EFFECTIVE ALGEBRAIC DEGENERACY

SIMONE DIVERIO, JOËL MERKER, AND ERWAN ROUSSEAU

ABSTRACT. We show that for every generic smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geqslant 2$, there exists a proper algebraic subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ which lies in Y, provided $\deg X \geqslant 2^{n^5}$.

Table of contents

1. INTRODUCTION

In 1979, Green and Griffiths [8] conjectured that every projective algebraic variety X of general type contains a certain *proper* algebraic *sub*variety $Y \subsetneqq X$ inside which all nonconstant entire holomorphic curves $f\colon \mathbb{C} \to X$ must necessarily lie.

A positive answer to this conjecture has been given for surfaces by McQuillan [11] under the assumption that the second Segre number $c_1^2 - c_2$ is positive. In the survey article [21] (*cf.* also [20]), Siu provided a beautiful strategy to establish algebraic degeneracy of entire holomorphic curves in generic hypersurfaces $X \subset \mathbb{P}^{n+1}$ of high degree larger than a certain $d_n \gg 1$, and also *Kobayashi-hyperbolicity* of such X's if d_n is even much higher.

Siu's strategy is based on two key steps: 1) the explicit construction, in projective coordinates, of global holomorphic jet differentials; 2) the deformation of such jet differentials by means of slanted vector fields having low pole order. The explicit construction of jet differentials can be seen as a replacement of the argument using Riemann-Roch which is known to be difficult to realize since it involves a control of the cohomology. The reason to perform explicit constructions is also a better access to the base-point set, in order to provide hyperbolicity instead of just algebraic degeneracy. Complete up-to-date survey considerations may further be found in [22, 4, 12, 5, 10, 25].

In this paper, we overcome the difficulty of the Riemann-Roch argument thanks to an alternative approach for Siu's first key step based on Demailly's bundle of invariant jets [4]. The advantage of this method is also that it usually yields better bounds on the degree. Indeed, after performing in Sections 4 and 5 below some explicit, delicate elimination computations, we finally obtain a lower bound on the degree $d_n = d(n)$ as an explicit function of n, for generic projective hypersurfaces of arbitrary dimension $n \geqslant 2$.

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d and of arbitrary dimension $n \geqslant 2$. If X is generic and if its degree satisfies the* effective *lower bound:*

$$d \geqslant 2^{n^5},$$

then there exists a proper *algebraic subvariety* $Y \subsetneqq X$ *such that every nonconstant entire holomorphic curve* $f\colon \mathbb{C} \to X$ *has image* $f(\mathbb{C})$ *contained in* Y.

As in [20, 21], we thereby confirm, for generic projective hypersurfaces of high degree, the Green-Griffiths-Lang conjecture. Even if our lower bound is far from the one $\deg X \geqslant n+3$ insuring general type, to our knowledge, Theorem 1.1 is, in this direction, the first n-dimensional result with, moreover, an explicit degree lower bound. In addition, as a byproduct of our constructions, the subvarieties absorbing the images of nonconstant entire curves vary as a holomorphic family with the generic projective hypersurface.

Two main ingredients enter our proof: 1) the existence of invariant jet differentials vanishing on an ample divisor in projective hypersurfaces of high degree, following [4, 6]; and Siu's second key step: 2) the global generation of a sufficiently high twisting of the tangent bundle to the so-called *manifold of vertical* n*-jets*, which is canonically associated to the universal family of projective hypersurfaces, following [21, 13].

The first ingredient dates back to the seminal work of Bloch [1], revisited by Green-Griffiths in [8], by Siu in [19, 22, 21] and by Demailly in [4]. Bloch's main philosophical idea is that global jet differentials vanishing on an ample divisor provide some algebraic differential equations that every entire holomorphic curve $f\colon \mathbb{C} \to X$ must satisfy. Five decades later, Green and Griffiths [8] modernized Bloch's concepts and established several results — still fundamental nowadays — about the geometry of entire curves.

Later on, Demailly [4] refined and enlarged the whole theory by defining jet differentials that are invariant under reparametrization of the source $\mathbb{C}$. Through this geometrically adequate, new point of view, one looks only at the conformal class of all entire curves. In [6, 7], the first-named author combined Demailly's approach with Trapani's [23] algebraic version of the holomorphic Morse inequalities, so as to construct global invariant jet differentials in *any* dimension $n \geqslant 2$. The first effective aspect of our proof is to make somewhat explicit such a construction.

Indeed, by following [6, 7], we consider a certain intersection product (*see* (10) and (13) below), the positivity of which yields — thanks to a suitable application of the holomorphic Morse inequalities — a lower bound for the (asymptotic) dimension of the space of global sections of a certain *weighted subbundle* of Demailly's full bundle $E_{n,m}T_X^*$ of invariant n-jet differentials. This intersection product lives in the cohomology algebra of the n-th projectivized jet bundle over X, a polynomial algebra in n indeterminates $u_1, u_2, \dots, u_n$ equipped with canonical, geometrically significant relations ([4, 6]). The u_i here are the first Chern classes of the successive (anti)tautological line bundles which arise during the projectivization process. The task of reducing the mentioned intersection product in terms of the Chern classes of T_X — after eliminating *all* the Chern classes living at each level of Demailly's tower — happens to be of high algebraic complexity, because four combinatorics are intertwined there: 1) presence of several relations shared by all the Chern classes of the lifted horizontal distributions; 2) Newton expansion of large n^2-powers; 3) differences of various binomial coefficients; 4) emergence of many Jacobi-Trudy determinants.

The second ingredient, *viz.* the *vertical jets*, comes from ideas developed for 1-jets by Voisin [24] in order to generalize works of Clemens [3] and Ein on the positivity of the canonical bundles of subvarieties of generic projective hypersurfaces of high degree. In [21], Siu showed how the corresponding *global generation property* for 1-jets devised by Voisin generalizes to the bundle of tangents to the space of vertical n-jets. Siu then established that one may use the available tangential generators, which are meromorphic vector fields with a certain *pole order* $c_n \geqslant 1$, so as to produce, by plain differentiation, many new algebraically independent invariant jet differentials when starting from just a single *nonzero* jet differential. At the end, one obtains in this way sufficiently many independent jet differentials, and this then forces entire curves to lie in a positive-codimensional subvariety $Y \subsetneqq X$.

This strategy was realized in details for 2-jets in dimension 2 by Păun [15] with pole order $c_2 = 7$, and similarly, for 3-jets in dimension 3 by the third-named author in [18] with $c_3 = 12$. In both works, global generation holds outside a certain exceptional set. The general case of n-jets in dimension n was performed recently by the second-named author in [13] with $c_n = \frac{n^2+5n}{2}$ and with a quite similar exceptional set. It then became clear, when [13] appeared, that Demailly's invariant jets combined with Siu's second key step could yield *weak* algebraic degeneracy (nonexistence of Zariski-dense entire curves) in *any* dimension $n \geqslant 2$. But to reach effectivity, it yet remained to perform what the present article is aimed at: taming somehow the complicated combinatorics of Demailly's tower. Furthermore, at the cost of increasing the pole order up to $c_n' = n^2 + 2n$, the exceptional set is shrunk to be just the set of singular jets ([13]), and then *strong effective* algebraic degeneracy is gained. This is Theorem 1.1.

As the effective lower bound $\deg X \geqslant 2^{n^5}$ of the main theorem above is not optimal, Sections 6 and 7 of the paper are intended to provide numerically better estimates in small dimensions. For surfaces, the best known effective lower bound for the degree is $d \geqslant 18$ ([15]), after $d \geqslant 21$ ([5]) and $d \geqslant 36$ ([12]). In [18], the third-named author obtained the first effective result for weak algebraic degeneracy of entire curves inside threefolds X of $\mathbb{P}^4$, whenever $\deg X \geqslant 593$.

Theorem 1.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d. If X is generic, then there exists a proper closed subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ contained in Y*

- *for* $\dim X = 3$, *whenever* $\deg X \geqslant 593$;
- *for* $\dim X = 4$, *whenever* $\deg X \geqslant 3203$;
- *for* $\dim X = 5$, *whenever* $\deg X \geqslant 35355$;
- *for* $\dim X = 6$, *whenever* $\deg X \geqslant 172925$.

The last three effective lower bounds in dimensions 4, 5 and 6 are entirely new. In dimension 3, our bound 593 is the same as in [18]. Indeed, an inspection of the exceptional set in [18] shows that the part of the degeneracy locus which may depend on f is in fact of codimension 2 (*cf.* [13]), and therefore is empty, thanks to Clemens' result [3] which excludes elliptic and rational curves. Using $c_4 = 18$ and $c_5 = 25$ instead of $c_4' = 24$ and $c_5' = 35$, we would have obtained the two lower bounds $\deg X \geqslant 2432$ and $\deg X \geqslant 25586$ which were announced in our first `arxiv.org` preprint and which insured only *weak* algebraic degeneracy (*cf.* [13]; using $c_6 = 33$ instead of $c_6' = 48$, the bound would be $\deg X \geqslant 120176$).

For dimensions 5 and 6, our strategy of proof is the same as for Theorem 1.1, except that we choose a numerically better weighted subbundle of Demailly's bundle of invariant jet differentials, exactly as in [6].

Quite differently, for dimensions 3 and 4, the construction of nonzero jet differentials is based on a *complete* algebraic description of the full Demailly bundles $E_{n,m}T_X^*$, $n = 3, 4$, due respectively to the third-named author ([16]) and to the second-named author ([14]), after Demailly [4] and Demailly-El Goul [5] for $n = 2$. The invariant theory approach requires finding the composition series of the $E_{n,m}T_X^*$, but this is understood only in dimensions 2, 3 and 4, because of the proliferation of secondary invariants — a well known phenomenon, *cf.* [14] and the references therein. Then by appropriately summing the Euler characteristics of the composing Schur bundles [16], taking account of the numerous syzygies shared by a collection of fundamental bi-invariants [14], one establishes the positivity of the Euler characteristics $\chi\big(E_{n,m}T_X^*\big)$ for $n = 3, 4$, at least asymptotically as m goes to infinity. Furthermore, realizing also in dimension 4 the strategy finalized in dimension 3 by the third-named author [17], we estimate from above the contribution of the even cohomology dimensions $h^{2i}\big(X, E_{n,m}T_X^*\big)$, thereby gaining a suitable lower bound for the dimension of the space $h^0\big(X, E_{n,m}T_X^*\big)$ of global sections. Such estimates are done by means of Demailly's [4] generalization of a vanishing theorem due to Bogomolov for the top cohomology, and also by means of the algebraic version of the weak holomorphic Morse inequalities for the intermediate cohomologies [17].

Even if the numerical bounds obtained in this way in dimensions 3 and 4 are better than the ones we obtained in all dimensions, the extreme intricacy of the algebras of invariants by reparametrization (*cf.* [14]) is the main obstacle to run the process in the higher dimensions $n \geqslant 5$. This was our central motivation to follow the strategy of [6, 7].

Acknowledgments. The first-named author warmly thanks Stefano Trapani for patiently listening all the details of the proof of the main theorem.

2. Preliminaries

2.1. **Jet differentials.** We briefly present here useful geometric concepts selected from the theory of Green-Griffiths' and Demailly's jets [8, 4] (*cf.* also [16, 6]). Let (X, V) be a *directed manifold*, *i.e.* a pair consisting of a complex manifold X together with a (not necessarily integrable) holomorphic subbundle $V \subset T_X$ of the tangent bundle to X. This category will be very useful later on, when we will consider the situation where X is the universal family of projective hypersurfaces of fixed degree and V the relative tangent bundle to the family. The bundle J_kV is the bundle of k-jets of germs of holomorphic curves $f\colon (\mathbb{C}, 0) \to X$ which are tangent to V, *i.e.*, such that $f'(t) \in V_{f(t)}$ for all t near 0, together with the projection map $f \mapsto f(0)$ onto X.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1\, t + a_2\, t^2 + \cdots + a_k\, t^k, \quad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2$$

of $(\mathbb{C}, 0)$, the composition law being taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ admits a natural fiberwise right action on J_kV which consists in reparametrizing k-jets of curves by such changes φ of parameters. In [13], one finds the multivariate FaÃă di Bruno formulae yielding explicit reparametrization for the so-called *absolute case* $V = T_X$. Moreover the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda\, t$ is a (non-normal) subgroup of $\mathbb{G}_k$ and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}_k' \ltimes \mathbb{H}$, where $\mathbb{G}_k'$ is the group of k-jets of biholomorphisms tangent to the identity, *i.e.* with $a_1 = 1$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot \big(f', f'', \ldots, f^{(k)}\big) = \big(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}\big). \tag{1}$$

As in [8], we introduce the *Green-Griffiths vector bundle* $E_{k,m}^{GG}V^* \to X$, the fibers of which are complex-valued polynomials $Q(f', f'', \ldots, f^{(k)})$ in the fibers of J_kV having weighted degree m with respect to the $\mathbb{C}^*$ action, namely such that:

$$Q\big(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}\big) = \lambda^m Q\big(f', f'', \ldots, f^{(k)}\big),$$

for all $\lambda \in \mathbb{C}^*$ and all $\big(f', f'', \ldots, f^{(k)}\big) \in J_kV$. Demailly refined this concept.

Definition 2.1 ([4])**.** The *bundle of invariant jet differentials of order* k *and weighted degree* m is the subbundle $E_{k,m}V^* \subset E_{k,m}^{GG}V^*$ of polynomial differential operators $Q(f', f'', \dots, f^{(k)})$ which are invariant under *arbitrary* changes of parametrization, *i.e.* which, for every $\varphi \in \mathbb{G}_k$, satisfy:

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \dots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m\, Q\big(f', f'', \dots, f^{(k)}\big).$$

Alternatively, $E_{k,m}V^* = \big(E_{k,m}^{GG}V^*\big)^{\mathbb{G}'_k}$ is the set of invariants of $E_{k,m}^{GG}V^*$ under the action of $\mathbb{G}'_k$.

We now define a filtration on $E_{k,m}^{GG}V^*$. A coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ having, for any s with $1 \leqslant s \leqslant k$, the partial weighted degrees $|\ell|_s := |\ell_1| + 2|\ell_2| + \cdots + s|\ell_s|$, into a new polynomial $\big((\Psi\circ f)^{(\bullet)}\big)^\ell$ in $(f', f'', \dots, f^{(k)})$, which has the same partial weighted degree of order s when $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise (use the chain rule). Hence, for each $s = 1, \dots, k$, we get a well defined decreasing filtration $F_s^\bullet$ on $E_{k,m}^{GG}V^*$ as follows:

$$F_s^p\big(E_{k,m}^{GG}V^*\big) = \left\{\begin{matrix} Q(f', f'', \dots, f^{(k)}) \in E_{k,m}^{GG}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix}\right\}, \quad \forall\, p \in \mathbb{N}.$$

The graded terms $\mathrm{Gr}_{k-1}^p\left(E_{k,m}^{GG}V^*\right)$ associated with the $(k-1)$-filtration $F_{k-1}^p(E_{k,m}^{GG}V^*)$ are the homogeneous polynomials $Q(f', f'', \dots, f^{(k)})$ all the monomials $(f^{(\bullet)})^\ell$ of which have partial weighted degree $|\ell|_{k-1} = p$; hence, their degree ℓ_k in $f^{(k)}$ is such that $m-p = k\ell_k$ and $\mathrm{Gr}_{k-1}^p(E_{k,m}^{GG}V^*) = 0$ unless $k|m-p$. Looking at the transition automorphisms of the graded bundle induced by the coordinate change $f \mapsto \Psi\circ f$, it turns out that $f^{(k)}$ transforms as an element of $V \subset T_X$ and, by means of a simple computation, one finds

$$\mathrm{Gr}_{k-1}^{m-k\ell_k}\left(E_{k,m}^{GG}V^*\right) = E_{k-1,m-k\ell_k}^{GG}V^* \otimes S^{\ell_k}V^*.$$

Combining all filtrations $F_s^\bullet$ together, we find inductively a filtration $F^\bullet$ on $E_{k,m}^{GG}V^*$ the graded terms of which are

$$\mathrm{Gr}^\ell\left(E_{k,m}^{GG}V^*\right) = S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*, \quad \ell \in \mathbb{N}^k,\ \ |\ell|_k = m.$$

Moreover ([4]), invariant jet differentials enjoy the natural induced filtration:

$$F_s^p(E_{k,m}V^*) = E_{k,m}V^* \cap F_s^p\big(E_{k,m}^{GG}V^*\big),$$

the associated graded bundle being, if we employ $(\bullet)^{\mathbb{G}'_k}$ to denote $\mathbb{G}'_k$-invariance:

$$\mathrm{Gr}^\bullet(E_{k,m}V^*) = \left(\bigoplus_{|\ell|_k=m} S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*\right)^{\mathbb{G}'_k}.$$

2.2. **Projectivized k-jet bundles.** Next, we recall briefly Demailly's construction [4] of the tower of projectivized bundles providing a (relative) smooth compactification of $J_k^{\mathrm{reg}}V/\mathbb{G}_k$, where $J_k^{\mathrm{reg}}V$ is the bundle of *regular k-jets tangent to V*, that is, k-jets such that $f'(0) \neq 0$.

Let (X, V) be a directed manifold, with $\dim X = n$ and rank $V = r$. With (X, V), we associate another directed manifold $(\widetilde{X}, \widetilde{V})$ where $\widetilde{X} = P(V)$ is the projectivized bundle of lines of V, $\pi\colon \widetilde{X} \to X$ is the natural projection and $\widetilde{V}$ is the subbundle of $T_{\widetilde{X}}$ defined fiberwise as

$$\widetilde{V}_{(x_0,[v_0])} \stackrel{\mathrm{def}}{=} \big\{\xi \in T_{\widetilde{X},(x_0,[v_0])} \mid \pi_*\xi \in \mathbb{C}\cdot v_0\big\},$$

for any $x_0 \in X$ and $v_0 \in T_{X,x_0} \setminus \{0\}$. We also have a "lifting" operator which assigns to a germ of holomorphic curve $f\colon (\mathbb{C},0) \to X$ tangent to V a germ of holomorphic curve $\widetilde{f}\colon (\mathbb{C},0) \to \widetilde{X}$ tangent to $\widetilde{V}$ in such a way that $\widetilde{f}(t) = (f(t), [f'(t)])$.

To construct the projectivized k-jet bundle we simply set inductively $(X_0, V_0) = (X, V)$ and $(X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1})$. Clearly rank $V_k = r$ and $\dim X_k = n + k(r-1)$. Of course, we have for each $k > 0$ a tautological line bundle $\mathcal{O}_{X_k}(-1) \to X_k$ and a natural projection $\pi_k\colon X_k \to X_{k-1}$. We call $\pi_{j,k}$ the composition of the projections $\pi_{j+1} \circ \cdots \circ \pi_k$, so that the total projection is given by $\pi_{0,k}\colon X_k \to X$. We have, for each $k > 0$, two short exact sequences

(2) $$0 \to T_{X_k/X_{k-1}} \to V_k \to \mathcal{O}_{X_k}(-1) \to 0,$$

(3) $$0 \to \mathcal{O}_{X_k} \to \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0.$$

Here, we also have an inductively defined k-lifting for germs of holomorphic curves such that $f_{[k]}\colon (\mathbb{C},0) \to X_k$ is obtained as $f_{[k]} = \widetilde{f}_{[k-1]}$.

Theorem 2.1 ([4]). *Suppose that* rank $V \geqslant 2$. *The quotient $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k \hookrightarrow X_k$ over X, which identifies $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k$ with X_k^{reg}, that is the set of points in X_k on the form $f_{[k]}(0)$ for some non singular k-jet f. In other words X_k is a relative compactification of $J_k^{\mathrm{reg}} V / \mathbb{G}_k$ over X. Moreover, one has the direct image formula:*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}\big(E_{k,m} V^*\big).$$

Next, we are in position to recall the fundamental application of jet differentials to Kobayashi-hyperbolicity and to Green-Griffiths algebraic degeneracy.

Theorem 2.2 ([8, 22, 4]). *Assume that there exist integers $k, m > 0$ and an ample line bundle $A \to X$ such that*

$$H^0\big(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\big) \simeq H^0\big(X, E_{k,m} V^* \otimes A^{-1}\big)$$

has non zero sections $\sigma_1, \dots, \sigma_N$ and let $Z \subset X_k$ be the base locus of these sections. Then every entire holomorphic curve $f\colon \mathbb{C} \to X$ tangent to V necessarily satisfies $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant differential equation P vanishing on an ample divisor, every entire holomorphic curve f must satisfy the algebraic differential equation $P\big(j^k f(t)\big) \equiv 0$. Furthermore, the same result also holds true for the bundle $E_{k,m}^{GG} T_X^*$.

2.3. **Existence of invariant jet differentials.** Now, we recall some results obtained by the first-named author in [7], concerning the existence of invariant jet differentials on projective hypersurfaces which generalized to all dimensions n previous works by Demailly [4] and of the third-named author [17].

