooth manifold N valued in some $\mathbb{K}$-vector space V, then $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$ for any two vector fields X and Y on M.

replace $\widetilde{X}_p$ by the constant vector (field) $\widehat{X}_p' := \omega_p^{-1}(\mathsf{x}')$ associated to an arbitrary $\mathsf{x}' \in \mathfrak{g}$:

$$\omega_{ph}\big((R_h)_{*p}\big(\omega_p^{-1}(\mathsf{x}')\big)\big) = \omega_{ph}\big((R_h)_{*p}\big(\widehat{X}_p'\big)\big) = \mathrm{Ad}(h^{-1})\big[\omega_p\big(\widehat{X}_p'\big)\big] = \mathrm{Ad}(h^{-1})[\mathsf{x}'],$$

whence equivalently if we apply ω_{ph}^{-1} to both extreme sides:

$$(R_h)_{*p}\big(\omega_p^{-1}(\mathsf{x}')\big) = \omega_{ph}^{-1}\big(\mathrm{Ad}(h^{-1})\,\mathsf{x}'\big).$$

If we now replace x' by just $\mathsf{Ad}(h)\,\mathsf{x}'$ here, we get the useful formula:

$$(R_h)_{*p}\big(\omega_p^{-1}\big(\mathrm{Ad}(h)\,\mathsf{x}'\big)\big) = \omega_{ph}^{-1}(\mathsf{x}'). \tag{32}$$

Thanks to this preliminary, we may now compute:

$$\begin{aligned}
\kappa(ph)(\mathsf{x}',\mathsf{x}'') &\overset{\text{def}}{=} \Omega_{ph}\big(\omega_{ph}^{-1}(\mathsf{x}'),\, \omega_{ph}^{-1}(\mathsf{x}'')\big) \\
\text{[apply (32)]} \qquad &= \Omega_{ph}\Big(\big(R_h)_{*p}\big(\omega_p^{-1}\big(\mathrm{Ad}(h)\,\mathsf{x}'\big)\big),\ (R_h)_{*p}\big(\omega_p^{-1}\big(\mathrm{Ad}(h)\,\mathsf{x}''\big)\big)\Big) \\
\text{[use (31)]} \qquad &= \mathrm{Ad}(h^{-1})\big[\Omega_p\big(\omega_p^{-1}\big(\mathrm{Ad}(h)\,\mathsf{x}'\big),\ \omega_p^{-1}\big(\mathrm{Ad}(h)\,\mathsf{x}''\big)\big)\big] \\
&\overset{\text{def}}{=} \mathrm{Ad}(h^{-1})\big[\kappa(p)\big(\mathrm{Ad}(h)\,\mathsf{x}',\, \mathrm{Ad}(h)\,\mathsf{x}''\big)\big],
\end{aligned}$$

which completes the verification of the first formula claimed in the proposition.

Next, let us assume that the translational element $h \in H$ is of the form $h_t := \exp(t\mathsf{y})$, for some $\mathsf{y} \in \mathfrak{h}$, where the parameter $t \in \mathbb{K}$ is small, so that we have, after multiplying both sides by $\mathrm{Ad}(h_t)$:

$$\mathrm{Ad}(h_t)\big[\kappa(ph_t)(\mathsf{x}',\mathsf{x}'')\big] = \kappa(p)\big(\mathrm{Ad}(h_t)\,\mathsf{x}',\, \mathrm{Ad}(h_t)\,\mathsf{x}''\big).$$

From the very definition of the vector field $Y^\dagger = \frac{d}{dt}\big|_0 \mathsf{R}_{\exp(t\mathsf{y})}$ on $\mathscr{P}$, it classically follows that for any function $f\colon \mathscr{P} \to \mathbb{C}$, one has:

$$\frac{d}{dt}\Big|_0 \Big(f\big(R_{\exp(t\mathsf{y})}(p)\big)\Big) = \frac{d}{dt}\Big|_0 \Big(f\big(p\exp(t\mathsf{y})\big)\Big) = (Y^\dagger f)(p).$$

Consequently, if we apply $\frac{d}{dt}\big|_0$ to the above identity, using $\mathrm{Ad}(h_0) = \mathrm{Id}$, using Leibniz' formula and using bilinearity of the curvature function, we get:

$$\begin{aligned}
\frac{d}{dt}\Big|_0 \mathrm{Ad}(\exp(t\mathsf{y}))\big[\kappa(p)(\mathsf{x}',\mathsf{x}'')\big] + (Y^\dagger\kappa)(p)(\mathsf{x}',\mathsf{x}'') &= \\
= \kappa(p)\big(\tfrac{d}{dt}\big|_0 \mathrm{Ad}(\exp(t\mathsf{y}))\,\mathsf{x}',\ \mathsf{x}''\big) &+ \kappa(p)\big(\mathsf{x}',\ \tfrac{d}{dt}\big|_0 \mathrm{Ad}(\exp(t\mathsf{y}))\,\mathsf{x}''\big).
\end{aligned}$$

Applying then twice within the right hand side the classical identity:

$$\frac{d}{dt}\Big|_0 \mathrm{Ad}\big(\exp(t\mathsf{y})\big)[\mathsf{z}] = [\mathsf{y},\mathsf{z}]_{\mathfrak{g}} = \mathrm{ad}(\mathsf{y})(\mathsf{z}),$$

which can also be thought of as defining the adjoint representation (restricted to the subalgebra $\mathsf{h} \subset \mathfrak{g}$):

$$\mathrm{ad}\colon \mathfrak{h} \longrightarrow \mathrm{End}(\mathfrak{g}), \qquad \mathsf{y} \longmapsto \big(\mathsf{z} \mapsto [\mathsf{y},\mathsf{z}]_{\mathfrak{g}}\big),$$

and putting $(Y^\dagger\kappa)$ single on the left, we finally obtain the second identity claimed in the proposition:

$$(Y^\dagger\kappa)(p)(\mathsf{x}',\mathsf{x}'') = -\big[\mathsf{y},\, \kappa(p)(\mathsf{x}',\mathsf{x}'')\big]_{\mathfrak{g}} + \kappa(p)\big([\mathsf{y},\mathsf{x}']_{\mathfrak{g}},\, \mathsf{x}''\big) + \kappa(p)\big(\mathsf{x}',\, [\mathsf{y},\mathsf{x}']_{\mathfrak{g}}\big).$$

In the case where H and G are connected and simply connected, a natural converse holds by integrating these differential relations, but as we will work with local Lie groups, we will not need such a converse. □

Computing this curvature function explicitly in coordinates and with a basis of $\mathfrak{g}$ will exhibit much more the algebraic complexity it can possess.

8.4. **Lie algebra bases.** In order to achieve the construction of Cartan connections for specific new geometric structures, one must work and compute effectively with some concrete bases for $\mathfrak{g}$ and $\mathfrak{h}$. Denote then by:

$$r := \dim_{\mathbb{K}} \mathfrak{g}, \qquad n := \dim_{\mathbb{K}} \big(\mathfrak{g}/\mathfrak{h}\big) \quad \text{and} \quad n - r = \dim_{\mathbb{K}} \mathfrak{h},$$

suppose $r \geqslant 2$, $n \geqslant 1$, $n - r \geqslant 1$ so that $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are all nonzero. For an adapted basis $(\mathsf{x}_k)_{1 \leqslant k \leqslant r}$ which may be thought of, in concrete examples, to enjoy some useful normalizations/simplification (think of root systems for semi-simple Lie algebras), we then have:

$$\begin{aligned} \mathfrak{g} &= \operatorname{Span}_{\mathbb{K}}\big(\mathsf{x}_1, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}, \dots, \mathsf{x}_r\big), \\ \mathfrak{h} &= \qquad\qquad\quad \operatorname{Span}_{\mathbb{K}}\big(\mathsf{x}_{n+1}, \dots, \mathsf{x}_r\big). \end{aligned}$$

Also, we may suppose that $(n - r)$ representatives in $\mathfrak{g}/\mathfrak{h}$ are just:

$$\mathsf{y}_1 := \mathsf{x}_{n+1}, \ \dots\dots, \ \mathsf{y}_{n-r} := \mathsf{x}_r,$$

and we shall have to remember the notational coincidences:

$$\mathsf{y}_j \equiv \mathsf{x}_{n+j} \qquad\qquad {\scriptstyle (j = 1 \cdots r - n)}.$$

Next, let $\mathfrak{g}^* = \operatorname{Lin}(\mathfrak{g}, \mathbb{C})$ denote the dual of the Lie algebra $\mathfrak{g}$, viewed as a plain vector space (it has no natural Lie bracket structure). If we introduce the basis of $\mathfrak{g}^*$:

$$\mathsf{x}_1^*, \dots, \mathsf{x}_n^*, \mathsf{x}_{n+1}^*, \dots, \mathsf{x}_r^*$$

which is dual to the previously fixed basis $\mathsf{x}_1, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}, \dots, \mathsf{x}_r$ of $\mathfrak{g}$, then by definition, for any $i, j = 1, \dots, n, n+1, \dots, r$, we have:

$$\mathsf{x}_i^*(\mathsf{x}_j) = \delta_j^i := \begin{cases} 1 \ \text{ if } \ i = j, \\ 0 \ \text{ otherwise}. \end{cases}$$

Of course, $\mathfrak{h}^*$ is then spanned by $\mathsf{x}_1^*, \dots, \mathsf{x}_{r-n}^*$.

For later use, we remind also that if E is a finite-dimensional $\mathbb{K}$-vector space and if ω^*, π^* are one-forms belonging to its dual $E^* = \operatorname{Lin}(E, \mathbb{C})$, then the two-form $\omega^* \wedge \pi^*$ acts on pairs $(e, f) \in E^2$ by definition as:

$$\omega^* \wedge \pi^*(e, f) \overset{\text{def}}{=} \omega^*(e)\, \pi^*(f) - \omega^*(f)\, \pi^*(e).$$

In particular, for any i_1, i_2 with $i_1 < i_2$ and for any j_1, j_2 without restriction, we have:

$$\begin{aligned} \mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^*(\mathsf{x}_{j_1}, \mathsf{x}_{j_2}) &= \mathsf{x}_{i_1}^*(\mathsf{x}_{j_1})\, \mathsf{x}_{i_2}^*(\mathsf{x}_{j_2}) - \mathsf{x}_{i_1}^*(\mathsf{x}_{j_2})\, \mathsf{x}_{i_2}^*(\mathsf{x}_{j_1}) \\ &= \delta_{j_1}^{i_1}\, \delta_{j_2}^{i_2} - \delta_{j_2}^{i_1}\, \delta_{j_1}^{i_2}. \end{aligned} \tag{33}$$

With any such choice of a basis, brackets of the Lie algebra $\mathfrak{g}$ are encoded by the so-called *structure constants*:

$$\big[\mathsf{x}_{k_1},\,\mathsf{x}_{k_2}\big]_{\mathfrak{g}}=\sum_{s=1}^{r}c_{k_1,k_2}^{s}\,\mathsf{x}_s$$
$$(k_1,k_2=1\cdots n,n+1,...,r).$$

Of course, the skew-symmetry and the Jacobi identity:

$$\begin{aligned}0&=[\mathsf{x}_{k_1},\mathsf{x}_{k_2}]_{\mathfrak{g}}+[\mathsf{x}_{k_1},\mathsf{x}_{k_2}]_{\mathfrak{g}}\\0&=\big[[\mathsf{x}_{k_1},\mathsf{x}_{k_2}]_{\mathfrak{g}},\mathsf{x}_{k_3}\big]_{\mathfrak{g}}+\big[[\mathsf{x}_{k_3},\mathsf{x}_{k_1}]_{\mathfrak{g}},\mathsf{x}_{k_2}\big]_{\mathfrak{g}}+\big[[\mathsf{x}_{k_2},\mathsf{x}_{k_3}]_{\mathfrak{g}},\mathsf{x}_{k_1}\big]_{\mathfrak{g}}\end{aligned}\tag{34}$$

read at the level of structure constants as:

$$\begin{aligned}0&=c_{k_1,k_2}^{s}+c_{k_2,k_1}^{s}\\0&=\sum_{s=1}^{r}\big(c_{k_1,k_2}^{s}\,c_{s,k_3}^{l}+c_{k_3,k_1}^{s}\,c_{s,k_2}^{l}+c_{k_2,k_3}^{s}\,c_{s,k_1}^{l}\big)\end{aligned}$$
$$(k_1,\,k_2,\,k_3,\,l=1\cdots r).$$

Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, one has $c_{k_1,k_2}^{s}=0$ whenever $n+1\leqslant k_1,k_2\geqslant r$, for any $s\leqslant n$. In terms of the generators $\mathsf{y}_j=\mathsf{x}_{n+j}$ of $\mathfrak{h}$, we may also write the Lie algebra structure of $\mathfrak{h}$ under the form:

$$\big[\mathsf{y}_{j_1},\,\mathsf{y}_{j_2}\big]_{\mathfrak{g}}=\sum_{t=1}^{r-n}d_{j_1,j_2}^{t}\,\mathsf{y}_t\qquad\qquad(j_1,j_2,t=1\cdots r-n),$$

where we admit the notational coincidences:

$$d_{j_1,j_2}^{t}=c_{n+j_1,n+j_2}^{n+t}\qquad\qquad(j_1,j_2,t=1\cdots r-n).$$

8.5. **Expansion of the $\mathfrak{g}$-valued Cartan connection form ω.** In terms of the basis $(\mathsf{x}_k)_{1\leqslant k\leqslant r}$ of $\mathfrak{g}$, the $\mathfrak{g}$-valued 1-form ω expands under the shape:

$$\omega=\sum_{k=1}^{r}\omega^k\,\mathsf{x}_k,$$

where the ω^k are *scalar*, *i.e.* $\mathbb{K}$-valued, 1-forms on $\mathscr{P}$. If ϖ is another $\mathfrak{g}$-valued 1-form that one wants to "wedge" with ω, it is natural to simultaneously "wedge" the 1-forms and let the Lie bracket act on the basis elements x_k as follows:

$$\begin{aligned}\omega\wedge_{\mathfrak{g}}\varpi&:=\Big(\sum_{k_1=1}^{r}\omega^{k_1}\,\mathsf{x}_{k_1}\Big)\wedge_{\mathfrak{g}}\Big(\sum_{k_2=1}^{r}\varpi^{k_2}\,\mathsf{x}_{k_2}\Big)\\&=\sum_{k_1=1}^{r}\sum_{k_2=1}^{r}\omega^{k_1}\wedge\varpi^{k_2}\,[\mathsf{x}_{k_1},\,\mathsf{x}_{k_2}]_{\mathfrak{g}}.\end{aligned}$$

We employ here the intuitively clear notation "$\wedge_{\mathfrak{g}}$" in order to expressly notify that both a wedge product and a Lie bracket act simultaneously. With this natural definition, it then follows from the classical formula:

$$(\omega^{k_1}\wedge\varpi^{k_2})\big(\widetilde{X},\widetilde{Y}\big)=\omega^{k_1}(\widetilde{X})\,\varpi^{k_2}(\widetilde{Y})-\omega^{k_1}(\widetilde{Y})\,\varpi^{k_2}(\widetilde{X})$$

that $\omega \wedge_{\mathfrak{g}} \varpi$ acts on pairs $(\widetilde{X}, \widetilde{Y})$ of vector fields on $\mathscr{P}$ just as:

$$\begin{aligned}(\omega \wedge_{\mathfrak{g}} \varpi)\big(\widetilde{X}, \widetilde{Y}\big) &= \sum_{k_1,k_2=1}^{r} \big(\omega^{k_1}(\widetilde{X})\,\varpi^{k_2}(\widetilde{Y}) - \omega^{k_2}(\widetilde{X})\,\varpi^{k_1}(\widetilde{Y})\big)\,[\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}} \\ &= \big[\omega(\widetilde{X}),\, \varpi(\widetilde{Y})\big]_{\mathfrak{g}} - \big[\omega(\widetilde{Y}),\, \varpi(\widetilde{X})\big]_{\mathfrak{g}}.\end{aligned}$$

In particular the connection form ω wedged with itself acts as:

$$\{\omega \wedge_{\mathfrak{g}} \omega\}\big(\widetilde{X}, \widetilde{Y}\big) = 2\,\big[\omega(\widetilde{X}),\, \omega(\widetilde{Y})\big]_{\mathfrak{g}},$$

from which we deduce the following alternative, equivalent expression of the curvature.

Lemma 8.7. *The curvature* 2*-form* $\Omega \in \Lambda^2\big(T^*\mathscr{P}, \mathfrak{g}\big)$ *writes shortly as:*

$$\Omega = d\omega + \tfrac{1}{2}\,\omega \wedge_{\mathfrak{g}} \omega,$$

and also in terms of a basis $(\mathsf{x}_k)_{1\leqslant k\leqslant r}$ *for* $\mathfrak{g}$*, just as:*

$$\Omega = \sum_{k=1}^{r} \Big\{ d\omega^k + \sum_{1\leqslant l_1<l_2\leqslant r} c_{l_1,l_2}^k\, \omega^{l_1} \wedge \omega^{l_2} \Big\}\, \mathsf{x}_k.$$

Proof. We compute Ω and reorganize its expression using the natural linearly independent frame $\omega^{l_1} \wedge \omega^{l_2}$, $l_1 < l_2$, on the space of 2-forms on $\mathscr{P}$:

$$\begin{aligned}\Omega &= d\omega + \tfrac{1}{2}\,\omega \wedge_{\mathfrak{g}} \omega \\ &= \sum_{k=1}^{r} d\omega^k\, \mathsf{x}_k + \tfrac{1}{2} \sum_{l_1=1}^{r} \sum_{l_2=1}^{r} \omega^{l_1} \wedge \omega^{l_2}\, [\mathsf{x}_{l_1}, \mathsf{x}_{l_2}]_{\mathfrak{g}} \\ &= \sum_{k=1}^{r} \Big\{ d\omega^k + \tfrac{1}{2} \sum_{l_1=1}^{r} \sum_{l_2=1}^{r} c_{l_1,l_2}^k\, \omega^{l_1} \wedge \omega^{l_2} \Big\}\, \mathsf{x}_k \\ \text{[use } c_{l_2,l_1}^k = -c_{l_1,l_2}^k\text{]} \qquad &= \sum_{k=1}^{r} \Big\{ d\omega^k + \sum_{1\leqslant l_1<l_2\leqslant r} c_{l_1,l_2}^k\, \omega^{l_1} \wedge \omega^{l_2} \Big\}\, \mathsf{x}_k,\end{aligned}$$

which is what was claimed. □

However, this representation of the curvature is not at all finalized for at least two reasons:

• each $d\omega^k$ should in principle be re-expressed in terms the basis $\big(\omega^{l_1} \wedge \omega^{l_2}\big)_{1\leqslant l_1<l_2\leqslant r}$ when one really wants to perform explicit computations;

• more deeply, any 2-form as the curvature Ω should be expressed as a 2-form in the coordinates (x, a) on $\mathscr{P}$ in order to view explicitly how the curvature depends upon some basic initial geometric data, namely upon the coefficients $\eta_{i,i'}(x)$ of a certain initial frame $(H_1, \dots, H_n)$ on M (*see* below) and also upon the coefficients $\upsilon_{j,j'}(a)$ of some explicit frame $(Y_1, \dots, Y_{n-r})$ of left-invariant vector fields on the (local) Lie group H.

Before entering further the construction of effective Cartan connections, we proceed to a preliminary investigation of what basic relations come at the level of only linear algebra.

8.6. **Back to the curvature function.** By definition, at any fixed point $p \in \mathscr{P}$, the curvature function $\kappa(p)$ happens to be a linear map from $\Lambda^2(\mathfrak{g}/\mathfrak{h})$ into $\mathfrak{g}$, whence, thanks to the canonical, classical identification $\mathrm{Lin}(E,F) \simeq E^* \otimes F$ which reads here as:

$$\begin{aligned}\mathrm{Lin}\big(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\,\mathfrak{g}\big) &\simeq \big(\Lambda^2\mathfrak{g}/\mathfrak{h}\big)^*\otimes\mathfrak{g}\\ &\simeq \Lambda^2(\mathfrak{g}^*/\mathfrak{h}^*)\otimes\mathfrak{g},\end{aligned}$$

we may express as follows the curvature function in terms of the basis elements $\mathsf{x}_1,\dots,\mathsf{x}_n,\mathsf{x}_{n+1},\dots,\mathsf{x}_r$ for $\mathfrak{g}$ and in terms of the basis (representatives) elements $\mathsf{x}_1^*,\dots,\mathsf{x}_n^*$ of $\mathfrak{g}^*/\mathfrak{h}^*$:

$$\boxed{\kappa(p) = \sum_{1\leqslant i_1<i_2\leqslant n}\,\sum_{k=1}^{r}\kappa_{i_1,i_2}^k(p)\;\mathsf{x}_{i_1}^*\wedge\mathsf{x}_{i_2}^*\otimes\mathsf{x}_k},$$

where the $p\mapsto\kappa_{i_1,i_2}^k(p)$ are a certain collection of $\mathbb{K}$-valued function on $\mathscr{P}$.

Now, we remind that Proposition 8.6 on p. 55 showed us that:

$$(Y^\dagger\kappa)(p)(\mathsf{x}',\mathsf{x}'') = -\big[\mathsf{y},\,\kappa(p)(\mathsf{x}',\mathsf{x}'')\big]_{\mathfrak{g}} + \kappa(p)\big([\mathsf{y},\mathsf{x}']_{\mathfrak{g}},\,\mathsf{x}''\big) + \kappa(p)\big(\mathsf{x}',\,[\mathsf{y},\mathsf{x}'']_{\mathfrak{g}}\big),$$

for any fundamental field $Y^\dagger = \frac{d}{dt}\big|_0 R_{\exp(t\mathsf{y})}$ on $\mathscr{P}$ associated to an arbitrary $\mathsf{y}\in\mathfrak{h}$. In terms of Lie algebra bases, it suffices to inspect what this formula gives when $\mathsf{x}'=\mathsf{x}_{i_1}$, when $\mathsf{x}''=\mathsf{x}_{i_2}$ for some $1\leqslant i_1<i_2\leqslant n$, and when $\mathsf{y}=\mathsf{y}_j=\mathsf{x}_{n+j}$ for some j with $1\leqslant j\leqslant r-n$. Plugging in then these values, we receive the equation:

$$\begin{aligned}\sum_{k=1}^r\big(Y^\dagger\kappa_{i_1,i_2}^k\big)(p)\,\mathsf{x}_k = -\sum_{k=1}^r\kappa_{i_1,i_2}^k(p)\,[\mathsf{y}_j,\,\mathsf{x}_k]_{\mathfrak{g}}+&\\ +\kappa(p)\big(\mathsf{x}_{i_1},[\mathsf{y}_j,\mathsf{x}_{i_2}]_{\mathfrak{g}}\big) - \kappa(p)\big(\mathsf{x}_{i_2},[\mathsf{y}_j,\mathsf{x}_{i_1}]_{\mathfrak{g}}\big).&\end{aligned}$$

On the first hand, using structure constants, we may easily compute the first (among three) terms appearing in the right-hand side:

$$-\sum_{k=1}^r\kappa_{i_1,i_2}^k(p)\,[\mathsf{x}_{n+j},\mathsf{x}_k]_{\mathfrak{g}} = -\sum_{k=1}^r\sum_{s=1}^r\kappa_{i_1,i_2}^k(p)\,c_{n+j,k}^s\,\mathsf{x}_s.$$

On the other hand, the two brackets appearing in the second line express themselves by means of structure constants just as follows modulo $\mathfrak{h}$:

$$\begin{aligned}[\mathsf{y}_j,\mathsf{x}_{i_1}]_{\mathfrak{g}} = [\mathsf{x}_{n+j},\mathsf{x}_{i_1}]_{\mathfrak{g}} = \sum_{k=1}^r c_{n+j,i_1}^k\,\mathsf{x}_k \equiv \sum_{i_3=1}^n c_{n+j,i_1}^{i_3}\,\mathsf{x}_{i_3}\mod\mathfrak{h},\\ [\mathsf{y}_j,\mathsf{x}_{i_2}]_{\mathfrak{g}} = [\mathsf{x}_{n+j},\mathsf{x}_{i_2}]_{\mathfrak{g}} = \sum_{k=1}^r c_{n+j,i_2}^k\,\mathsf{x}_k \equiv \sum_{i_3=1}^n c_{n+j,i_2}^{i_3}\,\mathsf{x}_{i_3}\mod\mathfrak{h}.\end{aligned}$$

We may hence derive that the entire second line above equals:

$$\sum_{i_3=1}^n c_{n+j,i_2}^{i_3}\,\kappa(p)(\mathsf{x}_{i_1},\mathsf{x}_{i_3}) - \sum_{i_3=1}^n c_{n+j,i_1}^{i_3}\,\kappa(p)(\mathsf{x}_{i_2},\mathsf{x}_{i_3}).$$

At this point however, the computation is not yet finished. Indeed, reminding the basic formula (33) on p. 57, we may pursue for instance the expansion of $\kappa(p)(\mathsf{x}_{i_1}, \mathsf{x}_{i_3})$ in the first sum:

$$\begin{aligned}\kappa(p)(\mathsf{x}_{i_1}, \mathsf{x}_{i_3}) &= \sum_{1\leqslant i_1'<i_2'\leqslant n} \sum_{k=1}^{r} \kappa_{i_1',i_2'}^{k}(p)\, \mathsf{x}_{i_1'}^{*} \wedge \mathsf{x}_{i_2'}^{*}(\mathsf{x}_{i_1}, \mathsf{x}_{i_3}) \otimes \mathsf{x}_k \\ &= \sum_{1\leqslant i_1'<i_2'\leqslant n} \sum_{k=1}^{r} \kappa_{i_1',i_2'}^{k}(p)\, \big[\delta_{i_1}^{i_1'}\delta_{i_3}^{i_2'} - \delta_{i_3}^{i_1'}\delta_{i_1}^{i_2'}\big]\, \mathsf{x}_k.\end{aligned}$$

Thanks to this simple formula, the first $\sum_{i_3}$ continues as (complete explanations follow just after):

$$\begin{aligned}\sum_{i_3=1}^{n} c_{n+j,i_2}^{i_3}\, \kappa(p)(\mathsf{x}_{i_1}, \mathsf{x}_{i_3}) &= \sum_{k=1}^{r} \bigg(\sum_{i_3=1}^{n} \sum_{1\leqslant i_1'<i_2'\leqslant n} c_{n+j,i_2}^{i_3}\, \kappa_{i_1',i_2'}^{k}(p)\big[\delta_{i_1}^{i_1'}\delta_{i_3}^{i_2'} - \delta_{i_3}^{i_1'}\delta_{i_1}^{i_2'}\big]\bigg)\, \mathsf{x}_k \\ &= \sum_{k=1}^{r} \bigg(\sum_{1\leqslant i_1'<i_2'\leqslant n} c_{n+j,i_2}^{i_2'}\, \kappa_{i_1',i_2'}^{k}(p)\, \delta_{i_1}^{i_1'} - \sum_{1\leqslant i_1'<i_2'\leqslant n} c_{n+j,i_2}^{i_1'}\, \kappa_{i_1',i_2'}^{k}(p)\, \delta_{i_1}^{i_2'}\bigg)\, \mathsf{x}_k \\ &= \sum_{k=1}^{r} \bigg(\sum_{i_2'=i_1+1}^{n} c_{n+j,i_2}^{i_2'}\, \kappa_{i_1,i_2'}^{k}(p) - \sum_{i_1'=1}^{i_1} c_{n+j,i_2}^{i_1'}\, \kappa_{i_1',i_1}^{k}(p)\bigg)\, \mathsf{x}_k \\ &= \sum_{k=1}^{r} \bigg(-\sum_{i'=1}^{i_1} c_{n+j,i_2}^{i'}\, \kappa_{i',i_1}^{k}(p) + \sum_{i'=i_1+1}^{n} c_{n+j,i_2}^{i'}\, \kappa_{i_1,i'}^{k}(p)\bigg)\, \mathsf{x}_k.\end{aligned}$$

From the first to the second line, the $\sum_{i_3}$ is automatically killed because of the presence of the two $\delta_{i_3}^{i_2'}$ and $-\delta_{i_3}^{i_1'}$. Then the two $\sum_{i_1'<i_2'}$ both reduce to a single sum, because of the presence of the two remaining $\delta_{i_1}^{i_1'}$ and $-\delta_{i_1}^{i_2'}$. Finally, the order of appearance of the two sums $\sum_{i_2'}$ and $-\sum_{i_1'}$ is interchanged and both summation indices are given the same name i'. Summing up all terms, we have shown the following

Proposition 8.8. *For every* $\mathsf{y} \in \mathfrak{h}$ *and every* $1 \leqslant i_1 < i_2 \leqslant n$, *one has:*

$$\begin{aligned}\sum_{k=1}^{r} \big(Y^{\dagger}\kappa_{i_1,i_2}^{k}\big)(p)\, \mathsf{x}_k = \sum_{k=1}^{r} \bigg(&\sum_{s=1}^{r} c_{n+j,s}^{k}\, \kappa_{i_1,i_2}^{s}(p) - \\ &- \sum_{i'=1}^{i_1} c_{n+j,i_2}^{i'}\, \kappa_{i',i_1}^{k}(p) + \sum_{i'=i_1+1}^{n} c_{n+j,i_2}^{i'}\, \kappa_{i_1,i'}^{k}(p) \\ &+ \sum_{i'=1}^{i_2} c_{n+j,i_1}^{i'}\, \kappa_{i',i_2}^{k}(p) - \sum_{i'=i_2+1}^{n} c_{n+j,i_1}^{i'}\, \kappa_{i_2,i'}^{k}(p)\bigg)\, \mathsf{x}_k,\end{aligned}$$

where $Y^{\dagger} = \widehat{Y} = \omega^{-1}(\mathsf{y})$ *is the constant field associated to* y. □

8.7. **Bianchi identity.** Looking at $\Omega = d\omega + \frac{1}{2}\, \omega \wedge_{\mathfrak{g}} \omega$, if one applies once more the exterior differential operator d, the term $dd\omega$ drops. This observation leads to Bianchi-type identities which were first understood by Christoffel and Lipschitz in the context of Riemannian geometry.

Theorem 8.1. *The curvature* 2*-form:*

$$\Omega = \sum_{k=1}^{r} \Omega^k \, \mathsf{x}_k = \sum_{k=1}^{r} \bigg\{ d\omega^k + \sum_{1 \leqslant l_1 < l_2 \leqslant r} c_{l_1,l_2}^k \, \omega^{l_1} \wedge \omega^{l_2} \bigg\} \mathsf{x}_k$$

satisfies the identity:

$$d\Omega = \Omega \wedge_{\mathfrak{g}} \omega,$$

that is to say in terms of the basis $(\mathsf{x}_k)_{1 \leqslant k \leqslant r}$ *of* $\mathfrak{g}$*:*

$$d\Omega^k = \sum_{i_1, i_2 = 1}^{r} c_{i_1, i_2}^k \, \Omega^{i_1} \wedge \omega^{i_2} \qquad\qquad (k = 1 \cdots r).$$

Proof. Applying, as we said, d to the definition $\Omega = d\omega + \frac{1}{2}\, \omega \wedge_{\mathfrak{g}} \omega$ of the curvature form, we get:

$$\begin{aligned} d\Omega &= \underline{dd\omega}_{\circ} + \tfrac{1}{2}\, d\omega \wedge_{\mathfrak{g}} \omega - \tfrac{1}{2}\, \omega \wedge_{\mathfrak{g}} d\omega \\ &= \tfrac{1}{2} \big(\Omega - \tfrac{1}{2}\, \omega \wedge_{\mathfrak{g}} \omega \big) \wedge_{\mathfrak{g}} \omega - \tfrac{1}{2}\, \omega \wedge_{\mathfrak{g}} \big(\Omega - \tfrac{1}{2}\, \omega \wedge_{\mathfrak{g}} \omega \big). \end{aligned}$$

Here, one has to mind the fact that, although the wedge product $\wedge$ between scalar one-forms is associative, the wedge-bracket product $\wedge_{\mathfrak{g}}$ is *not* (in general) associative, just because one does not have in general $\big[[\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}}, \mathsf{x}_{k_3} \big]_{\mathfrak{g}} = \big[\mathsf{x}_{k_1}, [\mathsf{x}_{k_2}, \mathsf{x}_{k_3}]_{\mathfrak{g}} \big]_{\mathfrak{g}}$. Fortunately, the Jacobi identity guarantees that the following is true.

Lemma 8.9. *For any* $\mathfrak{g}$*-valued* 1*-form* $\varpi = \sum_{k=1}^{r} \varpi^k \, \mathsf{x}_k$*, one has:*

$$0 = \big(\varpi \wedge_{\mathfrak{g}} \varpi \big) \wedge_{\mathfrak{g}} \varpi \qquad \text{and} \qquad 0 = \varpi \wedge_{\mathfrak{g}} \big(\varpi \wedge_{\mathfrak{g}} \varpi \big).$$

Proof. We check the first identity, the second one being similar. Expanding, we may write:

$$\begin{aligned} \big(\varpi \wedge_{\mathfrak{g}} \varpi \big) \wedge_{\mathfrak{g}} \varpi &= \bigg(\sum_{k_1=1}^{r} \varpi^{k_1} \, \mathsf{x}_{k_1} \wedge_{\mathfrak{g}} \sum_{k_2=1}^{r} \varpi^{k_2} \, \mathsf{x}_{k_2} \bigg) \wedge_{\mathfrak{g}} \sum_{k_3=1}^{r} \varpi^{k_3} \, \mathsf{x}_{k_3} \\ &= \bigg(\sum_{k_1, k_2 = 1}^{r} \varpi^{k_1} \wedge \varpi^{k_2} \, [\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}} \bigg) \wedge_{\mathfrak{g}} \sum_{k_3=1}^{r} \varpi^{k_3} \, \mathsf{x}_{k_3} \\ &= \sum_{k_1, k_2, k_3 = 1}^{r} \varpi^{k_1} \wedge \varpi^{k_2} \wedge \varpi^{k_3} \, \big[[\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}}, \mathsf{x}_{k_3} \big]_{\mathfrak{g}}. \end{aligned}$$

Clearly, since the three-terms wedge product $\varpi^{k_1} \wedge \varpi^{k_2} \wedge \varpi^{k_3}$ obviously vanishes as soon as at least two of the k_i's coincide, the triple sum $\sum_{k_1, k_2, k_3}$ decomposes plainly in six sub-sums:

$$\begin{aligned} \sum_{\substack{k_1, k_2, k_3 \\ \text{pairwise} \neq}} = \sum_{k_1 < k_2 < k_3} + \sum_{k_3 < k_1 < k_2} + \sum_{k_2 < k_3 < k_1} + \\ + \sum_{k_2 < k_1 < k_3} + \sum_{k_3 < k_2 < k_1} + \sum_{k_1 < k_3 < k_2}. \end{aligned}$$

Then each triple sum vanishes identically thanks to the Jacobi identity (34). □

Coming back to the $d\Omega$ we left above, the lemma shows that the two terms involving three times ω vanish and it remains:

$$\begin{aligned} d\Omega &= \tfrac{1}{2}\,\Omega \wedge_{\mathfrak{g}} \omega - \tfrac{1}{2}\,\omega \wedge_{\mathfrak{g}} \Omega \\ &= \Omega \wedge_{\mathfrak{g}} \omega, \end{aligned}$$

as claimed by the theorem. Expanding then this identity in bases and reorganizing, we get:

$$\begin{aligned} \sum_{k=1}^{r} d\Omega^k\, \mathsf{x}_k &= \Big(\sum_{i_1=1}^{r} \Omega^{i_1}\, \mathsf{x}_{i_1}\Big) \wedge_{\mathfrak{g}} \Big(\sum_{i_2=1}^{r} \Omega^{i_2}\, \mathsf{x}_{i_2}\Big) \\ &= \sum_{i_1,i_2=1}^{r} \Omega^{i_1} \wedge \omega^{i_2}\, [\mathsf{x}_{i_1}, \mathsf{x}_{i_2}]_{\mathfrak{g}} \\ &= \sum_{k=1}^{r} \Big(\sum_{i_1,i_2=1}^{r} c_{i_1,i_2}^{k}\, \Omega^{i_1} \wedge \omega^{i_2}\Big)\, \mathsf{x}_k \end{aligned}$$

that is to say, for every $k = 1, \ldots, r$:

$$d\Omega^k = \sum_{i_1,i_2=1}^{r} c_{i_1,i_2}^{k}\, \Omega^{i_1} \wedge \omega^{i_2},$$

which concludes the proof. □

The curvature function $p \mapsto \kappa(p)$ is a map $\mathscr{P} \to \mathscr{C}^2(\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$. A second version of Bianchi identities is as follows. Remind the operator $\partial\colon \mathscr{C}^2(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}) \longrightarrow \mathscr{C}^3(\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$.

