CARTAN EQUIVALENCES FOR 5-DIMENSIONAL CR-MANIFOLDS IN $\mathbb{C}^4$ BELONGING TO GENERAL CLASS III_1

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ABSTRACT. We reduce to various absolute parallelisms, namely to certain $\{e\}$-structures on manifolds of dimensions 7, 6, 5, the biholomorphic equivalence problem or the intrinsic CR equivalence problem for generic submanifolds M^5 in $\mathbb{C}^4$ of CR dimension 1 and of codimension 3 that are maximally minimal and are geometry-preserving deformations of one natural cubic model of Beloshapka, somewhere else called the General Class III_1 of 5-dimensional CR manifolds. Some inspiration links exist with the treatment of the General Class II previously done in 2007 by Beloshapka, Ezhov, Schmalz, and also with the classification of nilpotent Lie algebras due to Goze, Khakimdjanov, Remm.

1. INTRODUCTION

For *systematic completeness* of our study of CR manifolds having:

$$\text{dimension} \ \leqslant \ \mathbf{5},$$

in the case of a 3-dimensional Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$, we intentionally re-constructed in [66] an explicit $\{e\}$-structure on a certain natural 8-dimensional manifold $N^8 \longrightarrow M^3$, because our current (wide) goal is similarly to perform completely explicit constructions of $\{e\}$-structures for CR equivalences in the six General Classes:

$$\mathsf{I}, \quad \mathsf{II}, \quad \mathsf{III}_1, \quad \mathsf{III}_2, \quad \mathsf{IV}_1, \quad \mathsf{IV}_2,$$

of all the possibly existing embedded CR manifolds up to dimension 5 that were presented in [55].

The present memoir being specifically devoted to the:

$$\text{General Class } \mathsf{III}_1,$$

let us review at first the General Class I in order to appropriately recast the reader's thought in the right aspiration to mathematical unification.

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1.1. **Review of basic CR equivalences for Levi nondegenerate hypersurfaces** $M^3 \subset \mathbb{C}^2$**.** Let therefore $M^3 \subset \mathbb{C}^2$ be a (local) Levi-nondegenerate real hypersurface of class at least $\mathscr{C}^6$, graphed in coordinates:

$$(z,w) = \big(x+iy,\, u+iv\big)$$

as:

$$v = \varphi(x,y,u).$$

An elementary normalization ([52]) insures without loss of generality that:

$$\varphi = x^2 + y^2 + \mathrm{O}(3).$$

Starting with the (local) intrinsic generator for $T^{1,0}M$:

$$\mathscr{L} := \frac{\partial}{\partial z} - \frac{\varphi_z}{i+\varphi_u}\,\frac{\partial}{\partial u},$$

having conjugate:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} - \frac{\varphi_{\overline{z}}}{-\,i+\varphi_u}\,\frac{\partial}{\partial u},$$

which in turn generates $T^{0,1}M = \overline{T^{1,0}M}$, and introducing:

$$\mathscr{T} := i\,\big[\mathscr{L},\overline{\mathscr{L}}\big],$$

one gets by the Levi nondegeneracy assumption the natural frame:

$$\big\{\mathscr{T},\overline{\mathscr{L}},\mathscr{L}\big\},$$

for the complexified tangent bundle $\mathbb{C}\otimes_{\mathbb{R}} TM$.

We also introduce the dual coframe for $\mathbb{C}\otimes_{\mathbb{R}} T^*M$ consisting of three 1-forms denoted:

$$\big\{\rho_0,\,\overline{\zeta}_0,\,\zeta_0\big\},$$

namely satisfying by definition:

$$\begin{array}{lll}
\rho_0(\mathscr{T}) = 1 & \rho_0(\overline{\mathscr{L}}) = 0 & \rho_0(\mathscr{L}) = 0, \\
\overline{\zeta}_0(\mathscr{T}) = 0 & \overline{\zeta}_0(\overline{\mathscr{L}}) = 1 & \overline{\zeta}_0\big(\mathscr{L}\big) = 0, \\
\zeta_0(\mathscr{T}) = 0 & \zeta_0(\overline{\mathscr{L}}) = 0 & \zeta_0\big(\mathscr{L}\big) = 1.
\end{array}$$

Abbreviating next:

$$A := -\,\frac{\varphi_z}{i+\varphi_u},$$

so that:

$$\mathscr{T} = i\left[\frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u},\ \frac{\partial}{\partial \overline{z}} + \overline{A}\,\frac{\partial}{\partial u}\right],$$

the *Levi-factor* function:

$$\ell := i\,\big(\overline{A}_z + A\,\overline{A}_u - A_{\overline{z}} - \overline{A}\,A_u\big),$$

occurring in:

$$\mathscr{T} = \ell\,\frac{\partial}{\partial u}$$

is therefore *nowhere vanishing* by Levi nondegeneracy.

One also computes next:

$$\begin{aligned}\big[\mathscr{L},\,\mathscr{T}\big] &= \bigg[\frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u},\ \ell\,\frac{\partial}{\partial u}\bigg] \\ &= \Big(\ell_z + A\,\ell_u - \ell\,A_u\Big)\,\frac{\partial}{\partial u} \\ &= \frac{\ell_z + A\,\ell_u - \ell\,A_u}{\ell}\,\mathscr{T}.\end{aligned}$$

Then the appearing function:

$$P := \frac{\ell_z + A\,\ell_u - \ell\,A_u}{\ell},$$

happens to be the single one which enters the so-called *initial Darboux structure:*

$$\begin{aligned}&d\rho_0 = P\,\rho_0\wedge\zeta_0 + \overline{P}\,\rho_0\wedge\overline{\zeta}_0 + i\,\zeta_0\wedge\overline{\zeta}_0,\\ &d\overline{\zeta}_0 = 0,\\ &d\zeta_0 = 0.\end{aligned}$$

As explained in [53, 66] and as is quite also very well known, the initial ambiguity matrix group for (local) biholomorphic or CR equivalences of such hypersurfaces is:

$$\left\{\begin{pmatrix}\mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \mathsf{b} & 0 & \mathsf{a}\end{pmatrix} \in \mathsf{GL}_3(\mathbb{C}):\quad \mathsf{a}\in\mathbb{C},\ \mathsf{b}\in\mathbb{C}\right\},$$

just because through any extrinsic local biholomorphism (or through any intrinsic local CR-equivalence):

$$h\colon\quad M \longrightarrow M',$$

one has ([53]) for a certain coefficient-function a:

$$\begin{aligned}h'_*\big(\mathscr{L}'\big) &= a\,\mathscr{L},\\ h'_*\big(\overline{\mathscr{L}'}\big) &= \overline{a}\,\overline{\mathscr{L}},\end{aligned}$$

whence:

$$\begin{aligned} h'_*\big(\mathscr{T}'\big) &= h'_*\big(i\,\big[\mathscr{L}',\,\overline{\mathscr{L}}'\big]\big) \\ &= i\,\big[h'_*\big(\mathscr{L}'\big),\,h'_*\big(\overline{\mathscr{L}}'\big)\big] \\ &= i\,\big[a\,\mathscr{L},\ \overline{a}\,\overline{\mathscr{L}}\big] \\ &= a\overline{a}\,\underbrace{i\,\big[\mathscr{L},\,\overline{\mathscr{L}}\big]}_{=\mathscr{T}} + \underbrace{i\,a\,\mathscr{L}\big(\overline{a}\big)}_{=:\,\overline{b}}\cdot\overline{\mathscr{L}}\,\underbrace{-i\,\overline{a}\,\overline{\mathscr{L}}(a)}_{=:\,b}\cdot\mathscr{L}, \end{aligned}$$

so that setting:

$$b := -\,i\,\overline{a}\,\overline{\mathscr{L}}(a),$$

forgetting how this further coefficient-function is related to a, one obtains:

$$h'_*\big(\mathscr{T}'\big) = a\overline{a}\,\mathscr{T} + \overline{b}\,\overline{\mathscr{L}} + b\,\mathscr{L}.$$

Cartan's gist is to deal with the so-called *lifted coframe:*

$$\begin{pmatrix}\rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \mathsf{b} & 0 & \mathsf{a}\end{pmatrix}\begin{pmatrix}\rho_0\\ \overline{\zeta}_0\\ \zeta_0\end{pmatrix},$$

in the space of $\big(x, y, u, \mathsf{a}, \overline{\mathsf{a}}, \mathsf{b}, \overline{\mathsf{b}}\big)$.

In [66], after two absorbtions-normalizations and after one prolongation, the desired equivalence problem transforms to that of some — explicitly computed — eight-dimensional coframe:

$$\big\{\rho, \zeta, \overline{\zeta}, \alpha, \beta, \overline{\alpha}, \overline{\beta}, \delta\big\}$$

on a certain manifold $N^8 \longrightarrow M^3$ having $\{e\}$-structure equations:

$$\begin{aligned}
d\rho &= \alpha\wedge\rho + \overline{\alpha}\wedge\rho + i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta,\\
d\overline{\zeta} &= \overline{\beta}\wedge\rho + \overline{\alpha}\wedge\overline{\zeta},\\
d\alpha &= \delta\wedge\rho + 2\,i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta,\\
d\beta &= \delta\wedge\zeta + \beta\wedge\overline{\alpha} + \mathfrak{I}\,\overline{\zeta}\wedge\rho,\\
d\overline{\alpha} &= \delta\wedge\rho - 2\,i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\overline{\beta},\\
d\overline{\beta} &= \delta\wedge\overline{\zeta} + \overline{\beta}\wedge\alpha + \overline{\mathfrak{I}}\,\zeta\wedge\rho,\\
d\delta &= \delta\wedge\alpha + \delta\wedge\overline{\alpha} + i\,\beta\wedge\overline{\beta} + \mathfrak{T}\,\rho\wedge\zeta + \overline{\mathfrak{T}}\,\rho\wedge\overline{\zeta},
\end{aligned}$$

with the single primary complex invariant:

$$\begin{aligned}\mathfrak{I} := \frac{1}{6}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\Big(-\,2\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big)\big) + 3\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big)\big) - 7\,\overline{P}\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big) + \\ +\,4\,\overline{P}\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big) - \mathscr{L}\big(\overline{P}\big)\,\overline{\mathscr{L}}\big(\overline{P}\big) + 2\,\overline{P}\,\overline{P}\,\mathscr{L}\big(\overline{P}\big)\Big),\end{aligned}$$

and with one secondary invariant:

$$\mathfrak{T} = \frac{1}{\overline{\mathsf{a}}}\left(\overline{\mathscr{L}}(\overline{\mathfrak{I}}) - \overline{P}\,\overline{\mathfrak{I}}\right) - i\,\frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{\mathfrak{I}}.$$

1.2. **Explicitness obstacle.** At the level of the function P, the completely explicit formulas for $\mathfrak{I}$, for $\mathfrak{T}$ and for the 1-forms constituting the $\{e\}$-structure remain writable on an article, *but not so anymore when one expresses everything back in terms of the graphing function* φ.

Indeed, the real and imaginary parts Δ_1 and Δ_2 in:

$$\mathfrak{I} = \frac{4}{\mathsf{a}\overline{\mathsf{a}}^3}\big(\boldsymbol{\Delta}_1 + i\,\boldsymbol{\Delta}_4\big)$$

have numerators containing respectively ([58]):

1 553 198 and **1 634 457**

monomials in the differential ring in $\binom{6+3}{3} - 1 = 83$ variables:

$$\mathbb{Z}\big[\varphi_x, \varphi_y, \varphi_{x^2}, \varphi_{y^2}, \varphi_{u^2}, \varphi_{xy}, \varphi_{xu}, \varphi_{yu}, \ldots\ldots, \varphi_{x^6}, \varphi_{y^6}, \varphi_{u^6}, \ldots\big].$$

Hence contrary to the general case where $\varphi = \varphi(x, y, u)$ does depend upon the '*CR-transversal*' variable u, in the so-called *rigid case* (often useful as a case of study-exploration) where $\varphi = \varphi(x, y)$ is independent of u so that:

$$P = \frac{\varphi_{zz\overline{z}}}{\varphi_{z\overline{z}}},$$

one realizes that $\mathfrak{I}$ is rather easily writable:

$$\begin{aligned}\mathfrak{I}\Big|_{\substack{\text{rigid}\\\text{case}}} = \frac{1}{6}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\bigg(&\frac{\varphi_{z^2\overline{z}^4}}{\varphi_{z\overline{z}}} - 6\,\frac{\varphi_{z^2\overline{z}^3}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^2} - \frac{\varphi_{z\overline{z}^4}\,\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^2} - 4\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^2} +\\ &+ 10\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^3} + 15\,\frac{\left(\varphi_{z\overline{z}^2}\right)^2\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^3} - 15\,\frac{\left(\varphi_{z\overline{z}^2}\right)^3\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^4}\bigg),\end{aligned}$$

and this therefore shows that *there is a tremendous explosion of computational complexity when one passes from the rigid case to the general case*.

Consequently, one expects an even much deeper computational complexity when one addresses the question of passing to CR manifolds of higher dimensions.

1.3. **Embedded CR submanifolds of CR dimension** 1 **and nilpotent Lie algebras.** Consider now a general sufficiently smooth generic submanifold:

$$M^{2+d} \subset \mathbb{C}^{1+d}$$

having CR dimension 1 and real codimension $d \geqslant 1$. According to the background article [55], the core bundle is:

$$T^{1,0}M := T^{1,0}\mathbb{C}^{1+n} \cap \big(\mathbb{C} \otimes_{\mathbb{R}} TM\big).$$

To simplify (in fact, just a bit) the mathematical discussions, we shall assume throughout that M is *real analytic*.

The question is: to understand the possible initial geometries of such CR manifolds, at least at a Zariski-generic point. This question is not yet completely solved, because it opens up infinitely many branches of classification.

Yet, one can present well known general considerations which were already transparently explained in Sophus Lie's original writings (*cf. e.g.* [19, 20, 21]), though not targetly in a CR context.

Introduce the subdistributions of $\mathbb{C}\otimes_{\mathbb{R}} TM$:

$$D_{\mathbb{C}}^1 M := T^{1,0}M + T^{0,1}M,$$

$$D_{\mathbb{C}}^2 M := \mathsf{Span}_{\mathscr{C}^\omega(M)}\Big(D_{\mathbb{C}}^1 M + \big[T^{1,0}M,\, D_{\mathbb{C}}^1 M\big] + \big[T^{0,1}M,\, D_{\mathbb{C}}^1 M\big]\Big),$$

$$D_{\mathbb{C}}^3 M := \mathsf{Span}_{\mathscr{C}^\omega(M)}\Big(D_{\mathbb{C}}^2 M + \big[T^{1,0}M,\, D_{\mathbb{C}}^2 M\big] + \big[T^{0,1}M,\, D_{\mathbb{C}}^2 M\big]\Big),$$

and generally:

$$D_{\mathbb{C}}^{k+1} M := \mathsf{Span}_{\mathscr{C}^\omega(M)}\Big(D_{\mathbb{C}}^k M + \big[T^{1,0}M,\, D_{\mathbb{C}}^k M\big] + \big[T^{0,1}M,\, D_{\mathbb{C}}^k M\big]\Big).$$

By passing to some appropriate Zariski-open subset of M, one may assume as is known that all these $D^k M$ become true complex *vector subbundles* of $\mathbb{C}\otimes_{\mathbb{R}} TM$ having increasing ranks:

$$2 = r_1(M) < r_2(M) < \cdots < r_{k_M}(M) = r_{k_M+1}(M) = r_{k_M+2}(M) = \cdots ,$$

until a first and final stabilization occurs. So we will admit that the M we consider enjoy constancies of such ranks, and of several other invariants (finite in number) which might happen to pop up later on.

As is known too, in the very special circumstance where:

$$2 = r_1(M) = r_2(M) = r_3(M) = \cdots ,$$

namely when:

$$\big[T^{1,0}M,\, T^{0,1}M\big] \,\subset\, T^{1,0}M + T^{0,1}M,$$

the real analytic CR-generic submanifold $M^{2+d}\subset\mathbb{C}^{1+d}$ is, locally in some small neighborhood of each of its points, biholomorphic to $\mathbb{C}\times\mathbb{R}^d$, a degenerate case rapidly set away. Sometimes, one also says that M is *Levi-flat*, or equivalently (just when the CR dimension equals 1), that M is *holomorphically degenerate* ([42]).

In fact, whenever the maximal possible rank:

$$r_{k_M}(M) = r_{k_M+1}(M) = r_{k_M+2}(M) = \cdots \,<\, 2+d = \dim_{\mathbb{R}}\, M,$$

is still smaller than the dimension of M (not necessarily equal to 2), one realizes that M is similarly, locally in a neighborhood of a Zariski-generic point, biholomorphic to a CR-generic submanifold of $\mathbb{C}^{1+d}$ which is contained in $2 + d - r_{k_M}(M)$ straight real hyperplanes in transverse intersection, a case which is also degenerate hence is also disregarded, just because it essentially comes down to the case of CR-generic submanifolds having smaller dimension that $2 + d$.

So the question is: to understand all the possible geometries of such CR manifolds $M^{2+d} \subset \mathbb{C}^{1+d}$ of CR dimension 1 that have the so-called constant nonholonomic property that:

$$D_{\mathbb{C}}^{k_M} M = \mathbb{C} \otimes_{\mathbb{R}} TM.$$

In the Several Complex Variables literature, such CR manifolds happen to be those called *minimal in the sense of Tumanov*, or equivalently *of finite type in the sense of Bloom-Graham*, and they happen to necessarily be also simultaneously holomorphically nondegenerate too (an implication which is true only in CR dimension 1, as is easily checked).

Beloshapka and his students, *e.g.* Shananina, Mamai, Kossovskiy and others, have put some emphasis on the study of such a class of CR manifolds, notably in the search for *nice* models which would potentially reveal new Cartan geometries.

The truth is that this research field, like the one of hyperbolic groups in the sense of Gromov, is *per se* opened to infinitely many untamable branches of complexity, for one soon realizes after a moment of reflection that the Lie algebras of infinitesimal CR automorphisms (assuming for simplicity that everything is real analytic) are deeply related to the classification of *nilpotent Lie algebras*, an area which is known to be very rich and very infinite, as Lie himself understood more than one century ago (*see* Chapter 28 in the English translation [47] of Volume I of the *Theorie der Transformationsgruppen*). Section 6 here is devoted to review the easiest part (only up to dimension 5) of the deep nilpotent Lie algebra classification theorems of Goze, Khakimdjanov, Remm up to dimension 8 ([29, 30, 31]), which already shows up an exploding ramification of very many branches.

In dimension 4, there is a single irreducible nilpotent Lie algebra:

$$\mathfrak{n}_4^1: \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3, \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4. \end{cases}$$

Correspondingly, as Beloshapka discovered in 1997 ([4]), a real analytic 4-dimensional local CR-generic submanifold $M^4 \subset \mathbb{C}^3$ of codimension 2

whose complex tangent bundle satisfies:

$$\mathbb{C}\otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M,\, T^{0,1}M] + \big[T^{1,0}M, [T^{1,0}M,\, T^{0,1}M]\big] + \\ + \big[T^{0,1}M, [T^{1,0}M,\, T^{0,1}M]\big]$$

may always be represented, in suitable holomorphic coordinates:

$$\big(z, w_1, w_2\big) = \big(z, u_1 + iv_1, u_2 + iv_2\big)$$

by two complex defining equations of the specific form:

$$v_1 = z\overline{z} + \mathrm{O}_4(x, y, u_1, u_2),$$
$$v_2 = z\overline{z}\big(z + \overline{z}\big) + \mathrm{O}_4(x, y, u_1, u_2).$$

Since then, such CR manifolds have been intensively studied, by Beloshapka-Ezhov-Schmalz who constructed a canonical Cartan connection ([9]) and who generalized Pinchuk-Vitushkin's germ extension phenomenon ([8]), by Gammel-Kossovskiy ([24]), and by Beloshapka-Kossovskiy who provided a final complete classification ([10]).

In [55], one refers to the:

General Class II:

$M^4 \subset \mathbb{C}^3$ with $\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}],\ [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\}$ constituting a frame for $\mathbb{C}\otimes_{\mathbb{R}} TM$.

The next natural General Class ([55]) is the:

General Class III_1:

$M^5 \subset \mathbb{C}^4$ with $\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}],\ [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]],\ [\overline{\mathscr{L}},\, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\}$ constituting a frame for $\mathbb{C}\otimes_{\mathbb{R}} TM$,

and it is intrinsically related to the irreducible 5-dimensional nilpotent Lie algebra (labeled here in the notation of Goze-Remm):

$$\mathfrak{n}_5^4\colon \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3, \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4, \\ [\mathsf{x}_2, \mathsf{x}_3] = \mathsf{x}_5. \end{cases}$$

Three years ago, we started to study CR equivalences of such CR manifolds belonging to the General Class III_1, trying in the first months to directly construct a Cartan connection as did Beloshapka-Ezhov-Schmalz for the General Class II. But inspired by Chern's seminal 1939 paper on equivalences of third order ordinary differential equations under contact transformations, we realized that it would be better to perform at first a pure exploration of the problem by employing the powerful tools of Cartan's method of equivalence, in order to avoid as much as possible those errors of understanding that are caused by a too quick belief that certain features would somewhat easily generalize.

Because we have not been since then aware of any other paper or preprint or author having attacked the same problem, we decided to wait until the study reached a point of maturity in which everything could be presented in full computational details, whatever complexity the theory has.

Of course in such a General Class III_1, it is known that the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ in coordinates:

$$\big(z, w_1, w_2, w_3\big) = \big(z,\, u_1 + iv_1,\, u_2 + iv_2,\, u_3 + iv_3\big)$$

was also discovered by Beloshapka:

$$\begin{aligned} v_1 &= z\overline{z}, \\ v_2 &= z\overline{z}\,\big(z + \overline{z}\big), \\ v_3 &= z\overline{z}\big(-\,i\,z + i\,\overline{z}\big). \end{aligned}$$

But the Cartan invariants of the geometry-preserving deformations of such a model have apparently never been studied, and such a study is the main goal of the present memoir. Granted that the general class IV_1 is already well studied since Chern-Moser ([15]), a forthcoming paper by Samuel Pocchiola will soon treat the General Class III_2 (as presented in [55]), thus closing up the study of CR equivalences of CR manifolds up to dimension 5 (an overall systematic review is planned to appear at the end).

With J being the standard complex structure of $T\mathbb{C}^4$, one sets as usual ([55]):

$$T^cM := TM \cap J(TM),$$

or equivalently:

$$T^cM := \operatorname{Re} T^{1,0}M.$$

Our first elementary result appears in Section 5, *cf.* also [52].

Proposition 1.1. *Every real analytic 5-dimensional local CR-generic submanifold $M^5 \subset \mathbb{C}^4$ of codimension 3 which is maximally minimal, namely which satisfies:*

$$\begin{aligned} &D^1M = T^cM \qquad \textit{has rank } 2, \\ &D^2M = T^cM + [T^cM, T^cM] \qquad \textit{has rank } 3, \\ &D^3M = T^cM + [T^cM, T^cM] + \big[T^cM, [T^cM, T^cM]\big] \qquad \textit{has maximal possible rank } 5, \end{aligned}$$

may be represented, in suitable holomorphic coordinates (z, w_1, w_2, w_3)*, by three complex defining equations of the specific form:*

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z} + \Pi_1\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}) + \Pi_2\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \\ w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}) + \Pi_3\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \end{cases}$$

where the three remainders Π_1, Π_2, Π_3 *are all an* $\mathrm{O}(|z|^4) + z\overline{z}\,\mathrm{O}(|w|)$.

Conversely, for any choice of three such analytic functions enjoying these conditions, the zero-locus of the three equations above represents a real analytic 5-dimensional local CR-generic submanifold $M^5 \subset \mathbb{C}^4$ *of codimension* 3 *which is maximally minimal.* □

Next, a general $(1,0)$ holomorphic vector field:

$$\mathsf{X} = Z(z,w)\,\frac{\partial}{\partial z} + W^1(z,w)\,\frac{\partial}{\partial w_2} + W^2(z,w)\,\frac{\partial}{\partial w_2} + W^3(z,w)\,\frac{\partial}{\partial w_3}$$

is an infinitesimal CR automorphism of Beloshapka's cubic model M^5_{c} if by definition $\mathsf{X} + \overline{\mathsf{X}}$ is tangent to M^5_{c}. By analyzing in great details the system of linear partial differential equations satisfied by the unknown functions Z, W^1, W^2, W^3, we obtain the second already known:

Proposition 1.2. *The Lie algebra* $\mathfrak{aut}_{CR}(M) = 2\,\mathrm{Re}\,\mathfrak{hol}(M)$ *of the infinitesimal CR automorphisms of the* 5*-dimensional* 3*-codimensional CR-generic model cubic* $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ *represented by the three real graphed equations:*

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}), \\ w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}), \end{cases}$$

is 7*-dimensional and it is generated by the* $\mathbb{R}$*-linearly independent real parts of the following seven* $(1,0)$ *holomorphic vector fields:*

$$\begin{aligned} T &:= \partial_{w_1}, \\ S_1 &:= \partial_{w_2}, \\ S_2 &:= \partial_{w_3}, \\ L_1 &:= \partial_z + (2iz)\,\partial_{w_1} + (2iz^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}, \\ L_2 &:= i\,\partial_z + (2z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2iz^2 - 4w_1)\,\partial_{w_3}, \\ D &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}, \\ R &:= iz\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}, \end{aligned}$$

having Lie bracket commutator table:

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	$*$	0	0	0	0	$3S_1$	S_2
T	$*$	$*$	0	$4S_2$	$4S_1$	$2T$	0
L_2	$*$	$*$	$*$	0	$-4T$	L_2	$-L_1$
L_1	$*$	$*$	$*$	$*$	0	L_1	L_2
D	$*$	$*$	$*$	$*$	$*$	0	0
R	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

One easily realizes that in the natural grading:

$$\begin{aligned}
\mathfrak{g}_{-3} &:= \mathsf{Span}_{\mathbb{R}}\big\langle S_1,\, S_2\big\rangle,\\
\mathfrak{g}_{-2} &:= \mathsf{Span}_{\mathbb{R}}\big\langle T\big\rangle,\\
\mathfrak{g}_{-1} &:= \mathsf{Span}_{\mathbb{R}}\big\langle L_1, L_2\big\rangle,\\
\mathfrak{g}_{0} &:= \mathsf{Span}_{\mathbb{R}}\big\langle D, R\big\rangle,
\end{aligned}$$

the above nilpotent Lie algebra $\mathfrak{n}_5^4$ is isomorphic to:

$$\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

as is well known in Tanaka's theory. Of course, a wealth of other correspondences between nilpotent Lie algebras and CR manifolds of CR dimension 1 certainly exist, but we skip entering this already much studied question in order to enter the core of a new systematic effective development of Cartan's equivalence method, an aspect which is, as we believe, not yet enough developed in the mathematical literature as far as hardest computations are concerned.

As it also appears independently in [53], we verify:

Proposition 1.3. *The initial ambiguity matrix associated to the local biholomorphic equivalence problem between the cubic 5-dimensional model CR-generic submanifold M^5_{c} and any other maximally minimal CR-generic 5-dimensional submanifolds $M'^5 \subset \mathbb{C}^4$ under local biholomorphic transformations is of the general form:*

$$\begin{pmatrix}
\mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & \overline{\mathsf{c}} & \overline{\mathsf{e}} & \overline{\mathsf{d}}\\
0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & \mathsf{c} & \mathsf{d} & \mathsf{e}\\
0 & 0 & \mathsf{a}\overline{\mathsf{a}} & \overline{\mathsf{b}} & \mathsf{b}\\
0 & 0 & 0 & \overline{\mathsf{a}} & 0\\
0 & 0 & 0 & 0 & \mathsf{a}
\end{pmatrix},$$

where $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{e}, \mathsf{d}$ are complex numbers. Moreover, the collection of all these matrices makes up a real 10-dimensional matrix Lie subgroup of $\mathsf{GL}_5(\mathbb{C})$.

In Section 12, as a preliminary to higher level computations, we perform the Cartan equivalence algorithm on the cubic model M^5_{c} and we obtain an $\{e\}$-structure on a 7-dimensional manifold of the form:

$$\begin{aligned}
d\sigma &= \big(2\,\alpha + \overline{\alpha}\big) \wedge \sigma + \rho \wedge \zeta, \\
d\overline{\sigma} &= \big(\alpha + 2\,\overline{\alpha}\big) \wedge \overline{\sigma} + \rho \wedge \overline{\zeta}, \\
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\
d\zeta &= \alpha \wedge \zeta, \\
d\overline{\zeta} &= \overline{\alpha} \wedge \overline{\zeta}, \\
d\alpha &= 0, \\
d\overline{\alpha} &= 0,
\end{aligned}$$

all structure functions being *constant*, and up to a very mild change of basis, these are nothing but the Maurer-Cartan equations on the 7-dimensional Lie group associated to the above 2-dimensional (semidirect product) extension of the nilpotent Lie algebra $\mathfrak{n}_5^4$. This performing of Cartan's method on the cubic model M^5_{c} has the virtue of setting up a kind of 'GPS' for orientation in the much deeper 'computational jungle' of the general case.

In fact, it is in Section 13, that we start out the main computations for general geometry-preserving deformations $M^5 \subset \mathbb{C}^4$ of the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ with a first, already computationally nontrivial, result.

Proposition 1.4. *For any local real analytic CR-generic submanifold* $M^5 \subset \mathbb{C}^4$ *which is represented near the origin as a graph:*

$$\begin{aligned}
v_1 &:= \varphi_1(x, y, u_1, u_2, u_3), \\
v_2 &:= \varphi_2(x, y, u_1, u_2, u_3), \\
v_3 &:= \varphi_3(x, y, u_1, u_2, u_3),
\end{aligned}$$

in coordinates:

$$\big(z, w_1, w_2, w_3\big) = \big(x + iy, u_1 + iv_1, u_2 + iv_2, u_3 + iv_3\big),$$

its fundamental intrinsic complex bundle $T^{0,1}M$ *is generated by:*

$$\boxed{\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \mathsf{A}_1\,\frac{\partial}{\partial \overline{w}_1} + \mathsf{A}_2\,\frac{\partial}{\partial \overline{w}_2} + \mathsf{A}_3\,\frac{\partial}{\partial \overline{w}_3},}$$

where:

$$\begin{aligned}
\mathsf{A}_1 &= \frac{\Lambda_1^1}{\Delta} + i\frac{\Lambda_2^1}{\Delta}, \\
\mathsf{A}_2 &= \frac{\Lambda_1^2}{\Delta} + i\frac{\Lambda_2^2}{\Delta}, \\
\mathsf{A}_3 &= \frac{\Lambda_1^3}{\Delta} + i\frac{\Lambda_2^3}{\Delta},
\end{aligned}$$

where:

$$\Delta = \sigma^2 + \tau^2,$$

with:

$$\begin{aligned}\sigma &= \varphi_{3u_3} + \varphi_{1u_1} + \varphi_{2u_2} - \varphi_{1u_2}\varphi_{3u_1}\varphi_{2u_3} - \varphi_{1u_3}\varphi_{2u_1}\varphi_{3u_2} + \varphi_{1u_2}\varphi_{2u_1}\varphi_{3u_3} - \\ &\quad - \varphi_{1u_1}\varphi_{2u_2}\varphi_{3u_3} + \varphi_{1u_1}\varphi_{2u_3}\varphi_{3u_2} + \varphi_{1u_3}\varphi_{3u_1}\varphi_{2u_2}, \\ \tau &= -1 + \varphi_{1u_1}\varphi_{2u_2} - \varphi_{2u_3}\varphi_{3u_2} - \varphi_{1u_3}\varphi_{3u_1} + \varphi_{2u_2}\varphi_{3u_3} - \varphi_{1u_2}\varphi_{2u_1} + \varphi_{1u_1}\varphi_{3u_3},\end{aligned}$$

and where:

$$\begin{aligned}\Lambda_1^1 = \big(&- \varphi_{3u_3}\varphi_{2x}\varphi_{1u_2} - \varphi_{1u_3}\varphi_{3y} + \varphi_{2u_2}\varphi_{1x}\varphi_{3u_3} + \varphi_{3u_3}\varphi_{1y} - \varphi_{1x} - \varphi_{2y}\varphi_{1u_2} + \\ &+ \varphi_{2u_3}\varphi_{3x}\varphi_{1u_2} + \varphi_{2u_2}\varphi_{1y} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3x} + \varphi_{2x}\varphi_{1u_3}\varphi_{3u_2}\big)\sigma + \\ &+ \big(\varphi_{1u_3}\varphi_{3x} - \varphi_{1y} + \varphi_{2x}\varphi_{1u_2} + \varphi_{2u_3}\varphi_{1u_2}\varphi_{3y} - \varphi_{2u_2}\varphi_{1x} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1y} - \\ &- \varphi_{3u_3}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3y} - \varphi_{3u_3}\varphi_{1u_2}\varphi_{2y} + \varphi_{1u_3}\varphi_{3u_2}\varphi_{2y} + \varphi_{2u_2}\varphi_{3u_3}\varphi_{1y}\big)\tau,\end{aligned}$$

with similar formulas for Λ_2^1, Λ_1^2, Λ_2^2, Λ_1^3, Λ_2^3.

Next, introduce the third vector field:

$$\mathscr{T} := i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big],$$

which is *real:*

$$\overline{\mathscr{T}} = \mathscr{T}.$$

Direct computations provide the three numerators in:

$$\boxed{\mathscr{T} = \frac{\Upsilon_1}{\Delta^3}\frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^3}\frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^3}\frac{\partial}{\partial u_3},}$$

namely:

$$\begin{aligned}\Upsilon_1 &= -\big(\Delta^2\Lambda_{2x}^1 - \Delta\Delta_x\Lambda_2^1 - \Delta^2\Lambda_{1y}^1 + \Delta\Delta_y\Lambda_1^1 + \Delta\Lambda_1^1\Lambda_{2u_1}^1 - \Delta\Lambda_2^1\Lambda_{1u_1}^1 - \Delta\Lambda_2^2\Lambda_{1u_2}^1 + \\ &\quad + \Delta_{u_2}\Lambda_1^1\Lambda_2^2 - \Delta\Lambda_2^3\Lambda_{1u_3}^1 + \Delta_{u_3}\Lambda_2^3\Lambda_1^1 + \Delta\Lambda_1^2\Lambda_{2u_2}^1 - \Delta_{u_2}\Lambda_1^2\Lambda_2^1 + \Delta\Lambda_1^3\Lambda_{2u_3}^1 - \Delta_{u_3}\Lambda_1^3\Lambda_2^1\big), \\ \Upsilon_2 &= -\big(\Delta^2\Lambda_{2x}^2 - \Delta\Delta_x\Lambda_2^2 + \Delta\Lambda_1^1\Lambda_{2u_1}^2 - \Delta_{u_1}\Lambda_1^1\Lambda_2^2 - \Delta^2\Lambda_{1y}^2 + \Delta\Delta_y\Lambda_1^2 - \Delta\Lambda_2^1\Lambda_{1u_1}^2 + \\ &\quad + \Delta_{u_1}\Lambda_2^1\Lambda_1^2 + \Delta\Lambda_1^2\Lambda_{2u_2}^2 - \Delta\Lambda_2^2\Lambda_{1u_2}^2 + \Delta\Lambda_1^3\Lambda_{2u_3}^2 - \Delta_{u_3}\Lambda_1^3\Lambda_2^2 - \Delta\Lambda_2^3\Lambda_{1u_3}^2 + \Delta_{u_3}\Lambda_2^3\Lambda_1^2\big), \\ \Upsilon_3 &= -\big(\Delta^2\Lambda_{2x}^3 - \Delta\Delta_x\Lambda_2^3 + \Delta\Lambda_1^1\Lambda_{2u_1}^3 - \Delta_{u_1}\Lambda_1^1\Lambda_2^3 - \Delta^2\Lambda_{1y}^3 + \Delta\Delta_y\Lambda_1^3 - \Delta\Lambda_2^1\Lambda_{1u_1}^3 + \\ &\quad + \Delta_{u_1}\Lambda_2^1\Lambda_1^3 - \Delta\Lambda_2^2\Lambda_{1u_2}^3 + \Delta_{u_2}\Lambda_2^2\Lambda_1^3 + \Delta\Lambda_1^3\Lambda_{2u_3}^3 - \Delta\Lambda_2^3\Lambda_{1u_3}^3 + \Delta\Lambda_1^2\Lambda_{2u_2}^3 - \Delta_{u_2}\Lambda_1^2\Lambda_2^3\big).\end{aligned}$$

Next, introduce the two Lie brackets of length three:

$$\begin{aligned}\mathscr{S} &:= \big[\mathscr{L}, \mathscr{T}\big], \\ \overline{\mathscr{S}} &:= \big[\overline{\mathscr{L}}, \mathscr{T}\big].\end{aligned}$$

Again, direct computations provide the expressions:

$$\mathscr{S} = \frac{\Gamma_1^1 - i\,\Gamma_2^1}{\Delta^5}\frac{\partial}{\partial u_1} + \frac{\Gamma_1^2 - i\,\Gamma_2^2}{\Delta^5}\frac{\partial}{\partial u_2} + \frac{\Gamma_1^3 - i\,\Gamma_2^3}{\Delta^5}\frac{\partial}{\partial u_3},$$

where, by allowing the two notational coincidences $x \equiv x_1$ and $y \equiv x_2$, the numerators are (for $i = 1, 2$):

$$\Gamma_i^1 = -2\big(\tfrac{1}{4}\Delta^2\Upsilon_{1x_i} - 3\Delta\Delta_{x_i}\Upsilon_1 + \Delta\Lambda_i^1\Upsilon_{1u_1} - 2\Delta_{u_1}\Lambda_i^1\Upsilon_1 - \Delta\Lambda_{iu_1}^1\Upsilon_1 - \Delta\Lambda_{iu_2}^1\Upsilon_2 + \\ + \Delta_{u_2}\Lambda_i^1\Upsilon_2 - \Delta\Lambda_{iu_3}^1\Upsilon_3 + \Delta_{u_3}\Lambda_i^1\Upsilon_3 + \Delta\Lambda_i^2\Upsilon_{1u_2} - 3\Delta_{u_2}\Lambda_i^2\Upsilon_1 + \Delta\Lambda_i^3\Upsilon_{1u_3} - 3\Delta_{u_3}\Lambda_i^3\Upsilon_1\big),$$

$$\Gamma_i^2 = -2\big(\Delta^2\Upsilon_{2x_i} - 3\Delta\Delta_{x_i}\Upsilon_2 + \Delta\Lambda_i^1\Upsilon_{2u_1} - 3\Delta_{u_1}\Lambda_i^1\Upsilon_2 - \Delta\Lambda_{iu_1}^2\Upsilon_1 + \Delta_{u_1}\Lambda_i^2\Upsilon_1 + \\ + \Delta\Lambda_i^2\Upsilon_{2u_2} - 2\Delta_{u_2}\Lambda_i^2\Upsilon_2 - \Delta\Lambda_{iu_2}^2\Upsilon_2 - \Delta\Lambda_{iu_3}^2\Upsilon_3 + \Delta_{u_3}\Lambda_i^2\Upsilon_3 + \Delta\Lambda_i^3\Upsilon_{2u_3} - 3\Delta_{u_3}\Lambda_i^3\Upsilon_2\big),$$

$$\Gamma_i^3 = -2\big(\Delta^2\Upsilon_{3x_i} - 3\Delta\Delta_{x_i}\Upsilon_3 + \Delta\Lambda_i^1\Upsilon_{3u_1} - 3\Delta_{u_1}\Lambda_i^1\Upsilon_3 + \Delta\Lambda_i^2\Upsilon_{3u_2} - 3\Delta_{u_2}\Lambda_i^2\Upsilon_3 - \\ - \Delta\Lambda_{iu_1}^3\Upsilon_1 + \Delta_{u_1}\Lambda_i^3\Upsilon_1 - \Delta\Lambda_{iu_2}^3\Upsilon_2 + \Delta_{u_2}\Lambda_i^3\Upsilon_2 + \Delta\Lambda_i^3\Upsilon_{3u_3} - 2\Delta_{u_3}\Lambda_i^3\Upsilon_3 - \Delta\Lambda_{iu_3}^3\Upsilon_3\big).$$

At this point, remind that quite straightforwardly from the definition, one has:

Proposition 1.5. *A real analytic CR-generic submanifold $M^5 \subset \mathbb{C}^4$ belongs to the General Class III_1 if and only if the collection of five vector fields:*

$$\big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\},$$

where:

$$\mathscr{T} := i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big], \\ \mathscr{S} := \big[\mathscr{L},\, \mathscr{T}\big], \\ \overline{\mathscr{S}} := \big[\overline{\mathscr{L}},\, \mathscr{T}\big],$$

makes up a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM^5$. □

Having 5 fields implies that there are in sum 10 pairwise Lie brackets. Thus, in order to determine the full Lie bracket structure, there remain 7 such brackets to be looked at.