Denote by $c_\bullet(E)$ the total Chern class of a vector bundle E. The two short exact sequences (2) and (3) give, for each $k > 0$, the following two formulae:

$$c_\bullet(V_k) = c_\bullet\big(T_{X_k/X_{k-1}}\big)\, c_\bullet\big(\mathcal{O}_{X_k}(-1)\big)$$
$$c_\bullet\big(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)\big) = c_\bullet\big(T_{X_k/X_{k-1}}\big),$$

so that by a plain substitution:

(4) $$c_\bullet(V_k) = c_\bullet\big(\mathcal{O}_{X_k}(-1)\big)\, c_\bullet\big(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)\big).$$

Let us call $u_j = c_1\big(\mathcal{O}_{X_j}(1)\big)$ and $c_l^{[j]} = c_l(V_j)$. With these notations, (4) becomes:

$$c_l^{[k]} = \sum_{s=0}^{l} \big[\tbinom{n-s}{l-s} - \tbinom{n-s}{l-s-1}\big] u_k^{l-s} \cdot \pi_k^* c_s^{[k-1]}, \quad 1 \leqslant l \leqslant r. \tag{5}$$

Since X_j is the projectivized bundle of line of V_{j-1}, we also have the polynomial relations

$$u_j^r + \pi_j^* c_1^{[j-1]} \cdot u_j^{r-1} + \cdots + \pi_j^* c_{r-1}^{[j-1]} \cdot u_j + \pi_j^* c_r^{[j-1]} = 0, \quad 1 \leqslant j \leqslant k. \tag{6}$$

After all, the cohomology ring of X_k is defined in terms of generators and relations as the polynomial algebra $H^\bullet(X)[u_1, \dots, u_k]$ with the relations (6) in which, using inductively (5), one may express in advance all the $c_l^{[j]}$ as certain polynomials with integral coefficients in the variables $u_1, \dots, u_j$ and $c_1(V), \dots, c_l(V)$. In particular, for the first Chern class of V_k, a simple explicit formula is available:

$$c_1^{[k]} = \pi_{0,k}^* c_1(V) + (r-1) \sum_{s=1}^{k} \pi_{s,k}^* \, u_s. \tag{7}$$

Also, it is classically known that the Chern classes $c_j(X)$ of a smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$ are polynomials in $d := \deg X$ and the hyperplane class $h := c_1\big(\mathcal{O}_{\mathbb{P}^{n+1}}(1)\big)$, *viz.* for $1 \leqslant j \leqslant n$:

$$c_j(X) = c_j(T_X) = (-1)^j \, h^j \, \sum_{i=0}^{j} (-1)^i \tbinom{n+2}{i} \, d^{j-i}. \tag{8}$$

Now, let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree $\deg X = d$ and consider, for all what follows in the sequel, the absolute case $V = T_X$ with jet order $k = n$ equal to the dimension. Given any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we define (*cf.* [4, 6]) the following line bundle $\mathcal{O}_{X_n}(\mathbf{a})$ on X_n:

$$\mathcal{O}_{X_n}(\mathbf{a}) = \pi_{1,n}^* \mathcal{O}_{X_1}(a_1) \otimes \pi_{2,n}^* \mathcal{O}_{X_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{X_n}(a_n).$$

Using the algebraic version — first appeared in Trapani's article [23] — of Demailly's holomorphic Morse inequalities, the first-named author showed in [7] that, in order to check the *bigness* of $\mathcal{O}_{X_n}(1)$, it suffices to show the *positivity*, for some $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ lying arbitrarily in the cone defined by:

$$a_1 \geqslant 3a_2, \dots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1, \tag{9}$$

of the following intersection product:

$$F^N - N \, F^{N-1} \cdot G,$$

where $N = \dim X_n = n^2$, and where the two bundles $F := \mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)$ and $G := \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)$ are both globally nef on X_n ([7], Proposition 2); here, $\mathcal{O}_X(1)$ is the hyperplane bundle over X and we abbreviate $|\mathbf{a}| := a_1 + \cdots + a_n$. In other words, we express $\mathcal{O}_{X_n}(\mathbf{a})$ as a "difference" $F \otimes G^{-1}$ between two nef line bundles over X_n:

$$\mathcal{O}_{X_n}(\mathbf{a}) = \big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big) \otimes \big(\pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{-1}.$$

Thus in sum, we have to find some $\mathbf{a} \in \mathbb{Z}^n$ lying in the cone (9) for which the concerned intersection product written in length:

$$\begin{aligned} &\big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ &\qquad - n^2 \big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|) \end{aligned} \tag{10}$$

is positive. This was done by the first-named author, and an application of the mentioned Morse inequalities yielded the following.

Theorem 2.3 ([7]). *Let $X \subset \mathbb{P}^{n+1}$ by a smooth complex hypersurface of degree $\deg X = d$ and fix any ample line bundle $A \to X$. Then, for jet order $k = n$ equal to the dimension, there exists a positive integer d_n such that the two isomorphic spaces of sections:*

$$H^0\big(X_n, \mathcal{O}_{X_n}(m) \otimes \pi_{0,n}^* A^{-1}\big) \simeq H^0\big(X, E_{n,m}T_X^* \otimes A^{-1}\big) \neq 0,$$

are nonzero*, whenever $d \geqslant d_n$ provided that m is large enough.*

It is also proved in [6] that for any jet order $k < n$ *smaller* than the dimension, no nonzero sections, though, are available: $H^0\big(X_k,\, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\big) = 0$; in fact, this vanishing property is used as a technical tool in the proof of Theorem 2.3.

In our applications, it will be crucial to be able to control in a more precise way the order of vanishing of these differential operators along the ample divisor. Thus, we shall need here a slightly different theorem, inspired from [21, 15, 18]. Recall at first that for X a smooth projective hypersurface of degree d in $\mathbb{P}^{n+1}$, the canonical bundle has the following expression in terms of the hyperplane bundle:

$$K_X \simeq \mathcal{O}_X(d - n - 2),$$

whence it is ample as soon as $d \geqslant n + 3$.

Theorem 2.4. *Let $X \subset \mathbb{P}^{n+1}$ by a smooth complex hypersurface of degree $\deg X = d$. Then, for all positive rational numbers δ small enough, there exists a positive integer d_n such that the space of twisted jet differentials:*

$$H^0\big(X_n, \mathcal{O}_{X_n}(m) \otimes \pi_{0,n}^* K_X^{-\delta m}\big) \simeq H^0\big(X, E_{n,m}T_X^* \otimes K_X^{-\delta m}\big) \neq 0,$$

is nonzero, whenever $d \geqslant d_{n,\delta}$ provided again that m is large enough and that δm is an integer.

Observe that all nonzero sections $\sigma \in H^0\big(X, E_{n,m}T_X^* \otimes K_X^{-\delta m}\big)$ then have vanishing order at least equal to $\delta m(d - n - 2)$, when viewed as sections of $E_{n,m}T_X^*$.

Proof of Theorem 2.4. For each weight $\mathbf{a} \in \mathbb{N}^n$ satisfying (9), we first of all express $\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* K_X^{-\delta|\mathbf{a}|}$ as the following difference of two nef line bundles:

$$\big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big) \otimes \big(\pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big)^{-1}.$$

In order to apply the algebraic holomorphic Morse inequalities to obtain the existence of sections for high powers, we are thus led to compute the following intersection product:

$$\begin{aligned}(11)\qquad & \big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ & - n^2 \big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \big(\pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big),\end{aligned}$$

and to decide whether it is positive. After reducing it in terms of the Chern classes of X, and then in terms of $d = \deg X$ using (8), this intersection product becomes a polynomial — difficult to compute explicitly, but effective aspects will start in Section 4 — in d of degree less than or equal to $n + 1$, having coefficients which are polynomials in $(\mathbf{a}, \delta)$ of bidegree $(n^2, 1)$, homogeneous in $\mathbf{a}$ or identically zero. Notice that for $\delta = 0$, the intersection product identifies with (10); we claim that there exists a weight $\mathbf{a}'$ such that (10) is positive. Thus by continuity, with the same choice of weight, for all $\delta > 0$ small enough, the leading coefficient still remains positive. So the polynomial in question again takes only positive values when $d \geqslant d_n$, for some (noneffective) d_n. Holomorphic Morse inequalities then insure the existence of nonzero sections.

Coming back to our claim, the argument is as follow. First of all, the three intersection products: (10), $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$ and $\big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^*\mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2}$, once evaluated with respect to the degree d of the hypersurface, are all polynomials in the variable d with coefficients in $\mathbb{Z}[a_1, \dots, a_n]$ of degree at most $n+1$ and the coefficients of d^{n+1} of the three expressions are the same (*cf.* Proposition 3 in [7]). Next, by Proposition 2 in [7], $\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^*\mathcal{O}_X(2|\mathbf{a}|)$ is nef if $\mathbf{a}$ satisfies (9); therefore the coefficient of d^{n+1} of its top self-intersection must be non-negative. Thus, by Lemma 1 in [7], in order to find a weight $\mathbf{a}'$ in the cone defined by (9) as in the claim, it suffices to show that this coefficient is not an identically zero polynomial in $\mathbb{Z}[a_1, \dots, a_n]$. So, we have to prove that it contains at least one non-zero monomial: but by Lemma 3 in [7], the coefficient of its monomial $a_1^n \cdot a_2^n \cdots a_n^n$ is $(n^2)!/(n!)^n$ and we are done (*cf.* also Subsection 4.4). □

2.4. **Global generation of the tangent bundle to the variety of vertical jets.** We now briefly present the second ingredient, as said in the Introduction. Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d^n}$ be the universal family of projective n-dimensional hypersurfaces of degree d in $\mathbb{P}^{n+1}$; its parameter space is the projectivization $\mathbb{P}\big(H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))\big) = \mathbb{P}^{N_d^n}$, where $N_d^n = \binom{n+d+1}{d} - 1$. We have two canonical projections:

$$\begin{array}{ccc} & \mathcal{X} & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathbb{P}^{n+1} & & \mathbb{P}^{N_d^n}. \end{array}$$

Consider the relative tangent bundle $\mathcal{V} \subset T_{\mathcal{X}}$ with respect to the second projection $\mathcal{V} := \ker(\mathrm{pr}_2)_*$, and form the corresponding directed manifold $(\mathcal{X}, \mathcal{V})$. It is clear that $\mathcal{V}$ is integrable and that any entire holomorphic curve from $\mathbb{C}$ to $\mathcal{X}$ tangent to $\mathcal{V}$ has its image entirely contained in some fiber $\mathrm{pr}_2^{-1}(s) = X_s$, $s \in \mathbb{P}^{N_d^n}$.

Now, let $p\colon J_n\mathcal{V} \to \mathcal{X}$ be the bundle of n-jets of germs of holomorphic curves in $\mathcal{X}$ tangent to $\mathcal{V}$, the so-called *vertical jets*, and consider the subbundle $J_n^{\mathrm{reg}}\mathcal{V}$ of *regular n-jets* of maps $f\colon (\mathbb{C}, 0) \to \mathcal{X}$ tangent to $\mathcal{V}$ such that $f'(0) \neq 0$.

Theorem 2.5 ([13]). *The twisted tangent bundle to vertical n-jets:*

$$T_{J_n\mathcal{V}} \otimes p^*\mathrm{pr}_1^*\,\mathcal{O}_{\mathbb{P}^{n+1}}(n^2+2n) \otimes p^*\mathrm{pr}_2^*\,\mathcal{O}_{\mathbb{P}^{N_d^n}}(1)$$

is generated over $J_n^{\mathrm{reg}}\mathcal{V}$ by its global holomorphic sections. Moreover, one may choose such global generating vector fields to be invariant with respect to the reparametrization action of $\mathbb{G}_n$ on $J_n\mathcal{V}$.

This means that we have enough independent, global, invariant vector fields having *meromorphic* coefficients over $J_n\mathcal{V}$ in order to linearly generate the tangent space $T_{J_n\mathcal{V},j^n}$ at every arbitrary fixed regular jet $j^n \in J_n^{\mathrm{reg}}\mathcal{V}$. The poles of these vector fields occur only in the base variables of $\mathcal{X}$, but not in the vertical jet variables of positive differentiation order. *Most importantly*, the maximal pole order here is $\leqslant n^2+2n$, hence it is compensated by the first twisting $(\bullet) \otimes p^*\mathrm{pr}_1^*\,\mathcal{O}_{\mathbb{P}^{n+1}}(n^2+2n)$.

3. Algebraic degeneracy of entire curves

Now, we are fully in position to establish the *noneffective* version of Theorem 1.1. The proof (*cf.* the Introduction) incorporates two main ingredients: 1) the existence, already established by Theorem 2.4, of at least *one* nonzero global invariant jet differential vanishing on an ample divisor; 2) Theorem 2.5 just above to produce sufficiently many *new algebraically independent* jet differentials.

Theorem 3.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of arbitrary dimension $n \geqslant 2$. Then there exists a positive integer d_n such that whenever $\deg X \geqslant d_n$ and X is generic, there exists a* proper *algebraic subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ contained in Y.*

Proof. As above, consider the universal projective hypersurface $\mathbb{P}^{n+1} \xleftarrow{\mathrm{pr}_1} \mathcal{X} \xrightarrow{\mathrm{pr}_2} \mathbb{P}^{N_d^n}$ of degree d in $\mathbb{P}^{n+1}$. Observe that $X_s = \mathrm{pr}_2^{-1}(s)$ is a smooth projective hypersurface of $\mathbb{P}^{n+1}$ for generic $s \in \mathbb{P}^{N_d^n}$ and that $\mathcal{V} = \ker(\mathrm{pr}_2)_*$ restricted to X_s coincides with the tangent bundle to X_s. We infer therefore that:

$$H^0\big(X_s,\, E_{n,m}\mathcal{V}^* \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)\big|_{X_s}\big) \simeq H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big).$$

Thanks to Theorem 2.4, the latter space of sections is nonzero, for small rational $\delta > 0$, for $d \geqslant d_{n,\delta}$ and for m large enough, independently of s. Fix any $s_0 \in \mathbb{P}^{N_d^n}$ and pick a nonzero jet differential $P_0 \in H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m}\big)$. In order to employ the vector fields of Theorem 2.5, we must at first extend P_0 as a *holomorphic family* of nonzero jet differentials. Thus, we invoke the following classical extension result.

Theorem 3.2 ([9], p. 288)**.** *Let $\tau\colon \mathcal{Y} \to S$ be a flat holomorphic family of compact complex spaces and let $\mathcal{L} \to \mathcal{Y}$ be a holomorphic vector bundle. Then there exists a proper subvariety $Z \subset S$ such that for each $s_0 \in S \setminus Z$, the restriction map $H^0\big(\tau^{-1}(U_{s_0}), \mathcal{L}\big) \to H^0\big(\tau^{-1}(s_0), \mathcal{L}|_{\tau^{-1}(s_0)}\big)$ is onto, for some Zariski-dense open set $U_{s_0} \subset S$ containing s_0.*

We remark that this theorem implies that the weighted degree of the jet differential constructed above may be chosen to be independent of the hypersurface X_s of degree d. Now, we apply this statement $\tau = \mathrm{pr}_2$, to $\mathcal{Y} = \mathcal{X}$, to $S = \mathbb{P}^{N_d^n}$, to $\mathcal{L} = E_{n,m}\mathcal{V}^* \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)$ and we similarly denote by $Z \subset \mathbb{P}^{N_d^n}$ the embarrassing proper algebraic subvariety. The genericity of X assumed in the two theorems 1.1 and 3.1 will just consist in requiring that $s_0 \notin Z$ (notice *passim* that we do not have a constructive access to Z) and of course also, that s does not belong to the set for which X_s is singular.

We therefore obtain a holomorphic family of jet differentials:

$$P = \big\{P|_s \in H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)\big\}$$

parametrized by s with $P|_{s_0} = P_0 \not\equiv 0$ and vanishing on $K_{X_s}^{\delta m}$; for our purposes, it will suffice that s varies in some neighborhood of s_0.

Now, take a *nonconstant* entire holomorphic curve $f\colon \mathbb{C} \to \mathcal{X}$ tangent to $\mathcal{V}$. Since the distribution $\mathcal{V}$ has integral manifolds $\mathrm{pr}_2^{-1}(s) = X_s$, f maps $\mathbb{C}$ into some X_{s_0}, for some $s_0 \in \mathbb{P}^{N_d^n}$. Of course, we assume that $s_0 \notin Z$ and that X_{s_0} is non-singular. Consider now the zero-set locus

$$Y_{s_0} := \big\{x \in X_{s_0} \colon P|_{s_0}(x) = 0\big\},$$

where $P|_{s_0} \not\equiv 0$ vanishes as a section of the vector bundle $E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m}$. Then Y_{s_0} is a *proper algebraic subvariety* of X_{s_0}. We then claim that

$$f(\mathbb{C}) \subset Y_{s_0},$$

which will complete the proof of the theorem. (It will even come out that we obtain strong algebraic degeneracy of entire curves $f : \mathbb{C} \to X_s$ inside a $Y_s \subsetneqq X_s$ defined by $Y_s = \big\{x \in X : \ P|_s(x) = 0\big\}$ and parametrized by s near s_0.)

Reasoning by contradiction, suppose that there exists $t_0 \in \mathbb{C}$ with $f(t_0) \notin Y_{s_0}$. Consider the n-jet map $j^n f\colon \mathbb{C} \to J_n\mathcal{V}$ induced by f. If $j^n f(\mathbb{C})$ would be entirely contained in $J_n^{\mathrm{sing}}\,\mathcal{V} \overset{\mathrm{def}}{=} J_n\mathcal{V} \setminus J_n^{\mathrm{reg}}\,\mathcal{V}$, then f would be *constant*, since singular n-jets satisfy $f'(t) = 0$.

So necessarily $j^n f(\mathbb{C}) \not\subset J_n \mathcal{V}^{\rm sing}$, namely $f' \not\equiv 0$. Then by shifting a bit t_0 if necessary, we can assume that we in addition have $f'(t_0) \neq 0$, *viz.* $j^n f(t_0) \in J_n^{\rm reg} \mathcal{V}$.

Theorem 2.2 ensures that $P|_{s_0}\big(j^n\, f(t)\big) \equiv 0$. Denote $U := \mathbb{P}^{N_d^n} \setminus Z$.

We may now view the family $P = \{P|_s\}$ as being a holomorphic map

$$P\colon\, J_n\mathcal{V}\big|_{\mathrm{pr}_2^{-1}(U)} \longrightarrow p^*\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)\big|_{\mathrm{pr}_2^{-1}(U)}$$

which is polynomial of weighted degree m in the jet variables. Let V be any of the global invariant holomorphic vector fields on $J_n\mathcal{V}$ with values in $p^*\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}(n^2+2n)$ that were provided by Theorem 2.5. Then we observe that the Lie derivative $L_V P$ together with the natural duality pairing

$$\mathcal{O}_{\mathbb{P}^{n+1}}(p) \times \mathcal{O}_{\mathbb{P}^{n+1}}(-q) \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(p-q) \qquad\qquad (p, q \geqslant 1)$$

provides a new holomorphic map (notice the shift by n^2+2n):

$$L_V P\colon\, J_n\mathcal{V}\big|_{\mathrm{pr}_2^{-1}(U)} \longrightarrow p^*\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)+n^2+2n\big)\big|_{\mathrm{pr}_2^{-1}(U)},$$

again polynomial of weighted degree m in the jet variables, thus a new parameterized family of invariant jet differentials. In particular, the restriction $L_V P|_{s_0}$ of $L_V P$ to $\{s = s_0\}$ yields a *nonzero* global holomorphic section in

$$H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m} \otimes \mathcal{O}_{X_{s_0}}(n^2+2n)\big) = \\ = H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes \mathcal{O}_{X_{s_0}}(-\delta m(d-n-2)+n^2+2n)\big),$$

which is a global invariant jet differential on X_{s_0} vanishing on an ample divisor provided that $-\delta m(d-n-2)+n^2+2n$ *still remains negative*; therefore, if we ensure such a negativity (*see* below), Theorem 2.2 shows that $[L_V P|_{s_0}]\big(j^n f(t)\big) \equiv 0$. As a result, the n-jet of f now satisfies *two* global algebraic differential equations:

$$P_{s_0}\big(j^n f(t)\big) \equiv \big[L_V P|_{s_0}\big]\big(j^n f(t)\big) \equiv 0.$$

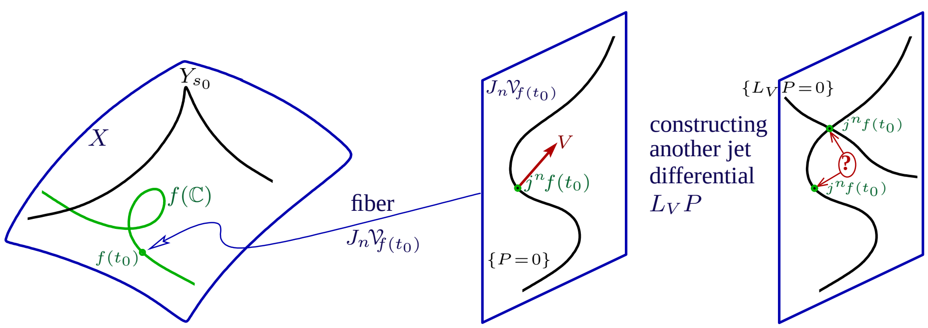

Fig. 1: Producing from P a new jet differential $L_V P$ having distinct zero locus in $J_n\mathcal{V}$

Heuristically (*cf.* the figure), if the fiber $J_n\mathcal{V}_{f(t_0)}$ would be, say, 2-dimensional, and if the intersection of $\{P_{s_0} = 0\}$ with $\{L_V P|_{s_0} = 0\}$, viewed in the fiber $J_n\mathcal{V}_{f(t_0)}$, would be a point *distinct from the original* $j^n f(t_0)$, we would get the sought contradiction. Now we realize this idea (*cf.* [21, 15, 18]) by producing enough new jet differential divisors whose intersection becomes *empty*.

Indeed, with t_0 such that $f(t_0) \notin Y_{s_0}$ and $j^n f(t_0) \in J_n^{\rm reg}\mathcal{V}$, and with W_i, V_j denoting some global meromorphic vector fields in

$$H^0\big(J_n\mathcal{V},\, T_{J_n\mathcal{V}} \otimes p^*\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^{n+1}}(n^2+2n) \otimes p^*\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{N_d^n}}(1)\big),$$

that are supplied by Theorem 2.5, we claim that the following two *evidently contradictory* conditions can be satisfied, and this will achieve the proof.

(i) For every $p \leqslant m$ and for arbitrary such fields $W_1, \dots, W_p$, the restriction $L_{W_p} \cdots L_{W_1} P\big|_{s_0}$ yields a nonzero global holomorphic section in

$$H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes \mathcal{O}_{X_{s_0}}(-\delta m(d-n-2) + p(n^2+2n))\big)$$

with the property that $\big[L_{W_p} \cdots L_{W_1} P\big]\big(s_0,\, j^n f(t)\big) \equiv 0$.

(ii) there exist some $p \leqslant m$ and some invariant fields $V_1, \dots, V_p$ such that $\big[L_{V_p} \cdots L_{V_1} P\big]\big(s_0,\, j^n f(t_0)\big) \neq 0$.

The first condition **(i)** will automatically be ensured by Theorem 2.2 provided the resulting jet differential still vanishes on an ample divisor, *i.e.* provided that

$$-\delta m(d-n-2) + p(n^2+2n) < 0$$

is still negative. But since p will be $\leqslant m$, it suffices that $-\delta m(d-n-2) + m(n^2+2n) < 0$, and then after erasing m, that:

$$d > \tfrac{n^2+2n}{\delta} + n + 2. \tag{12}$$

To get **(i)**, we first fix a rational $\delta > 0$ so that Theorem 2.4 gives a *nonzero* jet differential for any $d \geqslant d_{n,\delta}$, we increase (if necessary) this lower bound by taking account of (12), we construct the holomorphic family $P|_s$, and **(i)** holds.