Proposition 8.10. *For any three* $\mathsf{x}', \mathsf{x}'', \mathsf{x}''' \in \mathfrak{g}$*, one has at every point* $p \in \mathscr{P}$*:*

$$0 = (\partial\kappa)(p)(\mathsf{x}', \mathsf{x}'', \mathsf{x}''') + \sum_{\text{cycl}} \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}''),\, \mathsf{x}'''\big) + \sum_{\text{cycl}} \big(\widehat{X}'\kappa\big)(p)(\mathsf{x}'', \mathsf{x}'''),$$

that is to say in greater length:

$$\begin{aligned} 0 = {} & \big[\mathsf{x}', \kappa(p)(\mathsf{x}'', \mathsf{x}''')\big]_{\mathfrak{g}} - \big[\mathsf{x}'', \kappa(p)(\mathsf{x}', \mathsf{x}''')\big]_{\mathfrak{g}} + \big[\mathsf{x}''', \kappa(p)(\mathsf{x}', \mathsf{x}'')\big]_{\mathfrak{g}} - \\ & - \kappa(p)\big([\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}},\, \mathsf{x}'''\big) + \kappa(p)\big([\mathsf{x}', \mathsf{x}''']_{\mathfrak{g}},\, \mathsf{x}''\big) - \kappa(p)\big([\mathsf{x}'', \mathsf{x}''']_{\mathfrak{g}},\, \mathsf{x}'\big) + \\ & + \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}''),\, \mathsf{x}'''\big) - \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}'''),\, \mathsf{x}''\big) + \kappa(p)\big(\kappa(p)(\mathsf{x}'', \mathsf{x}'''),\, \mathsf{x}''\big) + \\ & + \big(\widehat{X}'\kappa\big)(p)(\mathsf{x}'', \mathsf{x}''') - \big(\widehat{X}''\kappa\big)(p)(\mathsf{x}', \mathsf{x}''') + \big(\widehat{X}'''\kappa\big)(p)(\mathsf{x}', \mathsf{x}''). \end{aligned}$$

Proof. By the Definition 8.4 of the curvature function and the Definition 8.2 of the curvature form, for any two $\mathsf{z}, \mathsf{x}''' \in \mathfrak{g}$, one has:

$$\begin{aligned} \kappa(p)(\mathsf{z}, \mathsf{x}''') &= \Omega_p\big(\widehat{Z}_p, \widehat{X}_p'''\big) \\ &= d\omega_p\big(\widehat{Z}_p, \widehat{X}_p'''\big) + \big[\omega_p(\widehat{Z}_p), \omega_p(\widehat{X}_p''')\big]_{\mathfrak{g}}. \end{aligned}$$

Replacing then $\widehat{Z}_p$ by $\big[\widehat{X}_p', \widehat{X}_p''\big]$, hence also z by $\omega_p\big(\big[\widehat{X}_p', \widehat{X}_p''\big]\big)$ gives:

$$\kappa(p)\big(\omega_p\big(\big[\widehat{X}_p', \widehat{X}_p''\big]\big),\, \mathsf{x}'''\big) = d\omega_p\big(\big[\widehat{X}_p', \widehat{X}_p''\big], \widehat{X}_p'''\big) + \big[\omega_p\big(\big[\widehat{X}_p', \widehat{X}_p''\big]\big), \mathsf{x}'''\big]_{\mathfrak{g}}.$$

Now, if we apply the Cartan formula to expand $d\omega_p$ in the right-hand side, we receive:

$$\kappa(p)\big(\omega_p\big([\widehat{X}_p', \widehat{X}_p'']\big),\, \mathsf{x}'''\big) =$$
$$= \underline{[\widehat{X}', \widehat{X}'']\big(\omega_p(\widehat{X}''')\big)}_{\circ} - \widehat{X}'''\big(\omega_p([\widehat{X}_p', \widehat{X}_p''])\big) - \omega_p\big([[\widehat{X}_p', \widehat{X}_p''],\, \widehat{X}_p''']\big) + \big[\omega_p\big([\widehat{X}_p', \widehat{X}_p'']\big), \mathsf{x}'''\big]_{\mathfrak{g}},$$

and the first term in the right-hand side vanishes, because $\omega_p(\widehat{X}''') = \mathsf{x}'''$ is constant. Next, both in the left-hand side and in the right-hand side, we replace thrice from Lemma 8.5

$$\omega_p\big([\widehat{X}_p', \widehat{X}_p'']\big) = [\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}} - \kappa(p)(\mathsf{x}', \mathsf{x}''),$$

and we receive:

$$\kappa(p)\big([\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}, \mathsf{x}'''\big) - \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}''), \mathsf{x}'''\big) =$$
$$= -\underline{\widehat{X}'''\big([\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}\big)}_{\circ} + \big(\widehat{X}'''\kappa\big)(p)(\mathsf{x}', \mathsf{x}'') - \omega_p\big([[\widehat{X}_p', \widehat{X}_p''],\, \widehat{X}_p''']\big)$$
$$+ \big[[\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}, \mathsf{x}'''\big]_{\mathfrak{g}} - \big[\kappa(p)(\mathsf{x}', \mathsf{x}''), \mathsf{x}'''\big]_{\mathfrak{g}},$$

Putting all the six remaining terms in the right-hand side and reorganizing their order of appearance, we get an identity in which:

$$0 = \big[\mathsf{x}''',\, \kappa(p)(\mathsf{x}', \mathsf{x}'')\big]_{\mathfrak{g}} - \kappa(p)\big([\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}, \mathsf{x}'''\big) + \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}''), \mathsf{x}'''\big) +$$
$$+ \big(\widehat{X}'''\kappa\big)(p)(\mathsf{x}', \mathsf{x}'') - \omega_p\big([[\widehat{X}_p', \widehat{X}_p''],\, \widehat{X}_p''']\big) + \big[[\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}}, \mathsf{x}'''\big]_{\mathfrak{g}},$$

when one sums over all cyclic permutations of $\{\mathsf{x}', \mathsf{x}'', \mathsf{x}'''\}$, the last two terms disappear thanks to the Jacobi identity for vector fields (fifth term) and thanks to the Jacobi identity within $\mathfrak{g}$ (sixth term) so that we obtain the formula stated. □

Corollary 8.11. *For any three* $\mathsf{y}', \mathsf{y}'', \mathsf{y}''' \in \mathfrak{h}$*, one has:*

$$(\partial\kappa)(p)\big(\mathsf{x}' + \mathsf{y}', \mathsf{x}'' + \mathsf{y}'', \mathsf{x}''' + \mathsf{y}'''\big) = (\partial\kappa)(p)\big(\mathsf{x}', \mathsf{x}'', \mathsf{x}'''\big)$$

so that $\partial\kappa$ *is well defined in the space of* 3*-cochains* $\mathscr{C}^3(\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$.

Proof. By symmetry, it suffices to check this when $\mathsf{y}'' = \mathsf{y}''' = 0$ with y' simply denoted y. But then, applying twice the *expanded* second formula of the proposition and subtracting, noticing that $\widehat{X' + Y} = \widehat{X}' + Y^{\dagger}$, we obtain:

$$0 \overset{?}{=} \big[\mathsf{y},\, \kappa(p)(\mathsf{x}'', \mathsf{x}''')\big]_{\mathfrak{g}} - \kappa(p)\big([\mathsf{y}, \mathsf{x}'']_{\mathfrak{g}}, \mathsf{x}'''\big) + \kappa(p)\big([\mathsf{y}, \mathsf{x}''']_{\mathfrak{g}}, \mathsf{x}''\big) + \big(Y^{\dagger}\kappa\big)(p)(\mathsf{x}'', \mathsf{x}'''),$$

which is coherent with the second formula of Proposition 8.6. □

Proposition 8.12. ([8, 14]) *When the Lie algebra* $\mathfrak{g} = \mathfrak{g}_{-a} \oplus \cdots \oplus \mathfrak{g}_b$ *is graded as on p. 105 with:*

$$\mathfrak{g}_- := \mathfrak{g}_{-a} \oplus \cdots \oplus \mathfrak{g}_{-1},$$
$$\mathfrak{h} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_b,$$

if one decomposes the curvature function $\kappa = \sum_{h \in \mathbb{Z}} \kappa_{[h]}$ *in homogeneous components* $\kappa_{[h]}$*, then the differential of every such a* $\kappa_{[h]}$ *is uniquely determined by* *the lower homogeneous components* $\kappa_{[h']}$, *with* $h' \leqslant h-1$, *through the specific formula*:

$$\partial_{[h]}\big(\kappa_{[h]}\big)(\mathsf{x}',\mathsf{x}'',\mathsf{x}''') = -\sum_{\rm cycl}\sum_{h'=1}^{h-1}\Big(\kappa_{[h-h']}\big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[h']}(\mathsf{x}',\mathsf{x}'')\big),\,\mathsf{x}'''\big)\Big)- \\ -\sum_{\rm cycl}\big(\widehat{X}'\kappa_{[h+|\mathsf{x}'|]}\big)(\mathsf{x}'',\mathsf{x}'''),$$

in which $\mathsf{proj}_{\mathfrak{g}_-}(\mathsf{z})$ *denotes the* $\mathfrak{g}_-$*-component of an element* $\mathsf{z}\in\mathfrak{g}$, *while* $|\mathsf{x}'|$ *denotes the grade of* x'.

Proof. Back to the preceding proposition, it suffices to pick, in the expression:

$$-\sum_{\rm cycl}\kappa(p)\big(\kappa(p)(\mathsf{x}',\mathsf{x}'')\ \mathrm{mod}\,\mathfrak{h},\ \mathsf{x}'''\big)-\sum_{\rm cycl}\big(\widehat{X}'\kappa\big)(p)(\mathsf{x}'',\mathsf{x}''')$$

only the components that are of homogeneous degree h. □

8.8. **Normality.** We can now present a consequence of the above graded Bianchi-Tanaka identities that will be important for the construction of a Cartan connection on strongly pseudoconvex hypersurfaces $M^3\subset\mathbb{C}^2$, to be achieved in an effective way in Section 9 below. When $\mathfrak{g}$ is semi-simple (*cf.* [8, 9]) with a grading $\mathfrak{g} = \mathfrak{g}_{-\mu}\oplus\cdots\oplus\mathfrak{g}_{\mu}$ of depth $\mu\geqslant 1$ ($a=b$ in this case), it is known (*cf.* [8], p. 492) that the curvature function:

$$\kappa = \kappa_{[1]}+\cdots+\kappa_{[3\mu]}$$

has components whose homogeneities $h\geqslant 1$ are all positive. Then a Cartan connection $\omega\colon T\mathscr{P}\to\mathfrak{g}$ is said to be *normal* (in the sense of Tanaka, *cf.* [36, 8]) if the codifferential operator annihilates all the (positive) homogeneous components of the curvature function, namely if:

$$0=\partial^*\big(\kappa_{[1]}\big)=\cdots=\partial^*\big(\kappa_{[3\mu]}\big).$$

Proposition 8.13. ([8, 14]) *If* $\mathfrak{g}$ *is semi-simple and if the Cartan connection* $\omega\colon T\mathscr{P}\to\mathfrak{g}$ *is normal, then the homogeneity of the first non-zero homogeneous component of the curvature function* κ *is greater than the homogeneity of the first non-zero homogeneous component of the second cohomology* $H^2(\mathfrak{g}_-,\mathfrak{g})$.

Proof. If, as is assumed, $H^2_{[h']}(\mathfrak{g}_-,\mathfrak{g})=0$ for all $h'\leqslant h$ with some $h\geqslant 1$, and if, by induction:

$$0=\kappa_{[1]}=\cdots=\kappa_{[h-1]},$$

then the graded Bianchi-Tanaka identities of Proposition 8.12 show immediately that:

$$\partial_{[h]}\big(\kappa_{[h]}\big)=0,$$

which just means that:

$$\kappa_{[h]}\in\ker\big(\partial_{[h]}\big).$$

Furthermore, by normality of the connection, we also have:

$$\kappa_{[h]}\in\ker\big(\partial^*_{[h]}\big).$$

In addition, according to Proposition 2.6 of [8], the semi-simplicity of $\mathfrak{g}$ insures that for every homogeneous degree $h \in \mathbb{Z}$, the h-homogeneous 2-cochain space (to which $\kappa_{[h]}$ belongs) splits up as:

$$\begin{aligned}\mathscr{C}^2_{[h]} &= \mathrm{im}\big(\partial_{[h]}\big) \oplus \ker\big(\partial^*_{[h]}\big) \\ &= \mathscr{B}^2_{[h]} \oplus \ker\big(\partial^*_{[h]}\big)\end{aligned}$$

But since $H^2_{[h]} = 0$ by assumption, which just means that we can replace $\mathscr{B}^2_{[h]} = \mathscr{Z}^2_{[h]} = \ker \partial_{[h]}$ here, we deduce that:

$$\kappa_{[h]} \in \ker\big(\partial_{[h]}\big) \oplus \ker\big(\partial^*_{[h]}\big) \qquad \text{and} \qquad \kappa_{[h]} \in \ker\big(\partial_{[h]}\big) \cap \ker\big(\partial^*_{[h]}\big),$$

which trivially implies that $\kappa_{[h]} = 0$, as desired. □

8.9. **Vector fields and one-forms.** A remark about notations is in order. As did Sophus Lie, we shall most often prefer to conduct computations in terms of vector fields (first-order derivations) instead of using Élie Cartan's exterior differential calculus. But as both points of view are complementary, we shall nonetheless *need at the same time* to consider differential forms, and the best way of remembering the duality between a frame and a coframe is to put a "$*$" as an upper index, as we already did for the x_k and their duals x_k^*. This is why we shall not use Élie Cartan's classical letters ω^i or ϖ^j to denote differential forms, except for $\omega\colon T\mathscr{P} \to \mathfrak{g}$. In fact, in the formalism we set up presently, Greek letters will be mostly reserved to denote *functions*, namely coefficients of vector fields or of 1-forms.

8.10. **Left-invariant Lie frame and Maurer coframe.** Now, as we agreed that our constructions shall be local, we must consider that the local lie group H not only comes with its abstract Lie algebra $\mathfrak{h}$, but also with a collection of left-invariant vector fields near its identity element. So, let us assume that H, of dimension n, is coordinatized by means of $(a_1, \dots, a_{r-n}) \in \mathbb{K}^{r-n}$, locally near the origin (its identity element) and that the Lie algebra of left-invariant vector fields on H is given by $r - n$ (left-invariant) vector fields:

$$Y_j = \sum_{j'=1}^{r-n} \upsilon_{j,j'}(a_1, \dots, a_{r-n}) \, \frac{\partial}{\partial a_{j'}} \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,r\,-\,n)}$$

having coefficients $\upsilon_{j,j'}(a)$ that are smooth or $\mathbb{K}$-analytic in a neighborhood of the origin in $\mathbb{K}^{r-n}$.

Dually, the Maurer-Cartan coframe on H consists of precisely the same number $r - n$ of (left-invariant) 1-forms:

$$Y_j^* = \sum_{j'=1}^{r-n} \upsilon_*^{j,j'}(a) \, da_{j'},$$

satisfying by definition:

$$Y^*_{j_2}(Y_{j_1}) = \delta^{j_2}_{j_1}.$$

Hence the $(r - n) \times (r - n)$ matrix of functions $\upsilon_*^{j,j'}(a)$ is the transpose-inverse of the matrix of functions $\upsilon_{j,j'}(a)$.

8.11. **Initial frame on the base manifold.** On the other hand, on the manifold M equipped with local coordinates $x = (x_1, \dots, x_n)$, we suppose that a certain frame is given which consists of $n = \dim_{\mathbb{K}} M$ vector fields:

$$H_i = \sum_{i'=1}^{n} \eta_{i,i'}(x) \frac{\partial}{\partial x_{i'}} \qquad\qquad (i = 1 \cdots n)$$

having coefficients $\eta_{i,i'}(x)$ that are smooth or $\mathbb{K}$-analytic in a neighborhood of the origin in $\mathbb{K}^n$.

Then the dual coframe consists of precisely the same number n of 1-forms:

$$H_i^* := \sum_{i'=1}^{n} \eta_*^{i,i'}(x)\, dx_{i'} \qquad\qquad (i = 1 \cdots n)$$

satisfying by definition:

$$H_{i_2}^*(H_{i_1}) = \delta_{i_1}^{i_2}.$$

Hence the $n \times n$ matrix of functions $\eta_*^{i,i'}(x)$ is the transpose-inverse of the matrix of functions $\eta_{i,i'}(x)$.

8.12. **Constant frame and coframe on $\mathscr{P}$.** After all, the coordinates on the total space $\mathscr{P}$ are just:

$$(x, a) = (x_1, \dots, x_n, a_1, \dots, a_{r-n}).$$

From now on, in accordance to the convention made a while ago, we shall denote by:

$$\widehat{X}_1^*, \dots, \widehat{X}_n^*, \widehat{X}_{n+1}^*, \dots, \widehat{X}_r^*$$

the coframe on $\mathscr{P}$ which is dual to the frame:

$$\widehat{X}_1, \dots, \widehat{X}_n, \widehat{X}_{n+1}, \dots, \widehat{X}_r$$

made of all the constant fields $\widehat{X}_k := \omega^{-1}(\mathsf{x}_k)$ on $\mathscr{P}$, $k = 1, \dots, r$, so that by definition:

$$\widehat{X}_{k_1}^*(\widehat{X}_{k_2}) = \delta_{k_2}^{k_1} \qquad\qquad (k_1,\ k_2 = 1 \cdots r).$$

In fact, because for any $k_1 = 1, \dots, r$, one has:

$$\mathsf{x}_{k_2} = \omega(\widehat{X}_{k_2}) = \sum_{k_1=1}^{r} \omega^{k_1}(\widehat{X}_{k_2})\, \mathsf{x}_{k_1} \qquad \text{whence} \qquad \omega^{k_1}(\widehat{X}_{k_2}) = \delta_{k_2}^{k_1}$$

also holds, it follows that the $\widehat{X}_k$ were already known and that we must admit the coincidence of notation:

$$\omega^k \equiv \widehat{X}_k^* \qquad\qquad (k = 1 \cdots r).$$

Proposition 8.14. *The coefficients $\kappa_{i_1,i_2}^k(p)$ of the curvature function:*

$$\kappa(p) = \sum_{1 \leqslant i_1 < i_2 \leqslant n} \sum_{k=1}^{r} \kappa_{i_1,i_2}^k(p)\ \mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k$$

express explicitly as follows in terms the structure constants of $\mathfrak{g}$ *and in terms of the constant frame* $\widehat{X}_i$ *and coframe* $\widehat{X}_k^*$*:*

$$\boxed{\kappa_{i_1,i_2}^k(p) = c_{i_1,i_2}^k - \widehat{X}_k^*\big(\big[\widehat{X}_{i_1}, \widehat{X}_{i_2}\big]\big)}.$$

Proof. It suffices to apply ω_p^{-1} to the two extreme sides of:

$$\begin{aligned}
\sum_{k=1}^r \kappa_{i_1,i_2}^k(p)\,\mathsf{x}_k &= \kappa(p)(\mathsf{x}_{i_1}, \mathsf{x}_{i_2}) \\
\text{[apply Lemma 8.5]} \qquad &= [\mathsf{x}_{i_1}, \mathsf{x}_{i_2}]_{\mathfrak{g}} - \omega_p\big([\omega_p^{-1}(\mathsf{x}_{i_1}),\, \omega_p^{-1}(\mathsf{x}_{i_2})]_{\mathfrak{g}}\big) \\
&= \sum_{k=1}^r c_{i_1,i_2}^k\,\mathsf{x}_k - \omega_p\big(\big[\widehat{X}_{i_1}, \widehat{X}_{i_2}\big]\big) \\
&= \omega_p\Big(\sum_{k=1}^r c_{i_1,i_2}^k\,\widehat{X}_k - \big[\widehat{X}_{i_1}, \widehat{X}_{i_2}\big]\Big),
\end{aligned}$$

which simply yields:

$$\sum_{k=1}^r \kappa_{i_1,i_2}^k(p)\,\widehat{X}_k = \sum_{k=1}^r c_{i_1,i_2}^k\,\widehat{X}_k - \big[\widehat{X}_{i_1}, \widehat{X}_{i_2}\big].$$

Finally, the $\widehat{X}_k$-component of the right-hand side is immediately obtained by applying $\widehat{X}_k^*$ to both sides, and this delivers the expression boxed in the lemma. □

9. Effective Construction of a Normal, Regular Cartan-Tanaka Connection

9.1. **General form of the unknown Cartan connection frame.** Again with an $M^3 \subset \mathbb{C}^2$ being a Levi nondegenerate $\mathscr{C}^6$-smooth real hypersurface which is an arbitrary deformation of the Heisenberg sphere equipped with three real coordinates (x, y, u) as above, suppose (in advance) that we may construct a Cartan connection:

$$\omega_p\colon\ \ T_p\mathscr{P} \longrightarrow \mathfrak{g} \qquad\qquad {\scriptstyle (p\in\mathscr{G})}$$

on a certain (local) P-principal bundle $\mathscr{P}$ locally equal to the product $M^3 \times P$, where P is the unique (local, connected) five-dimensional Lie group associated to the *abstract* Lie algebra:

$$\mathfrak{p} := \operatorname{Span}_{\mathbb{R}}\big(\mathsf{d},\, \mathsf{r},\, \mathsf{i}_1,\, \mathsf{i}_2,\, \mathsf{j}\big)$$

corresponding to the five generators D, R, I_1, I_2, J of the Lie isotropy algebra of the origin $0 \in \mathbb{H}^3$, namely where the brackets between the abstract d, r, i_1, i_2, j are assumed to be exactly the same as those already shown between D, R, I_1, I_2, J, namely $[\mathsf{d}, \mathsf{i}_1] = \mathsf{i}_1$, *etc.*

At first, one needs to make explicit some corresponding five left-invariant vector fields on the (local) Lie group P. Let P be equipped with five real coordinates denoted by (a, b, c, d, e). Then one can simply take exactly the same five left-invariant vector fields as those shown in [14], namely:

$$\begin{aligned}
D &:= -a\,\frac{\partial}{\partial a} - b\,\frac{\partial}{\partial b} - c\,\frac{\partial}{\partial c} - d\,\frac{\partial}{\partial d} - 2e\,\frac{\partial}{\partial e} \\
R &:= -b\,\frac{\partial}{\partial a} + a\,\frac{\partial}{\partial b} + d\,\frac{\partial}{\partial c} - c\,\frac{\partial}{\partial d} \\
I_1 &:= \frac{\partial}{\partial a} - b\,\frac{\partial}{\partial e} \\
I_2 &:= \frac{\partial}{\partial b} + a\,\frac{\partial}{\partial e} \\
J &:= \tfrac{1}{2}\,\frac{\partial}{\partial e},
\end{aligned}$$

for one verifies that the Lie brackets between these vector fields are precisely the same as before, namely:

	D	R	I_1	I_2	J
D	0	0	I_1	I_2	$2\,J$
R	$*$	0	$-I_2$	I_1	0
I_1	$*$	$*$	0	$4\,J$	0
I_2	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	0

so that they indeed form a basis for the Lie algebra $\mathfrak{p}$ of the abstract Lie group P. Then according to property **(ii)** that a Cartan connection should enjoy on the product $\mathscr{P} = M^3 \times P$ which is naturally equipped with the eight real coordinates:

$$(x, y, u, a, b, c, d, e) =: \text{an arbitrary point}\, p \in \mathscr{P},$$

it follows that some five corresponding generating *vertical constant* fields must necessarily be of the plain form:

$$\begin{aligned}
\widehat{D}\big|_p &:= \omega_p^{-1}(\mathsf{d}) \equiv D \\
\widehat{R}\big|_p &:= \omega_p^{-1}(\mathsf{r}) \equiv R \\
\widehat{I_1}\big|_p &:= \omega_p^{-1}(\mathsf{i}_1) \equiv I_1 \\
\widehat{I_2}\big|_p &:= \omega_p^{-1}(\mathsf{i}_2) \equiv I_2 \\
\widehat{J}\big|_p &:= \omega_p^{-1}(\mathsf{j}) \equiv J,
\end{aligned}$$

where we identify vector fields on the (x, y, u, a, b, c, d, e)-space to vector fields on the (a, b, c, d, e)-space. We notice passim and for coherence of thought that the vanishing of the curvature in vertical bi-directions just means here that brackets between these five fields should correspond, through ω, to abstract brackets within the Lie algebra $\mathfrak{p}$, say for instance:

$$\big[\widehat{D},\, \widehat{I_1}\big] = \big[\omega^{-1}(\mathsf{d}),\, \omega^{-1}(\mathsf{i}_1)\big] = \omega^{-1}\big([\mathsf{d},\, \mathsf{i}_1]_{\mathfrak{p}}\big) = \omega^{-1}\big(\mathsf{i}_1\big) = \widehat{I_1},$$

and this is clearly the case by the unique choice $\widehat{D} \equiv D$ and $\widehat{I_1} \equiv I_1$ made at the moment. Only the three vector fields:

$$\widehat{T} := \omega^{-1}(\mathsf{t}), \qquad \widehat{H}_1 := \omega^{-1}(\mathsf{h}_1), \qquad \widehat{H}_2 := \omega^{-1}(\mathsf{h}_2)$$

are really unknown, and their determination should be subjected to specific constraints that shall be presented right below.

In summary, in order to construct the sought Cartan connection, we must find certain functions $\alpha_{\cdot\cdot}$ of the eight variables (x, y, u, a, b, c, d, e) as coefficients for the lifted horizontal vector fields:

$$\begin{cases} \widehat{T} := \alpha_{tt}\, T + \alpha_{th_1}\, H_1 + \alpha_{th_2}\, H_2 + \alpha_{td}\, D + \alpha_{tr}\, R + \alpha_{ti_1}\, I_1 + \alpha_{ti_2}\, I_2 + \alpha_{tj}\, J \\ \widehat{H}_1 := \qquad \alpha_{h_1h_1}\, H_1 + \alpha_{h_1h_2}\, H_2 + \alpha_{h_1d}\, D + \alpha_{h_1r}\, R + \alpha_{h_1i_1}\, I_1 + \alpha_{h_1i_2}\, I_2 + \alpha_{h_1j}\, J \\ \widehat{H}_2 := \qquad \alpha_{h_2h_1}\, H_1 + \alpha_{h_2h_2}\, H_2 + \alpha_{h_2d}\, D + \alpha_{h_2r}\, R + \alpha_{h_2i_1}\, I_1 + \alpha_{h_2i_2}\, I_2 + \alpha_{h_2j}\, J, \end{cases}$$

while, as we said, the lifts of the vertical vector fields identify to the vertical vector fields themselves:

$$\begin{cases} \widehat{D} \equiv D = -a\,\frac{\partial}{\partial a} - b\,\frac{\partial}{\partial b} - c\,\frac{\partial}{\partial c} - d\,\frac{\partial}{\partial d} - 2e\,\frac{\partial}{\partial e} \\ \widehat{R} \equiv R = -b\,\frac{\partial}{\partial a} + a\,\frac{\partial}{\partial b} + d\,\frac{\partial}{\partial c} - c\,\frac{\partial}{\partial d} \\ \widehat{I_1} \equiv I_1 = \frac{\partial}{\partial a} - b\,\frac{\partial}{\partial e} \\ \widehat{I_2} \equiv I_2 = \frac{\partial}{\partial b} + a\,\frac{\partial}{\partial e} \\ \widehat{J} \equiv J = \frac{1}{2}\,\frac{\partial}{\partial e}. \end{cases}$$

9.2. **Conditions for the determination of the Cartan connection.** We have to determine appropriate functions $\alpha_{\bullet\bullet}$ of the eight coordinates (x, u, v, a, b, c, d, e) in a neighborhood of the origin in such a way that the following four conditions are satisfied.

(c1) For any $X = D, R, I_1, I_2, J$ and any $Y = H_1, H_2, T$ with corresponding $\mathsf{x} = \mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}$ and $\mathsf{y} = \mathsf{h}_1, \mathsf{h}_2, \mathsf{t}$, one should have:

$$\big[\widehat{X}, \widehat{Y}\big] = \widehat{[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}}, \tag{35}$$

or equivalently in terms of the $\mathfrak{g}$-valued one-form:

$$\big[\omega^{-1}(\mathsf{x}),\, \omega^{-1}(\mathsf{y})\big] = \omega^{-1}\big([\mathsf{x},\, \mathsf{y}]_{\mathfrak{g}}\big).$$

As is known (*see e.g.* [12] page 3), if Lie groups are assumed to be connected (ours are, because we suppose they are local), this condition is equivalent to the equivariancy $R_h^*(\omega) = \mathrm{Ad}(h^{-1}) \circ \omega$ that ω should enjoy under right translations by elements $h \in H$ (Section 8).

(c2) For each $p \in \mathscr{P}$, the map $\omega_p \colon T_p\mathscr{P} \to \mathfrak{g}$ should be an isomorphism. We postpone the checking of this property to the end of all computations, but at least here, we may observe that this property is equivalent to the fact that the eight (local) vector fields $\widehat{T}$, $\widehat{H}_1$, $\widehat{H}_2$, $\widehat{D}$, $\widehat{R}$, $\widehat{I_1}$, $\widehat{I_2}$, $\widehat{J}$ constitute a *frame* near the origin (linear independency). Furthermore, since T, H_1, H_2 live in the (x, u, v)-space and since D, R, I_1, I_2, J already make a frame in the (a, b, c, d, e)-space, this property is equivalent to the fact that the (T, H_1, H_2)-components of $\widehat{T}$, $\widehat{H}_1$, $\widehat{H}_2$ are independent, namely:

$$\alpha_{tt}\big(\alpha_{h_1h_1}\alpha_{h_2h_2} - \alpha_{h_1h_2}\alpha_{h_2h_1}\big)$$

should not vanish in a neighborhood of 0.

(c3) The obtained Cartan connection ω should be *normal*, namely the codifferential operator ∂^* should annihilate all homogeneous curvature components, *i.e.*:

$$\partial^*_{[h]}\big(\kappa_{[h]}\big) \equiv 0 \qquad\qquad \text{for } \ h = 1, \ldots, 5.$$

Similarly as for condition **(c2)**, we shall also examine this condition at the end of our main calculations.

(c4) The connection should be *regular*, that is to say, all curvatures of negative homogeneities should vanish. In fact, possible homogeneities of 2-cochains are just $0, 1, 2, 3, 4, 5$ (Subsection 5.1), and we will see that making $\kappa^{[0]} = 0$ is the easiest thing. Furthermore, thanks to the fact that second cohomologies vanish up to homogeneity $h = 3$ according to the table on p. 37), the Proposition 8.13 insures that the first nonzero homogeneous curvature components can only be $\kappa_{[4]}$, whence something a bit better than regularity will in a certain sense hold freely.

In fact, the process of construction (*cf.* [14]) will mainly consist in annihilating as many curvatures components as possible, and without calling to ∂^*, we will be able to annihilate $\kappa_{[0]}$ (easiest thing), $\kappa_{[1]}$, $\kappa_{[2]}$ and $\kappa_{[3]}$ by an appropriate progressive building of ω which requires somewhat hard elimination computations.

9.3. **Explicit (sought) dual coframe.** Before beginning by carefully inspecting how to fulfill the first, main condition **(c1)**, we still need further preliminaries.

On the (x, y, u)-space, it is clear that the three vector fields H_1, H_2 and T make a frame, for we remember that:

$$H_1\big|_0 = \tfrac{\partial}{\partial x}\big|_0, \qquad H_2\big|_0 = \tfrac{\partial}{\partial y}\big|_0, \qquad T\big|_0 = \tfrac{\partial}{\partial u}\big|_0,$$

and consequently, there exists a well defined *dual coframe* which is composed of three one-forms H_1^*, H_2^* and T^* satisfying by definition:

$$H_1^*(H_1) = 1, \qquad H_1^*(H_2) = 0, \qquad H_1^*(T) = 0, \qquad etc.$$

At the moment, we shall not consider it to be necessary to express explicitly H_1^*, H_2^* and T^* in terms of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}u$, leaving such a task to a computer at the very end of all our constructions.