The first group of four Lie brackets is:

$$\big[\mathscr{L},\, \mathscr{S}\big], \qquad \big[\overline{\mathscr{L}},\, \mathscr{S}\big], \qquad \big[\mathscr{L},\, \overline{\mathscr{S}}\big], \qquad \big[\overline{\mathscr{L}},\, \overline{\mathscr{S}}\big].$$

Lemma 1.6. *There are six complex-valued functions P, Q, R, A, B, C defined on M such that:*

$$\big[\mathscr{L},\, \mathscr{S}\big] = P\,\mathscr{T} + Q\,\mathscr{S} + R\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\, \mathscr{S}\big] = A\,\mathscr{T} + B\,\mathscr{S} + C\,\overline{\mathscr{S}}, \\ \big[\mathscr{L},\, \overline{\mathscr{S}}\big] = \overline{A}\,\mathscr{T} + \overline{C}\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\, \overline{\mathscr{S}}\big] = \overline{P}\,\mathscr{T} + \overline{R}\,\mathscr{S} + \overline{Q}\,\overline{\mathscr{S}}.$$

Moreover, the two brackets:

$$\big[\overline{\mathscr{L}}, \mathscr{S}\big] = \big[\mathscr{L}, \overline{\mathscr{S}}\big],$$

are real and equal. In particular, A is a real-valued function and $C = \overline{B}$.

So we have:

$$\boxed{\begin{aligned}\big[\mathscr{L},\,\mathscr{S}\big] &= P\,\mathscr{T} + Q\,\mathscr{S} + R\,\overline{\mathscr{S}},\\ \big[\overline{\mathscr{L}},\,\mathscr{S}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}},\\ \big[\mathscr{L},\,\overline{\mathscr{S}}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}},\\ \big[\overline{\mathscr{L}},\,\overline{\mathscr{S}}\big] &= \overline{P}\,\mathscr{T} + \overline{R}\,\mathscr{S} + \overline{Q}\,\overline{\mathscr{S}}.\end{aligned}}$$

From Section 3 of [54], we know that the full expansion of the above numerator differential polynomials Υ_1, Υ_2, Υ_3, Γ_i^1, Γ_i^2, Γ_i^3 involve dozens of millions of monomials in the $\mathbf{3}\cdot\mathbf{55}$ partial derivatives:

$$\Big(\varphi_{1,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{2,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{3,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}}\Big)_{1\leqslant j+k+l_1+l_2+l_3\leqslant 3},$$

of the three graphing functions φ_1, φ_2, φ_3. *A fortiori*, the numerators of the five functions:

P,	Q,	R,
A,	B,	

would involve even much more monomials, without there being any reasonable hope to deal with them systematically on any currently available mostly powerful computer machine.

There is therefore some unavoidable practical necessity of lowering the ambition of complete expliciteness by raising up the level of calculations up to the so-denoted five fundamental functions P, Q, R, A, B.

It yet remains to compute the 3 among 10 structure Lie brackets:

$$\big[\mathscr{T},\,\mathscr{S}\big],\qquad\qquad \big[\mathscr{T},\,\overline{\mathscr{S}}\big],\qquad\qquad \big[\mathscr{S},\,\overline{\mathscr{S}}\big].$$

A careful systematic inspection of various Jacobi identities provides their expressions in terms of the five fundamental functions P, Q, R, A, B.

Lemma 1.7. *The coefficients of the two Lie brackets:*

$$\begin{aligned}\big[\mathscr{T},\,\mathscr{S}\big] &= E\,\mathscr{T} + F\,\mathscr{S} + G\,\overline{\mathscr{S}},\\ \big[\mathscr{T},\,\overline{\mathscr{S}}\big] &= \overline{E}\,\mathscr{T} + \overline{G}\,\mathscr{S} + \overline{F}\,\overline{\mathscr{S}},\end{aligned}$$

are three complex-valued functions E, F, G *which can be expressed as follows in terms of* P, Q, R, A, B *and their first-order frame derivatives:*

$$\begin{aligned}E &= -i\,\overline{\mathscr{L}}(P) - i\,A\,Q - i\,\overline{P}\,R + i\,\mathscr{L}(A) + i\,B\,P + i\,A\,\overline{B},\\ F &= -i\,\overline{\mathscr{L}}(Q) - i\,R\,\overline{R} + i\,A + i\,\mathscr{L}(B) + i\,B\,\overline{B},\\ G &= -i\,P - i\,\overline{B}\,Q - i\,R\,\overline{Q} - i\,\overline{\mathscr{L}}(R) + i\,B\,R + i\,\overline{B}\overline{B} + i\,\mathscr{L}(\overline{B}),\end{aligned}$$

while the coefficients of the last, tenth structure bracket:

$$\big[\mathscr{S},\,\overline{\mathscr{S}}\big] = i\,J\,\mathscr{T} + K\,\mathscr{S} - \overline{K}\,\overline{\mathscr{S}},$$

are one complex-valued function K and one real-valued function J which can be expressed as follows in terms of P, Q, R, A, B and their frame derivatives up to order 2:

$$\begin{aligned}
-2\,J = & -\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(P)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(A)\big) - \mathscr{L}\big(\mathscr{L}(\overline{P})\big) \\
& - Q\,\overline{\mathscr{L}}(A) - 2\,A\,\overline{\mathscr{L}}(Q) - R\,\overline{\mathscr{L}}(\overline{P}) - 2\,\overline{P}\,\overline{\mathscr{L}}(R) - 2\,AR\overline{R} - 2\,P\overline{P} - \overline{BPQ} - \overline{PQ}R- \\
& - \overline{R}\mathscr{L}(P) - 2\,P\,\mathscr{L}(\overline{R}) - \overline{Q}\,\mathscr{L}(A) - 2\,A\mathscr{L}(\overline{Q}) - PQ\overline{R} - BP\overline{Q}+ \\
& + 2\,P\overline{\mathscr{L}}(B) + B\overline{\mathscr{L}}(P) + 2\,A\overline{\mathscr{L}}(\overline{B}) + \overline{B}\,\overline{\mathscr{L}}(A) + 2\,A\,\mathscr{L}(B) + 2\,AA + 2\,AB\overline{B} + 2\,\overline{P}\,\mathscr{L}(\overline{B})+ \\
& + B\overline{P}R + \overline{BBP} + B\,\mathscr{L}(A) + \overline{B}\,\mathscr{L}(\overline{P}) + BBP + \overline{B}P\overline{R},
\end{aligned}$$

$$\begin{aligned}
2i\,K = & -\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(Q)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(B)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(B)\big) - \mathscr{L}\big(\mathscr{L}(\overline{R})\big)- \\
& - 2\,\overline{R}\,\overline{\mathscr{L}}(R) - R\,\overline{\mathscr{L}}(\overline{R}) - B\,\overline{\mathscr{L}}(Q) - BR\overline{R} - 2\,P\overline{R} - \overline{Q}R\overline{R} - 2\,\mathscr{L}(\overline{P}) - \overline{R}\,\mathscr{L}(Q)- \\
& - 2\,Q\,\mathscr{L}(\overline{R}) - \overline{Q}\,\mathscr{L}(B) - 2\,B\,\mathscr{L}(\overline{Q}) - A\overline{Q} - \overline{P}Q - QQ\overline{R} - BQ\overline{Q}+ \\
& + 2\,\overline{\mathscr{L}}(A) + \overline{B}\,\overline{\mathscr{L}}(B) + 2\,B\,\overline{\mathscr{L}}(\overline{B}) + 3\,B\,\mathscr{L}(B) + 3\,AB + BBQ + 2\,BB\overline{B} + 2\,\overline{R}\,\mathscr{L}(\overline{B})+ \\
& + \overline{BBR} + \overline{B}\,\mathscr{L}(\overline{R}) + \overline{BP} + Q\,\overline{\mathscr{L}}(B).
\end{aligned}$$

Crucially too, since no existing computer machine is powerful enough to compute everything in terms of the three graphing functions φ_1, φ_2, φ_3, one must possess some means of knowing as many as possible of the differential-algebraic relations that these 5 further coefficient-functions E, F, G, J, K share in common with P, Q, R, A, B. An inspection of higher, length-six, Jacobi identities already explored in [58] provides 5 such relations labeled $\overset{1}{=}$, $\overset{2}{=}$, $\overset{3}{=}$, $\overset{4}{=}$, $\overset{5}{=}$, which will appear to be very useful and important when performing Cartan's method later on.

$$\begin{aligned}
0 \overset{1}{\equiv} & \, 2\,\mathscr{L}(\overline{\mathscr{L}}(P)) - \mathscr{L}(\mathscr{L}(A)) - \overline{\mathscr{L}}(\mathscr{L}(P))- \\
& 2\,P\mathscr{L}(B) - B\mathscr{L}(P) - 2\,A\mathscr{L}(\overline{B}) - \overline{B}\mathscr{L}(A) + P\overline{\mathscr{L}}(Q) + A\mathscr{L}(Q)+ \\
& + 2\,Q\mathscr{L}(A) - Q\overline{\mathscr{L}}(P) + A\overline{\mathscr{L}}(R) + 2\,R\mathscr{L}(\overline{P}) + \overline{P}\mathscr{L}(R) - R\overline{\mathscr{L}}(A)- \\
& - PB\overline{B} - A\overline{B}^2 + PBQ + 2\,AQ\overline{B} - AQ^2 - 2\,ABR + 2\,RP\overline{R} + 2\,AR\overline{Q} - QR\overline{P} - R\overline{B}\,\overline{P},
\end{aligned}$$

$$\begin{aligned}
0 \overset{2}{\equiv} & \, 2\,\mathscr{L}(\overline{\mathscr{L}}(Q)) - \mathscr{L}(\mathscr{L}(B)) - \overline{\mathscr{L}}(\mathscr{L}(Q))- \\
& - 2\,\mathscr{L}(A) - 2\,B\mathscr{L}(\overline{B}) - \overline{B}\mathscr{L}(B) + B\overline{\mathscr{L}}(R) + 2\,R\mathscr{L}(\overline{R}) + \overline{R}\mathscr{L}(R) - R\overline{\mathscr{L}}(B) + \overline{\mathscr{L}}(P)+ \\
& + 2\,R\overline{P} + BQ\overline{B} - A\overline{B} - B\overline{B}^2 + AQ + QR\overline{R} + 2\,BR\overline{Q} - 2\,B^2R - R\overline{B}\,\overline{R},
\end{aligned}$$

$$\begin{aligned}
0 \overset{3}{\equiv} & \, 2\,\mathscr{L}(\overline{\mathscr{L}}(R)) - \mathscr{L}(\mathscr{L}(\overline{B})) - \overline{\mathscr{L}}\big(\mathscr{L}(R)\big)- \\
& - 3\,\overline{B}\mathscr{L}(\overline{B}) + \overline{B}\mathscr{L}(Q) + 2\,Q\mathscr{L}(\overline{B}) - 2\,R\mathscr{L}(B) - B\mathscr{L}(R) + R\overline{\mathscr{L}}(Q) + \overline{B}\,\overline{\mathscr{L}}(R)+ \\
& + 2\,R\mathscr{L}(\overline{Q}) + \overline{Q}\mathscr{L}(R) - Q\overline{\mathscr{L}}(R) - \overline{\mathscr{L}}\mathscr{L}(R) - R\overline{\mathscr{L}}(\overline{B}) + \mathscr{L}(P)+ \\
& + 2\,Q\overline{B}^2 - QP - Q^2\overline{B} - \overline{B}^3 + P\overline{B} - 2\,AR - 2\,BR\overline{B} + BQR + 2\,R^2\overline{R} + R\overline{B}\,\overline{Q} - QR\overline{Q},
\end{aligned}$$

$$\begin{aligned}
0 \overset{4}{\equiv} & -3\overline{\mathscr{L}}(\mathscr{L}(A)) - \mathscr{L}(\mathscr{L}(\overline{P})) + 3\,\mathscr{L}(\overline{\mathscr{L}}(A)) + \overline{\mathscr{L}}(\overline{\mathscr{L}}(P))- \\
& - 2\,A\mathscr{L}(\overline{Q}) - \overline{Q}\mathscr{L}(A) + 3\,B\mathscr{L}(A) + 3\,\overline{B}\mathscr{L}(\overline{P}) - 3\,B\overline{\mathscr{L}}(P) - 3\,\overline{B}\,\overline{\mathscr{L}}(A) + 2\,A\overline{\mathscr{L}}(Q)-
\end{aligned}$$

$$
\begin{aligned}
&+ Q\overline{\mathscr{L}}(A) - 2\,P\mathscr{L}(\overline{R}) - \overline{R}\mathscr{L}(P) + 2\,\overline{P}\,\overline{\mathscr{L}}(R) + R\overline{\mathscr{L}}(\overline{P}) - \\
&- BP\overline{Q} + 3\,B^2P + 2\,A\overline{B}\,\overline{Q} - 2\,BQA - 3\,\overline{B}^2\overline{P} + Q\overline{B}\,\overline{P} - PQ\overline{R} + 3\,P\overline{B}\,\overline{R} - 3\,BR\overline{P} + R\overline{P}\,\overline{Q}, \\
0 \overset{5}{\equiv}\; & -3\,\overline{\mathscr{L}}(\mathscr{L}(B)) + 3\,\mathscr{L}(\overline{\mathscr{L}}(B)) + \overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) - \mathscr{L}(\mathscr{L}(\overline{R})) + \\
&+ 3\,B\mathscr{L}(B) - 3\,\overline{B}\,\overline{\mathscr{L}}(B) + Q\overline{\mathscr{L}}(B) - B\overline{\mathscr{L}}(Q) - 2\,Q\mathscr{L}(\overline{R}) - \overline{R}\mathscr{L}(Q) - \\
&- 2\,B\mathscr{L}(\overline{Q}) - \overline{Q}\mathscr{L}(B) + 3\,\overline{B}\mathscr{L}(\overline{R}) + 2\,\overline{R}\,\overline{\mathscr{L}}(R) + R\overline{\mathscr{L}}(\overline{R}) - 2\,\mathscr{L}(\overline{P}) - \\
&- Q\overline{P} - A\overline{Q} - BQ\overline{Q} + 3\,AB + 3\,\overline{B}\,\overline{P} + 2\,B\overline{B}\,\overline{Q} + B^2Q - Q^2\overline{R} + 4\,Q\overline{B}\,\overline{R} - \\
&- 3\,BR\overline{R} - 3\,\overline{B}^2\overline{R} + R\overline{Q}\,\overline{R}.
\end{aligned}
$$

Higher length 7 or 8 Jacobi relations should also be useful if one wants to explore deeper the problem, but even when admitting to compute everything only in terms of P, Q, R, A, B, the formulas again show off a striking tendency to striking swelling.

In any case, introducing the coframe of 1-forms generating the complexified cotangent bundle $\mathbb{C} \otimes_{\mathbb{R}} T^*M$:

$$
\big\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\big\} \quad \text{which is dual to the frame} \quad \big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\},
$$

namely which satisfy by definition:

$$
\begin{array}{lllll}
\overline{\sigma_0}(\overline{\mathscr{S}}) = 1 & \overline{\sigma_0}(\mathscr{S}) = 0 & \overline{\sigma_0}(\mathscr{T}) = 0 & \overline{\sigma_0}(\overline{\mathscr{L}}) = 0 & \overline{\sigma_0}\big(\mathscr{L}\big) = 0, \\
\sigma_0(\overline{\mathscr{S}}) = 0 & \sigma_0(\mathscr{S}) = 1 & \sigma_0(\mathscr{T}) = 0 & \sigma_0(\overline{\mathscr{L}}) = 0 & \sigma_0\big(\mathscr{L}\big) = 0, \\
\rho_0(\overline{\mathscr{S}}) = 0 & \rho_0(\mathscr{S}) = 0 & \rho_0(\mathscr{T}) = 1 & \rho_0(\overline{\mathscr{L}}) = 0 & \rho_0\big(\mathscr{L}\big) = 0, \\
\overline{\zeta_0}(\overline{\mathscr{S}}) = 0 & \overline{\zeta_0}(\mathscr{S}) = 0 & \overline{\zeta_0}(\mathscr{T}) = 0 & \overline{\zeta_0}(\overline{\mathscr{L}}) = 1 & \overline{\zeta_0}\big(\mathscr{L}\big) = 0, \\
\zeta_0(\overline{\mathscr{S}}) = 0 & \zeta_0(\mathscr{S}) = 0 & \zeta_0(\mathscr{T}) = 0 & \zeta_0(\overline{\mathscr{L}}) = 0 & \zeta_0\big(\mathscr{L}\big) = 1,
\end{array}
$$

one determine its Darboux structure by reading vertically a convenient auxiliary array:

$$
\begin{array}{lcccccccccccl}
 & & \overline{\mathscr{S}} & & \mathscr{S} & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & & \\
 & & \boxed{d\overline{\sigma_0}} & & \boxed{d\sigma_0} & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & & \\
\big[\overline{\mathscr{S}}, \mathscr{S}\big] & = & \overline{K}\cdot\overline{\mathscr{S}} & + & -K\cdot\mathscr{S} & + & -i\,J\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\sigma_0} \\
\big[\overline{\mathscr{S}}, \mathscr{T}\big] & = & -\overline{F}\cdot\overline{\mathscr{S}} & + & -\overline{G}\cdot\mathscr{S} & + & -\overline{E}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\rho_0} \\
\big[\overline{\mathscr{S}}, \overline{\mathscr{L}}\big] & = & -\overline{Q}\cdot\overline{\mathscr{S}} & + & -\overline{R}\cdot\mathscr{S} & + & -\overline{P}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\overline{\zeta_0}} \\
\big[\overline{\mathscr{S}}, \mathscr{L}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\zeta_0} \\
\big[\mathscr{S}, \mathscr{T}\big] & = & -G\cdot\overline{\mathscr{S}} & + & -F\cdot\mathscr{S} & + & -E\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\rho_0} \\
\big[\mathscr{S}, \overline{\mathscr{L}}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{S}, \mathscr{L}\big] & = & -R\cdot\overline{\mathscr{S}} & + & -Q\cdot\mathscr{S} & + & -P\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\zeta_0} \\
\big[\mathscr{T}, \overline{\mathscr{L}}\big] & = & -\overline{\mathscr{S}} & + & 0 & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{T}, \mathscr{L}\big] & = & 0 & + & -\mathscr{S} & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\zeta_0} \\
\big[\overline{\mathscr{L}}, \mathscr{L}\big] & = & 0 & + & 0 & + & i\,\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\zeta_0}\wedge\zeta_0}
\end{array}
$$

and this provides (minding an overall minus sign due to duality):

$$\boxed{\begin{aligned}
d\overline{\sigma}_0 &= -\overline{K}\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{F}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{Q}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+\overline{B}\cdot\overline{\sigma}_0\wedge\zeta_0+\\
&\qquad+G\cdot\sigma_0\wedge\rho_0+\overline{B}\cdot\sigma_0\wedge\overline{\zeta}_0+R\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\overline{\zeta}_0,\\
d\sigma_0 &= K\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{G}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{R}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+B\cdot\overline{\sigma}_0\wedge\zeta_0+\\
&\qquad+F\cdot\sigma_0\wedge\rho_0+B\cdot\sigma_0\wedge\overline{\zeta}_0+Q\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\zeta_0,\\
d\rho_0 &= i\,J\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{E}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{P}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+A\cdot\overline{\sigma}_0\wedge\zeta_0+\\
&\qquad+E\cdot\sigma_0\wedge\rho_0+A\cdot\sigma_0\wedge\overline{\zeta}_0+P\cdot\sigma_0\wedge\zeta_0-i\,\overline{\zeta}_0\wedge\zeta_0,\\
d\overline{\zeta}_0 &= 0,\\
d\zeta_0 &= 0.
\end{aligned}}$$

Recall that exactly as for the model reviewed above, the initial ambiguity matrix associated to the equivalence problem under local biholomorphic transformations for maximally minimal CR-generic 3-codimensional submanifolds $M^5\subset\mathbb{C}^4$ is of the general form:

$$\begin{pmatrix}
\mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & \overline{\mathsf{c}} & \overline{\mathsf{e}} & \overline{\mathsf{d}}\\
0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & \mathsf{c} & \mathsf{d} & \mathsf{e}\\
0 & 0 & \mathsf{a}\overline{\mathsf{a}} & \overline{\mathsf{b}} & \mathsf{b}\\
0 & 0 & 0 & \overline{\mathsf{a}} & 0\\
0 & 0 & 0 & 0 & \mathsf{a}
\end{pmatrix},$$

where a, b, c, e, d are complex numbers. The so-called *lifted coframe* is then (one must transpose the above matrix):

$$\begin{pmatrix}\overline{\sigma}\\ \sigma\\ \rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} := \begin{pmatrix}
\mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0\\
0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0\\
\overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0\\
\overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\
\overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a}
\end{pmatrix}\begin{pmatrix}\overline{\sigma_0}\\ \sigma_0\\ \rho_0\\ \overline{\zeta_0}\\ \zeta_0\end{pmatrix},$$

that is to say:

$$\begin{aligned}
\overline{\sigma} &= \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}}\,\overline{\sigma_0},\\
\sigma &= \mathsf{a}\mathsf{a}\overline{\mathsf{a}}\,\sigma_0,\\
\rho &= \overline{\mathsf{c}}\,\overline{\sigma_0}+\mathsf{c}\,\sigma_0+\mathsf{a}\overline{\mathsf{a}}\,\rho_0,\\
\overline{\zeta} &= \overline{\mathsf{e}}\,\overline{\sigma_0}+\mathsf{d}\,\sigma_0+\overline{b}\,\rho_0+\overline{\mathsf{a}}\,\overline{\zeta_0},\\
\zeta &= \overline{\mathsf{d}}\,\overline{\sigma_0}+\mathsf{e}\,\sigma_0+\mathsf{b}\,\rho_0+\mathsf{a}\,\zeta_0.
\end{aligned}$$

To launch Cartan's method, with the Maurer-Cartan forms (not writing their conjugates):

$$
\begin{aligned}
\alpha_1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\
\alpha_2 &:= \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} - \frac{\mathsf{c}\, d\mathsf{a}}{\mathsf{a}^3\overline{\mathsf{a}}} - \frac{\mathsf{c}\, d\overline{\mathsf{a}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}, \\
\alpha_3 &:= -\frac{\mathsf{c}\, d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \left(\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\right) d\mathsf{a} + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\, d\mathsf{e}, \\
\alpha_4 &:= \frac{d\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2} - \frac{\overline{\mathsf{c}}\, d\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^3} + \left(\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right) d\mathsf{a}, \\
\alpha_5 &:= \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}} - \frac{\mathsf{b}\, d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}},
\end{aligned}
$$

one computes the complete structure equations:

$$
\begin{aligned}
d\sigma = \big(2\,\alpha_1 + \overline{\alpha}_1\big) \wedge \sigma + \\
+ U_1\,\sigma\wedge\overline{\sigma} + U_2\,\sigma\wedge\rho + U_3\,\sigma\wedge\zeta + U_4\,\sigma\wedge\overline{\zeta} + \\
+ U_5\,\overline{\sigma}\wedge\rho + U_6\,\overline{\sigma}\wedge\zeta + U_7\,\overline{\sigma}\wedge\overline{\zeta} + \\
+ \rho\wedge\zeta,
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
d\rho = \alpha_2\wedge\sigma + \overline{\alpha}_2\wedge\overline{\sigma} + \alpha_1\wedge\rho + \overline{\alpha}_1\wedge\rho + \\
+ V_1\,\sigma\wedge\overline{\sigma} + V_2\,\sigma\wedge\rho + V_3\,\sigma\wedge\zeta + V_4\,\sigma\wedge\overline{\zeta} + \\
+ \overline{V_2}\,\overline{\sigma}\wedge\rho + \overline{V_4}\,\overline{\sigma}\wedge\zeta + \overline{V_3}\,\overline{\sigma}\wedge\overline{\zeta} + \\
+ V_8\,\rho\wedge\zeta + \overline{V_8}\,\rho\wedge\overline{\zeta} + \\
+ i\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = \alpha_3\wedge\sigma + \alpha_4\wedge\overline{\sigma} + \alpha_5\wedge\rho + \alpha_1\wedge\zeta + \\
+ W_1\,\sigma\wedge\overline{\sigma} + W_2\,\sigma\wedge\rho + W_3\,\sigma\wedge\zeta + W_4\,\sigma\wedge\overline{\zeta} + \\
+ W_5\,\overline{\sigma}\wedge\rho + W_6\,\overline{\sigma}\wedge\zeta + W_7\,\overline{\sigma}\wedge\overline{\zeta} + \\
+ W_8\,\rho\wedge\zeta + W_9\,\rho\wedge\overline{\zeta} + \\
+ W_{10}\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

expressed in terms of the lifted coframe; the explicit expressions of the initial torsion coefficients U_i, V_j, W_k are not reviewed in this introductory presentation, but they appear in Section 15.

Lemma 1.8. *In the first loop of Cartan's method, five essential linear combinations of torsion coefficients appear:*

$$\begin{aligned}
U_5 &= \frac{1}{\overline{\mathsf{a}}^2}\,\overline{G} - \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^2}\,B - \frac{\overline{\mathsf{b}}}{\overline{\mathsf{a}}^3}\,\overline{R} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2},\\
U_6 &= \frac{1}{\overline{\mathsf{a}}}\,B - \frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2},\\
U_7 &= \frac{\mathsf{a}}{\overline{\mathsf{a}}^2}\,\overline{R},\\
U_3 + \overline{U}_4 - 3\,V_8 &= \frac{1}{\mathsf{a}}\,Q - 4\,\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{1}{\mathsf{a}}\,\overline{B} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}},\\
\overline{U}_4 - V_8 - \overline{W}_{10} &= \frac{1}{\mathsf{a}}\,\overline{B} - \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}}.
\end{aligned}$$

Precisely, this means that assigning the values:

$$\begin{aligned}
0 &= U_5,\\
0 &= U_6,\\
1 &= U_7,\\
0 &= U_3 + \overline{U}_4 - 3\,V_8,\\
0 &= \overline{U}_4 - V_8 - \overline{W}_{10},
\end{aligned}$$

one can use these 5 equations to *normalize* some of the group parameters.

Here, the third equation plays a special role, because when the function $R \not\equiv 0$ is not identically zero, one can then *normalize the special parameter* a *lying on the diagonal of the matrix group*.

In this case, it is rather easy to realize that one can construct an absolute parallelism on the basis M^5, which is 5-dimensional (more will be said in a while).

Thus, we distinguish two branches in Cartan's method:

☐ $R \equiv 0$,

☐ $R \not\equiv 0$,

and we begin by exploring the first one.

1.4. **The branch** $R = 0$**.** In this case, the above 5 normalizable expressions reduce to 3 and we perform the normalizations:

$$\boxed{\mathsf{c} := \mathsf{a}\overline{\mathsf{a}}\,\overline{B}.}$$

$$\boxed{\mathsf{b} := \mathsf{a}\,\big(-i\,B + \tfrac{i}{3}\,\overline{Q}\big).}$$

$$\boxed{\mathsf{d} = \overline{\mathsf{a}}\,\big(-\,i\,\mathscr{L}\big(\overline{B}\big) + i\,P + \tfrac{2i}{3}\,\overline{B}\,Q\big).}$$

Lemma 1.9. *Within the branch $R \equiv 0$, after determining the three group parameters* b, c, d *accordingly, the further essential torsion functions become:*

$$
\begin{aligned}
&V_1, \qquad V_3, \qquad V_4, \\
&W_5, \qquad W_7, \\
&X_1 := U_2 + 2\,W_8 + \overline{W}_8, \qquad X_2 := \overline{U}_1 + V_2 + \overline{W}_6,
\end{aligned}
$$

and V_4 enables one to determine:

$$
\boxed{\mathsf{e} := \mathsf{a} \cdot \big(i\,\overline{\mathscr{L}}(\overline{B}) - i\,A - 2i\,B\overline{B} + \tfrac{i}{3}\,\overline{B}\,\overline{Q}\big),}
$$

while in fact:

$$
\boxed{V_3 \equiv 0,} \qquad \boxed{W_7 \equiv 0,} \qquad \boxed{X_2 \equiv 0,}
$$

and while:

$$
\begin{aligned}
V_1 &= \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\Big(\text{long polynomial in the } \big\{\mathscr{L}, \overline{\mathscr{L}}\big\}\text{-derivatives of } P,\ Q,\ R,\ A,\ B\Big), \\
W_5 &= \frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\Big(\text{long polynomial in the } \big\{\mathscr{L}, \overline{\mathscr{L}}\big\}\text{-derivatives of } P,\ Q,\ R,\ A,\ B\Big), \\
W_9 &= \frac{i}{9\overline{\mathsf{a}}^2}\Big(18\,\overline{\mathscr{L}}(B) - 3\,\overline{\mathscr{L}}(\overline{Q}) - 9\,\overline{P} - 12\,B\overline{Q} + 9\,B^2 + \overline{Q}^2\Big), \\
X_1 &= -\frac{i}{9\mathsf{a}\overline{\mathsf{a}}}\Big(6\,\overline{\mathscr{L}}(Q) + 6\,\mathscr{L}(\overline{Q}) - 18\,\mathscr{L}(B) - 18\,\overline{\mathscr{L}}(\overline{B}) + \\
&\qquad + 27\,B\overline{B} - 6\,BQ - 6\,\overline{B}\,\overline{Q} + 2\,Q\overline{Q} + 9\,A\Big).
\end{aligned}
$$

In the next loop, new essential torsion coefficients appear:

$$
\begin{aligned}
&W_1, \qquad W_2, \qquad W_4, \\
&Y := V_2 - W_3 + \overline{W}_6,
\end{aligned}
$$

with:

$$
\begin{aligned}
W_1 &= \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\Big(\text{very long polynomial in the } \big\{\mathscr{L}, \overline{\mathscr{L}}\big\}\text{-derivatives of } P,\ Q,\ R,\ A,\ B\Big), \\
W_2 &= \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\Big(\text{very long polynomial in the } \big\{\mathscr{L}, \overline{\mathscr{L}}\big\}\text{-derivatives of } P,\ Q,\ R,\ A,\ B\Big),
\end{aligned}
$$

but we easily show that:

$$
\boxed{Y \equiv 0,}
$$

while after some demanding computational explorations:

$$
W_4 = -\frac{6\,\overline{B} - 2\,Q}{5\,\mathsf{a}}\,W_9 + \frac{\overline{Q}}{5\overline{\mathsf{a}}}\,X_1 + \frac{3}{5\,\mathsf{a}}\mathscr{L}\big(W_9\big) - \frac{3}{5\overline{\mathsf{a}}}\overline{\mathscr{L}}\big(X_1\big).
$$

In quite summarized words, our first main theorem is as follows (more details about the ramification of possible $\{e\}$-structures appear in the last two sections of the memoir, and more is also said about explicit curvatures).

Theorem 1.1. *Within the branch $R = 0$, the extrinsic biholomorphic equivalence problem or the intrinsic CR-equivalence problem for real analytic CR-generic submanifolds $M^5 \subset \mathbb{C}^4$ that are maximally minimal in the above sense, or equivalently that belong to the General Class III_1, reduces to various absolute parallelisms namely to $\{e\}$-structures on certain manifolds of dimension 6, or directly on the 5-dimensional basis M, unless all existing potentially normalizable torsion coefficients vanish identically, in which case M is (locally) biholomorphic to Beloshapka's cubic model M^5_{c} with a characterization of such a condition being explicit in terms of the five fundamental functions P, Q, R, A, B.*

1.5. **The branch $R \not\equiv 0$.** In this branch, introducing a (locally defined, after relocalization near points where $R \neq 0$ is truly non-vanishing) real analytic function $\mathbf{A}_0$ satisfying:

$$\frac{(\mathbf{A}_0)^2}{\overline{\mathbf{A}}_0} = R,$$

one can immediately also normalize a, already during the first loop:

$$\boxed{\begin{aligned} &\mathsf{a} := \mathbf{A}_0, \\ &\mathsf{b} := \mathbf{A}_0 \big(- i\, B + \tfrac{i}{3}\, \overline{Q} \big), \\ &\mathsf{c} := \mathbf{A}_0\, \overline{\mathbf{A}}_0\, B, \\ &\mathsf{d} := \overline{\mathbf{A}}_0 \Big(i\, \overline{\mathscr{L}}(R) - i\, \mathscr{L}(\overline{B}) + \tfrac{4i}{3}\, \overline{Q} R + iP + \tfrac{2i}{3}\, \overline{B} Q - 2i\, BR \Big). \end{aligned}} \tag{2}$$

Quite similarly to the branch $R \equiv 0$, the last group parameter e can also be normalized:

$$\boxed{\mathsf{e} = -\tfrac{i}{3}\, \mathbf{A}_0 \big(6\, B \overline{B} - 3\, \overline{\mathscr{L}}(\overline{B}) + 3\, A - \overline{B}\, \overline{Q} \big),}$$

just because the essential torsion coefficient V_4 stays essentially the same.

Our second and last main theorem therefore is as follows.

Theorem 1.2. *Within the branch $R \not\equiv 0$, the extrinsic biholomorphic equivalence or the intrinsic CR-equivalence problem for real analytic CR-generic submanifolds $M^5 \subset \mathbb{C}^4$ that are maximally minimal in the above sense, or else that belong to the General Class III_1, always reduces to an absolute parallelisms on the 5-dimensional basis M.*

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2. Standard geometry of real affine planes in $\mathbb{C}^N$

2.1. **Standard complex structure on $T\mathbb{C}^N$.** Let $N \geqslant 1$ be a positive integer and consider the complex Euclidean space $\mathbb{C}^N$, equipped with the canonical coordinates $(z_1, z_2, \dots, z_N)$, where the complex numbers:

$$z_k = x_k + i\, y_k,$$

for $k = 1, 2, \dots, N$, belong to $\mathbb{C}$, with:

$$x_k = \mathrm{Re}\, z_k \qquad \text{and} \qquad y_k = \mathrm{Im}\, z_k.$$

The (real) tangent bundle $T\mathbb{C}^N$ is generated by the obvious frame constituted by the $2N$ basic vector fields:

$$\frac{\partial}{\partial x_1},\ \ \frac{\partial}{\partial y_1},\ \ \frac{\partial}{\partial x_2},\ \ \frac{\partial}{\partial y_2},\ \ \cdots\cdots,\ \ \frac{\partial}{\partial x_N},\ \ \frac{\partial}{\partial y_N}.$$

At an arbitrary fixed point $p \in \mathbb{C}^N$, a general vector tangent to $\mathbb{C}^N$ therefore writes:

$$\mathsf{vect}_p = \mathsf{x}_1 \, \frac{\partial}{\partial x_1}\Big|_p + \mathsf{y}_1 \, \frac{\partial}{\partial y_1}\Big|_p + \cdots\cdots + \mathsf{x}_N \, \frac{\partial}{\partial x_N}\Big|_p + \mathsf{y}_N \, \frac{\partial}{\partial y_N}\Big|_p,$$

where $\mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{x}_N, \mathsf{y}_N$ are free real numbers.