To establish **(ii)**, we choose local coordinates:

$$\big(s, z, z', \dots, z^{(n)}\big) \in \mathbb{C}^{N_d^n} \times \mathbb{C}^n \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n$$

on $J_n\mathcal{V}$ near $\big(s_0, j^n f(t_0)\big)$, where $z \in \mathbb{C}^n$ provides some local coordinates on X_s for any fixed s near s_0, and where $\big(z', \dots, z^{(n)}\big)$ are the jet coordinates associated with z. We also choose a local trivialization of the line bundle $K_{X_s}^{-\delta m}$. Then our holomorphic family of jet differentials $P|_s \in H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)$ writes locally as a weighted m-homogeneous jet-polynomial:

$$P = \sum_{|i_1| + \cdots + n|i_n| = m} q_{i_1,\dots,i_n}(s,z)\,(z')^{i_1} \cdots (z^{(n)})^{i_n},$$

where $i_1, \dots, i_n \in \mathbb{N}^n$ and where the $q_{i_1,\dots,i_n}(s,z)$ are holomorphic near $(s_0, f(t_0))$. Locally, the proper subvariety $Y_{s_0} \subset X$ is represented as the common zero-locus:

$$Y_{s_0} = \big\{z \in X_{s_0} \colon q_{i_1,\dots,i_n}(s_0, z) = 0,\ \ \forall\, i_1, \dots, i_n\big\}.$$

By our assumption that $f(t_0) \not\in Y_{s_0}$, there exist $i_1^0, \dots, i_n^0 \in \mathbb{N}^n$ such that $q_{i_1^0,\dots,i_n^0}\big(s_0, f(t_0)\big) \neq 0$. If we make the translational change of jet coordinates $\overline{z}' := z' - f'(t_0), \dots, \overline{z}^{(n)} := z^{(n)} - f^{(n)}(t_0)$, our jet-polynomial transfers to:

$$\overline{P} = \sum_{|i_1| + \cdots + n|i_n| \leqslant m} \overline{q}_{i_1,\dots,i_n}(s,z)\,(\overline{z}')^{i_1} \cdots (\overline{z}^{(n)})^{i_n},$$

(notice "$\leqslant m$") with new coefficients $\overline{q}_{i_1,\dots,i_n}(s,z)$ that depend linearly upon the old ones and polynomially upon $\big(f'(t_0), \dots, f^{(n)}(t_0)\big)$. Again, there exist $\overline{i}_1^0, \dots, \overline{i}_n^0 \in \mathbb{N}^n$ such that $\overline{q}_{\overline{i}_1^0,\dots,\overline{i}_n^0}\big(s_0, f(t_0)\big) \neq 0$, because otherwise the two jet-polynomials $P\big|_{s_0, f(t_0)}$ and $\overline{P}\big|_{s_0, f(t_0)}$ would be both identically zero.

Since $j^n f(t_0) \in J_n^{\mathrm{reg}}\mathcal{V}$, by the property 2.5 of generation by global sections, we get that for every k with $1 \leqslant k \leqslant n$ and for every i with $1 \leqslant i \leqslant n$, there exists an invariant vector field V_i^k with

$$V_i^k\big|_{(s_0, \overline{j}^n f(t_0))} = \frac{\partial}{\partial \overline{z}_i^{(k)}}\Big|_{(s_0, \overline{j}^n f(t_0))},$$

where we have denoted the translated central jet by $\overline{j}^n f(t_0) := \big(f(t_0), 0, \dots, 0\big)$.

To achieve the proof of **(ii)**, we may suppose that for every integer p with $p < |\overline{i}_1^0| + |\overline{i}_2^0| + \cdots + |\overline{i}_n^0|$, whence $p < |\overline{i}_1^0| + 2\,|\overline{i}_2^0| + \cdots + n\,|\overline{i}_n^0| = m$, and for every p invariant vector fields $W_1, \dots, W_p$, one has $\big[W_1 \cdots W_p\, \overline{P}\,\big]\big(s_0, \overline{j}^n f(t_0)\big) = 0$, since if any such an expression is already $\neq 0$, **(ii)** would be got gratuitously. Thanks to the global generation Theorem 2.5, this vanishing property then holds for any vector fields W_i involving all the possible differentiations $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}'}$, $\dots$, $\frac{\partial}{\partial \overline{z}^{(n)}}$. Then under this assumption, the contribution of the remainder differentiations present in V_i^k after $\partial / \partial \overline{z}_i^{(k)}\big|_{(s_0, \overline{j}^n f(t_0))}$ will vanish at the point $\big(s_0, \overline{j}^n f(t_0)\big)$ when performing any multi-derivation of length equal to $|\overline{i}_1^0| + \cdots + |\overline{i}_n^0|$, hence if we write in length $\overline{i}_k^0 = \big(\overline{i}_{k,1}^0, \dots, \overline{i}_{k,n}^0\big) \in \mathbb{N}^n$ all the multiindices present in the specific coefficient $\overline{q}_{\overline{i}_1^0, \dots, \overline{i}_n^0}$, it follows that:

$$
\begin{aligned}
\big[V^n_{\overline{i}^0_{n,n}} \cdots V^n_{\overline{i}^0_{n,1}} \cdots\cdots V^1_{\overline{i}^0_{1,n}} \cdots V^1_{\overline{i}^0_{1,n}}\, \overline{P}\,\big]\big(s_0,\, \overline{j}^n f(t_0)\big) &= \\
= \Big[\frac{\partial}{\partial \overline{z}^{(n)}_{\overline{i}^0_{n,n}}} \cdots \frac{\partial}{\partial \overline{z}^{(n)}_{\overline{i}^0_{n,1}}} \cdots\cdots \frac{\partial}{\partial \overline{z}^{(1)}_{\overline{i}^0_{1,n}}} \cdots \frac{\partial}{\partial \overline{z}^{(1)}_{\overline{i}^0_{1,1}}} \overline{P}\,\Big]&\big(s_0,\, f(t_0), 0, \dots, 0\big) \\
= \overline{i}^0_{n,n}! \cdots \overline{i}^0_{n,1}! \cdots\cdots \overline{i}^0_{1,n}! \cdots \overline{i}^0_{1,1}!\; \overline{q}_{\overline{i}^0_1, \dots, \overline{i}^0_n}&\big(s_0,\, f(t_0)\big) \neq 0,
\end{aligned}
$$

which is nonzero. Thus **(ii)** holds and the proof of Theorem 3.1 is complete. Theorem 3.1 being *not* effective regarding the condition $d \geqslant d_n$, the next two Sections 4 and 5 are devoted to the proof of the effective main Theorem 1.1. □

4. Effectiveness of the degree lower bound

It is known (*cf.* [19, 4, 25, 21, 16, 6, 14]) that reaching an explicit lower bound degree $\deg X \geqslant d_n$ both for Green-Griffiths algebraic degeneracy and for Kobayashi hyperbolicity (in nonoptimal degree) still remained an open question in arbitrary dimension n, due to the existence of *substantial algebraic obstacles*. In order to render somewhat explicit the lower bound d_n of Theorem 3.1, one has to expand the n^2-powered intersection product (11) and then to reduce it as an explicit polynomial $\mathsf{P}_{\mathbf{a},\delta}(d)$, as was foreseen in the proof of Theorem 2.4. To this aim, one should descend Demailly's tower *step by step*, each time using the two relations (5) and (6). As a matter of fact, one must perform some numerous, explicit eliminations and substitutions and thereby tame the exponential growth of computations. At several places, we shall leave aside optimality of majorations in order to reach the neat announced lower bound 2^{n^5}.

4.1. Reduction of the basic intersection product.

We remind from Theorem 2.4 that, in order to produce a global invariant jet differential with controlled vanishing order on hypersurfaces X whose degree $d \geqslant d_n$ would be bounded from below by an effectively known function $d_n = d(n)$ of n, we should ensure *in an effective way* the positivity of the intersection product:

$$
\begin{aligned}
&\big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\
&\qquad - n^2 \big(\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \big(\pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big),
\end{aligned}
$$

for a certain n-tuple of integers $\mathbf{a} = \mathbf{a}(n) \in \mathbb{N}^n$ belonging to the cone (9) (with $k = n$) which would depend *effectively* upon n, and for a certain rational number $\delta = \delta(n) > 0$ which would also depend *effectively* upon n.

As in [7], denote $u_\ell = c_1\big(\mathcal{O}_{X_\ell}(1)\big)$ for $\ell = 1, \dots, n$, denote $c_k = c_k(T_X)$ for $k = 1, \dots, n$, and $h = c_1\big(\mathcal{O}_X(1)\big)$. With these standard notations, the intersection product we have to evaluate becomes:

$$\tag{13} \Pi_\delta := \big(a_1u_1 + \cdots + a_nu_n + 2|\mathbf{a}|h\big)^{n^2} - \\ - n^2\big(a_1u_1 + \cdots + a_nu_n + 2|\mathbf{a}|h\big)^{n^2-1} \cdot \big(2|\mathbf{a}|h - \delta|\mathbf{a}|c_1\big);$$

here and from now on, admitting a slight abuse of notation which will greatly facilitate the reading of formal computations, *we systematically omit every pull-back symbol* $\pi_{j,k}^*(\bullet)$. After elimination and reduction using the relations (5) and (6) (*see* below), our intersection product gives in principle a polynomial (difficult to compute, *see* the end of the paper) of degree $\leqslant n{+}1$ with respect to $d = \deg X$, which is affine in δ, and all of which coefficients are homogeneous polynomials in $\mathbf{a}$ of degree n^2. Thus, let us call it:

$$\mathsf{P}_{\mathbf{a},\delta}(d) = \mathsf{P}_{\mathbf{a}}(d) + \delta\,\mathsf{P}'_{\mathbf{a}}(d) = \sum_{k=0}^{n+1} \mathsf{p}_{k,\mathbf{a}}\, d^k + \delta \sum_{k=0}^{n+1} \mathsf{p}'_{k,\mathbf{a}}\, d^k.$$

Now, suppose in advance that we have an effective control, through explicit inequalities, of all the coefficients $\mathsf{p}_{k,\mathbf{a}} \in \mathbb{Z}$ and $\mathsf{p}'_{k,\mathbf{a}} \in \mathbb{Z}$ of both $\mathsf{P}_{\mathbf{a}}$ and $\mathsf{P}'_{\mathbf{a}}$, and more precisely, that we already know inequalities of the type:

$$|\mathsf{p}_{k,\mathbf{a}}| \leqslant \mathsf{E}_k \quad (k=0,\dots,n), \qquad \mathsf{p}_{n+1,\mathbf{a}} \geqslant \mathsf{G}_{n+1}, \qquad |\mathsf{p}'_{k,\mathbf{a}}| \leqslant \mathsf{E}'_k \quad (k=0,\dots,n,n+1),$$

with the $\mathsf{E}_k \in \mathbb{N}$, with $\mathsf{G}_{n+1} \in \mathbb{N}\setminus\{0\}$ and with the $\mathsf{E}'_k \in \mathbb{N}$ all depending upon n only. According to the proof of Theorem 2.4, a good choice of weight $\mathbf{a}$ indeed makes $\mathsf{p}_{n+1,\mathbf{a}}$ positive; we will see below that $\mathsf{p}'_{n+1,\mathbf{a}}$ is then necessarily negative.

If we now set $\delta := \frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}$ so that δ also depends *a posteriori* explicitly upon n, the leading d^{n+1}-coefficient of $\mathsf{P}_{\mathbf{a},\delta}$ becomes positive and bounded from below:

$$\mathsf{p}_{n+1,\mathbf{a}} + \delta\,\mathsf{p}'_{n+1,\mathbf{a}} = \mathsf{p}_{n+1,\mathbf{a}} - \delta\,\big|\mathsf{p}'_{n+1,\mathbf{a}}\big| \geqslant \mathsf{G}_{n+1} - \tfrac{1}{2}\tfrac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\,\mathsf{E}'_{n+1} = \tfrac{1}{2}\,\mathsf{G}_{n+1}.$$

The largest real root of a polynomial $\mathsf{a}_{n+1}\,d^{n+1} + \mathsf{a}_n\,d^n + \cdots + \mathsf{a}_0$ having integer coefficients and positive leading coefficient $\mathsf{a}_{n+1} \geqslant 1$ may be checked to be less than $1 + (\mathsf{a}_n + \cdots + \mathsf{a}_0)\big/\mathsf{a}_{n+1}$; instead of the finer bound $2\max_{0\leqslant j\leqslant n}\big(\frac{|\mathsf{a}_j|}{|\mathsf{a}_{n+1}|}\big)^{1/n+1-j}$, we use this easier-to-write-down majoration because at the end of Section 4, this will make no difference in reaching the bound $\deg X \geqslant 2^{n^5}$ of Theorem 1.1. Applied to our situation:

Lemma 4.1. *If one chooses* $\delta := \frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}$, *then the intersection product* $\sum_{k=0}^{n+1}\big(\mathsf{p}_{k,\mathbf{a}} + \delta\,\mathsf{p}'_{k,\mathbf{a}}\big)\,d^k$ *has positive leading coefficient* $\mathsf{p}_{n+1,\mathbf{a}} + \delta\,\mathsf{p}'_{n+1,\mathbf{a}} \geqslant \frac{1}{2}\,\mathsf{G}_{n+1}$ *and has other coefficients enjoying the majorations:*

$$\Big|\mathsf{p}_{k,\mathbf{a}} + \delta\,\mathsf{p}'_{k,\mathbf{a}}\Big| \leqslant \mathsf{E}_k + \tfrac{1}{2}\tfrac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\,\mathsf{E}'_k \qquad (k=0,\dots,n),$$

and therefore it takes only positive values for all degrees

$$d \geqslant 1 + \Big(\mathsf{E}_n + \cdots + \mathsf{E}_0 + \tfrac{1}{2}\tfrac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\big\{\mathsf{E}'_n + \cdots + \mathsf{E}'_0\big\}\Big)\Big/\tfrac{1}{2}\,\mathsf{G}_{n+1} =: d_n^1. \quad \square$$

Thus, this d_n^1 will be effectively known in terms of n when E_k, G_{n+1}, E'_k will be so. In order to have not only the existence of global invariant jet differentials with controlled vanishing order, but also algebraic degeneracy, we have also to take account of condition (12), and this condition now reads:

$$d \geqslant 1 + n + 2 + 2\,(n^2 + 2n)\,\frac{\mathsf{E}'_{n+1}}{\mathsf{G}_{n+1}} =: d_n^2.$$

In conclusion, we would obtain the *effective* estimate of Theorem 1.1 provided we compute the bounds E_k, G_{n+1}, E'_k in terms of n and provided we establish that:

$$2^{n^5} \geqslant \max\big\{d_n^1, d_n^2\big\} =: d_n. \tag{14}$$

4.2. **Expanding the intersection product.** By expanding the n^2- and the (n^2-1)-powers, the intersection product Π_δ in (13) writes as a certain sum, with coefficients being polynomials in $\mathbb{Z}\big[a_1, \dots, a_n, \delta\big]$, of monomials in the present Chern classes that are of the general form:

$$h^l u_1^{i_1} \cdots u_n^{i_n} \quad \text{or} \quad h^l c_1 u_1^{j_1} \cdots u_n^{j_n},$$

where $l + i_1 + \cdots + i_n = n^2$ or $l + 1 + j_1 + \cdots + j_n = n^2$.

Lemma 4.2 ([4, 6])**.** *After several elimination computations which take account of the relations* (5) *and* (6)*, any such monomial reduces to a certain polynomial in* $\mathbb{Z}\big[h, c_1, \dots, c_n\big]$ *which is homogeneous of degree* $n = \dim X$*, if* h *is assigned the weight* 1 *and each* c_k *receives the weight* k*. Furthermore, after a last substitution by means of* (8) *which uses* $h^n \equiv \int_X h^n = d = \deg X$*, the polynomial in question becomes a plain polynomial in* $\mathbb{Z}[d]$ *of degree* $\leqslant n+1$*.* □

We illustrate with $h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n}$ three fundamental processes of reduction that will be intensively used. Recall that any *sub*monomial $h^l u_1^{i_1} \cdots u_\ell^{i_\ell} = \pi_{0,\ell}^*(h^l)\pi_{1,\ell}^*(u_1^{i_1}) \cdots u_\ell^{i_\ell}$ denotes a differential form living X_ℓ and that $\dim X_\ell = n + \ell(n-1)$. Such a form is of bidegree (p,p) where $p = l + i_1 + \cdots + i_\ell$. We shall allow the (slight) abuse of language to say that p itself is the *degree* of a (p,p)-form.

At first, if $i_n \leqslant n-2$, then $l + i_1 + \cdots + i_{n-1} \geqslant n^2 - n + 2 = 1 + \dim_{\mathbb{C}} X_{n-1}$, whence the (sub)form $h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}$ which lives on X_{n-1} annihilates, as then does $h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n}$ too. We call this (straightforward) first kind of reduction process:

"vanishing for degree-form reasons",

and we symbolically point out the annihilating subform by underlining it with a small circle appended, *viz.*:

$$\underline{h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}}_{\circ}\, u_n^{i_n} = 0 \qquad \text{when } \ i_n \leqslant n-2.$$

This will greatly improve readability of elimination computations below.

Secondly, in the case where $i_n = n-1$, using an appropriate version of the Fubini theorem and taking account of the fact that $\int_{\text{fiber}} u_n^{n-1} = \int_{\mathbb{P}^{n-1}} u_n^{n-1} = 1$, where all the fibers of $\pi_{n-1,n} : X_n \to X_{n-1}$ are $\simeq \mathbb{P}^{n-1}(\mathbb{C})$ ([4, 18, 6, 7]), we may simplify as follows our monomial:

$$h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \underline{u_n^{n-1}}_{\int} = h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \cdot 1 = h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}.$$

We shall call this second kind of reduction process:

"fiber-integration".

The third process of course consists in substituting the two relations (5) and (6) as many times as necessary. With $r = n$ and without any $\pi_{j,k}^*(\bullet)$, they now read:

$$c_j^{[\ell]} = \sum_{k=0}^{j} \lambda_{j,j-k} \cdot c_k^{[\ell-1]} \big(u_\ell\big)^{j-k}, \tag{15}$$

where $1 \leqslant j,\, \ell \leqslant n$, with the conventions $c_0^{[\ell]} = 1$ and $c_j^{[0]} = c_j$, where we set

$$\lambda_{j,j-k} := \binom{n-k}{j-k} - \binom{n-k}{j-k-1} = \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!},$$

and also, with upper indices of u_ℓ denoting exponents:

$$u_\ell^n = -c_1^{[\ell-1]}\, u_\ell^{n-1} - c_2^{[\ell-1]}\, u_\ell^{n-2} - \cdots - c_{n-1}^{[\ell-1]}\, u_\ell - c_n^{[\ell-1]}. \tag{16}$$

Estimating the coefficient of d^{n+1}. Our first main task is to reach a lower bound $\mathsf{G}_{n+1} - \delta\, \mathsf{E}'_{n+1}$ for the coefficient of d^{n+1} in Π_δ, and this cannot be straightforward, because there are *very numerous* monomials in the expansion of Π_δ. In a first reading, one might jump directly to Subsection 4.4 just after Proposition 4.1. Here is an initial observation.

Lemma 4.3 ([7]). *Assume $l + i_1 + \cdots + i_n = n^2$ or $l + 1 + j_1 + \cdots + j_n = n^2$. Then as soon as $l \geqslant 1$, one has:*

$$0 = \mathsf{coeff}_{d^{n+1}}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big] \quad \text{and} \quad 0 = \mathsf{coeff}_{d^{n+1}}\big[h^l c_1 u_1^{j_1} \cdots u_n^{j_n}\big].$$

Proof. Indeed, after reduction of either u-monomial in terms of the Chern classes c_k of the base, one obtains a sum with integer coefficients of terms of the form:

$$h^l c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}$$

with $l + \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$. But then if we replace the Chern classes by their expressions (8) in terms of h and of the degree, we get:

$$\begin{aligned}
\mathsf{coeff}_{d^{n+1}}\big[h^l c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^{\lambda_1+\cdots+\lambda_n}\, h^n \cdot d^{\lambda_1+\lambda_2+\cdots+\lambda_n} + \mathsf{l.o.t}\big] \\
&= \mathsf{coeff}_{d^{n+1}}\big[(-1)^{\lambda_1+\cdots+\lambda_n}\, d \cdot d^{\lambda_1+\lambda_2+\cdots+\lambda_n} + \mathsf{l.o.t}\big] \\
&= 0,
\end{aligned}$$

since $1 + \lambda_1 + \lambda_2 + \cdots + \lambda_n \leqslant l + \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$. □

As a result, a glance at (13) immediately shows that:

$$\mathsf{coeff}_{d^{n+1}}\big[\Pi_\delta\big] = \mathsf{coeff}_{d^{n+1}}\Big[\big(a_1u_1 + \cdots + a_nu_n\big)^{n^2} + \delta|\mathbf{a}|c_1\big(a_1u_1 + \cdots + a_nu_n\big)^{n^2-1}\Big].$$

4.3. **Reverse lexicographic ordering for the u-monomials.** We order the collection of all homogeneous monomials $u_1^{i_1} \cdots u_n^{i_n}$ with $i_1 + \cdots + i_n = n^2$ appearing in the expansion of $\big(a_1u_1 + \cdots + a_nu_n\big)^{n^2}$ above by declaring that the monomial $u_1^{i_1} \cdots u_n^{i_n}$ is *smaller*, for the *reverse lexicographic ordering*, than another monomial $u_1^{j_1} \cdots u_n^{j_n}$, again of course with $j_1 + \cdots + j_n = n^2$, if:

$$\left\{\begin{array}{l}
\qquad i_n > j_n \\
\text{or if } \; i_n = j_n \; \text{ but } \; i_{n-1} > j_{n-1} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\text{or if } \; i_n = j_n,\, \ldots,\, i_3 = j_3 \; \text{ but } \; i_2 > j_2.
\end{array}\right.$$

Observe that $i_n = j_n, \ldots, i_2 = j_2$ implies $i_1 = j_1$. An equivalent language says that the multiindices themselves are ordered in this way:

$$(i_1, \ldots, i_n) <_{\mathsf{revlex}} (j_1, \ldots, j_n).$$

Proposition 4.1. *The coefficient of d^{n+1} in any monomial $u_1^{i_1} \cdots u_n^{i_n}$ which is* larger *than $u_1^n \cdots u_n^n$ is zero:*

$$\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_n^{i_n}\big] = 0 \quad \text{for any} \quad (i_1, \ldots, i_n) >_{\mathsf{revlex}} (n, \ldots, n).$$

Proof. Thus, assume $(i_1, \ldots, i_n) >_{\mathsf{revlex}} (n, \ldots, n)$. Firstly, if $i_n = n$, the claimed vanishing property is in all concerned subcases yielded by **(iii)** of the lemma just below. Secondly, if $i_n = n-1$, an integration on the fiber of $\pi_{n-1,n} : X_n \to X_{n-1}$ replaces u_n^{n-1} by the constant $+1$, hence we are left with $u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}$ and **(i)** of the same lemma then yields the conclusion. Thirdly and lastly, if $i_n \leqslant n-2$, then the form $u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}$ vanishes identically for degree-form reasons. Thus, granted the lemma, the proposition is proved. □

Lemma 4.4. *The coefficient of d^{n+1} in all the following four sorts of u-monomials is equal to zero:*

- **(i)** $u_1^{i_1} \cdots u_k^{i_k}$ *for any* $k \leqslant n-1$ *and any* $i_1, \ldots, i_k$ *with* $i_1 + \cdots + i_k = n + k(n-1)$;
- **(ii)** $(c_1)^{n-k}\, u_1^{i_1} \cdots u_k^{i_k}$ *for any* $k \leqslant n-1$, *and any* $i_1, \ldots, i_k$ *with* $i_k \leqslant n-1$ *and* $i_1 + \cdots + i_k = kn$;
- **(iii)** $u_1^{i_1} \cdots u_l^{i_l}\, u_{l+1}^n \cdots u_n^n$ *for any* $l \leqslant n$, *any* $i_1, \ldots, i_l$ *with* $i_l \leqslant n-1$ *and* $i_1 + \cdots + i_l = ln$;
- **(iv)** $c_1 u_1^{i_1} \cdots u_l^{i_l} u_{l+1}^n \cdots u_{n-1}^n$ *for any* $l \leqslant n-1$, *any* $i_l \leqslant n-1$, *any* $i_1, \ldots, i_l$ *with* $i_1 + \cdots + i_l = ln$.