Now, our eight unknown vector fields $\widehat{T} = \omega^{-1}(\mathsf{t})$, $\ldots$, $\widehat{J} = \omega^{-1}(\mathsf{j})$ on the (x, y, u, a, b, c, d, e)-space read as linear combinations:

$$\begin{aligned}
\widehat{T} &= \alpha_{tt}\, T + \ \alpha_{th_1} H_1 + \ \alpha_{th_2} H_2 + \ \alpha_{td}\, D + \ \alpha_{tr}\, R + \ \alpha_{ti_1} I_1 + \ \alpha_{ti_2} I_2 + \ \alpha_{tj}\, J \\
\widehat{H}_1 &= \qquad\quad \alpha_{h_1h_1} H_1 + \alpha_{h_1h_2} H_2 + \alpha_{h_1d}\, D + \alpha_{h_1r}\, R + \alpha_{h_1i_1} I_1 + \alpha_{h_1i_2} I_2 + \alpha_{h_1j}\, J \\
\widehat{H}_2 &= \qquad\quad \alpha_{h_2h_1} H_1 + \alpha_{h_2h_2} H_2 + \alpha_{h_2d}\, D + \alpha_{h_2r}\, R + \alpha_{h_2i_1} I_1 + \alpha_{h_2i_2} I_2 + \alpha_{h_2j}\, J \\
\widehat{D} &= D \\
\widehat{R} &= R \\
\widehat{I}_1 &= I_1 \\
\widehat{I}_2 &= I_2 \\
\widehat{J} &= J
\end{aligned}$$

of the eight fields $T, \ldots, J$ with certain 22 unknown coefficients $\alpha_{tt}, \ldots, \alpha_{h2j}$, where we use *letters instead of numbers* as lower indices, the logic of indexing being clearly visible. We need to know explicitly the dual (unknown) coframe:

$$\widehat{T}^*, \quad \widehat{H}_1^*, \quad \widehat{H}_2^*, \quad \widehat{D}^*, \quad \widehat{R}^*, \quad \widehat{I}_1^*, \quad \widehat{I}_2^*, \quad \widehat{J}^*,$$

and the task is simple. Indeed, we recall the elementary fact that, in a standard vector space $\mathrm{Span}_{\mathbb{R}}(e_1, e_2, \ldots, e_n)$, the dual of an arbitrary frame:

$$v_k := \sum_{i=1}^n \alpha_{ki}\, e_i \qquad\qquad {\scriptstyle (k=1\cdots n)}$$

is a coframe of the form:

$$v_l^* := \sum_{j=1}^n \beta_{lj}\, e_j^* \qquad\qquad {\scriptstyle (l=1\cdots n)},$$

where the matrix $(\beta_{lj})_{1\leqslant l\leqslant n}^{1\leqslant j\leqslant n}$ is just the *transpose-inverse* of the initial matrix $(\alpha_{ki})_{1\leqslant k\leqslant n}^{1\leqslant i\leqslant n}$. Leaving it to a computer to invert and to transpose the above 8×8 matrix (*see* [2]) whose determinant is clearly equal to:

$$\alpha_{tt} = \big(\alpha_{h_1h_1}\alpha_{h_2h_2} - \alpha_{h_1h_2}\alpha_{h_2h_1}\big),$$

we find without much pain expressions of the form:

$$\begin{aligned}
\widehat{T}^* &= \beta_{tt}\, T^* \\
\widehat{H}_1^* &= \beta_{h_1t}\, T^* + \beta_{h_1h_1}\, H_1^* + \beta_{h_1h_2}\, H_2^* \\
\widehat{H}_2^* &= \beta_{h_2t}\, T^* + \beta_{h_2h_1}\, H_1^* + \beta_{h_2h_2}\, H_2^* \\
\widehat{D}^* &= \beta_{dt}\, T^* + \beta_{dh_1}\, H_1^* + \beta_{dh_2}\, H_2^* + D^* \\
\widehat{R}^* &= \beta_{rt}\, T^* + \beta_{rh_1}\, H_1^* + \beta_{rh_2}\, H_2^* + R^* \\
\widehat{I}_1^* &= \beta_{i_1t}\, T^* + \beta_{i_1h_1}\, H_1^* + \beta_{i_1h_2}\, H_2^* + I_1^* \\
\widehat{I}_2^* &= \beta_{i_2t}\, T^* + \beta_{i_2h_1}\, H_1^* + \beta_{i_2h_2}\, H_2^* + I_2^* \\
\widehat{J}^* &= \beta_{jt}\, T^* + \beta_{jh_1}\, H_1^* + \beta_{jh_2}\, H_2^* + J^*,
\end{aligned}$$

where the 22 coefficients $\beta_{tt}, \ldots, \beta_{jh_2}$ express themselves rationally and explicitly in terms of the α's right as follows:

$$\beta_{tt} := \frac{1}{\alpha_{tt}},$$

$$\beta_{h_1t} := \frac{-\alpha_{th_1}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \qquad \beta_{h1h1} := \frac{\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{h_1h_2} := \frac{-\alpha_{h_2h_1}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \qquad \beta_{h_2t} := \frac{\alpha_{th_1}\alpha_{h_1h_2}-\alpha_{th_2}\alpha_{h_1h1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{h_2h_1} := \frac{-\alpha_{h_1h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \qquad \beta_{h_2h_2} := \frac{\alpha_{h_1h_1}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{dt} := \frac{-\alpha_{th_1}\alpha_{h_1h_2}\alpha_{h_2d}+\alpha_{th_1}\alpha_{h_1d}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2d}\alpha_{h_1h_1}-\alpha_{th_2}\alpha_{h_2h_1}\alpha_{h_1d}-\alpha_{td}\alpha_{h_1h_1}\alpha_{h_2h_2}+\alpha_{td}\alpha_{h_1h_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{dh_1} := \frac{\alpha_{h_1h_2}\alpha_{h_2d}\alpha_{tt}-\alpha_{h_1d}\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \qquad \beta_{dh_2} := \frac{-\alpha_{h_2d}\alpha_{h_1h_1}\alpha_{tt}+\alpha_{h_2h_1}\alpha_{h_1d}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{rt} := \frac{-\alpha_{th_1}\alpha_{h_1h_2}\alpha_{h_2r}+\alpha_{th_1}\alpha_{h_1r}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2r}\alpha_{h_1h_1}-\alpha_{th_2}\alpha_{h_2h_1}\alpha_{h_1r}-\alpha_{tr}\alpha_{h_1h_1}\alpha_{h_2h_2}+\alpha_{tr}\alpha_{h_1h_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{rh_1} := \frac{\alpha_{h_1h_2}\alpha_{h_2r}\alpha_{tt}-\alpha_{h_1r}\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \qquad \beta_{rh_2} := \frac{-\alpha_{h_2r}\alpha_{h_1h_1}\alpha_{tt}+\alpha_{h_2h_1}\alpha_{h_1r}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{i_1t} := \frac{-\alpha_{th_1}\alpha_{h_1h_2}\alpha_{h_2i_1}+\alpha_{th_1}\alpha_{h_1i_1}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2i_1}\alpha_{h_1h_1}-\alpha_{th_2}\alpha_{h_2h_1}\alpha_{h_1i_1}-\alpha_{ti_1}\alpha_{h_1h_1}\alpha_{h_2h_2}+\alpha_{ti_1}\alpha_{h_1h_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$
$$\beta_{i_1h_1} := \frac{\alpha_{h_1h_2}\alpha_{h_2i_1}\alpha_{tt}-\alpha_{h_1i_1}\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \quad \beta_{i_1h_2} := \frac{-\alpha_{h_2i_1}\alpha_{h_1h_1}\alpha_{tt}+\alpha_{h_2h_1}\alpha_{h_1i_1}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{i_2t} := \frac{-\alpha_{th_1}\alpha_{h_1h_2}\alpha_{h_2i_2}+\alpha_{th_1}\alpha_{h_1i_2}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2i_2}\alpha_{h_1h_1}-\alpha_{th_2}\alpha_{h_2h_1}\alpha_{h_1i_2}-\alpha_{ti_2}\alpha_{h_1h_1}\alpha_{h_2h_2}+\alpha_{ti_2}\alpha_{h_1h_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$
$$\beta_{i_2h_1} := \frac{\alpha_{h_1h_2}\alpha_{h_2i_2}\alpha_{tt}-\alpha_{h_1i_2}\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \quad \beta_{i_2h_2} := \frac{-\alpha_{h_2i_2}\alpha_{h_1h_1}\alpha_{tt}+\alpha_{h_2h_1}\alpha_{h_1i_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

$$\beta_{jt} := \frac{-\alpha_{th_1}\alpha_{h_1h_2}\alpha_{h_2j}+\alpha_{th_1}\alpha_{h_1j}\alpha_{h_2h_2}+\alpha_{th_2}\alpha_{h_2j}\alpha_{h_1h_1}-\alpha_{th_2}\alpha_{h_2h_1}\alpha_{h_1j}-\alpha_{tj}\alpha_{h_1h_1}\alpha_{h_2h_2}+\alpha_{tj}\alpha_{h_1h_2}\alpha_{h_2h_1}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$
$$\beta_{jh_1} := \frac{\alpha_{h_1h_2}\alpha_{h_2j}\alpha_{tt}-\alpha_{h_1j}\alpha_{h_2h_2}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})}, \quad \beta_{jh_2} := \frac{-\alpha_{h_2j}\alpha_{h_1h_1}\alpha_{tt}+\alpha_{h_2h_1}\alpha_{h_1j}\alpha_{tt}}{\alpha_{tt}(\alpha_{h_1h_1}\alpha_{h_2h_2}-\alpha_{h_1h_2}\alpha_{h_2h_1})},$$

9.4. **Expressions of the $\alpha_{\bullet\bullet}$ in terms of fiber variables.** Of the mentioned four conditions, we will start by examining thoroughly **(c1)**, namely (35). By expressing, in terms of the basic frame T, H_1, H_2, D, R, I_1, I_2, J, each one of the $5 \times 3 = 15$ equalities of the form $[\widehat{X}, \widehat{Y}] = \widehat{[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}}$ for $\mathsf{x} = \mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}$ and for $\mathsf{y} = \mathsf{t}, \mathsf{h}_1, \mathsf{h}_2$, we get at each time eight scalar equations (sometimes only seven, because one equation may reduce to just $0 = 0$). To set up these scalar equations, it suffices to expand any bracket of the form $\big[\widehat{V},\, \alpha W\big]$ just as $\widehat{V}(\alpha)\, W + \alpha\, \big[\widehat{V},\, \widehat{W}\big]$, to remind that any T, H_1, H_2 trivially commutes with any D, R, I_1, I_2, J and to expand any occurring Lie bracket between D, R, I_1, I_2, J by means of the table on p. 69. Then the 15×8 equations in question read as follows, where we list the coefficients of T, of H_1, of H_2, of D, of R, of I_1, of I_2, of J rigorously in this order (incidentally, we drop exactly $2 \times 5 = 10$ trivial equations $0 = 0$ which appear due to the fact that $\widehat{H}_1$ and $\widehat{H}_2$ have zero T-component by our above assumption):

(1) $\boxed{[\widehat{D},\, \widehat{T}] + 2\widehat{T} = 0}$:

$$\widehat{D}(\alpha_{tt}) + 2\alpha_{tt} = 0, \quad \widehat{D}(\alpha_{th_1}) + 2\alpha_{th_1} = 0, \quad \widehat{D}(\alpha_{th_2}) + 2\alpha_{th_2} = 0, \quad \widehat{D}(\alpha_{td}) + 2\alpha_{td} = 0,$$
$$\widehat{D}(\alpha_{tr}) + 2\alpha_{tr} = 0, \quad \widehat{D}(\alpha_{ti_1}) + 3\alpha_{ti_1} = 0, \quad \widehat{D}(\alpha_{ti_2}) + 3\alpha_{ti_2} = 0, \quad \widehat{D}(\alpha_{tj}) + 4\alpha_{tj} = 0.$$

(2) $\boxed{[\widehat{D},\, \widehat{H}_1] + \widehat{H}_1 = 0}$:

$$\widehat{D}(\alpha_{h_1h_1}) + \alpha_{h_1h_1} = 0, \quad \widehat{D}(\alpha_{h_1h_2}) + \alpha_{h_1h_2} = 0, \quad \widehat{D}(\alpha_{h_1d}) + \alpha_{h_1d} = 0, \quad \widehat{D}(\alpha_{h_1r}) + \alpha_{h_1r} = 0,$$
$$\widehat{D}(\alpha_{h_1i_1}) + 2\alpha_{h_1i_1} = 0, \quad \widehat{D}(\alpha_{h_1i_2}) + 2\alpha_{h_1i_2} = 0, \quad \widehat{D}(\alpha_{h_1j}) + 3\alpha_{h_1j} = 0.$$

(3) $\boxed{[\widehat{D},\, \widehat{H}_2] + \widehat{H}_2 = 0}$:

$$\widehat{D}(\alpha_{h_2h_1}) + \alpha_{h_2h_1} = 0, \quad \widehat{D}(\alpha_{h_2h_2}) + \alpha_{h_2h_2} = 0, \quad \widehat{D}(\alpha_{h_2d}) + \alpha_{h_2d} = 0, \quad \widehat{D}(\alpha_{h_2r}) + \alpha_{h_2r} = 0,$$
$$\widehat{D}(\alpha_{h_2i_1}) + 2\alpha_{h_2i_1} = 0, \quad \widehat{D}(\alpha_{h_2i_2}) + 2\alpha_{h_2i_2} = 0, \quad \widehat{D}(\alpha_{h_2j}) + 3\alpha_{h_2j} = 0.$$

(4) $\boxed{[\widehat{R},\, \widehat{T}] = 0}$:

$$\widehat{R}(\alpha_{tt}) = 0, \quad \widehat{R}(\alpha_{th_1}) = 0, \quad \widehat{R}(\alpha_{th_2}) = 0, \quad \widehat{R}(\alpha_{td}) = 0, \quad \widehat{R}(\alpha_{tr}) = 0,$$
$$\widehat{R}(\alpha_{ti_1}) + \alpha_{ti_2} = 0, \quad \widehat{R}(\alpha_{ti_2}) - \alpha_{ti_1} = 0, \quad \widehat{R}(\alpha_{tj}) = 0.$$

(5) $\boxed{[\widehat{R},\,\widehat{H}_1]+\widehat{H}_2=0}$:

$$\widehat{R}(\alpha_{h_1h_1})+\alpha_{h_2h_1}=0,\quad \widehat{R}(\alpha_{h_1h_2})+\alpha_{h_2h_2}=0,\quad \widehat{R}(\alpha_{h_1d})+\alpha_{h_2d}=0,$$
$$\widehat{R}(\alpha_{h_1r})+\alpha_{h_2r}=0,\quad \widehat{R}(\alpha_{h_1i_1})+\alpha_{h_1i_2}+\alpha_{h_2i_1}=0,\quad \widehat{R}(\alpha_{h_1i_2})-\alpha_{h_1i_1}+\alpha_{h_2i_2}=0,$$
$$\widehat{R}(\alpha_{h_1j})+\alpha_{h_2j}=0.$$

(6) $\boxed{[\widehat{R},\,\widehat{H}_2]-\widehat{H}_1=0}$:

$$\widehat{R}(\alpha_{h_2h_1})-\alpha_{h_1h_1}=0,\quad \widehat{R}(\alpha_{h_2h_2})-\alpha_{h_1h_2}=0,\quad \widehat{R}(\alpha_{h_2d})-\alpha_{h_1d}=0,$$
$$\widehat{R}(\alpha_{h_2r})-\alpha_{h_1r}=0,\quad \widehat{R}(\alpha_{h_2i_1})+\alpha_{h_2i_2}-\alpha_{h_1i_1}=0,\quad \widehat{R}(\alpha_{h_2i_2})-\alpha_{h_2i_1}-\alpha_{h_1i_2}=0,$$
$$\widehat{R}(\alpha_{h_2j})-\alpha_{h_1j}=0.$$

(7) $\boxed{[\widehat{I}_1,\,\widehat{T}]+\widehat{H}_1=0}$:

$$\widehat{I}_1(\alpha_{tt})=0,\quad \widehat{I}_1(\alpha_{th_1})+\alpha_{h_1h_1}=0,\quad \widehat{I}_1(\alpha_{th_2})+\alpha_{h_1h_2}=0,\quad \widehat{I}_1(\alpha_{td})+\alpha_{h_1d}=0,$$
$$\widehat{I}_1(\alpha_{tr})+\alpha_{h_1r}=0,\quad \widehat{I}_1(\alpha_{ti_1})-\alpha_{td}+\alpha_{h_1i_1}=0,\quad \widehat{I}_1(\alpha_{ti_2})+\alpha_{tr}+\alpha_{h_1i_2}=0,$$
$$\widehat{I}_1(\alpha_{tj})+4\alpha_{ti_2}+\alpha_{h_1j}=0.$$

(8) $\boxed{[\widehat{I}_1,\,\widehat{H}_1]+6\widehat{R}=0}$:

$$\widehat{I}_1(\alpha_{h_1h_1})=0,\quad \widehat{I}_1(\alpha_{h_1h_2})=0,\quad \widehat{I}_1(\alpha_{h_1d})=0,\quad \widehat{I}_1(\alpha_{h_1r})+6=0,$$
$$\widehat{I}_1(\alpha_{h_1i_1})-\alpha_{h_1d}=0,\quad \widehat{I}_1(\alpha_{h_1i_2})+\alpha_{h_1r}=0,\quad \widehat{I}_1(\alpha_{h_1j})+4\alpha_{h_1i_2}=0.$$

(9) $\boxed{[\widehat{I}_1,\,\widehat{H}_2]-2\widehat{D}=0}$:

$$\widehat{I}_1(\alpha_{h_2h_1})=0,\quad \widehat{I}_1(\alpha_{h_2h_2})=0,\quad \widehat{I}_1(\alpha_{h_2d})-2=0,\quad \widehat{I}_1(\alpha_{h_2r})=0,$$
$$\widehat{I}_1(\alpha_{h_2i_1})-\alpha_{h_2d}=0,\quad \widehat{I}_1(\alpha_{h_2i_2})+\alpha_{h_2r}=0,\quad \widehat{I}_1(\alpha_{h_2j})+4\alpha_{h_2i_2}=0.$$

(10) $\boxed{[\widehat{I}_2,\,\widehat{T}]+\widehat{H}_2=0}$:

$$\widehat{I}_2(\alpha_{tt})=0,\quad \widehat{I}_2(\alpha_{th_1})+\alpha_{h_2h_1}=0,\quad \widehat{I}_2(\alpha_{th_2})+\alpha_{h_2h_2}=0,\quad \widehat{I}_2(\alpha_{td})+\alpha_{h_2d}=0,$$
$$\widehat{I}_2(\alpha_{tr})+\alpha_{h_2r}=0,\quad \widehat{I}_2(\alpha_{ti_1})-\alpha_{tr}+\alpha_{h_2i_1}=0,\quad \widehat{I}_2(\alpha_{ti_2})-\alpha_{td}+\alpha_{h_2i_2}=0,$$
$$\widehat{I}_2(\alpha_{tj})-4\alpha_{ti_1}+\alpha_{h_2j}=0.$$

(11) $\boxed{[\widehat{I}_2,\,\widehat{H}_1]+2\widehat{D}=0}$:

$$\widehat{I}_2(\alpha_{h_1h_1})=0,\quad \widehat{I}_2(\alpha_{h_1h_2})=0,\quad \widehat{I}_2(\alpha_{h_1d})+2=0,\quad \widehat{I}_2(\alpha_{h_1r})=0,$$
$$\widehat{I}_2(\alpha_{h_1i_1})-\alpha_{h_1r}=0,\quad \widehat{I}_2(\alpha_{h_1i_2})-\alpha_{h_1d}=0,\quad \widehat{I}_2(\alpha_{h_1j})-4\alpha_{h_1i_1}=0.$$

(12) $\boxed{[\widehat{I}_2,\,\widehat{H}_2]+6\widehat{R}=0}$:

$$\widehat{I}_2(\alpha_{h_2h_1})=0,\quad \widehat{I}_2(\alpha_{h_2h_2})=0,\quad \widehat{I}_2(\alpha_{h_2d})=0,\quad \widehat{I}_2(\alpha_{h_2r})+6=0,$$
$$\widehat{I}_2(\alpha_{h_2i_1})-\alpha_{h_2r}=0,\quad \widehat{I}_2(\alpha_{h_2i_2})-\alpha_{h_2d}=0,\quad \widehat{I}_2(\alpha_{h_2j})-4\alpha_{h_2i_1}=0.$$

(13) $\boxed{[\widehat{J},\,\widehat{T}]+\widehat{D}=0}$:

$$\widehat{J}(\alpha_{tt})=0,\quad \widehat{J}(\alpha_{th_1})=0,\quad \widehat{J}(\alpha_{th_2})=0,\quad \widehat{J}(\alpha_{td})+1=0,$$
$$\widehat{J}(\alpha_{tr})=0,\quad \widehat{J}(\alpha_{ti_1})=0,\quad \widehat{J}(\alpha_{ti_2})=0,\quad \widehat{J}(\alpha_{tj})-2\alpha_{td}=0,$$

(14) $\boxed{[\widehat{J},\,\widehat{H}_1]+\widehat{I}_1=0}$:

$$\widehat{J}(\alpha_{h_1h_1})=0,\quad \widehat{J}(\alpha_{h_1h_2})=0,\quad \widehat{J}(\alpha_{h_1d})=0,\quad \widehat{J}(\alpha_{h_1r})=0,$$
$$\widehat{J}(\alpha_{h_1i_1})+1=0,\quad \widehat{J}(\alpha_{h_1i_2})=0,\quad \widehat{J}(\alpha_{h_1j})-2\,\alpha_{h_1d}=0.$$

(15) $\boxed{[\widehat{J},\,\widehat{H}_2]+\widehat{I}_2=0}$:

$$\widehat{J}(\alpha_{h_2h_1})=0,\quad \widehat{J}(\alpha_{h_2h_2})=0,\quad \widehat{J}(\alpha_{h_2d})=0,\quad \widehat{J}(\alpha_{h_2r})=0,$$
$$\widehat{J}(\alpha_{h_2i_1})=0,\quad \widehat{J}(\alpha_{h_2i_2})+1=0,\quad \widehat{J}(\alpha_{h_2j})-2\,\alpha_{h_2d}=0.$$

We therefore get a system of precisely 110 first-order partial differential equations having the twenty-two unknowns α_{tt}, ..., α_{h_2j} in the space (x,u,v,a,b,c,d,e), and the differentiations only involve the five partial derivatives $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$, $\frac{\partial}{\partial c}$, $\frac{\partial}{\partial d}$, $\frac{\partial}{\partial e}$. Importantly, this system is of first order and is linear, hence this is the reason why its (large) solution set could be found rather quickly by means of Maple ([2]). The 22 appearing functions $\delta_1,\dots,\delta_{22}$ will be determined later: they now constitute the only remaining unknown part of $\alpha_{tt},\dots,\alpha_{h_2j}$.

Lemma 9.1. *The general solution of the above system* **(1)**, **(2)**, ..., **(15)** *of 110 partial differential equations is polynomial of degree* $\leqslant 4$ *with respect to the five vertical variables* a, b, c, d, e *of the principal bundle* P, *and it involves 22 coefficients* $\delta_1(x,y,z)$, ..., $\delta_{22}(x,y,z)$ *that are arbitrary (smooth) functions of the horizontal variables* (x,y,z)*:*

$$\begin{aligned}
\alpha_{tt}&=(c^2+d^2)\,\delta_{22},\\
\alpha_{th_1}&=-(ad+bc)\,\delta_{13}+(ac-bd)\,\delta_{14}+(c^2+d^2)\,\delta_{21},\\
\alpha_{th2}&=-(ad+bc)\,\delta_{11}+(ac-bd)\,\delta_{12}+(c^2+d^2)\,\delta_{20}\\
\alpha_{td}&=(-\tfrac14 bc-\tfrac14 ad)\,\delta_1+(\tfrac14 ac-\tfrac14 bd)\,\delta_2+(\tfrac14 c^2+\tfrac14 d^2)\,\delta_{15}-2e,\\
\alpha_{tr}&=(\tfrac14 c^2+\tfrac14 d^2)\,\delta_4+(\tfrac12 ac-\tfrac12 bd)\,\delta_7+(\tfrac12 c^2+\tfrac12 d^2)\,\delta_9-(\tfrac12 ad+\tfrac12 bc)\,\delta_{10}+\\
&\qquad+(\tfrac12 c^2+\tfrac12 d^2)\,\delta_{19}+3b^2+3a^2,
\end{aligned}$$

$$\begin{aligned}
\alpha_{ti_1}=&-(\tfrac14 a^2d+\tfrac14 abc)\,\delta_1+(-\tfrac14 abd+\tfrac14 a^2c)\,\delta_2+(\tfrac1{24}d^3+\tfrac1{24}c^2d)\,\delta_3+(\tfrac18 d^2b+\tfrac18 bc^2)\,\delta_4+\\
&+(\tfrac18 c^3+\tfrac18 cd^2)\,\delta_5+(\tfrac18 ac^2+\tfrac18 ad^2)\,\delta_6+(-\tfrac12 db^2+\tfrac12 bca)\,\delta_7-(\tfrac14 dcb+\tfrac14 ad^2)\,\delta_8+\\
&+(\tfrac14 bc^2+\tfrac12 d^2b-\tfrac14 dca)\,\delta_9-(\tfrac12 bda+\tfrac12 cb^2)\,\delta_{10}+(\tfrac14 d^2a+\tfrac14 ac^2)\,\delta_{15}+\\
&+(\tfrac14 c^3+\tfrac14 cd^2)\,\delta_{16}+(\tfrac14 c^2d+\tfrac14 d^3)\,\delta_{17}+(\tfrac12 d^2b+\tfrac12 bc^2)\,\delta_{19}+2a^2b+2b^3,
\end{aligned}$$

$$\begin{aligned}
\alpha_{ti_2}=&-(\tfrac14 bda+\tfrac14 cb^2)\,\delta_1+(-\tfrac14 db^2+\tfrac14 bca)\,\delta_2+(\tfrac1{24}c^3+\tfrac1{24}cd^2)\,\delta_3+(-\tfrac18 ac^2-\tfrac18 ad^2)\,\delta_4+\\
&+(-\tfrac18 c^2d-\tfrac18 d^3)\,\delta_5+(\tfrac18 d^2b+\tfrac18 bc^2)\,\delta_6+(-\tfrac12 a^2c+\tfrac12 bda)\,\delta_7-(\tfrac14 dca+\tfrac14 bc^2)\,\delta_8+\\
&+(-\tfrac12 ac^2+\tfrac14 dcb-\tfrac14 d^2a)\,\delta_9+(\tfrac12 a^2d+\tfrac12 bca)\,\delta_{10}+(\tfrac14 bc^2+\tfrac14 d^2b)\,\delta_{15}-\\
&-(\tfrac14 c^2d+\tfrac14 d^3)\,\delta_{16}+(\tfrac14 c^3+\tfrac14 cd^2)\,\delta_{17}+(-\tfrac12 ac^2-\tfrac12 d^2a)\,\delta_{19}-2a^3-2ab^2,
\end{aligned}$$

$$
\begin{aligned}
\alpha_{tj} = & -(acb+ade)\,\delta_1+(-edb+eca)\,\delta_2+(cb^2a+ca^3-db^3-dba^2)\,\delta_7+\\
& +(-d^2ab+c^2ab-dcb^2+dca^2)\,\delta_8+(-2dcba+c^2a^2+d^2b^2)\,\delta_9\\
& +(-da^3-db^2a-cb^3-cba^2)\,\delta_{10}+(ed^2+ec^2)\,\delta_{15}+\\
& +(ad^3+c^3b+dac^2+cbd^2)\,\delta_{16}+(-cad^2+bd^3+dbc^2-c^3a)\,\delta_{17}+\\
& +(d^4+c^4+2d^2c^2)\,\delta_{18}+(b^2c^2+d^2b^2+c^2a^2+a^2d^2)\,\delta_{19}+6a^2b^2-\\
& -4e^2+3a^4+3b^4,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_1h_1} &= (d)\,\delta_{13}-(c)\,\delta_{14},\\
\alpha_{h_1h_2} &= (d)\,\delta_{11}-(c)\,\delta_{12},\\
\alpha_{h_1d} &= (\tfrac14 d)\,\delta_1-(\tfrac14 c)\,\delta_2-2b,\\
\alpha_{h1r} &= -(\tfrac12 c)\,\delta_7+(\tfrac12 d)\,\delta_{10}-6a,\\
\alpha_{h_1i_1} &= (\tfrac14 ad)\,\delta_1-(\tfrac14 ac)\,\delta_2-(\tfrac18 c^2+\tfrac18 d^2)\,\delta_6-(\tfrac12 bc)\,\delta_7+(\tfrac14 d^2)\,\delta_8+(\tfrac14 cd)\,\delta_9+\\
&\qquad +(\tfrac12 bd)\,\delta_{10}-4ab-2e,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_1i_2} &= (\tfrac14 bd)\delta_1-(\tfrac14 bc)\,\delta_2-(\tfrac18 c^2+\tfrac18 d^2)\,\delta_4+(\tfrac12 ac)\,\delta_7+(\tfrac14 cd)\,\delta_8-(\tfrac14 d^2)\,\delta_9-\\
&\qquad -(\tfrac12 ad)\,\delta_{10}+3a^2-b^2,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_1j} = & (de)\,\delta_1-(ce)\,\delta_2-(\tfrac16 c^3+\tfrac16 cd^2)\,\delta_3+(\tfrac12 ac^2+\tfrac12 d^2a)\,\delta_4+(\tfrac12 d^3+\tfrac12 c^2d)\,\delta_5-\\
& -(\tfrac12 d^2b+\tfrac12 bc^2)\,\delta_6-(a^2c+cb^2)\,\delta_7+(-dca+d^2b)\,\delta_8+(dcb+d^2a)\,\delta_9+\\
& +(a^2d+db^2)\,\delta_{10}-8be-4a^3-4ab^2,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_2h_1} &= (c)\,\delta_{13}+(d)\,\delta_{14},\\
\alpha_{h_2h_2} &= (c)\,\delta_{11}+(d)\,\delta_{12},\\
\alpha_{h_2d} &= (\tfrac14 c)\,\delta_1+(\tfrac14 d)\,\delta_2+2a,\\
\alpha_{h_2r} &= (\tfrac12 d)\,\delta_7+(\tfrac12 c)\,\delta_{10}-6b,\\
\alpha_{h_2i_1} &= (\tfrac14 ac)\,\delta_1+(\tfrac14 ad)\,\delta_2+(\tfrac18 c^2+\tfrac18 d^2)\,\delta_4+(\tfrac12 bd)\,\delta_7+(\tfrac14 cd)\,\delta_8+(\tfrac14 c^2)\,\delta_9+\\
&\qquad (\tfrac12 bc)\,\delta_{10}-3b^2+a^2,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_1i_2} &= (\tfrac14 cb)\,\delta_1+(\tfrac14 bd)\,\delta_2-(\tfrac18 c^2+\tfrac18 d^2)\,\delta_6-(\tfrac12 ad)\,\delta_7+(\tfrac14 c^2)\,\delta_8-(\tfrac14 dc)\,\delta_9-\\
&\qquad -(\tfrac12 ac)\,\delta_{10}+4ab-2e,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_2j} = & (ce)\,\delta_1+(ed)\,\delta_2+(\tfrac16 d^3+\tfrac16 c^2d)\,\delta_3+(\tfrac12 bc^2+\tfrac12 d^2b)\,\delta_4+(\tfrac12 c^3+\tfrac12 cd^2)\,\delta_5+\\
& +(\tfrac12 d^2a+\tfrac12 ac^2)\,\delta_6+(db^2+da^2)\,\delta_7+(-ac^2+dcb)\,\delta_8+(bc^2+dca)\,\delta_9+\\
& +(ca^2+cb^2)\,\delta_{10}-4a^2b+8ae-4b^3,
\end{aligned}
$$

where the coefficients δ_k are only with respect to the horizontal coordinates x, y, u and are independent of the fibre variables.

If we would assume that the functions $\alpha_{\bullet\bullet}$ would be independent of the horizontal coordinates, *i.e.* that the functions δ_k would be constant, then condition **(c1)** would hold. However, a major problem would occur when we would try to fulfill the normality condition **(c3)**. Hence, the dependence of the $\alpha_{\bullet\bullet}$'s with respect to horizontal coordinates should better be determined by annihilating all curvatures $\kappa^{[h]} \equiv 0$ of homogeneities $h = 0, 1, 2, 3$. We will achieve this task in the next subsections and it will require quite hard elimination computations.

9.5. **Graded differential structure.** Now, the choice of the seven coefficients $\alpha_{tt}, \alpha_{th_1}, \alpha_{th_2}, \alpha_{h_1h_1}, \alpha_{h_1h_2}, \alpha_{h_2h_1}, \alpha_{h_2h_2}$ is governed by the geometry[6] of the graded tangent bundle $T^cM \oplus (TM/T^cM)$. In fact, the five coefficients α_{tt} must be fiber coordinates with respect to some fixed trivialization of $T^cM \oplus (TM/T^cM)$, which is a principal bundle. Given our two fixed complex-tangent local vector fields $H_1, H_2 \in \Gamma(T^cM)$ spanning T^cM, the $(\partial_x, \partial_y, \partial_u)$-part of a first lift $\widehat{H}_1$ must take account, in terms of the coordinates (a, b, c, d, e) of the principal bundle, of the non-uniqueness of the choice of a first vector field in $\Gamma(T^cM)$. Thus, the $(\partial_x, \partial_y, \partial_u)$-part of $\widehat{H}_1$ must be of the form $c\, H_1 + d\, H_2$. Next, the $(\partial_x, \partial_y, \partial_u)$-part of $\widehat{H}_2$ must be equal to $J(c\, H_1 + d\, H_2) = -d\, H_1 + c\, H_2$. With this, the coefficient α_{tt} must be equal to the $(\partial_x, \partial_y, \partial_u)$-part of $\big[\widehat{H}_1,\, \widehat{H}_2\big]$, whence $\alpha_{tt} = c^2 + d^2$ necessarily. Finally, the choice of the $(\partial_x, \partial_y, \partial_u)$-part of $\widehat{T}$, as a section of the quotient TM/T^cM, can still be made up to an arbitrary linear combination $-a\, H_1 - b\, H_2$, whence $\alpha_{th_1} = bd - ac$ and $\alpha_{th_2} = -ad - bc$. In summary, for geometric reasons, we must have:

$$
\begin{cases}
\quad \alpha_{tt} = c^2 + d^2, & \alpha_{th_1} = bd - ac, \qquad \alpha_{th_2} = -ad - bc, \\
\alpha_{h_1h_1} = c, & \alpha_{h_1h_2} = d, \\
\alpha_{h_2h_1} = -d, & \alpha_{h_2h_2} = c,
\end{cases}
$$

which means, equivalently, that seven of the functions $\delta_{\bullet\bullet}$ are already completely determined.

$$
\delta_{11} = \delta_{22} = 1, \quad \delta_{12} = \delta_{13} = \delta_{20} = \delta_{21} = 0, \quad \delta_{14} = -1.
$$

The remaining 15 undetermined coefficient functions δ_k will be determined progressively (and uniquely) by subjecting them to the conditions that the wanted Cartan connection be normal and regular. In particular they should be determined such that all curvatures $\kappa_{[h]}$ in the four homogeneities $h = 0, 1, 2, 3$ should be zero. The next subsections are devoted to inspecting these conditions, until we examine homogeneities $h = 4, 5$, for which we shall take account of the Bianchi-Tanaka identities, too.

9.6. **Brackets between horizontal vector fields.** For our access to Cartan curvatures, we need as a tool to compute the Lie brackets between the three horizontal vector fields $\widehat{H_1}$, $\widehat{H}_2$, $\widehat{T}$ whose coefficients $\alpha_{tt}, \dots, \alpha_{h2j}$, still unknown, are to be determined so as to simplify curvatures. Thus three brackets $\big[\widehat{H}_1,\, \widehat{H}_2\big]$, $\big[\widehat{H}_1,\, \widehat{T}\big]$,

[6] The authors would like to thank Ben McLaughlin and Gerd Schmalz for their helpful explanations in this regard.