Now, according to a standard definition, the *complex structure* J is the endomorphism of $T\mathbb{C}^N$ which — in purely real language — indicates how such a vector, when written using complex notation:

$$\mathsf{vect}_p = \big(\mathsf{x}_1 + i\, \mathsf{y}_1, \dots, \mathsf{x}_N + i\, \mathsf{y}_N\big),$$

is transformed after multiplication by $\sqrt{-1}$, namely:

$$i\, \mathsf{vect}_p = \big(- \mathsf{y}_1 + i\, \mathsf{x}_1, \dots, -\mathsf{y}_N + i\, \mathsf{x}_N\big),$$

that is to say, in terms of the basic frame (without restricting the considerations at a fixed point), J is defined by:

$$J\big(\tfrac{\partial}{\partial x_k}\big) := \tfrac{\partial}{\partial y_k} \qquad\qquad {\scriptstyle (k=1\cdots N)},$$
$$J\big(\tfrac{\partial}{\partial y_k}\big) := -\tfrac{\partial}{\partial x_k} \qquad\qquad {\scriptstyle (k=1\cdots N)}.$$

Of course, one has:

$$J^2 = -\,\mathsf{Id}.$$

In the case $N = 1$ of classical Complex Analysis, J identifies with $\frac{\pi}{2}$-rotation of tangent vectors.

2.2. **Study of real affine subspaces of $\mathbb{C}^N$.** Next, given a real affine subspace $H_p \subset T_p\mathbb{C}^N$, a real vector $\mathsf{vect}_p \in H_p$ will be a vector belonging to some complex affine line — ***i.e.*** to some one-dimensional $\mathbb{C}$-linear affine subspace of $\mathbb{C}^N$ — if and only if $J(\mathsf{vect}_p)$ also belongs to H_p. Further, using $J^2 = -\mathrm{Id}$, one easily convinces oneself that the intersection:

$$H_p \cap J(H_p) =: H_p^c$$

is the *largest* J-invariant real linear subspace of H_p, hence also, it is the largest space in H_p on which one can define a true structure of a *complex* vector space. In fact, it suffices to define the complex scalar multiplication of an arbitrary vector simply by:

$$(a + i\,b)\,\mathsf{vect}_p := a\,\mathsf{vect}_p + b\,J(\mathsf{vect}_p),$$

and one then verifies at once that this indeed equips H_p^c with a structure of vector space over $\mathbb{C}$. In particular, H_p^c is even-dimensional. The upper index c reminds that H_p^c is usually called the *complex subspace* of H_p.

On the other hand and somehow 'dually', the sum:

$$H_p^{i_c} = H_p + J(H_p)$$

is the smallest J-invariant real linear subspace of $T_p\mathbb{C}^N$ in which H_p is contained. Again, this space possesses a $\mathbb{C}$-linear structure, hence it is even-dimensional too. The upper index i_c reminds that $H_p^{i_c}$ is usually called the *intrinsic complexification* of H_p.

Thus in summary, we have the two inclusions:

$$H_p^c \subset H_p \subset H_p^{i_c}$$

inside $T_p\mathbb{C}^N$.

Concerning dimensions, we must introduce appropriate notations. We will call $2n$ the real dimension $\dim_{\mathbb{R}} H_p^c$ and $2d$ the excess dimension of $H_p^{i_c}$ over H_p^c, namely:

$$2n := \dim_{\mathbb{R}} H_p^c, \qquad\qquad 2n + 2d := \dim_{\mathbb{R}} H_p^{i_c},$$

with $n \geqslant 0$ and $d \geqslant 0$ in full generality. From the definitions, it follows that there are exactly d linearly independent vectors $\mathsf{v}_1, \dots, \mathsf{v}_p$ in H_p that are also independent from H_p^c such that the d (linearly independent) vectors $J(\mathsf{v}_1), \dots, J(\mathsf{v}_d)$ are *not* contained in H_p, from which it follows that:

$$H_p^{i_c} = H_p \oplus \mathbb{R}J(\mathsf{v}_1) \oplus \cdots \oplus \mathbb{R}J(\mathsf{v}_d).$$

With these notations, we therefore have:

$$\dim_{\mathbb{R}} H_p^c = 2n, \qquad \dim_{\mathbb{R}} H_p = 2n + d, \qquad \dim_{\mathbb{R}} H_p^{i_c} = 2n + 2d.$$

Of course, $n + d \leqslant N$. Obviously, the (real) codimension of H_p in $T_p\mathbb{C}^N$ is then equal to:

$$\operatorname{codim}_{\mathbb{R}} H_p = 2N - 2n - d.$$

Geometrically speaking, one should view our initial general real vector subspace H_p as sitting inside the complex vector subspace $H_p^{i_c} \simeq \mathbb{C}^{n+d}$ and also as containing within itself the complex vector subspace $H_p^c \simeq \mathbb{C}^n$. In fact and more precisely, the true model geometric picture is:

$$\mathbb{C}^n \subset \mathbb{R}^{2n+d} \subset \mathbb{C}^{n+d} \subset \mathbb{C}^N,$$

as the following elementary lemma ([17, 11]), which we will reprove, confirms.

Lemma 2.1. *With $H_p \subset T_p\mathbb{C}^N$ as above being an arbitrary real affine subspace, there exist N affine* complex *coordinates:*

$$\big(z_1, \dots, z_n, w_1, \dots, w_d, t_1, \dots, t_{N-n-d}\big)$$

centered at p such that H_p is represented by the following combination of real and complex Cartesian linear equations:

$$0 = \operatorname{Im} w_1 = \cdots = \operatorname{Im} w_d \quad \text{ and } \quad 0 = t_1 = \cdots = t_{N-n-d}.$$

Proof. Firstly, one brings the complex-linear space $H_p^{i_c} \simeq \mathbb{C}^{n+d}$ just to $\{t_1 = \cdots = t_{N-n-d} = 0\}$ using a complex-linear straightening. Secondly, inside this space $\mathbb{C}^{n+d}$, using another complex-linear straightening, one brings $H_p^c \simeq \mathbb{C}^n$ to just $\{w_1 = \cdots = w_d = 0\}$ so that H_p^c is then spanned by the $z_1, \dots, z_n$ directions. Thirdly, and lastly, again inside the $\mathbb{C}^{n+d} = H_p^{i_c}$, the affine subspace H_p under study is spanned by the $z_1, \dots, z_n$ directions and by yet d real directions lying in $\{0\} \times \mathbb{C}^d$, hence H_p is represented by d linearly independent Cartesian equations without any z that are necessarily of the form:

$$0 = \sum_{k=1}^{d} \alpha_{j,k}\, u_k + \sum_{k=1}^{d} \beta_{j,k}\, v_k \qquad\qquad {\scriptstyle (j = 1 \cdots d)},$$

where $w_k =: u_k + i\, u_k$, for certain real constants $\alpha_{j,k}$ and $\beta_{j,k}$. But such equations are visibly equivalent to:

$$0 = \sum_{k=1}^{d} \mathrm{Im}\big[(\beta_{j,k} + i\,\alpha_{j,k})(u_k + i\,v_k)\big] \qquad {\scriptstyle (j=1\cdots d)},$$

hence it suffices to make the $\mathbb{C}$-linear change of coordinates:

$$w_j' = \sum_{k=1}^{d} \big(\beta_{j,k} + i\,\alpha_{j,k}\big)\, w_k \qquad {\scriptstyle (j=1\cdots d)}$$

— the determinant is again nonzero — in order to represent H_p inside $H_p^{i_c} \simeq \mathbb{C}^{n+d}$ by just $0 = \mathrm{Im}\, w_1' = \cdots = \mathrm{Im}\, w_d'$, as was to be proved. □

From this linear normalization lemma, we clearly see that the quotient real vector space $T_p\mathbb{C}^N \big/ H_p$ — which somehow represents the 'external', 'normal' space — decomposes as a direct sum of a complex vector space $T_p\mathbb{C}^N \big/ H_p^{i_c} \simeq \mathbb{C}^{N-n-d}$ plus a real vector space $H_p^{i_c}/H_p \simeq \mathbb{R}^d$, while the original linear subspace H_p also decomposes as a direct sum of a complex vector space $H_p^c \simeq \mathbb{C}^n$ plus a real vector space $H_p \big/ H_p^c \simeq \mathbb{R}^d$, the two complex dimensions $N-n-d$ and n being in general distinct. We may therefore express a bit differently the view that $H_p \simeq \mathbb{C}^n \times \mathbb{R}^d \times \{0\}^{N-d-d}$.

Lemma 2.2. *To any arbitrary real affine subspace $H_p \subset T_p\mathbb{C}^N$ are simultaneously associated its largest complex subspace H_p^c and its intrinsic complexification $H_p^{i_c}$ satisfying:*

$$H_p^c = H_p \cap J(H_p) \subset H_p \subset H_p + J(H_p) = H_p^{i_c},$$

where the two extra dimensions $\dim_{\mathbb{R}} \big(H_p/H_p^c\big)$ *and* $\dim_{\mathbb{R}} \big(H_p^{i_c}/H_p\big)$ *coincide (underlined terms):*

$$(3) \qquad \begin{cases} \dim_{\mathbb{R}} H_p = \dim_{\mathbb{R}} H_p^c + \underline{\big(\dim_{\mathbb{R}} H_p - \dim_{\mathbb{R}} H_p^c\big)} \\ \dim_{\mathbb{R}} H_p^{i_c} = \dim_{\mathbb{R}} H_p + \underline{\big(\dim_{\mathbb{R}} H_p - \dim_{\mathbb{R}} H_p^c\big)}. \end{cases} \qquad \square$$

In fact, this coincidence:

$$\dim_{\mathbb{R}} H_p - \dim_{\mathbb{R}} H_p^c = \dim_{\mathbb{R}} H_p^{i_c} - \dim_{\mathbb{R}} H_p$$

can also be seen by observing that the complex structure induces a general isomorphism:

$$J\colon\, H_p \big/ H_p^c \stackrel{\simeq}{\longrightarrow} H_p^{i_c} \big/ H_p,$$

because quotients match through: $J(H_p^c) \subset H_p$, and because J is invertible, for $J^2 = -\mathrm{Id}$.

At present, let us say in advance that we shall mainly study *generic* spaces, in the following sense:

Definition 2.3. The above arbitrary real vector subspace $H_p \subset T_p\mathbb{C}^N$ is said to be:

- *totally real* if $H_p^c = H_p \cap JH_p = \{0\}$, that is to say, if $n = 0$;
- *generic* if $H_p^{i_c} = H_p + JH_p = T_p\mathbb{C}^N$, that is to say, if $n + d = N$;
- *maximally real* if it is both totally real and generic, that is to say, if $n = 0$ and if $d = N$.

We notice passim that some constraints on (co)dimensions exist. Indeed, when H_p is totally real, one has $n = 0$, whence $0 + d \leqslant N$ and hence $\operatorname{codim}_{\mathbb{R}} H_p = 2N - 0 - d \geqslant N$. But if H_p is in addition maximally real, one has $0 + d = N$, whence $\operatorname{codim}_{\mathbb{R}} H_p = N$ exactly. Finally, when H_p is generic, its real codimension:

$$2N - 2n - d = 2(n + d) - 2n - d = d$$

is simply equal to the dimension d of its purely real part $H_p / H_p^c \simeq \mathbb{R}^d$, and one should remember this fact, which can also be seen by reminding that the complex structure induces a general isomorphism $J \colon H_p / H_p^c \xrightarrow{\simeq} H_p^{i_c} / H_p$, in which $H_p^{i_c} / H_p$ becomes $T_p\mathbb{C}^N / H_p$ when H_p is generic. In addition and for later use, we specify explicitly the form of the defining Cartesian equations of a generic affine subspace.

Corollary 2.4. *Let $H_p \subset T_p\mathbb{C}^{n+d}$ as above be an arbitrary real affine space which is generic and d-codimensional in $\mathbb{C}^N = \mathbb{C}^{n+d}$. Then there exist $n+d$ affine* complex *coordinates:*

$$\big(z_1, \dots, z_n,\, w_1, \dots, w_d\big) = \big(x_1 + iy_1, \dots, x_n + iy_n,\, u_1 + iv_1, \dots, u_d + iv_d\big)$$

centered at p such that H_p is represented by the following d real equations:

$$0 = \operatorname{Im} w_1 = \cdots = \operatorname{Im} w_d. \quad \square$$

In such a concrete coordinate representation in which:

$$\begin{aligned}
H_p &= \mathbb{R}\tfrac{\partial}{\partial x_1}\big|_p \oplus \mathbb{R}\tfrac{\partial}{\partial y_1}\big|_p \oplus \cdots \oplus \mathbb{R}\tfrac{\partial}{\partial x_n}\big|_p \oplus \mathbb{R}\tfrac{\partial}{\partial y_n}\big|_p \oplus \mathbb{R}\tfrac{\partial}{\partial u_1}\big|_p \oplus \cdots \oplus \mathbb{R}\tfrac{\partial}{\partial u_d}\big|_p, \\
H_p^c &= \mathbb{R}\tfrac{\partial}{\partial x_1}\big|_p \oplus \mathbb{R}\tfrac{\partial}{\partial y_1}\big|_p \oplus \cdots \oplus \mathbb{R}\tfrac{\partial}{\partial x_n}\big|_p \oplus \mathbb{R}\tfrac{\partial}{\partial y_n}\big|_p, \\
H_p / H_p^c &= \mathbb{R}\tfrac{\partial}{\partial u_1}\big|_p \oplus \cdots \oplus \mathbb{R}\tfrac{\partial}{\partial u_d}\big|_p, \\
T_p\mathbb{C}^{n+d} / H_p &= \mathbb{R}\tfrac{\partial}{\partial v_1}\big|_p \oplus \cdots \oplus \mathbb{R}\tfrac{\partial}{\partial v_d}\big|_p,
\end{aligned}$$

one sees at once how J induces an isomorphism $H_p / H_p^c \longrightarrow T_p\mathbb{C}^{n+d} / H_p$.

On the other hand, when $n + d \leqslant N - 1$ in full generality, so that H_p is *not* generic, the complex-codimensional part $T_p\mathbb{C}^N / H_p^{i_c} \simeq \mathbb{C}^{N-n-d}$ is nontrivial, but one easily convinces oneself that H_p becomes truly generic when it is viewed *inside* its intrinsic complexification $H_p^{i_c} \simeq \mathbb{C}^{n+d}$. Thus, from the point of view of understanding the position of a real linear space within a complex linear space, one may just drop the $\mathbb{C}^N/\mathbb{C}^{N-n-d}$ and view directly H_p sitting as a generic subspace of its intrinsic complexification $H_p^{i_c} \simeq \mathbb{C}^{n+d}$.

2.3. **Complexifications.** By tensoring with $\mathbb{C}$ the real tangent bundle $T\mathbb{C}^N$, we get the complex vector bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} T\mathbb{C}^N = T\mathbb{C}^N \oplus i\, T\mathbb{C}^N,$$

whose fiber $\simeq \mathbb{C}^N$ at an arbitrary point p of $\mathbb{C}^N$ consists of all possible linear combinations:

$$\mathsf{a}_1 \frac{\partial}{\partial x_1}\Big|_p + \mathsf{b}_1 \frac{\partial}{\partial y_1}\Big|_p + \cdots + \mathsf{a}_N \frac{\partial}{\partial x_N}\Big|_p + \mathsf{b}_N \frac{\partial}{\partial y_N}\Big|_p$$

with free *complex* coefficients a_k, b_k.

Introduce also the following basic *holomorphic* and *antiholomorphic* vector fields, for $k = 1, \dots, N$:

$$\frac{\partial}{\partial z_k} := \frac{1}{2}\bigg(\frac{\partial}{\partial x_k} - i\,\frac{\partial}{\partial y_k}\bigg) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_k} := \frac{1}{2}\bigg(\frac{\partial}{\partial x_k} + i\,\frac{\partial}{\partial y_k}\bigg),$$

with of course inversely:

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \overline{z}_k} \qquad \text{and} \qquad \frac{\partial}{\partial y_k} = i\,\frac{\partial}{\partial z_k} - i\,\frac{\partial}{\partial \overline{z}_k}.$$

Then $\mathbb{C} \otimes_{\mathbb{R}} T\mathbb{C}^N$ happens to decompose as the direct sum of two specific *holomorphic* and *antiholomorphic* bundles:

$$\mathbb{C} \otimes_{\mathbb{R}} T\mathbb{C}^N =: T^{1,0}\mathbb{C}^N \oplus T^{0,1}\mathbb{C}^N,$$

whose fibers at an arbitrary point $p \in \mathbb{C}^N$ are defined by:

$$T_p^{1,0}\mathbb{C}^N := \mathrm{Span}_{\mathbb{C}}\bigg(\frac{\partial}{\partial z_1}\Big|_p, \dots, \frac{\partial}{\partial z_N}\Big|_p\bigg),$$
$$T_p^{0,1}\mathbb{C}^N := \mathrm{Span}_{\mathbb{C}}\bigg(\frac{\partial}{\partial \overline{z}_1}\Big|_p, \dots, \frac{\partial}{\partial \overline{z}_N}\Big|_p\bigg).$$

Since $\frac{\partial}{\partial z_k} = \frac{\partial}{\partial x_k} - iJ\big(\frac{\partial}{\partial x_k}\big)$ for $k = 1, \dots, N$, one observes that one may also write:

$$T_p^{1,0}\mathbb{C}^N = \mathrm{Span}_{\mathbb{C}}\big\{\mathsf{v}_p - iJ(\mathsf{v}_p) \colon \mathsf{v}_p \in T_p\mathbb{C}^N\big\},$$
$$T_p^{0,1}\mathbb{C}^N = \mathrm{Span}_{\mathbb{C}}\big\{\mathsf{v}_p + iJ(\mathsf{v}_p) \colon \mathsf{v}_p \in T_p\mathbb{C}^N\big\}.$$

Now, let $H_p \subset T_p\mathbb{C}^N$ be an arbitrary vector space as before. In accordance with what precedes, the tensored complexification $\mathbb{C} \otimes_{\mathbb{R}} H_p^c$ of its maximal J-invariant subspace $H^c = H_p \cap J(H_p)$ decomposes as a direct sum:

$$\mathbb{C} \otimes_{\mathbb{R}} H_p^c = H_p^{1,0} \oplus H_p^{0,1},$$

where, quite similarly:

$$
H_p^{1,0} := \big\{ \mathsf{v}_p - iJ(\mathsf{v}_p) \colon \mathsf{v}_p \in H_p^c \big\},
\qquad (4)
$$
$$
H_p^{0,1} := \big\{ \mathsf{v}_p + iJ(\mathsf{v}_p) \colon \mathsf{v}_p \in H_p^c \big\}.
$$

On the other hand, in coordinates $(z_1, \dots, z_n, w_1, \dots, w_d, t_1, \dots, t_{N-n-d})$ as in Lemma 2.1 above, we have concretely:

$$
H_p^{1,0} = \mathrm{Span}_{\mathbb{C}} \bigg(\frac{\partial}{\partial z_1}\Big|_p, \dots, \frac{\partial}{\partial z_n}\Big|_p \bigg),
$$
$$
H_p^{0,1} = \mathrm{Span}_{\mathbb{C}} \bigg(\frac{\partial}{\partial \overline{z}_1}\Big|_p, \dots, \frac{\partial}{\partial \overline{z}_n}\Big|_p \bigg).
$$

3. Zariski-Generic CR behavior of real analytic submanifolds in $\mathbb{C}^N$

3.1. **Admitted analyticity assumption.** Now, consider an arbitrary connected differentiable *submanifold* M of $\mathbb{C}^N$, not necessarily straight as were the affine spaces H_p above. In order to understand the interactions between the *real differentiable structure* of M and the *complex structure* J of $\mathbb{C}^N$, one should study the way how the real tangent planes T_pM behave with respect to J when p varies in M.

However, although $\dim_{\mathbb{R}} T_pM$ is constant — by definition of a real manifold —, it is not at all true that in general, the complex-tangent planes:

$$
T_p^c M := T_pM \cap J(T_pM)
$$

have constant dimensions as p varies in M.

As in the ancient works of Sophus Lie and Élie Cartan and because we shall mainly study Lie groups in CR geometry, we shall assume that M and all the subsequently appearing geometric objects are *real analytic*, and there is a strong reason for this choice: this will insure, among other things, that for every point $p \in M \backslash \Sigma$ not lying in a certain closed *thin* subset $\Sigma \subsetneqq M$, the dimension of $T_pM \cap J(T_pM)$ will be constant.

Precisely and concretely, the real analyticity assumption — to be held throughout this memoir — will thus be the following. Let c be an integer with $0 \leqslant c \leqslant 2N$ and use the coordinates $(z_1, \dots, z_N) \equiv (x_1, y_1, \dots, x_N, y_N)$ on $\mathbb{C}^N$.

Definition 3.1. A *real analytic submanifold* of $\mathbb{C}^N$ of codimension c is a closed subset $M \subset \mathbb{C}^N$ having the property that for every point $p \in M$, there exists an open (small) neighborhood $\mathbb{U}_p$ of p in $\mathbb{C}^N$ and there exist c *real analytic* functions $\rho_1(x,y), \dots, \rho_c(x,y)$ defined and converging in $\mathbb{U}_p$

having independent real differentials:

$$c = \mathrm{rk}_{\mathbb{R}} \begin{pmatrix} \frac{\partial \rho_1}{\partial x_1} & \frac{\partial \rho_1}{\partial y_1} & \cdots & \frac{\partial \rho_1}{\partial x_N} & \frac{\partial \rho_1}{\partial y_N} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \frac{\partial \rho_c}{\partial x_1} & \frac{\partial \rho_c}{\partial y_1} & \cdots & \frac{\partial \rho_c}{\partial x_N} & \frac{\partial \rho_c}{\partial y_N} \end{pmatrix} (x,y)$$

at every point $(x,y) \in \mathbb{U}_p$, such that $M \cap \mathbb{U}_p$ consists of exactly the points (x,y) which satisfy the c Cartesian equations:

$$0 = \rho_1(x,y) = \cdots = \rho_c(x,y).$$

As is well known, the rank assumption is precisely the one which insures that the zero-set is *geometrically smooth*, *i.e.* is a manifold. Of course, the neighborhoods $\mathbb{U}_p$ associated to points $p \in M$ may be assumed to be plain small balls in which all the Taylor series:

$$\rho_j(x,y) = \sum_{\alpha \in \mathbb{N}^N, \, \beta \in \mathbb{N}^N} \rho_{j,\alpha,\beta} \, x^\alpha \, y^\beta \qquad\qquad {\scriptstyle (j = 1 \cdots N)}$$

of the functions ρ_j converge normally. Furthermore, as M is possibly global in $\mathbb{C}^N$, one must be able to compare two systems of c defining equations inside overlapping balls.

Lemma 3.2. *Whenever a point* p *belongs to two such neighborhoods* $\mathbb{U}_p'$ *with local defining functions* $\rho_1'(x,y), \dots, \rho_c'(x,y)$ *and* $\mathbb{U}_p''$ *with local defining functions* $\rho_1''(x,y), \dots, \rho_c''(x,y)$*, there exists a nonempty open subneighborhood* $\mathbb{V}_p \subset \mathbb{U}_p' \cap \mathbb{U}_p''$ *and there exists an invertible* $c \times c$ *matrix* $A = \big(a_{jk}(x,y)\big)_{1 \leqslant j \leqslant c}^{1 \leqslant k \leqslant c}$ *of analytic functions in* $\mathbb{V}_p$ *such that:*

$$\rho_j''(x,y) = \sum_{k=1}^{c} a_{j,k}(x,y) \, \rho_k'(x,y),$$

or equivalently:

$$\rho_j'(x,y) = \sum_{k=1}^{c} a_{j,k}^{-1}(x,y) \, \rho_k''(x,y),$$

for every $(x,y) \in \mathbb{V}_p$.

Proof. Leaving out the technical details, the main reason why this is true is the following. After a straightening, $M \cap \mathbb{U}_p'$ is defined by:

$$0 = s_1 = \cdots = s_c,$$

in some real coordinates $(s_1, \dots, s_c, s_{c+1}, \dots, s_{2N})$ on $\mathbb{C}^N \cong \mathbb{R}^{2N}$ with p being the origin. Then any other local system of defining equations $0 = \rho_1(s) = \cdots = \rho_c(s)$ for M in a possibly smaller subset $\mathbb{V}_p \subset \mathbb{U}_p'$ must be so that:

$$0 \equiv \rho_j(0, \dots, 0, s_{c+1}, \dots, s_{2N}) \qquad\qquad {\scriptstyle (j = 1 \cdots d)},$$

whence it immediately follows that the power series of each ρ_j writes under the form:

$$\rho_j(s) = s_1\, a_{j,1}(s) + \cdots + s_c\, a_{j,c}(s) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

for some remainder power series $a_{j,1}(s), \ldots, a_{j,c}(s)$. But the assumption that the full Jacobian matrix $\big(\frac{\partial \rho_j}{\partial s_k}(s)\big)_{1\leqslant k\leqslant 2N}^{1\leqslant j\leqslant c}$ has rank c at every $s \in \mathbb{U}_p'$ and the condition $T_pM = \{0 = s_1 = \cdots = s_c\}$ imply that in fact, already the leftmost $c \times c$ minor is nonzero: $\det\big(\frac{\partial \rho_j}{\partial s_k}(0)\big)_{1\leqslant k\leqslant c}^{1\leqslant j\leqslant c} \neq 0$. It therefore follows by applying the operators $\frac{\partial}{\partial s_k}\big|_{s=0}$ to the $\rho_j(s)$ written above that $\det\big(a_{j,k}(0)\big)_{1\leqslant j\leqslant c}^{1\leqslant k\leqslant c} \neq 0$, whence by continuity the same determinant $\det\big(a_{j,k}(s)\big)_{1\leqslant j\leqslant c}^{1\leqslant k\leqslant c}$ does not vanish for s near the origin, and this completes the essence of the argument. □

3.2. **Generic constancy of complex tangential data.** As we said by anticipation, outside some thin real analytic subset, the geometric behavior is always fine.

Proposition 3.3. *Let M be an arbitrary* connected *real analytic submanifold of $\mathbb{C}^N$. Then there exist two integers $n_M \geqslant 0$ and $d_M \geqslant 0$ with $n_M + d_M \leqslant N$, and there exists a proper real analytic subset $\Sigma \subsetneqq M$ such that, for every point $p \in M \backslash \Sigma$ not lying in Σ, the space:*

$$T_p^c M := T_pM \cap J(T_pM)$$

has constant real dimension $2n_M$, and such that in addition, the space:

$$T_p^{i_c} M := T_pM + J(T_pM)$$

also has constant real dimension, equal to $2n_M + 2d_M$.

Usually, the first space $T_p^c M$ is called the *complex tangent plane* to M at p, while no special name is given, in the literature, to the intrinsic complexification $T_p^{i_c} M$ of T_pM. Thus, for all p not in Σ, the tangent plane T_pM behaves constantly with respect to J, like an affine space as studied previously. Now, before entering the proof, what is a real analytic subset, and why is it thin?

3.3. **Basic structure of real analytic subsets of $\mathbb{C}^N$.** By passing to the charts of an atlas, it suffices to consider the case where M is some real Euclidean space $\mathbb{R}^e$.

Definition 3.4. A *real analytic subset* of $\mathbb{R}^e$, $e \geqslant 1$, equipped with coordinates $(s_1, \ldots, s_e)$ is a *closed* subset $\Sigma \subset \mathbb{R}^e$ having the property that for every point $p \in \Sigma$, there exists an open (small) neighborhood $\mathbb{U}_p$ of p in $\mathbb{R}^e$ and there exists a finite number of real analytic functions:

$$\rho_1(s_1, \dots, s_e), \dots\dots, \rho_c(s_1, \dots, s_e)$$

defined in $\mathbb{U}_p$ such that $\Sigma \cap \mathbb{U}_p$ consists of exactly the points (x, y) which satisfy the c Cartesian equations:

$$0 = \rho_1(s_1, \dots, s_e) = \cdots = \rho_c(s_1, \dots, s_e).$$

Notice that no rank condition is required on the differentials of the ρ_j, so that Σ is allowed to have completely arbitrary singularities. The closedness condition must be emphasized ([39, 75]).

Thus, to define a real analytic subset of a connected real analytic (abstract) manifold M, one sets up the same definition, intrinsically to $M \cong \mathbb{R}^{\dim M}$, the quantities $(s_1, \dots, s_{\dim M})$ being any system of local real analytic coordinates on M. However, in the context we will be dealing with in this memoir, only *local* analytic geometric objects will be studied, sitting inside some fixed small ball of $\mathbb{C}^N$, with all concerned power series converging normally in such a small ball.

A real analytic subset $\Sigma \subset M$ is said to be *proper* if it is not equal to the whole of M. As M was assumed to be connected, it happens that any such proper $\Sigma \subset M$ may be shown to be closed and nowhere dense, so that $M \backslash \Sigma$ is open with $\overline{M \backslash \Sigma} = M$. In fact, more is true, because as is well known, every real analytic subset may be *stratified*.

Definition 3.5. ([39, 75]) A *stratification* of a real analytic subset $\Sigma \subset M$ of some real analytic manifold M is a collection of geometrically smooth real analytic submanifolds S_α of M, for α running in some index set A, which constitutes a *partition* of M:

$$\bigcup_{\alpha \in A} S_\alpha = \Sigma, \qquad \text{with} \qquad \emptyset = S_\alpha \cap S_\beta \quad \text{for } \alpha \neq \beta,$$

which is in addition *locally finite*, namely satisfies for every compact subset $K \Subset M$:

$$\mathsf{Card}\, \big\{ \alpha \in A \colon\; S_\alpha \cap K \neq \emptyset \big\} < \infty,$$

and lastly, which satisfies the so-called *frontier condition*:

if $S_\alpha \neq S_\beta$ and $S_\alpha \cap \overline{S_\beta} \neq \emptyset$, then $S_\alpha \subset \overline{S_\beta}$ and $\dim S_\alpha \leqslant \dim S_\beta - 1$.

We shall admit without proof the next classical result of stratifiability, *see* [75] and the references therein. Importantly, it implies that the complement $M \backslash \Sigma$ of a proper real analytic subset is open, for Σ then is a locally finite union of submanifolds of M of dimensions equal to 1, 2, $\dots$ up to at most $n - 1$.

Theorem 3.1. *Any real analytic subset of a real analytic manifold admits a stratification.* □

Definition 3.6. A *Zariski-generic* pointwise property on a real analytic manifold M is meant a property that holds true at every point $p \in M \backslash \Sigma$ outside some *proper* real analytic subset $\Sigma \subset M$.

The precise terminology *Zariski-generic* is chosen in order to avoid confusion with the notion of *CR-generic* submanifold of $\mathbb{C}^N$ (*see* below) — some authors used the term *generating* in the past.

3.4. **Ranks and generic ranks of matrices and mappings.** As a second preliminary before entering the proof of Proposition 3.3, we now study the (generic) ranks of real analytic matrices and of real analytic mappings, some two useful model cases in which some exceptional real analytic subset naturally appear.

Let $e \geqslant 1$, let $(s_1, \dots, s_e)$ be the canonical coordinates on $\mathbb{R}^e$, let $a \geqslant 1$, let $b \geqslant 1$ and consider an $a \times b$ matrix:

$$\Psi(s) = \big(\psi_j^k(s)\big)_{1\leqslant j\leqslant a}^{1\leqslant k\leqslant b}$$

of functions $\psi_j^k(s)$ that are real analytic in some small neighborhood of the origin in $\mathbb{R}^e$. For every integer r such that $1 \leqslant r \leqslant \min(a, b)$, one may form the collection of all $r \times r$ determinants (minors) that are extracted from this matrix:

$$\Psi_{j_1,\dots,j_r}^{k_1,\dots,k_r}(s) := \left| \begin{matrix} \psi_{j_1}^{k_1}(s) & \cdots & \psi_{j_1}^{k_r}(s) \\ \cdot\cdot & \cdots & \cdot\cdot \\ \psi_{j_r}^{k_1}(s) & \cdots & \psi_{j_r}^{k_r}(s) \end{matrix} \right| \qquad \begin{matrix} (1 \leqslant k_1 < \cdots < k_r \leqslant b), \\ (1 \leqslant j_1 < \cdots < j_r \leqslant a). \end{matrix}$$

Starting from the largest possible size $r := \min(a, b)$, if all these determinants are identically zero (as functions of the variables $s_1, \dots, s_e$), then one passes from the size r to the lower size $r-1$, one forms all the $(r-1)\times(r-1)$ minors, and one tests again whether they all vanish identically or not, and so on.

Then by definition, the *generic rank* r^* of the matrix-valued function $\Psi(s)$ is the largest integer r having the property that at least one $r \times r$ minor is not identically zero, while all higher minors are identically zero. One has $r^* = 0$ if and only if all entry functions $\psi_j^k(s)$ are identically zero (uninteresting case), and otherwise, one has in full generality $1 \leqslant r^* \leqslant \min(a, b)$. Most importantly, if one introduces the locus:

$$\Sigma := \Big\{ s \in \mathbb{R}^e : \ \Psi_{j_1,\dots,j_{r^*}}^{k_1,\dots,k_{r^*}}(s) = 0 \colon \forall\, j_1, \dots, j_{r^*}, \ \forall\, k_1, \dots, k_{r^*} \Big\}$$

of the points s at which all the $r^* \times r^*$ minors vanish, then this locus clearly is a *proper* real analytic subset of $\mathbb{R}^e$, for at least one function $\Psi_{j_1,\dots,j_{r^*}}^{k_1,\dots,k_{r^*}}$ is not identically zero.

Furthermore and by construction, at every point $s \in \mathbb{R}^e \backslash \Sigma$, at least one $r^* \times r^*$ minor is nonzero, and because all minors of higher size vanish identically by definition of r^*, we deduce the following remarkable property: *at every point $s \in \mathbb{R}^e \backslash \Sigma$ near the origin, the rank of the matrix-valued function $\Psi(s)$ is maximal, equal to its generic rank r^*.* A particular case is when $r^* = a = b$, so that the square matrix $\Psi(s)$ is invertible at every point $s \in \mathbb{R}^e \backslash \Sigma$ near the origin.

Conceptionally speaking, the *generic rank* of a matrix-valued function is equal to its rank *at a generic point*. What matters for us is that the exceptional real analytic set of 'bad' points is explicitly described as the zero-set of a collection of minors, which are real analytic functions concretely known in terms of the initial data $\psi_j^k(s)$.

These considerations apply directly to the study of the (generic) rank of any local real analytic map:

$$(s_1, \dots, s_e) \longmapsto \big(\Phi_1(s_1, \dots, s_e), \, \dots, \, \Phi_b(s_1, \dots, s_e)\big),$$

for the rank of this map at any point s is equal to the rank at that point s of its associated Jacobian matrix:

$$\begin{pmatrix} \frac{\partial \Phi_1}{\partial s_1} & \cdots & \frac{\partial \Phi_1}{\partial s_e} \\ \cdot\cdot & \cdots & \cdot\cdot \\ \frac{\partial \Phi_b}{\partial s_1} & \cdots & \frac{\partial \Phi_b}{\partial s_e} \end{pmatrix} (s).$$

If the real analytic objects are globally defined, one verifies that the exceptional real analytic subsets defined in two coordinate charts match together and we get the following basic useful observation.

Lemma 3.7. *Given any matrix of real analytic functions defined on a real analytic manifold, or given any real analytic mapping between two real analytic manifolds, the set of points where its rank is maximal, equal to the generic rank, is the complement of a certain* proper *real analytic subset, which may be empty, and in any case, which may be explicitly described in terms of the matrix, or in terms of the mapping.* □

3.5. **Proof of Proposition 3.3.** We can now prove the proposition left above. In the first part of the proof, we work locally, and in the second part, we show how to glue the local reasonings.

Near an arbitrary point $p \in M$, the real analytic submanifold $M \subset \mathbb{C}^N$ is represented by $c \leqslant N$ real analytic Cartesian equations of the form:

$$0 = \rho_1(x, y) = \cdots = \rho_c(x, y),$$

in coordinates $(z_1, \dots, z_N) = (x_1 + iy_1, \dots, x_N + iy_N)$ vanishing at p, and the local geometric smoothness ('manifoldness') of M amounts to the assumption that the $c \times N$ Jacobian matrix:

$$\begin{pmatrix} \frac{\partial \rho_1}{\partial x_1} & \frac{\partial \rho_1}{\partial y_1} & \cdots & \frac{\partial \rho_1}{\partial x_N} & \frac{\partial \rho_1}{\partial y_N} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \frac{\partial \rho_c}{\partial x_1} & \frac{\partial \rho_c}{\partial y_1} & \cdots & \frac{\partial \rho_c}{\partial x_N} & \frac{\partial \rho_c}{\partial y_N} \end{pmatrix} (x,y)$$

has rank c everywhere near the origin. Then a vector based at any point of coordinates (x, y) lying close to the origin:

$$\mathsf{v}\big|_{(x,y)} = \mathsf{x}_1 \frac{\partial}{\partial x_1} + \mathsf{y}_1 \frac{\partial}{\partial y_1} + \cdots + \mathsf{x}_N \frac{\partial}{\partial x_N} + \mathsf{y}_N \frac{\partial}{\partial y_N}\Big|_{(x,y)}$$

belongs to the tangent space $T_{(x,y)}M$ if and only the column vector ${}^\tau(\mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{x}_N, \mathsf{y}_N)$ belongs to the *kernel* of this Jacobian matrix.