Proof. Property **(i)** is established in Section 3 of [7]. So **(i)** holds.

Applying (15) written for $j = 1$, namely $c_1^{[\ell]} = c_1^{[\ell-1]} + (n-1)\, u_\ell$, we get:

$$c_1^{[\ell]} = c_1 + (n-1)\, u_1 + \cdots + (n-1)\, u_\ell. \tag{17}$$

To begin with, we start from **(i)** for $k = n-1$, $i_{n-1} = n$ and $i_1 + \cdots + i_{n-2} = n + (n-1)(n-1) - i_{n-1} = n^2 - 2n + 1$ arbitrary, namely:

$$0 = \mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n \big].$$

Next, thanks to (16), we may replace in this equality u_{n-1}^n by $-c_1^{[n-2]} u_{n-1}^{n-1} - c_2^{[n-2]} u_{n-1}^{n-2} - \cdots - c_n^{[n-2]}$:

$$\begin{aligned}
0 &= \mathsf{coeff}_{d^{n+1}} \Big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big(- c_1^{[n-2]} u_{n-1}^{n-1} - \underline{c_2^{[n-2]} u_{n-1}^{n-2} - \cdots - c_n^{[n-2]}}_{\circ} \big) \Big] \\
&= \mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big(- c_1^{[n-2]}\, u_{n-1}^{n-1} \big) \big] \qquad \text{[degree-form reasons]} \quad \text{[use (17)]} \\
&= \mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big(- c_1 - \underline{(n-1)u_1 - \cdots - (n-1)u_{n-2}}_{\circ} \big)\, u_{n-1}^{n-1} \big] \\
&= \mathsf{coeff}_{d^{n+1}} \big[- c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{n-1} \big] \qquad \text{[apply (i) again]},
\end{aligned}$$

and we therefore get **(ii)** for $k = n-1$ when $i_{n-1} = n-1$. But in all the other remaining cases when $i_{n-1} \leqslant n-2$, then by the assumption that the sum of the indices i_l is equal to $(n-1)n$:

$$i_1 + \cdots + i_{n-2} \geqslant (n-1)n - (n-2) = n^2 - 2n + 2 = \dim X_{n-2},$$

and consequently, the degree of the form $c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}}$ is $\geqslant 1 + \dim X_{n-2}$, whence this form vanishes identically. Thus **(ii)** is proved completely for $k = n-1$.

Next, consider **(iii)** for $l = n$. If $i_n \leqslant n-2$, then by degree-form reasons $0 \equiv u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}$, whence $\mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} \big] = 0$ gratuitously. So we assume $i_n = n-1$. But then $i_1 + \cdots + i_{n-1} = n^2 - n + 1$, hence **(i)** applies to give:

$$\begin{aligned}
0 &= \mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \big] \qquad \text{[reconstitute hidden integration of } u_n^{n-1}\text{]} \\
&= \mathsf{coeff}_{d^{n+1}} \big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{n-1} \big],
\end{aligned}$$

and therefore this proves **(iii)** completely for $l=n$. But we also get at the same time the property **(iii)** for $l=n-1$. Indeed, with $i_1+\cdots+i_{n-1}=(n-1)n$ and with $i_{n-1}\leqslant n-1$, we may reduce, using (16):

$$\begin{aligned} u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}u_n^n &= u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}\big[-c_1^{[n-1]}u_n^{n-1}-\underline{c_2^{[n-1]}u_n^{n-2}-\cdots-c_n^{[n-1]}}_{\circ}\big] \\ &= u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}\big[-c_1^{[n-1]}u_n^{n-1}\big] \qquad \text{[degree-form reasons]} \qquad \text{[use (17)]} \\ &= u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}\big[-c_1-(n-1)u_1-\cdots-(n-1)u_{n-1}\big] \end{aligned}$$

Thanks to **(i)**, after expansion, the pure u-monomials give no contribution to d^{n+1}, and consequently:

$$\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}u_n^n\big]=\mathsf{coeff}_{d^{n+1}}\big[-c_1u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}\big]=0,$$

where the last equality holds true thanks to the property **(ii)** already proved for $k=n-1$. Thus **(iii)** is completely proved for $l=n$ and for $l=n-1$.

Lastly, we just observe that **(iv)** for $l=n-1$ coincides with **(ii)** for $k=n-1$. In summary, we have completed a first loop of proofs.

Consider now the second loop. We start from **(ii)** for $k=n-1$ (already got) with $i_{n-1}=n-1$ and with $i_{n-2}=n$, so that $i_1+\cdots+i_{n-3}=(n-1)n-i_{n-2}-i_{n-1}=n^2-3n+1$, and then we compute:

$$\begin{aligned} 0 &= \mathsf{coeff}_{d^{n+1}}\big[c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}u_{n-2}^n\underline{u_{n-1}^{n-1}}_{\int}\big] \qquad \text{[fiber-integration]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}\big(-c_1^{[n-3]}u_{n-2}^{n-1}-\underline{c_2^{[n-3]}u_{n-2}^{n-2}-\cdots-c_n^{[n-3]}}_{\circ}\big)\big] \qquad \text{[use (16)]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}\big(-c_1^{[n-3]}\big)u_{n-2}^{n-1}\big] \qquad \text{[degree-form reasons]} \qquad \text{[use (17)]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}\big(-c_1-\underline{(n-1)u_1-\cdots-(n-1)u_{n-3}}_{\circ}\big)u_{n-2}^{n-1}\underline{u_{n-1}^{n-1}}_{\int}\big] \\ &= \mathsf{coeff}_{d^{n+1}}\big[-c_1c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}u_{n-2}^{n-1}\underline{u_{n-1}^{n-1}}_{\int}\big] \qquad \text{[apply (ii) for } k=n-1 \text{ again]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[-c_1c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}u_{n-2}^{n-1}\big] \qquad \text{[fiber-integration]}, \end{aligned}$$

where we have reintroduced u_{n-1}^{n-1} (artificially) in the fourth line, so as to apply **(ii)** for $k=n-1$ (got). As a result of the last obtained equation, we have gained **(ii)** for $k=n-2$ when $i_{n-2}=n-1$, but since when $i_{n-2}\leqslant n-2$, the form $c_1c_1u_1^{i_1}\cdots u_{n-3}^{i_{n-3}}$ vanishes identically for degree reasons, we finally have fully established **(ii)** for $k=n-2$.

Next, we look at **(iii)** for $l=n-2$. Then $i_1+\cdots+i_{n-2}=(n-2)n$ with $i_{n-2}\leqslant n-1$. So we ask whether the following coefficient vanishes:

$$\begin{aligned} &\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}u_{n-1}^nu_n^n\big]= \\ &=\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}u_{n-1}^n\big(c_1-\underline{(n-1)u_1-\cdots-(n-1)u_{n-1}}_{\circ}\big)\big] \\ &=\mathsf{coeff}_{d^{n+1}}\big[-c_1u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}u_{n-1}^n\big] \\ &=\mathsf{coeff}_{d^{n+1}}\big[-c_1u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}\big(-c_1-\underline{(n-1)u_1-\cdots-(n-1)u_{n-2}}\big)u_{n-1}^{n-1}\big] \\ &=\mathsf{coeff}_{d^{n+1}}\big[c_1c_1u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}\underline{u_{n-1}^{n-1}}_{\int}\big] \\ &=0, \end{aligned}$$

and in fact, this coefficient vanishes actually, thanks to **(ii)** for $k=n-2$ seen a moment ago. This therefore proves **(iii)** for $l=n-2$ completely.

Finally, consider **(iv)** for $l=n-2$. Then $i_1+\cdots+i_{n-2}=(n-2)n$ and $i_{n-2}\leqslant n-1$. But coming back to the third line of the equations just above, where $i_{n-2}\leqslant n-1$ too, we have in fact already implicitly proved that:

$$0 = \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n\big],$$

and this is **(iv)** for $l = n-2$. Thus, the second loop is completed, and the general induction, similar, is now intuitively clear. □

Corollary 4.1. *The coefficient of d^{n+1} in any monomial $c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}$ with $1 + j_1 + \cdots + j_{n-1} + j_n = n^2$ which is larger than $c_1 u_1^n \cdots u_{n-1}^n u_n^{n-1}$ is zero:*

$$\mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}\big] = 0, \qquad \text{for any} \quad (j_1, \dots, j_{n-1}, j_n) >_{\mathsf{revlex}} (n, \dots, n, n-1).$$

Furthermore:

$$\begin{aligned}\mathsf{coeff}_{d^{n+1}}\big[u_1^n \cdots u_{n-1}^n u_n^n\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^n (c_1)^n\big] = +1.\\ \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^n \cdots u_{n-1}^n u_n^{n-1}\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^{n-1} (c_1)^n\big] = -1.\end{aligned}$$

Proof. The first claim is just a rephrasing of the property **(iv)** of the lemma, after one notices that $c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}$ vanishes identically for degree reasons when $j_n \leqslant n-2$, while the term $u_n^{n-1} = u_n^{j_n}$ disappears after fiber integration when $j_n = n-1$. The identities stated just after now have obvious proofs. □

4.4. **Minorating** $\mathsf{coeff}_{d^{n+1}}\big[\Pi\big]$**.** Let us decompose the intersection product Π_δ defined by (13) as $\Pi + \delta \Pi'$, where:

$$\begin{aligned}\Pi &:= \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2} - n^2 h\big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2-1}\, 2|\mathbf{a}|,\\ \Pi' &:= n^2 c_1 \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2-1}\, |\mathbf{a}|.\end{aligned}$$

The (ineffective) Lemma 4.2 insures that the reduction of Π in terms of $d = \deg X$ is a certain polynomial:

$$\mathsf{P}_{\mathbf{a}}(d) = \sum_{k=0}^{n+1} \mathsf{p}_{k,\mathbf{a}}\, d^k,$$

having certain coefficients $\mathsf{p}_{k,\mathbf{a}} \in \mathbb{Z}\big[a_1, \dots, a_n\big]$. Moreover, Lemma 4.3 showed that positive powers of h do not contribute to the leading coefficient, whence:

$$\begin{aligned}\mathsf{p}_{n+1,\mathbf{a}} = \mathsf{coeff}_{d^{n+1}}\big[\Pi\big] &= \mathsf{coeff}_{d^{n+1}}\Big[\big(a_1 u_1 + \cdots + a_n u_n\big)^{n^2}\Big]\\ &= \mathsf{coeff}_{d^{n+1}}\Big[\big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2}\Big].\end{aligned}$$

By Proposition 2 in [7], the bundle:

$$\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|)$$

is nef whenever $\mathbf{a}$ belongs to the cone defined by (9), therefore its top self-intersection must be non-negative. Thus, once this top self-intersection is evaluated in term of the degree d of the hypersurface, its dominating coefficient must be non-negative, too. In other words we must have:

$$\mathsf{p}_{n+1,\mathbf{a}} \geqslant 0.$$

But from the corollary just above, we know that $\mathsf{p}_{n+1,\mathbf{a}} \in \mathbb{Z}[\mathbf{a}]$ is not identically zero, for it incorporates at least the nonzero (central) monomial:

$$\mathsf{coeff}_{d^{n+1}}\Big[\tfrac{n^2!}{n!\cdots n!}\, a_1^n \cdots a_n^n\, u_1^n \cdots u_n^n\Big] = \tfrac{n^2!}{n!\cdots n!}\, a_1^n \cdots a_n^n.$$

Then, in order to capture a weight $\mathbf{a}$ for which $\mathsf{p}_{n+1,\mathbf{a}} > 0$, we at first observe that the cube of $\mathbb{N}^n$ having edges of length n^2 which consists of all integers $(a_1, \ldots, a_n)$ satisfying the inequalities:

$$1 \leqslant a_n \leqslant 1 + n^2, \quad 3n^2 \leqslant a_{n-1} \leqslant (3+1)n^2, \quad (3^2+3)n^2 \leqslant a_{n-2} \leqslant (3^2+3+1)n^2$$
$$\ldots, \quad (3^{n-1} + \cdots + 3)n^2 \leqslant a_1 \leqslant (3^{n-1} + \cdots + 3 + 1)n^2$$

is visibly contained in the cone in question:

$$a_n \geqslant 1, \quad a_{n-1} \geqslant 2a_n, \quad a_{n-2} \geqslant 3a_{n-1}, \ldots, a_1 \geqslant 3a_2.$$

We now claim that there exists at least one n-tuple of integers $\mathbf{a}^* = (a_1^*, \ldots, a_n^*)$ belonging to this cube with the property that $\mathsf{p}_{n+1,\mathbf{a}^*}$ is nonzero, and hence:

$$\mathsf{p}_{n+1,\mathbf{a}^*} \geqslant 1 =: \mathsf{G}_{n+1},$$

so that we can take 1 as the minorant introduced at the beginning. Indeed, $\mathsf{p}_{n+1,\mathbf{a}}$ is a homogeneous polynomial of degree n^2 to which an elementary lemma applies.

Lemma 4.5. *Let $\mathsf{q} = \mathsf{q}(b_1, \ldots, b_\nu) \in \mathbb{Z}\big[b_1, \ldots, b_\nu\big]$ be a polynomial of degree $c \geqslant 1$. Then q can vanish at all points of a cube of integers having edges of length equal to its degree c only when it is identically zero.*

Proof. Expand $\mathsf{q} = \sum_{k_1=0}^{c} b_1^{k_1}\, \mathsf{q}_{k_1}(b_2, \ldots, b_\nu)$, recognize a $(c+1) \times (c+1)$ Van der Monde determinant, deduce that each $\mathsf{q}_{k_1}(b_2, \ldots, b_\nu)$ vanishes at all points of a similar cube in a space of dimension $\nu - 1$, and terminate by induction. □

4.5. **Majorating the other coefficients** $\mathsf{coeff}_{d^k}\big[\Pi\big]$**.** Now, for such an $\mathbf{a}^*$ which is not very precisely located in the cube, we nevertheless have the effective control, which is useful below:

$$\max_{1 \leqslant i \leqslant n} a_i^* = a_1^* = \tfrac{3^n - 1}{2}\, n^2 \leqslant \tfrac{3^n}{2}\, n^2.$$

From now on, we shall simply denote $\mathbf{a}^*$ by $\mathbf{a}$. At present, for any integer k with $0 \leqslant k \leqslant n$, let us denote by $\mathsf{D}_k(n)$ any available bound (*see* in advance Theorem 5.1) in terms of n only for the maximal absolute value of the coefficient of d^k in all monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ with $l + i_1 + \cdots + i_n = n^2$, namely:

$$\max_{l + i_1 + \cdots + i_n = n^2} \big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant \mathsf{D}_k(n).$$

Then for any k with $0 \leqslant k \leqslant n$, we now aim at estimating from above the coefficient of d^k in our intersection product Π, using two new lemmas and starting from its expansion, all terms of which we shall have to control:

$$\begin{aligned} &\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| \leqslant \\ &\leqslant \sum_{l + i_1 + \cdots + i_n = n^2} \tfrac{n^2!}{l!\, i_1! \cdots i_n!} \cdot (2|\mathbf{a}|)^l a_1^{i_1} \cdots a_n^{i_n} \cdot \big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| + \\ &+ \sum_{l + j_1 + \cdots + j_n = n^2 - 1} n^2\, \tfrac{(n^2-1)!}{l!\, j_1! \cdots j_n!} \cdot 2|\mathbf{a}|(2|\mathbf{a}|)^l a_1^{j_1} \cdots a_n^{j_n} \cdot \big|\mathsf{coeff}_{d^k}\big[h h^l u_1^{j_1} \cdots u_n^{j_n}\big]\big|. \end{aligned}$$

Lemma 4.6. *Let $l, i_1, \ldots, i_n \in \mathbb{N}$ satisfying $l + i_1 + \cdots + i_n = n^2$ and let $l, j_1, \ldots, j_n \in \mathbb{N}$ satisfying $l + j_1 + \cdots + j_n = n^2 - 1$. Then:*

$$\tfrac{n^2!}{l!\, i_1! \cdots i_n!} \leqslant (n+1)^{n^2} \quad \textit{and:} \quad n^2\, \tfrac{(n^2-1)!}{l!\, j_1! \cdots j_n!} \leqslant (n+1)^{n^2+1}.$$

Furthermore, the number of summands in $\sum_{l+i_1+\cdots+i_n=n^2}$ *and the number of summands in* $\sum_{l+j_1+\cdots+j_n=n^2-1}$*, which are both plain binomial coefficients, enjoy the following two elementary majorations:*

$$\frac{(n^2+n)!}{n^2!\,n!}\leqslant 4\,n^{2n-1} \qquad \textit{and:} \qquad \frac{(n^2-1+n)!}{(n^2-1)!\,n!}\leqslant 2\,n^{2n-1}.$$

Proof. Indeed, any multinomial coefficient $\frac{n^2!}{l!\,i_1!\cdots i_n!}$ is less than or equal to the sum of all multinomial coefficients $(1+1+\cdots+1)^{n^2}=(n+1)^{n^2}$. At the same time, we deduce: $n^2\,\frac{(n^2-1)!}{l!j_1!\cdots j_n!}=n^2(n+1)^{n^2-1}\leqslant(n+1)^{n^2+1}$.

For the second claim, we as a preliminary have:

$$\frac{(n^2+n-1)!}{n^2!\,(n-1)!}=\frac{(n^2+1)\cdots(n^2+n-1)}{1\,\cdots\,(n-1)}\leqslant\frac{(n^2+n^2)\cdots(n^2+n^2)}{(n-1)!}=\frac{2^{n-1}\,n^{2n-2}}{(n-1)!}\leqslant 2\,n^{2n-2},$$

since $2^{n-1}\leqslant 2\,(n-1)!$ for any $n\geqslant 1$. Consequently, we deduce:

$$\frac{(n^2+n)!}{n^2!\,n!}=\frac{(n^2+n-1)!}{n^2!\,(n-1)!}\cdot\frac{(n^2+n)}{n}\leqslant 2\,n^{2n-2}\cdot(n+\tfrac{1}{n})\leqslant 4\,n^{2n-1},$$

and similarly: $\frac{(n^2-1+n)!}{(n^2-1)!\,n!}\leqslant\frac{(n^2+n-1)!}{n^2!\,(n-1)!}\cdot\frac{n^2}{n}\leqslant 2\,n^{2n-2}\cdot n=2\,n^{2n-1}$. □

Lemma 4.7. *For any* $l,i_1,\dots,i_n\in\mathbb{N}$ *satisfying* $l+i_1+\cdots+i_n=n^2$*, one has:*

$$(2|\mathbf{a}|)^l a_1^{i_1}\cdots a_n^{i_n}\leqslant n^{3n^2}\,3^{n^3}.$$

Proof. Indeed, we majorate each a_i by $|\mathbf{a}|$ and $|\mathbf{a}|=a_1+\cdots+a_n$ by na_1, and also l by n^2, so that $(2|\mathbf{a}|)^l a_1^{i_1}\cdots a_n^{i_n}\leqslant 2^{n^2}\big(na_1\big)^{n^2}$ and we apply $a_1\leqslant\frac{3^n}{2}\,n^2$. □

Thanks to these two lemmas, we may perform majorations:

$$\begin{aligned}\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| &\leqslant 4\,n^{2n-1}\cdot(n+1)^{n^2}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}_k(n)+\\ &\quad+2\,n^{2n-1}\cdot(n+1)^{n^2+1}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}_k(n)\\ &\leqslant 6\,n^{2n-1}\cdot(n+1)^{n^2+1}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}_k(n) \qquad {\scriptstyle (k=0,\dots,n)}.\end{aligned}$$

Lemma 4.8. *For any exponent* k *with* $0\leqslant k\leqslant n$*, one has:*

$$\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big|\leqslant 6\,n^{2n-1}\cdot(n+1)^{n^2}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}_k(n). \quad \square$$

To conclude these estimates, for any integer $k=0,1,\dots,n,n+1$, let us denote by $\mathsf{D}'_k(n)$ any available majorant for all the monomials appearing in Π':

$$\max_{1+l+j_1+\cdots+j_n=n^2}\big|\mathsf{coeff}_{d^k}\big[c_1h^lu_1^{j_1}\cdots u_n^{j_n}\big]\big|\leqslant\mathsf{D}'_k(n).$$

Lemma 4.9. *For any exponent* k *with* $0\leqslant k\leqslant n+1$*, one has:*

$$\big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big|\leqslant n^{2n-1}\cdot(n+1)^{n^2+1}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}'_k(n).$$

Proof. Indeed, one performs the similar majorations:

$$\begin{aligned}&\big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big|\leqslant\\ &\leqslant\sum_{l+j_1+\cdots+j_n=n^2-1} n^2\,\frac{(n^2-1)!}{l!\,j_1!\cdots j_n!}\cdot|\mathbf{a}|(2|\mathbf{a}|)^l a_1^{j_1}\cdots a_n^{j_n}\cdot\big|\mathsf{coeff}_{d^k}\big[c_1h^lu_1^{j_1}\cdots u_n^{j_n}\big]\big|\\ &\leqslant 2\,n^{2n-1}\cdot(n+1)^{n^2+1}\cdot\tfrac{1}{2}\,n^{3n^2}\,3^{n^3}\cdot\mathsf{D}'_k(n)\\ &\leqslant n^{2n-1}\cdot(n+1)^{n^2+1}\cdot n^{3n^2}\,3^{n^3}\cdot\mathsf{D}'_k(n),\end{aligned}$$

hence the bound we obtain is exactly the same, up to the factor 6. □

4.6. **Final effective estimations.** We can now explain how to achieve the proof of Theorem 1.1. At first, we shall realize in Section 5 that both constant coefficients $\mathsf{coeff}_{d^0}\big[\Pi\big] = \mathsf{coeff}_{d^0}\big[\Pi'\big] = 0$ vanish, hence $\mathsf{D}_0(n) = \mathsf{D}_0'(n) = 0$ works. Most importantly, we shall establish in Section 5 that one may choose:

$$\mathsf{D}_1(n) = \cdots = \mathsf{D}_n(n) = \mathsf{D}_1'(n) = \cdots = \mathsf{D}_n'(n) = \mathsf{D}_{n+1}'(n) = n^{4n^3} 2^{n^4}.$$