$\big[\widehat{H}_2,\,\widehat{T}\big]$ must be considered, and they all are of the general form:

$$\begin{aligned}
&\big[\alpha\,T+\beta\,H_1+\gamma\,H_2+\delta\,D+\rho\,R+\lambda\,I_1+\mu\,I_2+\nu\,J,\\
&\qquad\qquad\alpha'\,T+\beta'\,H_1+\gamma'\,H_2+\delta'\,D+\rho'\,R+\lambda'\,I_1+\mu'\,I_2+\nu'\,J\big]=\\
&=\big[\alpha\,T+\beta\,H_1+\gamma\,H_2,\ \ \alpha'\,T+\beta'\,H_1+\gamma'\,H_2\big]+\\
&\quad+\big[\alpha\,T+\beta\,H_1+\gamma\,H_2,\ \ \delta'\,D+\rho'\,R+\lambda'\,I_1+\mu'\,I_2+\nu'\,J\big]+\\
&\quad+\big[\delta\,D+\rho\,R+\lambda\,I_1+\mu\,I_2+\nu\,J,\ \ \alpha'\,T+\beta'\,H_1+\gamma'\,H_2\big]+\\
&\quad+\big[\delta\,D+\rho\,R+\lambda\,I_1+\mu\,I_2+\nu\,J,\ \ \delta'\,D+\rho'\,R+\lambda'\,I_1+\mu'\,I_2+\nu'\,J\big].
\end{aligned}$$

Applying bilinearity, any obtained bracket of the form $\big[\phi\,X,\,\psi Y\big]$ then expands as:

$$\big[\phi\,X,\,\psi Y\big]=\phi\psi\,[X,\,Y]+\phi X(\psi)\,Y-\psi Y(\phi)\,X,$$

and for the brackets in lines 2 and 3, all first terms $\phi\psi\,[X,Y]$ vanish. After a reorganization using the commutator rules between H_1, H_2, T and those between D, R, I_1, I_2, J, we obtain that this general bracket equals:

$$\begin{aligned}
&\Big(\alpha T(\alpha')+\beta H_1(\alpha')+\gamma H_2(\alpha')-\alpha' T(\alpha)-\beta' H_1(\alpha)-\gamma' H_2(\alpha)-\\
&-(\alpha\beta'-\alpha'\beta)\Phi_1-(\alpha\gamma'-\alpha'\gamma)\Phi_2+4(\beta\gamma'-\beta'\gamma)+\\
&+\delta D(\alpha')+\rho R(\alpha')+\lambda I_1(\alpha')+\mu I_2(\alpha')+\nu J(\alpha')-\delta' D(\alpha)-\rho' R(\alpha)-\lambda' I_1(\alpha)-\mu' I_2(\alpha)-\nu' J(\alpha)\Big)\,T+\\
&+\Big(\alpha T(\beta')+\beta H_1(\beta')+\gamma H_2(\beta')-\alpha' T(\beta)-\beta' H_1(\beta)-\gamma' H_2(\beta)+\\
&+\delta D(\beta')+\rho R(\beta')+\lambda I_1(\beta')+\mu I_2(\beta')+\nu J(\beta')-\delta' D(\beta)-\rho' R(\beta)-\lambda' I_1(\beta)-\mu' I_2(\beta)-\nu' J(\beta)\Big)\,H_1+\\
&+\Big(\alpha T(\gamma')+\beta H_1(\gamma')+\gamma H_2(\gamma')-\alpha' T(\gamma)-\beta' H_1(\gamma)-\gamma' H_2(\gamma)+\\
&+\delta D(\gamma')+\rho R(\gamma')+\lambda I_1(\gamma')+\mu I_2(\gamma')+\nu J(\gamma')-\delta' D(\gamma)-\rho' R(\gamma)-\lambda' I_1(\gamma)-\mu' I_2(\gamma)-\nu' J(\gamma)\Big)\,H_2+\\
&+\Big(\alpha T(\delta')+\beta H_1(\delta')+\gamma H_2(\delta')-\alpha' T(\delta)-\beta' H_1(\delta)-\gamma' H_2(\delta)+\\
&+\delta D(\delta')+\rho R(\delta')+\lambda I_1(\delta')+\mu I_2(\delta')+\nu J(\delta')-\delta' D(\delta)-\rho' R(\delta)-\lambda' I_1(\delta)-\mu' I_2(\delta)-\nu' J(\delta)\Big)\,D+\\
&+\Big(\alpha T(\rho')+\beta H_1(\rho')+\gamma H_2(\rho')-\alpha' T(\rho)-\beta' H_1(\rho)-\gamma' H_2(\rho)+\\
&+\delta D(\rho')+\rho R(\rho')+\lambda I_1(\rho')+\mu I_2(\rho')+\nu J(\rho')-\delta' D(\rho)-\rho' R(\rho)-\lambda' I_1(\rho)-\mu' I_2(\rho)-\nu' J(\rho)\Big)\,R+\\
&+\Big(\alpha T(\lambda')+\beta H_1(\lambda')+\gamma H_2(\lambda')-\alpha' T(\lambda)-\beta' H_1(\lambda)-\gamma' H_2(\lambda)+\\
&+\delta D(\lambda')+\rho R(\lambda')+\lambda I_1(\lambda')+\mu I_2(\lambda')+\nu J(\lambda')-\delta' D(\lambda)-\rho' R(\lambda)-\lambda' I_1(\lambda)-\mu' I_2(\lambda)-\nu' J(\lambda)+\\
&+\delta\lambda'-\delta'\lambda-\rho\mu'+\rho'\mu\Big)\,I_1+\\
&+\Big(\alpha T(\mu')+\beta H_1(\mu')+\gamma H_2(\mu')-\alpha' T(\mu)-\beta' H_1(\mu)-\gamma' H_2(\mu)+\\
&+\delta D(\mu')+\rho R(\mu')+\lambda I_1(\mu')+\mu I_2(\mu')+\nu J(\mu')-\delta' D(\mu)-\rho' R(\mu)-\lambda' I_1(\mu)-\mu' I_2(\mu)-\nu' J(\mu)+\\
&+\delta\mu'-\delta'\mu-\rho\lambda'+\rho'\lambda\Big)\,I_2+\\
&+\Big(\alpha T(\nu')+\beta H_1(\nu')+\gamma H_2(\nu')-\alpha' T(\nu)-\beta' H_1(\nu)-\gamma' H_2(\nu)+\\
&+\delta D(\nu')+\rho R(\nu')+\lambda I_1(\nu')+\mu I_2(\nu')+\nu J(\nu')-\delta' D(\nu)-\rho' R(\nu)-\lambda' I_1(\nu)-\mu' I_2(\nu)-\nu' J(\nu)+\\
&+\delta\nu'-\delta'\nu-4\lambda\mu'-4\lambda'\mu\Big)\,J.
\end{aligned}$$

As will be useful later, in order to get for instance $\big[\widehat{H}_1, \widehat{T}\big]$, it suffices to replace in this expression $\alpha, \beta, \ldots, \nu$ by $0, \alpha_{h_1 h 1}, \ldots, \alpha_{h_1 j}$ and $\alpha', \beta', \ldots, \nu'$ by $\alpha_{tt}, \alpha_{th_1}, \ldots, \alpha_{tj}$.

Now we are ready to start the computation of the curvature components. To do this, recall at first that the curvature function κ as an element of the space $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$, splits up in components of various homogeneities. In the case of our Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

the minimal homogeneity occurs when we one considers the value of κ on $(\mathsf{h}_1, \mathsf{h}_2)$ in $\mathfrak{g}_{-2} = \mathfrak{g}_{-1-1+0}$. So the minimal homogeneity of κ is zero. On the other hand, the maximal homogeneity occurs when one considers the value of κ on $(\mathsf{h}_i, \mathsf{t})$, $i = 1, 2$, in $\mathfrak{g}_2 = \mathfrak{g}_{-1-2+5}$. Hence the maximal homogeneity is five. Now let, $\kappa_{q_j}^{p_{j_1} p_{j_2}}$ be the coefficient of q_j in $\kappa(\mathsf{p}_{j_1}, \mathsf{p}_{j_2})$, for $\mathsf{p}_{j_1} \in \mathfrak{g}_{j_1}, \mathsf{p}_{j_2} \in \mathfrak{g}_{j_2}$ and $\mathsf{q}_j \in \mathfrak{g}_j$, where naturally $j_1, j_2 < 0$. Hence if $h = j - (j_1 + j_2)$, then clearly the h-homogeneous component of the value of $\kappa(\mathsf{p}_{j_1}, \mathsf{p}_{j_2})$ is:

$$\kappa_{[h]}(\mathsf{p}_{j_1}, \mathsf{p}_{j_2}) = \sum_{h = j - (j_1 + j_2)} \kappa_{q_j}^{p_{j_1} p_{j_2}} \mathsf{q}_j.$$

Any coefficient $\kappa_{q_j}^{p_{j_1} p_{j_2}}$ is called a *curvature coefficient of homogeneity h*. From Proposition 8.14, we know that every occurring curvature coefficient $\kappa_{q_j}^{p_{j_1} p_{j_2}}$ is equal to:

$$\begin{aligned}\kappa_{q_j}^{p_{j_1} p_{j_2}} &= \widehat{Q}_j^*\big([\omega^{-1}\mathsf{p}_{j_1}, \omega^{-1}\mathsf{p}_{j_2}] - \omega^{-1}[\mathsf{p}_{j_1}, \mathsf{p}_{j_2}]\big) \\ &= \widehat{Q}_j^*\big([\widehat{P}_{j_1}, \widehat{P}_{j_2}] - \widehat{[\mathsf{p}_{j_1}, \mathsf{p}_{j_2}]_{\mathfrak{g}}}\big).\end{aligned}$$

Let us list the curvature coefficients corresponding to our sought Cartan connection, according to their homogeneities:

$\boxed{0}$ $\kappa_t^{h_1 h_2} = \widehat{T}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{2}$ $\kappa_{h_2}^{h_1 t} = \widehat{H}_2^*([\widehat{H}_1, \widehat{T}])$	$\boxed{3}$ $\kappa_r^{h_2 t} = \widehat{R}^*([\widehat{H}_2, \widehat{T}])$
$\boxed{1}$ $\kappa_{h_1}^{h_1 h_2} = \widehat{H}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{2}$ $\kappa_{h_1}^{h_2 t} = \widehat{H}_1^*([\widehat{H}_2, \widehat{T}])$	$\boxed{4}$ $\kappa_j^{h_1 h_2} = \widehat{J}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$
$\boxed{1}$ $\kappa_{h_2}^{h_1 h_2} = \widehat{H}_2^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{2}$ $\kappa_{h_2}^{h_2 t} = \widehat{H}_2^*([\widehat{H}_2, \widehat{T}])$	$\boxed{4}$ $\kappa_{i_1}^{h_1 t} = \widehat{I}_1^*([\widehat{H}_1, \widehat{T}])$
$\boxed{1}$ $\kappa_t^{h_1 t} = \widehat{T}^*([\widehat{H}_1, \widehat{T}])$	$\boxed{3}$ $\kappa_{i_1}^{h_1 h_2} = \widehat{I}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{4}$ $\kappa_{i_2}^{h_1 t} = \widehat{I}_2^*([\widehat{H}_1, \widehat{T}])$
$\boxed{1}$ $\kappa_t^{h_2 t} = \widehat{T}^*([\widehat{H}_2, \widehat{T}])$	$\boxed{3}$ $\kappa_{i_2}^{h_1 h_2} = \widehat{I}_2^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{4}$ $\kappa_{i_1}^{h_2 t} = \widehat{I}_1^*([\widehat{H}_2, \widehat{T}])$
$\boxed{2}$ $\kappa_d^{h_1 h_2} = \widehat{D}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{3}$ $\kappa_d^{h_1 t} = \widehat{D}^*([\widehat{H}_1, \widehat{T}])$	$\boxed{4}$ $\kappa_{i_2}^{h_2 t} = \widehat{I}_2^*([\widehat{H}_2, \widehat{T}])$
$\boxed{2}$ $\kappa_r^{h_1 h_2} = \widehat{R}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	$\boxed{3}$ $\kappa_d^{h_2 t} = \widehat{D}^*([\widehat{H}_2, \widehat{T}])$	$\boxed{5}$ $\kappa_j^{h_1 t} = \widehat{J}^*([\widehat{H}_1, \widehat{T}])$
$\boxed{2}$ $\kappa_{h_1}^{h_1 t} = \widehat{H}_1^*([\widehat{H}_1, \widehat{T}])$	$\boxed{3}$ $\kappa_r^{h_1 t} = \widehat{R}^*([\widehat{H}_1, \widehat{T}])$	$\boxed{5}$ $\kappa_j^{h_2 t} = \widehat{J}^*([\widehat{H}_2, \widehat{T}])$

From now on, our aim will be to compute all of these 24 curvature coefficients and to determine the functions $\alpha_{\bullet\bullet}$ in conformity with the properties they are subjected to.

9.7. **Homogeneity 0.** In this homogeneity, we encounter only one curvature coefficient:

$$\kappa_t^{h_1 h_2} = \widehat{T}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T}) = \frac{1}{\alpha_{tt}}(4\alpha_{h_1 h_1}\alpha_{h_2 h_2} - 4\alpha_{h_1 h_2}\alpha_{h_2 h_1}) - 4,$$

which is in fact the t-component of the value of the curvature on $(\mathsf{h}_1, \mathsf{h}_2)$ as a $\mathfrak{g}$-valued bilinear map. In order to satisfy the regularity condition **(c4)**, this curvature component should vanish. Hence we should have:

$$\alpha_{tt} = \alpha_{h_1h_1}\alpha_{h_2h_2} - \alpha_{h_1h_2}\alpha_{h_2h_1}.$$

But this equality is automatically satisfied, as one sees by coming back to the expressions of $\alpha_{\bullet\bullet}$ introduced after Lemma 9.1. Therefore, the desired condition $\kappa_{[0]} = 0$ holds.

9.8. **Homogeneity 1.** In this homogeneity, we have four curvature coefficients, while, according to the regularity condition **(c4)**, all of them should vanish. A detailed and latex-ed calculation gives:

$$\begin{aligned}
\kappa_{h_1}^{h_1h_2} &= \widehat{H}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T}) \\
&= \alpha_{h_1r} + \beta_{h_1h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \\
&\quad - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \\
&\quad - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{h_1h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \\
&\quad + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \\
&\quad - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{h_1t}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big).
\end{aligned}$$

$$\begin{aligned}
\kappa_{h_2}^{h_1h_2} &= \widehat{H}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T}) \\
&= -\alpha_{h_1d} + \beta_{h_2h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \\
&\quad - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \\
&\quad + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{h_2h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \\
&\quad - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \\
&\quad - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{h_2t}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big).
\end{aligned}$$

$$\begin{aligned}
\kappa_t^{h_1t} &= \widehat{T}^*([\widehat{H}_1, \widehat{T}]) \\
&= -2\alpha_{h_1d} + \beta_{tt}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 + \alpha_{h_1h_1}\underline{H_1(\alpha_{tt})}_\circ - 4\alpha_{h_1h_2}\alpha_{th_1} + \\
&\quad + \alpha_{h_1h_2}\alpha_{tt}\Phi_2 + \alpha_{h_1h_2}\underline{H_2(\alpha_{tt})}_\circ\big).
\end{aligned}$$

$$\begin{aligned}
\kappa_t^{h_2t} &= \widehat{T}^*([\widehat{H}_2, \widehat{T}]) \\
&= -2\alpha_{h_2d} + \beta_{tt}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1\alpha_{h_2h_1}\underline{H_1(\alpha_{tt})}_\circ - 4\alpha_{h_2h_2}\alpha_{th_1} + \\
&\quad + \alpha_{h_2h_2}\alpha_{tt}\Phi_2 + \alpha_{h_2h_2}\underline{H_2(\alpha_{tt})}_\circ\big).
\end{aligned}$$

Here remind that in Lemma 9.1, we saw the exact expressions of the functions $\alpha_{tt}, \alpha_{th_1}, \alpha_{th_2}, \alpha_{h_1h_1}, \alpha_{h_1h_2}, \alpha_{h_2h_1}, \alpha_{h_2h_2}$ and visibly, they were independent of the horizontal coordinates. Hence the value of the horizontal vector fields H_1, H_2 and T on these functions vanish, as is made visible by a specific underlining in the above calculations. Moreover, we have replaced the the values of the vector fields $\widehat{D}, \widehat{R}, \widehat{I}_1, \widehat{I}_2, \widehat{J}$ on the concerned functions $\alpha_{\bullet\bullet}$ by just using the 110 equations stated before Lemma 9.1. Furthermore, we notice here that the functions $\beta_{\bullet\bullet}$ are the coefficients of the dual basis introduced in Subsection 9.3. After simplifying carefully these four expressions, we get:

$$\begin{aligned}
k_{h_1}^{h_1h_2} &= \big(-4\alpha_{th_1}c - 4\alpha_{th_2}d + (\alpha_{h_1r}+\alpha_{h_2d})c^2 + (\alpha_{h_1r}+\alpha_{h_2d})d^2\big)/\big(c^2+d^2\big)\\
k_{h_2}^{h_1h_2} &= -\big(4\alpha_{th_2}c - 4\alpha_{th_1}d + (-\alpha_{h_2r}+\alpha_{h_1d})c^2 + (-\alpha_{h_2r}+\alpha_{h_1d})d^2\big)/\big(c^2+d^2\big)\\
k_t^{h_1t} &= \big(4\alpha_{th_2}c - 2\alpha_{h_1d}c^2 + \Phi_1c^3 - 4\alpha_{th_1}d - 2\alpha_{h_1d}d^2 + \Phi_1cd^2 + \Phi_2d^3 + \Phi_2c^2d\big)/\big(c^2+d^2\big)\\
k_t^{h_2t} &= -\big(4\alpha_{th_1}c + 2\alpha_{h_2d}c^2 - \Phi_2c^3 + 4\alpha_{th_2}d + 2\alpha_{h_2d}d^2 + \Phi_1d^3 + \Phi_1c^2d - \Phi_2cd^2\big)/\big(c^2+d^2\big).
\end{aligned}$$

Now as said, all these curvature coefficients should vanish. Looking at the above expressions, we realize that one encounters exactly six undetermined functions α_{h_1d}, α_{h_2d}, α_{h_1r}, $\alpha_{h_2r}, \alpha_{th_1}$ and α_{th_2}. But if we replace $\alpha_{th_1} = bd - ac$ and $\alpha_{th_2} = -ad - bc$ by the values which were already ascribed in Subsection 9.5), we are left with exactly four $\alpha_{\bullet\bullet}$. We therefore see that for these four curvatures to vanish, it is necessary and sufficient that:

$$\begin{aligned}
\alpha_{h_1d} &= -2b + \tfrac{1}{2}\Phi_1c + \tfrac{1}{2}\Phi_2d, \quad & \alpha_{h_2d} &= 2a + \tfrac{1}{2}\Phi_2c - \tfrac{1}{2}\Phi_1d,\\
\alpha_{h_1r} &= -6a - \tfrac{1}{2}\Phi_2c + \tfrac{1}{2}\Phi_1d, \quad & \alpha_{h_2r} &= -6b + \tfrac{1}{2}\Phi_1c + \tfrac{1}{2}\Phi_2d.
\end{aligned}$$

Due to the existence of the functions Φ_1 and Φ_2 in the last four expressions, one recognizes that the four functions $\alpha_{h_1,d}$, $\alpha_{h_2,d}$, $\alpha_{h_1,r}$, $\alpha_{h_2,r}$ really depend on the horizontal coordinates for $i = 1, 2$.

On the other hand, these four functions α_{h_1d}, α_{h_2d}, α_{h_1r}, α_{h_2r} should also be of the form introduced in Lemma 9.1, namely we should have:

$$\left[\begin{aligned}
&\boxed{\alpha_{h_1d}}: \quad -2b + \tfrac{1}{2}\Phi_1c + \tfrac{1}{2}\Phi_2d = -\tfrac{1}{4}c\delta_2 - 2b + \tfrac{1}{4}d\delta_1,\\
&\boxed{\alpha_{h_2d}}: \quad 2a + \tfrac{1}{2}\Phi_2c - \tfrac{1}{2}\Phi_1d = (\tfrac{1}{4}c)\,\delta_1 + (\tfrac{1}{4}d)\,\delta_2 + 2a,\\
&\boxed{\alpha_{h_1r}}: \quad -6a - \tfrac{1}{2}\Phi_2c + \tfrac{1}{2}\Phi_1d = -(\tfrac{1}{2}c)\,\delta_7 + (\tfrac{1}{2}d)\,\delta_{10} - 6a\\
&\boxed{\alpha_{h_2r}}: \quad -6b + \tfrac{1}{2}\Phi_1c + \tfrac{1}{2}\Phi_2d = (\tfrac{1}{2}d)\,\delta_7 + (\tfrac{1}{2}c)\,\delta_{10} - 6b.
\end{aligned}\right.$$

By plain identification, these four equations immediately determine the values of four of the functions $\delta_{\bullet}$ as follows:

$$\delta_1 = 2\Phi_2, \qquad \delta_2 = -2\Phi_1, \qquad \delta_7 = \Phi_2, \qquad \delta_{10} = \Phi_1.$$

9.9. **Homogeneity 2.** In this homogeneity, we encounter the following six curvature coefficients that should be annihilated in order to satisfy the regularity condition **(c4)**:

$$
\begin{aligned}
\kappa_d^{h_1h_2} &= \widehat{D}^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) \\
&= 2\alpha_{h_1i_1} - \widehat{H}_2(a_{h_1d}) + \alpha_{h_1h_2}H_2(ah2d) + \alpha_{h_1h_1}H_1(\alpha_{h_2d}) + \beta_{dh_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \\
&- \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{dh_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \\
&- \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \\
&- \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{dt}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_r^{h_1h_2} &= \widehat{R}^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) \\
&= -6\alpha_{h_1i_2} - \widehat{H}_2(\alpha_{h_1r}) + \alpha_{h_1h_2}H_2(\alpha_{h_2r}) + \alpha_{h_1h_1}H_1(\alpha_{h_2r}) + \beta_{rh_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \\
&- \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{rh_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \\
&- \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \\
&- \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{rt}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{h_1}^{h_1t} &= \widehat{H}_1^*([\widehat{H}_1,\widehat{T}]) \\
&= -\alpha_{h_1i_1} + \beta_{h_1h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_1})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ\big) + \\
&+ \beta_{h_1h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{th_2})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \\
&- \alpha_{tt}\underline{T(\alpha_{h_1h_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \\
&- \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{h_1t}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_1h_2}\alpha_{th_1} + \alpha_{h_1h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{h_2}^{h_1t} &= \widehat{H}_2^*([\widehat{H}_1,\widehat{T}]) \\
&= -\alpha_{h_1i_2} + \beta_{h_2h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_1})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ\big) + \\
&+ \beta_{h_2h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{th_2})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \\
&- \alpha_{tt}\underline{T(\alpha_{h_1h_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \\
&- \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{h_2t}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_1h_2}\alpha_{th_1} + \alpha_{h_1h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{h_1}^{h_2t} &= \widehat{H}_1^*([\widehat{H}_2, \widehat{T}]) \\
&= -\alpha_{h_2i_1} + \beta_{h_1h_1}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_1})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_1})}_{\alpha_{h_1h_1}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_1})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_1})}_\circ + \\
&+ \alpha_{h_2h_1}\underline{H_2(\alpha_{th_1})}_\circ\big) + \beta_{h_1h_2}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_2})}_\circ + \alpha_{h_2h_2}\underline{H_2(\alpha_{th_2})}_\circ - \\
&- \alpha_{th_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_2})}_{-\alpha_{h_2h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_2})}_{\alpha_{h_1h_2}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_2})}_\circ - \\
&- \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_2})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_2})}_\circ\big) + \beta_{h_1t}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_2h_2}\alpha_{th_1} + \alpha_{h_2h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{h_2}^{h_2t} &= \widehat{H}_2^*([\widehat{H}_2, \widehat{T}]) \\
&= -\alpha_{h_2i_2} + \beta_{h_2h_1}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_1})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_1})}_{\alpha_{h_1h_1}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_1})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_1})}_\circ + \\
&+ \alpha_{h_2h_1}\underline{H_2(\alpha_{th_1})}_\circ\big) + \beta_{h_2h_2}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_2})}_\circ + \alpha_{h_2h_2}\underline{H_2(\alpha_{th_2})}_\circ - \\
&- \alpha_{th_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_2})}_{-\alpha_{h_2h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_2})}_{\alpha_{h_1h_2}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_2})}_\circ - \\
&- \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_2})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_2})}_\circ\big) + \beta_{h_2t}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_2h_2}\alpha_{th_1} + \alpha_{h_2h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}
$$

Before simplifying the above expressions, we notice that the expressions of the four functions α_{h_1d}, α_{h_2d}, α_{h_1r}, α_{h_2r} obtained in the previous subsection enable us to identify how the three horizontal vector fields H_1, H_2, T act on them as first-order differential operators. Because these vector fields only differentiate with respect to (x, y, u) and not with respect to (a, b, c, d, e), we have, for any for $Y = H_1, H_2, T$:

$$
\begin{aligned}
&Y(a_{h_1d}) = \tfrac{1}{2}Y(\Phi_1)c + \tfrac{1}{2}Y(\Phi_2)d - 2b, \qquad Y(a_{h_2d}) = \tfrac{1}{2}(\Phi_2)c - \tfrac{1}{2}Y(\Phi_1)d + 2a, \\
&Y(a_{h_1r}) = -\tfrac{1}{2}Y(\Phi_2)c + \tfrac{1}{2}Y(\Phi_1)d - 6a, \quad Y(a_{h_2r}) = \tfrac{1}{2}Y(\Phi_1)c + \tfrac{1}{2}Y(\Phi_2)d - 6b.
\end{aligned}
$$

Moreover, we can find the expressions of $\widehat{H}_2(\alpha_{h_1d})$ and $\widehat{H}_2(\alpha_{h_1r})$ which appear above in the expressions of the two curvature coefficients $\kappa_d^{h_1h_2}$, $\kappa_r^{h_1h_2}$, coming back to the definition of the vector field $\widehat{H}_2$ in Subsection 9.3:

$$
\begin{aligned}
\widehat{H}_2(\alpha_{h_1d}) &= \alpha_{h_2h_1}H_1(\alpha_{h_1d}) + \alpha_{h_2h_2}H_2(\alpha_{h_1d}) + \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1d})}_{-\alpha_{h_1d}} + \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1d})}_{-\alpha_{h_2d}} + \alpha_{h_2i_2}\underbrace{\widehat{I}_2(\alpha_{h_1d})}_{-2} + \\
&\quad + \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1d})}_\circ + \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1d})}_\circ \\
&\quad - 2\alpha_{h_2i_2} + (\tfrac{1}{2}H_2(\Phi_1) - \tfrac{1}{2}\Phi_1\Phi_2)c^2 + (-\tfrac{1}{2}H_1(\Phi_2) + \tfrac{1}{2}\Phi_1\Phi_2)d^2 + 16ab - 2\Phi_2ad + \\
&\quad + 4\Phi_2bc - 2\Phi_1ac - 4\Phi_1bd + (\tfrac{1}{2}\Phi_1^2 + \tfrac{1}{2}H_2(\Phi_2) - \tfrac{1}{2}\Phi_2^2 - \tfrac{1}{2}H_1(\Phi_1))cd
\end{aligned}
$$

$$
\begin{aligned}
\widehat{H}_2(\alpha_{h_1r}) &= \alpha_{h_2h_1}H_1(\alpha_{h_1r}) + \alpha_{h_2h_2}H_2(\alpha_{h_1r}) + \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1r})}_{-\alpha_{h_1r}} + \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1r})}_{-\alpha_{h_2r}} + \alpha_{h_2i_1}\underbrace{\widehat{I}_1(\alpha_{h_1r})}_{-6} + \\
&\quad + \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1r})}_\circ + \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1r})}_\circ \\
&\quad = -6\alpha_{h_2i_1} + (-\tfrac{1}{2}H_2(\Phi_2) - \tfrac{1}{4}\Phi_1^2 + \tfrac{1}{4}\Phi_2^2)c^2 + (\tfrac{1}{4}\Phi_1^2 - \tfrac{1}{4}\Phi_1^2 - \tfrac{1}{2}H_1(\Phi_1))d^2 + (-\Phi_1\Phi_2 + \\
&\quad + \tfrac{1}{2}H_1(\Phi_2) + \tfrac{1}{2}H_2(\Phi_1))cd + 12a^2 - 36b^2 + 4\Phi_2ac - 4\Phi_1ad + 6\Phi_1bc + 6\Phi_2bd.
\end{aligned}
$$

Thanks to these preparations, we are now in a position to simplify the six curvature coefficients of homogeneity 2, and careful calculations give at the end:

$$\begin{aligned}
\kappa_d^{h_1h_2} &= 2\alpha_{h_2i_2} - 4\alpha_{td} + 2\alpha_{h_1i_1} + \tfrac{1}{2}\big(-H_2(\Phi_1) + H_1(\Phi_2)\big)c^2 + \tfrac{1}{2}\big(-H_2(\Phi_1) + H_1(\Phi_2)\big)d^2 + \\
&\quad + 2\big(\Phi_1 bd - \Phi_2 bc - \Phi_1 ac - \Phi_2 ad\big), \\
\kappa_r^{h_1h_2} &= -6\alpha_{h_1i_2} + 6\alpha_{h_2i_1} - 4\alpha_{tr} + \tfrac{1}{2}\big(H_2(\Phi_2) + H_1(\Phi_1)\big)c^2 + \tfrac{1}{2}\big(H_2(\Phi_2) + (H_1)\Phi_1\big)d^2 + \\
&\quad + 24a^2 + 24b^2 + 2\Phi_2 ac - 2\Phi_1 bc - 2\Phi_1 ad - 2\Phi_2 bd, \\
\kappa_{h_1}^{h_1t} &= \alpha_{td} - \alpha_{h_1i_1} + \Phi_1 ac + \Phi_2 ad - 4ab, \\
\kappa_{h_2}^{h_1t} &= \alpha_{tr} - \alpha_{h_1i_2} + \Phi_1 bc + \Phi_2 bd - 4b^2, \\
\kappa_{h_1}^{h_2t} &= -\alpha_{tr} - \alpha_{h_2i_1} + \Phi_2 ac - \Phi_1 ad + 4a^2, \\
\kappa_{h_2}^{h_2t} &= \alpha_{td} - \alpha_{h_2i_2} + \Phi_2 bc - \Phi_1 bd + 4ab.
\end{aligned}$$

Inspecting these six equations, we see that there are exactly six undetermined functions α_{td}, α_{tr}, $\alpha_{h_1i_1}$, $\alpha_{h_1i_2}$,$\alpha_{h_2i_1}$, $\alpha_{h_2i_2}$. Thus, we might to fully annihilate the curvature of homogeneity 2 by solving this system of six equations in six unknowns. But unfortunately, the solution set of this system, as it is written here, happens to be empty!

Temporarily, let us omit the first equation involving the curvature coefficient $\kappa_d^{h_1h_2}$ and let us solve the remaining system of five equations with six unknowns. The obtained solution set of this system reveals that vanishing of this new system is independent of the function $\alpha_{h_2i_2}$. Consequently, $\alpha_{h_2i_2}$ is free and we then determine the remaining five functions α_{td}, α_{tr}, $\alpha_{h_1i_1}$, $\alpha_{h_1i_2}$, $\alpha_{h_2i_1}$ according to this subsystem of five equations.

On the other hand, reminding Lemma 9.1, $\alpha_{h_2i_2}$ should be of the form:

$$\alpha_{h_2i_2} = (-\tfrac{1}{8}c^2 - \tfrac{1}{8}d^2)\delta_6 + \tfrac{1}{4}\delta_8 c^2 - \tfrac{1}{4}\delta_9 dc - \tfrac{1}{2}\Phi_1 ac + 4ab - \tfrac{1}{2}\Phi_2 ad + \tfrac{1}{2}\Phi_2 bc - \tfrac{1}{2}\Phi_1 bd - 2e.$$

and furthermore, the obtained values of the five functions in question should identify to:

$$\begin{aligned}
\alpha_{td} &= \alpha_{h_2i_2} - \Phi_2 bc + \Phi_1 bd - 4ab \\
&= -\tfrac{1}{8}(c^2 + d^2)\delta_6 + \tfrac{1}{4}\delta_8 c^2 - \tfrac{1}{4}\delta_9 cd + \tfrac{1}{2}\Phi_1 bd - \tfrac{1}{2}\Phi_2 bc - \tfrac{1}{2}\Phi_1 ac - \tfrac{1}{2}\Phi_2 ad - 2e, \\
\alpha_{tr} &= 3(a^2 + b^2) + \tfrac{1}{32}\big(H_1(\Phi_1) + H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1) + H_2(\Phi_2)\big)d^2 + \\
&\quad + \tfrac{1}{2}\big(\Phi_2 ac - \Phi_1 bc - \Phi_2 bd - \Phi_1 ad\big), \\
\alpha_{h_1i_1} &= \alpha_{h_2i_2} + \Phi_1 ac - \Phi_2 bc + \Phi_1 bd - 8ab + \Phi_2 ad \\
&= (-\tfrac{1}{8}c^2 - \tfrac{1}{8}d^2)\delta_6 + \tfrac{1}{4}\delta_8 c^2 - \tfrac{1}{4}\delta_9 cd + \tfrac{1}{2}\Phi_1 bd - \tfrac{1}{2}\Phi_2 bc + \tfrac{1}{2}\Phi_1 ac + \tfrac{1}{2}\Phi_2 ad - 4ab - 2e, \\
\alpha_{h_1i_2} &= 3a^2 - b^2 + \tfrac{1}{32}\big(H_1(\Phi_1) + H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1) + H_2(\Phi_2)\big)d^2 + \\
&\quad + \tfrac{1}{2}\big(\Phi_2 ac + \Phi_2 bd + \Phi_1 bc - \Phi_1 ad\big), \\
\alpha_{h_2i_1} &= a^2 - 3b^2 + \big(-\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2)\big)c^2 + \big(-\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2)\big)d^2 + \\
&\quad + \tfrac{1}{2}\Phi_2 bd + \tfrac{1}{2}\Phi_1 bc + \tfrac{1}{2}\Phi_2 ac - \tfrac{1}{2}\Phi_1 ad,
\end{aligned}$$

after the necessary simplifications. These five equations guarantee the vanishing of the five curvature coefficients $\kappa_r^{h_1h_2}$, $\kappa_{h_1}^{h_1t}$, $\kappa_{h_2}^{h_1t}$, $\kappa_{h_1}^{h_2t}$, but still, it remains to also annihilate the first, left aside, curvature coefficient $\kappa_d^{h_1h_2}$. But replacing the above expressions in the expression of this curvature coefficient, we get:

$$\kappa_d^{h_1h_2} = \big(-\tfrac{1}{2}H_2(\Phi_1) + \tfrac{1}{2}H_1(\Phi_2)\big)c^2 + \big(-\tfrac{1}{2}H_2(\Phi_1) + \tfrac{1}{2}H_1(\Phi_2)\big)d^2.$$

Fortunately, this last expression vanishes thanks to the fact that the two functions $H_1(\Phi_2) = H_2(\Phi_1)$ are equal, which was already seen in Lemma 6.3.