On the other hand, the J-rotated vector:

$$J\big(\mathsf{v}|_{(x,y)}\big) = -\mathsf{y}_1 \frac{\partial}{\partial x_1} + \mathsf{x}_1 \frac{\partial}{\partial y_1} + \cdots - \mathsf{y}_N \frac{\partial}{\partial x_N} + \mathsf{x}_N \frac{\partial}{\partial y_N}\Big|_{(x,y)}$$

stays tangent to M at (x, y) if and only if it also belongs to the same kernel. Equivalently, the initial vector $\mathsf{v}|_{x,y}$ belongs to the kernel of the associated $c \times N$ auxiliary matrix:

$$\begin{pmatrix} \frac{\partial \rho_1}{\partial y_1} & -\frac{\partial \rho_1}{\partial x_1} & \cdots & \frac{\partial \rho_1}{\partial y_N} & -\frac{\partial \rho_1}{\partial x_N} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \frac{\partial \rho_c}{\partial y_1} & -\frac{\partial \rho_c}{\partial x_1} & \cdots & \frac{\partial \rho_c}{\partial y_N} & -\frac{\partial \rho_c}{\partial x_N} \end{pmatrix} (x,y).$$

In sum, such a general vector $\mathsf{v}|_{(x,y)}$ belongs to the complex tangent plane:

$$T^c_{(x,y)}M = T_{(x,y)}M \cap J(T_{(x,y)}M)$$

if and only if the column vector ${}^\tau(\mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{x}_N, \mathsf{y}_N)$ belongs to the kernel of the $2c \times 2N$ matrix (we now use index notation to denote partial derivatives):

$$\begin{pmatrix} \rho_{1,x_1} & \rho_{1,y_1} & \cdots & \rho_{1,x_N} & \rho_{1,y_N} \\ \rho_{1,y_1} & -\rho_{1,x_1} & \cdots & \rho_{1,y_N} & -\rho_{1,x_N} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \rho_{c,x_1} & \rho_{c,y_1} & \cdots & \rho_{c,x_N} & \rho_{c,y_N} \\ \rho_{c,y_1} & -\rho_{c,x_1} & \cdots & \rho_{c,y_N} & -\rho_{c,x_N} \end{pmatrix} (x,y).$$

The kernel of this matrix is necessarily even-dimensional, because $T^c_{(x,y)}M$ itself — a complex vector space — is even-dimensional, as we already know. Thus in the process of forming minors, we may restrict attention to minors of size $(2j) \times (2j)$, from $j = \max(2c, 2N) = 2N$, downwards to $j = 2$, and skip identically vanishing minors until some minor which is not identically zero is found. Denote then by $2N - 2n_M$ the largest even integer $2j$ such that there exists a $(2j) \times (2j)$ minor which is not identically zero, and define Σ to be the real analytic subset which is the zero-set of all minors of size $(2N-2n_M)\times(2N-2n_M)$, namely:

$$\Sigma\colon\qquad 0=\Delta_1(x,y)=\cdots=\Delta_K(x,y),$$

where the number K of such minors is just equal to the binomial product $\binom{2N}{2N-2n_M}\binom{2c}{2N-2n_M}$.

Then by construction, a point (x,y) does not belong to Σ if and only if $T^c_{(x,y)}M$ has maximal possible dimension $2n_M$:

$$\dim_{\mathbb{R}} T^c_{(x,y)}M = 2\,n_M, \qquad\qquad \forall\,(x,y)\notin\Sigma,$$

this is the first property claimed by Proposition 3.3.

Next, if we set:

$$\begin{aligned} d_M &:= \dim_{\mathbb{R}} T_{(x,y)}M - \dim_{\mathbb{R}}\, T^c_{(x,y)}M \\ &= 2N - c - 2\,n_M, \end{aligned}$$

we deduce from an application of the second formula in equation (3) above that the complex dimension:

$$\begin{aligned} \dim_{\mathbb{C}} T^{i_c}_{(x,y)}M &= \tfrac{1}{2}\,\dim_{\mathbb{R}} T^{i_c}_{(x,y)}M \\ &= \tfrac{1}{2}\,\big[\dim_{\mathbb{R}} T_pM + (\dim_{\mathbb{R}} T_pM - \dim_{\mathbb{R}} T^c_pM)\big] \\ &= \tfrac{1}{2}\,\big[2N-c+(2N-c-2n_M)\big)\big] \\ &= 2N-c-n_M \\ &= n_M + d_M \end{aligned}$$

is also constant, as was claimed by the second property of Proposition 3.3.

In order to glue these local reasonings, we observe that if M is represented by two systems of equations $\rho'_j=0$, $j=1,\dots,c$ and $\rho''_j=0$, $j=1,\dots,c$, so that, according to Lemma 3.2, one has $\rho''=A\,\rho'$ for some invertible matrix A, then the Jacobian matrix of ρ'', after restriction to M, is equal to A times the Jacobian matrix of ρ'. A similar relation holds between the two associated auxiliary matrices as above, and it follows from the theory of matrices that the two zero-sets of minors coincide. □.

Scholium 3.8. In the proof of Proposition (3.3), Σ is exactly the set of points $p\in M$ at which $\dim T^c_pM\neq n_M$, and in fact, this dimension can only increase:

$$\Sigma=\big\{p\in M\colon\;\; \dim T^c_pM\geqslant 1+n_M\big\},$$

whence the open subset $M\backslash\Sigma$ gathers exactly all (generic) points of M at which a constant J-tangential behavior holds. □

3.6. **Reduction of real analytic local CR submanifolds to CR-generic submanifolds.** The preceding considerations showed that it is justified to delineate the following (classical) concepts.

Definition 3.9. A real analytic submanifold $M \subset \mathbb{C}^N$ is said to be:

- *totally real* if $T_p^c M = T_p M \cap J(T_p M) = \{0\}$ is null, at *every* point $p \in M$;
- *holomorphic* if $T_p M = J(T_p M)$ is fully complex, at *every* point $p \in M$;
- *CR-generic* if $T_p M + J(T_p M) = T_p \mathbb{C}^N$ generates the whole ambient tangent space, at *every* point $p \in M$;
- *Cauchy-Riemann* — *CR* for short — if the complex dimension of $T_p^c M$ is *constant*, as p varies in M, namely equal to a certain fixed integer n_M.

Obviously, a totally real or holomorphic manifold M is Cauchy-Riemann. Also a CR-generic M is CR too, because the dimension formula:

$$\dim_{\mathbb{R}}(H + G) = \dim_{\mathbb{R}} H + \dim_{\mathbb{R}} G - \dim_{\mathbb{R}}(H \cap G)$$

for vector subspaces applied to $H := T_p M$ and to $G := J(T_p M)$ having the same dimension yields if one assumes $T_p M + J(T_p M) = T_p \mathbb{C}^N$:

$$\dim_{\mathbb{R}} \big(T_p M \cap J(T_p M)\big) = 2 \dim_{\mathbb{R}} T_p M - 2N,$$

which is indeed constant independently of the base point (we already saw this argument in Lemma 2.2 and after Definition 2.3).

The concept of CR submanifold of $\mathbb{C}^N$ embraces that of totally real, holomorphic and CR-generic submanifolds, but the next proposition (*see* [11] for a proof), shows that after possibly passing to a smaller $\mathbb{C}^N$, every local real analytic CR submanifold becomes in fact *CR-generic*.

Proposition 3.10. *Every connected real analytic CR submanifold of $M \subset \mathbb{C}^N$ of any CR dimension:*

$$n_M = \mathsf{rank}\,\big(TM \cap J(TM)\big)$$

and of any intrinsic real codimension:

$$d_M := \dim_{\mathbb{R}} M - 2\, n_M,$$

is locally contained in a certain uniquely defined smallest 'germ' of holomorphic submanifold M^{i_c} spread along M:

$$M \subset M^{i_c} \subset \mathbb{C}^N,$$

called its intrinsic complexification which has complex dimension equal to:

$$\dim_{\mathbb{C}} M^{i_c} = n_M + d_M,$$

and in addition, M is CR-generic within its intrinsic complexification:

$$T_pM + J(T_pM) = T_pM^{i_c} \qquad\qquad {\scriptstyle (p \in M)}.$$

Furthermore, at every point $p \in M$, there exist centered affine holomorphic coordinates:

$$\begin{aligned}&\big(z_1,\dots,z_{n_M},w_1,\dots,w_{d_M},t_1,\dots,t_{N-n_M-d_M}\big) \in \mathbb{C}^{n_M} \times \mathbb{C}^{d_M} \times \mathbb{C}^{N-n_M-d_M},\\ &z_1 = x_1 + i\,y_1,\dots\dots,z_{n_M} = x_{n_M} + i\,y_{n_M},\\ &w_1 = u_1 + i\,v_1,\dots\dots,w_{d_M} = u_{d_M} + i\,v_{d_M},\end{aligned}$$

vanishing at that point in which M is locally represented as the zero-locus of $d_M + 2(N - n_M - d_M)$ real Cartesian equations of the form:

$$\left[\begin{aligned} v_1 &= \varphi_1(x,y,u),\\ &\cdots\cdots\cdots\cdots\cdots,\\ v_{n_M} &= \varphi_{n_M}(x,y,u),\end{aligned}\right. \qquad \left[\begin{aligned} t_1 &= \Psi_1(z,w),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ t_{N-n_M-d_M} &= \Psi_{N-n_M-d_M}(z,w),\end{aligned}\right.$$

— implicitly, one takes the real and the imaginary parts of each one of the $(N - n_M - d_M)$ complex equations of the second group —, in which the $\varphi_j(x,y,u)$ are local real analytic functions while the $\Psi_k(z,w)$ are local holomorphic *functions. In such a representation, the holomorphic submanifold M^{i_c} is locally represented as the zero-locus of the second group of* holomorphic *equations.*

Lastly, by performing the natural holomorphic *change of coordinates:*

$$t_1' := t_1 - \Psi_1(z,w),\dots\dots\dots,t_{N-n_M-d_M}' := t_{N-n_M-d_M} - \Psi_{N-n_M-d_M}(z,w),$$

one straightens out the intrinsic complexification M^{i_c} locally to become the complex $(n_M + d_M)$-dimensional complex Euclidean space:

$$\big\{0 = t_1' = \cdots = t_{N-n_M-d_M}'\big\},$$

so that the original CR submanifold $M \subset \mathbb{C}^N$ may be viewed as sitting in the new complex Euclidean space $\mathbb{C}^{n_M+d_M}$ of smaller dimension and as being CR-generic *there.* □

These facts then justifies that the equivalence problem under biholomorphic mappings for arbitrary real analytic submanifolds of $\mathbb{C}^N$ — understood mainly at Zariski-generic points similarly as was the case in Sophus Lie's and Élie Cartan's works and as is usual in its contemporary prolongations as well — comes down to studying *CR-generic submanifolds* in some appropriate $\mathbb{C}^N$.

Of course, when M is of null codimension, $M \equiv \mathbb{C}^N$ locally, and no equivalence problem exists. Also, in the case where the CR dimension n_M of M is null, only one local model exists.

Proposition 3.11. *Every CR-generic real analytic submanifold $M \subset \mathbb{C}^N$ which is totally real, namely of CR dimension $n_M = 0$, is locally biholomorphically equivalent to a real N-dimensional hyperplane,* e.g. *to $\mathbb{R}^N$ sitting in $\mathbb{R}^N + i\,\mathbb{R}^N$.* □

Thus, we have now fully justified the (known) fact that only the consideration of CR-generic submanifolds that have *positive* codimension and *positive* CR dimension opens up mathematical problems.

In this memoir, we will mainly consider the case of CR dimension 1, and the equivalence problem will appear to be not at all completely settled when the codimension is high.

4. CR-Generic real analytic submanifolds $M \subset \mathbb{C}^{n+d}$: real and complex

4.1. **Real and complex local equations for CR-generic submanifolds.**

Consider therefore a local *real analytic* submanifold $M \subset \mathbb{C}^N$ of positive (real) codimension $d \geqslant 1$ which is *CR-generic* in the sense that its tangent planes:

$$T_pM + J(T_pM) = T_p\mathbb{C}^N \qquad {\scriptstyle (p \in M)}$$

generate the whole ambient tangent plane $T_p\mathbb{C}^N$ over complex numbers, and which has *positive* CR dimension:

$$n := \dim_{\mathbb{R}} \big(T_pM \cap J(T_pM)\big).$$

Thus, $N = n + d$, and the letter N will not be used anymore.

In any system of affine holomorphic coordinates:

$$\begin{aligned}(z, w) = \big(z_1, \dots, z_n,\, w_1, \dots, w_d\big),&\\ z_1 = x_1 + i\,y_1, \dots\dots\dots, z_n = x_n + i\,y_n,&\\ w_1 = u_1 + i\,v_1, \dots\dots\dots, w_d = u_d + i\,y_d,&\end{aligned}$$

centered at one reference point $p_0 \in M$ — the associated *origin* 0 — and for which:

$$T_{p_0}M = \big\{\mathrm{Im}\, w_j = 0\colon\; j = 1, \dots, d\big\},$$

the CR-generic submanifold $M \subset \mathbb{C}^{n+d}$ is locally represented by d *real analytic* equations of the form:

$$v_j = \varphi_j(x, y, u) \qquad {\scriptstyle (j = 1 \cdots d)}. \tag{5}$$

Viewed geometrically, M is a graph over its d-codimensional plane $T_{p_0}M \subset T_{p_0}\mathbb{C}^{n+d}$, with of course the property that the first order jet of each graphing function φ_j vanishes at the origin:

$$0 = \varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_k}\varphi_j(0) = \partial_{u_{j'}}\varphi_j(0) \qquad {\scriptstyle (k=1\cdots n;\ j,\, j'=1\cdots d)}.$$

Let us rewrite these d Cartesian real equations as:

$$\tfrac{w_j - \overline{w}_j}{2\,i} = \varphi_j\Big(\tfrac{z+\overline{z}}{2},\, \tfrac{z-\overline{z}}{2\,i},\, \tfrac{w+\overline{w}}{2}\Big) \qquad {\scriptstyle (j=1\cdots d)}.$$

Since the right-hand sides φ_j are all an $\mathrm{O}(2)$, we can then apply the analytic implicit function theorem in order to solve these equations for the d variables w_j, $j = 1, \dots, d$. Performing this, we obtain a collection of d equations of the shape:

$$w_j = \Theta_j\big(z,\, \overline{z},\, \overline{w}\big) \qquad {\scriptstyle (j=1\cdots d)},$$

whose right-hand side power series converge of course near the origin:

$$(0,0,0) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d.$$

Since $d\varphi(0) = 0$, one has $\Theta = -\overline{w} + \text{order } 2 \text{ terms}$. In fact, the functions Θ_j are *analytic* with respect to their variables $(z, \overline{z}, \overline{w})$, hence they expand in convergent Taylor series, say under the form:

$$\Theta_j\big(z,\, \overline{z},\, \overline{w}\big) = \sum_{\substack{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^n,\, \gamma\in\mathbb{N}^d \\ |\alpha|+|\beta|+|\gamma|\geqslant 1}} \Theta_{j,\alpha,\beta,\gamma}\, z^\alpha\, \overline{z}^\beta\, \overline{w}^\gamma \in \mathbb{C}\big\{z,\, \overline{z},\, \overline{w}\big\},$$

the coefficients $\Theta_{j,\alpha,\beta,\gamma} \in \mathbb{C}^d$ being in general *non-real complex numbers*, because of the presence of $i = \sqrt{-1}$ in the rewritten Cartesian equations. These sorts of complex equations will appear to be more convenient to deal with in the sequel, *cf.* [42, 44, 45, 48, 49], hence let us explain in which precise, rigorous sense they are equivalent to the original ones $v_j = \varphi_j(x, y, u)$.

Initially, our functions $\varphi_j(x, y, u)$ were in fact all *real-valued*, namely, in their Taylor series:

$$\varphi_j(x,y,u) = \sum_{\substack{\alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^n,\ \gamma\in\mathbb{N}^d \\ |\alpha|+|\beta|+|\gamma|\geqslant 2}} \varphi_{j,\alpha,\beta,\gamma}\, x^\alpha\, y^\beta\, u^\gamma \qquad {\scriptstyle (j=1\cdots d)},$$

the appearing Taylor coefficients were all *real*:

$$\varphi_{j,\alpha,\beta,\gamma} \in \mathbb{R}.$$

This feature may be expressed under the form:

$$\overline{\varphi_j(x,y,u)} \equiv \sum_{|\alpha|+|\beta|+|\gamma|\geqslant 1} \overline{\varphi_{j,\alpha,\beta,\gamma}}\, \overline{x}^\alpha\, \overline{y}^\beta\, \overline{u}^\gamma \equiv \sum_{|\alpha|+|\beta|+|\gamma|\geqslant 1} \varphi_{j,\alpha,\beta,\gamma}\, x^\alpha y^\beta u^\gamma \equiv \varphi_j(x,y,u)$$

of equations that are identically satisfied in the ring $\mathbb{C}\{x, y, u\}$.

Natural principle for the conjugation of Taylor series. *With* $t = (t_1, \dots, t_c) \in \mathbb{C}^c$ *being complex-valued variables, given an arbitrary complex Taylor series:*

$$\Phi(t) = \Phi(t_1, \dots, t_c) = \sum_{(\gamma_1, \dots, \gamma_c) \in \mathbb{N}^c} \Phi_{\gamma_1, \dots, \gamma_c} \, (t_1)^{\gamma_1} \cdots (t_c)^{\gamma_c} = \sum_{\gamma \in \mathbb{N}^c} \Phi_\gamma \, t^\gamma,$$

convergent or not, and having complex coefficients:

$$\Phi_{\gamma_1, \dots, \gamma_c} \in \mathbb{C},$$

one defines the new Taylor series:

$$\overline{\Phi}(t) := \sum_{\gamma \in \mathbb{N}^c} \overline{\Phi_\gamma} \, t^\gamma$$

by conjugating only *its complex coefficients, so that the conjugation operator (overline) can be applied independently and separately over functions and over variables as shown by the functional identity:*

$$\overline{\Phi(t_1, \dots, t_c)} \equiv \overline{\Phi}(\overline{t_1}, \dots, \overline{t_c}).$$

But now, coming back to the d *complex* equations:

$$w_j = \Theta_j\big(z, \overline{z}, \overline{w}\big) \qquad {\scriptstyle (j = 1 \cdots d)},$$

which we obtained through the implicit function theorem, how can they represent a *real* d-codimensional submanifold of $\mathbb{C}^{n+d}$? For in principle, they provide not d, but $2\,d$ real equations:

$$\begin{aligned} 0 &= \mathrm{Re}\big[w_j - \Theta_j\big(z, \overline{z}, \overline{w}\big)\big] \qquad {\scriptstyle (j = 1 \cdots d)}, \\ 0 &= \mathrm{Im}\big[w_j - \Theta_j\big(z, \overline{z}, \overline{w}\big)\big] \qquad {\scriptstyle (j = 1 \cdots d)}, \end{aligned}$$

which is twice what is appropriate. Fortunately — *cf.* [44], § 2.1.13 —, the complex power series $\Theta_j\big(z, \overline{z}, \overline{w}\big)$ are *not* arbitrary, they keep a track of reality.

Theorem 4.1. *The* d *complex analytic power series:*

$$\Theta_j\big(z, \overline{z}, \overline{w}\big) = \sum_{|\alpha| + |\beta| + |\gamma| \geqslant 1} \Theta_{j, \alpha, \beta, \gamma} \, z^\alpha \, \overline{z}^\beta \, \overline{w}^\gamma$$

together with their respective complex conjugate series:

$$\overline{\Theta}_j = \overline{\Theta}_j\big(\overline{z}, z, w\big) = \sum_{|\alpha| + |\beta| + |\gamma| \geqslant 1} \overline{\Theta_{j, \alpha, \beta, \gamma}} \, \overline{z}^\alpha \, z^\beta \, w^\gamma \in \mathbb{C}\big\{\overline{z}, \, z, \, w\big\}^d$$

satisfy the two — equivalent by conjugation — collections of d functional equations:

$$
(6)\qquad
\begin{aligned}
\overline{w}_j &\equiv \overline{\Theta}_j\big(\overline{z}, z, \Theta(z, \overline{z}, \overline{w})\big) && {\scriptstyle (j\,=\,1\,\cdots\,d)},\\
w_j &\equiv \Theta_j\big(z, \overline{z}, \overline{\Theta}(\overline{z}, z, w)\big) && {\scriptstyle (j\,=\,1\,\cdots\,d)},
\end{aligned}
$$

identically in $\mathbb{C}\big\{z, \overline{z}, \overline{w}\big\}$ and in $\mathbb{C}\big\{\overline{z}, z, w\big\}$, respectively.

Conversely, given any collection of d local analytic power series:

$$
\Theta_j\big(z, \overline{z}, \overline{w}\big) = \sum_{|\alpha|+|\beta|+|\gamma|\geqslant 1} \Theta_{j,\alpha,\beta,\gamma}\, z^\alpha\, \overline{z}^\beta\, \overline{w}^\gamma \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}
$$

having complex coefficients $\Theta_{j,\alpha,\beta,\gamma} \in \mathbb{C}$ and satisfying:

$$
\Theta_j = -\,\overline{w}_j + \text{second order terms},
$$

which, in conjunction with their conjugates $\overline{\Theta_j}\big(\overline{z}, z, w\big)$, satisfy this pair of (equivalent) functional equations, then the two zero-sets:

$$
\big\{0 = -\,w + \Theta\big(z,\, \overline{z},\, \overline{w}\big)\big\} \qquad \text{and} \qquad \big\{0 = -\,\overline{w} + \overline{\Theta}\big(\overline{z},\, z,\, w\big)\big\}
$$

coincide and define a local CR-generic d-codimensional real analytic submanifold passing through the origin in $\mathbb{C}^{n+d}$. □

In fact, one may also show ([44, 45]) that there is an invertible $d \times d$ matrix $a(z, w, \overline{z}, \overline{w})$ of analytic functions defined near the origin such that one has:

$$
w - \Theta(z, \overline{z}, \overline{w}) \equiv a(z, w, \overline{z}, \overline{w})\, \big[\overline{w} - \overline{\Theta}(\overline{z}, z, w)\big],
$$

identically in $\mathbb{C}\{z, w, \overline{z}, \overline{w}\}^d$, whence the coincidence of the two zero-sets immediately follows, but we will not need this.

4.2. **Rigid CR-generic submanifolds.** Sometimes, it is advisable to restrict attention to those CR-generic submanifolds, usually called *rigid*, for which the right-hand side graphing functions are *all independent of the variables* $(u_1, \ldots, u_d)$:

$$
v_j = \varphi_j(x, y) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.
$$

In this case, the associated complex defining equations are most simply computed:

$$
w_j = \overline{w}_j + 2i\, \Phi_j\big(z, \overline{z}\big) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},
$$

with:

$$
\Phi_j\big(z, \overline{z}\big) := \varphi_j(x, y) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.
$$

4.3. **Existence of normal coordinates.** Up to now, we have made a distinction between writing *complex analytic* functions like the $\Theta_j\big(z, \overline{z}, \overline{w}\big)$, and writing *real analytic* functions like the $\varphi_j(x, y, u)$ using the real and imaginary parts x and y of z. But since $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$, we can also consider that the latter functions:

$$\varphi_j\big(\tfrac{z+\overline{z}}{2},\, \tfrac{z+\overline{z}}{2},\, u\big)$$

after expansion in convergent Taylor series and reorganization of its monomials, depend on $\big(z, \overline{z}, u\big)$. (Observe *passim* that this rewriting would fail if φ_j were only smooth.) Hence by a slight abuse of notation, we will sometimes accept to also write $\varphi_j(z, \overline{z}, u)$ instead of $\varphi_j(x, y, u)$.

Theorem 4.2. *Let $M \subset \mathbb{C}^{n+d}$ be a local real analytic CR-generic submanifold, let $p_0 \in M$ be one of its points, and assume it to be represented, in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ centered at p_0, simultaneously by d real defining equations and by d complex defining equations of the form:*

$$\begin{aligned} v_j &= \varphi_j\big(z, \overline{z}, u\big) && \scriptstyle (j = 1 \cdots d), \\ w_j &= \Theta_j\big(z, \overline{z}, \overline{w}\big) && \scriptstyle (j = 1 \cdots d). \end{aligned}$$

Then there exists a local biholomorphic change of coordinates $h\colon (z, w) \longmapsto (z', w')$ fixing the origin and having the specific property of leaving unchanged the z-coordinates:

$$z' = z, \qquad w' = g(z, w),$$

such that the image $M' := h(M)$ — again a CR-generic submanifold of $\mathbb{C}^{n+d}$ passing through the origin — has new real and complex defining equations:

$$\begin{aligned} v_j' &= \varphi_j'\big(z', \overline{z}', u'\big) && \scriptstyle (j = 1 \cdots d), \\ w_j' &= \Theta_j'\big(z', \overline{z}', \overline{w}'\big) && \scriptstyle (j = 1 \cdots d) \end{aligned}$$

with right-hand side graphing functions becoming identically zero whenever one of its arguments z' or $\overline{z}'$ is null:

$$\begin{aligned} 0 &\equiv \varphi_j'\big(0, \overline{z}', u'\big) \equiv \varphi_j'\big(z', 0, u'\big) && \scriptstyle (j = 1 \cdots d), \\ 0 &\equiv \Theta_j'\big(0, \overline{z}', \overline{w}'\big) \equiv \Theta_j'\big(z', 0, \overline{w}'\big) && \scriptstyle (j = 1 \cdots d). \end{aligned}$$

4.4. **Fundamental $(1, 0)$ and $(0, 1)$ fields in terms of real defining equations.** As above, let $M \subset \mathbb{C}^{n+d}$ be a real analytic CR-generic submanifold of positive codimension $d \geqslant 1$ and of positive CR dimension $n \geqslant 1$. Since the real dimension of $T_p^c M = T_p M \cap J(T_p M)$ is constantly equal to n at every point of M, the collection of complex-tangential subplanes $\big(T_p^c M\big)_{p \in M}$ organizes coherently as a real vector subbundle of TM having rank $2\,n$.

Furthermore, to keep track on M of the basic holomorphic and antiholomorphic tangent vectors which exist on $\mathbb{C}^{n+d}$:

$$\begin{aligned}&\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n},\ \frac{\partial}{\partial w_1},\dots,\frac{\partial}{\partial w_d},\\ &\frac{\partial}{\partial \overline{z}_1},\dots,\frac{\partial}{\partial \overline{z}_n},\ \frac{\partial}{\partial \overline{w}_1},\dots,\frac{\partial}{\partial \overline{w}_d},\end{aligned}$$

it is natural, in view of the definitions (4) of $T_p^{1,0}M$ and of $T_p^{0,1}M$, to define, at every point $p \in M$, two *complex* vector subspaces:

$$\begin{aligned}T_p^{1,0}M &:= \mathsf{Span}_{\mathbb{C}}\big\{\mathsf{v}_p - iJ(\mathsf{v}_p)\colon\ \mathsf{v}_p \in T_p^cM\big\} \subset T_p^{1,0}M,\\ T_p^{0,1}M &:= \mathsf{Span}_{\mathbb{C}}\big\{\mathsf{v}_p + iJ(\mathsf{v}_p)\colon\ \mathsf{v}_p \in T_p^cM\big\} \subset T_p^{0,1}M\end{aligned}$$

that are of course conjugate to each other:

$$T_p^{0,1}M = \overline{T_p^{1,0}M}.$$

One then easily convinces oneself that, as p varies on M, these two collections of spaces organize coherently as two complex vector bundles on M of rank n, and that one may also define them as being:

$$\begin{aligned}T^{1,0}M &:= T^{1,0}\mathbb{C}^{n+d} \cap \big(\mathbb{C}\otimes_{\mathbb{R}} TM\big),\\ T^{0,1}M &:= T^{0,1}\mathbb{C}^{n+d} \cap \big(\mathbb{C}\otimes_{\mathbb{R}} TM\big).\end{aligned}$$

Visibly also, both of them are complex vector *subbundles* of:

$$\mathbb{C}\otimes_{\mathbb{R}} T\mathbb{C}^{n+d}.$$

In all what follows, working with a given local real analytic CR-generic submanifold represented as above, it will be necessary to express explicitly two (conjugate) bases for these two fundamental vector bundles.

Thus, with some usual Cartesian equations $v_j = \varphi_j(x,y,u)$ in which $T_{p_0}M = \big\{\mathrm{Im}\,w = 0\big\}$, we have of course at the origin:

$$T_{p_0}^{1,0}M = \mathsf{Span}_{\mathbb{C}}\bigg(\frac{\partial}{\partial z_1}\bigg|_0,\ \dots,\ \frac{\partial}{\partial z_n}\bigg|_0\bigg).$$

It follows geometrically that a local basis of $(1,0)$-vector fields tangent to M, namely a *local frame* for the antiholomorphic tangent bundle $T^{0,1}M$, will necessarily be of the form:

$$\overline{\mathscr{L}}_k := \frac{\partial}{\partial \overline{z}_k} + \sum_{l=1}^{d} \mathsf{A}_{k,l}(x,y,u)\,\frac{\partial}{\partial \overline{w}_l} \qquad\qquad {\scriptstyle (k\,=\,1\,\cdots\,n)},$$

for certain uniquely defined real analytic functions $\mathsf{A}_{k,l}(x,y,u)$ that may be computed elementarily. Indeed, the condition that these $\overline{\mathscr{L}}_k$ be tangent to

M, namely tangent to the zero-locus of the d graphed equations:

$$\frac{w_j-\overline{w}_j}{2\,i} = \varphi_j\big(x,y,\tfrac{w+\overline{w}}{2}\big) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

writes down as:

$$0 = \overline{\mathscr{L}_k}\Big[-\ \frac{w_j}{2\,i} + \frac{\overline{w}_j}{2\,i} + \varphi_j\big(x,y,\tfrac{w+\overline{w}}{2}\big)\Big] \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.$$

Equivalently, this condition amounts to requiring that, for every fixed $k = 1,\dots,n$, the following d affine-linear equations are identically satisfied on M by the d unknowns $\mathsf{A}_{k,1},\dots,\mathsf{A}_{k,d}$:

$$0 = \tfrac{1}{2\,i}\,\mathsf{A}_{k,j} + \varphi_{j,\overline{z}_k} + \tfrac{1}{2}\,\mathsf{A}_{k,1}\,\varphi_{j,u_1} + \cdots + \tfrac{1}{2}\,\mathsf{A}_{k,d}\,\varphi_{j,u_d} \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.$$

Thanks to the fact that by assumption all the $\varphi_{j,\overline{z}_k}$ and all the $\varphi_{j,u_{j'}}$ vanish at the origin — the central point —, a unique local solution exists for each k, which, in abbreviated matrix notation writes, shortly:

$$\mathsf{A}_k = 2\,\big(i\,I_{d\times d} - \varphi_u\big)^{-1}\cdot\varphi_{\overline{z}_k},$$

where the $d\times d$ matrix:

$$\varphi_u = \big(\varphi_{j,u_{j'}}\big)_{1\leqslant j\leqslant d}^{1\leqslant j'\leqslant d}$$

has row index j, where $\varphi_{\overline{z}_k}$ is the $d\times 1$ matrix $(\varphi_{j,\overline{z}_k})_{1\leqslant j\leqslant d}$ — a column vector —, and where:

$$\mathsf{A}_k = \big(\mathsf{A}_{k,j}\big)_{1\leqslant j\leqslant d}.$$

Of course, this solution is real analytic in a (possibly shrunk) neighborhood of the origin.

We notice here that these vector fields $\overline{\mathscr{L}_k}$ are considered *extrinsically*, namely they involve the extra vector fields $\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_d}$ living in $\mathbb{C}^{n+d}$ that are *not* tangent to M. In order to get *intrinsic* sections of $T^{0,1}M$, since M is naturally equipped with the coordinates (x,y,u), we must naturally drop the $\frac{\partial}{\partial v_j}$ and we obtain:

$$\overline{\mathscr{L}_k}\big|_M = \frac{\partial}{\partial\overline{z_k}} + \sum_{l=1}^d \tfrac{1}{2}\,\mathsf{A}_{k,l}\,\frac{\partial}{\partial u_l} \qquad\qquad {\scriptstyle (k\,=\,1\,\cdots\,n)}.$$

Of course, a local frame for $T^{1,0}M$ in a neighborhood of the origin is obtained by plain complex conjugation:

$$\mathscr{L}_k\big|_M = \frac{\partial}{\partial z_k} + \sum_{l=1}^d \tfrac{1}{2}\,\overline{\mathsf{A}_{k,l}}\,\frac{\partial}{\partial u_l} \qquad\qquad {\scriptstyle (k\,=\,1\,\cdots\,n)}.$$

4.5. **Holomorphic and antiholomorphic tangent vector fields.** When computing the coefficients $\mathsf{A}_{k,l}$, a somehow unpleasant matrix inversion was needed at the moment, and this happens to cause some differential algebra swelling troubles as soon as one enters a more-in-depth study of the equivalence problem, *cf.* what will follow. On the other hand, when dealing with the (equivalent) complex defining equations:

$$\overline{w}_j = \overline{\Theta}_j\big(\overline{z}, z, w\big) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

it is clear that the conditions of tangency:

$$\begin{aligned} 0 &= \overline{\mathscr{L}_k}\Big[-\,\overline{w}_j + \overline{\Theta}_j\big(\overline{z}, z, w\big)\Big] \\ &= -\,\mathsf{A}_{k,j} + \frac{\partial\overline{\Theta}_j}{\partial\overline{z}_k}\big(\overline{z}, z, w\big) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)} \end{aligned}$$

solves straightforwardly, and we deduce that in such a representation, the bundle $T^{1,0}M$ and its conjugate $T^{0,1}M$ are generated, respectively, by the two collections of mutually independent $(1,0)$ and $(0,1)$ vector fields:

$$\begin{aligned} \mathscr{L}_k &= \frac{\partial}{\partial z_k} + \sum_{j=1}^{d}\frac{\partial\Theta_j}{\partial z_k}\big(z, \overline{z}, \overline{w}\big)\,\frac{\partial}{\partial w_j} \qquad\qquad {\scriptstyle (k\,=\,1\,\cdots\,n)}, \\ \overline{\mathscr{L}}_k &= \frac{\partial}{\partial \overline{z}_k} + \sum_{j=1}^{d}\frac{\partial\overline{\Theta}_j}{\partial \overline{z}_k}\big(\overline{z}, z, w\big)\,\frac{\partial}{\partial \overline{w}_j} \qquad\qquad {\scriptstyle (k\,=\,1\,\cdots\,n)}. \end{aligned}$$

Then with such a pair of frames, it becomes immediately easier to compute somewhat explicitly some iterated brackets like for instance:

$$\big[\mathscr{L}_{k_1}, \overline{\mathscr{L}}_{k_2}\big], \qquad \big[\mathscr{L}_{k_1}, [\mathscr{L}_{k_2}, \overline{\mathscr{L}}_{k_3}]\big],$$

while this task is much more difficult when using the real representation with the coefficients:

$$\mathsf{A}_k = 2\,\big(i\,I_{d\times d} - \varphi_u\big)^{-1}\cdot\varphi_{\overline{z}_k}.$$

But before pushing further the general theory, it is now great time to exhibit some paradigmatic examples.

5. HEISENBERG SPHERE IN $\mathbb{C}^2$ AND BELOSHAPKA'S HIGHER DIMENSIONAL MODELS

5.1. **Heisenberg sphere in $\mathbb{C}^2$ and its deformations.** With $n=1$, $d=1$ and $(z,w)\in\mathbb{C}^2$, it is known that the complex equation:

$$w = \Theta\big(z, \overline{z}, \overline{w}\big)$$

of any real analytic hypersurface $M^3\subset\mathbb{C}^2$ which satisfies:

$$T^{1,0}M + T^{0,1}M + \big[T^{1,0}M,\, T^{0,1}M\big] = \mathbb{C}\otimes_{\mathbb{R}} TM$$

— such are usually called *Levi nondegenerate* —, may be brought to the form:

$$w - \overline{w} = 2i\, z\overline{z} + \text{terms of order} \geqslant 3.$$

Moreover, for *any* remainder of order $\geqslant 3$, the obtained hypersurface is Levi nondegenerate.

A concrete proof consists in examining the first-order terms in the Taylor series of the graphing function:

$$\tfrac{w-\overline{w}}{2\,i} = \varphi(x,y,u) = \alpha\, z^2 + c\, z\overline{z} + \overline{\alpha}\,\overline{z}^2 + \mathrm{O}_3\big(z,\overline{z}\big) + u\,\mathrm{O}_1\big(x,y,u\big),$$

with $\alpha \in \mathbb{C}$ and $c \in \mathbb{R}$. If the coordinates are already normal in the sense of Theorem 4.2, one has $\alpha = 0$, otherwise, a plain replacement of w by $w' := w - 2i\,\alpha\, z^2$ makes $\alpha = 0$. The constant c happens to be unremovable by means of local biholomorphic changes (invariance of the Levi form) of variables, and one makes $c = \pm 1$ by substituting $z' := c^{-1/2}\, z$, and lastly $c = 1$ by replacing $w' := \pm w$.