Taking $n^{4n^3} 2^{n^4}$ for granted, remind that with the above choice of weight $\mathbf{a}^*$ (now denoted $\mathbf{a}$), we ensure that:

$$\mathsf{coeff}_{d^{n+1}}\big[\Pi\big] = \mathsf{p}_{n+1,\mathbf{a}} \geqslant 1 =: \mathsf{G}_{n+1}.$$

From the preceding two lemmas, we therefore deduce that:

$$\begin{aligned} \big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| &\leqslant 6\, n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3} 2^{n^4} =: 6\, \mathsf{H}(n) \qquad (k = 1 \cdots n) \\ \big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big| &\leqslant n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3} 2^{n^4} =: \mathsf{H}(n) \qquad (k = 1 \cdots n+1). \end{aligned}$$

so that, coming back to the beginning of Section 4, we may choose $\mathsf{E}_0 = \mathsf{E}_0' = 0$ (since $\mathsf{D}_0(n) = \mathsf{D}_0'(n) = 0$) and also explicitly in terms of n:

$$\begin{aligned} \mathsf{E}_1 = \cdots = \mathsf{E}_n &= 6\, \mathsf{H}(n) \\ \mathsf{E}_1' = \cdots = \mathsf{E}_n' = \mathsf{E}_{n+1}' &= \mathsf{H}(n). \end{aligned}$$

Coming back to the definition of d_n^1, d_n^2 given at the end of Lemma 4.1 and just after, we may now majorate:

$$\begin{aligned} d_n^1 &\leqslant 1 + \big(n\, 6\, \mathsf{H}(n) + \tfrac{n+1}{2}\big) \big/ \tfrac{1}{2} =: \widetilde{d}_n^1, \\ d_n^2 &\leqslant 1 + n + 2 + 2\,(n^2 + 2n)\, \mathsf{H}(n) =: \widetilde{d}_n^2. \end{aligned}$$

Notice that $\widetilde{d}_n^2 \geqslant \widetilde{d}_n^1$ as soon as $n \geqslant 3$. Finally, by comparing the growth of all terms in $\mathsf{H}(n)$ as $n \to \infty$, one sees that 2^{n^4} dominates and hence that the following inequality:

$$\widetilde{d}_n^2 = 1 + n + 2 + 2\,(n^2 + 2n) \cdot n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3} 2^{n^4} \leqslant 2^{n^5},$$

holds for all large n. However, any symbolic computer shows that for $n = 2, 3, 4$, one in fact has $\widetilde{d}_2^2 > 2^{2^5}$, $\widetilde{d}_3^2 > 2^{3^5}$, $\widetilde{d}_4^2 > 2^{4^5}$, while $\widetilde{d}_5^2 < 2^{n^5}$ and $\widetilde{d}_n^2 \ll 2^{n^5}$ for $n = 6, 7, 8, 9$ so that $\widetilde{d}_n^2 < 2^{n^5}$ holds for any $n \geqslant 5$ by an elementary inspection of the function $n \mapsto \widetilde{d}_n^2$. Fortunately, the three left cases $n = 2$, $n = 3$ and $n = 4$ of Theorem 1.1 are covered, firstly for the classical surface case $n = 2$ by, say [5] in which $\deg X \geqslant 21$ with $21 \ll 2^{2^5}$, and secondly for $n = 3$ and $n = 4$ by our second Theorem 1.2, because $2^{3^5} \gg 593$ and $2^{4^5} \gg 3203$. So we conclude that if we take for granted: 1) that one may take all the $\mathsf{D}_k(n)$ and all the $\mathsf{D}_k'(n)$ equal to $n^{4n^3} 2^{n^4}$, a technical and crucial statement to which Section 5 below is entirely devoted; and 2) that Theorem 1.2 is got, an effective statement to which the two Sections 6 and 7 below are devoted, then the proof of our main Theorem 1.1 is to be considered as complete, and finally, the neat uniform degree bound $\deg X \geqslant 2^{n^5}$ works in all dimensions $n \geqslant 2$. □

5. Estimations of the quantities $\mathsf{D}_k(n)$ and $\mathsf{D}_k'(n)$

To complete our program, it now remains only to capture somewhat effective upper bounds $\mathsf{D}_k(n)$, $0 \leqslant k \leqslant n$ and $\mathsf{D}_k'(n)$, $0 \leqslant k \leqslant n+1$.

Theorem 5.1. *With $n \geqslant 2$, for any $l, i_1, \dots, i_n \in \mathbb{N}$ with $l + i_1 + \cdots + i_n = n^2$ and any $l, j_1, \dots, j_n \in \mathbb{N}$ with $1 + l + j_1 + \cdots + j_n = n^2$, one has:*

$$0 = \mathsf{coeff}_{d^0}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big] = \mathsf{coeff}_{d^0}\big[c_1 h^l u_1^{j_1} \cdots u_n^{j_n}\big].$$

Moreover and above all, for every $k = 1, \dots, n+1$, the following uniform effective upper bound holds:

$$\big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1}\cdots u_n^{i_n}\big]\big| \leqslant n^{4n^3} 2^{n^4},$$
$$\big|\mathsf{coeff}_{d^k}\big[c_1 h^l u_1^{j_1}\cdots u_n^{j_n}\big]\big| \leqslant n^{4n^3} 2^{n^4}.$$

In other words, in the above notations, one may choose $\mathsf{D}_0(n) = \mathsf{D}_0'(n) = 0$ and $\mathsf{D}_k(n) = \mathsf{D}_k'(n) = n^{4n^3}2^{n^4}$ for $k = 1, \dots, n+1$.

5.1. **Jacobi-Trudy determinants.** One key observation towards these estimations is that the reduction process from one level to the lower level in Demailly's tower involves Jacobi-Trudy determinants in the Chern classes of the lower level in question.

Definition 5.1. At any level ℓ with $0 \leqslant \ell \leqslant n-1$ and for any J with $0 \leqslant J \leqslant n + \ell(n-1) = \dim X_\ell$, we define the corresponding *Jacobi-Trudy determinant*:

$$\mathcal{C}_J^\ell := \begin{vmatrix} c_1^{[\ell]} & c_2^{[\ell]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

where, again by convention, we set any $c_k^{[\ell]} := 0$ as soon as $k \geqslant n+1$; by convention also, $\mathcal{C}_J^\ell := 0$ is set to zero when $J > \dim X_\ell$ and when $J < 0$; lastly, we set $\mathcal{C}_0^\ell := 1$.

Expanding the determinant $\mathcal{C}_J^\ell$ along its first line, and expanding again the obtained block-determinants, one easily convinces oneself of the induction formulae:

$$\mathcal{C}_J^\ell = c_1^{[\ell]}\,\mathcal{C}_{J-1}^\ell - c_2^{[\ell]}\,\mathcal{C}_{J-2}^\ell + c_3^{[\ell]}\,\mathcal{C}_{J-3}^\ell - \cdots, \tag{18}$$

the last term in this expansion being either $(-1)^{n-1}\,c_n^{[\ell]}\,\mathcal{C}_{J-n}^\ell$ when $J \geqslant n$ or else $(-1)^{J-1}\,c_J^{[\ell]}\,\mathcal{C}_0^\ell$ when $J < n$.

In the proof of Theorem 5.1, the study of the monomials $u_1^{i_1}\cdots u_n^{i_n}$ will appear *a posteriori* to be exactly the same as the study of the monomials $h^l u_1^{i_1}\cdots u_n^{i_n}$ and $c_1 h^l u_1^{j_1}\cdots u_n^{j_n}$.

Generally speaking, fixing ℓ with $1 \leqslant \ell \leqslant n$ and exponents $i_1, \dots, i_\ell \in \mathbb{N}$ satisfying $i_1 + \cdots + i_\ell = n + \ell(n-1) = \dim X_\ell$, let us therefore study the reduction, in term of the degree d of X, of the specific monomial $u_1^{i_1}\cdots u_{\ell-1}^{i_{\ell-1}} u_\ell^{i_\ell}$. We write it as $\Omega_K^{\ell-1} u_\ell^{i_\ell}$, where $\Omega_K^{\ell-1} := u_1^{i_1}\cdots u_{\ell-1}^{i_{\ell-1}}$ is a (K,K)-form living on $X_{\ell-1}$ with $K + i_\ell = n + \ell(n-1)$.

If $i_\ell \leqslant n-2$, then $\Omega_K^{\ell-1}$ vanishes form degree-form reasons. If $i_\ell = n-1$, then a fiber-integration gives $\Omega_K^{\ell-1}\underline{u_\ell^{n-1}}_{\int} = \Omega_K^{\ell-1}\cdot 1 = \Omega_K^{\ell-1}\mathcal{C}_0^{\ell-1}$.

Lemma 5.1. *For any ℓ with $1 \leqslant \ell \leqslant n$, given any (K,K)-form $\Omega_K^{\ell-1}$ at level $\ell-1$ and any integer i_ℓ with $i_\ell \geqslant n-1$ and $i_\ell + K = \dim X_\ell$, the reduction of $\Omega_K^{\ell-1}\,u_\ell^{i_\ell}$ down to level $\ell-1$ precisely reads:*

$$\begin{aligned}\Omega_K^{\ell-1} u_\ell^{i_\ell} &= (-1)^{i_\ell - n+1}\,\Omega_K^{\ell-1} \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & \cdots & c_{i_\ell-n+1}^{[\ell-1]} \\ 1 & c_1^{[\ell-1]} & \cdots & c_{i_\ell-n}^{[\ell-1]} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix} \\ &= (-1)^{i_\ell-n+1}\,\Omega_K^{\ell-1}\,\mathcal{C}_{i_\ell-n+1}^{\ell-1}.\end{aligned}$$

Proof. Assume first that $i_\ell = n$ and use (16) to get:

$$\begin{aligned}\Omega_K^{\ell-1}\, u_\ell^n &= -\Omega_K^{\ell-1}\, c_1^{[\ell-1]}\, \underline{u_\ell^{n-1}}_{\smallint} - \underline{\Omega_K^{\ell-1} c_2^{[\ell-1]}}_{\circ}\, u_\ell^{n-2} - \cdots - \underline{\Omega_K^{\ell-1} c_n^{[\ell-1]}}_{\circ} \\ &= -\Omega_K^{\ell-1}\, \mathcal{C}_1^{\ell-1}.\end{aligned}$$

Reasoning by induction, assume now that the lemma holds for all i'_ℓ with $n \leqslant i'_\ell \leqslant i_\ell$ for some $i_\ell \geqslant n$. Take an arbitrary (L, L)-form $\Omega_L^{\ell-1}$ on $X_{\ell-1}$ with $L + i_\ell + 1 = \dim X_\ell$, multiply (16) by $\Omega_L^{\ell-1}\, u_\ell^{i_\ell+1-n}$ to get:

$$\begin{aligned}\Omega_L^{\ell-1}\, u_\ell^{i_\ell+1} &= -\Omega_L^{\ell-1}\Big(c_1^{[\ell-1]}\, u_\ell^{i_\ell} + c_2^{[\ell-1]}\, u_\ell^{i_\ell-1} + c_3^{[\ell-1]}\, u_\ell^{i_\ell-2} + \cdots\Big) \\ &= (-1)^{1+i_\ell-n+1}\, \Omega_L^{\ell-1}\Big(c_1^{[\ell-1]}\, \mathcal{C}_{i_\ell-n+1}^{\ell-1} - c_2^{[\ell-1]}\, \mathcal{C}_{i_\ell-n}^{\ell-1} + c_3^{[\ell-1]}\, \mathcal{C}_{i_\ell-n-1}^{\ell-1} - \cdots\Big) \\ &= (-1)^{i_\ell+1-n+1}\, \Omega_L^{\ell-1}\, \mathcal{C}_{i_\ell+1-n+1}^{\ell-1},\end{aligned}$$

thanks to (18), which gives the claimed reduction for the exponent $i_\ell + 1$. □

Applying this lemma to the monomial $u_1^{i_1} \cdots u_\ell^{i_\ell} u_{\ell+1}^{i_{\ell+1}}$, we thus reduce it to

$$u_1^{i_1} \cdots u_\ell^{i_\ell} u_{\ell+1}^{i_{\ell+1}} = (-1)^{i_{\ell+1}-n+1}\, u_1^{i_1} \cdots u_\ell^{i_\ell}\, \mathcal{C}_{i_{\ell+1}-n+1}^{\ell}\,.$$

To obtain effective estimations, we will need to further reduce such a Jacobi-Trudy determinant $\mathcal{C}_{i_{\ell+1}-n+1}^{\ell}$ from level ℓ down to level $\ell - 1$. A whole program begins. In the application we have in mind, one should think that $\Omega_K^\ell = (-1)^{i_{\ell+1}-n+1}\, u_1^{i_1} \cdots u_\ell^{i_\ell}$ and that $J = i_{\ell+1} - n + 1$.

Lemma 5.2. *At an arbitrary level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathcal{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K, K)-form on X_ℓ whose degree K satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the reduction of $\Omega_K^\ell \mathcal{C}_J^\ell$ down to level $\ell - 1$ relies upon the following formulae:*

$$\Omega_K^\ell \mathcal{C}_J^\ell = \Omega_K^\ell\big[\mathcal{C}_J^{\ell-1} + \mathcal{C}_0^\ell \mathsf{A}_J^\ell + \mathcal{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathcal{C}_{J-1}^\ell \mathsf{A}_1^\ell\big],$$

in which, for any k with $1 \leqslant k \leqslant J$, one has set:

$$\mathsf{A}_k^\ell := \mathsf{X}_1^\ell \mathcal{C}_{k-1}^{\ell-1} - \mathsf{X}_2^\ell \mathcal{C}_{k-2}^{\ell-1} + \cdots + (-1)^{k-1} \mathsf{X}_k^\ell \mathcal{C}_0^{\ell-1},$$

where the X-terms here gather all the terms after $c_j^{[\ell-1]}$ in a convenient rewriting of (15) *under the following form:*

$$c_j^{[\ell]} = c_j^{[\ell-1]} + \underbrace{\lambda_{j,1}\, c_{j-1}^{[\ell-1]} u_\ell + \lambda_{j,2}\, c_{j-2}^{[\ell-1]} u_\ell^2 + \cdots + \lambda_{j,j}\, u_\ell^j}_{\stackrel{\text{def}}{=} \mathsf{X}_j^\ell},$$

with the convention that $\mathsf{X}_j^\ell = 0$ for any $j \geqslant n+1$.

Proof. Naturally, we should expand the Jacobi-Trudy determinant in question after inserting in it the relation (15). This is based on linear algebra considerations and we shall drop Ω_K^ℓ in the computations.

More precisely, let us write down the determinant $\mathcal{C}_J^\ell$ we have to expand:

$$\mathcal{C}_J^\ell = \begin{vmatrix} c_1^{[\ell]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} = \begin{vmatrix} \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0+1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}$$

by emphasizing the induction on ℓ which represents its first column naturally as the sum of two columns. As already devised, we expand it by linearity, getting:

$$\mathcal{C}_J^\ell = \begin{vmatrix} \mathsf{X}_1^\ell & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} + \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

and just afterwards immediately, we expand the first determinant along its first column, while at the same time, in the second column of the second determinant, we again emphasize the induction on ℓ:

$$\mathcal{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathcal{C}_{J-1}^\ell + \begin{vmatrix} c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}.$$

Next, we similarly expand by linearity the obtained determinant, realizing again that its second column is a sum of two columns:

$$\begin{aligned} \mathcal{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathcal{C}_{J-1}^\ell &+ \begin{vmatrix} c_1^{[\ell-1]} & \mathsf{X}_2^\ell & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & \mathsf{X}_1^\ell & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\ &+ \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}, \end{aligned}$$

and evidently again, we must expand the first obtained determinant along its second column, getting:

$$\begin{aligned} \mathcal{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathcal{C}_{J-1}^\ell - \mathsf{X}_2^\ell \cdot \begin{vmatrix} 1 & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} &+ \mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\ &+ \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & \mathsf{X}_3^\ell + c_3^{[\ell-1]} & c_4^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ 0 & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}, \end{aligned}$$

and we are supposed to iterate once again the same two processes:

$$
\begin{aligned}
\mathcal{C}_J^\ell &= \mathsf{X}_1^\ell \cdot \mathcal{C}_{J-1}^\ell - \mathsf{X}_2^\ell \cdot 1 \cdot \mathcal{C}_{J-2}^\ell + \mathsf{X}_1^\ell \cdot \mathcal{C}_1^{\ell-1} \cdot \mathcal{C}_{J-2}^\ell \\
&\quad + \mathsf{X}_3^\ell \cdot \begin{vmatrix} 1 & c_1^{[\ell-1]} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
&\quad - \mathsf{X}_2^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
&\quad + \mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} \\ 1 & c_1^{[\ell-1]} \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
&\quad + \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & c_3^{[\ell-1]} & \mathsf{X}_4^\ell + c_4^{[\ell-1]} & c_5^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & c_2^{[\ell-1]} & \mathsf{X}_3^\ell + c_3^{[\ell-1]} & c_4^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ 0 & 0 & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ 0 & 0 & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-4}^{[\ell]} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}.
\end{aligned}
$$

At this point where things start to become clearer, we make the following general observation. Consider the determinant that one obtains after a finite number of steps:

$$
\begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & \cdots & c_{k-1}^{[\ell-1]} & \mathsf{X}_k^\ell + c_k^{[\ell-1]} & c_{k+1}^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & \cdots & c_{k-2}^{[\ell-1]} & \mathsf{X}_{k-1}^\ell + c_{k-1}^{[\ell-1]} & c_k^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-k+2}^{[\ell]} \\ 0 & 0 & \cdots & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-k+1}^{[\ell]} \\ 0 & 0 & \cdots & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-k}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},
$$

where the central-looking column is the k-th one, for some k with $1 \leqslant k \leqslant J$. Write this determinant as a sum of two determinants by linearity, and expand the first obtained determinant, let us call it Δ_k, along its k-th column in which are present all the X_k^ℓ's. We thus get that the first determinant is equal to:

$$\begin{aligned}\Delta_k := (-1)^{k+1}\,\mathsf{X}_k^\ell \cdot &\begin{vmatrix} 1 & \cdots & c_{k-2}^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \cdot \mathcal{C}_{J-k}^\ell \\ & + (-1)^{k+2}\,\mathsf{X}_{k-1}^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & * & \cdots & * \\ 0 & 1 & \cdots & c_{k-3}^{[\ell-1]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \cdot \mathcal{C}_{J-k}^\ell \\ & + (-1)^{k+3}\,\mathsf{X}_{k-2}^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & * & \cdots & * \\ 1 & c_1^{[\ell-1]} & * & \cdots & * \\ 0 & 0 & 1 & \cdots & c_{k-4}^{[\ell-1]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \cdot \mathcal{C}_{J-k}^\ell \\ & + \cdots + (-1)^{k+k}\,\mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & \cdots & c_{k-1}^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix} \cdot \mathcal{C}_{J-k}^\ell,\end{aligned}$$

while the second determinant is of the same kind as the one we started with, except that the X's are now located in the $(k+1)$-th column. Thus after mild simplifications, what we called the first determinant equals:

$$\begin{aligned}\Delta_k &= (-1)^{k+1}\,\mathsf{X}_k^\ell \cdot 1 \cdot \mathcal{C}_{J-k}^\ell + (-1)^{k+2}\,\mathsf{X}_{k-1}^\ell \cdot \mathcal{C}_1^{\ell-1} \cdot \mathcal{C}_{J-k}^\ell + \\ &\quad + (-1)^{k+3}\,\mathsf{X}_{k-2}^\ell \cdot \mathcal{C}_2^{\ell-1} \cdot \mathcal{C}_{J-k}^\ell + \cdots + \mathsf{X}_1^\ell \cdot \mathcal{C}_{k-1}^{\ell-1} \cdot \mathcal{C}_{J-k}^\ell \\ &= \mathsf{A}_k^\ell \mathcal{C}_{J-k}^\ell.\end{aligned}$$

In conclusion, the initial Jacobi-Trudy determinant $\mathcal{C}_J^\ell$ we started with now equals:

$$\mathcal{C}_J^\ell = \Delta_1 + \cdots + \Delta_k + \cdots + \Delta_J + \begin{vmatrix} c_1^{[\ell-1]} & \cdots & c_J^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix},$$

where the last written determinant, equal to $\mathcal{C}_J^{\ell-1}$ and living at the $(\ell-1)$-th level, is the remainder determinant after all X-terms are removed by expansion. Summing the $\Delta_k = \mathsf{A}_k^\ell\, \mathcal{C}_{J-k}^\ell$, we obtain the formula announced in the lemma. □

As J varies, the formulae given by this lemma:

$$\mathcal{C}_J^\ell = \mathcal{C}_J^{\ell-1} + \mathcal{C}_0^\ell \mathsf{A}_J^\ell + \mathcal{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathcal{C}_{J-1}^\ell \mathsf{A}_1^\ell,$$

are still imperfect, for their right-hand sides still involve Jacobi-Trudy determinants at the level ℓ. So necessarily, we must perform further reductions.

Lemma 5.3. *For any J with $0 \leqslant J \leqslant \dim X_\ell$ and any ℓ with $1 \leqslant \ell \leqslant n$, one has:*

$$\mathcal{C}_J^\ell = \sum_{j=0}^{J} \mathcal{C}_{J-j}^{\ell-1} \bigg(\sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \bigg),$$

with the convention that for $j=0$, the empty sum in parentheses equals 1.