Now, we are sure that the above determination of the functions α_{td}, α_{tr}, $\alpha_{h_1 i_1}$, $\alpha_{h_1 i_2}$, $\alpha_{h_2 i_1}$, $\alpha_{h_2 i_2}$ annihilates all the curvature components of homogeneity two, which is what was announced in the regularity condition **(c4)**. Lastly, we also have to take care of the condition **(c1)**. Similarly as in homogeneity one, reminding Lemma 9.1, an identification gives:

$$
\left[
\begin{aligned}
&\boxed{\alpha_{td}}: \ (-\tfrac{1}{8}c^2 - \tfrac{1}{8}d^2)\delta_6 + \tfrac{1}{4}\delta_8 c^2 - \tfrac{1}{4}\delta_9 cd + \tfrac{1}{2}\Phi_1 bd - \tfrac{1}{2}\Phi_2 bc - \tfrac{1}{2}\Phi_1 ac - \tfrac{1}{2}\Phi_2 ad - 2e \\
&\qquad = 2(-\tfrac{1}{4}bc - \tfrac{1}{4}ad)\,\Phi_2 - 2(\tfrac{1}{4}ac - \tfrac{1}{4}bd)\,\Phi_1 + (\tfrac{1}{4}c^2 + \tfrac{1}{4}d^2)\,\delta_{15} - 2e, \\
&\boxed{\alpha_{tr}}: \ 3a^2 + 3b^2 + \big(\tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2)\big)c^2 + \big(\tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}\Phi_2 ac - \tfrac{1}{2}\Phi_1 bc - \\
&\qquad - \tfrac{1}{2}\Phi_2 bd - \tfrac{1}{2}\Phi_1 ad \\
&\qquad = (\tfrac{1}{4}c^2 + \tfrac{1}{4}d^2)\,\delta_4 + (\tfrac{1}{2}ac - \tfrac{1}{2}bd)\,\Phi_2 + (\tfrac{1}{2}c^2 + \tfrac{1}{2}d^2)\,\delta_9 - (\tfrac{1}{2}ad + \tfrac{1}{2}bc)\,\Phi_1 + (\tfrac{1}{2}c^2 + \tfrac{1}{2}d^2)\,\delta_{19} + 3b^2 + 3a^2, \\
&\boxed{\alpha_{h_1 i_1}}: \ (-\tfrac{1}{8}c^2 - \tfrac{1}{8}d^2)\delta_6 + \tfrac{1}{4}\delta_8 c^2 - \tfrac{1}{4}\delta_9 cd + \tfrac{1}{2}\Phi_1 bd - \tfrac{1}{2}\Phi_2 bc + \tfrac{1}{2}\Phi_1 ac + \tfrac{1}{2}\Phi_2 ad - 4ab - 2e \\
&\qquad = 2(\tfrac{1}{4}ad)\,\Phi_2 + 2(\tfrac{1}{4}ac)\,\Phi_1 - (\tfrac{1}{8}c^2 + \tfrac{1}{8}d^2)\,\delta_6 - (\tfrac{1}{2}bc)\,\Phi_2 + (\tfrac{1}{4}d^2)\,\delta_8 + (\tfrac{1}{4}cd)\,\delta_9 + (\tfrac{1}{2}bd)\,\Phi_1 - 4ab - 2e, \\
&\boxed{\alpha_{h_1 i_2}}: \ 3a^2 - b^2 + \big(\tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2)\big)c^2 + \big(\tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}\Phi_2 ac + \tfrac{1}{2}\Phi_2 bd + \\
&\qquad \tfrac{1}{2}\Phi_1 bc - \tfrac{1}{2}\Phi_1 ad \\
&\qquad = 2(\tfrac{1}{4}bd)\Phi_2 + 2(\tfrac{1}{4}bc)\,\Phi_1 - (\tfrac{1}{8}c^2 + \tfrac{1}{8}d^2)\,\delta_4 + (\tfrac{1}{2}ac)\,\Phi_2 + (\tfrac{1}{4}cd)\,\delta_8 - (\tfrac{1}{4}d^2)\,\delta_9 - (\tfrac{1}{2}ad)\,\Phi_1 + 3a^2 - b^2, \\
&\boxed{\alpha_{h_2 i_1}}: \ a^2 - 3b^2 + \big(-\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2)\big)c^2 + \big(-\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}\Phi_2 bd + \tfrac{1}{2}\Phi_1 bc + \\
&\qquad \tfrac{1}{2}\Phi_2 ac - \tfrac{1}{2}\Phi_1 ad \\
&\qquad = 2(\tfrac{1}{4}ac)\,\Phi_2 - 2(\tfrac{1}{4}ad)\,\Phi_1 + (\tfrac{1}{8}c^2 + \tfrac{1}{8}d^2)\,\delta_4 + (\tfrac{1}{2}bd)\,\Phi_2 + (\tfrac{1}{4}cd)\,\delta_8 + (\tfrac{1}{4}c^2)\,\delta_9 + (\tfrac{1}{2}bc)\,\Phi_1 - 3b^2 + a^2.
\end{aligned}
\right.
$$

The right hand sides of these equations are the expressions of the mentioned functions as in Lemma 9.1 after the possible simplifications. Inspection of this system shows that it will be satisfied whenever one has:

$$
\left[
\begin{aligned}
&\boxed{\alpha_{td}}: \ -\tfrac{1}{8}\delta_6 + \tfrac{1}{4}\delta_8 = \tfrac{1}{4}\delta_{15}, \quad -\tfrac{1}{8}\delta_6 = \tfrac{1}{4}\delta_{15}, \quad \tfrac{1}{4}\delta_9 = 0, \\
&\boxed{\alpha_{tr}}: \ \tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2) = \tfrac{1}{4}\delta_4 + \tfrac{1}{2}\delta_9 + \tfrac{1}{2}\delta_{19}, \\
&\boxed{\alpha_{h_1 i_1}}: \ -\tfrac{1}{8}\delta_6 + \tfrac{1}{4}\delta_8 = -\tfrac{1}{8}\delta_6, \quad -\tfrac{1}{8}\delta_6 = \tfrac{1}{8}\delta_6 + \tfrac{1}{4}\delta_8, \quad -\tfrac{1}{4}\delta_9 = \tfrac{1}{4}\delta_9, \\
&\boxed{\alpha_{h_1 i_2}}: \ \tfrac{1}{32}H_1(\Phi_1) + \tfrac{1}{32}H_2(\Phi_2) = -\tfrac{1}{8}\delta_4, \\
&\boxed{\alpha_{h_2 i_1}}: \ -\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2) = \tfrac{1}{8}\delta_4 + \tfrac{1}{4}\delta_9, \quad -\tfrac{1}{32}H_1(\Phi_1) - \tfrac{1}{32}H_2(\Phi_2) = \tfrac{1}{8}\delta_4, \quad \tfrac{1}{4}\delta_8 = 0.
\end{aligned}
\right.
$$

One immediately checks that this system has the following solution set which guarantees that our computations in this homogeneity are in direction of satisfying both of the conditions **(c1)** and **(c4)**:

$$
\delta_6 = \delta_8 = \delta_9 = \delta_{15} = 0, \quad \delta_4 = -\tfrac{1}{4}H_1(\Phi_1) - \tfrac{1}{4}H_2(\Phi_2), \quad \delta_{19} = \frac{3}{16}H_1(\Phi_1) + \frac{3}{16}H_2(\Phi_2).
$$

Finally we obtain:

$$\begin{aligned}
\alpha_{td} &= \tfrac{1}{2}(bd-ac)\Phi_1 - \tfrac{1}{2}\Phi_2(bc+ad)\Phi_2 - 2e,\\
\alpha_{tr} &= \tfrac{1}{32}\big(H_1(\Phi_1)H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 - \tfrac{1}{2}(ad+bc)\Phi_1 + \tfrac{1}{2}(ac-bd)\Phi_2 + 3a^2 + 3b^2,\\
\alpha_{h_1i_1} &= \tfrac{1}{2}(bd+ac)\Phi_1 - \tfrac{1}{2}(bc-ad)\Phi_2 - 4ab - 2e,\\
\alpha_{h_1i_2} &= \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + 3a^2 - b^2,\\
\alpha_{h_2i_1} &= -\tfrac{1}{32}H_1\big((\Phi_1)+H_2(\Phi_2)\big)c^2 - \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + a^2 - 3b^2,\\
\alpha_{h_2i_2} &= -\tfrac{1}{2}(ac+bd)\Phi_1 - \tfrac{1}{2}(ad-bc)\Phi_2 + 4ab - 2e.
\end{aligned}$$

9.10. **Homogeneity 3.** In this homogeneity, we have exactly six curvature coefficients:

$$\begin{aligned}
\kappa_{i_1}^{h_1h_2} &= \widehat{I}_1^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T})\\
&= -\widehat{H}_2(\alpha_{h_1i_1}) + \alpha_{h_1h_2}H_2(\alpha_{h_2i_1}) + \alpha_{h_1h_1}H_1(\alpha_{h_2i_1}) + \beta_{i_1h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ -\\
&\quad - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ -\\
&\quad - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{i_1h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ -\\
&\quad - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} -\\
&\quad - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \underline{\alpha_{h_2j}\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{i_1t}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_{i_2}^{h_1h_2} &= \widehat{I}_2^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T})\\
&\quad - \alpha_{h_1j} - \widehat{H}_2(\alpha_{h_1i_2}) + \alpha_{h_1h_2}H_2(\alpha_{h_2i_2}) + \alpha_{h_1h_1}H_1(\alpha_{h_2i_2}) + \beta_{i_2h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ -\\
&\quad - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ -\\
&\quad - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ\big) + \beta_{i_2h_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ -\\
&\quad - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} -\\
&\quad - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \underline{\alpha_{h_2j}\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{i_2t}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_1t} &= \widehat{D}^*([\widehat{H}_1,\widehat{T}]) = -\alpha_{h_1j} - \widehat{T}(\alpha_{h_1d}) + \alpha_{h_1h_2}H_2(\alpha_{td}) + \alpha_{h_1h_1}H_1(\alpha_{td}) + \beta_{dh_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_1})}_\circ -\\
&\quad - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_1})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ -\\
&\quad - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{tr}\underline{\widehat{R}(\alpha_{h_1h_1})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ\big) - \beta_{dh_2}\big(\alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} + \alpha_{h_1h_1}\underline{H_1(\alpha_{th_2})}_\circ -\\
&\quad + \alpha_{th_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{th_2})}_\circ + \alpha_{th_2}\underline{H_2(\alpha_{h_1h_2})}_\circ + \alpha_{tt}\underline{T(\alpha_{h_1h_2})}_\circ + \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ +\\
&\quad + \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{dt}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_1h_2}\alpha_{th_1} + \alpha_{h_1h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_2t} &= \widehat{D}^*([\widehat{H}_2,\widehat{T}]) = -\alpha_{h_2j} - \widehat{T}(\alpha_{h_2d}) + \alpha_{h_2h_2}H_2(\alpha_{td}) + \alpha_{h_2h_1}H_1(\alpha_{td}) + \beta_{dh_1}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_1})}_\circ - \\
&- \alpha_{th_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_1})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_1})}_{\alpha_{h_1h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_1})}_\circ\big) - \beta_{dh_2}\big(\alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_2})}_{-\alpha_{h_2h_2}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_2})}_{\alpha_{h_1h_2}} - \alpha_{h_2h_1}\underline{H_1(\alpha_{th_2})}_\circ + \\
&+ \alpha_{th_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{th_2})}_\circ + \alpha_{th_2}\underline{H_2(\alpha_{h_2h_2})}_\circ + \alpha_{tt}\underline{T(\alpha_{h_2h_2})}_\circ + \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_2})}_\circ + \\
&+ \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_2})}_\circ + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_2})}_\circ\big) + \beta_{dt}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_2h_2}\alpha_{th_1} + \alpha_{h_2h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_r^{h_1t} &= \widehat{R}^*([\widehat{H}_1,\widehat{T}]) = -\widehat{T}(\alpha_{h_1r}) + \alpha_{h_1h_2}H_2(\alpha_{tr}) + \alpha_{h_1h_1}H_1(\alpha_{tr}) + \beta_{rh_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_1})}_\circ - \\
&- \alpha_{th_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_1})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ\big) + \beta_{rh_2}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} - \\
&- \alpha_{th_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{th_2})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_2})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \\
&- \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{rt}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_1h_2}\alpha_{th_1} + \alpha_{h_1h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_r^{h_2t} &= \widehat{R}^*([\widehat{H}_2,\widehat{T}]) = -\widehat{T}(\alpha_{h_2r}) + \alpha_{h_2h_2}H_2(\alpha_{tr}) + \alpha_{h_2h_1}H_1(\alpha_{tr}) + \beta_{rh_1}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_1})}_\circ - \\
&- \alpha_{th_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_1})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_1})}_\circ - \\
&- \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_1})}_{\alpha_{h_1h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_1})}_\circ\big) + \beta_{rh_2}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_2})}_{-\alpha_{h_2h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_2})}_{\alpha_{h_1h_2}} - \\
&- \alpha_{th_1}\underline{H_1(\alpha_{h_2h_2})}_\circ + \alpha_{h_2h_2}\underline{H_2(\alpha_{th_2})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_2h_2})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_2})}_\circ - \\
&- \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_2})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_2})}_\circ\big) + \beta_{rt}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_2h_2}\alpha_{th_1} + \alpha_{h_2h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}$$

According to the expressions of the functions $\alpha_{\bullet\bullet}$, obtained in homogeneity two and according to the properties of the horizontal vector field $Y = H_1, H_2, T$ we have:

$$\begin{aligned}
Y(\alpha_{td}) &= -\frac{1}{2}(bc+ad)\,Y(\Phi_2) - \frac{1}{2}(ac-bd)\,Y(\Phi_1), \\
Y(\alpha_{tr}) &= \tfrac{1}{32}\big[Y(H_1(\Phi_1)) + Y(H_2(\Phi_2))\big]c^2 + \tfrac{1}{32}\big[Y(H_1(\Phi_1)) + Y(H_2(\Phi_2))\big]d^2 + \tfrac{1}{2}Y(\Phi_2)ac - \\
&- \tfrac{1}{2}Y(\Phi_1)bc - \tfrac{1}{2}Y(\Phi_2)bd - \tfrac{1}{2}Y(\Phi_1)ad, \\
Y(\alpha_{h_1i_1}) &= \tfrac{1}{2}(bd+ac)Y(\Phi_1) + \tfrac{1}{2}(ad-bc)Y(\Phi_2), \\
Y(\alpha_{h_1i_2}) &= \tfrac{1}{32}\big[Y(H_1(\Phi_1)) + Y(H_2(\Phi_2))\big]c^2 + \tfrac{1}{32}\big[Y(H_1(\Phi_1)) + Y(H_2(\Phi_2))\big]d^2 + \tfrac{1}{2}Y(\Phi_2)ac + \\
&+ \tfrac{1}{2}Y(\Phi_2)bd + \tfrac{1}{2}Y(\Phi_1)bc - \tfrac{1}{2}Y(\Phi_1)ad, \\
Y(\alpha_{h_2i_1}) &= -\tfrac{1}{32}(Y(H_1(\Phi_1)) + Y(H_2(\Phi_2)))c^2 - \tfrac{1}{32}(Y(H_1(\Phi_1)) + Y(H_2(\Phi_2)))d^2 + \tfrac{1}{2}Y(\Phi_2)bd + \\
&+ \tfrac{1}{2}Y(\Phi_1)bc + \tfrac{1}{2}Y(\Phi_2)ac - \tfrac{1}{2}Y(\Phi_1)ad, \\
Y(\alpha_{h_2i_2}) &= -\tfrac{1}{2}Y(\Phi_1)ac - \tfrac{1}{2}Y(\Phi_2)ad + \tfrac{1}{2}Y(\Phi_2)bc - \tfrac{1}{2}Y(\Phi_1)bd.
\end{aligned}$$

Furthermore, we can simplify the following functions which appear just above by using the 110 equations introduced before Lemma 9.1:

$$\begin{aligned}
\widehat{H}_2(\alpha_{h_1i_1}) &= \alpha_{h_2h_1}H_1(\alpha_{h_1i_1}) + \alpha_{h_2h_2}H_2(\alpha_{h_1i_1}) + \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1i_1})}_{-2\alpha_{h_1i_1}} + \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1i_1})}_{-\alpha_{h_1i_2}-\alpha_{h_2i_1}} + \\
&+ \alpha_{h_2i_1}\underbrace{\widehat{I}_1(\alpha_{h_1i_1})}_{\alpha_{h_1d}} + \alpha_{h_2i_2}\underbrace{\widehat{I}_2(\alpha_{h_1i_1})}_{\alpha_{h_1r}} + \alpha_{h_2j}\underbrace{\widehat{J}(\alpha_{h_1i_1})}_{-1}
\end{aligned}$$

$$\widehat{H}_2(\alpha_{h_1i_2}) = \alpha_{h_2h_1}H_1(\alpha_{h_1i_2}) + \alpha_{h_2h_2}H_2(\alpha_{h_1i_2}) + \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1i_2})}_{-2\alpha_{h_1i_2}} + \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1i_2})}_{\alpha_{h_1i_1}-\alpha_{h_2i_2}} + $$
$$+ \alpha_{h_2i_1}\underbrace{\widehat{I}_1(\alpha_{h_1i_2})}_{-\alpha_{h_1r}} + \alpha_{h_2i_2}\underbrace{\widehat{I}_2(\alpha_{h_1i_2})}_{\alpha_{h_1d}} + \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1i_2})}_{\circ},$$

$$\widehat{T}(\alpha_{h_1d}) = \alpha_{tt}T(\alpha_{h_1d}) + \alpha_{th_1}H_1(\alpha_{h_1d}) + \alpha_{th_2}H_2(\alpha_{h_1d}) + \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1d})}_{-\alpha_{h_1d}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1d})}_{-\alpha_{h_2d}} + \alpha_{ti_2}\underbrace{\widehat{I}_2(\alpha_{h_1d})}_{-2} +$$
$$+ \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1d})}_{\circ} + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1d})}_{\circ},$$

$$\widehat{T}(\alpha_{h_2d}) = \alpha_{tt}T(\alpha_{h_2d}) + \alpha_{th_1}H_1(\alpha_{h_2d}) + \alpha_{th_2}H_2(\alpha_{h_2d}) + \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2d})}_{-\alpha_{h_2d}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2d})}_{\alpha_{h_1d}} + \alpha_{ti_1}\underbrace{\widehat{I}_1(\alpha_{h_2d})}_{2} +$$
$$+ \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2d})}_{\circ} + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2d})}_{\circ},$$

$$\widehat{T}(\alpha_{h_1r}) = \alpha_{tt}T(\alpha_{h_1r}) + \alpha_{th_1}H_1(\alpha_{h_1r}) + \alpha_{th_2}H_2(\alpha_{h_1r}) + \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1r})}_{-\alpha_{h_1r}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1r})}_{-\alpha_{h_2r}} + \alpha_{ti_1}\underbrace{\widehat{I}_1(\alpha_{h_1r})}_{-6} +$$
$$+ \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1r})}_{\circ} + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1r})}_{\circ},$$

$$\widehat{T}(\alpha_{h_2r}) = \alpha_{tt}T(\alpha_{h_2r}) + \alpha_{th_1}H_1(\alpha_{h_2r}) + \alpha_{th_2}H_2(\alpha_{h_2r}) + \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2r})}_{-\alpha_{h_2r}} + \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2r})}_{\alpha_{h_1r}} + \alpha_{ti_2}\underbrace{\widehat{I}_2(\alpha_{h_2r})}_{-6} +$$
$$+ \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2r})}_{\circ} + \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2r})}_{\circ}.$$

Before replacing the above expressions in the curvature coefficients in order to simplify, we should be aware of the following fact, which helps us to substitute the vector field T in terms of the two basic sections H_1 and H_2 of T^cM:

$$T(\Phi_i) = 4[H_1, H_2](\Phi_i) = 4\big[H_1(H_2((\Phi_i))) - H_2(H_1((\Phi_i)))\big], \qquad i = 1, 2.$$

This is natural and this helps us to get more simplified expressions. Thus, replacing the above expressions in the curvature coefficients of homogeneity 3 and carefully simplifying, we get:

$$\begin{aligned}
\kappa_{i_1}^{h_1h_2} &= \alpha_{h_2j} - 4\alpha_{ti_1} + \tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1 - H_1(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1))\big]c^3 + \\
&+ \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]d^3 + 12a^2b - 2\Phi_2ce + 12b^3 - \\
&- 3\Phi_1a^2c - 3\Phi_1b^2c + 2\Phi_1de - 3\Phi_2a^2d - 3\Phi_2b^2d - 8ae + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + \\
&+ H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^2d + \tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1 - \\
&- H_1(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1))\big]cd^2 + \tfrac{3}{8}\big[H_1\Phi_1 + H_2(\Phi_2)\big]bc^2 + \tfrac{3}{8}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]bd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_{i_2}^{h_1h_2} &= -\alpha_{h_1j} - 4\alpha_{ti_2} + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^3 - \\
&- \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2 + \tfrac{1}{32}\big[H_1(H_1(\Phi_1)) - H_2(\Phi_2)\Phi_1 - H_1(\Phi_1)\Phi_1 + H_1(H_2(\Phi_2))\big]d^3 + \\
&+ 2\Phi_1ce - 8be - 3\Phi_2a^2c - 3\Phi_2b^2c + 2\Phi_2de + 3\Phi_1a^2d + 3\Phi_1b^2d - 12a^3 - 12ab^2 - \\
&- \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ac^2 + \tfrac{1}{32}\big[H_1(H_1(\Phi_1)) - H_2(\Phi_2)\Phi_1 - H_1(\Phi_1)\Phi_1 + H_1(H_2)\Phi_2\big]c^2d + \\
&+ \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]cd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_1t} &= 2\alpha_{ti_2} - \alpha_{h_1j} - \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^3 + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]d^3 + \\
&+ 2\Phi_1ce - 8be + 2\Phi_2de + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]c^2d - \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]cd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_2t} &= -2\alpha_{ti_1} - \alpha_{h_2j} + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]c^3 + \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]d^3 + \\
&+ 2\Phi_2ce - 2\Phi_1de + 8ae + \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^2d + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]cd^2,
\end{aligned}$$

$$\kappa_r^{h_1t} = 6\alpha_{ti_1} + \tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 - 4H_2(H_1(\Phi_2)) + 5H_1(H_2(\Phi_2)) + \tfrac{1}{16}H_1(H_1(\Phi_1))\big]c^3 - \\ - \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bd^2 - \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) - \\ - H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]c^2d + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) - \\ - H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]d^3 - 12a^2b - 12b^3 + 3\Phi_1a^2c + 3\Phi_1cb^2 + 3\Phi_2a^2d + 3\Phi_2b^2d + \\ + \tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 - 4H_2(H_1(\Phi_2)) + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]d^2c$$

$$\kappa_r^{h_2t} = 6\alpha_{ti_2} - \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_2 + 4H_1(H_1(\Phi_2)) - H_2(H_2(\Phi_2))\big]c^3 + \\ + \tfrac{1}{32}\big[-5H_1(H_2(\Phi_2)) + 4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_1\big]d^3 + 3ca^2\Phi_2 + \\ + 3cb^2\Phi_2 - 3a^2d\Phi_1 - 3b^2d\Phi_1 + 12a^3 + 12ab^2 + \tfrac{3}{8}\big[H_2\Phi_2 + H_1(\Phi_1)big]c^2a + \tfrac{1}{32}\big[-5H_1(H_2(\Phi_2)) + \\ + 4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_1\big]c^2d + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) - \\ - H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]cd^2 + \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2.$$

We therefore see here exactly four undetermined functions α_{ti_1}, α_{ti_2}, α_{h_1j}, α_{h_2j} within these six expressions. Although the number (six) of equations is greater than the number (four) of unknowns, we can annihilate all the six curvature coefficients by making the following appropriate determinations:

$$\alpha_{ti_1} = \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]d^3 + \\ + \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]c^3 + \\ + \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]cd^2 + \\ + \tfrac{1}{16}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + \\ + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^2d + \tfrac{1}{16}\big[H_2(\Phi_2) + H_1\Phi_1\big]bd^2 + \\ + \tfrac{1}{2}\big[-\Phi_1a^2c + 4b^3 - \Phi_1b^2c + 4ba^2 - \Phi_2b^2d - \Phi_2a^2d\big],$$

$$\alpha_{ti_2} = \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^3 - \\ - \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ac^2 - \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ad^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - \\ - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]cd^2 + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - \\ - H_2(\Phi_2)\Phi_1 + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]c^2d + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 + \\ + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]d^3 - \tfrac{1}{2}\big[\Phi_2a^2c + \Phi_2b^2c - \Phi_1b^2d - \Phi_1a^2d - 4ab^2 + 4a^3\big],$$

$$\alpha_{h_1j} = \tfrac{1}{96}\big[-H_2(H_2(\Phi_2) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + 7H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]c^3 - \\ + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^2d + \\ + \tfrac{1}{96}\big[-H_2(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + \tfrac{7}{16}H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]cd^2 + \\ + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]d^3 - \\ - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ac^2 - \tfrac{1}{8}\big[\frac{1}{8}H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2 - \Phi_2a^2c - \Phi_2b^2c + 2\Phi_1ce - \\ - 8be + 2\Phi_2de + \Phi_1b^2d + \Phi_1a^2d - 4ab^2 - 4a^3,$$

$$\alpha_{h_2j} = \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^3 + \\ - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + 8H_1(H_1(\Phi_2)) - \\ - 7H_2(H_1(\Phi_1))\big]d^3 - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - \\ - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]cd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + \\ + 8H_1(H_1(\Phi_2)) - 7H_2(H_1(\Phi_1))\big]c^2d + \Phi_1a^2c - 2\Phi_1de + \Phi_2b^2d + \Phi_2a^2d + 8ae4b^3 + \Phi_1b^2c - \\ - 4a^2b + 2\Phi_2ce,$$

Lastly, reminding Lemma 9.1, we must have (after possible simplifications) by identification:

$$\left[\begin{aligned}
\alpha_{ti_1} &= -2(\tfrac{1}{4}a^2d+\tfrac{1}{4}abc)\,\Phi_2 - 2(-\tfrac{1}{4}abd+\tfrac{1}{4}a^2c)\,\Phi_1 + (\tfrac{1}{24}d^3+\tfrac{1}{24}c^2d)\,\delta_3 - \\
&\quad - \frac{1}{4}(\tfrac{1}{8}d^2b+\tfrac{1}{8}bc^2)\,(H_1(\Phi_1)+H_2(\Phi_2)) + (\tfrac{1}{8}c^3+\tfrac{1}{8}cd^2)\,\delta_5 + (-\tfrac{1}{2}db^2+\tfrac{1}{2}bca)\,\Phi_2 - (\tfrac{1}{2}bda+\tfrac{1}{2}cb^2)\,\Phi_1 + \\
&\quad + (\tfrac{1}{4}c^3+\tfrac{1}{4}cd^2)\,\delta_{16} + (\tfrac{1}{4}c^2d+\tfrac{1}{4}d^3)\,\delta_{17} + \frac{3}{16}(\tfrac{1}{2}d^2b+\tfrac{1}{2}bc^2)\,(H_1(\Phi_1)+H_2(\Phi_2)) + 2a^2b + 2b^3, \\
\alpha_{ti_2} &= -2(\tfrac{1}{4}bda+\tfrac{1}{4}cb^2)\,\Phi_2 - 2(-\tfrac{1}{4}db^2+\tfrac{1}{4}bca)\,\Phi_1 + (\tfrac{1}{24}c^3+\tfrac{1}{24}cd^2)\,\delta_3 - \frac{1}{4}(-\tfrac{1}{8}ac^2-\tfrac{1}{8}ad^2)\,(H_1(\Phi_1) + \\
&\quad + H_2(\Phi_2)) + (-\tfrac{1}{8}c^2d-\tfrac{1}{8}d^3)\,\delta_5 + (-\tfrac{1}{2}a^2c+\tfrac{1}{2}bda)\,\Phi_2 + (\tfrac{1}{2}a^2d+\tfrac{1}{2}bca)\,\Phi_1 - (\tfrac{1}{4}c^2d+\tfrac{1}{4}d^3)\,\delta_{16} + \\
&\quad + (\tfrac{1}{4}c^3+\tfrac{1}{4}cd^2)\,\delta_{17} + \frac{3}{16}(-\tfrac{1}{2}ac^2-\tfrac{1}{2}d^2a)\,(H_1(\Phi_1)+H_2(\Phi_2)) - 2a^3 - 2ab^2, \\
\alpha_{h_1j} &= 2(de)\,\Phi_2 + 2(ce)\,\Phi_1 - (\tfrac{1}{6}c^3+\tfrac{1}{6}cd^2)\,\delta_3 - \frac{1}{4}(\tfrac{1}{2}ac^2+\tfrac{1}{2}d^2a)\,(H_1(\Phi_1)+H_2(\Phi_2)) + \\
&\quad + (\tfrac{1}{2}d^3+\tfrac{1}{2}c^2d)\,\delta_5 - (a^2c+cb^2)\,\Phi_2 + (a^2d+db^2)\,\Phi_1 - 8be - 4a^3 - 4ab^2, \\
\alpha_{h_2j} &= 2(ce)\,\Phi_2 - 2(ed)\,\Phi_1 + (\tfrac{1}{6}d^3+\tfrac{1}{6}c^2d)\,\delta_3 - \frac{1}{4}(\tfrac{1}{2}bc^2+\tfrac{1}{2}d^2b)\,(H_1(\Phi_1)+H_2(\Phi_2)) + \\
&\quad + (\tfrac{1}{2}c^3+\tfrac{1}{2}cd^2)\,\delta_5 + (db^2+da^2)\,\Phi_2 + (ca^2+cb^2)\,\Phi_1 - 4a^2b + 8ae - 4b^3.
\end{aligned}\right.$$

These equations will be satisfied if and only if the functions $\delta_\bullet$ are determined as follows:

$$\begin{aligned}
\delta_3 &= \tfrac{1}{2}H_1(H_1(\Phi_2)) - \tfrac{1}{16}\Phi_2H_2(\Phi_2) - \tfrac{7}{16}H_2(H_1(\Phi_1)) + \tfrac{1}{16}H_2(H_2(\Phi_2)) - \tfrac{1}{16}\Phi_2H_1(\Phi_1), \\
\delta_5 &= -\tfrac{1}{48}\Phi_1H_2(\Phi_2) - \tfrac{7}{48}H_1(H_2(\Phi_2)) + \tfrac{1}{48}H_1(H_1(\Phi_1)) - \tfrac{1}{48}\Phi_1H_1(\Phi_1) + \tfrac{1}{6}H_2(H_1(\Phi_2)), \\
\delta_{16} &= \tfrac{1}{32}\Phi_1H_1(\Phi_1) - \tfrac{1}{32}H_1(H_1(\Phi_1)) + \tfrac{1}{32}\Phi_1H_2(\Phi_2) - \tfrac{1}{32}H_1(H_2(\Phi_2)), \\
\delta_{17} &= \tfrac{1}{32}\Phi_2H_1(\Phi_1) - \tfrac{1}{32}H_2(H_2(\Phi_2)) + \tfrac{1}{32}\Phi_2H_2(\Phi_2) - \tfrac{1}{32}H_2(H_1(\Phi_1)).
\end{aligned}$$

9.11. **Homogeneity 4.** In this homogeneity, we see exactly five curvature coefficients:

$$\begin{aligned}
\kappa_j^{h_1h_2} &= \widehat{J}^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) - \widehat{H}_2(\alpha_{h_1j}) + \alpha_{h_1h_2}H_2(\alpha_{h_2j}) + \alpha_{h_1h_1}H_1(\alpha_{h_2j}) + \beta_{jh_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_1})}_\circ - \\
&\quad - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ + \\
&\quad + \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_1})}_\circ - \alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}}\big) + \beta_{jh_2}\big(-\alpha_{h_2d}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{h_2r}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} + \\
&\quad + \alpha_{h_1h_1}\underline{H_1(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_1}\underline{H_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_1h_2}\underline{H_2(\alpha_{h_2h_2})}_\circ - \alpha_{h_2h_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \\
&\quad - \alpha_{h_2i_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{h_2i_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{h_2j}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{jt}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2} - 4\alpha_{h_1h_2}\alpha_{h_2h_1}\big),
\end{aligned}$$

$$\begin{aligned}
\kappa_{i_1}^{h_1t} &= \widehat{I}_1^*([\widehat{H}_1,\widehat{T}]) - \widehat{T}(\alpha_{h_1i_1}) + \alpha_{h_1h_2}H_2(\alpha_{ti_1}) + \alpha_{h_1h_1}H_1(\alpha_{ti_1}) + \beta_{i_1h_1}\big(\alpha_{h_1h_1}\underline{H_1(\alpha_{th_1})}_\circ - \\
&\quad - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_1})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_1})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_1})}_\circ - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_1})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_1})}_\circ - \\
&\quad - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_1})}_{-\alpha_{h_1h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_1})}_{-\alpha_{h_2h_1}}\alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_1})}_\circ\big) + \beta_{i_1h_2}(-\alpha_{td}\underbrace{\widehat{D}(\alpha_{h_1h_2})}_{-\alpha_{h_1h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_1h_2})}_{-\alpha_{h_2h_2}} + \\
&\quad + \alpha_{h_1h_1}\underline{H_1(\alpha_{th_2})}_\circ - \alpha_{th_1}\underline{H_1(\alpha_{h_1h_2})}_\circ + \alpha_{h_1h_2}\underline{H_2(\alpha_{th_2})}_\circ - \alpha_{th_2}\underline{H_2(\alpha_{h_1h_2})}_\circ - \alpha_{tt}\underline{T(\alpha_{h_1h_2})}_\circ - \\
&\quad - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_1h_2})}_\circ - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_1h_2})}_\circ - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_1h_2})}_\circ\big) + \beta_{i_1t}\big(4\alpha_{h_1h_1}\alpha_{th_2} + \alpha_{h_1h_1}\alpha_{tt}\Phi_1 - \\
&\quad - 4\alpha_{h_1h_2}\alpha_{th_1} + \alpha_{h_1h_2}\alpha_{tt}\Phi_2\big),
\end{aligned}$$

$$
\begin{aligned}
\kappa_{i_2}^{h_1 t} &= \widehat{I}_2^*([\widehat{H}_1, \widehat{T}]) = -\widehat{T}(\alpha_{h_1 i_2}) + \alpha_{h_1 h_2} H_2(\alpha_{t i_2}) + \alpha_{h_1 h_1} H_1(\alpha_{t i_2}) + \beta_{i_2 h_1}\big(\alpha_{h_1 h_1} \underline{H_1(\alpha_{t h_1})}_\circ - \\
&- \alpha_{t h_1} \underline{H_1(\alpha_{h_1 h_1})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_1 h_1})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_1 h_1})}_\circ - \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_1 h_1})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_1 h_1})}_\circ - \\
&- \alpha_{td} \underbrace{\widehat{D}(\alpha_{h_1 h_1})}_{-\alpha_{h_1 h_1}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_1 h_1})}_{-\alpha_{h_2 h_1}} \alpha_{tj} \underline{\widehat{J}(\alpha_{h_1 h_1})}_\circ\big) + \beta_{i_2 h_2}(-\alpha_{td} \underbrace{\widehat{D}(\alpha_{h_1 h_2})}_{-\alpha_{h_1 h_2}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_1 h_2})}_{-\alpha_{h_2 h_2}} + \\
&+ \alpha_{h_1 h_1} \underline{H_1(\alpha_{t h_2})}_\circ - \alpha_{t h_1} \underline{H_1(\alpha_{h_1 h_2})}_\circ + \alpha_{h_1 h_2} \underline{H_2(\alpha_{t h_2})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_1 h_2})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_1 h_2})}_\circ - \\
&- \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_1 h_2})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_1 h_2})}_\circ - \alpha_{tj} \underline{\widehat{J}(\alpha_{h_1 h_2})}_\circ\big) + \beta_{i_2 t}\big(4\alpha_{h_1 h_1} \alpha_{t h_2} + \alpha_{h_1 h_1} \alpha_{tt} \Phi_1 - \\
&- 4\alpha_{h_1 h_2} \alpha_{t h_1} + \alpha_{h_1 h_2} \alpha_{tt} \Phi_2\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{i_1}^{h_2 t} &= \widehat{I}_1^*([\widehat{H}_2, \widehat{T}]) - \widehat{T}(\alpha_{h_2 i_1}) + \alpha_{h_2 h_2} H_2(\alpha_{t i_1}) + \alpha_{h_2 h_1} H_1(\alpha_{t i_1}) + \beta_{i_1 h_1}\big(\alpha_{h_2 h_1} \underline{H_1(\alpha_{t h_1})}_\circ - \\
&- \alpha_{t h_1} \underline{H_1(\alpha_{h_2 h_1})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_2 h_1})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_2 h_1})}_\circ - \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_2 h_1})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_2 h_1})}_\circ - \\
&- \alpha_{td} \underbrace{\widehat{D}(\alpha_{h_2 h_1})}_{-\alpha_{h_2 h_1}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_2 h_1})}_{\alpha_{h_1 h_1}} - \alpha_{tj} \underline{\widehat{J}(\alpha_{h_2 h_1})}_\circ + \alpha_{h_2 h_1} \underline{H_2(\alpha_{t h_1})}_\circ\big) + \beta_{i_1 h_2}(\alpha_{h_2 h_1} \underline{H_1(\alpha_{t h_2})}_\circ - \\
&- \alpha_{t h_1} \underline{H_1(\alpha_{h_2 h_2})}_\circ + \alpha_{h_2 h_2} \underline{H_2(\alpha_{t h_2})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_2 h_2})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_2 h_2})}_\circ - \\
&- \alpha_{td} \underbrace{\widehat{D}(\alpha_{h_2 h_2})}_{-\alpha_{h_2 h_2}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_2 h_2})}_{\alpha_{h_1 h_2}} - \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_2 h_2})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_2 h_2})}_\circ - \alpha_{tj} \underline{\widehat{J}(\alpha_{h_2 h_2})}_\circ\big) + \\
&+ \beta_{i_1 t}\big(4\alpha_{h_2 h_1} \alpha_{t h_2} + \alpha_{h_2 h_1} \alpha_{tt} \Phi_1 - 4\alpha_{h_2 h_2} \alpha_{t h_1} + \alpha_{h_2 h_2} \alpha_{tt} \Phi_2\big),
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{i_2}^{h_2 t} &= \widehat{I}_2^*([\widehat{H}_2, \widehat{T}]) = -\widehat{T}(\alpha_{h_2 i_2}) + \alpha_{h_2 h_2} H_2(\alpha_{t i_2}) + \alpha_{h_2 h_1} H_1(\alpha_{t i_2}) + \beta_{i_2 h_1}\big(\alpha_{h_2 h_1} \underline{H_1(\alpha_{t h_1})}_\circ - \\
&- \alpha_{t h_1} \underline{H_1(\alpha_{h_2 h_1})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_2 h_1})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_2 h_1})}_\circ - \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_2 h_1})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_2 h_1})}_\circ - \\
&- \alpha_{td} \underbrace{\widehat{D}(\alpha_{h_2 h_1})}_{-\alpha_{h_2 h_1}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_2 h_1})}_{\alpha_{h_1 h_1}} - \alpha_{tj} \underline{\widehat{J}(\alpha_{h_2 h_1})}_\circ + \alpha_{h_2 h_1} \underline{H_2(\alpha_{t h_1})}_\circ\big) + \beta_{i_2 h_2}(\alpha_{h_2 h_1} \underline{H_1(\alpha_{t h_2})}_\circ - \\
&- \alpha_{t h_1} \underline{H_1(\alpha_{h_2 h_2})}_\circ + \alpha_{h_2 h_2} \underline{H_2(\alpha_{t h_2})}_\circ - \alpha_{t h_2} \underline{H_2(\alpha_{h_2 h_2})}_\circ - \alpha_{tt} \underline{T(\alpha_{h_2 h_2})}_\circ - \\
&- \alpha_{td} \underbrace{\widehat{D}(\alpha_{h_2 h_2})}_{-\alpha_{h_2 h_2}} - \alpha_{tr} \underbrace{\widehat{R}(\alpha_{h_2 h_2})}_{\alpha_{h_1 h_2}} - \alpha_{t i_1} \underline{\widehat{I}_1(\alpha_{h_2 h_2})}_\circ - \alpha_{t i_2} \underline{\widehat{I}_2(\alpha_{h_2 h_2})}_\circ - \alpha_{tj} \underline{\widehat{J}(\alpha_{h_2 h_2})}_\circ\big) + \\
&+ \beta_{i_2 t}\big(4\alpha_{h_2 h_1} \alpha_{t h_2} + \alpha_{h_2 h_1} \alpha_{tt} \Phi_1 - 4\alpha_{h_2 h_2} \alpha_{t h_1} + \alpha_{h_2 h_2} \alpha_{tt} \Phi_2\big),
\end{aligned}
$$