Next, the related '*model*' is the one for which the remainder is identically, namely the so-called *Heisenberg sphere* $\mathbb{H}^3 \subset \mathbb{C}^2$ having quadratic equation:

$$w - \overline{w} = 2i\, z\overline{z}.$$

On the other hand, it is known that the unit real 3-sphere S^3 in $\mathbb{C}^2$ having equation:

$$1 = z'\overline{z}' + w'\overline{w}'$$

plays the remarkable rôle, in CR geometry, of being the universal model in $\mathbb{C}^2$. But in fact, $S^3 \setminus \{p_\infty\}$ with $p_\infty := (0,-1)$, is biholomorphic, through the so-called *Cayley transform*:

$$(z,w) \longmapsto \Big(\tfrac{-\,4z}{4i+w},\ \tfrac{4-iw}{4i+w}\Big) =: (z',w')$$

to the above *Heisenberg sphere*.

5.2. **Beloshapka's cubic fourfold in $\mathbb{C}^3$.** Assuming that the CR dimension $n = 1$ is smallest possible, the next example of a '*universal*' model was introduced in codimension $d = 2$ by Beloshapka in [4]. In coordinates (z, w_1, w_2), it is the cubic:

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}). \end{cases}$$

Here is the way it may be introduced. Consider the two graphed equations of a generic $M^4 \subset \mathbb{C}^3$:

$$\begin{cases} v_1 = \varphi_1(x,y,u_1,u_2), \\ v_2 = \varphi_2(x,y,u_1,u_2), \end{cases}$$

assume that the coordinates are normal, and find the 'simplest possible' model. The first equation is brought to the Heisenberg form. Next, one looks at the first terms in the Taylor series of the second equation:

$$\begin{cases} v_1 = z\overline{z}, \\ v_2 = a\, z\overline{z} + \beta\, z^2\overline{z} + \overline{\beta}\, \overline{z}^2 z + \underbrace{z\overline{z}\,\Big[\mathrm{O}_2\big(z,\overline{z}\big) + u_1\cdot \text{remainder} + u_2 \cdot \text{remainder}\Big]}_{\text{all monomials are of weighted order} \geqslant 4}, \end{cases}$$

with $a \in \mathbb{R}$ and $\beta \in \mathbb{C}$; notice that, since the coordinates are normal, namely since $0 \equiv \varphi_2\big(0, \overline{z}, u_1, u_2\big) \equiv \varphi_2\big(z, 0, u_1, u_2\big)$, and since $z \in \mathbb{C}$ is a *single* complex variable, all monomials in the Taylor series must be divisible by $z\overline{z}$. If one then assigns natural weights to the variables:

$$\text{weight}(z) := 1, \qquad \text{weight}(w_1) := 2, \qquad \text{weight}(w_2) := 3,$$

all the remainder terms are of weighted order $\geqslant 4$, hence they may be dropped if one just seeks a simple model with at most *cubic* terms:

$$\begin{cases} v_1 = z\overline{z}, \\ v_2 = a\, z\overline{z} + \beta\, z^2\overline{z} + \overline{\beta}\, \overline{z}^2 z. \end{cases}$$

Here of course, a plain subtraction $w_2' := w_2 - a\, w_1$ makes $a = 0$. Next, it is natural to assume that $\beta \neq 0$ (otherwise, there is degeneration), and replacing z by $\lambda\, z$ with a $\lambda \in \mathbb{C}$ satisfying $\beta\, \lambda^2 \overline{\lambda} = 1$, one arrives at the so-called *Beloshapka cubic*:

$$\begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + \overline{z}^2 z. \end{cases}$$

Its geometry-preserving deformations may be introduced in a coordinate-invariant manner as follows (we skip the proof, because full details of a more substantial case will be provided in a while).

Proposition 5.1. *A real analytic 4-dimensional local CR-generic submanifold $M^4 \subset \mathbb{C}^3$ of codimension 2 whose complex tangent bundle satisfies the two equivalent conditions:*

$$\begin{aligned} TM &= T^cM + [T^cM,\, T^cM] + \big[T^cM, [T^cM,\, T^cM]\big], \\ \mathbb{C} \otimes_{\mathbb{R}} TM &= T^{1,0}M + T^{0,1}M + [T^{1,0}M,\, T^{0,1}M] + \big[T^{1,0}M, [T^{1,0}M,\, T^{0,1}M]\big] + \\ &\qquad + \big[T^{0,1}M, [T^{1,0}M,\, T^{0,1}M]\big] \end{aligned}$$

may always be represented, in suitable holomorphic coordinates (z, w_1, w_2) by two complex defining equations of the specific form:

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z} + \mathrm{O}_{\text{weighted}}(3), \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}\big(z + \overline{z}\big) + \mathrm{O}_{\text{weighted}}(4). \qquad \square \end{cases}$$

5.3. **Coordinatewise introduction of a cubic model $M^5 \subset \mathbb{C}^4$.** Consider now a real analytic, five-dimensional local real analytic CR submanifold $M^5 \subset \mathbb{C}^4$ which is CR-generic, hence of CR dimension 1, and let $p_0 \in M^5$ be one of its points. There are holomorphic coordinates:

$$\big(z, w_1, w_2, w_3\big) = \big(x + iy,\ u_1 + iv_1,\ u_2 + iv_2,\ u_3 + iv_3\big)$$

vanishing at p_0 in which M^5 can be represented as a graph of the form:

$$\left[\begin{aligned} \tfrac{w_1-\overline{w}_1}{2i} &= v_1 = \varphi_1(x,y,u_1,u_2,u_3) = \psi_1(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ \tfrac{w_2-\overline{w}_2}{2i} &= v_2 = \varphi_2(x,y,u_1,u_2,u_3) = \psi_2(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ \tfrac{w_3-\overline{w}_3}{2i} &= v_3 = \varphi_3(x,y,u_1,u_2,u_3) = \psi_3(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \end{aligned}\right.$$

over its tangent plane:

$$T_{p_0}M^5 = \big\{v_1 = v_2 = v_3 = 0\big\}$$

by means of three real analytic graphing functions φ_1, φ_2, φ_3 which vanish, together with all their first order derivatives at the origin. Here by reality of right-hand sides, the pure $(z,\overline{z})$ functions must satisfy:

$$\overline{\psi}_j(\overline{z}, z) \equiv \psi_j(z,\overline{z}) \qquad\qquad {\scriptstyle (j=1,2,3)},$$

identically in $\mathbb{C}\{z,\overline{z}\}$. Replacing if necessary w_j by the new holomorphic variable $w_j - 2i\,\psi_j(z,0)$, we may assume that $\psi_j(z,\overline{z}) = \mathrm{O}(z\overline{z})$, whence there are constants $c_j \in \mathbb{R}$ and $\alpha_j \in \mathbb{C}$ such that the terms of order $\leqslant 3$ look like:

$$\psi_j(z,\overline{z}) = c_j\, z\overline{z} + \alpha_j\, z^2\overline{z} + \overline{\alpha}_j\, \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) \qquad\qquad {\scriptstyle (j=1,2,3)}.$$

In fact, since the coordinates may freely be assumed to be normal in the sense of Theorem 4.2, we can even assume that the remainders $\mathrm{O}_4\big(z,\overline{z}\big) = z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big)$ are divisible by $z\overline{z}$.

Now, we shall make the following first (among two) nondegeneracy assumption:

Hypothesis 1: At least one of the above three real constants c_1, c_2, c_3 is nonzero, say $c_1 \neq 0$.

Under this assumption, by replacing w_1 by $\frac{1}{c_1}\, w_1$, we arrange $c_1 = 1$, and then, by replacing w_2 by $w_2 - c_2\, w_1$ and w_3 by $w_3 - c_3\, w_1$, we come to $c_2 = c_3 = 0$. Let us therefore rewrite the three equations as follows, using the same letters α_j which may have changed in the process:

$$\left[\begin{aligned} v_1 &= z\overline{z} + \alpha_1\, z^2\overline{z} + \overline{\alpha}_1\, \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ v_2 &= \qquad\ \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ v_3 &= \qquad\ \alpha_3\, z^2\overline{z} + \overline{\alpha}_3\, \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3). \end{aligned}\right.$$

Hypothesis 2: The two complex numbers $\alpha_2, \alpha_3 \in \mathbb{C}$ above are $\mathbb{R}$-linearly independent.

Hence firstly, $\alpha_2 \neq 0$ is nonzero and replacing z by $\lambda\, z$ with a $\lambda \in \mathbb{C}$ satisfying $1 = \alpha_2\, \lambda^2\, \overline{\lambda}$, we arrange $\alpha_2 = 1$ (to keep $v_1 = z\overline{z} + \mathrm{O}(3)$, it suffices to simultaneously replace w_1 by $\frac{1}{\lambda\overline{\lambda}}\, w_1$).

Secondly, writing $\alpha_3 = \alpha_3' + i\, \alpha_3''$, we may replace w_3 by $w_3 - \alpha_3'\, w_2$ to arrange that $\alpha_3 = i\alpha_3''$ becomes purely imaginary. Then $\alpha_3'' \neq 0$ too by Hypothesis 2, and replacing w_3 by $\frac{1}{-\alpha_3''}\, w_3$, we arrive at $\alpha_3 = -i$.

Thirdly and Lastly, writing $\alpha_1 = \alpha_1' + i\alpha_1''$ and replacing w_1 by $w_1 - \alpha_1'\, w_2 + \alpha_1''\, w_3$, we come to $\alpha_1 = 0$, whence the three equations of M^5 have been reduced to the following initial general form:

$$\left[\begin{aligned} \tfrac{w_1-\overline{w}_1}{2i} &= z\overline{z} + && + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ \tfrac{w_2-\overline{w}_2}{2i} &= && z^2\overline{z} + \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3), \\ \tfrac{w_3-\overline{w}_3}{2i} &= && -i\, z^2\overline{z} + i\, \overline{z}^2 z + \mathrm{O}_4(z,\overline{z}) + \mathrm{O}(u_1) + \mathrm{O}(u_2) + \mathrm{O}(u_3). \end{aligned}\right.$$

If we assume that all remainders vanish, we get the following model of cubic 5-dimensional real algebraic CR-generic submanifold of $\mathbb{C}^4$:

$$\left[\begin{aligned} \tfrac{w_1-\overline{w}_1}{2i} &= z\overline{z}, \\ \tfrac{w_2-\overline{w}_2}{2i} &= z^2\overline{z} + \overline{z}^2 z, \\ \tfrac{w_3-\overline{w}_3}{2i} &= -i\, z^2\overline{z} + i\, \overline{z}^2 z. \end{aligned}\right.$$

It generalizes both the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$ of CR dimension 1 having defining equation:

$$w - \overline{w} = 2i\, z\overline{z},$$

and Beloshapka's four-dimensional cubic $\mathbb{B}^4 \subset \mathbb{C}^3$ of CR dimension 1 having the two defining equations:

$$\left[\begin{aligned} w_1 - \overline{w}_1 &= 2i\, z\overline{z} \\ w_2 - \overline{w}_2 &= 2i\, z\overline{z}(z + \overline{z}). \end{aligned}\right.$$

5.4. **Invariant introduction of the cubic model $M^5 \subset \mathbb{C}^4$.** The two hypotheses made above about the CR-generic submanifold $M^5 \subset \mathbb{C}^4$ can be reformulated in a way which shows well that it is completely invariant and independent of any choice of coordinates.

Indeed, for a general CR-generic n-dimensional and d-codimensional $M \subset \mathbb{C}^{n+d}$, one may look at all the iterated brackets between the local sections of the complex tangent bundle:

$$T^c M = TM \cap JTM,$$

which is here of real rank 2, since M is of CR dimension 1. More precisely, one introduces the subsequent subdistributions:

$$D^1M := T^cM, \qquad D^2M := \mathrm{Span}_{\mathscr{C}^\omega(M)}\big(D^1M + \big[T^cM,\, D^1M\big]\big),$$
$$D^3M := \mathrm{Span}_{\mathscr{C}^\omega(M)}\big(D^2M + \big[T^cM,\, D^2M\big]\big), \qquad \dots$$

of TM that are linearly generated, over the algebra $\mathscr{C}^\omega(M)$ of real analytic functions on M, by all possible iterated Lie brackets between the local sections of T^cM, with of course $D^1M \subset D^2M \subset D^3M \subset \cdots$.

Both in the ancient Lie-Cartan theory and in the more recent field of subRiemannian geometry (*see e.g.* the survey [3]), it is usual to assume *strong uniformity*, namely that for each $i = 1, 2, 3, \dots$, the dimensions of the D^i_pM are all locally constant as p runs in M, hence are fully constant if M is thought of as being localized around one of its points (and connected too, as will always be assumed implicitly). So, all D^iM are subbundles of TM. Furthermore, it is natural to assume in addition that M is *minimal* (*see* [76, 45, 56]), namely that:

$$D^iM = TM \quad \text{for all } \ i \geqslant i^* \ \text{ large enough}.$$

Lastly, as a first step in the study of such differential structures, it is also natural to assume that the ranks $r_1(M)$, $r_2(M)$, $r_3(M)$, ..., of the subbundles D^1M, D^2M, D^3M, ..., increase as much as possible. We propose to say simply that a CR-generic M for which the $r_i(M)$ increase maximally is *maximally minimal*. There is a general problem, that we will not touch here, of describing the structure of all possible maximally minimal CR manifolds, and the concept of free Lie algebra ([27, 28, 65]) is concerned.

In our case of a CR-generic 3-codimensional $M^5 \subset \mathbb{C}^4$, because T^cM^5 has rank:

$$r_1(M^5) = 2\,\mathrm{CRdim}\,M^5 = 2,$$

and because the Lie bracket is skew-symmetric, D^2M^5 can at most be of rank 3. Bracketing then D^2M^5 with some two linearly independent local sections of T^cM^5 can at most yield two more independent vector fields, so $r_3(M^5)$ can at most be equal to $5 = \dim M^5$. Thus on the agreement that the $r_i(M^5)$ are maximal possible, it suffices in fact to jump only up to level $i = 3$ to reach *minimality*, namely:

$$D^3M^5 = TM^5.$$

We will therefore study the class of real analytic 5-dimensional CR-generic 3-codimensional submanifolds $M^5 \subset \mathbb{C}^4$ for which:

$$\begin{aligned} r_1(M^5) &= 2 = 2\,\mathrm{CRdim}\,M^5, \\ r_2(M^5) &= 3, \\ r_3(M^5) &= 5 = \dim M^5. \end{aligned}$$

In other words and using the terminology introduced a moment ago, we will study *maximally minimal* real analytic CR-generic 3-codimensional submanifolds $M^5 \subset \mathbb{C}^4$.

For completeness and briefly, let us observe that a real analytic hypersurface $M^3 \subset \mathbb{C}^2$ is maximally minimal if and only if it is Levi nondegenerate (at every point). Furthermore, the so-called *Engel CR manifolds* $M^4 \subset \mathbb{C}^3$ of codimension 2 that are deformations of Beloshapka's cubic as stated in Proposition 5.1 are maximally minimal too, with:

$$\begin{aligned} r_1(M^4) &= 2 = 2\,\mathrm{CRdim}\,M^4, \\ r_2(M^4) &= 3, \\ r_3(M^4) &= 4 = \dim M^4. \end{aligned}$$

However, in this latter case, a specific phenomenon occurs, because the maximal freedom for iterated Lie brackets between two linearly independent sections ξ_1 and ξ_2 of T^cM^4 may (as already seen) yield in general *five* linearly independent vector fields (*see* also §2.1.3 p. 9 in [56]):

$$\xi_1, \qquad \xi_2, \qquad [\xi_1,\, \xi_2], \qquad \big[\xi_1,\, [\xi_1,\, \xi_2]\big], \qquad \big[\xi_2,\, [\xi_1,\, \xi_2]\big],$$

while the dimension of $M^4 \subset \mathbb{C}^3$ is equal to $4 < 5$, which imposes a constraint. This is the true reason why, in [7], there is a *distinguished* complex-tangential direction field ξ_0, namely a section of T^cM^4 such that $\big[\xi_0,\, [\xi_1,\, \xi_2]\big]$ is *not* linearly independent of ξ_1, ξ_2 and $[\xi_1,\, \xi_2]$. But in our case of a maximally minimal $M^5 \subset \mathbb{C}^4$, the 5-dimensionality of M drops such a dimensional constraint and hence, there is no distinguished direction field, which gives a situation somewhat analogous to the paradigmatic case of a Levi nondegenerate $M^3 \subset \mathbb{C}^2$.

Thus, let $M^5 \subset \mathbb{C}^4$ be a local 5-dimensional real analytic CR-generic submanifold having codimension 3. In suitable holomorphic coordinates (z, w_1, w_2, w_3) centered at some point of M^5, the three *complex defining equations* of M^5 are of the form:

$$\left[\begin{aligned} w_1 - \overline{w}_1 &= \Xi_1\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \\ w_2 - \overline{w}_2 &= \Xi_2\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \\ w_3 - \overline{w}_3 &= \Xi_3\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big), \end{aligned}\right. \tag{7}$$

where Ξ_1, Ξ_2 and Ξ_3 are analytic functions defined in a neighborhood of the origin in $\mathbb{C}^5$ such that the functions:

$$\Theta_j\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big) = \overline{w}_j + \Xi_j\big(z, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big) \qquad {\scriptstyle (j\,=\,1,2,3)}$$

are subjected to the reality condition (6).

5.5. **Initial normalization of defining equations.** Now, we want to interpret the condition of maximal minimality in terms of the three complex defining equations (7). As said above, M^5 is local, real analytic, connected and it passes through the origin. Clearly, a (single) vector field generating the bundle $T^{1,0}M^5$ of $(1,0)$-fields tangent to M^5 is:

$$\mathscr{L} := \frac{\partial}{\partial z} + \Xi_{1,z}\,\frac{\partial}{\partial w_1} + \Xi_{2,z}\,\frac{\partial}{\partial w_2} + \Xi_{3,z}\,\frac{\partial}{\partial w_3}.$$

Let $\overline{\mathscr{L}}$ denote the complex conjugate to $\mathscr{L}$:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \overline{\Xi}_{1,\overline{z}}\,\frac{\partial}{\partial \overline{w}_1} + \overline{\Xi}_{2,\overline{z}}\,\frac{\partial}{\partial \overline{w}_2} + \overline{\Xi}_{3,\overline{z}}\,\frac{\partial}{\partial \overline{w}_3},$$

which is also tangent to M^5, since it annihilates the three equations conjugate to (7) that are known to be equivalent to them, whence $\overline{\mathscr{L}}$ generates $T^{0,1}M^5$.

It is often easier and more natural to work with the *extrinsic complexification* M^{e_c} of M^5, having the same equations (7), but with $\overline{z}$, $\overline{w}_1$, $\overline{w}_2$ and $\overline{w}_3$ being considered as *new independent* complex variables that we will denote:

$$\big(\underline{z},\,\underline{w}_1,\,\underline{w}_2,\,\underline{w}_3\big)$$

as in Subsection 7.5 below. Then according to this subsection, two equivalent collections of d Cartesian equations of this extrinsic complexification — which is now a true holomorphic submanifold of $\mathbb{C}^4 \times \mathbb{C}^4 = \mathbb{C}^8$ equipped with coordinates $\big(z, w_1, w_2, w_3, \underline{z}, \underline{w}_1, \underline{w}_2, \underline{w}_3\big)$ — are:

$$\begin{aligned} w_j - \underline{w}_j &= \Xi_j\big(z, \underline{z}, \underline{w}_1, \underline{w}_2, \underline{w}_3\big) && {\scriptstyle (j\,=\,1,2,3)}, \\ \underline{w}_j - w_j &= \overline{\Xi}_j\big(\underline{z}, z, w_1, w_2, w_3\big) && {\scriptstyle (j\,=\,1,2,3)}. \end{aligned}$$

Furthermore, the extrinsic complexifications of the $(1,0)$ and of the $(0,1)$ vector fields are:

$$\begin{aligned} \mathscr{L} &:= \frac{\partial}{\partial z} + \sum_{l=1}^{3} \Xi_{l,z}\big(z, \underline{z}, \underline{w}_1, \underline{w}_2, \underline{w}_3\big)\,\frac{\partial}{\partial w_l}, \\ \underline{\mathscr{L}} &:= \frac{\partial}{\partial \underline{z}} + \sum_{l=1}^{3} \overline{\Xi}_{l,\underline{z}}\big(\underline{z}, z, w_1, w_2, w_3\big)\,\frac{\partial}{\partial \underline{w}_l}. \end{aligned}$$

Lastly, since M^{e_c} is a 5-dimensional holomorphic submanifold of $\mathbb{C}^8$ and since it has two equivalent 3-tuples of Cartesian equations, two equivalent local frames for its holomorphic tangent bundle near the origin are:

$$\Big(\mathscr{L},\ \underline{\mathscr{L}},\ \tfrac{\partial}{\partial \underline{w}_1},\ \tfrac{\partial}{\partial \underline{w}_2},\ \tfrac{\partial}{\partial \underline{w}_3}\Big) \qquad \text{and} \qquad \Big(\underline{\mathscr{L}},\ \mathscr{L},\ \tfrac{\partial}{\partial w_1},\ \tfrac{\partial}{\partial w_2},\ \tfrac{\partial}{\partial w_3}\Big).$$

Lemma 5.2. *The local CR-generic 3-codimensional real analytic submanifold $M^5 \subset \mathbb{C}^4$ is maximally minimal at the origin if and only if the five vector fields:*

$$\mathscr{L}, \quad \underline{\mathscr{L}}, \quad \big[\mathscr{L}, \underline{\mathscr{L}}\big], \quad \big[\mathscr{L}, \big[\mathscr{L}, \underline{\mathscr{L}}\big]\big], \quad \big[\underline{\mathscr{L}}, \big[\mathscr{L}, \underline{\mathscr{L}}\big]\big]$$

are $\mathbb{C}$-linearly independent at the origin, hence at all points of M^{e_c} (which, again, is small and localized around the origin).

Proof. Indeed, to check this claim, one may consider:

$$L_1 := \operatorname{Re} \mathscr{L} \quad \text{and} \quad L_2 := \operatorname{Im} \mathscr{L}$$

as two sections generating $T^c M^5 = \operatorname{Re} T^{1,0} M^5$ and compare real and complex linear spans. □

At first, such an independency condition requires at least that the first three vector fields $\mathscr{L}$, $\underline{\mathscr{L}}$ and $\big[\mathscr{L}, \underline{\mathscr{L}}\big]$ are $\mathbb{C}$-linearly independent at the origin. But a direct computation yields the expression of the bracket:

(8)
$$\begin{aligned}
\big[\mathscr{L}, \underline{\mathscr{L}}\big] &= \bigg[\frac{\partial}{\partial z} + \sum_{l=1}^{3} \Xi_{l,z} \frac{\partial}{\partial w_l}, \ \frac{\partial}{\partial \underline{z}} + \sum_{j=1}^{3} \underline{\Xi}_{j,\underline{z}} \frac{\partial}{\partial \underline{w}_j}\bigg] \\
&= \sum_{j=1}^{3} \bigg(\underline{\Xi}_{j,\underline{z}z} + \sum_{l=1}^{3} \Xi_{l,z}\, \underline{\Xi}_{j,\underline{z}w_l}\bigg) \frac{\partial}{\partial \underline{w}_j} - \sum_{j=1}^{3} \bigg(\Xi_{j,z\underline{z}} + \sum_{l=1}^{3} \underline{\Xi}_{l,\underline{z}}\, \Xi_{j,z\underline{w}_l}\bigg) \frac{\partial}{\partial w_j},
\end{aligned}$$

and since $0 = \Theta_{l,z}(0) = \Theta_{l,\underline{z}}(0)$ for $l = 1, 2, 3$, we realize that these three vectors based at the origin have the following simple values:

$$\begin{aligned}
\mathscr{L}\big|_0 &= \tfrac{\partial}{\partial z}\big|_0, \\
\underline{\mathscr{L}}\big|_0 &= \tfrac{\partial}{\partial \underline{z}}\big|_0, \\
\big[\mathscr{L}, \underline{\mathscr{L}}\big]\big|_0 &= \underline{\Xi}_{1,\underline{z}z}(0)\, \tfrac{\partial}{\partial \underline{w}_1}\big|_0 + \underline{\Xi}_{2,\underline{z}z}(0)\, \tfrac{\partial}{\partial \underline{w}_2}\big|_0 + \underline{\Xi}_{3,\underline{z}z}(0)\, \tfrac{\partial}{\partial \underline{w}_3}\big|_0 - \\
&\quad - \Xi_{1,z\underline{z}}(0)\, \tfrac{\partial}{\partial w_1}\big|_0 - \Xi_{2,z\underline{z}}(0)\, \tfrac{\partial}{\partial w_2}\big|_0 - \Xi_{3,z\underline{z}}(0)\, \tfrac{\partial}{\partial w_3}\big|_0.
\end{aligned} \tag{9}$$

Without loss of generality, and in order to simplify a bit our next computations, we will assume that the coordinates (z, w_1, w_2, w_3) are normal from the beginning, namely that:

$$0 \equiv \Xi_j\big(0, \underline{z}, \underline{w}_1, \underline{w}_2, \underline{w}_3\big) \equiv \Xi_j\big(z, 0, \underline{w}_1, \underline{w}_2, \underline{w}_3\big) \qquad (j = 1, 2, 3). \tag{10}$$

It follows that each Ξ_j is a multiple of the product $z\underline{z}$, namely of the form $\Xi_j = z\underline{z}\, \Xi_j^{\sim}$, and in particular, is an $\mathrm{O}(z\overline{z})$; obviously, the same also holds true of each $\underline{\Xi}_j$. So necessarily, there can be only one second-order term in each Ξ_j and it is of the form $\lambda_j\, z\underline{z}$ for some $\lambda_j \in \mathbb{C}$, because only the constant in $\Xi_j^{\sim}$ is concerned. But the reality condition (6) implies that each

λ_j belongs to $i\,\mathbb{R}$, and hence we can write $\lambda_j = 2i\, c_j\, z\underline{z}$ for some $c_j \in \mathbb{R}$. We therefore get:

$$\begin{aligned} w_1 - \underline{w}_1 &= 2ic_1\, z\underline{z} + \mathrm{O}(3), \\ w_2 - \underline{w}_2 &= 2ic_2\, z\underline{z} + \mathrm{O}(3), \\ w_3 - \underline{w}_3 &= 2ic_3\, z\underline{z} + \mathrm{O}(3). \end{aligned}$$

After these preparations, looking at the three expressions (9), we deduce that for $\mathrm{Span}_{\mathbb{C}}\big(\mathscr{L}\big|_0,\ \underline{\mathscr{L}}\big|_0,\ [\mathscr{L},\, \underline{\mathscr{L}}]\big|_0\big)$ to be 3-dimensional, it is necessary and sufficient that at least one c_j be nonzero, let us say: $c_1 \neq 0$, after permuting the coordinates, if necessary. Replacing then z by $\sqrt{c_1}\, z$ (for some complex square root of c_1), we get $c_1 = 1$. But then in addition, replacing w_2 by $w_2 - c_2 w_1$ and w_3 by $w_3 - c_3 w_1$, we make $c_2 = c_3 = 0$. In summary, the equations of M^{e_c} receive the form:

$$\begin{aligned} w_1 - \underline{w}_1 &= 2i\, z\underline{z} + \mathrm{O}(3) = \Xi_1, \\ w_2 - \underline{w}_2 &= \qquad\quad\ \ \mathrm{O}(3) = \Xi_2, \\ w_3 - \underline{w}_3 &= \qquad\quad\ \ \mathrm{O}(3) = \Xi_3, \end{aligned}$$

and the coordinates are still normal. Now, we must examine the $\mathrm{O}(3)$ terms.

The first length-three bracket may be computed completely and the result appears to be a long four-lines expression:

$$\begin{aligned} &[\mathscr{L},\ [\mathscr{L},\, \underline{\mathscr{L}}]] = \\ &= \Big[\frac{\partial}{\partial z} + \sum_{k=1}^3 \Xi_{k,z}\,\frac{\partial}{\partial w_k},\ \ \sum_{j=1}^3 \Big(\underline{\Xi}_{j,\underline{z}z} + \sum_{l=1}^3 \Xi_{l,z}\,\underline{\Xi}_{j,\underline{z}w_l}\Big)\frac{\partial}{\partial \underline{w}_j} - \sum_{j=1}^3 \Big(\Xi_{j,z\underline{z}} + \sum_{l=1}^3 \underline{\Xi}_{l,\underline{z}}\,\Xi_{j,z\underline{w}_l}\Big)\frac{\partial}{\partial w_j}\Big] = \\ &= \sum_{j=1}^3 \Big(\underline{\Xi}_{j,\underline{z}zz} + \sum_{l=1}^3 \big(\Xi_{l,zz}\,\underline{\Xi}_{j,\underline{z}w_l} + \Xi_{l,z}\,\underline{\Xi}_{j,\underline{z}w_l z}\big) + \sum_{k=1}^3 \Xi_{k,z}\,\underline{\Xi}_{j,\underline{z}zw_k} + \sum_{k=1}^3\sum_{l=1}^3 \Xi_{k,z}\Big(\Xi_{l,zw_k}\,\underline{\Xi}_{j,\underline{z}w_l} + \Xi_{l,z}\,\underline{\Xi}_{j,\underline{z}w_l w_k}\Big)\Big)\frac{\partial}{\partial \underline{w}_j} - \\ &- \sum_{j=1}^3 \Big(\Xi_{j,z\underline{z}z} + \sum_{l=1}^3 \big(\underline{\Xi}_{l,\underline{z}z}\,\Xi_{j,z\underline{w}_l} + \underline{\Xi}_{l,\underline{z}}\,\Xi_{j,z\underline{w}_l z}\big) + \sum_{k=1}^3 \Xi_{k,z}\,\Xi_{j,z\underline{z}w_k} + \sum_{k=1}^3\sum_{l=1}^3 \Xi_{k,z}\Big(\underline{\Xi}_{l,\underline{z}w_k}\,\Xi_{j,z\underline{w}_l} + \underline{\Xi}_{l,\underline{z}}\,\Xi_{j,z\underline{w}_l w_k}\Big)\Big)\frac{\partial}{\partial w_j} - \\ &- \sum_{k=1}^3 \Big(\sum_{j=1}^3 \Big(\underline{\Xi}_{j,\underline{z}z} + \sum_{l=1}^3 \Xi_{l,z}\,\underline{\Xi}_{j,\underline{z}w_l}\Big)\,\Xi_{k,z\underline{w}_j}\Big)\frac{\partial}{\partial w_k} + \\ &+ \sum_{k=1}^3 \Big(\sum_{j=1}^3 \Big(\Xi_{j,z\underline{z}} + \sum_{l=1}^3 \underline{\Xi}_{l,\underline{z}}\,\Xi_{j,z\underline{w}_l}\Big)\,\Xi_{k,zw_j}\Big)\frac{\partial}{\partial w_k}. \end{aligned} \tag{11}$$

But the normality conditions (10) entail that:

$$0 = \Xi_{j,z}(0) = \Xi_{l,zz}(0) = \Xi_{k,z\underline{w}_j}(0) = \Xi_{k,zw_j}(0),$$

and consequently, the value at zero of this length-three bracket is just equal to:

$$[\mathscr{L},\ [\mathscr{L},\, \underline{\mathscr{L}}]]\big|_0 = \sum_{j=1}^3 \underline{\Xi}_{j,\underline{z}zz}(0)\,\frac{\partial}{\partial \underline{w}_j} - \sum_{j=1}^3 \Xi_{j,z\underline{z}z}(0)\,\frac{\partial}{\partial w_j}.$$

In a completely similar way, we also get:

$$\big[\underline{\mathscr{L}},\ \big[\mathscr{L},\,\underline{\mathscr{L}}\big]\big]\big|_0=\sum_{j=1}^{3}\underline{\Xi}_{j,\underline{z}z\underline{z}}(0)\,\frac{\partial}{\partial\underline{w}_j}-\sum_{j=1}^{3}\Xi_{j,z\underline{z}\underline{z}}(0)\,\frac{\partial}{\partial w_j}.$$

Now, looking at those third-order terms in the three equations of M^5 which only involve z and $\underline{z}$, we may write:

$$\begin{aligned}
w_1-\underline{w}_1&=2i\,z\underline{z}+\alpha\,z^2\underline{z}+\widetilde{\alpha}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|)\\
w_2-\underline{w}_2&=\qquad\quad\ \beta\,z^2\underline{z}+\widetilde{\beta}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|)\\
w_3-\underline{w}_3&=\qquad\quad\ \gamma\,z^2\underline{z}+\widetilde{\gamma}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|),
\end{aligned}$$

for some complex constants $\alpha,\widetilde{\alpha},\beta,\widetilde{\beta},\gamma,\widetilde{\gamma}\in\mathbb{C}$. But we claim that $\widetilde{\alpha}=-\overline{\alpha}$, that $\widetilde{\beta}=-\overline{\beta}$ and that $\widetilde{\gamma}=-\overline{\gamma}$, in fact. Indeed, the reality condition may be inspected by conjugating-complexifying the above equations, which yields:

$$\begin{aligned}
\underline{w}_1-w_1&=-2i\,\underline{z}z+\overline{\alpha}\,\underline{z}^2z+\overline{\widetilde{\alpha}}\,z^2\underline{z}+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|)\\
\underline{w}_2-w_2&=\qquad\qquad\ \overline{\beta}\,\underline{z}^2z+\overline{\widetilde{\beta}}\,z^2\underline{z}+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|)\\
\underline{w}_3-w_3&=\qquad\qquad\ \overline{\gamma}\,\underline{z}^2z+\overline{\widetilde{\gamma}}\,z^2\underline{z}+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|),
\end{aligned}$$

which should yield equations that are *equivalent* the previous ones, and since the two remainders $\mathrm{O}(|z|^4)$ and $\mathrm{O}(|w|)$ visibly do not interfere at all, it follows by identifications of the monomials $z^2\underline{z}$ and $\underline{z}^2z$ that $\widetilde{\alpha}=-\overline{\alpha}$, that $\widetilde{\beta}=-\overline{\beta}$ and that $\widetilde{\gamma}=-\overline{\gamma}$, as was claimed. As a result, the equations of M^5 are:

$$\begin{aligned}
w_1-\underline{w}_1&=2i\,z\underline{z}+\alpha\,z^2\underline{z}-\overline{\alpha}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|),\\
w_2-\underline{w}_2&=\qquad\quad\ \beta\,z^2\underline{z}-\overline{\beta}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|),\\
w_3-\underline{w}_3&=\qquad\quad\ \gamma\,z^2\underline{z}-\overline{\gamma}\,\underline{z}^2z+\mathrm{O}(|z|^4)+z\underline{z}\,\mathrm{O}(|w|).
\end{aligned}$$

On the intrinsic complexification M^{e_c}, we now choose the five complex coordinates z, $\underline{z}$, $\underline{w}_1$, $\underline{w}_2$ and $\underline{w}_3$. But what has been already seen, we know that, in these five intrinsic coordinates of M^{e_c} — which means that we drop $\frac{\partial}{\partial w_1}$, $\frac{\partial}{\partial w_2}$ and $\frac{\partial}{\partial w_3}$ —, we have:

$$\mathscr{L}\big|_0=\tfrac{\partial}{\partial z}\big|_0,\qquad\quad \underline{\mathscr{L}}\big|_0=\tfrac{\partial}{\partial\underline{z}}\big|_0,\qquad\quad \big[\mathscr{L},\,\underline{\mathscr{L}}\big]\big|_0=\tfrac{\partial}{\partial\underline{w}_1}\big|_0.$$

On the other hand, also in the coordinates of M^{e_c}, we know that:

$$\begin{aligned}
\big[\mathscr{L},\ \big[\mathscr{L},\,\underline{\mathscr{L}}\big]\big]\big|_0&=\underline{\Xi}_{1,zz\underline{z}}(0)\,\tfrac{\partial}{\partial\underline{w}_1}\big|_0+\underline{\Xi}_{2,zz\underline{z}}(0)\,\tfrac{\partial}{\partial\underline{w}_2}\big|_0+\underline{\Xi}_{3,zz\underline{z}}(0)\,\tfrac{\partial}{\partial\underline{w}_3}\big|_0\\
&=\alpha\,\tfrac{\partial}{\partial\underline{w}_1}\big|_0+\beta\,\tfrac{\partial}{\partial\underline{w}_2}\big|_0+\gamma\,\tfrac{\partial}{\partial\underline{w}_3}\big|_0,
\end{aligned}$$

and quite similarly, that:

$$\begin{aligned}\big[\underline{\mathscr{L}},\ [\mathscr{L},\underline{\mathscr{L}}]\big]\big|_0 &= \underline{\Xi}_{1,\underline{z}z\underline{z}}(0)\,\frac{\partial}{\partial\underline{w}_1}\Big|_0+\underline{\Xi}_{2,\underline{z}z\underline{z}}(0)\,\frac{\partial}{\partial\underline{w}_2}\Big|_0+\underline{\Xi}_{3,\underline{z}z\underline{z}}(0)\,\frac{\partial}{\partial\underline{w}_3}\Big|_0\\ &= -\overline{\alpha}\,\frac{\partial}{\partial\underline{w}_1}\Big|_0-\overline{\beta}\,\frac{\partial}{\partial\underline{w}_2}\Big|_0-\overline{\gamma}\frac{\partial}{\partial\underline{w}_3}\Big|_0.\end{aligned}$$

Consequently, for these five complex vectors based at the origin to generate:

$$T_0M^{e_c}=\mathbb{C}\frac{\partial}{\partial z}\oplus\mathbb{C}\frac{\partial}{\partial\underline{z}}\oplus\mathbb{C}\frac{\partial}{\partial\underline{w}_1}\oplus\mathbb{C}\frac{\partial}{\partial\underline{w}_2}\oplus\mathbb{C}\frac{\partial}{\partial\underline{w}_3},$$

it is necessary and sufficient that the 2×2 determinant (we drop minus signs):

$$\begin{vmatrix}\beta & \gamma\\ \overline{\beta} & \overline{\gamma}\end{vmatrix}\neq 0$$

be nonzero. But things are not completely finished, for some pleasant simplifications are still available. At least, this last condition requires $\beta\neq 0$. Then we can make $\beta=2i$. Indeed, if we write $\beta=|\beta|\,e^{i\arg\beta}$, this can simply be done by just dilating z to $\lambda\,z$ with $\lambda:=i\,|\frac{2}{\beta}|^{1/3}\,e^{-i\arg\beta}$, and at the same time, we replace w_1 by $\frac{1}{\lambda\overline{\lambda}}\,w_1$ so as to keep unchanged the term $2i\,z\underline{z}$ of the first equation; in the process, α and γ change a bit, but we do not introduce a new name for these modified constants. Again, the 2×2 determinant must be nonzero. Since $\beta=2i$ is purely imaginary, this means that $\gamma=\gamma'+i\,\gamma''$ is not purely imaginary, namely that $\gamma'\neq 0$. Replacing w_3 by $w_3-\frac{\gamma''}{2}\,w_2$, one makes $\gamma''=0$, *i.e.* $\gamma=\gamma'$ with $\gamma'\neq 0$ real. But then a final dilation $z\mapsto\mu\,z$ for some appropriate $\mu\in\mathbb{C}$ makes $\gamma''=2$, while a simultaneous dilation of w_1 and of w_2 keeps unchanged the terms already simplified.