Proof. First, for $J = 0$, recall that by convention $\mathcal{C}_0^\ell = \mathcal{C}_0^{\ell-1} = 1$. Next, for $J = 1$, we start from the formula of the preceding lemma and we perform an evident computation:

$$\mathcal{C}_1^\ell = \mathcal{C}_1^{\ell-1} + \mathcal{C}_0^\ell \mathsf{A}_1^\ell = \mathcal{C}_1^{\ell-1}\Sigma_0^\ell(\mathsf{A}) + \mathcal{C}_0^{\ell-1}\Sigma_1^\ell(\mathsf{A}),$$

if, generally speaking, we denote for convenient abbreviation:

$$\Sigma_j^\ell(\mathsf{A}) := \sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell, \tag{19}$$

with of course $\Sigma_0^\ell(\mathsf{A}) = 1$. These $\Sigma_j^\ell(\mathsf{A})$ satisfy useful induction formulae:

$$\begin{aligned}
\Sigma_j^\ell(\mathsf{A}) &= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \sum_{\substack{k_1+k_2+\cdots+k_\nu=j \\ k_1,k_2,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \\
&= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \bigg(\mathsf{A}_1^\ell \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\substack{k_2,\dots,k_\nu=j-2 \\ k_2,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\substack{k_2+\cdots+k_\nu=1 \\ k_1,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \bigg) \\
&= \mathsf{A}_j^\ell + \mathsf{A}_1^\ell \sum_{\nu=2}^{j-1} \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\nu=2}^{j-2} \sum_{\substack{k_2+\cdots+k_\nu=j-2 \\ k_2,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\nu=2}^{2} \sum_{\substack{k_2=1 \\ k_2 \geqslant 1}} \mathsf{A}_{k_2}^\ell \\
&= \mathsf{A}_j^\ell \Sigma_0^\ell(\mathsf{A}) + \mathsf{A}_1^\ell\, \Sigma_{j-1}^\ell(\mathsf{A}) + \mathsf{A}_2^\ell \Sigma_{j-2}^\ell(\mathsf{A}) + \cdots + \mathsf{A}_{j-1}^\ell \Sigma_1^\ell(\mathsf{A}).
\end{aligned} \tag{20}$$

Next, for $J = 2$, starting again from the known (imperfect) formula and using what has just been seen:

$$\begin{aligned}
\mathcal{C}_2^\ell &= \mathcal{C}_2^{\ell-1} + \mathcal{C}_0^\ell \mathsf{A}_2^\ell + \mathcal{C}_1^\ell \mathsf{A}_1^\ell \\
&= \mathcal{C}_2^{\ell-1} + \mathcal{C}_0^{\ell-1} \mathsf{A}_2^\ell + \big[\mathcal{C}_1^{\ell-1}\Sigma_0^\ell(\mathsf{A}) + \mathcal{C}_0^{\ell-1}\Sigma_1^\ell(\mathsf{A})\big]\mathsf{A}_1^\ell \\
&= \mathcal{C}_2^{\ell-1}\Sigma_0^\ell(\mathsf{A}) + \mathcal{C}_1^{\ell-1}\big[\Sigma_0^\ell(\mathsf{A})\mathsf{A}_1^\ell\big] + \mathcal{C}_0^{\ell-1}\big[\Sigma_1^\ell(\mathsf{A})\mathsf{A}_1^\ell + \mathsf{A}_2^\ell\big] \\
&= \mathcal{C}_2^{\ell-1}\Sigma_0^\ell(\mathsf{A}) + \mathcal{C}_1^{\ell-1}\Sigma_1^\ell(\mathsf{A}) + \mathcal{C}_0^{\ell-1}\Sigma_2^\ell(\mathsf{A}).
\end{aligned}$$

Suppose now by induction that we have already proved that:

$$\mathcal{C}_{J'}^\ell = \mathcal{C}_{J'}^{\ell-1}\Sigma_0^\ell(\mathsf{A}) + \mathcal{C}_{J'-1}^{\ell-1}\Sigma_1^\ell(\mathsf{A}) + \mathcal{C}_{J'-2}^{\ell-1}\Sigma_2^\ell(\mathsf{A}) + \cdots + \mathcal{C}_0^{\ell-1}\Sigma_J^\ell(\mathsf{A}),$$

for all J' with $0 \leqslant J' \leqslant J$, for some $J \geqslant 2$. Then we apply the known general (imperfect) formula with J replaced by $J+1$ in it, and afterwards, we use the induction hypothesis, which gives:

$$
\begin{aligned}
\mathcal{C}_{J+1}^{\ell} &= \mathcal{C}_{J+1}^{\ell-1} + \mathcal{C}_0^{\ell}\mathsf{A}_{J+1}^{\ell} + \mathcal{C}_1^{\ell}\mathsf{A}_J^{\ell} + \cdots + \mathcal{C}_{J-1}^{\ell}\mathsf{A}_2^{\ell} + \mathcal{C}_J^{\ell}\mathsf{A}_1^{\ell} \\
&= \mathcal{C}_{J+1}^{\ell-1}\Sigma_0^{\ell}(\mathsf{A}) + \\
&\quad + \big[\mathcal{C}_0^{\ell-1}\Sigma_0^{\ell}(\mathsf{A})\big]\mathsf{A}_{J+1}^{\ell} + \\
&\quad + \big[\mathcal{C}_1^{\ell-1}\Sigma_0^{\ell}(\mathsf{A}) + \mathcal{C}_0^{\ell-1}\Sigma_1^{\ell}(\mathsf{A})\big]\mathsf{A}_J^{\ell} + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + \big[\mathcal{C}_{J-1}^{\ell-1}\Sigma_0^{\ell}(\mathsf{A}) + \mathcal{C}_{J-2}^{\ell-1}\Sigma_1^{\ell}(\mathsf{A}) + \mathcal{C}_{J-3}^{\ell-1}\Sigma_2^{\ell}(\mathsf{A}) + \cdots + \mathcal{C}_0^{\ell-1}\Sigma_{J-1}^{\ell}(\mathsf{A})\big]\mathsf{A}_2^{\ell} + \\
&\quad + \big[\mathcal{C}_J^{\ell-1}\Sigma_0^{\ell}(\mathsf{A}) + \mathcal{C}_{J-1}^{\ell-1}\Sigma_1^{\ell}(\mathsf{A}) + \mathcal{C}_{J-2}^{\ell-1}\Sigma_2^{\ell}(\mathsf{A}) + \cdots + \mathcal{C}_1^{\ell-1}\Sigma_{J-1}^{\ell}(\mathsf{A}) + \mathcal{C}_0^{\ell-1}\Sigma_J^{\ell}(\mathsf{A})\big]\mathsf{A}_1^{\ell}.
\end{aligned}
$$

A necessary and natural reorganization then gives:

$$
\begin{aligned}
\mathcal{C}_{J+1}^{\ell} &= \mathcal{C}_{J+1}^{\ell-1}\big[\Sigma_0(\mathsf{A})\big] + \\
&\quad + \mathcal{C}_J^{\ell-1}\big[\Sigma_0^{\ell}(\mathsf{A})\mathsf{A}_1^{\ell}\big] + \\
&\quad + \mathcal{C}_{J-1}^{\ell-1}\big[\Sigma_1^{\ell}(\mathsf{A})\mathsf{A}_1^{\ell} + \Sigma_0^{\ell}(\mathsf{A})\mathsf{A}_2^{\ell}\big] + \\
&\quad + \mathcal{C}_{J-2}^{\ell-1}\big[\Sigma_2^{\ell}(\mathsf{A})\mathsf{A}_1^{\ell} + \Sigma_1^{\ell}(\mathsf{A})\mathsf{A}_2^{\ell} + \Sigma_0^{\ell}(\mathsf{A})\mathsf{A}_3^{\ell}\big] + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + \mathcal{C}_0^{\ell-1}\big[\Sigma_J^{\ell}(\mathsf{A})\mathsf{A}_1^{\ell} + \Sigma_{J-1}^{\ell}(\mathsf{A})\mathsf{A}_2^{\ell} + \Sigma_{J-2}^{\ell}(\mathsf{A})\mathsf{A}_3^{\ell} + \cdots + \Sigma_0^{\ell}(\mathsf{A})\mathsf{A}_{J+1}^{\ell}\big] \\
&= \mathcal{C}_{J+1}^{\ell-1}\Sigma_0^{\ell}(\mathsf{A}) + \mathcal{C}_J^{\ell-1}\Sigma_1^{\ell}(\mathsf{A}) + \mathcal{C}_{J-1}^{\ell-1}\Sigma_2^{\ell}(\mathsf{A}) + \mathcal{C}_{J-2}^{\ell-1}\Sigma_3^{\ell}(\mathsf{A}) + \cdots + \mathcal{C}_0^{\ell-1}\Sigma_{J+1}^{\ell}(\mathsf{A}),
\end{aligned}
$$

where at the end, one applies the formulae (20) just seen. Notice *passim* that the number of terms in $\Sigma_j^{\ell}(\mathsf{A})$ is equal to 2^{j-1} for all $j \geqslant 1$. □

5.2. **Upper reduction operator.** The reduction process, after several elimination computations involving (15) and (16) and at the end (8), transforms a general monomial of the form $h^l u_1^{i_1} \cdots u_n^{i_n}$ with $l + i_1 + \cdots + i_n = n^2$ into a polynomial $\mathcal{R}\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big)$ of degree $\leqslant n+1$ in d, where the symbol "$\mathcal{R}$" stands for "*reduction*".

From now on, complete explicit algebraic computations will not be conducted anymore, and instead, to tame their complexity, *inequalities* will be dealt with.

For our majoration purposes, we now introduce an important *upper reduction operator* $\mathcal{R}^+$ which by definition, at each computational step of the reduction process, while going down in the Demailly's tower, always replaces any incoming sign "$-$" by a sign "$+$". Accordingly, for any two monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ and $h^{l'} u_1^{i_1'} \cdots u_n^{i_n'}$, we shall say that:

$$
\mathcal{R}^+\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big) \leqslant \mathcal{R}^+\big(h^{l'} u_1^{i_1'} \cdots u_n^{i_n'}\big),
$$

and write more briefly:

$$
h^l u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathcal{R}^+} h^{l'} u_1^{i_1'} \cdots u_n^{i_n'},
$$

if the corresponding two (upper) reduced polynomials $\sum_{k=0}^{n+1} \mathsf{p}_k \cdot d^k$ and $\sum_{k=0}^{n+1} \mathsf{p}_k' \cdot d^k$ have all their coefficients satisfying:

$$
\big(0 \leqslant\big)\ \mathsf{p}_k \leqslant \mathsf{p}_k' \qquad \text{for every } \ k = 0, 1, \ldots, n+1.
$$

Then obviously the absolute values of the coefficients of the reduction are smaller than the (nonnegative) coefficients of the upper reduction:

$$
\big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant \mathsf{coeff}_{d^k}\big[\mathcal{R}^+\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big)\big]\,.
$$

To obtain the desired bound $n^{4n^3}2^{n^4}$ we need to handle the Jacobi-Trudy determinants seen above. The following lemma will be useful.

Lemma 5.4. *For any $\lambda_1, \lambda_2, \dots, \lambda_n$ with $n = \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n$, one has:*

$$c_1^{\lambda_1} \big(\mathcal{C}_2^0\big)^{\lambda_2} \cdots \big(\mathcal{C}_n^0\big)^{\lambda_n} \leqslant_{\mathcal{R}^+} \mathcal{C}_n^0 .$$

Proof. An inspection of the determinant $\mathcal{C}_n^0$ shows that one may view all the pure monomials $c_1^{\lambda_1}$, $\big(\mathcal{C}_2^0\big)^{\lambda_2}$, ..., $\big(\mathcal{C}_k^0\big)^{\lambda_k}$ as diagonal subblocks of the corresponding sizes lying inside $\mathcal{C}_n^0$. Since the operator $\mathcal{R}^+$ expands the determinants and replaces all the minus signs by plus signs, it is then clear that there are more terms in the right-hand side than there are in the left-hand side, which completes the proof. □

The same arguments yield determinantal inequalities at any level.

Lemma 5.5. *For any two J_1, J_2 with $0 \leqslant J_1, J_2 \leqslant \dim X_\ell$ satisfying in addition $J_1 + J_2 \leqslant \dim X_\ell$, and for any j_1 with $0 \leqslant j_1 \leqslant n$ satisfying in addition $j_1 + J_2 \leqslant \dim X_\ell$, one has the two majorations:*

$$\mathcal{R}^+\big(\Omega_K^\ell \cdot \mathcal{C}_{J_1}^\ell \cdot \mathcal{C}_{J_2}^\ell\big) \leqslant \mathcal{R}^+\big(\Omega_K^\ell \cdot \mathcal{C}_{J_1+J_2}^\ell\big) \quad \textit{and} \quad \mathcal{R}^+\big(\Omega_K^\ell \cdot c_{j_1}^{[\ell]} \cdot \mathcal{C}_{J_2}^\ell\big) \leqslant \mathcal{R}^+\big(\Omega_K^\ell \cdot \mathcal{C}_{j_1+J_2}^\ell\big),$$

where Ω_K^ℓ is any (K,K)-form living on X_ℓ completing to $\dim X_\ell$ the degree, namely with $K + J_1 + J_2$ and with $K + j_1 + J_2$ both equal to $\dim X_\ell$.

If $J_1 + J_2 < 0$ or if $J_1 + J_2 > \dim X_\ell$, and if $j_1 + J_2 < 0$ or if $j_1 + J_2 > \dim X_\ell$, the two sides vanish in both inequalities, which hence hold without restriction.

Lemma 5.6. *These coefficients $\lambda_{j,j-k} = \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!}$ appearing in* (15) *satisfy the uniform majoration:*

$$\big|\lambda_{j,j-k}\big| \leqslant 2^n =: \lambda$$

expressed in terms of the dimension n only.

Proof. Indeed, the absolute value of the difference $\lambda_{j,j-k} = \lambda'_{j,j-k} - \lambda''_{j,j-k}$ of two nonnegative integers is less than the largest one, and we majorate any appearing binomial coefficient $\frac{n'!}{i'!\,(n'-i')!}$ or $\frac{n''!}{i''!\,(n''-i'')!}$ with $n' \leqslant n$ and $n'' \leqslant n$ plainly by 2^n. □

In the subsequent majorations, while applying the upper majoration operator $\mathcal{R}^+$, we shall also replace any incoming $\lambda_{j,j-k}$ by this majorant $\lambda = 2^n$. As a result, we define a generalized upper majoration operator "$\mathcal{R}_\lambda^+$" which both replaces any minus sign by a plus sign and any $\lambda_{j,j-k}$ by $\lambda = 2^n$.

Also, when executing inequalities, we shall sometimes not write the left differential form Ω_K^ℓ which completes to $\dim X_\ell$ the total degree of the considered forms, for one knows well now that forms to be reduced always have degree equal to the dimension of the level on which they sit, unless they vanish identically for degree-form reasons.

Lemma 5.7. *For all $k = 1, 2, \dots, n$, one has the $\mathcal{R}_\lambda^+$ majorations:*

$$\mathsf{A}_k^\ell \;\leqslant_{\mathcal{R}_\lambda^+}\; k\lambda\big(\mathcal{C}_{k-1}^{\ell-1} u_\ell + \mathcal{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + u_\ell^k\big).$$

Proof. Starting from the evident majoration of the X_j^ℓ that were defined at the end of Lemma 5.2:

$$\mathsf{X}_j^\ell \;\leqslant_{\mathcal{R}_\lambda^+}\; \lambda\big(c_{j-1}^{[\ell-1]} u_\ell + c_{j-2}^{[\ell-1]} u_\ell^2 + \cdots + u_\ell^j\big),$$

we may perform majorations of an arbitrary A_k^ℓ also defined there:

$$\begin{aligned}
\mathsf{A}_k^\ell &= \mathsf{X}_1^\ell \mathcal{C}_{k-1}^{\ell-1} - \mathsf{X}_2^\ell \mathcal{C}_{k-2}^{\ell-1} + \mathsf{X}_3^\ell \mathcal{C}_{k-3}^{\ell-1} - \cdots + (-1)^{k-1} \mathsf{X}_k^\ell \mathcal{C}_0^{\ell-1} \\
&\leqslant_{\mathcal{R}_\lambda^+} \big[\lambda u_\ell\big] \mathcal{C}_{k-1}^{\ell-1} + \big[\lambda\big(c_1^{[\ell-1]} u_\ell + u_\ell^2\big)\big] \mathcal{C}_{k-2}^{\ell-1} + \big[\lambda\big(c_2^{[\ell-1]} u_\ell + c_1^{[\ell-1]} u_\ell^2 + u_\ell^3\big)\big] \mathcal{C}_{k-3}^{\ell-1} + \\
&\qquad + \cdots + \big[\lambda\big(c_{k-1}^{[\ell-1]} u_\ell + \cdots + c_1^{[\ell-1]} u_\ell^{k-1} + u_\ell^k\big)\big] \mathcal{C}_0^{\ell-1} \\
&= \lambda \Big(u_\ell \big[\mathcal{C}_{k-1}^{\ell-1} + c_1^{[\ell-1]} \mathcal{C}_{k-2}^{\ell-1} + c_2^{[\ell-1]} \mathcal{C}_{k-3}^{\ell-1} + \cdots + c_{k-1}^{[\ell-1]} \mathcal{C}_0^{\ell-1} \big] + \\
&\quad + u_\ell^2 \big[\mathcal{C}_{k-2}^{\ell-1} + c_1^{[\ell-1]} \mathcal{C}_{k-3}^{\ell-1} + \cdots + c_{k-2}^{[\ell-1]} \mathcal{C}_0^{\ell-1} \big] + \\
&\quad + u_\ell^3 \big[\mathcal{C}_{k-3}^{\ell-1} + \cdots + c_{k-3}^{[\ell-1]} \mathcal{C}_0^{\ell-1} \big] + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + u_\ell^k \big[\mathcal{C}_0^{\ell-1} \big] \Big).
\end{aligned}$$

Now, we use the majoration of an arbitrary product of a Jacobi-Trudy determinant by a Chern class that was provided in advance by Lemma 5.5 to obtain:

$$\begin{aligned}
\mathsf{A}_k^\ell &\leqslant_{\mathcal{R}_\lambda^+} \lambda \Big(u_\ell \big[k \cdot \mathcal{C}_{k-1}^{\ell-1} \big] + u_\ell^2 \big[(k-1) \cdot \mathcal{C}_{k-2}^{\ell-1} \big] + \cdots + u_\ell^k \big[\mathcal{C}_0^{\ell-1} \big] \Big) \\
&\leqslant_{\mathcal{R}_\lambda^+} k\lambda \big(\mathcal{C}_{k-1}^{\ell-1} u_\ell + \mathcal{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + u_\ell^k \big),
\end{aligned}$$

as was to be proved. □

We now have to majorate conveniently the A-polynomials $\Sigma_j^\ell(\mathsf{A})$ defined by (19) in terms of Jacobi-Trudy determinants living at the inferior level $\ell - 1$, and in terms of u_ℓ, too. For this purpose, let us define what will play the role of a convenient majorant:

$$\Theta_k^\ell := \mathcal{C}_{k-1}^{\ell-1} u_\ell + \mathcal{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + \mathcal{C}_1^{\ell-1} u_\ell^{k-1} + u_\ell^k,$$

and let us keep in mind that the lemma just proved provided the majorations $\mathsf{A}_k^\ell \leqslant_{\mathcal{R}_\lambda^+} k\lambda\, \Theta_k^\ell$. To majorate products of A_k^ℓ's, we majorate products of Θ_k^ℓ's.

Lemma 5.8. *For any $k_1, k_2, \dots, k_\nu$ with $k_1, k_2, \dots, k_\nu \geqslant 1$ whose sum $k_1 + k_2 + \cdots + k_\nu = j$ equals j, one has the majoration:*

$$\Theta_{k_1}^\ell \Theta_{k_2}^\ell \cdots \Theta_{k_\nu}^\ell \leqslant_{\mathcal{R}_\lambda^+} k_1 k_2 \cdots k_\nu\, \Theta_{k_1+k_2+\cdots+k_\nu}^\ell.$$

Proof. In greater length, the considered product writes:

$$\big(\mathcal{C}_{k_1-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_1}\big)\big(\mathcal{C}_{k_2-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_2}\big) \cdots \big(\mathcal{C}_{k_\nu-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_\nu}\big),$$

and the total number of terms, after expansion, is hence clearly $\leqslant k_1 k_2 \cdots k_\nu$. Using the already known inequality $\mathcal{C}_{J_1}^{\ell-1} \cdot \mathcal{C}_{J_2}^{\ell-1} \leqslant_{\mathcal{R}_\lambda^+} \mathcal{C}_{J_1+J_2}^{\ell-1}$, we may majorate as follows any monomial appearing after expansion:

$$\mathcal{C}_{k_1'}^{\ell-1} \mathcal{C}_{k_2'}^{\ell-1} \cdots \mathcal{C}_{k_\nu'}^{\ell-1} u_\ell^{k''} \leqslant_{\mathcal{R}_\lambda^+} \mathcal{C}_{k_1'+\cdots+k_\nu'}^{\ell-1} u_\ell^{k''},$$

where $k_1' + k_2' + \cdots + k_\nu' + k'' = k_1 + k_2 + \cdots + k_\nu = j$ of course, which completes the proof. □

At last, we can state and prove the main useful majoration proposition which will enable us to achieve the proof of Theorem 5.1, *cf.* the program launched just before Lemma 5.2.