As for the preceding homogeneities, it is possible to simplify further these curvature coefficients by applying the horizontal vector fields $Y = H_1, H_2, T$ on the expressions of $\alpha_{h_1 j}$, $\alpha_{h_2 j}$, $\alpha_{t i_1}$, $\alpha_{t i_2}$ already determined. For this, we employ the following equality from elementary differential geometry:

$$
Y(fg) = fY(g) + gY(f), \qquad {\scriptstyle f,g \in C^\infty(M)}.
$$

Moreover, we can compute the values of the vector fields $\widehat{H}_1$, $\widehat{H}_2$, $\widehat{T}$ on the functions $\alpha_{\bullet\bullet}$ that are visible in the above curvature coefficients. After that, we will be ready to simplify the expressions of the curvature coefficients of homogeneity four. The obtained expressions are a bit too long for them to appear here, *see* instead [2]. On the other hand, inspection of the set of functions $\alpha_{\bullet\bullet}$ shows that the only undetermined one is α_{tj}. This function appears in the expressions of β_{jt} (*see* subsection 9.3), visible in $\kappa_j^{h_1 h_2}$ and then we may annihilate at least this curvature coefficient by choosing:

$$\begin{aligned}
\alpha_{tj} &= 3a^4+3b^4-4e^2-\Phi_1a^2bc+ca\Phi_2b^2-\Phi_1ab^2d-\Phi_2a^2bd-2\Phi_2bce-2\Phi_1ace-2\Phi_2ade+2\Phi_1bde-\\
&-\Phi_1a^3d+\Phi_2a^3c-\Phi_1b^3c-\Phi_2b^3d+6a^2b^2+\big[\tfrac{3}{16}H_1(\Phi_1)+\tfrac{3}{16}H_2(\Phi_2)\big]b^2d^2+\\
&+\big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1-\tfrac{11}{3072}H_2({\Phi_2}^2)+\tfrac{1}{384}{\Phi_2}^2H_2(\Phi_2)-\tfrac{11}{3072}H_1(\Phi_1^2)+\\
&+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))+\tfrac{1}{384}{\Phi_2}^2H_1(\Phi_1)-\\
&-\tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{384}{\Phi_1}^2H_2(\Phi_2)-\\
&-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]d^4+\\
&+\big[-\tfrac{11}{768}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{7}{192}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{192}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{192}H_1(H_1(H_1(\Phi_1)))+\\
&+\tfrac{1}{24}H_1(H_2(H_1(\Phi_2)))-\tfrac{1}{96}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{32}H_1(H_2(\Phi_2))\Phi_1+\tfrac{1}{192}\Phi_2^2H_1(\Phi_1)-\tfrac{7}{192}H_1(H_1(H_2(\Phi_2)))+\\
&+\tfrac{1}{192}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{1536}H_1(\Phi_1^2)-\tfrac{1}{24}\Phi_2H_1(H_1(\Phi_2))-\tfrac{11}{1536}H_2(\Phi_2^2)+\tfrac{1}{32}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{96}H_1(H_1(\Phi_1))\Phi_1+\\
&+\tfrac{1}{192}\Phi_1^2H_2(\Phi_2)+\tfrac{1}{192}\Phi_1^2H_1(\Phi_1)-\tfrac{1}{24}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{24}H_2(H_1(H_1(\Phi_2)))\big]c^2d^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\\
&+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bcd^2+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\\
&-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]acd^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\\
&+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]ad^3+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]ac^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2d^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]bd^3+\\
&+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bc^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2c^2+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]b^2c^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-\\
&-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]dbc^2+\tfrac{1}{32}\big[-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))+H_1(\Phi_1)\Phi_1\big]ac^2d+\\
&+\big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1-\tfrac{11}{3072}H_2(\Phi_2^2)+\tfrac{1}{384}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{3072}H_1(\Phi_1^2)+\\
&+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))+\tfrac{1}{384}\Phi_2^2H_1(\Phi_1)-\\
&-\tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{384}\Phi_1^2H_2(\Phi_2)-\\
&-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]c^4.
\end{aligned}$$

However, this choice does not annihilate the remaining four curvature coefficients. By a careful examination (either by hand or with the help of a computer), we realize that the remaining four curvature coefficients have the following expressions:

$$\begin{aligned}
\kappa_{i_1}^{h_1t} &= \mathbf{\Delta_1}d^4+(\mathbf{\Delta_2}-\mathbf{\Delta_1})c^4+\mathbf{\Delta_2}c^2d^2+\mathbf{\Delta_3}c^3d+\mathbf{\Delta_3}cd^3,\\
\kappa_{i_2}^{h_1t} &= \mathbf{\Delta_4}d^4+(\mathbf{\Delta_3}+\mathbf{\Delta_4})c^4+(\mathbf{\Delta_3}+2\mathbf{\Delta_4})c^2d^2+(2\mathbf{\Delta_1}-\mathbf{\Delta_2})c^3d+(2\mathbf{\Delta_1}-\mathbf{\Delta_2})cd^3,\\
\kappa_{i_1}^{h_2t} &= -\kappa_{i_2}^{h_1t}-\widehat{R}(\kappa_{i_1}^{h_1t}),\\
\kappa_{i_2}^{h_2t} &= \kappa_{i_1}^{h_1t}-\widehat{R}(\kappa_{i_2}^{h_1t}),
\end{aligned}$$

where:

$$\begin{aligned}
\mathbf{\Delta_1} := \tfrac{1}{384}\big[&-20\Phi_2H_1(H_1(\Phi_2))-(H_1(\Phi_1))^2-2\Phi_2^2H_1(\Phi_1)+8H_1(H_2(H_1(\Phi_2)))+2\Phi_1^2H_1(\Phi_1)-\\
&-7H_1(H_1(H_2(\Phi_2)))-4\Phi_1H_2(H_1(\Phi_2))+H_1(H_1(H_1(\Phi_1)))+\Phi_1H_1(H_2(\Phi_2))+\\
&+23\Phi_2H_2(H_1(\Phi_1))+(H_2(\Phi_2))^2-3\Phi_1H_1(H_1(\Phi_1))+3\Phi_2H_2(H_2(\Phi_2))-2\Phi_2^2H_2(\Phi_2)-\\
&-17H_2(H_2(H_1(\Phi_1)))+2\Phi_1^2H_2(\Phi_2)+16H_2(H_1(H_1(\Phi_2)))-H_2(H_2(H_2(\Phi_2)))\big],
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Delta_2} := \tfrac{1}{384}\big[&24H_1(H_2(H_1(\Phi_2)))-24\Phi_1H_2(H_1(\Phi_2))+24\Phi_1H_1(H_2(\Phi_2))+24H_2(H_1(H_1(\Phi_2)))-\\
&-24H_1(H_1(H_2(\Phi_2)))-24\Phi_2H_1(H_1(\Phi_2))+24\Phi_2H_2(H_1(\Phi_1))-24H_2(H_2(H_1(\Phi_1)))\big],
\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_3} := \tfrac{1}{384}\big[& -2H_2(H_1(H_1(\Phi_1))) + 8H_1(H_1(H_1(\Phi_2))) - 2\Phi_1\Phi_2H_1(\Phi_1) - 8\Phi_1\Phi_2H_2(\Phi_2)- \\ & -2H_1(H_2(H_2(\Phi_2))) - 10H_2(H_1(H_2(\Phi_2))) - 16\Phi_1H_1(H_1(\Phi_2)) + 8H_1(\Phi_2)H_2(\Phi_2)+ \\ & +6\Phi_1H_2(H_2(\Phi_2)) + 8H_2(H_2(H_1(\Phi_2))) + 22\Phi_2H_1(H_2(\Phi_2)) - 16\Phi_2H_2(H_1(\Phi_2))+ \\ & +22\Phi_1H_2(H_1(\Phi_1)) - 10H_1(H_2(H_1(\Phi_1))) + 4H_1(\Phi_1)H_1(\Phi_2) + 6\Phi_2H_1(H_1(\Phi_1))\big], \\ \mathbf{\Delta_4} := \tfrac{1}{384}\big[& 4\Phi_1H_1(H_1(\Phi_2)) - 2H_1(\Phi_2)H_2(\Phi_2) - 2H_1(\Phi_1)H_1(\Phi_2) + 13H_2(H_1(H_2(\Phi_2)))- \\ & -3H_1(H_2(H_2(\Phi_2))) - 3\Phi_2H_1(H_1(\Phi_1)) - 15\Phi_2H_1(H_2(\Phi_2)) + 4\Phi_1\Phi_2H_1(\Phi_1)- \\ & -8H_2(H_2(H_1(\Phi_2))) - 3H_1(H_2(H_1(\Phi_1))) + 12\Phi_2H_2(H_1(\Phi_2)) - 3\Phi_1H_2(H_2(\Phi_2))- \\ & -7\Phi_1H_2(H_1(\Phi_1)) + 4\Phi_1\Phi_2H_2(\Phi_2) + 5H_2(H_1(H_1(\Phi_1)))\big].\end{aligned}$$

Lemma 9.2. *One in fact has, identically as functions of* (x, y, u)*:*

$$\boxed{0 \equiv \Delta_2} \qquad \text{and} \qquad \boxed{0 \equiv \Delta_3 + 2\,\Delta_4}.$$

Proof. These two nontrivial relations were already prepared in advance, *cf.* the Corollary 7.2. □

Furthermore, by taking account of the relations listed in Proposition 7.1 and of $H_1(\Phi_2) = H_2(\Phi_1)$, one sees that the expressions of the two remaining functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$ of (x, y, u) can be given better, completely symmetric forms, as is stated by the following summarizing proposition.

Proposition 9.3. *The four remaining curvature coefficients of homogeneity* $h = 4$ *express explicitly as follows:*

$$\begin{aligned}\kappa_{i_1}^{h_1 t} &= -\,\mathbf{\Delta_1}\,c^4 - 2\,\mathbf{\Delta_4}\,c^3d - 2\,\mathbf{\Delta_4}\,cd^3 + \mathbf{\Delta_1}\,d^4, \\ \kappa_{i_2}^{h_1 t} &= -\,\mathbf{\Delta_4}\,c^4 + 2\,\mathbf{\Delta_1}\,c^3d + 2\,\mathbf{\Delta_1}\,cd^3 + \mathbf{\Delta_4}\,d^4, \\ \kappa_{i_1}^{h_2 t} &= \kappa_{i_2}^{h_1 t}, \\ \kappa_{i_2}^{h_2 t} &= -\,\kappa_{i_1}^{h_1 t},\end{aligned}$$

where the two functions $\mathbf{\Delta_1}$ *and* $\mathbf{\Delta_4}$ *of the three horizontal variables* (x, y, u) *have the following explicit expressions:*

$$\boxed{\begin{aligned}\mathbf{\Delta_1} = \tfrac{1}{384}\Big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\,H_1(H_2(H_1(\Phi_2))) - 11\,H_2(H_1(H_2(\Phi_1)))+ \\ & +6\,\Phi_2\,H_2(H_1(\Phi_1)) - 6\,\Phi_1\,H_1(H_2(\Phi_2)) - 3\,\Phi_2\,H_1(H_1(\Phi_2)) + 3\,\Phi_1\,H_2(H_2(\Phi_1))- \\ & -\,3\,\Phi_1\,H_1(H_1(\Phi_1)) + 3\,\Phi_2\,H_2(H_2(\Phi_2)) - 2\,\Phi_1\,H_1(\Phi_1) + 2\,\Phi_2\,H_2(\Phi_2)- \\ & -\,2\,(\Phi_2)^2\,H_1(\Phi_1) + 2\,(\Phi_1)^2\,H_2(\Phi_2) - 2\,(\Phi_2)^2\,H_2(\Phi_2) + 2\,(\Phi_1)^2\,H_1(\Phi_1)\Big], \\ \mathbf{\Delta_4} = \tfrac{1}{384}\Big[& -\,3\,H_2(H_1(H_2(\Phi_2))) - 3\,H_1(H_2(H_1(\Phi_1))) + 5\,H_1(H_2(H_2(\Phi_2))) + 5\,H_2(H_1(H_1(\Phi_1)))+ \\ & +4\,\Phi_1\,H_1(H_1(\Phi_2)) + 4\,\Phi_2\,H_2(H_1(\Phi_2)) - 3\,\Phi_2\,H_1(H_1(\Phi_1)) - 3\,\Phi_1\,H_2(H_2(\Phi_2))- \\ & -\,7\,\Phi_2\,H_1(H_2(\Phi_2)) - 7\,\Phi_1\,H_2(H_1(\Phi_1)) - 2\,H_1(\Phi_1)\,H_1(\Phi_2) - 2\,H_2(\Phi_2)\,H_2(\Phi_1)+ \\ & +4\,\Phi_1\Phi_2\,H_1(\Phi_1) + 4\,\Phi_1\Phi_2\,H_2(\Phi_2)\Big].\end{aligned}}$$

Proof. As said, one uses the relations listed in Proposition 7.1 until formal expressions show up symmetries. □

In the next subsection, we will establish that the obtained Cartan connection is actually normal. Up to know, all the functions $\alpha_{\bullet\bullet}$ are determined, but still there is last function of type $\delta_{\bullet}$, namely δ_{18} which is yet undetermined. To determine it it suffices to equate the above expression of α_{tj} to the corresponding one in Lemma 9.1, after making the possible simplification:

$$\left[\begin{aligned}\alpha_{tj} &= (c^4+2c^2d^2+d^4)\delta_{18}+6a^2b^2-2\Phi_2bce-2\Phi_1ace-\Phi_1a^2bc+\Phi_2ab^2c-\Phi_2a^2bd-2\Phi_2ade-\\ &-\Phi_1ab^2d+2\Phi_1bde+3a^4+3b^4-4e^2-\Phi_1b^3c+\Phi_2a^3c-\Phi_1a^3d-\Phi_2b^3d+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2c^2+\\ &+\tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_2+H_2(H_1(\Phi_1))+H_2(H_2(\Phi_2))-H_2(\Phi_2)\Phi_2\big]ac^3+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]b^2c^2+\\ &+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))\big]bc^3+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]b^2d^2+\\ &+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2d^2+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_2-H_2(H_2(\Phi_2))+H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))\big]bd^3+\\ &+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))\big]ad^3+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_2-H_2(H_2(\Phi_2))+\\ &+H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))\big]bc^2d+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))\big]ac^2d+\\ &+\tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_2+H_2(H_1(\Phi_1))+H_2(H_2(\Phi_2))-H_2(\Phi_2)\Phi_2\big]acd^2+\tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1-\\ &-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))\big]bcd^2,\end{aligned}\right.$$

By identification, this equation holds when one makes the following assignment:

$$\begin{aligned}\delta_{18} &= \tfrac{1}{64}\Phi_2H_2(H_1(\Phi_1))-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}\Phi_1H_1(H_1(\Phi_1))-\tfrac{11}{3072}H_1(\Phi_1^2)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))-\\ &-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))+\tfrac{1}{384}\Phi_2^2H_2(\Phi_2)-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))-\tfrac{1}{192}\Phi_2H_2(H_2(\Phi_2))+\tfrac{1}{64}\Phi_1H_1(H_2(\Phi_2))+\\ &+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)-\tfrac{11}{3072}H_2(\Phi_2^2)+\tfrac{1}{384}\Phi_2^2H_1(\Phi_1)-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\\ &+\tfrac{1}{384}\Phi_1^2H_2(\Phi_2)+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1))).\end{aligned}$$

9.12. **Homogeneity 5.** In this homogeneity, it is possible to express curvature coefficients in terms of the curvature coefficients of lower homogeneities. More precisely, consider the restricted h-homogeneous differential operators:

$$\partial_{[h]}\colon\quad \mathscr{C}^2_{[h]}(\mathfrak{g}_-,\mathfrak{g})\to\mathscr{C}^3_{[h]}(\mathfrak{g}_-,\mathfrak{g}).$$

We will use the graded Bianchi-Tanaka identities of Proposition 8.12 to identify $\partial\kappa^{(5)}$ in terms of the lower components of κ.

In homogeneity 5, we encounter two curvature coefficients $\kappa_j^{h_1t}$ and $\kappa_j^{h_2t}$ and according to the notations introduced in Section 5, we have:

$$\kappa_{[5]}=\kappa_j^{h_1t}\,\mathsf{h}_1^*\wedge\mathsf{t}^*\otimes\mathsf{j}+\kappa_j^{h_2t}\,\mathsf{h}_2^*\wedge\mathsf{t}^*\otimes\mathsf{j}. \tag{36}$$

Applying the differential gives:

$$\begin{aligned}\partial\kappa_{[5]}(\mathsf{h}_1,\mathsf{h}_2,\mathsf{t}) &= \big[\mathsf{h}_1,\underbrace{\kappa_{[5]}(\mathsf{h}_2,\mathsf{t})}_{\kappa_j^{h_2t}\mathsf{j}}\big]-\big[\mathsf{h}_2,\underbrace{\kappa_{[5]}(\mathsf{h}_1,\mathsf{t})}_{\kappa_j^{h_1t}\mathsf{j}}\big]+\big[\mathsf{t},\underline{\kappa_{[5]}(\mathsf{h}_1,\mathsf{h}_2)}_{\circ}\big]-\\ &-\big[\kappa_{[5]}(\underbrace{[\mathsf{h}_1,\mathsf{h}_2]}_{\mathsf{t}},\mathsf{t})\big]+\big[\kappa_{[5]}(\underline{[\mathsf{h}_1,\mathsf{t}]}_{\circ},\mathsf{h}_2)\big]-\big[\kappa_{[5]}(\underline{[\mathsf{h}_2,\mathsf{t}]}_{\circ},\mathsf{h}_1)\big]\\ &=\kappa_j^{h_2t}\,\mathsf{i}_1-\kappa_j^{h_1t}\,\mathsf{i}_2+0+0+0+0.\end{aligned}\tag{37}$$

On the other hand, the graded Bianchi-Tanaka identities of Proposition 8.12 assert that:

$$\partial\kappa_{[5]}(\mathsf{h}_1,\mathsf{h}_2,\mathsf{t}) = \underbrace{-\sum_{j=1}^{4}\kappa_{[5-j]}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[j]}(\mathsf{h}_1,\mathsf{h}_2)\big),\mathsf{t}\Big) - \big(\widehat{T}\underline{\kappa_{[3]}}_{\circ}\big)(\mathsf{h}_1,\mathsf{h}_2) -}_{-P_1}$$
$$\underbrace{-\sum_{j=1}^{4}\kappa_{[5-j]}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[j]}(\mathsf{t},\mathsf{h}_1)\big),\mathsf{h}_2\Big) - \big(\widehat{H}_2\kappa_{[4]}\big)(\mathsf{t},\mathsf{h}_1) -}_{-P_2}$$
$$\underbrace{-\sum_{j=1}^{4}\kappa_{[5-j]}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[j]}(\mathsf{h}_2,\mathsf{t})\big),\mathsf{h}_1\Big) - \big(\widehat{H}_1\kappa_{[4]}\big)(\mathsf{h}_2,\mathsf{t})\,.}_{-P_3}$$

Let us compute for example the last term P_3, taking account of the vanishing of the curvature components $\kappa_{[1]}$, $\kappa_{[2]}$ and $\kappa_{[3]}$. At first we compute its $\sum_{j=1}^{4}$ part:

$$\boxed{j=1}\qquad \kappa_{[4]}\Big(\underline{\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[1]}(\mathsf{h}_2,\mathsf{t})\big)}_{\circ},\ \mathsf{h}_1\Big) = 0,$$
$$\boxed{j=2}\qquad \kappa_{[3]}\Big(\underline{\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[2]}(\mathsf{h}_2,\mathsf{t})\big)}_{\circ},\ \mathsf{h}_1\Big) = 0,$$
$$\boxed{j=3}\qquad \kappa_{[2]}\Big(\underline{\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[3]}(\mathsf{h}_2,\mathsf{t})\big)}_{\circ},\ \mathsf{h}_1\Big) = 0,$$
$$\boxed{j=4}\qquad \underline{\kappa_{[1]}}_{\circ}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa_{[4]}(\mathsf{h}_2,\mathsf{t})\big),\mathsf{h}_1\Big) = 0.$$

So the $\sum_{j=1}^{4}$ part of P_3 is zero. For the remaining part, reminding that:

(38)

$$\kappa_{[4]} = \kappa_{i_1}^{h_1t}\,\mathsf{h}_1^*\wedge\mathsf{t}^*\otimes\mathsf{i}_1 + \kappa_{i_1}^{h_2t}\,\mathsf{h}_2^*\wedge\mathsf{t}^*\otimes\mathsf{i}_1 + \kappa_{i_2}^{h_1t}\,\mathsf{h}_1^*\wedge\mathsf{t}^*\otimes\mathsf{i}_2 + \kappa_{i_2}^{h_2t}\,\mathsf{h}_2^*\wedge\mathsf{t}^*\otimes\mathsf{i}_2,$$

it is clear that:

$$\kappa_{[4]}(\mathsf{h}_2,\mathsf{t}) = \kappa_{i_1}^{h_2t}\,\mathsf{i}_1 + \kappa_{i_2}^{h_2t}\,\mathsf{i}_2,$$

whence:

$$\big(\widehat{H}_1\kappa_{[4]}\big)(\mathsf{h}_2,\mathsf{t}) = \widehat{H}_1\big(\kappa_{i_1}^{h_2t}\big)\,\mathsf{i}_1 + \widehat{H}_1\big(\kappa_{i_2}^{h_2t}\big)\,\mathsf{i}_2,$$

which is the expression of P_3. Similar computations provide:

$$P_1 = 0, \qquad P_2 = -\widehat{H}_2\big(\kappa_{i_1}^{h_1t}\big)\,\mathsf{i}_1 - \widehat{H}_2\big(\kappa_{i_2}^{h_1t}\big)\,\mathsf{i}_2,$$

and consequently:

(39)

$$\begin{aligned}\partial\kappa_{[5]}(\mathsf{h}_1,\mathsf{h}_2,\mathsf{t}) &= -\big(P_1+P_2+P_3\big)\\ &= \Big(\widehat{H}_2\big(\kappa_{i_1}^{h_1t}\big) - \widehat{H}_1\big(\kappa_{i_1}^{h_2t}\big)\Big)\,\mathsf{i}_1 + \Big(\widehat{H}_2\big(\kappa_{i_2}^{h_1t}\big) - \widehat{H}_1\big(\kappa_{i_2}^{h_2t}\big)\Big)\,\mathsf{i}_2.\end{aligned}$$

Now comparison of (37) and (39) implies that:

$$\begin{aligned}\kappa_j^{h_1t} &= \widehat{H}_1\big(\kappa_{i_2}^{h_2t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1t}\big),\\ \kappa_j^{h_2t} &= -\widehat{H}_1\big(\kappa_{i_1}^{h_2t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1t}\big).\end{aligned}$$

9.13. **Conclusion.** A review of the results obtained so far shows that the only non-zero curvature coefficients are:

$$\boxed{\text{Hom 4}} \quad \kappa_{i_1}^{h_1 t}, \quad \kappa_{i_2}^{h_1 t}, \quad \kappa_{i_1}^{h_2 t}, \quad \kappa_{i_2}^{h_2 t};$$

$$\boxed{\text{Hom 5}} \quad \kappa_{j}^{h_1 t}, \quad \kappa_{j}^{h_2 t}.$$

All these curvature coefficients can be expressed as the combinations of $\kappa_{i_1}^{h_1 t}$ and $\kappa_{i_2}^{h_1 t}$ and the values of the constant vector fields on them. These two curvature coefficients are called *essential curvatures*. A Cartan geometry is homogeneous if and only if all of its essential curvatures vanish ([37, 14]). Hence a consequence of our results is the following

Theorem 9.1. *The Cartan geometry associated to any strongly pseudoconvex deformation $M^3 \subset \mathbb{C}^2$ the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$ having curvature function equal to:*

$$\begin{aligned}
\kappa &= \kappa_{[4]} + \kappa_{[5]} \\
&= \kappa_{i_1}^{h_1 t}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1 t}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2 t}\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \\
&\quad + \kappa_{i_2}^{h_2 t}\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1 t}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t}\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},
\end{aligned} \tag{40}$$

with:

$$\begin{aligned}
\kappa_{i_1}^{h_2 t} &= \kappa_{i_2}^{h_1 t}, \\
\kappa_{i_2}^{h_2 t} &= -\,\kappa_{i_1}^{h_1 t}, \\
\kappa_{j}^{h_1 t} &= \widehat{H}_1\big(\kappa_{i_2}^{h_2 t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1 t}\big), \\
\kappa_{j}^{h_2 t} &= -\widehat{H}_1\big(\kappa_{i_1}^{h_2 t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1 t}\big),
\end{aligned}$$

is locally homogeneous if and only if its two essential curvatures:

$$\begin{aligned}
\kappa_{i_1}^{h_1 t} &= -\,\mathbf{\Delta_1}\, c^4 - 2\,\mathbf{\Delta_4}\, c^3 d - 2\,\mathbf{\Delta_4}\, c d^3 + \mathbf{\Delta_1}\, d^4, \\
\kappa_{i_2}^{h_1 t} &= -\,\mathbf{\Delta_4}\, c^4 + 2\,\mathbf{\Delta_1}\, c^3 d + 2\,\mathbf{\Delta_1}\, c d^3 + \mathbf{\Delta_4}\, d^4
\end{aligned}$$

vanish identically; equivalently, the following two explicit *functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$ of only the three horizontal variables (x, y, u):*

$$\begin{aligned}
\mathbf{\Delta_1} = \tfrac{1}{384}\Big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\, H_1(H_2(H_1(\Phi_2))) - 11\, H_2(H_1(H_2(\Phi_1))) + \\
& + 6\,\Phi_2\, H_2(H_1(\Phi_1)) - 6\,\Phi_1\, H_1(H_2(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_2)) + 3\,\Phi_1\, H_2(H_2(\Phi_1)) - \\
& -\, 3\,\Phi_1\, H_1(H_1(\Phi_1)) + 3\,\Phi_2\, H_2(H_2(\Phi_2)) - 2\,\Phi_1\, H_1(\Phi_1) + 2\,\Phi_2\, H_2(\Phi_2) - \\
& -\, 2\,(\Phi_2)^2\, H_1(\Phi_1) + 2\,(\Phi_1)^2\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_2(\Phi_2) + 2\,(\Phi_1)^2\, H_1(\Phi_1)\Big],
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Delta_4} = \tfrac{1}{384}\Big[& -\, 3\, H_2(H_1(H_2(\Phi_2))) - 3\, H_1(H_2(H_1(\Phi_1))) + 5\, H_1(H_2(H_2(\Phi_2))) + 5\, H_2(H_1(H_1(\Phi_1))) + \\
& + 4\,\Phi_1\, H_1(H_1(\Phi_2)) + 4\,\Phi_2\, H_2(H_1(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_1)) - 3\,\Phi_1\, H_2(H_2(\Phi_2)) - \\
& -\, 7\,\Phi_2\, H_1(H_2(\Phi_2)) - 7\,\Phi_1\, H_2(H_1(\Phi_1)) - 2\, H_1(\Phi_1)\, H_1(\Phi_2) - 2\, H_2(\Phi_2)\, H_2(\Phi_1) + \\
& + 4\,\Phi_1\Phi_2\, H_1(\Phi_1) + 4\,\Phi_1\Phi_2\, H_2(\Phi_2)\Big]
\end{aligned}$$

vanish identically. □

Up to now, we have achieved all the necessary computations. We saw that the regularity condition **(c4)** is satisfied automatically and condition **(c1)** holds by applying Lemma 9.1 in the computations of various homogeneities. The only remaining task is to verify that both the isomorphism condition **(c2)** and the normality condition **(c3)** hold. We inspect these two final conditions via the following propositions, respectively:

Proposition 9.4. *For any element $p = (a, b, c, d, e, x, y, u)$ of $\mathscr{G}$, the $\mathfrak{g}$-valued map $\omega_p \colon T_p\mathscr{P} \longrightarrow \mathfrak{g}$ is an isomorphism.*

Proof. According to the expressions of $\widehat{T}, \dots, \widehat{J}$ as components of ω_p^{-1}, the matrix corresponding to ω_p^{-1} is:

$$\begin{pmatrix} \alpha_{tt} & \alpha_{th_1} & \alpha_{th_2} & \alpha_{td} & \alpha_{tr} & \alpha_{ti_1} & \alpha_{ti_2} & \alpha_{tj} \\ 0 & \alpha_{h_1h_1} & \alpha_{h_1h_2} & \alpha_{h_1d} & \alpha_{h_1r} & \alpha_{h_1i_1} & \alpha_{h_1i_2} & \alpha_{h_1j} \\ 0 & \alpha_{h_2h_1} & \alpha_{h_2h_2} & \alpha_{h_2d} & \alpha_{h_2r} & \alpha_{h_2i_1} & \alpha_{h_2i_2} & \alpha_{h_2j} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and it is has determinant:

$$\alpha_{tt}\big(\alpha_{h_1h_1}\alpha_{h_2h_2} - \alpha_{h_2h_1}\alpha_{h_1h_2}\big) = (c^2 + d^2)^2,$$

which is nonzero by assumption. □

Proposition 9.5. *The Cartan connection constructed in the preceding paragraphs in a completely effective way is normal.*

Proof. According to (40), the t-, h_1-, h_2-, d- and r-components of the Cartan curvature κ vanish together. Vanishing of its t, h_1 and h_2-components means that this curvature is torsion free. Moreover, the d- and r-components of κ constitute its $\mathfrak{g}_0$-component and consequently $\kappa_{[0]} \equiv 0$ by construction. Therefore the Cartan connection is normal according to Definition 1.6.7 page 128 of [9]. □

10. General Formulas for the Second Cohomology of Graded Lie Algebras

Throughout cohomology considerations, the ground field $\mathbb{K}$ will be a commutative field of characteristic zero, while in most expected applications to exterior differential systems, $\mathbb{K}$ will be either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

10.1. **Arbitrary abstract Lie algebra.**

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ of dimension $r \geqslant 2$ containing a Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ of dimension n with $1 \leqslant n \leqslant r-1$, so that $[\mathfrak{g}_-, \mathfrak{g}_-]_{\mathfrak{g}} \subset \mathfrak{g}_-$. Let $\mathsf{x}_1, \dots, \mathsf{x}_n$ be an arbitrary but fixed basis of $\mathfrak{g}_-$ which is completed by means of vectors $\mathsf{x}_{n+1}, \dots, \mathsf{x}_r$ to produce a basis of $\mathfrak{g}$. To any such pair of bases are associated the so-called *structure constants* $c_{k_1,k_2}^s \in \mathbb{K}$ encoding the Lie bracket:

$$\big[\mathsf{x}_{k_1},\, \mathsf{x}_{k_2}\big]_{\mathfrak{g}} = \sum_{s=1}^{r} c_{k_1,k_2}^s\, \mathsf{x}_s \qquad (k_1,\, k_2 = 1 \cdots n,\, n+1, \dots, r), \tag{41}$$

and because $[\mathfrak{g}_-, \mathfrak{g}_-]_{\mathfrak{g}} \subset \mathfrak{g}_-$, we must naturally have $c_{k_1,k_2}^s = 0$ for $s = n+1, \dots, r$ whenever $1 \leqslant k_1, k_2 \leqslant n = \dim \mathfrak{g}_-$. Furthermore, one may adopt the convention that $c_{k_1,k_2}^s = 0$ whenever one does not have $1 \leqslant k_1, k_2 \leqslant r$ and $1 \leqslant s \leqslant r$. Of course, the skew-symmetry and the Jacobi identity:

$$\begin{aligned}
0 &= [\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}} + [\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}} \\
0 &= \big[[\mathsf{x}_{k_1}, \mathsf{x}_{k_2}]_{\mathfrak{g}}, \mathsf{x}_{k_3}\big]_{\mathfrak{g}} + \big[[\mathsf{x}_{k_3}, \mathsf{x}_{k_1}]_{\mathfrak{g}}, \mathsf{x}_{k_2}\big]_{\mathfrak{g}} + \big[[\mathsf{x}_{k_2}, \mathsf{x}_{k_3}]_{\mathfrak{g}}, \mathsf{x}_{k_1}\big]_{\mathfrak{g}}
\end{aligned}$$

read at the level of structure constants as:

$$\begin{aligned}
0 &= c_{k_1,k_2}^s + c_{k_2,k_1}^s \\
0 &= \sum_{s=1}^{r} \big(c_{k_1,k_2}^s\, c_{s,k_3}^l + c_{k_3,k_1}^s\, c_{s,k_2}^l + c_{k_2,k_3}^s\, c_{s,k_1}^l\big) \\
&\qquad (k_1,\, k_2,\, k_3,\, l = 1 \cdots r).
\end{aligned}$$

10.2. **Exterior algebra.** Given any integer $\ell \geqslant 1$, consider the ℓ-th exterior power $\Lambda^\ell \mathfrak{g}_-$. Whenever $\ell \leqslant n$, it is a nonzero vector space generated over $\mathbb{K}$ by the basis consisting of the $\binom{n}{\ell} = \frac{n!}{\ell!\,(n-\ell)!}$ linearly independent ℓ-fold exterior products:

$$\big(\mathsf{x}_{j_1} \wedge \mathsf{x}_{j_2} \wedge \cdots \wedge \mathsf{x}_{j_\ell}\big)_{1 \leqslant j_1 < j_2 < \cdots < j_\ell \leqslant n},$$

while $\Lambda^\ell \mathfrak{g}_- = \{0\}$ for all $\ell \geqslant n+1$. Next, let $\mathfrak{g}^* = \mathrm{Lin}(\mathfrak{g}, \mathbb{K})$ denote the dual of the Lie algebra $\mathfrak{g}$, viewed as a plain vector space (it has no natural Lie bracket structure). If we introduce the basis:

$$\mathsf{x}_1^*, \dots, \mathsf{x}_n^*, \mathsf{x}_{n+1}^*, \dots, \mathsf{x}_r^*$$

of $\mathfrak{g}^*$ which is dual to the previously fixed basis $\mathsf{x}_1, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}, \dots, \mathsf{x}_r$ of $\mathfrak{g}$, then by definition, for any $i, j = 1, \dots, n, n+1, \dots, r$, we have:

$$\mathsf{x}_i^*(\mathsf{x}_j) = \delta_j^i := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}$$

For any $\ell \geqslant 1$, let us define (*cf.* [17], Chap. 3) the space $\mathscr{C}^\ell(\mathfrak{g}_-, \mathfrak{g})$ of ℓ-*cochains* as the space of linear maps from $\Lambda^k \mathfrak{g}_-$ to $\mathfrak{g}$, that is to say:

$$\mathscr{C}^\ell(\mathfrak{g}_-, \mathfrak{g}) \overset{\text{def}}{=} \mathrm{Lin}\big(\Lambda^\ell \mathfrak{g}_-,\, \mathfrak{g}\big).$$

Thanks to the canonical identifications:

$$\begin{aligned}
\mathrm{Lin}\big(\Lambda^\ell \mathfrak{g}_-,\, \mathfrak{g}\big) &\simeq \big(\Lambda^\ell \mathfrak{g}_-\big)^* \otimes \mathfrak{g} \\
&\simeq \Lambda^\ell \mathfrak{g}_-^* \otimes \mathfrak{g},
\end{aligned}$$

valid for any ℓ, an arbitrary ℓ-cochain writes, in term of bases for $\mathfrak{g}_-^*$ and for $\mathfrak{g}$, under the general form:

$$\Phi = \sum_{1\leqslant i_1<i_2<\cdots<i_\ell\leqslant n} \sum_{l=1}^{r} \phi^k_{i_1,i_2,\dots,i_\ell} \left(\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \wedge \cdots \wedge \mathsf{x}^*_{i_\ell}\right) \otimes \mathsf{x}_k,$$

where the $\phi^k_{i_1,i_2,\dots,i_\ell}$ are coefficients in the ground field $\mathbb{K}$.