What we have seen so far can be summarized in the following basic statement.

Proposition 5.3. *Every real analytic 5-dimensional local CR-generic submanifold $M^5\subset\mathbb{C}^4$ of codimension 3 which is maximally minimal, namely satisfies:*

$D^1M=T^cM$ *has rank* 2,

$D^2M=T^cM+[T^cM,T^cM]$ *has rank* 3,

$D^3M=T^cM+[T^cM,T^cM]+\big[T^cM,[T^cM,T^cM]\big]$ *has maximal possible rank* 5,

may be represented, in suitable holomorphic coordinates (z,w_1,w_2,w_3)*, by three complex defining equations of the specific form:*

$$\text{(12)}\qquad\begin{cases}w_1-\overline{w}_1=2i\,z\overline{z}+\Pi_1\big(z,\overline{z},\overline{w}_1,\overline{w}_2,\overline{w}_3\big),\\ w_2-\overline{w}_2=2i\,z\overline{z}(z+\overline{z})+\Pi_2\big(z,\overline{z},\overline{w}_1,\overline{w}_2,\overline{w}_3\big),\\ w_3-\overline{w}_3=2\,z\overline{z}(z-\overline{z})+\Pi_3\big(z,\overline{z},\overline{w}_1,\overline{w}_2,\overline{w}_3\big),\end{cases}$$

where the three remainders Π_1, Π_2 *and* Π_3 *are all an* $\mathrm{O}(|z|^4) + z\overline{z}\,\mathrm{O}(|w|)$ *and satisfy both the reality condition* (6) *and the two normality conditions:*

$$0 \equiv \Pi_j\big(0, \overline{z}, \overline{w}_1, \overline{w}_2, \overline{w}_3\big) \equiv \Pi_j\big(z, 0, \overline{w}_1, \overline{w}_2, \overline{w}_3\big) \qquad (j = 1,\, 2,\, 3).$$

Conversely, for any choice of three such analytic functions enjoying these conditions, the zero-locus of the three equations (12) *above represents a real analytic 5-dimensional local CR-generic submanifold* $M^5 \subset \mathbb{C}^4$ *of codimension* 3 *which is maximally minimal.* □

Since it is visibly natural to assign the weights:

$$\text{weight}(z) := 1, \qquad \text{weight}(w_1) := 2, \qquad \text{weight}(w_2) := 3, \qquad \text{weight}(w_3) := 3,$$

one can write more-in-brief that these remainders are:

$$\Pi_1 = \mathrm{O}_{\text{weighted}}(4), \qquad \Pi_2 = \mathrm{O}_{\text{weighted}}(4), \qquad \Pi_3 = \mathrm{O}_{\text{weighted}}(4).$$

5.6. **Shananina's and Mamai's models.** Still in CR dimension $n = 1$, what happens for higher codimensions $d \geqslant 4$? The above class of maximally minimal CR-generic $M^5 \subset \mathbb{C}^4$ already appears among the first members of Shananina's ([70]) and Mamai's ([41]) lists. Since the principal goal of the present memoir is to apply Cartan's method of equivalence to these $M^5 \subset \mathbb{C}^4$ after having set up firm, conceptual and computational grounds, let us briefly review some of these results.

In codimension $d = 4$, after reductions and simplifications that are similar to the one explained above, one adds either an equation of the form:

$$v_4 = z^3\overline{z} + \overline{z}^3 z + c\, z^2\overline{z}^2 + \mathrm{O}_{\text{weighted}}(5),$$

where the constant $c \in \mathbb{R}$ happens to be a *nonremovable* parameter, or an equation of the form:

$$v_4 = z^2\overline{z}^2 + \mathrm{O}_{\text{weighted}}(5).$$

Open Problem 5.4. For these classes of 4-codimensional nondegenerate real analytic CR-generic submanifolds $M^6 \subset \mathbb{C}^5$ of CR dimension 1, perform Cartan's equivalence method in order to construct an absolute parallelism on certain appropriate principal bundles, and compute explicitly the related biholomorphic invariants.

We hope to come to that in a future publication, the interest being that it would be the first instance — in CR geometry — of a study of geometry-preserving deformations of models in which a non-removable (real) parameter exists ([38]).

In codimension $d = 5$, after reductions and simplifications that are similar to the ones explained above, one comes ([70]) either to:

$$\begin{cases} v_4 = z^3\overline{z} + \overline{z}^3 z + b\, z^2\overline{z}^2 + \mathrm{O}_{\mathsf{weighted}}(5), \\ v_5 = -i\, z^3\overline{z} + i\,\overline{z}^3 z + c\, z^2\overline{z}^2 + \mathrm{O}_{\mathsf{weighted}}(5), \end{cases}$$

for some two nonremovable real constants $b, c \in \mathbb{R}$, or to:

$$\begin{cases} v_4 = z^3\overline{z} + \overline{z}^3 z + \mathrm{O}_{\mathsf{weighted}}(5), \\ v_5 = z^2\overline{z}^2 + \mathrm{O}_{\mathsf{weighted}}(5). \end{cases}$$

In codimension $d = 6$, since the real vector space of real quartic polynomials in $(z, \overline{z})$ that are divisible by the product $z\overline{z}$ is 3-dimensional, generated by $\mathrm{Re}\, z^3\overline{z}$, $\mathrm{Im}\, z^3\overline{z}$ and $z^2\overline{z}^2$, easy $\mathbb{R}$-linear transformations in the (w_4, w_5, w_6)-space yield that the natural polynomial CR-generic model $M^7 \subset \mathbb{C}^6$ has equations:

$$\begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + \overline{z}^2 z, \\ v_3 = -\,i\, z^2\overline{z} + i\,\overline{z}^2 z, \end{cases} \qquad \begin{cases} v_4 = z^3\overline{z} + \overline{z}^3 z, \\ v_5 = -\,i\, z^3\overline{z} + i\,\overline{z}^3 z, \\ v_6 = z^2\overline{z}^2. \end{cases}$$

Similar, more refined and nonrigid models, exists in the higher codimensions $7 \leqslant d \leqslant 12$, *see* [70, 41] where the Lie algebra of infinitesimal CR automorphisms of the corresponding models is also presented. Of course, performing effectively Cartan's method for all these geometries would be 'fantastic'.

6. Symbol algebra and nilpotent Lie algebras up to dimension 5

6.1. **Zariski-generic invariants of completely non-holonomic complex-tangential distributions.** Now, coming back to a general real analytic CR-generic $M \subset \mathbb{C}^{n+d}$ which is connected, for every open subset $U \subset M$, denote by:

$$\Gamma\big(U, T^c M\big)$$

the $\mathscr{C}^\omega(U)$-module of sections of T^cM over U, namely complex tangent vector fields on U. Also, let $\Gamma\big(T^cM\big)$ denote the sheaf of (local) sections of T^cM.

Next, set $D^1M := \Gamma\big(T^cM\big)$, set:

$$D^2M := D^1M + \big[D^1M,\, D^1M\big]$$

(usual Lie brackets between vector fields) and inductively, define for every integer $k \geqslant 2$:

$$D^{k+1}M := D^kM + \big[D^1M,\, D^kM\big].$$

In this way, one obtains a nested family of sheaves of local sections of TM:

$$\Gamma\big(T^cM\big) = D^1M \subset D^2M \subset \cdots \subset \Gamma\big(D^{k-1}M\big) \subset \Gamma\big(D^kM\big) \subset \cdots \subset \Gamma\big(TM\big).$$

We shall assume that the complex-tangential bundle T^cM is *completely nonholonomic* in the sense that at every point $p \in M$, there exists an integer $k(p) \geqslant 1$ and there exists an open neighborhood U_p of p in M such that:

$$\Gamma\big(U_p,\ D^{k(p)}M\big) \equiv \Gamma\big(U_p, TM\big);$$

this condition means that the collection of all possible Lie brackets between complex-tangential fields up to a sufficiently high order generate the whole complex tangent bundle to M near p.

Another way of expressing this is as follows. At an arbitrary fixed point $p \in M$, introduce the vector subspaces of T_pM:

$$D^1M(p) := \big\{X_1(p) \colon\ X_1 \text{ is a local section of } D^1M \text{ near } p\big\} = T_p^cM,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D^kM(p) := \big\{X_k(p) \colon\ X_k \text{ is a local section of } D^kM \text{ near } p\big\},$$

which are of course nested:

$$D^1M(p) \subset D^2M(p) \subset \cdots \subset D^{k-1}M(p) \subset D^kM(p) \subset \cdots .$$

Then this sequence grows with k, hence if one introduces a notation for their dimensions:

$$\mathbf{n}_k(p) := D^kM(p),$$

one has the inequalities:

$$2n = \mathbf{n}_1(p) \leqslant \mathbf{n}_2(p) \leqslant \cdots \leqslant \mathbf{n}_k(p) \leqslant \mathbf{n}_{k+1}(p) \leqslant \cdots .$$

Further, the assumption that T^cM is completely nonholonomic reformulates as the existence, at every point p, of an integer $\mathsf{h}(p) \geqslant 2$ such that:

$$\mathbf{n}_{\mathsf{h}(p)}(p) = 2n + d = \dim_{\mathbb{R}} T_pM.$$

By technically analyzing the ranks in a system of local charts on M in terms of matrices written in coordinates, one may establish:

Proposition 6.1. *Under the assumption that the CR-generic submanifold $M \subset \mathbb{C}^{n+d}$ is connected and real analytic, there exists a proper exceptional real analytic subset of M, an integer $\mathsf{h} \geqslant 2$ and integers:*

$$2n = \mathbf{n}_1 < \mathbf{n}_2 < \cdots < \mathbf{n}_{\mathsf{h}} = 2n + d$$

with the properties that the degree of non-holonomy:

$$\mathsf{h}(p) = \mathsf{h}$$

is constant at every point outside the exceptional set, and that moreover the gained dimensions:

$$\mathbf{n}_1(p) = n_1, \quad \mathbf{n}_2(p) = \mathbf{n}_2, \quad \ldots\ldots, \quad \mathbf{n}_{\mathsf{h}(p)}(p) = \mathbf{n}_{\mathsf{h}},$$

are also constant at every point outside the exceptional set. □

In other words, the sheaves $D^k M$ are true *real vector subbundles* of TM. Denote by $D^k M(p) \subset T_p M$ their fibers at points $p \in M$.

By double induction on $k \geqslant 1$ and on $l \geqslant 1$, one verifies using the Jacobi identity ([74, 77]) that:

$$\big[D^k M,\, D^l M\big] \subset D^{k+l} M.$$

Also, if on an open subset $U \subset M \backslash \Sigma$, one has for a certain integer k the one-step stabilization:

$$\Gamma\big(U, D^{k+1} M\big) = \Gamma\big(U, D^k M\big),$$

then this stabilization is inherited by higher order bundles:

$$\Gamma\big(U, D^{k+l} M\big) = \Gamma\big(U, D^k M\big),$$

for every $l \geqslant 1$.

6.2. **The Tanaka symbol Lie algebra.** Now, at any point $p \in M \backslash \Sigma$, introduce for any integer $k \geqslant 2$ the quotient spaces — mind the negative lower indices —:

$$\mathfrak{g}_{-k}(p) := D^k(p) \big/ D^{k-1}(p),$$

together with the canonical projections:

$$\mathsf{proj}_k(p)\colon \quad D^k(p) \longrightarrow D^k(p) \big/ D^{k-1}(p).$$

Definition 6.2. At a Zariski-generic point $p \in M \backslash \Sigma$, the *Tanaka symbol Lie algebra* is the graded sum of the quotient vector spaces:

$$\mathfrak{m}(p) := \bigoplus_{k=1}^{k=\mathsf{h}_M} \mathfrak{g}_{-k}(p)$$

associated to the filtration of the vector subbundles over $M \backslash \Sigma$:

$$D^1 M \subset D^2 M \subset \cdots \subset D^{k-1} M \subset D^k M \subset \cdots .$$

Thanks to taking quotients, a natural Lie bracket $[\cdot, \cdot]$ exists on $\mathfrak{m}(p)$, which justifies the name '*algebra*', and it is defined as follows. Take two elements:

$$\mathsf{x} \in \mathfrak{g}_{-k}(p) \qquad \text{and} \qquad \mathsf{y} \in \mathfrak{g}_{-l}(p)$$

in two different quotients. Both have representatives:

$$\widetilde{\mathsf{x}} \in D^k M(p) \qquad \text{and} \qquad \widetilde{\mathsf{y}} \in D^l M(p),$$

which are plain tangent vectors in T_pM. Take any two local vector fields defined in some open neighborhood U_p of p in M:

$$\widetilde{X} \in \Gamma\big(U_p, D^kM(p)\big) \qquad \text{and} \qquad \widetilde{Y} \in \Gamma\big(U_p, D^lM(p)\big)$$

which 'extend' these two fixed vectors in the sense that:

$$\widetilde{X}\big|_p = \widetilde{\mathsf{x}} \qquad \text{and} \qquad \widetilde{Y}\big|_p = \widetilde{\mathsf{y}}.$$

Next, compute the usual Lie bracket between these two vector fields, and then by definition, the Lie bracket between the two original elements is its projection:

$$\begin{aligned}[\mathsf{x}, \mathsf{y}] &:= \mathsf{proj}_{k+l}(p)\Big(\underbrace{\big[\widetilde{X}, \widetilde{Y}\big]\big|_p}_{\in D^{k+l}M(p)}\Big) \\ &\in \mathfrak{g}_{-k-l}(p).\end{aligned}$$

One verifies ([74], Lemma 1.1; [77], pp. 420–421) that the result is independent of the choices made and that:

Proposition 6.3. *Endowed with this bracket operation, the Tanaka symbol Lie algebra $\mathfrak{m}(p)$ for $p \in M\backslash\Sigma$ becomes a nilpotent graded Lie algebra with:*

$$\dim_{\mathbb{R}} \mathfrak{m}(p) = \dim_{\mathbb{R}} M$$

which in addition is generated by $\mathfrak{g}_{-1}(p)$:

$$\mathfrak{g}_{-k}(p) = \big[\mathfrak{g}_{-1}(p),\ \mathfrak{g}_{-k+1}(p)\big]. \quad \square$$

Conversely ([74, 77]), it is elementary to verify that:

Theorem 6.1. *Let a nilpotent Lie algebra:*

$$\mathfrak{m} = \bigoplus_{k=1}^{k=\mu} \mathfrak{g}_{-k}$$

be graded:

$$\big[\mathfrak{g}_{-k},\ \mathfrak{g}_{-l}\big] \subset \mathfrak{g}_{-k-l} \qquad (\mathfrak{g}_{-\nu} = 0 \ \text{for} \ \nu \geqslant \mu + 1),$$

and satisfy the generating condition:

$$\mathfrak{g}_{-k-1} = \big[\mathfrak{g}_{-1},\ \mathfrak{g}_{-k}\big].$$

Then in the unique simply connected Lie group $G(\mathfrak{m})$ associated with $\mathfrak{m}$, the distribution $D^1 \subset TG(\mathfrak{m})$ corresponding to $\mathfrak{g}_{-1}$ is completely nonholonomic and its derived distributions have Lie bracket structure isomorphic to that of $\mathfrak{m}$. $\square$

Lastly, without writing out a proof which could require extended tedious technical considerations, we would like to emphasize that, on an arbitrary real analytic CR-generic submanifold $M \subset \mathbb{C}^{n+d}$, the behavior of iterated Lie brackets between sections of the complex-tangential distribution happens to be constant at a Zariski-generic point. We give a precise name Σ_{CR} to the appearing exceptional set, for a second one $\Sigma_{\sf Segre}$ will be introduced in Theorem 8.2, and from the point of view of studying the biholomorphic equivalence problem at Zariski-generic points of CR-generic real analytic submanifolds, one naturally has to avoid the *union*:

$$\Sigma_{CR} \cup \Sigma_{\sf Segre}.$$

Theorem 6.2. *Under the assumption that the CR-generic submanifold $M \subset \mathbb{C}^{n+d}$ is connected real analytic and that its complex-tangential distribution T^cM is completely non-holonomic at a Zariski-generic point, there exists a certain proper real analytic subset:*

$$\Sigma_{\sf CR} \subset M$$

such that the Tanaka symbol Lie algebras $\mathfrak{m}(p)$ of T^cM are all isomorphic — *hence have same dimensional growths — at every point:*

$$p \in M \backslash \Sigma_{CR}. \quad \square$$

For the production of model generic submanifolds which would complement Beloshapka's approach, one could classify in small dimensions those Lie algebras that are possible Tanaka symbol Lie algebras for the distribution of complex-tangential planes in a CR-generic $M \subset \mathbb{C}^{n+d}$.

A more general problem concerns the classification, up to isomorphisms, of nilpotent Lie algebras, without the generating condition. They have been classified up to dimensions 7 and 8 and we now review the concepts and the classification up to dimension 5.

6.3. **Isomorphic finite-dimensional Lie algebras.** Let $\mathfrak{g}$ be a real or complex abstract Lie algebra of finite dimension $r < \infty$, equipped with a Lie bracket operator denoted as usual by $[\cdot, \cdot]$, or sometimes by $[\cdot, \cdot]_{\mathfrak{g}}$ when a precision is needed. Two Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ of the same dimension $r = \widetilde{r}$ are said to be *isomorphic* if there is a linear isomorphism $\phi\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ which transfers properly the Lie bracket structure, namely which satisfies:

$$\phi\big([{\sf x}, {\sf y}]_{\mathfrak{g}}\big) = [\phi({\sf x}), \phi({\sf y})]_{\widetilde{\mathfrak{g}}}$$

for any two elements ${\sf x}, {\sf y} \in \mathfrak{g}$.

Following Lie (Chap. 17 in [19]), this abstract definition of isomorphism can be made more effective by introducing some two linear bases ${\sf x}_1$, ${\sf x}_2$, ..., ${\sf x}_r$ and $\widetilde{\sf x}_1, \widetilde{\sf x}_2, \dots, \widetilde{\sf x}_r$ of $\mathfrak{g}$ and of $\widetilde{\mathfrak{g}}$, so that one can write:

$$\mathfrak{g} = \mathbb{C}{\sf x}_1 \oplus \mathbb{C}{\sf x}_2 \oplus \cdots \oplus \mathbb{C}{\sf x}_r \quad \text{and} \quad \widetilde{\mathfrak{g}} = \mathbb{C}\widetilde{\sf x}_1 \oplus \mathbb{C}\widetilde{\sf x}_2 \oplus \cdots \oplus \mathbb{C}\widetilde{\sf x}_r,$$

as plain vector spaces. Then the datum of such two Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ in terms of such kinds of specific bases is clearly equivalent to the datum of the so-called *structure constants* c_{jk}^s and $\widetilde{c}_{jk}^s$ which appear in all possible brackets:

$$[\mathsf{x}_j,\,\mathsf{x}_k] = \sum_{s=1}^r c_{jk}^s\,\mathsf{x}_s \quad\text{and}\quad [\widetilde{\mathsf{x}}_j,\,\widetilde{\mathsf{x}}_k] = \sum_{s=1}^r c_{jk}^s\,\widetilde{\mathsf{x}}_s \qquad {\scriptstyle (j,\,k\,=\,1\,\cdots\,r)}.$$

With these notations, the two (arbitrary) r-dimensional Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ happen to be isomorphic if and only if one has:

$$\phi\big([\mathsf{x}_j, \mathsf{x}_k]_{\mathfrak{g}}\big) = \big([\phi(\mathsf{x}_j),\,\phi(\mathsf{x}_k)]_{\widetilde{\mathfrak{g}}}\big),$$

for any two integers $j, k = 1, 2, \dots, r$. Of course, there are constants ϕ_{jr} so that $\phi(\mathsf{x}_j) = \sum_{s=1}^r \phi_{jr}\,\widetilde{\mathsf{x}}_r$.

We evaluate firstly the left-hand side:

$$\phi\big([\mathsf{x}_j, \mathsf{x}_k]_{\mathfrak{g}}\big) = \phi\bigg(\sum_{t=1}^r c_{jk}^t\,\mathsf{x}_t\bigg) = \sum_{t=1}^r c_{jk}^t\,\phi(\mathsf{x}_t) = \sum_{s=1}^r \bigg(\underline{\sum_{t=1}^r \phi_{ts}\,c_{jk}^t}\bigg)\,\widetilde{x}_s,$$

and secondly, we do the same for the right-hand side:

$$\begin{aligned}\big([\phi(\mathsf{x}_j),\,\phi(\mathsf{x}_k)]_{\widetilde{\mathfrak{g}}}\big) &= \bigg[\sum_{l=1}^r \phi_{jl}\,\widetilde{\mathsf{x}}_l,\ \sum_{m=1}^r \phi_{km}\,\widetilde{\mathsf{x}}_m\bigg]_{\widetilde{\mathfrak{g}}} \\ &= \sum_{l=1}^r \sum_{m=1}^r \phi_{jl}\,\phi_{km}\,[\widetilde{\mathsf{x}}_l,\,\widetilde{\mathsf{x}}_m]_{\widetilde{\mathfrak{g}}} \\ &= \sum_{s=1}^r \bigg(\underline{\sum_{l=1}^r \sum_{m=1}^r \phi_{jl}\,\phi_{km}\,\widetilde{c}_{lm}^s}\bigg)\,\widetilde{\mathsf{x}}_s.\end{aligned}$$

The two terms that have been underlined therefore identify for any $j, k, s = 1, 2, \dots, r$, and we thus get the family of relations:

$$\sum_{t=1}^r \phi_{ts}\,c_{jk}^t = \sum_{l=1}^r \sum_{m=1}^r \phi_{jl}\,\phi_{km}\,\widetilde{c}_{lm}^s \qquad {\scriptstyle (j,\,k,\,s\,=\,1\,\cdots\,r)}. \tag{13}$$

In order to finish the computation, introduce the inverse matrix $\phi^{-1}(\widetilde{\mathsf{x}}_j) = \sum_{l=1}^r \phi_{jl}^{-1}\,\mathsf{x}_l$, with hence the basic, defining properties that:

$$\sum_{l=1}^r \phi_{jl}\,\phi_{lk}^{-1} = \delta_{jk} \qquad \text{and inversely:} \qquad \sum_{l=1}^r \phi_{jl}^{-1}\,\phi_{lk} = \delta_{jk},$$

for any $j, k = 1, 2, \dots, r$. Thus, we may multiply (13) by ϕ_{su}^{-1}, where u is arbitrary between 1 and r, and sum with respect to s, which yields:

$$c_{jk}^u = \sum_{t=1}^r \delta_{tu}\, c_{jk}^t = \sum_{s=1}^r \sum_{t=1}^r \phi_{ts}\, \phi_{su}^{-1}\, c_{jk}^t = \sum_{l=1}^r \sum_{m=1}^r \sum_{s=1}^r \phi_{jl}\, \phi_{km}\, \phi_{su}^{-1}\, \widetilde{c}_{lm}^s \qquad (j,\, k,\, u = 1 \cdots r).$$

This is the way how the two collection of structure constants must be related when there exists a Lie algebra isomorphism between $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ and we summarize as follows the result gained.

Proposition 6.4. ([19, 51], Chap. 17) *Consider two arbitrary real or complex Lie algebras having the same dimension r which are described by generators:*

$$\mathfrak{g} = \mathrm{Span}\big(\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_r\big) \qquad \text{and} \qquad \mathfrak{g}' = \mathrm{Span}\big(\widetilde{\mathsf{x}}_1, \widetilde{\mathsf{x}}_2, \dots, \widetilde{\mathsf{x}}_r\big)$$

and whose Lie bracket structures, in terms of their respective generators x_k and $\widetilde{\mathsf{x}}_k$, read:

$$[\mathsf{x}_j, \mathsf{x}_k] = \sum_{s=1}^r c_{jk}^s\, \mathsf{x}_s \qquad \text{and} \qquad [\widetilde{\mathsf{x}}_j, \widetilde{\mathsf{x}}_k] = \sum_{s=1}^r \widetilde{c}_{jk}^s\, \widetilde{\mathsf{x}}_s,$$

where the c_{jk}^s and the $\widetilde{c}_{jk}^s$ are constants subjected to skew symmetry and to Jacobi identity:

$$\begin{cases} 0 = c_{jk}^s + c_{kj}^s, \\ 0 = \sum_{s=1}^r \big(c_{kl}^s\, c_{js}^m + c_{jk}^s\, c_{ls}^m + c_{lj}^s\, c_{ks}^m\big) \end{cases} \qquad \text{and} \qquad \begin{cases} 0 = \widetilde{c}_{jk}^s + \widetilde{c}_{kj}^s, \\ 0 = \sum_{s=1}^r \big(\widetilde{c}_{kl}^s\, \widetilde{c}_{js}^m + \widetilde{c}_{jk}^s\, \widetilde{c}_{ls}^m + \widetilde{c}_{lj}^s\, \widetilde{c}_{ks}^m\big). \end{cases}$$

Then $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are isomorphic as Lie algebras if and only it is possible to find a system of r^2 real or complex numbers ϕ_{lm} with nonzero $\det(\phi_{lm})_{1 \leqslant l \leqslant r}^{1 \leqslant m \leqslant r} \neq 0$ such that the two collections of constant structures exchange through the following formulas:

$$c_{jk}^s = \sum_{l=1}^r \sum_{m=1}^r \sum_{t=1}^r \phi_{jl}\, \phi_{km}\, \phi_{ts}^{-1}\, \widetilde{c}_{lm}^t,$$

where j, k, s are arbitrary integers between 1 and r, or equivalently, through the inverse formulas:

$$\widetilde{c}_{jk}^s = \sum_{l=1}^r \sum_{m=1}^r \sum_{t=1}^r \phi_{jl}^{-1}\, \phi_{km}^{-1}\, \phi_{ts}\, c_{lm}^t.$$

6.4. **Decreasing weak derived sequence.** Again, let $\mathfrak{g}$ be a real or complex abstract Lie algebra of finite dimension $r < \infty$, equipped with a Lie bracket operator denoted as usual by $[\cdot, \cdot]$. Introduce the following sequence of subspaces of $\mathfrak{g}$:

$$\begin{cases} \mathscr{N}^{-1}(\mathfrak{g}) := \mathfrak{g}, \\ \mathscr{N}^{-2}(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}] = \big[\mathfrak{g}, \mathscr{N}^{-1}(\mathfrak{g})\big] \\ \mathscr{N}^{-k-1}(\mathfrak{g}) := \big[\mathfrak{g}, \mathscr{N}^{-k}(\mathfrak{g})\big] \qquad \text{for any } k \geqslant 2. \end{cases}$$

Then the arising *weak derived sequence*[1]:

$$\cdots \subset \mathscr{N}^{-k-1}(\mathfrak{g}) \subset \mathscr{N}^{-k}(\mathfrak{g}) \subset \cdots \subset \mathscr{N}^{-2}(\mathfrak{g}) \subset \mathscr{N}^{-1}(\mathfrak{g}) = \mathfrak{g}$$

is constituted only of *ideals* of $\mathfrak{g}$, namely of subalgebras $\mathfrak{n}$ of $\mathfrak{g}$ satisfying $[\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{n}$, as may be verified by induction using the Jacobi identity. Also, because $\mathfrak{g}$ is finite-dimensional, this sequence must stabilize after some steps, that is to say, there must exist an integer $k \geqslant 1$ such that $\mathscr{N}^{-k-1}(\mathfrak{g}) = \mathscr{N}^{-k}(\mathfrak{g})$.

Definition 6.5. A real or complex Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if its weak derived sequences ends up to zero, namely if there is an integer $-k$ such that:

$$\mathscr{N}^{-k}(\mathfrak{g}) = \{0\}.$$

When this occurs, the smallest integer $\mu + 1$ such that $\mathscr{N}^{-\mu-1}(\mathfrak{g}) = \{0\}$ is called the *nilindex* of $\mathfrak{g}$, while the integer μ, the largest such that $\mathscr{N}^{-\mu}(\mathfrak{g}) \neq 0$, is called the *kind* of $\mathfrak{g}$ (*cf.* [74]).

The nilindex of a nilpotent Lie algebra $\mathfrak{g}$ is of course $\leqslant \dim \mathfrak{g} + 1$.

As a basic example, a Lie algebra is Abelian, namely $[\mathfrak{g}, \mathfrak{g}] = 0$, if and only if it is nilpotent with nilindex smallest possible, equal to 2. Another example, in dimension $r = 3$, is the Heisenberg Lie algebra $\mathrm{Span}_{\mathbb{C}}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3)$, having structure:

$$[\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3, \qquad [\mathsf{x}_1, \mathsf{x}_3] = 0, \qquad [\mathsf{x}_2, \mathsf{x}_3] = 0.$$

A fundamental tool for the classification of nilpotent Lie algebras is the so-called classical *theorem of Engel* ([34]), which states that a finite-dimensional $\mathfrak{g}$ is nilpotent if and only if, for every $\mathsf{x} \in \mathfrak{g}$, the associated adjoint endomorphism:

$$\begin{aligned} \mathrm{ad}(\mathsf{x})\colon \quad \mathfrak{g} &\longrightarrow \mathfrak{g} \\ \mathsf{y} &\longmapsto \big[\mathsf{x}, \mathsf{y}\big] \end{aligned}$$

[1] It should be distinguished from the *strong derived sequence* of iterated commutators starting also with $\mathscr{R}^{-2}(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$ but continuing with $\mathscr{R}^{-k-1}(\mathfrak{g}) := \big[\mathscr{R}^{-k}(\mathfrak{g}), \mathscr{R}^{-k}(\mathfrak{g})\big]$, so that $\mathscr{R}^{-k}(\mathfrak{g}) \subset \mathscr{N}^{-k}(\mathfrak{g})$ for any $k \geqslant -1$.

is nilpotent in $\mathrm{End}(\mathfrak{g})$, namely $\mathrm{ad}(\mathsf{x})^{\circ s} \equiv 0$ for all s large enough. It is well known, then, that in fact $\mathrm{ad}(\mathsf{x})^{\circ \dim \mathfrak{g}} \equiv 0$, uniformly for all x belonging to the nilpotent $\mathfrak{g}$. But since $\mathrm{ad}(\mathsf{x})(\mathsf{x}) = [\mathsf{x}, \mathsf{x}] = 0$, which shows that x is an eigenvector with zero eigenvalue, in the Jordan bloc decomposition by invariant linear subspaces, the largest dimension of an invariant subspace on which $\mathrm{ad}(\mathsf{x})$ is not identically zero is in any case $\leqslant \dim \mathfrak{g} - 1$. It follows that:

$$\mathrm{ad}(\mathsf{x})^{\circ(\dim \mathfrak{g} - 1)} \equiv 0 \qquad\qquad (\mathsf{x} \in \mathfrak{g}).$$

Definition 6.6. If $\mathfrak{g}$ is nilpotent, the smallest integer $s \leqslant \dim \mathfrak{g} - 1$ such that $\mathrm{ad}(\mathsf{x})^{\circ s} \equiv 0$ for all $\mathsf{x} \in \mathfrak{g}$ is called the *nilpotency order* of $\mathfrak{g}$. A nilpotent complex Lie algebra of dimension r is said to be *filiform* if its nilpotency order equals $\dim \mathfrak{g} - 1$.

For instance, the four-dimensional Lie algebra $\mathrm{Span}_{\mathbb{C}}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4)$ having structure:

$$[\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3, \qquad\qquad [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4$$

is filiform: look at $\mathrm{ad}(\mathsf{x}_1)$.

Definition 6.7. Let $\mathfrak{g}$ be a nilpotent complex Lie algebra of dimension n. For every $\mathsf{x} \in \mathfrak{g}$, let $c(\mathsf{x})$ be the decreasing sequence of the dimensions of Jordan blocs of the nilpotent endomorphism $\mathrm{ad}(\mathsf{x})$. The *characteristic sequence* of $\mathfrak{g}$ (Goze's invariant) is the sequence:

$$c(\mathfrak{g}) := \max \big\{ c(\mathsf{x}) \colon \; \mathsf{x} \in \mathfrak{g} \backslash [\mathfrak{g}, \mathfrak{g}] \big\}.$$

The characteristic sequence then obviously constitutes a *partition* of r and will be written as $c(\mathfrak{g}) = (\ell_1, \ell_2, \dots, \ell_r)$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_r \geqslant 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = r$. A nilpotent Lie algebra $\mathfrak{g}$ is filiform if and only if $c(\mathfrak{g}) = (r - 1, 1, 0, \dots, 0)$.

6.5. **Classification of complex nilpotent Lie algebras up to dimension five.** In the next paragraphs, we follow Goze's classification results [29, 30] closely. We now consider an arbitrary r-dimensional nilpotent *complex* Lie algebra $\mathfrak{g}$ of dimension $r \leqslant 5$ with generators $\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_r$.

Dimension 1: There exists only one nilpotent Lie algebra of dimension 1: the Abelian Lie algebra, and we will denote it by: $\mathfrak{a}_1$.

Dimension 2: In dimension 2, there is no indecomposable complex nilpotent Lie algebra, only $\mathfrak{a}_2 := \mathfrak{a}_1 \oplus \mathfrak{a}_1$ exists.

Dimension 3: In dimension 3, leaving aside $\mathfrak{a}_3 := \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$, there exists a single indecomposable complex nilpotent Lie algebra, the Heisenberg Lie algebra:

$$\boxed{\mathfrak{n}_3^1 \colon \qquad [\mathsf{x}_1, \, \mathsf{x}_2] = \mathsf{x}_3}\,.$$

By convention here, brackets that are not written are implicitly assumed to be zero. We shall observe later that $\mathfrak{n}_1^3$ is the Tanaka symbol algebra of any Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$.

Dimension 4: In dimension 4, leaving aside the two decomposable nilpotent complex Lie algebras,

$$\mathfrak{a}_4 := \mathfrak{a}_1^{\oplus 4} \qquad \text{and} \qquad \mathfrak{a}_1 \oplus \mathfrak{n}_1^3,$$

there again exists only a single indecomposable complex nilpotent Lie algebra, whose structure is:

$$\boxed{\mathfrak{n}_4^1 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4. \end{cases}}$$

Dimension 5: Next, in dimension $r = 5$, leaving aside the three decomposable nilpotent complex Lie algebras:

$$\mathfrak{a}_5 := \mathfrak{a}_1^{\oplus 5}, \qquad \mathfrak{n}_3^1 \oplus \mathfrak{a}_2 \qquad \text{and} \qquad \mathfrak{n}_4^1 \oplus \mathfrak{a}_1,$$

there exist six mutually nonisomorphic nilpotent complex Lie algebras that are gathered as follows according to their respective Goze invariants.

☐ $c(\mathfrak{g}) = (4,1)$ (filiform case):

$$\boxed{\mathfrak{n}_5^1 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4 \\ [\mathsf{x}_1, \mathsf{x}_4] = \mathsf{x}_5 \end{cases}} \qquad \boxed{\mathfrak{n}_5^2 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4 \\ [\mathsf{x}_1, \mathsf{x}_4] = \mathsf{x}_5 \\ [\mathsf{x}_2, \mathsf{x}_3] = \mathsf{x}_5 \end{cases}}$$

☐ $c(\mathfrak{g}) = (3,1,1)$:

$$\boxed{\mathfrak{n}_5^3 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4 \\ [\mathsf{x}_2, \mathsf{x}_5] = \mathsf{x}_4 \end{cases}} \qquad \boxed{\mathfrak{n}_5^4 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4 \\ [\mathsf{x}_2, \mathsf{x}_3] = \mathsf{x}_5 \end{cases}}$$

☐ $c(\mathfrak{g}) = (2,2,1)$:

$$\boxed{\mathfrak{n}_5^5 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_4] = \mathsf{x}_5 \end{cases}}$$

☐ $c(\mathfrak{g}) = (2,1,1,1)$:

$$\boxed{\mathfrak{n}_5^5 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_4, \mathsf{x}_5] = \mathsf{x}_3 \end{cases}}$$

6.6. **Graded nilpotent Lie algebras.** On p. 11 of [74], one finds up to dimension 5 the possible dimensional growths of graded nilpotent Lie algebras which come from the Tanaka symbol of a 2-dimensional distribution, but without the Lie bracket structure. In fact, the corresponding structures may be read off from the above list:

□ in dimension 3, with dimensional growth $(2, 1)$, one finds $\mathfrak{n}_3^1$;

□ in dimension 4, with dimensional growth $(2, 1, 1)$, one finds $\mathfrak{n}_4^1$;

□ in dimension 5, with dimensional growth $(2, 1, 2)$, one finds $\mathfrak{n}_5^4$ which corresponds to our cubic $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$;

□ in dimension 5, with growth vector $(2, 1, 1, 1)$, one finds $\mathfrak{n}_5^1$, and this would correspond to a yet unstudied class of CR-generic submanifolds $M^5 \subset \mathbb{C}^4$.