Proposition 5.1. *At any level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathcal{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K, K)-form on X_ℓ the degree K of which satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the upper reduction $\mathcal{R}_\lambda^+(\bullet)$ of $\Omega_K^\ell \mathcal{C}_J^\ell$ in which any incoming $\lambda_{j,j-k}$ is replaced by*

$\lambda = 2^n \geqslant \big|\lambda_{j,j-k}\big|$ *enjoys the following majoration in the right-hand side of which, notably, all the appearing Jacobi-Trudy determinants live at level* $\ell - 1$:

$$\Omega_K^\ell \mathcal{C}_J^\ell \ \leqslant_{\mathcal{R}_\lambda^+} \ J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} \cdot \Omega_K^\ell \left[\mathcal{C}_J^{\ell-1} + \mathcal{C}_{J-1}^{\ell-1} u_\ell + \cdots + \mathcal{C}_1^{\ell-1} u_\ell^{J-1} + u_\ell^J \right].$$

Proof. Recall that

$$\mathcal{C}_J^\ell = \sum_{j=1}^J \mathcal{C}_{J-j}^\ell \, \Sigma_j^\ell(\mathsf{A}) = \sum_{j=0}^J \mathcal{C}_{J-j}^{\ell-1} \bigg(\sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \bigg).$$

Using the last two lemmas, we deduce that for any $k_1, \dots, k_\nu \geqslant 1$ with $k_1 + \cdots + k_\nu$ the sum of which $k_1 + \cdots + k_\nu$ equals j, we have the majoration:

$$\begin{aligned} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \ &\leqslant_{\mathcal{R}_\lambda^+} \ k_1 \cdots k_\nu \, \lambda^\nu \, \Theta_{k_1}^\ell \cdots \Theta_{k_\nu}^\ell && \text{[Lemma 5.7]} \\ &\leqslant_{\mathcal{R}_\lambda^+} \ \big(k_1 \cdots k_\nu\big)^2 \, \lambda^\nu \, \Theta_{k_1+\cdots+k_\nu}^\ell && \text{[Lemma 5.8]} \\ &\leqslant_{\mathcal{R}_\lambda^+} \ j^{2j} \, \lambda^j \, \Theta_j^\ell. \end{aligned}$$

Since there are $2^{j-1} \leqslant 2^j$ terms in the sum $\sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}}$, we receive the useful majoration:

$$\begin{aligned} \Sigma_j^\ell(\mathsf{A}) &= \sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \\ &\leqslant_{\mathcal{R}_\lambda^+} \ 2^j \, j^{2j} \, \lambda^j \, \Theta_j^\ell. \end{aligned}$$

In conclusion, starting from Lemma 5.3 and using Lemma 5.5, we may lastly perform the following (not optimal) majoration:

$$\begin{aligned} \mathcal{C}_J^\ell &= \mathcal{C}_J^{\ell-1} + \mathcal{C}_{J-1}^{\ell-1}\Sigma_1^\ell(\mathsf{A}) + \mathcal{C}_{J-2}^{\ell-1}\Sigma_2^\ell(\mathsf{A}) + \cdots + \mathcal{C}_{J-j}^{\ell-1}\Sigma_j^\ell(\mathsf{A}) + \cdots + \mathcal{C}_0^{\ell-1}\Sigma_J^\ell(\mathsf{A}) \\ &\leqslant_{\mathcal{R}_\lambda^+} \ \mathcal{C}_J^{\ell-1} + \mathcal{C}_{J-1}^{\ell-1} \, 2^1 1^2 \lambda^1 \big[u_\ell\big] + \mathcal{C}_{J-2}^{\ell-1} 2^2 2^4 \lambda^2 \big[\mathcal{C}_1^{\ell-1} u_\ell + u_\ell^2\big] \\ &\qquad + \cdots + \mathcal{C}_{J-j}^{\ell-1} \, 2^j j^{2j} \lambda^j \big[\mathcal{C}_{j-1}^{\ell-1} u_\ell + \cdots + u_\ell^j\big] \\ &\qquad + \cdots + \mathcal{C}_0^{\ell-1} \, 2^J J^{2J} \lambda^J \big[\mathcal{C}_{J-1}^{\ell-1} u_\ell + \cdots + u_\ell^J\big] \\ &\leqslant_{\mathcal{R}_\lambda^+} \ 2^1 1^2 \lambda^1 \big[\mathcal{C}_J^{\ell-1} + \mathcal{C}_{J-1}^{\ell-1} u_\ell\big] + 2^2 2^4 \lambda^2 \big[\mathcal{C}_{J-1}^{\ell-1} u_\ell + \mathcal{C}_{J-2}^{\ell-1} u_\ell^2\big] \\ &\qquad + \cdots + 2^j j^{2j} \lambda^j \big[\mathcal{C}_{J-1}^{\ell-1} u_\ell + \cdots + \mathcal{C}_{J-j}^{\ell-1} u_\ell^j\big] \\ &\qquad + \cdots + 2^J J^{2J} \lambda^J \big[\mathcal{C}_{J-1}^{\ell-1} u_\ell + \cdots + u_\ell^J\big] \\ &\leqslant_{\mathcal{R}_\lambda^+} \ J \cdot 2^J \cdot J^{2J} \cdot \lambda^J \left[\mathcal{C}_J^{\ell-1} + \mathcal{C}_{J-1}^{\ell-1} u_\ell + \mathcal{C}_{J-2}^{\ell-1} u_\ell^2 + \cdots + \mathcal{C}_1^{\ell-1} u_\ell^{J-1} + u_\ell^J\right], \end{aligned}$$

where the introduction of supplementary terms in the brackets aims at producing a uniform right-hand side. □

5.3. **Proof of Theorem 5.1.** The vanishing of the d^0-coefficient comes from the fact that after reduction to the ground level $\ell = 0$, one gets a sum of homogeneous monomials of the form $h^l c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}$ with $l + \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$, and then after expressing each c_k in terms of d through (8), one always has the power $h^n = d$ of h in factor.

Notice that the integer J of the Proposition 5.1 will always be less than or equal to $\dim X_{n-1} = n^2 - n + 1$. To simplify the computations and to receive at the end as simple majorants as possible, we shall apply the following elementary majoration, using $J \leqslant n^2 - n + 1$:

$$\begin{aligned} J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} &= 2^{(n+1)J} \cdot J^{2J+1} \\ &\leqslant 2^{n^3+1} \, (n^2 - n + 1)^{2n^2-2n+3} \\ &\leqslant 2^{n^3} \big(n^2\big)^{2n^2}, \end{aligned}$$

because $2\,(n^2 - n + 1)^{2n^2-2n+3} \leqslant 2\,(n^2)^{2n^2-2n+3} \leqslant (n^2)^{2n^2}$ for any $n \geqslant 2$ (an assumption of Theorem 5.1). Let us temporarily denote this bound by:

$$\mathsf{N} := 2^{n^3} \, n^{4n^2}.$$

As expected, we can now perform a uniform upper majoration of an arbitrary monomial $u_1^{i_1} \cdots u_n^{i_n}$ with $i_1 + \cdots + i_n = n^2$ down to level $\ell = 0$ as follows:

$$\begin{aligned} u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \mathcal{C}_{i_n-n+1}^{n-1} \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{i_{n-1}} \big[\mathcal{C}_{i_n-n+1}^{n-2} + \mathcal{C}_{i_n-n}^{n-2} u_{n-1} \\ &\quad + \cdots + \mathcal{C}_1^{n-2} u_{n-1}^{i_n-n} + u_{n-1}^{i_n-n+1} \big] \qquad \text{[Proposition 5.1]} \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\underline{\mathcal{C}_{i_n-n+1}^{n-2} u_{n-1}^{i_{n-1}} + \cdots}_{\circ} \\ &\quad + \mathcal{C}_{i_{n-1}+i_n-2n+2}^{n-2} \, \underline{u_{n-1}^{n-1}}_{\int} + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1} \big] \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathcal{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathcal{C}_{i_{n-1}+i_n-2n+1}^{n-2} \, u_{n-1}^n \\ &\quad + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1} \big] \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathcal{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathcal{C}_{i_{n-1}+i_n-2n+1}^{n-2} \, \mathcal{C}_1^{n-2} \\ &\quad + \cdots + \mathcal{C}_{i_{n-1}+i_n-2n+2}^{n-2} \big] \qquad \text{[Lemma 5.1]} \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \mathsf{N} \, n^2 \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \, \mathcal{C}_{i_{n-1}+i_n-2n+2}^{n-2} \qquad \text{[Lemma 5.5]} \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \big(\mathsf{N} \, n^2\big)^2 \cdot u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \, \mathcal{C}_{i_{n-2}+i_{n-1}+i_n-3n+3}^{n-3} \qquad \text{[induction]} \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \big(\mathsf{N} \, n^2\big)^3 \cdot u_1^{i_1} \cdots u_{n-4}^{i_{n-4}} \, \mathcal{C}_{i_{n-3}+i_{n-2}+i_{n-1}+i_n-4n+4}^{n-4} \qquad \text{[induction].} \end{aligned}$$

In the third line, we exhibit the general case where i_{n-1} can be $< n-1$, we underline the terms vanishing for degree-form reasons and we point out the fiber-integration of u_{n-1}^{n-1}; when $i_{n-1} \geqslant n-1$, the underlined terms are absent. In the sixth line, we majorate plainly by n^2 the number of terms inside the brackets. (Recall that here by convention again, $\mathcal{C}_J^\ell = 0$ if either $J < 0$ or $J > \dim X_\ell$, so that some of the written $\mathcal{C}_J^\ell$ might well vanish, depending on $i_1, \dots, i_n$.) A now clear induction down to level $\ell = 1$ therefore yields:

$$\begin{aligned} u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} \leqslant_{\mathcal{R}_\lambda^+} \;& \big(\mathsf{N} \, n^2\big)^{n-2} \cdot u_1^{i_1} \, \mathcal{C}_{i_2+\cdots+i_n-(n-1)n+n-1}^1 \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \big(\mathsf{N} \, n^2\big)^{n-2} \cdot \mathsf{N} \cdot \big[\underline{\mathcal{C}_{2n-1-i_1}^0 + \cdots}_{\circ} + \\ & + \mathcal{C}_n^0 \underline{u_1^{n-1}}_{\int} + \cdots + u_1^{2n-1} \big] \\ \leqslant_{\mathcal{R}_\lambda^+} \;& \big(\mathsf{N} \, n^2\big)^{n-1} \, \mathcal{C}_n^0. \end{aligned}$$

It only remains to majorate $\mathcal{C}_n^0$. This last reduction using only (8) without any $\lambda_{j,j-k}$, let us denote by $\mathcal{R}_d^+$ the upper reduction operator restricted to level $\ell = 0$.

Lemma 5.9. *The $n \times n$ Jacobi-Trudy determinant $\mathcal{C}_0^n$ enjoys the majoration:*

$$\mathcal{C}_n^0 \;\leqslant_{\mathcal{R}_d^+}\; 2^{n^2+2n} \, n! \, n^n \, \big[d^{n+1} + d^n + \cdots + d \big].$$

Proof. The number of monomials in the universal $n \times n$ determinant $|a_i^j|$ is $\leqslant n!$ (and is $< n!$ when some a_i^j are zero). Hence:

$$\mathcal{C}_n^0 \leqslant_{\mathcal{R}_d^+} n! \max_{\lambda_1+2\lambda_2+\cdots+n\lambda_n=n} c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}.$$

The general binomial coefficient $\binom{n+2}{k}$ which appears in (8) is less than or equal to 2^{n+2}, so that:

$$c_j \leqslant_{\mathcal{R}_d^+} 2^{n+2} h^j \big[d^j + \cdots + d + 1\big].$$

We majorate as follows the products of these basic polynomials in d:

$$\big[d^{j_1} + \cdots + d + 1\big]\big[d^{j_2} + \cdots + d + 1\big] \leqslant_{\mathcal{R}_d^+} j_1 j_2 \big[d^{j_1+j_2} + \cdots + d + 1\big],$$

and we therefore deduce a majorant for the general homogeneous degree n monomial in the ground Chern classes:

$$\begin{aligned}
c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} &\leqslant_{\mathcal{R}_d^+} \big(2^{n+2}\big)^{\lambda_1+\lambda_2+\cdots+\lambda_n} 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n} h^{\lambda_1+2\lambda_2+\cdots+n\lambda_n} \\
&\qquad \cdot \big[d^{\lambda_1+2\lambda_2+\cdots+n\lambda_n} + \cdots + d + 1\big] \\
&\leqslant_{\mathcal{R}_d^+} \big(2^{n+2}\big)^n n^{\lambda_1+\lambda_2+\cdots+\lambda_n} h^n \big[d^n + \cdots + d + 1\big] \\
&\leqslant_{\mathcal{R}_d^+} 2^{n^2+2n} n^n d \big[d^n + \cdots + d + 1\big]
\end{aligned}$$

which completes the proof. □

Applying this lemma to the last obtained inequality:

$$u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathcal{R}_\lambda^+} (Nn^2)^{n-1} 2^{n^2+2n} n! \, n^n \cdot \big[d^{n+1} + d^n + \cdots + 1\big],$$

we then obtain the announced bound $n^{4n^3} 2^{n^4}$ as follows:

$$\begin{aligned}
\big|\mathsf{coeff}_{d^k}\big[u_1^{i_1} \cdots u_n^{i_n}\big]\big| &\leqslant \big(2^{n^3} n^{4n^2} n^2\big)^{n-1} 2^{n^2+2n} n! \, n^n \\
&\leqslant 2^{n^4-n^3+n^2+2n} n^{4n^3-4n^2+2n-2} n^n n^n \\
&\leqslant n^{4n^3} 2^{n^4}.
\end{aligned}$$

By an inspection of the final inequalities which enabled us to descend from the top of Demailly's tower to its ground level, one easily convinces oneself that the monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ and $c_1 h^l u_1^{j_1} \cdots u_n^{j_n}$ satisfy exactly the same upper bound reduction:

$$\begin{aligned}
h^l u_1^{i_1} \cdots u_n^{i_n} &\leqslant_{\mathcal{R}_\lambda^+} \big(N n^2\big)^{n-1} \mathcal{C}_n^0 \quad \text{and} \\
c_1 h^l u_1^{j_1} \cdots u_n^{j_n} &\leqslant_{\mathcal{R}_\lambda^+} \big(N n^2\big)^{n-1} \mathcal{C}_n^0,
\end{aligned}$$

since the forms h^l and $c_1 h^l$ do intervene only at the very end of the process. This completes the proof of Theorem 5.1. At the same time, the proof of Theorem 1.1 can be considered as complete, as soon as we take for granted Theorem 1.2, as was already explained at the end of Section 4. □

6. EFFECTIVE BOUNDS IN DIMENSIONS 2, 3 AND 4 THROUGH THE INVARIANT THEORY APPROACH

The goal of this section is to obtain the effective bound $\deg X^4 \geqslant 3203$ of Theorem 1.2 in dimension $n = 4$ which insures strong algebraic degeneracy of entire curves inside a generic projective four-fold $X^4 \subset \mathbb{P}^5$. As was said in the Introduction, our reasonings will be based on a complete knowledge ([14]) of the full algebra $\bigoplus_{m\geqslant 0} E_{4,m} T^*_{X^4,x_0}$ of germs of invariant 4-jet differentials at a point $x_0 \in X^4$, which, unfortunately, is still unavailable at present times for jets of order $k \geqslant n$ in the higher dimensions $n = 5, 6, 7, \dots$ (remind that by Theorem 1.1 in [17] and by Theorem 1 in [6], $H^0\big(X^n,\, E_{k,m}T^*_{X^n}\big) = 0$ whenever $k \leqslant n-1$). The so obtained bound $\deg X^4 \geqslant 3203$ happens to be sharper than the one $\deg X^4 \geqslant 6527$ that one would obtain using the intersection product (11). For completeness and in parallel, we also recall what happens in the lower dimensions 2 ([4, 5]) and 3 ([16, 17]).

6.1. **Algebras of bi-invariant k-jet differentials.** Let $(x_1, \dots, x_n)$ be local coordinates centered at some point $x_0 \in X$ and let $f = (f_1, \dots, f_n) \colon (\mathbb{C}, 0) \to (X, x_0)$ be a germ of holomorphic curve. For each fixed l-th jet level ($1 \leqslant l \leqslant k$) over x_0, the constant matrices $v = (v_i^j)_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n}$ in $\mathrm{GL}_n(\mathbb{C})$ act in a natural way on the n jet coordinates $(f_1^{(l)}, \dots, f_n^{(l)})$ simply by:

$$v \cdot f^{(l)} := \Big(\textstyle\sum_j v_i^j f_j^{(l)}\Big)_{1\leqslant i\leqslant n}.$$

In order to know what is the precise decomposition of $\mathrm{Gr}^\bullet(E_{k,m}T^*_X)$ as a direct sum of Schur bundles $\Gamma^{(\ell_1,\dots,\ell_n)}T^*_X$, the classical representation theory of $\mathrm{GL}_n(\mathbb{C})$ tells us that one should look at jet polynomials $Q(f', f'', \dots, f^{(k)})$ that are not only invariant under reparametrization in the sense of Definition 2.1, but also invariant under the action of the full unipotent subroup $\mathrm{U}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ consisting of matrices with 1 on the diagonal, 0 above the diagonal, and arbitrary complex number below the diagonal; background information may be found in [4, 16, 17, 14]. Accordingly, one may define the algebra of *bi-invariant* k-jet polynomials in dimension n:

$$\mathsf{UE}_k^n := \Big(\bigoplus_{m\geqslant 0} E_{k,m}T^*_{X^n, x_0}\Big)^{\mathrm{U}_n(\mathbb{C})}.$$

This algebra does not depend on the base point $x_0 \in X$. We shall employ the abbreviations $\Delta_{i_1,i_2}^{(\alpha),(\beta)} := f_{i_1}^{(\alpha)} f_{i_2}^{(\beta)} - f_{i_2}^{(\alpha)} f_{i_1}^{(\beta)}$ for 2×2 determinants, and similarly $\Delta_{i_1,i_2,i_3}^{(\alpha),(\beta),(\gamma)}$ for the analogous 3×3 determinants. The upper indices of all the appearing 16 bi-invariants f_1', Λ^3, Λ^5, Λ^7, D^6, D^8, N^{10}, W^{10}, M^8, E^{10}, L^{12}, Q^{14}, R^{15}, U^{17}, V^{19} and X^{21} below just denote their weighted degree m.

Theorem 6.1. *The following three algebraic descriptions hold.*

- [4] *In dimension* 2*, one has:* $\mathsf{UE}_2^2 = \mathbb{C}\big[f_1',\, \Lambda^3\big]$*, where* $\Lambda^3 := \Delta_{1,2}^{','' } = f_1' f_2'' - f_2' f_1''$ *is the two-dimensional Wronskian.*
- [16] *In dimension* 3*, one has:*

$$\mathsf{UE}_3^3 = \mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, D^6\big],$$

where $\Lambda^5 := \Delta_{1,2}^{','''} f_1' - 3\,\Delta_{1,2}^{',''} f_1''$ *and where* $D^6 := \Delta_{1,2,3}^{','','''}$ *is the three-dimensional Wronskian.*
- [14] *In dimension* 4*, one has:*

$$\mathsf{UE}_4^4 = \mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, \Lambda^7,\, D^6,\, D^8,\, N^{10},\, W^{10},\, M^8,\, E^{10},\, L^{12},$$
$$Q^{14},\, R^{15},\, U^{17},\, V^{19},\, X^{21}\big] \Big/ \textit{a certain ideal of } 41 \textit{ relations},$$

where:

$$\Lambda^7 = \Delta_{1,2}^{',''''}\, f_1' f_1' + \Delta_{1,2}^{'','''}\, f_1' f_1' - 10\,\Delta_{1,2}^{','''}\, f_1' f_1'' + 15\,\Delta_{1,2}^{',''}\, f_1'' f_1'',$$
$$D^8 = \Delta_{1,2,3}^{',''',''''}\, f_1' - 3\,\Delta_{1,2,3}^{','',''''}\, f_1'',$$
$$N^{10} = \Delta_{1,2,3}^{',''',''''}\, f_1' f_1' - 3\,\Delta_{1,2,3}^{','',''''}\, f_1' f_1'' + 4\,\Delta_{1,2,3}^{','',''' }\, f_1' f_1''' + 3\,\Delta_{1,2,3}^{','','''}\, f_1'' f_1'',$$

where $W^{10} = \Delta^{','',''','''''}_{1,2,3,4}$ *is the four-dimensional Wronskian and where the eight remaining bi-invariants defined by:*

$$M^8 := \frac{-5\,\Lambda^5\Lambda^5+3\,\Lambda^3\Lambda^7}{f_1'f_1'} \qquad E^{10} := \frac{-6\,\Lambda^5\,D^6+3\,\Lambda^3\,D^8}{f_1'}, \qquad L^{12} := \frac{-\Lambda^7D^6+5\,\Lambda^3N^{10}}{f_1'},$$
$$Q^{14} := \frac{\Lambda^7D^8-10\,\Lambda^5N^{10}}{f_1'}, \qquad R^{15} := \frac{D^8D^8-12\,D^6N^{10}}{f_1'}, \qquad U^{17} := \frac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'},$$
$$V^{19} := \frac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'}, \qquad X^{21} := \frac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'}$$

happen all to be true polynomials *in* $\mathbb{C}\big[f_1', \dots, f_4''''\big]$*, and where an explicit Gröbner basis, with respect to the pure lexicographic term-order* $f_1' < \Lambda^3 < \cdots < X^{21}$*, for the ideal of relations that they share, is provided in §11 of* [14].

For instance, the first three relations among the 41 written just before the theorem of §11 in [14] are:

$$0 \overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7 - f_1'f_1'M^8,$$
$$0 \overset{2}{\equiv} -2\,\Lambda^5D^6 + \Lambda^3D^8 - \tfrac{1}{3}f_1'\,E^{10},$$
$$0 \overset{3}{\equiv} -\Lambda^7D^6 + 5\,\Lambda^3N^{10} - f_1'L^{12}.$$

Although the complexity of the algebra of bi-invariants increases dramatically as soon as $n \geqslant 4$, one finds in [14] a complete algorithm which generates all bi-invariants together with all the relations that they share, this in arbitrary dimension $n \geqslant 1$ and for arbitrary jet order $k \geqslant 1$.

6.2. **Schur bundle decompositions.** In dimension 3, there are no relations between the four basic bi-invariants f_1', Λ^3, Λ^5 and D^6 and we hence clearly have:

$$\big(E_{k,m}T^*_{X^n,x_0}\big)^{\mathrm{U}_3(\mathbb{C})} = \mathrm{Span}_{\mathbb{C}}\Big\{(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d : a,b,c,d \in \mathbb{N},\ a+3b+5c+6d = m\Big\}.$$

Then to any such general monomial $(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d$ having weighted degree $m = a+3b+5c+6d$, the representation theory of $\mathrm{GL}_n(\mathbb{C})$ tells us that there corresponds the Schur bundle:

$$\Gamma^{(a+b+2c+d,\ b+c+d,\ d)}T^*_X,$$

just because the diagonal 3×3 matrices $t = \mathrm{diag}(t_1,t_2,t_3)$ act as: $t\cdot f_i^{(\lambda)} := t_i\, f_i^{(\lambda)}$, whence:

$$t\cdot f_1' = t_1f_1', \qquad t\cdot\Lambda^3 = t_1t_2\,\Lambda^3, \qquad t\cdot\Lambda^5 = t_1t_1t_2\,\Lambda^5, \qquad t\cdot D^6 = t_1t_2t_3\,D^6,$$

so that indeed the three exponents of the t_i in:

$$t\cdot(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d = t_1^{a+b+2c+d}\ t_2^{b+c+d}\ t_3^d\ (f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d$$

indicate the three corresponding integers in $\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T^*_X$. The same elementary process enables one, in dimensions 2 and 4, to immediately deduce from the preceding statement the following important decomposition theorem for the graded bundle $\mathrm{Gr}^\bullet E_{k,m}T^*_{X^n}$ associated to $E_{k,m}T^*_{X^n}$, which is valuable without assuming that X is projective.

Theorem 6.2. *Let* X *be a compact complex manifold and let* $m \in \mathbb{N}$.

- [4] *If* $\dim X = 2$ *then*

$$\mathrm{Gr}^\bullet\, E_{2,m}T^*_X = \bigoplus_{a+3b=m} \Gamma^{(a+b,\ b)}T^*_X.$$

- [16] *If* $\dim X = 3$ *then*
$$\mathrm{Gr}^\bullet\, E_{3,m} T_X^* = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\ b+c+d,\ d)} T_X^*.$$
- [14] *If* $\dim X = 4$ *then*
$$\mathrm{Gr}^\bullet\, E_{4,m} T_X^* = \bigoplus_{\substack{(a,b,c,d,e,f,g,h,i,j,k,l,m',n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ o+3a+5b+7c+6d+8e+10f+8g+10h+12i+14j+15k+17l+19m'+21n+10p=m}}$$
$$\Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T_X^*,$$
where the 41 subsets $\square_i$, $i = 1, 2, \dots, 41$, *of* $\mathbb{N}^{14} \ni (a, b, \dots, l, m', n)$ *are explicitly defined in §12 of* [14].