On the other hand, without any reference to bases, we recall from basic algebra that a ℓ-cochain $\Phi \in \mathscr{C}^\ell(\mathfrak{g}_-,\mathfrak{g})$ may be seen either as being a $\mathfrak{g}$-valued linear map acting on exterior ℓ-vectors $\mathsf{z}_1 \wedge \mathsf{z}_2 \wedge \cdots \wedge \mathsf{z}_\ell$ with the z_i belonging to $\mathfrak{g}_-$, or equivalently as being a *multilinear* map from the ℓ-fold product $\mathfrak{g}_- \times \mathfrak{g}_- \times \cdots \times \mathfrak{g}_-$ to $\mathfrak{g}$ which has the property that:

$$\Phi\big(\mathsf{z}_{\sigma(1)}, \mathsf{z}_{\sigma(2)}, \dots, \mathsf{z}_{\sigma(\ell)}\big) = (-1)^{\mathrm{sgn}(\sigma)}\, \Phi\big(\mathsf{z}_1, \mathsf{z}_2, \dots, \mathsf{z}_\ell\big)$$

for every permutation σ of $\{1,2,\dots,\ell\}$. This last property is easily seen to be equivalent to the property that the value of Φ vanishes as soon as two at least of its arguments coincide.

10.3. **Differential complex and cohomology.** From ℓ-cochains to $(\ell+1)$-cochains, there is a canonical *boundary operator*:

$$\partial^\ell\colon\quad \mathscr{C}^\ell\big(\mathfrak{g}_-,\,\mathfrak{g}\big) \longrightarrow \mathscr{C}^{\ell+1}\big(\mathfrak{g}_-,\,\mathfrak{g}\big).$$

which, to any $\Phi \in \mathscr{C}^\ell\big(\mathfrak{g}_-,\mathfrak{g}\big)$ associates a $(\ell+1)$-cochain $\partial^\ell\Phi$ whose action on any collection of $\ell+1$ vectors $\mathsf{z}_0, \mathsf{z}_1, \dots, \mathsf{z}_\ell$ of $\mathfrak{g}_-$ is defined by the specific formula:
(42)

$$\begin{aligned}(\partial^\ell\Phi)\big(\mathsf{z}_0,\mathsf{z}_1,\dots,\mathsf{z}_\ell\big) := & \sum_{i=0}^{\ell} (-1)^i \big[\mathsf{z}_i,\, \Phi(\mathsf{z}_0,\dots,\widehat{\mathsf{z}}_i,\dots,\mathsf{z}_\ell)\big]_{\mathfrak{g}} + \\ & + \sum_{0\leqslant i<j\leqslant\ell} (-1)^{i+j}\, \Phi\big([\mathsf{z}_i,\mathsf{z}_j]_{\mathfrak{g}}, \mathsf{z}_0,\dots,\widehat{\mathsf{z}}_i,\dots,\widehat{\mathsf{z}}_j,\dots,\mathsf{z}_n\big),\end{aligned}$$

where, as usual, $\widehat{\mathsf{z}}_l$ means removal of the term z_l; first, second and third cohomology spaces (*see* below) associated to this specific differential ∂ occur naturally as providing one-dimensional extensions of Lie algebras (H^1), as parametrizing infinitesimal deformations of Lie algebras (H^2), as obstruction to their deformations (H^3), or as the algebraic skeleton of curvatures of Cartan connections on principal bundles (H^2 and Bianchi-type identities). Let us check that $\partial^\ell\Phi$ really belongs to $\mathscr{C}^{\ell+1}(\mathfrak{g}_-,\mathfrak{g})$.

The so-defined action of $\partial^\ell\Phi$ is clearly linear with respect to each argument z_i, for $i = 0,1,\dots,\ell$. Furthermore, from the assumption that Φ vanishes when two of its arguments coincide, one immediately infers that $(\partial^\ell\Phi)(\mathsf{z}_0,\mathsf{z}_1,\dots,\mathsf{z}_\ell) = 0$ vanishes as soon as $\mathsf{z}_{i_1} = \mathsf{z}_{i_2}$ for at least two distinct indices $i_1 \neq i_2$, whence by a standard elementary reasoning, we have the skew-symmetry:

$$(\partial^\ell\Phi)\big(\mathsf{z}_{\sigma(0)},\mathsf{z}_{\sigma(1)},\dots,\mathsf{z}_{\sigma(\ell)}\big) = (-1)^{\mathrm{sgn}(\sigma)}\, (\partial^\ell\Phi)\big(\mathsf{z}_0,\mathsf{z}_1,\dots,\mathsf{z}_\ell\big),$$

for every permutation σ of $\{0, 1, \ldots, \ell\}$. Consequently, $(\partial^\ell \Phi)$ effectively identifies to a certain linear map:

$$\partial^\ell \Phi \colon \quad \Lambda^{\ell+1} \mathfrak{g}_- \longrightarrow \mathfrak{g}$$

from the $(\ell+1)$-th exterior product of $\mathfrak{g}_-$ into $\mathfrak{g}$, namely it truly is a $(\ell+1)$-cochain. Precisely, for any element $\mathsf{z}_0 \wedge \mathsf{z}_1 \wedge \cdots \wedge \mathsf{z}_\ell \in \Lambda^{\ell+1} \mathfrak{g}_-$ with $\mathsf{z}_i \in \mathfrak{g}_-$, one simply sets:

$$(\partial^\ell \Phi)\big(\mathsf{z}_0 \wedge \mathsf{z}_1 \wedge \cdots \wedge \mathsf{z}_\ell\big) := (\partial^\ell \Phi)\big(\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_\ell\big).$$

On the other hand, it is usually left as an exercise to verify that from any level ℓ to the level $\ell + 2$, one has $\partial^{\ell+1}(\partial^\ell \Phi) \equiv 0$, so that the datum:

$$0 \xrightarrow{\partial^0} \mathscr{C}^1 \xrightarrow{\partial^1} \mathscr{C}^2 \xrightarrow{\partial^2} \cdots \xrightarrow{\partial^{n-2}} \mathscr{C}^{n-1} \xrightarrow{\partial^{n-1}} \mathscr{C}^n \xrightarrow{\partial^n} 0 \tag{43}$$

forms what is called a *complex*, namely the composition $\partial^{\ell+1} \circ \partial^\ell = 0$ from any $\mathscr{C}^\ell$ to $\mathscr{C}^{\ell+2}$ always vanishes. Equivalently, one has:

$$\mathrm{im}\big(\partial^{\ell-1} \colon \mathscr{C}^{\ell-1} \to \mathscr{C}^\ell\big) \subset \ker\big(\partial^\ell \colon \mathscr{C}^\ell \to \mathscr{C}^{\ell+1}\big),$$

and the classical terminology is to call:

$$\mathscr{Z}^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big) := \ker\big(\partial^\ell \colon \mathscr{C}^\ell \to \mathscr{C}^{\ell+1}\big)$$

the *space of cocycles* of order ℓ, and also to call:

$$\mathscr{B}^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big) := \mathrm{im}\big(\partial^{\ell-1} \colon \mathscr{C}^{\ell-1} \to \mathscr{C}^\ell\big)$$

the *space of coboundaries* of order ℓ, which is thus always a vector subspace of $\mathscr{Z}^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big)$.

Definition 10.1. The quotient space:

$$H^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big) := \frac{\mathscr{Z}^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big)}{\mathscr{B}^\ell\big(\mathfrak{g}_-, \mathfrak{g}\big)}$$

is called the ℓ-th cohomology space of $\mathfrak{g}_-$ in $\mathfrak{g}$.

For applications to either deformations of Lie algebras or to the explicit constructions of Cartan connections, we will mainly be interested in computing the second cohomology:

$$H^2\big(\mathfrak{g}_-, \mathfrak{g}\big) = \frac{\mathscr{Z}^2\big(\mathfrak{g}_-, \mathfrak{g}\big)}{\mathscr{B}^2\big(\mathfrak{g}_-, \mathfrak{g}\big)},$$

which is a plain finite-dimensional vector space over $\mathbb{K}$, the complexity of which will depend on the geometric situation under study.

10.4. **Basis for 2-cochains.** As we already saw above, a 2-cochain writes, in term of bases, under the general form (we shall from now on regularly omit the parentheses in $(\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^*) \otimes \mathsf{x}_k$):

$$\Phi = \sum_{1 \leqslant i_1 < i_2 \leqslant n} \sum_{k=1}^{r} \phi_{i_1, i_2}^k \, \mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k,$$

where the $\phi^k_{i_1,i_2} \in \mathbb{K}$ are arbitrary coefficients, namely it is a linear combination of the $\frac{n(n-1)}{2}\, r$ basic elements:

$$\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k \qquad (1 \leqslant i_1 < i_2 \leqslant n\,;\; k = 1, ..., n, n{+}1, ..., r),$$

which visibly form a basis of $\Lambda^2 \mathfrak{g}^*_- \otimes \mathfrak{g}$. We remind that if E is a finite-dimensional $\mathbb{K}$-vector space and if ω^*, π^* are one-forms belonging to its dual $E^* = \mathrm{Lin}(E, \mathbb{K})$, then the two-form $\omega^* \wedge \pi^*$ acts on pairs $(e, f) \in E^2$ by definition as:

$$\omega^* \wedge \pi^*(e, f) \overset{def}{=} \omega^*(e)\, \pi^*(f) - \omega^*(f)\, \pi^*(e).$$

In particular, for any i_1, i_2 with $i_1 < i_2$ and for any j_1, j_2 without restriction, we have:

$$\begin{aligned} \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2}(\mathsf{x}_{j_1}, \mathsf{x}_{j_2}) &= \mathsf{x}^*_{i_1}(\mathsf{x}_{j_1})\, \mathsf{x}^*_{i_2}(\mathsf{x}_{j_2}) - \mathsf{x}^*_{i_1}(\mathsf{x}_{j_2})\, \mathsf{x}^*_{i_2}(\mathsf{x}_{j_1}) \\ &= \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\, \delta^{i_2}_{j_1}. \end{aligned} \tag{44}$$

However, we observe *passim* that the second product of Kronecker deltas necessarily vanishes whenever $j_1 < j_2$, a natural restriction we will sometimes make, though not always.

10.5. **Boundary of a basic 2-cochain.** According to the definition (42), for any triple of indices j_1, j_2, j_3 with $1 \leqslant j_1 < j_2 < j_3 \leqslant n$, we have:

$$\begin{aligned} (\partial^2 \Phi)\big(\mathsf{x}_{j_1} \wedge \mathsf{x}_{j_2} \wedge \mathsf{x}_{j_3}\big) &= (\partial^2 \Phi)\big(\mathsf{x}_{j_1}, \mathsf{x}_{j_2}, \mathsf{x}_{j_3}\big) \\ &= \big[\mathsf{x}_{j_1},\, \Phi(\mathsf{x}_{j_2}, \mathsf{x}_{j_3})\big]_{\mathfrak{g}} - \big[\mathsf{x}_{j_2},\, \Phi(\mathsf{x}_{j_1}, \mathsf{x}_{j_3})\big]_{\mathfrak{g}} + \big[\mathsf{x}_{j_3},\, \Phi(\mathsf{x}_{j_1}, \mathsf{x}_{j_2})\big]_{\mathfrak{g}} - \\ &\quad - \Phi\big([\mathsf{x}_{j_1},\, \mathsf{x}_{j_2}]_{\mathfrak{g}},\, \mathsf{x}_{j_3}\big) + \Phi\big([\mathsf{x}_{j_1},\, \mathsf{x}_{j_3}]_{\mathfrak{g}},\, \mathsf{x}_{j_2}\big) - \Phi\big([\mathsf{x}_{j_2},\, \mathsf{x}_{j_3}]_{\mathfrak{g}},\, \mathsf{x}_{j_1}\big). \end{aligned}$$

Let us hence apply this formula to any basic 2-cochain $\Phi^{i_1,i_2}_k = \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k$ and perform a few natural computational transformations, using the Lie algebra structure (41) and applying formulas (44):

$$\begin{aligned} &\partial^2(\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)\big(\mathsf{x}_{j_1} \wedge \mathsf{x}_{j_2} \wedge \mathsf{x}_{j_3}\big) = \\ &= \big[\mathsf{x}_{j_1},\, (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)(\mathsf{x}_{j_2}, \mathsf{x}_{j_3})\big]_{\mathfrak{g}} - \big[\mathsf{x}_{j_2},\, (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)(\mathsf{x}_{j_1}, \mathsf{x}_{j_3})\big]_{\mathfrak{g}} + \big[\mathsf{x}_{j_3},\, (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)(\mathsf{x}_{j_1}, \mathsf{x}_{j_2})\big]_{\mathfrak{g}} - \\ &\quad - (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)\big([\mathsf{x}_{j_1},\, \mathsf{x}_{j_2}]_{\mathfrak{g}},\, \mathsf{x}_{j_3}\big) + (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)\big([\mathsf{x}_{j_1},\, \mathsf{x}_{j_3}]_{\mathfrak{g}},\, \mathsf{x}_{j_2}\big) - (\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k)\big([\mathsf{x}_{j_2},\, \mathsf{x}_{j_3}]_{\mathfrak{g}},\, \mathsf{x}_{j_1}\big) \\ &= \big[\mathsf{x}_{j_1},\, \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\, \mathsf{x}_k\big]_{\mathfrak{g}} - \big[\mathsf{x}_{j_2},\, \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\, \mathsf{x}_k\big]_{\mathfrak{g}} + \big[\mathsf{x}_{j_3},\, \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, \mathsf{x}_k\big]_{\mathfrak{g}} - \\ &\quad - \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k\bigg(\sum_{l=1}^r c^l_{j_1,j_2}\, \mathsf{x}_l,\; \mathsf{x}_{j_3}\bigg) + \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k\bigg(\sum_{l=1}^r c^l_{j_1,j_3}\, \mathsf{x}_l,\; \mathsf{x}_{j_2}\bigg) - \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k\bigg(\sum_{l=1}^r c^l_{j_2,j_3}\, \mathsf{x}_l,\; \mathsf{x}_{j_1}\bigg) \\ &= \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\, [\mathsf{x}_{j_1}, \mathsf{x}_k]_{\mathfrak{g}} - \delta^{i_1}_{j_1}\delta^{i_2}_{j_3}\, [\mathsf{x}_{j_2}, \mathsf{x}_k]_{\mathfrak{g}} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, [\mathsf{x}_{j_3}, \mathsf{x}_k]_{\mathfrak{g}} - \\ &\quad - \bigg(\sum_{l=1}^r c^l_{j_1,j_2}\, \big(\delta^{i_1}_l\, \delta^{i_2}_{j_3} - \delta^{i_2}_l\, \delta^{i_1}_{j_3}\big) + \sum_{l=1}^r c^l_{j_1,j_3}\, \big(\delta^{i_1}_l\, \delta^{i_2}_{j_2} - \delta^{i_2}_l\, \delta^{i_1}_{j_2}\big) - \sum_{l=1}^r c^l_{j_2,j_3}\, \big(\delta^{i_1}_l\, \delta^{i_2}_{j_1} - \delta^{i_2}_l\, \delta^{i_1}_{j_1}\big)\bigg)\, \mathsf{x}_k \\ &= \sum_{l=1}^r \Big(c^l_{j_1,k}\, \delta^{i_1}_{j_2}\delta^{i_2}_{j_3} - c^l_{j_2,k}\, \delta^{i_1}_{j_1}\delta^{i_2}_{j_3} + c^l_{j_3,k}\, \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Big)\, \mathsf{x}_l + \\ &\quad + \big(- c^{i_1}_{j_1,j_2}\, \delta^{i_2}_{j_3} + c^{i_2}_{j_1,j_2}\, \delta^{i_1}_{j_3} + c^{i_1}_{j_1,j_3}\, \delta^{i_2}_{j_2} - c^{i_2}_{j_1,j_3}\, \delta^{i_1}_{j_2} - c^{i_1}_{j_2,j_3}\, \delta^{i_2}_{j_1} + c^{i_2}_{j_2,j_3}\, \delta^{i_1}_{j_1}\big)\, \mathsf{x}_k. \end{aligned}$$

At this point, in order to reach a neat formula, let us replace x_k in the last line by $\sum_{l=1}^r \delta_k^l\, \mathsf{x}_l$ and reorganize:

$$
\begin{aligned}
\big(\partial^2 \mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k\big)\big(\mathsf{x}_{j_1}\wedge \mathsf{x}_{j_2}\wedge \mathsf{x}_{j_3}\big) = \sum_{l=1}^r \Big(& c_{j_1,k}^l\,\delta_{j_2}^{i_1}\delta_{j_3}^{i_2} - c_{j_2,k}^l\,\delta_{j_1}^{i_1}\delta_{j_3}^{i_2} + c_{j_3,k}^l\,\delta_{j_1}^{i_1}\delta_{j_2}^{i_2} + \\
& + \delta_k^l \big(- c_{j_1,j_2}^{i_1}\,\delta_{j_3}^{i_2} + c_{j_1,j_2}^{i_2}\,\delta_{j_3}^{i_1} + c_{j_1,j_3}^{i_1}\,\delta_{j_2}^{i_2} - c_{j_1,j_3}^{i_2}\,\delta_{j_2}^{i_1} - c_{j_2,j_3}^{i_1}\,\delta_{j_1}^{i_2} + c_{j_2,j_3}^{i_2}\,\delta_{j_1}^{i_1}\big)\Big)\,\mathsf{x}_l.
\end{aligned}
$$

Since we may *a priori* represent without arguments this 3-cochain $\partial^2\big(\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k\big)$ as:

$$
\partial^2\big(\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k\big) = \sum_{1\leqslant j_1<j_2<j_3\leqslant n}\ \sum_{l=1}^r \mathsf{coefficient}_{k;j_1,j_2,j_3}^{i_1,i_2;l}\cdot \mathsf{x}_{j_1}^*\wedge \mathsf{x}_{j_2}^*\wedge \mathsf{x}_{j_3}^*\otimes \mathsf{x}_l,
$$

the preceding computations precisely give us that:

$$
\begin{aligned}
\mathsf{coefficient}_{k;j_1,j_2,j_3}^{i_1,i_2;l} = & \ c_{j_1,k}^l\,\delta_{j_2}^{i_1}\delta_{j_3}^{i_2} - c_{j_2,k}^l\,\delta_{j_1}^{i_1}\delta_{j_3}^{i_2} + c_{j_3,k}^l\,\delta_{j_1}^{i_1}\delta_{j_2}^{i_2} - \\
& - c_{j_1,j_2}^{i_1}\,\delta_k^l\delta_{j_3}^{i_2} + c_{j_1,j_2}^{i_2}\,\delta_k^l\delta_{j_3}^{i_1} + c_{j_1,j_3}^{i_1}\,\delta_k^l\delta_{j_2}^{i_2} - c_{j_1,j_3}^{i_2}\,\delta_k^l\delta_{j_2}^{i_1} - c_{j_2,j_3}^{i_1}\,\delta_k^l\delta_{j_1}^{i_2} + c_{j_2,j_3}^{i_2}\,\delta_k^l\delta_{j_1}^{i_1}.
\end{aligned}
$$

As a result, we obtain explicitly that:

$$
\begin{aligned}
\partial^2\big(\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k\big) = & \sum_{1\leqslant j_1<j_2<j_3\leqslant n}\ \sum_{l=1}^r \mathsf{x}_{j_1}^*\wedge \mathsf{x}_{j_2}^*\wedge \mathsf{x}_{j_3}^*\otimes \mathsf{x}_l\Big(c_{j_1,k}^l\,\delta_{j_2}^{i_1}\delta_{j_3}^{i_2} - c_{j_2,k}^l\,\delta_{j_1}^{i_1}\delta_{j_3}^{i_2} + c_{j_3,k}^l\,\delta_{j_1}^{i_1}\delta_{j_2}^{i_2} - \\
& - \delta_k^l\big[c_{j_1,j_2}^{i_1}\,\delta_{j_3}^{i_2} + c_{j_1,j_2}^{i_2}\,\delta_{j_3}^{i_1} + c_{j_1,j_3}^{i_1}\,\delta_{j_2}^{i_2} - c_{j_1,j_3}^{i_2}\,\delta_{j_2}^{i_1} - c_{j_2,j_3}^{i_1}\,\delta_{j_1}^{i_2} + c_{j_2,j_3}^{i_2}\,\delta_{j_1}^{i_1}\big]\Big) \\
& \qquad (1\leqslant i_1<i_2\leqslant n\,;\ k=1,\dots,n,n{+}1,\dots r).
\end{aligned}
$$

10.6. **Boundary of a general 2-cochain.** Next, with:

$$
\Phi = \sum_{1\leqslant i_1<i_2\leqslant n}\ \sum_{k=1}^r \phi_{i_1,i_2}^k\, \mathsf{x}_{i_1}^*\wedge \mathsf{x}_{i_2}^*\otimes \mathsf{x}_k,
$$

we may now compute $\partial^2\Phi$ by linearity:

$$
\begin{aligned}
\partial^2\Phi = & \sum_{1\leqslant i_1<i_2\leqslant n}\ \sum_{k=1}^r \phi_{i_1,i_2}^k\,\partial^2\big(\mathsf{x}_{i_1}^*\wedge \mathsf{x}_{i_2}^*\otimes \mathsf{x}_k\big) \\
= & \sum_{1\leqslant i_1<i_2\leqslant n}\ \sum_{k=1}^r \phi_{i_1,i_2}^k \sum_{1\leqslant j_1<j_2<j_3\leqslant n}\ \sum_{l=1}^r \mathsf{x}_{j_1}^*\wedge \mathsf{x}_{j_2}^*\wedge \mathsf{x}_{j_3}^*\otimes \mathsf{x}_l\Big(c_{j_1,k}^l\,\delta_{j_2}^{i_1}\delta_{j_3}^{i_2} - c_{j_2,k}^l\,\delta_{j_1}^{i_1}\delta_{j_3}^{i_2} + c_{j_3,k}^l\,\delta_{j_1}^{i_1}\delta_{j_2}^{i_2} - \\
& - \delta_k^l\big[c_{j_1,j_2}^{i_1}\,\delta_{j_3}^{i_2} + c_{j_1,j_2}^{i_2}\,\delta_{j_3}^{i_1} + c_{j_1,j_3}^{i_1}\,\delta_{j_2}^{i_2} - c_{j_1,j_3}^{i_2}\,\delta_{j_2}^{i_1} - c_{j_2,j_3}^{i_1}\,\delta_{j_1}^{i_2} + c_{j_2,j_3}^{i_2}\,\delta_{j_1}^{i_1}\big]\Big) \\
= & \sum_{1\leqslant j_1<j_2<j_3\leqslant n}\ \sum_{l=1}^r \mathsf{x}_{j_1}^*\wedge \mathsf{x}_{j_2}^*\wedge \mathsf{x}_{j_3}^*\otimes \mathsf{x}_l\Big(\sum_{k=1}^r \big(c_{j_1,k}^l\,\phi_{j_2,j_3}^k - c_{j_2,k}^l\,\phi_{j_1,j_3}^k + c_{j_3,k}^l\,\phi_{j_1,j_2}\big) + \\
& + \sum_{1\leqslant i_1<i_2\leqslant n} \phi_{i_1,i_2}^l\big(- c_{j_1,j_2}^{i_1}\,\delta_{j_3}^{i_2} + c_{j_1,j_2}^{i_2}\,\delta_{j_3}^{i_1}\big) + \sum_{1\leqslant i_1<i_2\leqslant n} \phi_{i_1,i_2}^l\big(c_{j_1,j_3}^{i_1}\,\delta_{j_2}^{i_2} - c_{j_1,j_3}^{i_2}\,\delta_{j_2}^{i_1}\big) + \\
& + \sum_{1\leqslant i_1<i_2\leqslant n} \phi_{i_1,i_2}^l\big(- c_{j_2,j_3}^{i_1}\,\delta_{j_1}^{i_2} + c_{j_2,j_3}^{i_2}\,\delta_{j_1}^{i_1}\big).
\end{aligned}
$$

At this point, we must finish the computation of the three sums appearing in the last two lines. In fact, any general triangle-like sum of the form:

$$
\sum_{1\leqslant i_1<i_2\leqslant n} \mu_{i_1,i_2}\,\big(- \nu^{i_1}\,\delta_{j_3}^{i_2} + \nu^{i_2}\,\delta_{j_3}^{i_1}\big),
$$

where the $\mu_{\cdot,\cdot}$, $\nu^{\cdot}$ are indexed numbers, has the property that its general term within parentheses is zero unless $i_2 = j_3$ or $i_1 = j_3$, whence it decomposes symbolically just as two simple sums:

$$\sum_{1\leqslant i_1<i_2\leqslant n} = \sum_{i_1=1}^{j_3-1}\Big|_{i_2=j_3} + \sum_{i_2=j_3+1}^{n}\Big|_{i_1=j_3},$$

so that the sum in question expands as:

$$-\sum_{i_1=1}^{j_3-1}\mu_{i_1,j_3}\,\nu^{i_1} + \sum_{i_2=j_3+1}^{n}\mu_{j_3,i_2}\,\nu^{i_2}.$$

Applying this formula, we may finish the computation of the three sums mentioned above and they are equal to:

$$-\sum_{i_1=1}^{j_3-1} c_{j_1,j_2}^{i_1}\,\phi_{j_1,j_3}^{l} + \sum_{i_2=j_3+1}^{n} c_{j_1,j_2}^{i_2}\,\phi_{j_3,i_2}^{l} + \sum_{i_1=1}^{j_2-1} c_{j_1,j_3}^{i_1}\,\phi_{i_1,j_2}^{l} - \sum_{i_2=j_2+1}^{n} c_{j_1,j_3}^{i_2}\,\phi_{j_2,i_2}^{l} - \\ -\sum_{i_1=1}^{j_1-1} c_{j_2,j_3}^{i_1}\,\phi_{i_1,j_1}^{l} + \sum_{i_2=j_1+1}^{n} c_{j_2,j_3}^{i_2}\,\phi_{j_1,i_2}.$$

As a result, we may explicitly characterize the condition that Φ be a 2-*cocycle*, stating the initial hypothesis for self-contentness reasons.

Proposition 10.2. *Let $\mathfrak{g}$ be an r-dimensional Lie algebra over $\mathbb{K}$, let $\mathfrak{g}_- \subset \mathfrak{g}$ be a proper n-dimensional Lie subalgebra with $2 \leqslant n \leqslant r-1$, and let $\mathsf{x}_1, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}, \dots, \mathsf{x}_r$ be a basis of $\mathfrak{g}$, its first n terms $\mathsf{x}_1, \dots, \mathsf{x}_n$ simultaneously constituting a basis of $\mathfrak{g}_-$ so that the $\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k$ with $1 \leqslant i_1 < i_2 \leqslant n$ and $k = 1, \dots, n, n+1, \dots, r$ make a naturally associated basis for $\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}$. Then a general 2-cochain in $\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}$:*

$$\Phi = \sum_{1\leqslant i_1<i_2\leqslant n}\sum_{k=1}^{r}\phi_{i_1,i_2}^{k}\,\mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k$$

having arbitrary undetermined coefficients $\phi_{i_1,i_2}^{k} \in \mathbb{K}$ is a cocycle, namely satisfies $\partial^2\Phi = 0$, if and only if the following $r\binom{n}{3}$ linear equations hold:

$$0 = \sum_{k=1}^{r}\Big(c_{j_1,k}^{l}\,\phi_{j_2,j_3}^{k} - c_{j_2,k}^{l}\,\phi_{j_1,j_3}^{k} + c_{j_3,k}^{l}\,\phi_{j_1,j_2}^{k}\Big) - \\ -\sum_{i_1=1}^{j_3-1} c_{j_1,j_2}^{i_1}\,\phi_{i_1,j_3}^{l} + \sum_{i_2=j_3+1}^{n} c_{j_1,j_2}^{i_2}\,\phi_{j_3,i_2}^{l} + \sum_{i_1=1}^{j_2-1} c_{j_1,j_3}^{i_1}\,\phi_{i_1,j_2}^{l} - \\ - \sum_{i_2=j_2+1}^{n} c_{j_1,j_3}^{i_2}\,\phi_{j_2,i_2}^{l} - \sum_{i_1=1}^{j_1-1} c_{j_2,j_3}^{i_1}\,\phi_{i_1,j_1}^{l} + \sum_{i_2=j_1+1}^{n} c_{j_2,j_3}^{i_2}\,\phi_{j_1,i_2}^{l}$$

$$(1 \leqslant j_1 < j_2 < j_3 \leqslant n\,;\ l = 1, \dots n, n+1, \dots, r),$$

where the $c_{j,k}^{s}$ are the structure constants: $[\mathsf{x}_j, \mathsf{x}_k]_{\mathfrak{g}} = \sum_{s=1}^{r} c_{j,k}^{s}\,\mathsf{x}_s$.

10.7. **Basis for** 1**-cochains.** Now, we want to characterize, in terms of the structure constants of $\mathfrak{g}$, the condition that the 2-cochain Φ identifies to the differential $\partial^1\Psi$ of a 1-cochain $\Psi \in \mathscr{C}^1(\mathfrak{g}_-,\, \mathfrak{g})$. Let us therefore write down such a general 1-cochain:

$$\Psi = \sum_{1\leqslant i\leqslant n}\sum_{k=1}^{r} \psi_k^i\, \mathsf{x}_i^*\otimes \mathsf{x}_k,$$

as being the linear combination, with arbitrary coefficients $\psi_k^i \in \mathbb{K}$, of the $n\cdot n$ basic 1-cochains:

$$\mathsf{x}_i^*\otimes\mathsf{x}_k \qquad\qquad (i,\, j=1\cdots n)$$

which clearly form a basis of $\mathscr{C}^1(\mathfrak{g}_-,\mathfrak{g}) = \mathfrak{g}_-^*\otimes\mathfrak{g}$ over $\mathbb{K}$.

10.8. **Boundary of a basic** 1**-cochain.** Applying the definition (42), the differential ∂^1 acts as follows on such a general 1-cochain:

$$(\partial^1\Psi)\big(\mathsf{x}_{j_1},\mathsf{x}_{j_2}\big) = \big[\mathsf{x}_{j_1},\, \Psi(\mathsf{x}_{j_2})\big]_{\mathfrak{g}} - \big[\mathsf{x}_{j_2},\, \Psi(\mathsf{x}_{j_1})\big]_{\mathfrak{g}} - \Psi\big([\mathsf{x}_{j_1},\, \mathsf{x}_{j_2}]_{\mathfrak{g}}\big).$$

Applying this formula to the basic forms, we may compute for any two indices j_1, j_2 with $1\leqslant j_1 < j_2 \leqslant n$:

$$\begin{aligned}\big(\partial^1(\mathsf{x}_i^*\otimes\mathsf{x}_k)\big)\big(\mathsf{x}_{j_1},\,\mathsf{x}_{j_2}\big) &= \big[\mathsf{x}_{j_1},\, \delta_{j_2}^i\,\mathsf{x}_k\big]_{\mathfrak{g}} - \big[\mathsf{x}_{j_2},\, \delta_{j_1}^i\,\mathsf{x}_k\big]_{\mathfrak{g}} - \mathsf{x}_i^*\otimes\mathsf{x}_k\Big(\sum_{1\leqslant s\leqslant n} c_{j_1,j_2}^s\,\mathsf{x}_s\Big)\\ &= \sum_{s=1}^{r}\mathsf{x}_s\Big(-\, c_{k,j_1}^s\delta_{j_2}^i + c_{k,j_2}^s\delta_{j_1}^i - c_{j_1,j_2}^i\delta_k^s\Big).\end{aligned}$$

This means that we have obtained the following representation of the differentials of all basic 1-cochains:

$$\partial^1\big(\mathsf{x}_i^*\otimes\mathsf{x}_k\big) = \sum_{1\leqslant j_1<j_2\leqslant n}\sum_{s=1}^{r}\mathsf{x}_{j_1}^*\wedge\mathsf{x}_{j_2}^*\otimes\mathsf{x}_s\Big(-\, c_{k,j_1}^s\delta_{j_2}^i + c_{k,j_2}^s\delta_{j_1}^i - c_{j_1,j_2}^i\delta_k^s\Big)$$

$$(1\leqslant i\leqslant n\,;\ k=1,\dots,n,n{+}1,\dots,r).$$

10.9. **Boundary of a general** 1**-cochain.** Thanks to these formulas, we may then compute $\partial^1\Psi$ by linearity:

$$\begin{aligned}\partial^1\Psi &= \sum_{1\leqslant i\leqslant n}\sum_{k=1}^{r}\psi_k^i\,\partial^1\big(\mathsf{x}_i^*\otimes\mathsf{x}_k\big)\\ &= \sum_{1\leqslant i\leqslant n}\sum_{k=1}^{r}\psi_k^i\sum_{1\leqslant j_1<j_2\leqslant n}\sum_{s=1}^{r}\mathsf{x}_{j_1}^*\wedge\mathsf{x}_{j_2}^*\otimes\mathsf{x}_s\Big(-\, c_{k,j_1}\delta_{j_2}^i + c_{k,j_2}^s\delta_{j_1}^i - c_{j_1,j_2}^i\delta_k^s\Big)\\ &= \sum_{1\leqslant j_1<j_2\leqslant n}\sum_{s=1}^{r}\mathsf{x}_{j_1}^*\wedge\mathsf{x}_{j_2}^*\otimes\mathsf{x}_s\Big(\sum_{1\leqslant i\leqslant n}\sum_{k=1}^{r} -\psi_k^i c_{k,j_1}^s\delta_{j_2}^i + \psi_k^i c_{k,j_2}^s\delta_{j_1}^{i_1} - \psi_k^i c_{j_1,j_2}^i\delta_k^s\Big)\\ &= \sum_{1\leqslant j_1<j_2\leqslant n}\sum_{s=1}^{r}\mathsf{x}_{j_1}^*\wedge\mathsf{x}_{j_2}^*\otimes\mathsf{x}_s\Big(-\sum_{k=1}^{r}\psi_k^{j_2}c_{k,j_1}^s + \sum_{k=1}^{r}\psi_k^{j_1}c_{k,j_2}^s - \sum_{1\leqslant i\leqslant n}\psi_s^i c_{j_1,j_2}^i\Big).\end{aligned}$$

As a result, we may explicitly characterize the condition that Φ be a 2-*coboundary*.