Open Problem 6.8. Classify, in small dimensions, real Lie algebras:

$$\mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

that are graded, nilpotent and satisfy the generating condition:

$$\mathfrak{g}_{-k-1} = \big[\mathfrak{g}_{-1},\, \mathfrak{g}_{-k}\big].$$

Apply the gained classifications to coordinate-independent production of model CR-generic submanifolds of complex Euclidean spaces.

7. Infinitesimal CR automorphisms: $\mathfrak{aut}_{CR}(M) = \mathrm{Re}(\mathfrak{hol}(M))$

7.1. **Extrinsic holomorphic definition.** According to a standard, important definition ([72, 4, 7]), a *(local) infinitesimal CR-automorphism of* M is a $(1, 0)$ vector field having *holomorphic* coefficients:

$$\mathsf{X} = \sum_{k=1}^{n} Z^k(z, w)\frac{\partial}{\partial z_k} + \sum_{j=1}^{d} W^j(z, w)\frac{\partial}{\partial w_j} \tag{14}$$

the real part of which:

$$\mathrm{Re}\, X = \tfrac{1}{2}\big(\mathsf{X} + \overline{\mathsf{X}}\big)$$

which is tangent to M. Importantly, one should notice here that, contrary to the $(1, 0)$ generators $\mathscr{L}_k$ of $T^{1,0}M$, such an X is supposed to have purely holomorphic coefficients, whereas the coefficients $\frac{\partial \Theta_j}{\partial z_k}(z, \overline{z}, \overline{w})$ of the $\mathscr{L}_k$ are — most often — neither purely holomorphic, nor purely antiholomorphic, but only real analytic.

This condition of tangency, much studied by Beloshapka and his school, will be explored in depth below because knowing all such X is the same as knowing the *CR symmetries* of M, and this knowledge lies in the heart of the problem of classifying local analytic CR manifolds up to biholomorphisms.

By integration, the *real* flow:

$$(t, z, w) \longmapsto \exp(t\,\mathsf{X})(z, w) \qquad\qquad (t \in \mathbb{R} \text{ small})$$

constitutes a local one-parameter group of local biholomorphisms of $\mathbb{C}^n$, and because X is tangent to M, this flow leaves M invariant, that is to say: through this flow, points of M are transferred to points of M (more details may be found in [72]). We note *passim* that this real flow coincides with restricting the consideration of the *complex* (holomorphic) flow:

$$(\tau, z, w) \longmapsto \exp(\tau\,\mathsf{X})(z, w) \qquad\qquad (\tau \in \mathbb{C} \text{ small})$$

to a *real* time parameter $\tau := t \in \mathbb{R}$. Conversely, one may show:

Lemma 7.1. *If $M \subset \mathbb{C}^{n+d}$ is a CR-generic submanifold and if $(z, w) \longmapsto \phi_t(z, w)$ is a local one-parameter group of* holomorphic *self-transformations of $\mathbb{C}^{n+d}$ which stabilizes M locally, then the vector field:*

$$\tfrac{d}{dt}\Big|_0 \big(\phi_t(z, w)\big)$$

has holomorphic coefficients and its real part is tangent to M. □

From fundamental facts of of Lie theory, if $\mathfrak{hol}(M)$ is finite-dimensional, then necessarily, it constitutes a *real* Lie algebra, namely if $\mathsf{X}_1, \dots, \mathsf{X}_r$ denote any basis of $\mathfrak{hol}(M)$, there are *real* structure constants $c_{jk}^s \in \mathbb{R}$ such that:

$$\big[\mathsf{X}_j, \mathsf{X}_k\big] = \sum_{s=1}^{r} c_{jk}^s\, \mathsf{X}_s. \tag{15}$$

For an explicitly given $M \subset \mathbb{C}^{n+d}$, determining a basis of the Lie algebra $\mathfrak{hol}(M)$ is a natural problem for which some systematic procedures exist (*see* below). The groundbreaking works of Sophus Lie and his collaborators, Friedrich Engel, Georg Scheffers and others showed that the most fundamental question in concern here is to draw lists of possible Lie algebras $\mathfrak{hol}(M)$ which would classify possible M's according to their CR symmetries, cf. [19, 21, 51, 50].

7.2. **Intrinsic CR definition.** On the other hand, if one prefers to view the CR-generic manifold M in a purely intrinsic way, one may consider the local group $\mathrm{Aut}_{CR}(M)$ of automorphisms of the CR structure, namely of local $\mathscr{C}^\infty$ diffeomorphisms $g \colon M \to M$ (close to the identity mapping) which satisfy:

$$dg_p(T_p^c M) = T_{g(p)}^c M \qquad \text{and} \qquad dg_p\big(J(\mathsf{v}_p)\big) = J_{g(p)}\big(dg_p(\mathsf{v}_p)\big)$$

at any point $p \in M$ and for any vector $\mathsf{v}_p \in T_p^c M$. In other words, g belongs to $\mathrm{Aut}_{CR}(M)$ if and only if it is a (local) CR-diffeomorphism of M, namely a diffeomorphism which respects the (intrinsic) CR structure of M.

As did Lie most of the time in his original theory ([19, 21]), we shall consider only a neighborhood of the identity mapping, hence all our groups will be *local Lie groups*; the reader is referred to [61, 51] for fundamentals about local Lie groups in general, especially concerning the fact that it is essentially useless to point out open sets and domains in which mappings and transformations are defined, some superfluous details we shall dispense ourselves with.

Accordingly, let:

$$\mathfrak{aut}_{CR}(M)$$

denote the collection of all (real) vector fields Y on M the flow of which $(t, p) \mapsto \exp(t\mathsf{Y})(p)$ becomes a local CR diffeomorphism of M. When $\mathrm{Aut}_{CR}(M)$ is a finite-dimensional Lie group, $\mathfrak{aut}_{CR}(M)$ is just its Lie algebra. The principles and the proof of the following assertion date back to Sophus Lie's monographs.

Lemma 7.2. ([19], Chap. 8) *A local real analytic vector field* Y *on* M *belongs to* $\mathfrak{aut}_{CR}(M)$*, if and only if for every local section* L *of the complex tangent bundle* T^cM*, the Lie bracket* $[\mathsf{Y},\, L]$ *is again a section of* T^cM*.* □

7.3. **Coincidence between extrinsic and intrinsic CR automorphisms.** In all cases which are of interest, namely when M is nondegenerate in a sense that we will make precise just later, such real analytic flows $(t, p) \mapsto \exp(t\mathsf{Y})(p)$ happen to extend as local *biholomorphic* maps from a neighborhood of M in $\mathbb{C}^{n+d}$. In all these cases which cover a broad universe of yet unstudied CR structures, one has the fundamental relation:

$$\boxed{\mathfrak{aut}_{CR}(M) = \mathrm{Re}\big(\mathfrak{hol}(M)\big)},$$

where both sides are finite-dimensional, spanned by vector fields whose coefficients are expandable in converging power series. Thus, one may work exclusively with the *holomorphic* vector fields generating $\mathfrak{hol}(M)$, as we will do from now on. And in any case, there will be no confusion to call an *infinitesimal CR automorphism* either the holomorphic vector field $\mathsf{X} \in \mathfrak{hol}(M)$ or its real part $\frac{1}{2}(\mathsf{X} + \overline{\mathsf{X}}) \in \mathfrak{aut}_{CR}(M)$.

Since holomorphic vector fields obviously commute with antiholomorphic vector fields, we deduce from (15) that when $\mathfrak{hol}(M) = \mathbb{R}\mathsf{X}_1 \oplus \cdots \oplus \mathbb{R}\mathsf{X}_r$ is r-dimensional, the real parts of the X_j which generate $\mathfrak{aut}_{CR}(M)$ simply have the same (real) structure constants:

$$\begin{aligned} \big[\mathsf{X}_j + \overline{\mathsf{X}}_j,\, \mathsf{X}_k + \overline{\mathsf{X}}_k\big] &= \big[\mathsf{X}_j,\, \mathsf{X}_k\big] + \big[\overline{\mathsf{X}}_j,\, \overline{\mathsf{X}}_k\big] \\ &= \sum_{s=1}^{r} c_{jk}^s \left(\mathsf{X}_s + \overline{\mathsf{X}}_s\right). \end{aligned} \tag{16}$$

7.4. **Isotropy Lie subalgebras.** At a fixed point $p \in M$, one may consider the Lie subalgebras $\mathfrak{hol}(M,p)$ of $\mathfrak{hol}(M)$ and $\mathfrak{aut}_{CR}(M,p)$ of $\mathfrak{aut}_{CR}(M)$ consisting of vector fields whose values vanish at p. These are the Lie algebra of the subgroups $\mathrm{Hol}(M,p)$ of $\mathrm{Hol}(M)$ and $\mathrm{Aut}_{CR}(M,p)$ of $\mathrm{Aut}_{CR}(M)$ consisting of maps that fix the point p. One has $\mathfrak{aut}_{CR}(M,p) = \mathrm{Re}\big(\mathsf{hol}(M,p)\big)$.

7.5. **Extrinsic complexification.** As is known in local CR geometry, it is natural to introduce new independent complex variables $(\underline{z}, \underline{w}) \in \mathbb{C}^n \times \mathbb{C}^d$ — underlining here should *not* be confused with complex conjugating — and to define the so-called *extrinsic complexification* M^{e_c} of M as being the *holomorphic* d-codimensional submanifold of $\mathbb{C}^{n+d} \times \mathbb{C}^{n+d}$ equipped with the $2n + 2d$ coordinates $(z, w, \underline{z}, \underline{w})$ which is defined by the d holomorphic equations:

$$w_j = \Theta_j\big(z, \underline{z}, \underline{w}\big) \qquad\qquad {\scriptstyle (j = 1 \cdots d)}.$$

Notice that the replacement of $(\overline{z}, \overline{w})$ by $(\underline{z}, \underline{w})$ in the Taylor series of Θ is really meaningful:

$$\Theta_j(z, \underline{z}, \underline{w}) := \sum_{\alpha \in \mathbb{N}^n,\, \beta \in \mathbb{N}^n,\, \gamma \in \mathbb{N}^d} \Theta_{j,\alpha,\beta,\gamma}\, z^\alpha\, \underline{z}^\beta\, \underline{w}^\gamma,$$

thanks to the fact that the series in question converges locally.

Equivalently, M^{e_c} is defined by the d equations:

$$\underline{w}_j = \overline{\Theta}_j\big(\underline{z}, z, w\big) \qquad\qquad {\scriptstyle (j = 1 \cdots d)}.$$

Then M is recovered from M^{e_c} by just replacing these independent variables $(\underline{z}, \underline{w})$ by the original conjugates $(\overline{z}, \overline{w})$.

Of course, the extrinsic complexifications of the $(1,0)$ and of the $(0,1)$ tangent vector fields are:

$$\mathscr{L}_k^{e_c} = \frac{\partial}{\partial z_k} + \sum_{j=1}^d \frac{\partial \Theta_j}{\partial z_k}(z, \underline{z}, \underline{w})\, \frac{\partial}{\partial w_j} \qquad\qquad {\scriptstyle (k = 1 \cdots n)},$$

$$\underline{\mathscr{L}}_k^{e_c} = \frac{\partial}{\partial \underline{z}_k} + \sum_{j=1}^d \frac{\partial \overline{\Theta}_j}{\partial \underline{z}_k}(\underline{z}, z, w)\, \frac{\partial}{\partial \underline{w}_j} \qquad\qquad {\scriptstyle (k = 1 \cdots n)}.$$

Lastly, we shall constantly use the following standard uniqueness principle.

Lemma 7.3. *With a CR-generic real analytic $M \subset \mathbb{C}^{n+d}$ as above, consider a complex-valued converging power series:*

$$\Phi = \Phi\big(z, w, \overline{z}, \overline{w}\big) = \sum_{\alpha \in \mathbb{N}^n,\, \beta \in \mathbb{N}^d,\, \gamma \in \mathbb{N}^n,\, \delta \in \mathbb{N}^d} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \overline{z}^\gamma\, \overline{w}^\delta$$

in $\mathbb{C}\{z, w, \overline{z}, \overline{w}\}$ *having complex coefficients* $\Phi_{\alpha,\beta,\gamma,\delta} \in \mathbb{C}$*. Then the following four properties are equivalent:*

- Φ *takes only the value zero when the point* (z, w) *varies on* $M \subset \mathbb{C}^n$*;*
- *the extrinsic complexification of* Φ*:*

$$\Phi^{e_c} = \Phi^{e_c}\big(z, w, \underline{z}, \underline{w}\big) := \sum_{\alpha,\,\beta,\,\gamma,\,\delta} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \underline{z}^\gamma\, \underline{w}^\delta$$

  takes only the value zero when the point $(z, w, \underline{z}, \underline{w})$ *varies on* $M^{e_c} \subset \mathbb{C}^{2n+2d}$*;*
- *after replacing* $\underline{w}$ *by* $\overline{\Theta}(\underline{z}, z, w)$ *in the extrinsic complexification* Φ^{e_c} *of* Φ*, the result is an identically zero series in* $\mathbb{C}\{\underline{z}, z, w\}$*:*

$$0 \equiv \sum_{\alpha,\,\beta,\,\gamma,\,\delta} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, w^\beta\, \underline{z}^\gamma\, \big[\overline{\Theta}(\underline{z}, z, w)\big]^\delta;$$

- *after replacing* w *by* $\Theta\big(z, \underline{z}, \underline{w}\big)$ *in the extrinsic complexification* Φ^{e_c}*, the result is an identically zero power series in* $\mathbb{C}\{z, \underline{z}, \underline{w}\}^d$*, namely:*

$$0 \equiv \sum_{\alpha,\,\beta,\,\gamma,\,\delta} \Phi_{\alpha,\beta,\gamma,\delta}\, z^\alpha\, \big[\Theta(z, \underline{z}, \underline{w})\big]^\beta\, \underline{z}^\gamma\, \underline{w}^\delta. \quad \square$$

7.6. **Tangency equations for the determination of $\mathfrak{aut}_{CR}(M)$.** In order to compute $\mathfrak{hol}(M)$ for an explicitly given CR-generic submanifold $M \subset \mathbb{C}^{n+d}$, it is most convenient to work with the extrinsic complexification of its complex defining equations:

$$w_j = \Theta_j(z, \underline{z}, \underline{w}) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}. \tag{17}$$

We will assume that M is *rigid*, in the sense that its real defining equations:

$$v_j = \varphi_j(x, y) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}$$

do *not* depend upon the variables $u = (u_1, \ldots, u_d)$. Two justifications of this simplification are: 1) the explicit computations presented in Section 10 concern our cubic model $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$, and this model is rigid; 2) the presentation of general formulas for the determination of infinitesimal CR automorphisms with non necessarily rigid CR-generic real analytic $M \subset \mathbb{C}^{n+d}$ has already been made in [1].

So let M be rigid and write the defining equations of its extrinsic complexification M^{e_c} as:

$$w_j = \underline{w}_j + 2i\, \Phi_j\big(z, \underline{z}\big) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

with the slight change of notation:

$$\Phi_j\big(z,\overline{z}\big) \equiv \varphi_j(x,y).$$

The extrinsic complexification:

$$\mathsf{X}+\underline{\mathsf{X}} = \sum_{k=1}^n Z^k(z,w)\,\frac{\partial}{\partial z_k}+\sum_{j=1}^d W^j(z,w)\,\frac{\partial}{\partial w_j}+\sum_{k=1}^n \overline{Z}^k\big(\underline{z},\underline{w}\big)\,\frac{\partial}{\partial \underline{z}_k}+\sum_{l=1}^d \overline{W}^l\big(\underline{z},\underline{w}\big)\,\frac{\partial}{\partial \underline{w}_l}$$

of (twice) the real part $X+\overline{X}$ of a $(1,0)$ vector field having holomorphic coefficients:

$$\mathsf{X} = \sum_{k=1}^n Z^k(z,w)\,\frac{\partial}{\partial z_k}+\sum_{j=1}^d W^j(z,w)\,\frac{\partial}{\partial w_j}$$

is tangent to M^{e_c} if and only if it annihilates its equations *identically on restriction to them* (an application of the uniqueness Lemma 7.3 is required to pass from M to M^{e_c}):

$$0 \equiv \big(\mathsf{X}+\underline{\mathsf{X}}\big)\Big[w_j-\underline{w}_j-2i\,\Phi_j\big(z,\underline{z}\big)\Big]\Big|_{M^{e_c}} \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.$$

Since restricting to M^{e_c} simply means replacing w by $\underline{w}+2i\,\Phi(z,\underline{z})$, these equations write out in greater length:

$$0 \equiv \Big[W^j(z,w)-2i\sum_{k=1}^n Z^k(z,w)\,\frac{\partial \Phi_j}{\partial z_k}\big(z,\underline{z}\big)-$$
$$-\overline{W}^j\big(\underline{z},\underline{w}\big)-2i\sum_{k=1}^n \overline{Z}^k\big(\underline{z},\underline{w}\big)\,\frac{\partial \Phi_j}{\partial \underline{z}_k}\big(z,\underline{z}\big)\Big]\Big|_{w=\underline{w}+2i\,\Phi(z,\underline{z})} \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},$$

that is to say, after really performing the mentioned replacement of w:

$$0 \equiv W^j\big(z,\underline{w}+2i\,\Phi(z,\underline{z})\big)-2i\sum_{k=1}^n Z^k\big(z,\underline{w}+2i\,\Phi(z,\underline{z})\big)\,\frac{\partial \Phi_j}{\partial z_k}\big(z,\underline{z}\big)-$$
$$-\overline{W}^j\big(\underline{z},\underline{w}\big)-2i\sum_{k=1}^n \overline{Z}^k\big(\underline{z},\underline{w}\big)\,\frac{\partial \Phi_j}{\partial \underline{z}_k}\big(z,\underline{z}\big) \qquad\qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.$$

Now, if we expand in partial Taylor series with respect to z and to $\underline{z}$ all the coefficients of X and $\underline{\mathsf{X}}$:

$$\left[\begin{aligned} Z^k(z,w) &= \sum_{\alpha\in\mathbb{N}^n} z^\alpha\,Z^{k,\alpha}(w), \\ W^j(z,w) &= \sum_{\alpha\in\mathbb{N}^n} z^\alpha\,W^{j,\alpha}(w), \end{aligned}\right. \qquad\qquad \left[\begin{aligned} \overline{Z}^k\big(\underline{z},\underline{w}\big) &= \sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\overline{Z}^{k,\beta}\big(\underline{w}\big), \\ \overline{W}^j\big(\underline{z},\underline{w}\big) &= \sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\overline{W}^{j,\beta}\big(\underline{w}\big), \end{aligned}\right.$$

the obtained equations rewrite under the form:

$$\begin{aligned}
0 \equiv \sum_{\alpha\in\mathbb{N}^n} & z^\alpha\, W^{j,\alpha}\big(\underline{w} + 2i\,\Phi(z,\underline{z})\big) - \\
& - 2i\sum_{k=1}^n \sum_{\alpha\in\mathbb{N}^n} z^\alpha\, Z^{k,\alpha}\big(\underline{w} + 2i\,\Phi(z,\underline{z})\big)\,\frac{\partial\Phi_j}{\partial z_k}\big(z,\underline{z}\big) - \\
& - \sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\overline{W}^{j,\beta}\big(\underline{w}\big) - \\
& - 2i\sum_{k=1}^n \sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\overline{Z}^{k,\beta}\big(\underline{w}\big)\,\frac{\partial\Phi_j}{\partial \underline{z}_k}\big(z,\underline{z}\big) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.
\end{aligned}$$

Next, applying the general Taylor expansion:

$$\Lambda\big(\underline{w} + 2i\,\Phi(z,\underline{z})\big) = \sum_{\gamma\in\mathbb{N}^d} \frac{\partial^{|\gamma|}\Lambda}{\partial\underline{w}^\gamma}\,\frac{1}{\gamma!}\,\big(2i\,\Phi(z,\underline{z})\big)^\gamma,$$

we continue to expand as follows our d equations:

$$\begin{aligned}
0 \equiv \sum_{\alpha\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d} & \frac{1}{\gamma!}\,z^\alpha\,\big(2i\,\Phi(z,\underline{z})\big)^\gamma\,\frac{\partial^{|\gamma|}W^{j,\alpha}}{\partial\underline{w}^\gamma}\big(\underline{w}\big) - \\
& - 2i\sum_{k=1}^n\sum_{\alpha\in\mathbb{N}^n}\sum_{\gamma\in\mathbb{N}^d} \frac{1}{\gamma!}\,z^\alpha\,\big(2i\,\Phi(z,\underline{z})\big)^\gamma\,\frac{\partial\Phi_j}{\partial z_k}\big(z,\underline{z}\big)\,\frac{\partial^{|\gamma|}Z^{k,\alpha}}{\partial\underline{w}^\gamma}\big(\underline{w}\big) - \\
& - \sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\overline{W}^{j,\beta}\big(\underline{w}\big) - \\
& - 2i\sum_{k=1}^n\sum_{\beta\in\mathbb{N}^n} \underline{z}^\beta\,\frac{\partial\Phi_j}{\partial\underline{z}_k}\big(z,\underline{z}\big)\,\overline{Z}^{k,\beta}\big(\underline{w}\big) \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)}.
\end{aligned}$$

After performing a complete expansion in powers of $\big(z,\overline{z}\big)$, we obtain a certain family of linear expressions in the partial derivatives of the unknown sub-coefficients $Z^{k,\alpha}(w)$, $W^{j,\alpha}(w)$ of the infinitesimal CR automorphism X — and of their complexifications as well $\overline{Z}^{k,\beta}\big(\underline{w}\big)$, $\overline{W}^{j,\beta}\big(\underline{w}\big)$ —:

$$\begin{aligned}
0 \equiv \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} & z^\alpha\underline{z}^\beta\cdot \\
\cdot\,\text{Linear-Expression}_{j,\alpha,\beta} & \left(\frac{\partial^{|\gamma'|}W^{j',\alpha'}}{\partial\underline{w}^{\gamma'}}\big(\underline{w}\big),\ \ \frac{\partial^{|\gamma'|}Z^{k',\alpha'}}{\partial\underline{w}^{\gamma'}}\big(\underline{w}\big),\ \ \overline{W}^{j',\alpha'}\big(\underline{w}\big),\ \ \overline{Z}^{k',\beta'}\big(\underline{w}\big)\right) \\
& \qquad {\scriptstyle (j\,=\,1\,\cdots\,d)},
\end{aligned}$$

the coefficients of these linear expressions being certain functions of $\underline{w}$ which depend exclusively upon the defining equation of M^{e_c}; more advanced information on the latter coefficients is provided in [1] in the general (non-rigid) case.

In summary, by identifying to zero the coefficients of all the monomials $z^\alpha \overline{z}^\beta$ in the above identically satisfied d equations, we have proved that X is an infinitesimal CR automorphism of a rigid real analytic CR-generic submanifold if and only some (infinite) system of linear partial differential equations is satisfied:

$$0 = \text{Linear-Expression}_{j,\alpha,\beta}\bigg(\frac{\partial^{|\gamma'|}W^{j',\alpha'}}{\partial \underline{w}^{\gamma'}}\big(\underline{w}\big),\ \ \frac{\partial^{|\gamma'|}Z^{k',\alpha'}}{\partial \underline{w}^{\gamma'}}\big(\underline{w}\big),\ \ \overline{W}^{j',\alpha'}\big(\underline{w}\big),\ \ \overline{Z}^{k',\beta'}\big(\underline{w}\big)\bigg)$$
$$(j=1\cdots d\,;\ \alpha\in\mathbb{N}^n\,;\ \beta\in\mathbb{N}^n).$$

by an infinite number of unknown functions of $\underline{w}$. In Section 10, we shall illustrate these general considerations by providing the full details of the computation of $\mathfrak{aut}_{CR}\big(M_{\sf c}^5\big)$ for our model 5-cubic.

It is worth noting that, jointed with Amir Hashemi and Benyamin M.-Alizadeh, recently we have prepared two articles concerning the computations of the Lie algebras of infinitesimal CR-automorphisms. In [67], we have employed applicable techniques of *differential algebra* to provide an effective algorithm to treat systematically solving the PDE systems arising among the computations. Moreover in [68], it is provided a powerful and fast algorithm to perform computations in the case of (parametric and non-parametric) homogeneous and weighted homogeneous CR-manifolds — as Beloshapka's models are. This algorithm employs just simple techniques of linear algebra instead of constructing and solving the mentioned PDE systems (*see* section 10). We also have implemented the designed algorithm by means of the effective tools of the new and modern concept *comprehensive Gröbner systems* for considering the parametric cases.

8. GEOMETRIC AND ANALYTIC INVARIANTS OF CR-GENERIC SUBMANIFOLDS $M \subset \mathbb{C}^{n+d}$

8.1. **Essential holomorphic dimension and Levi multitype.** Assume again that the CR-generic real analytic submanifold $M \subset \mathbb{C}^{n+d}$ is connected and let as before its extrinsic complexification M^{e_c} be represented by d holomorphic defining equations of the form:

$$w_j = \Theta_j\big(z, \underline{z}, \underline{w}\big) \qquad\qquad (j=1\cdots d).$$

For any integer $\kappa \in \mathbb{N}$, let us introduce, the *morphism of κ-th jets* of the holomorphic functions:

$$z \longmapsto \Theta_j\big(z, \underline{z}, \underline{w}\big)$$

with respect to only its n first variables $(z_1, \dots, z_n)$, that is to say precisely, the map:

$$\text{Segre-jet}_\kappa \colon \big(z, \underline{z}, \underline{w}\big) \longmapsto \bigg(z,\ \Big(\tfrac{1}{\beta!}\,\partial_z^\beta \Theta_j(z, \underline{z}, \underline{w})\Big)_{1 \leqslant j \leqslant d,\ \beta \in \mathbb{N}^n,\ |\beta| \leqslant \kappa}\bigg) \in \mathbb{C}^{n+d\binom{n+\kappa}{n}},$$

whose target components just collect all the partial derivatives of order $\leqslant \kappa$ of the Θ_j with respect to the z_k, adding the point z as its first n components.

It is established in [43, 44, 45] that the rank properties of this map — called there the morphism of κ-jets of complexified Segre varieties — are independent of the choice of the coordinates (z, w), hence also independent of the complex graphing functions Θ_j, and for this reason, this map gives access to *invariant properties* of the CR-generic submanifold M, namely to properties that are invariant under biholomorphisms of $\mathbb{C}^{n+d}$. Thus, we will not dwell on the invariant character of this map and just admit it here.

Clearly, the ranks at the origin 0 and the generic ranks increase with κ:

$$\text{rank}_0\big(\text{Segre-jet}_{\kappa+1}\big) \geqslant \text{rank}_0\big(\text{Segre-jet}_\kappa\big)$$
$$\text{genrank}\big(\text{Segre-jet}_{\kappa+1} \geqslant \text{genrank}\big(\text{Segre-jet}_\kappa\big).$$

Concretely, generic ranks are tested by examining all minors of the Jacobian matrix of this map. But before entering examination of minors, we mention the following elementary fact.

Lemma 8.1. ([43, 44, 45]) *If this generic rank does not increase stepwise at a certain jet level κ^*:*

$$\text{genrank}\big(\text{Segre-jet}_{\kappa^*+1}\big) = \text{genrank}\big(\text{Segre-jet}_{\kappa^*}\big),$$

then it remains constantly stabilized for any jet level $\kappa \geqslant \kappa^*$:

$$\text{genrank}\big(\text{Segre-jet}_\kappa\big) = \text{genrank}\big(\text{Segre-jet}_{\kappa^*}\big). \quad \square$$

Furthermore, the connectedness of M and the invariance of the Segre jet map imply (*ibidem*) that κ^* is the same at every point of M and in every system of coordinates.

Definition 8.2. Accordingly, denote now by κ_M the smallest integer κ such that the generic rank of the Segre jet map does not increase after κ, namely:

$$\text{genrank}\big(\text{Segre-jet}_{\kappa_M-1}\big) < \text{genrank}\big(\text{Segre-jet}_{\kappa_M}\big) = \text{genrank}\big(\text{Segre-jet}_{\kappa_M+1}\big) = \cdots.$$

Since generic ranks increase (strictly) from $\kappa = 0, 1, 2, \dots$ up to $\kappa = \kappa_M$, this integer is always bounded above by (a better bound follows below):

$$\kappa_M \leqslant 2n + d.$$

Before explaining the geometric meaning of this maximal generic rank, let us make the following simple observation. Since the map:

$$\mathbb{C}^d \ni \underline{w} \longmapsto \Theta\big(z, \underline{z}, \underline{w}\big) \in \mathbb{C}^d$$

is already of rank d at every point $w \in \mathbb{C}^d$ near the origin because of (6), it follows immediately that the (generic) rank of the zero-th order Segre-jet map satisfies already:

$$\begin{aligned} n + d &= \mathsf{genrk}\big[(z, \underline{z}, \underline{w}) \longmapsto \big(z, \Theta(z, \underline{z}, \underline{w})\big)\big] \\ &= \mathsf{genrk}\big[\mathsf{Segre\text{-}jet}_0\big], \end{aligned}$$

and it follows *passim* that:

$$\kappa_M \leqslant n.$$

Definition 8.3. Accordingly, decompose as:

$$n + d + n_M$$

the maximal possible generic rank of the Segre jet maps, with a certain (nonnegative) integer:

$$n_M \leqslant n.$$

that is to say, the generic rank of Segre-jet$_{\kappa_M}$.

Most importantly, the following crucial statement shows that it is natural to call the integer:

$$n_M + d$$

the *essential holomorphic dimension* of M. Indeed, let $\Delta := \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ denote the unit disc in $\mathbb{C}$, considered as a (small) open piece of it.

Theorem 8.1. ([43, 44, 45]) *Locally in a neighborhood of a Zariski-generic point $p \in M$, the CR-generic submanifold $M \subset \mathbb{C}^{n+d}$ is biholomorphically equivalent to the product:*

$$\underline{M}_p \times \Delta^{n-n_M}$$

of a certain CR-generic submanifold $\underline{M}_p$ of codimension d in $\mathbb{C}^{n_M+d}$ by a complex polydisc Δ^{n-n_M}. In addition, no such $\underline{M}_p$ is biholomorphic to a product of Δ by a CR-generic submanifold in a complex Euclidean space of smaller dimension. □

So in this precise sense, the integer:

$$n_M + d \leqslant n + d$$

is the smallest possible integer such that, locally around a generic point, M looks like a CR-generic submanifold of $\mathbb{C}^{n_M+d}$, modulo an innocuous factor $\mathbb{C}^{n-n_M}$. Even more precisely, this means that *the defining functions of $\underline{M}_p$ are completely independent of the variables of the* $\mathbb{C}^{n-n_M}$, so that $\underline{M}_p$ essentially lives in the space of smaller dimension $\mathbb{C}^{n_M+d}$ (in some appropriately chosen holomorphic coordinates).

To finish, we review a more precise combinatorics of the generic ranks of the Segre-jet maps.

Theorem 8.2. ([43, 44, 45]) *Let $M \subset \mathbb{C}^{n+d}$ be a* connected *real analytic CR-generic submanifold of codimension $d \geqslant 1$ and of CR dimension $n \geqslant 1$. Then there exist well defined integers:*

$$\ell_M \geqslant 0, \quad \lambda_M^0 \geqslant 1, \quad \lambda_M^1 \geqslant 1, \ \ldots\ldots, \ \lambda_M^{\ell_M} \geqslant 1$$

and there exists a proper real analytic subset:

$$\Sigma_{\mathsf{Segre}}$$

of M such that for every point $p \in M \backslash \Sigma_{\mathsf{Segre}}$ and for every system of coordinates (z, w) vanishing at p in which M is represented by defining equations in the standard form:

$$w_j = \Theta_j\big(z, \overline{z}, \overline{w}\big) \qquad\qquad {\scriptstyle (j = 1 \cdots d)},$$

then the following three properties hold:

- $\ell_M \leqslant n_M$*;*
- $\lambda_M^0 = d$*;*
- *for every $\kappa = 0, 1, \ldots, \ell_M$, the mapping of k-th order jets of the κ-th Segre jet map:*

  $$\big(z, \underline{z}, \underline{w}\big) \longmapsto \left(z, \ \Big(\tfrac{1}{\beta!}\, \partial_z^\beta \Theta_j\big(z, \underline{z}, \underline{w}\big)\Big)_{1 \leqslant j \leqslant d, \ \beta \in \mathbb{N}^n, \ |\beta| \leqslant \kappa}\right),$$

  is equal to:

  $$n + \lambda_M^0 + \lambda_M^1 + \cdots + \lambda_M^\kappa$$

  at the origin $(0, 0)$.
- *the essential holomorphic dimension n_M of M is equal to:*

  $$n_M = d + \lambda_M^1 + \cdots + \lambda_M^{\ell_M};$$

8.2. **Generic constancy of CR-geometric invariants.** Recalling that our main objects of study are *completely arbitrary* connected real analytic d-codimensional CR-generic submanifolds $M \subset \mathbb{C}^{n+d}$, with $d \geqslant 1$ and $n \geqslant 1$, the two fundamental, preliminary, Theorems 6.2 and 8.2 have shown that, by avoiding certain two exceptional proper real analytic subsets — which might be complicated —:

$$\Sigma_{CR} \qquad \text{and} \qquad \Sigma_{\mathsf{Segre}},$$

namely by re-localizing the study only at points:

$$p \in M \big\backslash \big(\Sigma_{CR} \cup \Sigma_{\mathsf{Segre}}\big),$$

one comes to:

I: constancy and maximality of the Tanaka symbol Lie bracket structure;

II: constancy and maximality of the stepwise ranks of the Segre-jet at a Zariski-generic point, where by 'maximality', we of course mean complete non-holonomy and maximal possible essential holomorphic dimension $n_M = n$.

8.3. **The specificity of CR dimension** 1. Remarkably, there is only *one* variable z in CR dimension $n = 1$. Then we leave as an exercise to the reader to verify that the fulfillement of Condition II with $n_M = n$ is just equivalent to the requirement that in some d real equations for M centered at a point $p_0 \in M \backslash \Sigma_{\mathsf{Segre}}$:

$$v_j = \varphi_j(x, y, u) \qquad\qquad {\scriptstyle (j=1 \cdots d)},$$

for at least one index j_0, there is a *nonzero* quadratic monomial $c_{j_0}\, z\overline{z}$ in the Taylor series of φ_{j_0}. Hence in CR dimension $n = 1$, Condition II is almost automatically satisfied, while a truly infinite algebraic complexity happens to come from the first Condition I.

9. INEFFECTIVE ACCESS TO THE LOCAL LIE GROUP STRUCTURE

A fundamental theorem was established in [26], Theorem 4.1, but because it does not solve either the biholomorphic equivalence problem or the effective the description of CR automorphism groups, we want to briefly comment on its defects after restating it. A detailed definition of the concept of *local Lie group* based on Sophus Lie' original presentation appears in [51, 50].

Theorem 9.1. *Let $M \subset \mathbb{C}^{n+d}$ be a connected real analytic CR-generic submanifold of positive codimension $d \geqslant 1$ and of positive CR dimension $d \geqslant 1$ and assume it to be maximally non-holonomic and that its essential holomorphic dimension is the ambient one $n + d$. Then at every point:*

$$p \in M \Big\backslash \big(\Sigma_{CR} \cup \Sigma_{\mathsf{Segre}}\big)$$

in a neighborhood of which all basic CR-geometric invariants behave constantly, the abstract group of local biholomorphic transformations:

$$\mathsf{Hol}(M)$$

fixing M — but not necessarily p —, is a finite-dimensional Lie group of dimension bounded above by:

$$n\,\frac{(2nd+5n)!}{n!\;(2nd+4n)!}. \quad \square$$

Such a bound is directly related to jet determination of local biholomorphisms fixing M, and then at least two defects exist.

□ This bound on the jet order, or quite similarly, some other more recent refined bounds, are usually much above what is truly required — a price to pay for generality —, as the finer study of specific CR structures shows, *confer exempli gratia* the geometry-preserving deformations of the above 5-dimensional cubic model $M^5 \subset \mathbb{C}^4$.

□ Most generally, among jets less than say an optimal jet order bound for the determination of local biholomorphisms fixing M, not all jets below the bound are free, but many are dependent in terms of a specific set of independent jets, *confer* Section 1 in [46], and *confer* also well known features of (differential) Gröbner bases. Hence determination results by only one jet order, and the accompanying bound about the dimension of $\mathsf{Hol}(M)$, appear to be a rather rough approach of the reality.