6.3. **Euler-Poincaré characteristic of Schur bundles.** With $X = X^n \subset \mathbb{P}^{n+1}$ projective as before and with $c_j = c_j(T_X)$ for $j = 1, \dots, n$ being the Chern classes of T_X as in (8), a general asymptotic formula for the Euler-Poincaré characteristic of a Schur bundle is given in §13 of [14] (*see* also Theorem 4 in [2]), and for $n = 4$, this formula expands as:

$$\chi\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\, T_X^*\big) = \frac{c_1^4 - 3\,c_1^2c_2 + c_2^2 + 2\,c_1c_3 - c_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \end{vmatrix} +$$
$$+ \frac{c_1^2c_2 - c_2^2 - c_1c_3 + c_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \end{vmatrix} + \frac{-c_1c_3 + c_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} +$$
$$+ \frac{c_1c_3 - c_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \frac{c_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \end{vmatrix} + \mathrm{O}\big(|\ell|^9\big).$$

Of course, similar expanded — though shorter — formulae exist also in dimensions 2 and 3, *cf.* again §13 of [14].

6.4. **Riemann-Roch computations.** Recalling that the n-th power $h^n = d$ of the hyperplane class $h = c_1\big(\mathcal{O}_{\mathbb{P}^{n+1}}(1)\big)$ equals $\deg X$, the formulae (8) entail that any monomial $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}$ whose weighted homogeneous degree $\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n$ equals n is a polynomial in $\mathbb{Z}[d]$, as are c_1^4, $c_1^2c_2$, c_2c_2, c_1c_3 and c_4 above when $n = 4$. Basic additivity, *e.g.* in dimension 3:
$$\chi\big(X,\, E_{3,m}T_X^*\big) = \chi\big(X,\, \mathrm{Gr}^\bullet E_{3,m}T_X^*\big) = \sum_{a+3b+5c+6d=m} \chi\big(X,\, \Gamma^{(a+b+2c+d,\ b+c+d,\ d)} T_X^*\big)$$
enables one to deduce, by plain numerical summation and with some electronic assistance, the following three Euler-Poincaré characteristics, depending upon m and d only. We notice that the summation of the three attached remainders, *e.g.* of $\mathrm{O}\big(|\ell|^9\big)$ in dimension 4, only contributes up to a lower power of m, *e.g.* up to an $\mathrm{O}\big(m^{15}\big)$ in dimension 4.

Theorem 6.3. *Let* $X \subset \mathbb{P}^{n+1}$ *be a smooth hypersurface of degree* d.

- [4] *For* $n = 2$:

$$\chi\big(X, E_{2,m}T_X^*\big) = \frac{m^4}{648}\, d\,\big(4d^2 - 68d + 154\big) + \mathrm{O}(m^3).$$

- [16] *For* $n = 3$:

$$\chi\big(X,\, E_{3,m}T_X^*\big) = \frac{m^9}{81648 \times 10^6}\, d\,\big(389d^3 - 20739d^2 + 185559d - 358873\big) + \mathrm{O}(m^8).$$

- [14] *For* $n = 4$:

$$\begin{aligned}\chi\big(X,\, E_{4,m}T_X^*\big) = & \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ & \cdot \big(5004851113579703425623 5\, d^4 - \\ & \quad - 6170606622505955255988786\, d^3 - \\ & \quad - 9288869013541411538806247 04\, d + \\ & \quad + 1411704752502476621473639 41\, d^2 + \\ & \quad + 1624908955061039283976041114\big) \\ & + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

6.5. **The strategy of controlling the even cohomology dimensions.** Remember from Theorem 2.2 that the first step towards the algebraic degeneracy of entire curves $f : \mathbb{C} \to X$ consists in proving the existence of nonzero global sections in $H^0\big(X,\, E_{k,m}T_X^* \otimes A^{-1}\big)$, for some ample line bundle $A \to X$, *e.g.* $A = \mathcal{O}_X(1)$, and when A does not depend on m, the asymptotic cohomologies, as $m \to \infty$, of the two bundles $E_{k,m}T_X^*$ and $E_{k,m}T_X^* \otimes A^{-1}$ coincide. So a quite natural strategy, followed by the third-named author in [17], consists to rewrite the characteristic, say in dimension four: $\chi = h^0 - h^1 + h^2 - h^4$ under the form:

$$\begin{aligned} h^0 &= \chi + h^1 - h^2 + h^3 - h^4 \\ &\geqslant \chi \qquad - h^2 \qquad - h^4, \end{aligned}$$

and to control asymptotically the dimensions $h^{2i}_{k,m}$ of all the even cohomology groups $H^{2i}(X,\, E_{k,m}T_X^* \otimes A^{-1})$ by some vanishing theorem or by some appropriate inequalities which would then show that these $h^{2i}_{k,m}$ grow less rapidly than the characteristic $\chi_{k,m}$ as m tends to ∞. In dimensions 2 and 4, the controls of the top even cohomology dimensions h^2 and h^4 are obtained thanks to a vanishing theorem due to Demailly which generalized a theorem of Bogomolov.

Theorem 6.4 ([4]). *Let X be a projective algebraic manifold of dimension $n \geqslant 2$ and let L be a holomorphic line bundle over X. Assume that K_X is big and nef and let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ be a weight with $\mu_1 \geqslant \cdots \geqslant \mu_n$. If either L is pseudo-effective and $|\mu| = \mu_1 + \cdots + \mu_n > 0$, or L is big and $|\mu| \geqslant 0$, then*:

$$H^0\big(X,\, \Gamma^{(\mu_1,\dots,\mu_n)}T_X \otimes L^*\big) = 0.$$

Recall that if some μ_i is negative, we may use the identity:

$$\Gamma^{(\mu_1,\dots,\mu_n)}T_X^* = \Gamma^{(\mu_1+l,\dots,\mu_n+l)}T_X^* \otimes K_X^{-l}.$$

For instance in dimension 4, we observe that the above vanishing theorem implies that

$$h^4\big(X,\, E_{4,m}T_X^* \otimes A^{-1}\big) = 0,$$

for all m sufficiently large; indeed, we have:

$$h^4\big(X,\,\Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T_X^*\otimes A^{-1}\big)=h^0\big(X,\,\Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T_X\otimes A\otimes K_X\big)$$
$$=h^0\Big(X,\,\Gamma^{(\lambda_1-\nu,\lambda_2-\nu,\lambda_3-\nu,\lambda_4-\nu)}T_X\otimes {K_X}^{1-\nu}\otimes\mathcal{O}(A)\Big).$$

But ${K_X}^{\nu-1}\otimes A^{-1}$ is big for ν large enough and then the above theorem applies to provide the vanishing of h^4 as soon as:

$$|\lambda|-4\nu\geqslant 0,$$

which is satisfied for m large enough since one easily convinces oneself that $|\lambda|\geqslant\frac{4m}{10}$ in the dimension 4 case of Theorem 6.2.

However, it has been discovered by the third-named author [17] that already in dimension three, $H^2\big(X,\,E_{3,m}T_X^*\big)\neq 0$ does not vanish in general. Fortunately, a suitable majoration holds.

Theorem 6.5 ([17]). *Let X be a smooth hypersurface of degree d in $\mathbb{P}^4$. Then for $|\lambda|$ large enough:*

$$h^2\big(X,\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T_X^*\big)$$
$$\leqslant d(d+13)\frac{3(\lambda_1+\lambda_2+\lambda_3)^3}{2}(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)+\mathrm{O}(|\lambda|^5).$$

In dimension 4 the same proof provides the new estimate:

Theorem 6.6. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$. Then for $|\lambda|$ large enough, we have:*

$$\begin{aligned}
&h^2\big(X,\Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T_X^*\big)\\
&\quad\leqslant\frac{1}{80}\,d\,(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)\\
&\quad\cdot\big(\lambda_1+\lambda_2+\lambda_3+\lambda_4\big)^2\big[5\lambda_2\lambda_1d^2+132\lambda_2\lambda_1d+132\lambda_1\lambda_3d+5\lambda_2\lambda_3d^2\\
&\quad+132\lambda_2\lambda_4d+5\lambda_2d^2\lambda_4+132\lambda_1\lambda_4d+5\lambda_3\lambda_4d^2+5\lambda_1\lambda_3d^2\\
&\quad+132\lambda_3\lambda_4d+132\lambda_2\lambda_3d+1308\lambda_2\lambda_1+648\lambda_2^2+648\lambda_3^2\\
&\quad+72\lambda_3^2d+648\lambda_1^2+72\lambda_1^2d+1308\lambda_1\lambda_4+5\lambda_1d^2\lambda_4+1308\lambda_2\lambda_4\\
&\quad+1308\lambda_2\lambda_3+648\lambda_4^2+72\lambda_2^2d+1308\lambda_1\lambda_3+72\lambda_4^2d+1308\lambda_3\lambda_4\big]\\
&\quad+\mathrm{O}\big(|\lambda|^9\big).
\end{aligned}$$

Proof. We follow [17] pp. 335-36, summarizing the main arguments for the convenience of the reader. The proof is essentially, again, an application of holomorphic Morse inequalities and the reader will notice strong similarities with the arguments presented in section 2.

Let $Y:=Fl(T_X^*)$ be the flag manifold of T_X^* and let $\pi\colon Fl(T_X^*)\to X$ the natural projection. Let $\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a weight and $\mathcal{L}^\lambda$ the line bundle on Y associated to $\Gamma^\lambda T_X^*$ such that $\Gamma^\lambda T_X^*=\pi_*(\mathcal{L}^\lambda)$. By a theorem of Bott, these bundles have the same cohomology (*cf.* [17] p. 327) and therefore we are reduced to control the cohomology of a line bundle. To this aim, we write:

$$\mathcal{L}^\lambda=\big(\mathcal{L}^\lambda\otimes\pi^*\mathcal{O}_X(3\,|\lambda|)\big)\otimes\big(\pi^*\mathcal{O}_X(3\,|\lambda|)\big)^{-1}=F\otimes G^{-1},$$

with $F:=\mathcal{L}^\lambda\otimes\pi^*\mathcal{O}_X(3\,|\lambda|)$ and $G:=\pi^*\mathcal{O}_X(3\,|\lambda|)$. We observe that $\mathcal{L}^\lambda\otimes\pi^*\mathcal{O}_X(3\,|\lambda|)$ is positive because $T_X^*\otimes\mathcal{O}_X(2)$ is semi-positive ([17]).

Recall also ([17]) that we can write $K_Y = (\mathcal{L}^\sigma)^{-1} \otimes \pi^* K_X^5$ where $\sigma = (7,5,3,1)$. So we have:

$$F \otimes K_Y^{-1} = \mathcal{L}^{\lambda+\sigma} \otimes \pi^* \mathcal{O}_X(3\,|\lambda|) \otimes \pi^* K_X^{-5},$$

and we still have $F \otimes K_Y^{-1} > 0$ for $|\lambda|$ large enough. Indeed, as above

$$\mathcal{L}^{\lambda+\sigma} \otimes \pi^* \mathcal{O}_X\big(2\,|\lambda+\sigma|\big)$$

is semi-positive and therefore $F \otimes K_Y^{-1} > 0$ as soon as $|\lambda| > 5(d-6)+32$.

Now we take a smooth irreducible divisor D_1 in the linear system $|G|$. We have the exact sequence:

$$0 \longrightarrow \mathcal{O}_Y\big(F \otimes G^{-1}\big) \longrightarrow \mathcal{O}_Y(F) \longrightarrow \mathcal{O}_{D_1}(F) \longrightarrow 0,$$

and therefore in the associated long exact cohomology sequence:

$$\begin{aligned} 0 = H^i\big(Y, \mathcal{O}_Y(F)\big) \longrightarrow H^i\big(D_1, \mathcal{O}_{D_1}(F)\big) &\longrightarrow H^{i+1}\big(Y, \mathcal{O}_Y(F \otimes G^{-1})\big) \\ &\longrightarrow H^{i+1}\big(Y, \mathcal{O}_Y(F)\big) = 0, \end{aligned}$$

both the first and last terms vanish for any $i > 0$ by an application of the Kodaira vanishing theorem. So we deduce:

$$h^i\big(D_1,\, \mathcal{O}_{D_1}(F)\big) = h^{i+1}\big(Y, \mathcal{O}_Y(F \otimes G^{-1})\big).$$

Taking a second divisor $D_2 \in |G|$ intersecting properly D_1, using the adjunction formula and applying again the same argument as above on $Z := D_1 \cap D_2$ (*cf.* [17], pp. 335–336), we obtain:

$$h^2\big(Y, \mathcal{O}_Y(F \otimes G^{-1})\big) = h^1\big(D_1, \mathcal{O}_{D_1}(F)\big) \leqslant h^0\big(Z, \mathcal{O}_Z(F \otimes G^2)\big) = \chi\big(Z, \mathcal{O}_Z(F \otimes G^2)\big).$$

We finish by an explicit Riemann-Roch computation of $\chi\big(Z, \mathcal{O}_Z(F \otimes G^2)\big)$ as was detailed in [17] p. 336 and the proof is completed with the help of a computer. □

From such controls of higher cohomology groups, one deduces the existence of global algebraic differential equations canalizing all entire holomorphic maps. For the sake of completeness, we recall here what is known in dimensions 2 and 3.

Theorem 6.7. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and let A be any ample line bundle over X.*

- [4] *For* $n = 2$:

$$h^0\big(X,\, E_{2,m}T_X^* \otimes A^{-1}\big) \geqslant \frac{m^4}{648}\, d\,\big(4d^2 - 68d + 154\big) + \mathrm{O}(m^3);$$

- [17] *For* $n = 3$:

$$\begin{aligned} h^0\big(X,\, E_{3,m}T_X^* \otimes A^{-1}\big) \geqslant \frac{m^9}{408240000000} \cdot d \cdot \big(&1945\, d^3 - 103695\, d^2 \\ &- 7075491\, d - 105837083\big) + \mathrm{O}(m^8). \end{aligned}$$

In particular, if $d \geqslant 15$ (resp. $d \geqslant 97$) then $E_{2,m}T_X^* \otimes A^{-1}$ (resp. $E_{3,m}T_X^* \otimes A^{-1}$) admits non trivial sections for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

In dimension 4, we therefore present the following new result.

Theorem 6.8. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$ and let A be any ample line bundle over X. Then:*

$$\begin{aligned} & h^0(X, E_{4,m}T_X^* \otimes A^{-1}) \\ & \qquad \geqslant \frac{m^{16}}{131331783230389433321033564160000000000000} \cdot d \\ & \qquad\qquad \cdot \big[-867659678949860838548185438614 \\ & \qquad\qquad - 934880693607607850940593379216\, d \\ & \qquad\qquad - 136932726517733910329233 1439\, d^2 \\ & \qquad\qquad - 617060662250595525598878 6\, d^3 \\ & \qquad\qquad + 500485111357970342562 35\, d^4 \big] \\ & \qquad\qquad\qquad + O\big(m^{15}\big). \end{aligned}$$

In particular, if $d \geqslant 259$ then $E_{4,m}T_X^* \otimes A^{-1}$ admits non trivial sections for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

6.6. **Algebraic degeneracy.** According to Theorem 2.5, in dimension n, the maximal pole order of a meromorphic frame on the space of vertical n-jets of the universal hypersurface parametrizing all degree d hypersurfaces of $\mathbb{P}^{n+1}$ is equal to $n^2 + 2n$. Then one applies the same arguments as in [18], pp. 381–383 to the Schur bundle decomposition provided in [14] and one uses the majoration for the h^2 of an arbitrary Schur bundle explicited above. As a result, thanks to effective computations executed independently on two digital computers by the second and by the third named author using different codes, one obtains in dimension 4 the new effective lower bound $\deg X \geqslant 3203$ of Theorem 1.2.

7. EFFECTIVE ALGEBRAIC DEGENERACY IN DIMENSIONS 5 AND 6

Finally, for dimensions 5 and 6, we simply carry out the same strategy as in the general case, but with a choice of weight different from a^* introduced in Subsection 4.4. Our choice specific for these two dimensions are $\mathbf{a} = (54, 18, 6, 2, 1)$ and $\mathbf{a} = (162, 54, 18, 6, 2, 1)$, that is to say: the minimal choice in order to have relative nefness of the weighted (anti)tautological line bundle $\mathcal{O}_{X_n}(\mathbf{a})$, $n = 5, 6$ (*cf.* [4, 6]); also, we choose $\delta = \frac{5^2+2\cdot 5}{d-5-2}$ and $\delta = \frac{6^2+2\cdot 6}{d-6-2}$. The bound is then obtained thanks to computer calculations with GP/PARI, (*cf.* [6] for the code). The same method, in dimension 4 (resp. 3), would have produced $\deg X \geqslant 6527$ (resp. $\geqslant 1019$), less sharp than $\deg X \geqslant 3203$ (resp. $\geqslant 593$).

In dimension $n = 5$, here are the corresponding two polynomials $\mathsf{P}_{\mathbf{a}}(d)$ and $\mathsf{P}'_{\mathbf{a}}(d)$ the length of which confirms the incompressible complexity of the reduction process:

$$\begin{aligned} \mathsf{P}_{54,18,6,2,1}(d) = {} & 829705555252684668951323755447424\, d^6 - \\ & - 690923576923829601983160082796154 24\, d^5 - \\ & - 375919573131846296972181088319559277 44\, d^4 - \\ & - 216114449751608047695560783767127869958 4\, d^3 - \\ & - 207679317231737411175485558372431638061 44\, d^2 - \\ & - 237364617790381662461159583045518710563 84\, d. \end{aligned} \tag{21}$$

and:

$$
\begin{aligned}
\mathsf{P}'_{54,18,6,2,1}(d) = &-810649364923821805499061816503472 00\, d^6 - \\
&- 2561926552944387465736285101371322 7200\, d^5 - \\
&- 113836022401687725413740756664273577 8400\, d^4 - \\
&- 26494079429881985392011761627532406 34400\, d^3 + \\
&+ 70399558265933283202949942118101580 280800\, d^2 + \\
&+ 90355953106499854530169310985578945 008800\, d.
\end{aligned} \tag{22}
$$

We believe that the sequence of weights $\mathbf{a} = (2 \cdot 3^{n-2}, \ldots, 6, 2, 1)$ instead of a^* should work in any dimension, and that it should provide better effective estimates in all dimensions, though we suspect the bound should remain exponential. To conclude, we collect our three effective estimates in a comparative table

dim X	**Theorem 1.2**	**Theorem 1.1**
3	593	2^{3^5}
4	3203	2^{4^5}
5	35355	2^{5^5}
6	172925	2^{6^5}

References

[1] A. Bloch, *Sur les systèmes de fonctions uniformes satisfaisant à l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension*, J. de Math. **5** (1926), 19–66.

[2] P. Brückmann, *The Hilbert polynomial of the sheaf Ω^T of germs of T-symmetrical tensor differential forms on complete intersections*, Math. Ann. **307** (1997), 461–472.

[3] H. Clemens, *Curves on generic hypersurface*, Ann. Sci. Éc. Norm. Sup., **19** (1986), 629–636.

[4] J.-P. Demailly, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Proc. Sympos. Pure Math., vol. 62, Amer. Math.Soc., Providence, RI, 1997, 285–360.

[5] J.-P. Demailly, J. El Goul, *Hyperbolicity of generic surfaces of high degree in projective 3-space*, Amer. J. Math. **122** (2000), 515–546.

[6] S. Diverio, *Differential equations on complex projective hypersurfaces of low dimension*, Compos. Math. **144** (2008), no. 4, 920–932.

[7] ——, *Existence of global invariant jet differentials on projective hypersurfaces of high degree*, Math. Ann. **344** (2009), no. 2, 293–315.

[8] M. Green, P. Griffiths, *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*. The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), 41–74, Springer, New York-Berlin, 1980.

[9] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[10] S. Kobayashi, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften 318, Springer Verlag, Berlin, 1998, xiv+471 pp.

[11] M. McQuillan, *Diophantine approximation and foliations*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 121–174.

[12] ——, *Holomorphic curves on hyperplane sections of* 3*-folds*, Geom. Funct. Anal. **9** (1999), 370–392.

[13] J. Merker, *Low pole order frames on vertical jets of the universal hypersurface*, Ann. Inst. Fourier (Grenoble), **59** (2009), no. 3, 1077–1104.

[14] ——, *An algorithm to generate all polynomials in the k-jet of a holomorphic disc $D \to \mathbb{C}^n$ that are invariant under source reparametrization*, arxiv.org/abs/0808.3547/

[15] M. Păun, *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity*, Math. Ann. **340** (2008), 875–892.

[16] E. Rousseau, *Étude des jets de Demailly-Semple en dimension* 3, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 397–421.

[17] ——, *Équations différentielles sur les hypersurfaces de* $\mathbb{P}^4$, J. Math. Pures Appl. (9) **86** (2006), no. 4, 322–341.

[18] ——, *Weak analytic hyperbolicity of generic hypersurfaces of high degree in* $\mathbb{P}^4$, Annales Fac. Sci. Toulouse, **16** (2007), no. 2, 369–383.

[19] Y.-T. Siu, *Hyperbolicity problems in function theory*, in *Five decades as a mathematician and educator – on the 80^th^ birthday of professor Yung-Chow Wong*, ed. Kai-Yuen Chan and Ming-Chit Liu, World Scientific: Singapore, New Jersey, London, Hong-Kong, 1995, pp. 409–514.

[20] ——, *Some recent transcendental techniques in algebraic and complex geometry*. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 439–448, Higher Ed. Press, Beijing, 2002.

[21] ——, *Hyperbolicity in complex geometry*, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, 543–566.

[22] Y.-T. Siu, S.-K; Yeung, *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane*, Invent. Math. **124** (1996), 573–618.

[23] S. Trapani, *Numerical criteria for the positivity of the difference of ample divisors*. Math. Z. **219** (1995), no. 3, 387–401.

[24] C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **44** (1996), no. 1, 200–213. A correction: "*On a conjecture of Clemens on rational curves on hypersurfaces*", J. Diff. Geom. **49** (1998), no. 3, 601–611.

[25] ——, *On some problems of Kobayashi and Lang; algebraic approaches*, Current developments in mathematics, 2003, 53–125, Int. Press, Somerville, MA, 2003.

SIMONE DIVERIO — ISTITUTO "GUIDO CASTELNUOVO", SAPIENZA UNIVERSITÀ DI ROMA
Current address: Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, Paris
E-mail address: `diverio@math.jussieu.fr`

JOËL MERKER — DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE, PARIS
E-mail address: `merker@dma.ens.fr`

ERWAN ROUSSEAU — DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ LOUIS PASTEUR, STRASBOURG
E-mail address: `rousseau@math.unistra.fr`