Proposition 10.3. *Under the assumptions of Proposition 10.2, a* 2*-cochain:*

$$\Phi = \sum_{1\leqslant i_1<i_2\leqslant n} \sum_{k=1}^{r} \phi_{i_1,i_2}^k \, \mathsf{x}_{i_1}^* \wedge \mathsf{x}_{i_2}^* \otimes \mathsf{x}_k$$

is the boundary $\Phi = \partial^1 \Psi$ *of a* 1*-cochain:*

$$\Psi = \sum_{1\leqslant i\leqslant n} \sum_{k=1}^{r} \psi_k^i \, \mathsf{x}_i^* \otimes \mathsf{x}_k$$

if and only if all its coefficients ϕ_{l_1,l_2}^k *are uniquely determined as the following linear combinations of the* $\psi_\cdot^\cdot$*:*

$$\phi_{j_1,j_2}^s = -\sum_{k=1}^{r} \psi_k^{j_2} c_{k,j_1}^s + \sum_{k=1}^{r} \psi_k^{j_1} c_{k,j_2}^s - \sum_{1\leqslant i\leqslant n} \psi_s^i c_{j_1,j_2}^i$$

$$(1 \leqslant j_1 < j_2 \leqslant n\,;\ s = 1, \dots n, n{+}1, \dots, r).$$

10.10. **Combinatorial assumptions for a general grading.** Now, we want to show that the complexity of cohomological computations splits when the linear system of equations considered above may be decomposed as a direct sum of blocks of linear systems in smaller dimensions. The typical and quite general case where such a splitting is available holds when $\mathfrak{g}$ is endowed with the supplementary structure of a *grading* in the sense that $\mathfrak{g}$, viewed as a vector space, writes down as a direct sum:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}_{-a} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_b \\ &= \bigoplus_{-a\leqslant k\leqslant b} \mathfrak{g}_k\end{aligned}$$

of nonzero vector subspaces $\mathfrak{g}_k$, where $a \geqslant 1$ and $b \geqslant 0$ are certain integers, when one assumes that:

$$\big[\mathfrak{g}_{k_1},\, \mathfrak{g}_{k_2}\big]_{\mathfrak{g}} \subset \mathfrak{g}_{k_1+k_2},$$

for all $k_1, k_2 \in \mathbb{Z}$, after prolonging trivially $\mathfrak{g}_k := \{0\}$ for either $k \leqslant -a-1$ or $k \geqslant b+1$. In this setting, one naturally considers:

$$\mathfrak{g}_- := \mathfrak{g}_{-a} \oplus \cdots \oplus \mathfrak{g}_{-1}$$

for computing the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ in the sense of the preceding sections. As before, we shall denote:

$$r := \dim_{\mathbb{K}} \mathfrak{g} \qquad \text{and} \qquad n := \dim_{\mathbb{K}} \mathfrak{g}_-.$$

Working abstractly and in full generality, we shall not assume that $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ in the sense that $\mathfrak{g}_{-i-1} = \big[\mathfrak{g}_{-1},\, \mathfrak{g}_{-i}\big]_{\mathfrak{g}}$ for all $i = 1, \dots, \mu - 1$, an assumption which, however, comes naturally in Tanaka's theory of graded differential systems.

Now, we need more notation. For any k with $-a \leqslant k \leqslant b$, each $\mathfrak{g}_k$ has a certain positive dimension, call it:

$$d_{(k)} := \dim_{\mathbb{K}} \mathfrak{g}_k,$$

so that one naturally has:

$$r = d_{(-a)} + \cdots + d_{(-1)} + d_{(0)} + d_{(1)} + \cdots + d_{(b)},$$
$$n = d_{(-a)} + \cdots + d_{(-1)}.$$

Let us introduce, for each k with $-a \leqslant k \leqslant b$, an arbitrary fixed basis:

$$\big(\mathsf{x}_k^{i_k}\big)^{1 \leqslant i_k \leqslant d_{(k)}}$$

of the $\mathbb{K}$-vector subspace $\mathfrak{g}_k$ of $\mathfrak{g}$. The lower index k refers to the graded part $\mathfrak{g}_k$ to which all the $\mathsf{x}_k^{i_k}$ belong, for $i_k = 1, \ldots, d_{(k)}$. Accordingly, because $\big[\mathfrak{g}_{k_1}, \mathfrak{g}_{k_2}\big]_{\mathfrak{g}} \subset \mathfrak{g}_{k_1+k_2}$, the structure constants of $\mathfrak{g}$ are of a certain specific form such that:

$$\big[\mathsf{x}_{k_1}^{i_1}, \mathsf{x}_{k_2}^{i_2}\big]_{\mathfrak{g}} = \sum_{i'=1}^{d_{(k_1+k_2)}} c_{k_1,k_2}^{i_1,i_2,i'} \, \mathsf{x}_{k_1+k_2}^{i'}.$$

Furthermore, one may adopt the convention that $c_{k_1,k_2}^{i_1,i_2,i'} = 0$ whenever one does not simultaneously have $-a \leqslant k_1, k_2, k_1 + k_2 \leqslant b$, $1 \leqslant i_1 \leqslant d_{(k_1)}$, $1 \leqslant i_2 \leqslant d_{(k_2)}$ and $1 \leqslant i' \leqslant d_{(k_1+k_2)}$.

10.11. **Splitting of cochains, of cocycles, of coboundaries and of cohomologies according to homogeneity.** Each vector space $\mathscr{C}^\ell(\mathfrak{g}_-, \mathfrak{g})$ naturally splits into a direct sum of so-called *homogeneous cochains* as follows: an ℓ-cochain $\Phi \in \mathscr{C}^\ell(\mathfrak{g}_-, \mathfrak{g})$ is said to be *of homogeneity* a certain integer $h \in \mathbb{Z}$ whenever for any ℓ vectors:

$$\mathsf{z}_{i_1} \in \mathfrak{g}_{k_1}, \ldots\ldots, \mathsf{z}_{i_\ell} \in \mathfrak{g}_{k_\ell}$$

belonging to certain arbitrary but determined $\mathfrak{g}$-components, its value:

$$\Phi(\mathsf{z}_{i_1}, \ldots, \mathsf{z}_{i_\ell}) \in \mathfrak{g}_{i_1+\cdots+i_\ell+h}$$

belongs to the $(i_1 + \cdots + i_\ell + h)$-th component of $\mathfrak{g}$. In fact, one easily convinces oneself that any ℓ-cochain $\Phi \in \mathscr{C}(\mathfrak{g}_-, \mathfrak{g})$ splits as a direct sum of ℓ-cochains of fixed homogeneity:

$$\Phi = \cdots + \Phi^{[h-1]} + \Phi^{[h]} + \Phi^{[h+1]} + \cdots,$$

where we denote the completely h-homogeneous component of Φ just by $\Phi^{[h]}$. In other words:

$$\mathscr{C}^\ell(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{h \in \mathbb{Z}} \mathscr{C}^\ell_{[h]}(\mathfrak{g}_-, \mathfrak{g}),$$

where of course the spaces $\mathscr{C}^\ell_{[h]}(\mathfrak{g}_-, \mathfrak{g})$ reduce to $\{0\}$ for all large $|h|$. Furthermore, applying the definition (42), one verifies the important fact that ∂^ℓ respects homogeneity for all $\ell = 0, 1, \ldots, n$, that is to say, for any $h \in \mathbb{Z}$, one has $\partial^\ell(\mathscr{C}^\ell_{[h]}) \subset \mathscr{C}^{\ell+1}_{[h]}$, whence the complex (43) splits up as a direct sum of complexes:

$$0 \xrightarrow{\partial^0_{[h]}} \mathscr{C}^1 \xrightarrow{\partial^1_{[h]}} \mathscr{C}^2 \xrightarrow{\partial^2_{[h]}} \cdots \xrightarrow{\partial^{n-2}_{[h]}} \mathscr{C}^{n-1} \xrightarrow{\partial^{n-1}_{[h]}} \mathscr{C}^n \xrightarrow{\partial^n_{[h]}} 0$$

indexed by $h \in \mathbb{Z}$, where $\partial^\ell_{[h]}$ naturally denotes the restriction:

$$\partial^\ell_{[h]} := \partial^\ell\big|_{\mathscr{C}^\ell_{[h]}} : \mathscr{C}^\ell_{[h]} \longrightarrow \mathscr{C}^{\ell+1}_{[h]}.$$

Consequently, one may introduce the spaces of h-*homogeneous cocycles* of order ℓ:

$$\mathscr{Z}^{\ell}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big) := \ker\big(\partial^{\ell}_{[h]} \colon \mathscr{C}^{\ell}_{[h]} \to \mathscr{C}^{\ell+1}_{[h]}\big),$$

together with the spaces of h-*homogeneous coboundaries* of order ℓ:

$$\mathscr{B}^{\ell}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big) := \operatorname{im}\big(\partial^{\ell-1}_{[h]} \colon \mathscr{C}^{\ell-1}_{[h]} \to \mathscr{C}^{\ell}_{[h]}\big).$$

Definition 10.4. The quotient space:

$$H^{\ell}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big) := \frac{\mathscr{Z}^{\ell}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big)}{\mathscr{B}^{\ell}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big)}$$

is called the h-homogeneous ℓ-th cohomology space of $\mathfrak{g}_-$ in $\mathfrak{g}$.

In the sequel, we will mainly be interested in showing how to compute h-homogeneous second cohomologies:

$$H^{2}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big) = \frac{\mathscr{Z}^{2}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big)}{\mathscr{B}^{2}_{[h]}\big(\mathfrak{g}_-, \mathfrak{g}\big)},$$

so that the task of computing the full cohomology spaces:

$$H^2(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{h\in\mathbb{Z}} H^2_{[h]}(\mathfrak{g}_-, \mathfrak{g})$$

requires to deal with vector (sub)spaces of smaller dimensions.

10.12. **Collecting 1-cochains and 2-cochains according to constant homogeneity.** In terms of the bases $\mathsf{x}_l^{j_l}$ for $\mathfrak{g}_-$ and $\mathsf{x}_k^{i_k}$ for $\mathfrak{g}$, the collection $\mathsf{x}_l^{j_l *} \otimes \mathsf{x}_k^i$ where $-a \leqslant l \leqslant -1$, $j_l = 1, \dots, d_{(l)}$ and where $-a \leqslant k \leqslant b$, $i_k = 1, \dots, d_{(k)}$ makes an obvious basis over $\mathbb{K}$ of the space 1-cochains. Similarly, the $\mathsf{x}_{l_1}^{j_{l_1} *} \wedge \mathsf{x}_{l_2}^{j_{l_2} *} \otimes \mathsf{x}_k^{i_k}$, where either $-a \leqslant l_1 < l_2 \leqslant -1$ or $l_1 = l_2$ but $1 \leqslant j_{l_1} < j_{l_2} \leqslant d_{l_1} = d_{l_2}$, makes too a basis over $\mathbb{K}$ of the space of 2-cochains. Thus, if we mind the fact that any double sum $\sum_{k=-a}^{b} \sum_{i_k=1}^{d_{(k)}}$ may also be written without mentioning that the second index $i = i_k$ depends upon the first index k (provided the order of summation is not permuted), it follows that a general 1-cochain $\Psi \in \mathscr{C}^1(\mathfrak{g}_-, \mathfrak{g})$ and a general 2 cochain $\Psi \in \mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$ write, in terms of these natural bases, as the following linear combinations:

$$\Psi = \sum_{-a\leqslant l\leqslant -1} \sum_{j=1}^{d_{(l)}} \sum_{k=-a}^{b} \sum_{i=1}^{d_{(k)}} \psi_{l,j}^{k,i}\, \mathsf{x}_l^{j*} \otimes \mathsf{x}_k^i$$

$$\Phi = \sum_{-a\leqslant l_1< l_2\leqslant -1} \sum_{j_1=1}^{d_{(l_1)}} \sum_{j_2=1}^{d_{(l_2)}} \sum_{k=-a}^{b} \sum_{i=1}^{d_{(k)}} \phi_{l_1,j_1,l_2,j_2}^{k,i}\, \mathsf{x}_{l_1}^{j_1 *} \wedge \mathsf{x}_{l_2}^{j_2 *} \otimes \mathsf{x}_k^i +$$

$$+ \sum_{l=-a}^{-1} \sum_{1\leqslant j'<j''\leqslant d_{(l)}} \sum_{k=-a}^{b} \sum_{i=1}^{d_{(k)}} \phi_{l,j',j''}^{k,i}\, \mathsf{x}_l^{j'*} \wedge \mathsf{x}_l^{j''*} \otimes \mathsf{x}_k^i,$$

where the $\psi_\cdot^\cdot$ and the $\phi_\cdot^\cdot$ are arbitrary constants in $\mathbb{K}$. However, we must at first improve such a preliminary representation.

The homogeneity of any basic 1-cochain $\mathsf{x}_l^{j*} \otimes \mathsf{x}_k^i$ and of any basic 2-cochain $\mathsf{x}_{l_1}^{j_1*} \wedge \mathsf{x}_{l_2}^{j_2*} \otimes \mathsf{x}_k^i$ is clearly given by:

$$\begin{aligned} \text{homogeneity}\big(\mathsf{x}_l^{j*} \otimes \mathsf{x}_k^i\big) &= -l + k, \\ \text{homogeneity}\big(\mathsf{x}_{l_1}^{j_1*} \wedge \mathsf{x}_{l_2}^{j_2*} \otimes \mathsf{x}_k^i\big) &= -l_1 - l_2 + k, \end{aligned}$$

so that minimal values and maximal values of homogeneities are equal to:

$$\begin{aligned} &\text{1-cochains } \Psi : \quad -a+1 \leqslant \text{homogeneity} \leqslant a+b \\ &\text{2-cochains } \Phi : \quad -a+2 \leqslant \text{homogeneity} \leqslant 2a+b. \end{aligned}$$

Thus, if we denote by the letter h the homogeneity of a cochain, in order to split our two cochains:

$$\Psi = \sum_{h=-a+1}^{a+b} \Psi^{[h]} \quad \text{and} \quad \Phi = \sum_{h=-a+2}^{2a+b} \Phi^{[h]}$$

as cochains having constant homogeneity h (for any $h \in \mathbb{Z}$), and if we introduce the following two sets of integers:

$$\begin{aligned} \Delta_1^{[h]} &:= \big\{(l,j) \in \mathbb{Z} \times \mathbb{N} \colon -a \leqslant l \leqslant -1,\ j = 1, \dots, d_{(l)},\ -a \leqslant l + h \leqslant b\big\}, \\ \Delta_2^{[h]} &:= \big\{(l_1, j_1, l_2, j_2) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{N} \colon \\ &\qquad \begin{array}{l} -a \leqslant l_1 \leqslant -1,\ j_1 = 1, \dots, d_{(l_1)}, \\ -a \leqslant l_2 \leqslant -1,\ j_2 = 1, \dots, d_{(l_2)}, \end{array} -a \leqslant l_1 + l_2 + h \leqslant b\big\}, \end{aligned}$$

we can expand $\Psi^{[h]}$ for $-a+1 \leqslant h \leqslant a+b$ and $\Phi^{[h]}$ for $-a+2 \leqslant h \leqslant 2a+b$ as:

$$\begin{aligned} \Psi^{[h]} &= \sum_{(l,j) \in \Delta_1^{[h]}} \sum_{i=1}^{d_{(l+h)}} \psi_{l,j}^{l+h,i}\, \mathsf{x}_l^{j*} \otimes \mathsf{x}_{l+h}^i \qquad (-a+1 \leqslant h \leqslant a+b) \\ \Phi^{[h]} &= \sum_{\substack{(l_1,j_1,l_2,j_2) \in \Delta_2^{[h]} \\ (l_1,j_1) <_{\mathsf{lex}} (l_2,j_2)}} \sum_{k=1}^{d_{(l_1+l_2+h)}} \phi_{l_1,j_1,l_2,j_2}^{l_1+l_2+h,k}\, \mathsf{x}_{l_1}^{j_1*} \wedge \mathsf{x}_{l_2}^{j_2*} \otimes \mathsf{x}_{l_1+l_2+h}^k \qquad (-a+2 \leqslant h \leqslant 2a+b), \end{aligned}$$

where $(l_1, j_1) <_{\mathsf{lex}} (l_2, j_2)$ means either $l_1 < l_2$ or $l_1 = l_2$ but $j_1 < j_2$.

10.13. **Boundary of h-homogeneous 1-cochains.** Thus, fix a homogeneity h for a 1-cochain with $-a+1 \leqslant h \leqslant a+b$, let $(l,j) \in \Delta_1^{[h]}$, let also $(l_1, j_1, l_2, j_2) \in \Delta_2^{[h]}$

and compute the boundary of an arbitrary basic h-homogeneous 1-cochain:

$$\begin{aligned}\partial^1_{[h]}(\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h})(\mathsf{x}^{j_1}_{l_1}, \mathsf{x}^{j_2}_{l_2}) &= \big[\mathsf{x}^{j_1}_{l_1}, (\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h})(\mathsf{x}^{j_2}_{l_2})\big]_{\mathfrak{g}} - \big[\mathsf{x}^{j_2}_{l_2}, (\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h})(\mathsf{x}^{j_1}_{l_1})\big]_{\mathfrak{g}} - (\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h})\big(\big[\mathsf{x}^{j_1}_{l_1}, \mathsf{x}^{j_2}_{l_2}\big]_{\mathfrak{g}}\big)\\
&= [\mathsf{x}^{j_1}_{l_1}, \delta^l_{l_2}\delta^j_{j_2}\mathsf{x}^i_{l+h}]_{\mathfrak{g}} - [\mathsf{x}^{j_2}_{l_2}, \delta^l_{l_1}\delta^j_{j_1}\mathsf{x}^i_{l+h}]_{\mathfrak{g}} - \big(\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h}\big)\Big(\sum_{k=1}^{d_{(l_1+l_2)}} c^{j_1,j_2,k}_{l_1,l_2}\,\mathsf{x}^k_{l_1+l_2}\Big)\\
&= \delta^l_{l_2}\delta^j_{j_2}\sum_{k=1}^{d_{l_1+l+h}} c^{j_1,i,k}_{l_1,l+h}\,\mathsf{x}^k_{l_1+l+h} - \delta^l_{l_1}\delta^j_{j_1}\sum_{k=1}^{d_{(l_2+l+h)}} c^{j_2,i,k}_{l_2,l+h}\,\mathsf{x}^k_{l_2+l+h} - \big(\delta^l_{l_1+l_2}c^{j_1,j_2,j}_{l_1,l_2}\big)\,\mathsf{x}^i_{l+h}\\
&= \sum_{k=1}^{d_{(l_1+l_2+h)}} \big(\delta^l_{l_2}\delta^j_{j_2}c^{j_1,i,k}_{l_1,l+h} - \delta^l_{l_1}\delta^j_{j_1}c^{j_2,i,k}_{l_2,l+h} - \delta^l_{l_1+l_2}\delta^k_i c^{j_1,j_2,j}_{l_1,l_2}\big)\,\mathsf{x}^k_{l_1+l_2+h}.\end{aligned}$$

In other words, we have obtained the following representation for the differential of any basic h-homogeneous 1-cochain:

$$\begin{aligned}\partial^1_{[h]}\big(\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h}\big) = \sum_{\substack{(l_1,j_1,l_2,j_2)\in\Delta^{[h]}_2\\ (l_1,j_1)<_{\mathsf{lex}}(l_2,j_2)}} \sum_{k=1}^{d_{(l_1+l_2+h)}} &\big(\delta^l_{l_2}\delta^j_{j_2}c^{j_1,i,k}_{l_1,l+h} - \delta^l_{l_1}\delta^j_{j_1}c^{j_2,i,k}_{l_2,l+h} - \delta^l_{l_1+l_2}\delta^k_i c^{j_1,j_2,j}_{l_1,l_2}\big)\cdot\\
&\cdot \mathsf{x}^{j_1*}_{l_1} \wedge \mathsf{x}^{j_2*}_{l_2} \otimes \mathsf{x}^k_{l_1+l_2+h} \qquad \big((j,l)\in\Delta^{[h]}_1,\ i=1\cdots d_{(l+h)}\big).\end{aligned}$$

Now by linearity, we can compute the boundary of a general h-homogeneous 1-cochain:

$$\begin{aligned}\partial^1_{[h]}\Psi^{[h]} &= \sum_{(l,j)\in\Delta^{[h]}_1}\sum_{i=1}^{d_{(l+h)}} \psi^{l+h,i}_{l,j}\,\partial^1_{[h]}(\mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h})\\
&= \sum_{(l_1,j_1,l_2,j_2)\in\Delta^{[h]}_2}\sum_{k=1}^{d_{(l_1+l_2+h)}}\Bigg(\sum_{(l,j)\in\Delta^{[h]}_1}\sum_{i=1}^{d_{(l+h)}}\\
&\qquad \Big(\delta^l_{l_2}\delta^j_{j_2}c^{j_1,i,k}_{l_1,l+h} - \delta^l_{l_1}\delta^j_{j_1}c^{j_2,i,k}_{l_2,l+h} - \delta^l_{l_1+l_2}\delta^i_k c^{j_1,j_2,j}_{l_1,l_2}\Big)\psi^{l+h,i}_{l,j}\Bigg)\,\mathsf{x}^{j_1*}_{l_1} \wedge \mathsf{x}^{j_2*}_{l_2} \otimes \mathsf{x}^k_{l_1+l_2+h}.\end{aligned}$$

Proposition 10.5. *Under the above assumptions, an arbitrary h-homogeneous 2-cochain:*

$$\Phi^{[h]} = \sum_{\substack{(l_1,j_1,l_2,j_2)\in\Delta^{[h]}_2\\ (l_1,j_1)<_{\mathsf{lex}}(l_2,j_2)}} \sum_{k=1}^{d_{(l_1+l_2+h)}} \phi^{l_1+l_2+h,k}_{l_1,j_1,l_2,j_2}\,\mathsf{x}^{j_1*}_{l_1} \wedge \mathsf{x}^{j_2*}_{l_2} \otimes \mathsf{x}^k_{l_1+l_2+h}$$

with $-a+2 \leqslant h \leqslant 2a+b$ is the boundary $\Phi^{[h]} = \partial^1_{[h]}\Psi^{[h]}$ of a h-homogeneous 1-cochain:

$$\Psi^{[h]} = \sum_{(l,j)\in\Delta^{[h]}_1}\sum_{i=1}^{d_{(l+h)}} \psi^{l+h,i}_{l,j}\ \mathsf{x}^{j*}_l \otimes \mathsf{x}^i_{l+h}$$

if and only if its homogeneous degree h satisfies in fact $-a+1 \leqslant h \leqslant a+b$ and if all its coefficients $\phi^{l_1+l_2+h,k}_{(l_1,j_1,l_2,j_2)}$ are uniquely determined as the following linear *combinations of the* $\psi_\cdot^{\cdot}$:

$$\phi_{l_1,j_1,l_2,j_2}^{l_1+l_2+h,k} = \sum_{(l,j)\in\Delta_1^{[h]}} \sum_{i=1}^{d_{(l+h)}} \Big(\delta_{l_2}^{l}\,\delta_{j_2}^{j}\,c_{l_1,l+h}^{j_1,i,k} - \delta_{l_1}^{l}\,\delta_{j_1}^{j}\,c_{l_2,l+h}^{j_2,i,k} - \delta_{l_1+l_2}^{l}\,\delta_k^i\,c_{l_1,l_2}^{j_1,j_2,j}\Big)\psi_{l,j}^{l+h,i}.$$

10.14. **Boundary of h-homogeneous 2-cochains.** Next, for any h with $-a+2 \leqslant h \leqslant 2a+b$, for any $(l_1,j_1,l_2,j_2) \in \Delta_2^{[h]}$, for any $k = 1,\dots,d_{(l_1+l_2+h)}$ and for any $(l_1',j_1',l_2',j_2',l_3',j_3') \in \Delta_3^{[h]}$, belonging to the set:

$$\begin{aligned}\Delta_3^{[h]} := \big\{(l_1',j_1',l_2',j_2',l_3',j_3') \in \mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\colon \\ -a \leqslant l_1' \leqslant -1,\ j_1' = 1,\dots,d_{(l_1')}, \\ -a \leqslant l_2' \leqslant -1,\ j_2' = 1,\dots,d_{(l_2')}, \\ -a \leqslant l_3' \leqslant -1,\ j_3' = 1,\dots,d_{(l_3')},\quad -a \leqslant l_1'+l_2'+l_3'+h \leqslant b\big\},\end{aligned}$$

by applying the definitional formula (42), we obtain the value of the boundary of a basic 2-cochain:

$$\begin{aligned}\partial_{[h]}^2\big(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k}\big)\big(\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_2'}^{j_2'},\mathsf{x}_{l_3'}^{j_3'}\big) = \quad & \big[\mathsf{x}_{l_1'}^{j_1'},\,(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})(\mathsf{x}_{l_2'}^{j_2'},\mathsf{x}_{l_3'}^{j_3'})\big]_{\mathfrak{g}}- \\ -\big[\mathsf{x}_{l_2'}^{j_2'},\,(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})(\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_3'}^{j_3'})\big]_{\mathfrak{g}} &+ \big[\mathsf{x}_{l_3'}^{j_3'},\,(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})(\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_2'}^{j_2'})\big]_{\mathfrak{g}}- \\ -(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})\big([\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_2'}^{j_2'}]_{\mathfrak{g}},\,\mathsf{x}_{l_3'}^{j_3'}\big) &+ (\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})\big([\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_3'}^{j_3'}]_{\mathfrak{g}},\,\mathsf{x}_{l_2'}^{j_2'}\big)- \\ -(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})\big([\mathsf{x}_{l_2'}^{j_2'},\mathsf{x}_{l_3'}^{j_3'}]_{\mathfrak{g}},\,\mathsf{x}_{l_1'}^{j_1'}\big).&\end{aligned}$$

Let us focus attention on terms at the first and fourth lines, since other terms are obtained by obvious permutations of triples $\big((l_1',j_1'),(l_2',j_2'),(l_3',j_3')\big)$. The term in the first line continues as:

$$\begin{aligned}\big[\mathsf{x}_{l_1'}^{j_1'},\,(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})(\mathsf{x}_{l_2'}^{j_2'},\mathsf{x}_{l_3'}^{j_3'})\big]_{\mathfrak{g}} &= \big[\mathsf{x}_{l_1'}^{j_1'},\,\delta_{l_2'}^{l_1}\,\delta_{j_2'}^{j_1}\,\delta_{l_3'}^{l_2}\,\delta_{j_3'}^{j_2}\,\mathsf{x}_{l_1+l_2+h}\big]_{\mathfrak{g}} \\ &= \delta_{l_2'}^{l_1}\,\delta_{j_2'}^{j_1}\,\delta_{l_3'}^{l_2}\,\delta_{j_3'}^{j_2} \sum_{s=1}^{d_{(l_1'+l_1+l_2+h)}} c_{l_1',l_1+l_2+h}^{j_1',k,s}\,\mathsf{x}_{l_1+l_2+l_1'+h}^{s} \\ &= \sum_{s=1}^{d_{(l_1'+l_2'+l_3'+h)}} \Big(\delta_{l_2'}^{l_1}\,\delta_{j_2'}^{j_1}\,\delta_{l_3'}^{l_2}\,\delta_{j_3'}^{j_2}\,c_{l_1',l_1+l_2+h}^{j_1',k,s}\Big)\,\mathsf{x}_{l_1'+l_2'+l_3'+h}^{s}.\end{aligned}$$

The term in the fourth line continues as:

$$\begin{aligned}&-(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})\big([\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_2'}^{j_2'}]_{\mathfrak{g}},\,\mathsf{x}_{l_3'}^{j_3'}\big) = \\ &\qquad = -(\mathsf{x}_{l_1}^{j_1*}\wedge\mathsf{x}_{l_2}^{j_2*}\otimes\mathsf{x}_{l_1+l_2+h}^{k})\Big(\sum_{s=1}^{d_{(l_1'+l_2')}} c_{l_1',l_2'}^{j_1',j_2',s}\,\mathsf{x}_{l_1'+l_2'}^{s},\ \mathsf{x}_{l_3'}^{j_3'}\Big) \\ &\qquad = -\Big(\sum_{s=1}^{d_{(l_1'+l_2')}} c_{l_1',l_2'}^{j_1',j_2',s}\,\big(\delta_{l_1'+l_2'}^{l_1}\,\delta_s^{j_1}\,\delta_{l_3'}^{l_2}\,\delta_{j_3'}^{j_2} - \delta_{l_3'}^{l_1}\,\delta_{j_3'}^{j_1}\,\delta_{l_1'+l_2'}^{l_2}\,\delta_s^{j_2}\big)\Big)\,\mathsf{x}_{l_1+l_2+h}^{k} \\ &\qquad = \sum_{s=1}^{d_{(l_1'+l_2'+l_3'+h)}} \Big(\delta_k^s\,\big[-\delta_{l_1'+l_2'}^{l_1}\,\delta_{l_3'}^{l_2}\,\delta_{j_3'}^{j_2}\,c_{l_1',l_2'}^{j_1',j_2',j_1} + \delta_{l_3'}^{l_1}\,\delta_{j_3'}^{j_1}\,\delta_{l_1'+l_2'}^{l_2}\,c_{l_1',l_2'}^{j_1',j_2',j_2}\big]\Big)\,\mathsf{x}_{l_1'+l_2'+l_3'+h}^{s}.\end{aligned}$$

Finally, bringing back the two pairs of terms left above which are obtained by simple permutations, we obtain the value of the boundary of a basic 2-cochain:

$$
\begin{aligned}
&\partial^2_{[h]}\big(\mathsf{x}_{l_1}^{j_1 *}\wedge\mathsf{x}_{l_2}^{j_2 *}\otimes\mathsf{x}_{l_1+l_2+h}^{k}\big)\big(\mathsf{x}_{l_1'}^{j_1'},\mathsf{x}_{l_2'}^{j_2'},\mathsf{x}_{l_3'}^{j_3'}\big) = \\
&= \sum_{s=1}^{d_{(l_1'+l_2'+l_3'+h)}} \Big(\delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_1+l_2+h}^{j_1',k,s} - \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_2',l_1+l_2+h}^{j_2',k,s} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_3',l_1+l_2+h}^{j_3',k,s} + \\
&\qquad + \delta_k^s\big[-\delta_{l_1'+l_2'}^{l_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_2'}^{j_1',j_2',j_1} + \delta_{l_3'}^{l_1}\delta_{j_3'}^{j_1}\delta_{l_1'+l_2'}^{l_2}\, c_{l_1',l_2'}^{j_1',j_2',j_2} + \delta_{l_1'+l_3'}^{l_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_1',l_3'}^{j_1',j_3',j_1} - \\
&\qquad - \delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_1'+l_3'}^{l_2}\, c_{l_1',l_3'}^{j_1',j_3',j_2} - \delta_{l_2'+l_3'}^{l_1}\delta_{l_1'}^{l_2}\delta_{j_1'}^{j_2}\, c_{l_2',l_3'}^{j_2',j_3',j_1} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'+l_3'}^{l_2}\, c_{l_2',l_3'}^{j_2',j_3',j_2}\big]\Big)\,\mathsf{x}_{l_1'+l_2'+l_3'+h}^{s}.
\end{aligned}
$$

Equivalently:

$$
\begin{aligned}
&\partial^2_{[h]}\big(\mathsf{x}_{l_1}^{j_1 *}\wedge\mathsf{x}_{l_2}^{j_2 *}\otimes\mathsf{x}_{l_1+l_2+h}^{k}\big) = \sum_{(l_1',j_1',l_2',j_2',l_3',j_3')\in\Delta_3^{[h]}} \ \sum_{s=1}^{d_{(l_1'+l_2'+l_3'+h)}} \\
&\Big(\delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_1+l_2+h}^{j_1',k,s} - \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_2',l_1+l_2+h}^{j_2',k,s} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_3',l_1+l_2+h}^{j_3',k,s} + \\
&+ \delta_k^s\big[-\delta_{l_1'+l_2'}^{l_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_2'}^{j_1',j_2',j_1} + \delta_{l_3'}^{l_1}\delta_{j_3'}^{j_1}\delta_{l_1'+l_2'}^{l_2}\, c_{l_1',l_2'}^{j_1',j_2',j_2} + \delta_{l_1'+l_3'}^{l_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_1',l_3'}^{j_1',j_3',j_1} - \\
&- \delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_1'+l_3'}^{l_2}\, c_{l_1',l_3'}^{j_1',j_3',j_2} - \delta_{l_2'+l_3'}^{l_1}\delta_{l_1'}^{l_2}\delta_{j_1'}^{j_2}\, c_{l_2',l_3'}^{j_2',j_3',j_1} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'+l_3'}^{l_2}\, c_{l_2',l_3'}^{j_2',j_3',j_2}\big]\Big)\,\mathsf{x}_{l_1'}^{j_1' *}\wedge\mathsf{x}_{l_2'}^{j_2' *}\wedge\mathsf{x}_{l_3'}^{j_3' *}\otimes\mathsf{x}_{l_1'+l_2'+l_3'+h}^{s}
\end{aligned}
$$

Now, by linearity, we deduce the boundary $\partial^2_{[h]}\Phi^{[h]}$ of a general h-homogeneous 2-cochain $\Phi^{[h]}$, and this yields the following statement.

Proposition 10.6. *Under the above assumptions, the boundary of a general h-homogeneous 2-cochain:*

$$
\Phi^{[h]} = \sum_{\substack{(l_1,j_1,l_2,j_2)\in\Delta_2^{[h]} \\ (l_1,j_1)<_{\mathsf{lex}}(l_2,j_2)}} \ \sum_{k=1}^{d_{(l_1+l_2+h)}} \phi_{l_1,j_1,l_2,j_2}^{l_1+l_2+h,k}\ \mathsf{x}_{l_1}^{j_1 *}\wedge\mathsf{x}_{l_2}^{j_2 *}\otimes\mathsf{x}_{l_1+l_2+h}^{k},
$$

where h satisfies $-a+2\leqslant h\leqslant 2a+b$ which has arbitrary coefficients $\phi_{l_1,j_1,l_2,j_2}^{l_1+l_2+h,k}\in\mathbb{K}$, is a cocycle, namely satisfies $0=\partial^2_{[h]}\Phi^{[h]}$, if and only if all the following linear equations hold:

$$
\begin{aligned}
0 = &\sum_{\substack{(l_1,j_1,l_2,j_2)\in\Delta_2^{[h]} \\ (l_1,j_1)<_{\mathsf{lex}}(l_2,j_2)}} \ \sum_{k=1}^{d_{(l_1+l_2+h)}} \Big(\delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_1+l_2+h}^{j_1',k,s} - \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_2',l_1+l_2+h}^{j_2',k,s} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_3',l_1+l_2+h}^{j_3',k,s} + \\
&+ \delta_k^s\big[-\delta_{l_1'+l_2'}^{l_1}\delta_{l_3'}^{l_2}\delta_{j_3'}^{j_2}\, c_{l_1',l_2'}^{j_1',j_2',j_1} + \delta_{l_3'}^{l_1}\delta_{j_3'}^{j_1}\delta_{l_1'+l_2'}^{l_2}\, c_{l_1',l_2'}^{j_1',j_2',j_2} + \delta_{l_1'+l_3'}^{l_1}\delta_{l_2'}^{l_2}\delta_{j_2'}^{j_2}\, c_{l_1',l_3'}^{j_1',j_3',j_1} - \\
&- \delta_{l_2'}^{l_1}\delta_{j_2'}^{j_1}\delta_{l_1'+l_3'}^{l_2}\, c_{l_1',l_3'}^{j_1',j_3',j_2} - \delta_{l_2'+l_3'}^{l_1}\delta_{l_1'}^{l_2}\delta_{j_1'}^{j_2}\, c_{l_2',l_3'}^{j_2',j_3',j_1} + \delta_{l_1'}^{l_1}\delta_{j_1'}^{j_1}\delta_{l_2'+l_3'}^{l_2}\, c_{l_2',l_3'}^{j_2',j_3',j_2}\big]\Big)\,\phi_{l_1,j_1,l_2,j_2}^{l_1+l_2+h,k}.
\end{aligned}
$$

Further considerations accompanied with an algorithm using Gröbner bases may be found in [1].

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