10. Algebra of infinitesimal CR automorphisms of the cubic model $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$

10.1. **Tangency equations.** Consider the extrinsic complexification of the cubic five-dimensional model CR-generic submanifold $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$ defined in coordinates $(z, w_1, w_2, w_3, \underline{z}, \underline{w}_1, \underline{w}_2, \underline{w}_3)$ by the three holomorphic equations:

$$\begin{cases} w_1 - \underline{w}_1 = 2iz\underline{z}, \\ w_2 - \underline{w}_2 = 2iz\underline{z}(z+\underline{z}), \\ w_3 - \underline{w}_3 = 2z\underline{z}(z-\underline{z}). \end{cases}$$

A general $(1,0)$ holomorphic vector field:

$$\mathsf{X} = Z(z,w)\,\frac{\partial}{\partial z} + W^1(z,w)\,\frac{\partial}{\partial w_2} + W^2(z,w)\,\frac{\partial}{\partial w_2} + W^3(z,w)\,\frac{\partial}{\partial w_3}$$

is an infinitesimal CR automorphism of the cubic model if and only if its local holomorphic coefficients Z, W^1, W^2, W^3 and their conjugates $\overline{Z}$, $\overline{W}^1$, $\overline{W}^2$, $\overline{W}^3$ satisfy the following three equations:

$$(18)\quad 0 \equiv \big[W^1 - \overline{W}^1 - 2i\underline{z}Z - 2iz\overline{Z}\big]_{w=\underline{w}+2i\,\Phi(z,\underline{z})},$$

$$(19)\quad 0 \equiv \big[W^2 - \overline{W}^2 - 4iz\underline{z}Z - 2i\underline{z}^2Z - 2iz^2\overline{Z} - 4iz\underline{z}\overline{Z}\big]_{w=\underline{w}+2i\,\Phi(z,\underline{z})},$$

$$(20)\quad 0 \equiv \big[W^3 - \overline{W}^3 - 4z\underline{z}Z - 2z^2\overline{Z} + 2\underline{z}^2Z + 4z\underline{z}\overline{Z}\big]_{w=\underline{w}+2i\,\Phi(z,\underline{z})},$$

identically in $\mathbb{C}\{z, \underline{z}, \underline{w}\}$. Since these coefficient functions are analytic, we may expand them with respect to the powers of $z \in \mathbb{C}$:

$$Z(z,w) = \sum_{k\in\mathbb{N}} z^k\, Z_k(w_1,w_2,w_3) \qquad \text{and} \qquad W^i(z,w) = \sum_{k\in\mathbb{N}} z^k\, W_k^i(w_1,w_2,w_3).$$

Our current aim is to find closed polynomial expressions for these holomorphic functions $Z(z,w)$, $W^1(z,w)$, $W^2(z,w)$, $W^3(z,w)$ by analyzing this system of three identically satisfied equations.

Lemma 10.1. *The Taylor expansions with respect to z are relatively polynomial:*

$$
\begin{aligned}
Z(z,w) &= Z_0(w) + z\,Z_1(w) + z^2\,Z_2(w) + z^3\,Z_3(w),\\
W^1(z,w) &= W_0^1(w) + z\,W_1^1(w),\\
W^2(z,w) &= W_0^2(w) + z\,W_1^2(w) + z^2\,W_2^2(w),\\
W^3(z,w) &= W_0^3(w) + z\,W_1^3(w) + z^2\,W_2^3(w).
\end{aligned}
$$

Proof. After expansion, the first equation reads:

$$
\begin{aligned}
0 \equiv \sum_{k\in\mathbb{N}} z^k \Big[& W_k^1\big(\underline{w}_1 + 2iz\underline{z},\, \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\, \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}\,Z_k\big(\underline{w}_1 + 2iz\underline{z},\, \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\, \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big)\Big] + \\
+ \sum_{k\in\mathbb{N}} \underline{z}^k \Big[& -\overline{W}_k^1\big(\underline{w}_1,\underline{w}_2,\underline{w}_3\big) - 2iz\,\overline{Z}_k\big(\underline{w}_1,\underline{w}_2,\underline{w}_3\big)\Big].
\end{aligned}
$$

We may expand further each W_k^1 and each Z_k using simply the Taylor series formula for a general holomorphic $\Lambda = \Lambda(\underline{w}_1,\underline{w}_2,\underline{w}_3)$:

(21)

$$
\begin{aligned}
&\Lambda\big(\underline{w}_1 + 2iz\underline{z},\, \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\, \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) = \\
&\sum_{l_1,\,l_2,\,l_3\in\mathbb{N}} \Lambda_{\underline{w}_1^{l_1}\underline{w}_2^{l_2}\underline{w}_3^{l_3}}\big(\underline{w}_1,\underline{w}_2,\underline{w}_3\big)\frac{(2iz\underline{z})^{l_1}}{l_1!}\,\frac{(2iz^2\underline{z} + 2iz\underline{z}^2)^{l_2}}{l_2!}\,\frac{(2z^2\underline{z} - 2z\underline{z}^2)^{l_3}}{l_3!}\,.
\end{aligned}
$$

Chasing then the coefficient of $\underline{z}^k$ for every $k \geqslant 2$ in the equation obtained after such an (unwritten) expansion, we see that the first two lines give absolutely no contribution, and that from the third line, it only comes: $0 \equiv \overline{W}_k^1(\underline{w})$, which is what was claimed about the W_k^1: all of them vanish identically for every $k \geqslant 2$.

Next, chasing the coefficient of $z\underline{z}^{k'}$ for every $k' \geqslant 4$, we get $0 \equiv \overline{Z}_{k'}(\underline{w})$, which is what was claimed about the $Z_k(w)$.

The two remaining families of vanishing equations $0 \equiv \overline{W}_k^2(\underline{w}) \equiv \overline{W}_k^3(\underline{w})$ for $k \geqslant 3$ are obtained in a completely similar way by looking at the second tangency equation (19) and as well at the third (20). □

Granted this remarkable relative polynomialness, our next aim is to determine the expressions of the twelve holomorphic functions:

$$
\begin{array}{llll}
Z_0(w), & Z_1(w), & Z_2(w), & Z_3(w),\\
W_0^1(w), & W_1^1(w), & & \\
W_0^2(w), & W_1^2(w), & W_2^2(w), & \\
W_0^3(w), & W_1^3(w), & W_2^3(w) &
\end{array}
$$

of only the variables (w_1, w_2, w_3).

To begin with, let us replace the just obtained expressions of the four functions Z, W^1, W^2, W^3 in the fundamental equations (18), (19) and (20):

$$
\begin{aligned}
0 \equiv & W_0^1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
& + z\,W_1^1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - \overline{W}_0^1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \underline{z}\,\overline{W}_1^1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 2i\underline{z}\,Z_0\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}z\,Z_1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}z^2\,Z_2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}z^3\,Z_3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2iz\,\overline{Z}_0(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2iz\underline{z}\,\overline{Z}_1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2iz\underline{z}^2\,\overline{Z}_2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 2iz\underline{z}^3\,\overline{Z}_3(\underline{w}_1, \underline{w}_2, \underline{w}_3),
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
0 \equiv & W_0^2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
& + z\,W_1^2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
& + z^2W_2^2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - \overline{W}_0^2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \underline{z}\,\overline{W}_1^2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \underline{z}^2\overline{W}_2^2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 4i\underline{z}z\,Z_0\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 4i\underline{z}z^2\,Z_1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 4i\underline{z}z^3\,Z_2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 4i\underline{z}z^4\,Z_3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}^2\,Z_0\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}^2z\,Z_1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}^2z^2\,Z_2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2i\underline{z}^2z^3\,Z_3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 2iz^2\,\overline{Z}_0(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2iz^2\underline{z}\,\overline{Z}_1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2iz^2\underline{z}^2\,\overline{Z}_2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 2iz^2\underline{z}^3\,\overline{Z}_3(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 4iz\underline{z}\,\overline{Z}_0(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 4iz\underline{z}^2\,\overline{Z}_1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 4iz\underline{z}^3\,\overline{Z}_2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 4iz\underline{z}^4\,\overline{Z}_3(\underline{w}_1, \underline{w}_2, \underline{w}_3),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
0 \equiv & W_0^3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
& + z\,W_1^3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
& + z^2W_2^3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - \overline{W}_0^3(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \underline{z}\,\overline{W}_1^3(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \underline{z}^2\overline{W}_2^3(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
& - 4\underline{z}z\,Z_0\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 4\underline{z}z^2\,Z_1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
& - 4\underline{z}z^3\,Z_2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) -
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&- 4\underline{z}z^4\, Z_3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) - \\
&- 2z^2\, \overline{Z}_0(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2\underline{z}z^2\, \overline{Z}_1(\underline{w}_1, \underline{w}_2, \underline{w}_3) - 2\underline{z}^2 z^2\, \overline{Z}_2(\underline{w}_1, \underline{w}_2, \underline{w}_3) - \\
&- 2\underline{z}^3 z^2\, \overline{Z}_3(\underline{w}_1, \underline{w}_2, \underline{w}_3) + \\
&+ 2\underline{z}^2\, Z_0\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
&+ 2\underline{z}^2 z\, Z_1\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
&+ 2\underline{z}^2 z^2\, Z_2\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
&+ 2\underline{z}^2 z^3\, Z_3\big(\underline{w}_1 + 2iz\underline{z},\ \underline{w}_2 + 2iz^2\underline{z} + 2iz\underline{z}^2,\ \underline{w}_3 + 2z^2\underline{z} - 2z\underline{z}^2\big) + \\
&+ 4z\underline{z}\, \overline{Z}_0(\underline{w}_1, \underline{w}_2, \underline{w}_3) + 4z\underline{z}^2\, \overline{Z}_1(\underline{w}_1, \underline{w}_2, \underline{w}_3) + 4z\underline{z}^3\, \overline{Z}_2(\underline{w}_1, \underline{w}_2, \underline{w}_3) + \\
&+ 4z\underline{z}^4\, \overline{Z}_3(\underline{w}_1, \underline{w}_2, \underline{w}_3).
\end{aligned}
$$

Next, applying (21), we insert the corresponding Taylor series of the holomorphic functions $Z(z,w)$, $W^1(z,w)$, $W^2(z,w)$, $W^3(z,w)$ in the above expressions. Then we extract the coefficients of the monomials $z^\mu \underline{z}^\nu$ for small values of μ and ν, and these coefficients must all vanish identically.

First of all, for $(\mu,\nu) = (0,0)$ we get the following expressions in the cases of (22), (23), (24), respectively:

$$0 \equiv W_0^1(\underline{w}) - \overline{W}_0^1(\underline{w}), \tag{25}$$

$$0 \equiv W_0^2(\underline{w}) - \overline{W}_0^2(\underline{w}), \tag{26}$$

$$0 \equiv W_0^3(\underline{w}) - \overline{W}_0^3(\underline{w}). \tag{27}$$

This means that the functions $W_0^1(w), W_0^2(w), W_0^3(w)$ are all real, *i.e.* have real Taylor coefficients.

For $(\mu,\nu) = (1,0)$, equation (22) gives the equality:

$$0 \equiv W_1^1(\underline{w}) - 2i\overline{Z}_0(\underline{w}), \tag{28}$$

while we have the following quite advantageous vanishing result from considering the same exponents $(\mu,\nu) = (1,0)$ in (23) and in (24):

Lemma 10.2. *The holomorphic functions $W_1^2(w)$ and $W_1^3(w)$ vanish identically.*

Next, for $(\mu,\nu) = (2,0)$ no new useful information comes from (22), while from (23) and (24) we obtain:

$$0 \equiv W_2^2(\underline{w}) - 2i\overline{Z}_0(\underline{w}), \tag{29}$$

$$0 \equiv W_2^3(\underline{w}) - 2\overline{Z}_0(\underline{w}). \tag{30}$$

Next, for $(\mu,\nu) = (1,1)$ we obtained from the three mentioned equations:

$$0 \equiv W^1_{0\underline{w}_1}(\underline{w}) - Z_1(\underline{w}) - \overline{Z}_1(\underline{w}), \tag{31}$$

$$0 \equiv W^2_{0\underline{w}_1}(\underline{w}) - 2Z_0(\underline{w}) - 2\overline{Z}_0(\underline{w}), \tag{32}$$

$$0 \equiv iW^3_{0\underline{w}_1}(\underline{w}) - 2Z_0(\underline{w}) + 2\overline{Z}_0(\underline{w}), \tag{33}$$

and for $(\mu, \nu) = (2, 1)$ we get:

$$0 \equiv iW^1_{0\underline{w}_2}(\underline{w}) + W^1_{0\underline{w}_3}(\underline{w}) + iW^1_{1\underline{w}_1}(\underline{w}) - iZ_2(\underline{w}), \tag{34}$$

$$0 \equiv iW^2_{0\underline{w}_2}(\underline{w}) + W^2_{0\underline{w}_3}(\underline{w}) - 2iZ_1(\underline{w}) - i\overline{Z}_1(\underline{w}), \tag{35}$$

$$0 \equiv iW^3_{0\underline{w}_2}(\underline{w}) + W^3_{0\underline{w}_3}(\underline{w}) - 2Z_1(\underline{w}) - \overline{Z}_1(\underline{w}). \tag{36}$$

Now let us continue the process for $(\mu, \nu) = (3, 1)$. In this case, (22) and (24) do not provide any useful information, while the following equality can be obtained by inspecting (23):

$$0 \equiv W^2_{2\underline{w}_1}(\underline{w}) - 2Z_2(\underline{w}). \tag{37}$$

Lemma 10.3. *The holomorphic function $Z_2(w)$ vanishes identically.*

Proof. For $(\mu, \nu) = (2, 2)$, we get two equations from (23) and (24):

$$0 \equiv -W^2_{0\underline{w}_1{}^2}(\underline{w}) + 4Z_{0\underline{w}_1}(\underline{w}) - iZ_2(\underline{w}) - i\overline{Z}_2(\underline{w}), \tag{38}$$

$$0 \equiv -W^3_{0\underline{w}_1{}^2}(\underline{w}) - 4iZ_{0\underline{w}_1}(\underline{w}) - \overline{Z}_2(\underline{w}) + Z_2(\underline{w}). \tag{39}$$

By differentiating once equation (32) with respect to $\underline{w}_1$ and then replacing the value of $W^2_{0\underline{w}_1{}^2}(\underline{w})$ in (38), we obtain:

$$2\big(Z_{0\underline{w}_1}(\underline{w}) - \overline{Z}_{0\underline{w}_1}(\underline{w})\big) \equiv i\big(Z_2(\underline{w}) + \overline{Z}_2(\underline{w})\big). \tag{40}$$

One can apply the same line of reasoning to the equations (39) and (33), and obtain, respectively:

$$2\big(Z_{0\underline{w}_1}(\underline{w}) + \overline{Z}_{0\underline{w}_1}(\underline{w})\big) \equiv i\big(\overline{Z}_2(\underline{w}) - Z_2(\underline{w})\big). \tag{41}$$

Now comparison of (40) and (41) yields that:

$$0 \equiv i\overline{Z}_2(\underline{w}) - 2Z_{0\underline{w}_1}(\underline{w}). \tag{42}$$

On the other hand, according to (29) one can replace $W^2_{2\underline{w}_1}(\underline{w})$ by $2i\overline{Z}_{0\underline{w}_1}(\underline{w})$ in (37). Thus we have:

$$0 \equiv Z_2(\underline{w}) - i\overline{Z}_{0\underline{w}_1}(\underline{w}). \tag{43}$$

Now, comparing (42) and (43) immediately yield $Z_2(w) \equiv 0$, as desired. □

Now, equations (43), (28), (29), (30), (32) and (33) imply the identical vanishing of the the following six functions:

$$\begin{aligned} 0 \equiv\ & Z_{0\underline{w}_1}(\underline{w}) \equiv W^1_{1\underline{w}_1}(\underline{w}) \equiv W^2_{2\underline{w}_1}(\underline{w}) \equiv \\ \equiv\ & W^3_{2\underline{w}_1}(\underline{w}) \equiv W^2_{0\underline{w}_1{}^2}(\underline{w}) \equiv W^3_{0\underline{w}_1{}^2}(\underline{w}). \end{aligned} \tag{44}$$

In particular, the identical vanishing of the holomorphic function $Z_{0\underline{w}_1}$ implies a third advantageous fact.

Lemma 10.4. *The holomorphic function $Z_3(w)$ vanishes, identically.*

Proof. It suffices to look at the case $(\mu, \nu) = (1, 4)$ in (23):

$$0 \equiv \underline{Z_{0\underline{w}_1}(\underline{w})}_{\circ} + i\overline{Z}_3(\underline{w}).$$

□

Taking account of the three vanishing Lemmas 10.2, 10.3 and 10.4, the initial forms in 10.1 of three of our four functions simplify:

$$Z(z,w) = Z_0(w) + zZ_1(w), \tag{45}$$

$$W^2(z,w) = W_0^2(w) + z^2 W_2^2(w), \tag{46}$$

$$W^3(z,w) = W_0^3(w) + z^2 W_2^3(w). \tag{47}$$

Moreover, the equation (34) changes into the following form:

$$0 \equiv iW^1_{0\underline{w}_2}(\underline{w}) + W^1_{0\underline{w}_3}(\underline{w}). \tag{48}$$

Since the function $W_0^1(w)$ is real by (25), this last expression (48) together with its conjugate yield that $W^1_{0\underline{w}_2}(\underline{w})$ and $W^1_{0\underline{w}_3}(\underline{w})$ vanish together, *i.e.* $W_0^1(w)$ is independent of the variables w_2 and w_3. Using this fact and differentiating (31) once with respect to $\underline{w}_2$ and $\underline{w}_3$ implies that $Z_{1\underline{w}_i}(\underline{w}) + \overline{Z}_{1\underline{w}_i}(\underline{w})$ vanishes for $i = 2, 3$. In other words, $Z_{1\underline{w}_i}$ is a real function for $i = 2, 3$.

On the other hand, differentiating equations (35) and (36) with respect to $\underline{w}_2$ and to $\underline{w}_3$ yields that:

$$0 \equiv iW^2_{0\underline{w}_2\underline{w}_i} + W^2_{0\underline{w}_3\underline{w}_i} - iZ_{1\underline{w}_i}(\underline{w}), \tag{49}$$

$$0 \equiv iW^3_{0\underline{w}_2\underline{w}_i} + W^3_{0\underline{w}_3\underline{w}_i} - Z_{1\underline{w}_i}(\underline{w}) \qquad (i = 2, 3). \tag{50}$$

But as (26) and (27) meant, the two function $W_0^2(\underline{w})$ and $W_0^3(\underline{w})$ are real. Then according to equation (49), we have $W^2_{0\underline{w}_3\underline{w}_i}(\underline{w}) \equiv 0$ for $i = 2, 3$.

Now, using again (49) for $i = 3$ immediately implies that $Z_{1\underline{w}_3}(\underline{w}) \equiv 0$. In a similar way, equation (50) yields the vanishing of the two differentiated functions $W^3_{0\underline{w}_2\underline{w}_i}(\underline{w})$ and $Z_{1\underline{w}_2}(\underline{w})$. It follows from this fact together with the vanishing of $Z_{1\underline{w}_i}(\underline{w})$, $W^2_{0\underline{w}_3\underline{w}_i}(\underline{w})$ and $W^3_{0\underline{w}_2\underline{w}_i}(\underline{w})$ for $i = 2, 3$ that — *see* (49) and (50) again —:

$$0 \equiv Z_{1\underline{w}_i}(\underline{w}) \equiv W^k_{0\underline{w}_i\underline{w}_j}(\underline{w}) \qquad (i, j, k = 2, 3). \tag{51}$$

Furthermore, equation (22) gives the following equality after inspecting $(\mu, \nu) = (1, 3)$:

$$0 \equiv Z_{0\underline{w}_2}(\underline{w}) + iZ_{0\underline{w}_3}(\underline{w}). \tag{52}$$

Lemma 10.5. *The two holomorphic functions $Z_0(w)$ and $Z_1(w)$ are constant.*

Proof. Inspection of the fundamental equation (23) for $(\mu,\nu)=(3,2)$ and for $(\mu,\nu)=(2,3)$ respectively gives:

$$
\begin{aligned}
(53)\qquad 0 &\equiv -2W^2_{0\underline{w}_1\underline{w}_2}(\underline{w}) + 2iW^2_{0\underline{w}_1\underline{w}_3}(\underline{w}) + iW^2_{2\underline{w}_2} - \\
&- W^2_{2\underline{w}_3}(\underline{w}) + 4Z_{0\underline{w}_2}(\underline{w}) - 4iZ_{0\underline{w}_3}(\underline{w}) + 4Z_{1\underline{w}_1}(\underline{w}),\\
(54)\qquad 0 &\equiv -2W^2_{0\underline{w}_1\underline{w}_2}(\underline{w}) - 2iW^2_{0\underline{w}_1\underline{w}_3}(\underline{w}) + 6Z_{0\underline{w}_2}(\underline{w}) + \\
&+ 2iZ_{0\underline{w}_3}(\underline{w}) + 2Z_{1\underline{w}_1}(\underline{w}).
\end{aligned}
$$

Using (29) and (32), we can replace $W^2_{0\underline{w}_1\underline{w}_i}(\underline{w})$ by $2Z_{0\underline{w}_i}(\underline{w}) + 2\overline{Z}_{0\underline{w}_i}(\underline{w})$ and $W^2_{2\underline{w}_i}(\underline{w})$ by $2i\overline{Z}_{0\underline{w}_i}(\underline{w})$ for $i=2,3$. Moreover, according to (52), we can replace $\overline{Z}_{0\underline{w}_3}$ by $-i\overline{Z}_{0\underline{w}_2}$. These substitutions change (53) and (54) into:

$$
\begin{aligned}
(55)\qquad 0 &\equiv \overline{Z}_{0\underline{w}_2}(\underline{w}) - Z_{1\underline{w}_1}(\underline{w}),\\
(56)\qquad 0 &\equiv 2Z_{0\underline{w}_2}(\underline{w}) - 4\overline{Z}_{0\underline{w}_2}(\underline{w}) + Z_{1\underline{w}_1}(\underline{w}).
\end{aligned}
$$

Eliminating the function $Z_{1\underline{w}_1}(\underline{w})$ from these expressions implies that:

$$
(57)\qquad 0 \equiv 2Z_{0\underline{w}_2}(\underline{w}) - 3\overline{Z}_{0\underline{w}_2}(\underline{w}),
$$

and together with its conjugate, this last equation yields that the holomorphic function $Z_{0\underline{w}_2}(\underline{w}_2)$ vanishes identically.

Thanks to this, (52) immediately implies that $Z_{0\underline{w}_3}(w) \equiv 0$. Furthermore, we know also from (44) that $Z_{0\underline{w}_1}(w) \equiv 0$ and hence the holomorphic function $Z_0(w)$ is constant. On the other hand, according to (55) we have $Z_{1\underline{w}_1}(\underline{w}) \equiv 0$, which, together with (51) yields that the holomorphic function $Z_1(w)$ is constant too. □

According to the above lemma we have the following forms for the holomorphic functions $Z_0(w)$ and $Z_1(w)$:

$$
\begin{aligned}
(58)\qquad Z_1(w) &:= \mathsf{d} + i\mathsf{r},\\
(59)\qquad Z_0(w) &:= \mathsf{l}_1 + i\mathsf{l}_2,
\end{aligned}
$$

in terms of four complex numbers $\mathsf{d}, \mathsf{r}, \mathsf{l}_1$ and l_2. Now, (29) and (30) immediately imply that:

$$
\begin{aligned}
(60)\qquad W^2_2(w) &= 2\mathsf{l}_2 + 2i\mathsf{l}_1,\\
(61)\qquad W^3_2(w) &= 2\mathsf{l}_1 - 2i\mathsf{l}_2.
\end{aligned}
$$

Moreover, differentiating once (32) and (33) with respect to $\underline{w}_i, i=1,2,3$, yields that the holomorphic functions $W^k_{0\underline{w}_1\underline{w}_i}$ vanish for $k=2,3$. Hence taking account of (51), we have:

$$
(62)\qquad 0 \equiv W^k_{0\underline{w}_i\underline{w}_j} \qquad (i,j = 1,2,3, \quad k = 2,3).
$$

More precisely, the degrees of the functions $W_0^2(w)$ and $W_0^3(w)$ with respect to the variables $w_i, i = 1, 2, 3$ are equal to 1. Hence, according to (32), (33), (35) and (36), we can write:

$$W_0^2(w) := 4\mathsf{l}_1 w_1 + 3\mathsf{d} w_2 - \mathsf{r} w_3 + \mathsf{s}_1, \tag{63}$$

$$W_0^3(w) := 4\mathsf{l}_2 w_1 + \mathsf{r} w_2 + 3\mathsf{d} w_3 + \mathsf{s}_2, \tag{64}$$

with two complex numbers s_1 and s_1. Moreover, (28) and (59) help us to realize the expression of $W_1^1(w)$ as follow:

$$W_1^1(w) = 2\mathsf{l}_2 + 2i\mathsf{l}_1. \tag{65}$$

Additionally as we saw, the degree of the real holomorphic function $W_0^1(w)$ with respect to the variable w_1 is one (*compare* (31) with (58)) and also this function is independent of the variables w_2 and w_3. Hence we have the following expression for $W_0^1(w)$:

$$W_0^1(w) := 2\mathsf{d} w_1 + \mathsf{t}, \tag{66}$$

for an arbitrary complex number t.

Now, the above process has determined explicitly the expressions of the holomorphic functions $Z(z, w)$ and $W^i(z, w), i = 1, 2, 3$. Indeed, according to the obtained results we have exactly found *seven* real numerical constants:

$$\mathsf{d}, \quad \mathsf{r}, \quad \mathsf{l}_1, \quad \mathsf{l}_2, \quad \mathsf{t}, \quad \mathsf{s}_1, \quad \mathsf{s}_2,$$

which give us seven $\mathbb{R}$-linearly independent infinitesimal automorphisms of our model. More precisely, by verification of the results we could find the expressions of the desired holomorphic functions as follows:

$$\begin{aligned}
Z(z, w) &= Z_0(w) + Z_1(w)z = \mathsf{l}_1 + i\mathsf{l}_2 + (\mathsf{d} + i\mathsf{r})z, \\
W^1(z, w) &= W_0^1(w) + W_1^1(w)z = 2\mathsf{d} w_1 + \mathsf{t} + 2(\mathsf{l}_2 + i\mathsf{l}_1)z, \\
W^2(z, w) &= W_0^2(w) + W_2^2(w)z^2 = 4\mathsf{l}_1 w_1 + 3\mathsf{d} w_2 - \mathsf{r} w_3 + \mathsf{s}_1 + 2(\mathsf{l}_2 + i\mathsf{l}_1)z^2 \\
W^3(z, w) &= W_0^3(w) + W_2^3(w)z^2 = 4\mathsf{l}_2 w_1 + \mathsf{r} w_2 + 3\mathsf{d} w_3 + \mathsf{s}_2 + 2(\mathsf{l}_1 - i\mathsf{l}_2)z^2.
\end{aligned}$$

Hence we have the following *detailed* confirmation of one of Shananina's computations ([70]):

Proposition 10.6. *The Lie algebra* $\mathfrak{aut}_{CR}(M) = 2\,\mathrm{Re}\,\mathfrak{hol}(M)$ *of the infinitesimal CR automorphisms of the* 5*-dimensional* 3*-codimensional CR-generic model cubic* $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ *represented by the three real graphed equations:*

$$\begin{cases}
w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\
w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}), \\
w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}),
\end{cases}$$

is 7-dimensional and it is generated by the $\mathbb{R}$-linearly independent real parts of the following seven $(1,0)$ holomorphic vector fields:

$$
\begin{aligned}
T &:= \partial_{w_1}, \\
S_1 &:= \partial_{w_2}, \\
S_2 &:= \partial_{w_3}, \\
L_1 &:= \partial_z + (2iz)\,\partial_{w_1} + (2iz^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}, \\
L_2 &:= i\,\partial_z + (2z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2iz^2 - 4w_1)\,\partial_{w_3}, \\
D &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}, \\
R &:= iz\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned}
$$

Computing all commutators between any two of these seven generators of $\mathfrak{aut}_{CR}(M)$ gives the following complete table of Lie brackets:

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	$*$	0	0	0	0	$3S_1$	S_2
T	$*$	$*$	0	$4S_2$	$4S_1$	$2T$	0
L_2	$*$	$*$	$*$	0	$-4T$	L_2	$-L_1$
L_1	$*$	$*$	$*$	$*$	0	L_1	L_2
D	$*$	$*$	$*$	$*$	$*$	0	0
R	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

Moreover, we would like to observe that $\mathfrak{aut}_{CR}(M)$ is a 3-graded Lie algebra with nilpotent negative part, in the sense of Tanaka. More precisely, with the notation:

$$\mathfrak{g} := \mathfrak{aut}_{CR}(M),$$

if we further set:

$$
\begin{aligned}
\mathfrak{g}_{-3} &:= \mathsf{Span}_{\mathbb{R}}\big\langle S_1,\, S_2\big\rangle, \\
\mathfrak{g}_{-2} &:= \mathsf{Span}_{\mathbb{R}}\big\langle T\big\rangle, \\
\mathfrak{g}_{-1} &:= \mathsf{Span}_{\mathbb{R}}\big\langle L_1, L_2\big\rangle, \\
\mathfrak{g}_{0} &:= \mathsf{Span}_{\mathbb{R}}\big\langle D, R\big\rangle,
\end{aligned}
$$

then one readily checks:

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

Furthermore, with the convention that $\mathfrak{g}_k = \{0\}$ for either $k \leqslant -4$ or $k \geqslant 2$, one may then verify the property that:

$$[\mathfrak{g}_{k_1}, \mathfrak{g}_{k_2}] \subseteq \mathfrak{g}_{k_1+k_2},$$

for any two integers $k_1, k_2 \in \mathbb{Z}$. Accordingly, the Lie subalgebra:

$$\mathfrak{g}_{-} := \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is called the *Levi-Tanaka algebra* of the CR-manifold M. This subalgebra is isomorphic to the Lie algebra $\mathfrak{n}_5^4$ in Goze's classification presented on p. 68 above.

11. TANAKA PROLONGATIONS

In the former section, we computed the Levi-Tanaka algebra associated to our model cubic CR-manifold $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$. Generally, Tanaka's prolongation (*see* [74, 77]) of a graded Lie algebra:

$$\mathfrak{g}_- := \mathfrak{g}_{-\mu} \oplus \ldots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is an algebraic procedure to generate a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots$ that determines to a large extent the geometric properties of the CR structure (*see* [1] for a computational example). We want to show that the prolongation of our model Lie algebra $\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ regives $\mathfrak{g}_0 = \langle D, R \rangle$.

Consider therefore a finite-dimensional graded real Lie algebra indexed by negative integers:

$$\mathfrak{g}_- = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

satisfying $[\mathfrak{g}_{-l_1}, \mathfrak{g}_{-l_2}] \subset \mathfrak{g}_{-l_1-l_2}$ with the convention that $\mathfrak{g}_k = 0$ for $k \leqslant -\mu-1$. Following [74], $\mathfrak{g}_-$ will be said to be *of μ-th kind.* Assume that there is a complex structure $J \colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ such that $J^2 = -\mathsf{Id}$, whence $\mathfrak{g}_{-1}$ is even-dimensional and bears a natural structure of a complex vector space. Tanaka's prolongation of $\mathfrak{g}_-$ is an algebraic procedure which generates a certain larger graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots$$

in the following way.

By definition, the order-zero component $\mathfrak{g}_0$ consists of all linear endomorphisms $\mathsf{d} \colon \mathfrak{g}_- \to \mathfrak{g}_-$ which preserve gradation: $\mathsf{d}(\mathfrak{g}_k) \subset \mathfrak{g}_k$, which respect the complex structure: $\mathsf{d}(J\mathsf{x}) = J\mathsf{d}(\mathsf{x})$ for all $\mathsf{x} \in \mathfrak{g}_{-1}$ and which are *derivations*, namely satisfy $\mathsf{d}([\mathsf{y}, \mathsf{z}]) = [\mathsf{d}(\mathsf{x}), \mathsf{y}] + [\mathsf{x}, \mathsf{d}(\mathsf{y})]$ for every $\mathsf{y}, \mathsf{z} \in \mathfrak{g}_-$. Then the bracket between a $\mathsf{d} \in \mathfrak{g}_0$ and an $\mathsf{x} \in \mathfrak{g}_-$ is simply defined by $[\mathsf{d}, \mathsf{x}] := \mathsf{d}(\mathsf{x})$, while the bracket between *two* elements $\mathsf{d}', \mathsf{d}'' \in \mathfrak{g}_0$ is defined to be the commutator $\mathsf{d}' \circ \mathsf{d}'' - \mathsf{d}'' \circ \mathsf{d}'$ between endomorphisms. One checks at once that Jacobi relations hold, hence $\mathfrak{g}_- \oplus \mathfrak{g}_0$ becomes a true Lie algebra.

By contrast, for any $l \geqslant 1$, no constraint with respect to J is required. Assuming that the components $\mathfrak{g}_{l'}$ are already constructed for any $l' \leqslant l-1$, the l-th component $\mathfrak{g}_l$ of the prolongation consists of l-shifted graded linear

morphisms $\mathfrak{g}_- \to \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ that are derivations, namely:
(67)
$$\mathfrak{g}_l = \Big\{ \mathsf{d} \in \bigoplus_{k\leqslant -1} \mathsf{Lin}(\mathfrak{g}_k,\, \mathfrak{g}_{k+l})\colon \mathsf{d}([\mathsf{y},\, \mathsf{z}]) = [\mathsf{d}(\mathsf{y}),\, \mathsf{z}] + [\mathsf{y},\, \mathsf{d}(\mathsf{z})], \quad \forall\, \mathsf{y},\, \mathsf{z} \in \mathfrak{g}_- \Big\}.$$

Now, for $\mathsf{d} \in \mathfrak{g}_k$ and $\mathsf{e} \in \mathfrak{g}_l$, by induction on the integer $k + l \geqslant 0$, one defines the bracket $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_{k+l} \otimes \mathfrak{g}_-^*$ by:

(68) $$[\mathsf{d},\, \mathsf{e}](\mathsf{x}) = \big[[\mathsf{d},\, \mathsf{x}],\, \mathsf{e}\big] + \big[\mathsf{d},\, [\mathsf{e},\, \mathsf{x}]\big] \qquad \text{for } \mathsf{x} \in \mathfrak{g}_-.$$

One notes that, for $k = l = 0$, this definition coincides with the above one for $[\mathfrak{g}_0,\, \mathfrak{g}_0]$. It follows by induction ([74, 77]) that $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_{k+l}$ and that with this bracket, the sum $\mathfrak{g}_- \bigoplus_{k\geqslant 1} \mathfrak{g}_k$ becomes a graded Lie algebra, because the general Jacobi identity:

$$0 = \big[[\mathsf{d},\, \mathsf{e}],\, \mathsf{f}\big] + \big[[\mathsf{f},\, \mathsf{d}],\, \mathsf{e}\big] + \big[[\mathsf{e},\, \mathsf{f}],\, \mathsf{d}\big]$$

for $\mathsf{d} \in \mathfrak{g}_k$, $\mathsf{e} \in \mathfrak{g}_l$ and $\mathsf{f} \in \mathfrak{g}_m$ follows by definition when one of k, l, m is negative, and can be shown by induction on the integer $k+l+m \geqslant 0$ when all of k, l, m are nonnegative.

11.1. **Tanaka prolongation of the Levi-Tanaka algebra $\mathfrak{g}_-$.** Now let us find the Tanaka prolongation $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with $\mathfrak{g}_{-3} = < \mathsf{s}_1, \mathsf{s}_2 >$, with $\mathfrak{g}_{-2} = < \mathsf{t} >$, with $\mathfrak{g}_{-1} = < \mathsf{l}_1, \mathsf{l}_2 >$ and with the same table of commutators as presented in the former section.[2] According to the definition, the zero-order component $\mathfrak{g}_0$ of this Lie algebra is the subalgebra containing all derivations $\mathsf{d} = (\mathsf{d}_1, \mathsf{d}_2, \mathsf{d}_3)$ with $\mathsf{d}_i \in \mathsf{End}(\mathfrak{g}_{-i}, \mathfrak{g}_{-i})$, $i = 1, 2, 3$. Assume the value of the components of d on the basis elements as follows:

$$\mathsf{d}_1(\mathsf{l}_1) = r_1\mathsf{l}_1 + r_2\mathsf{l}_2, \quad \mathsf{d}_1(\mathsf{l}_2) = r_3\mathsf{l}_1 + r_4\mathsf{l}_2, \quad \mathsf{d}_2(\mathsf{t}) = k\mathsf{t}$$
$$\mathsf{d}_3(\mathsf{s}_1) = m_1\mathsf{s}_1 + m_2\mathsf{s}_2, \quad \mathsf{d}_3(\mathsf{s}_2) = m_3\mathsf{s}_1 + m_4\mathsf{s}_2,$$

for some nine real unknown constants. Preserving the complex structure J by d implies that (notice that $J(\mathsf{l}_1) = \mathsf{l}_2$):

$$r_1 = r_4, \quad r_3 = -r_2.$$

Furthermore, since d is a derivation we can obtain some other relations within the coefficients r_i, k, m_i, $i = 1, \ldots, 4$. At first, this property gives the following equality:

$$\begin{aligned}\mathsf{d}([\mathsf{l}_1, \mathsf{l}_2]) &= [\mathsf{d}(\mathsf{l}_1), \mathsf{l}_2] - [\mathsf{d}(\mathsf{l}_2), \mathsf{l}_1] \\ &= [r_1\mathsf{l}_1 + r_2\mathsf{l}_2, \mathsf{l}_2] - [r_3\mathsf{l}_1 + r_4\mathsf{l}_2, \mathsf{l}_1],\end{aligned}$$

which can be read as:

$$k\mathsf{t} = r_1\mathsf{t} + r_4\mathsf{t}$$

[2] For reasons of coherence — as will be realized at the end of this section —, our notations are quite similar to the ones of the preceding section.

or equivalently:

$$k = r_1 + r_4.$$

Applying the same to the value $\mathsf{d}([\mathsf{l}_1, \mathsf{t}])$ gives:

$$m_3 = r_3, \quad m_4 = r_4 + k.$$

Other values of d do not have new result and can be disregarded. It follows from these relations that $\mathfrak{g}_0$ is two-dimensional and is generated by two derivations:

$$\mathsf{d}: \quad \mathsf{l}_1 \mapsto \mathsf{l}_1, \quad \mathsf{l}_2 \mapsto \mathsf{l}_2, \quad \mathsf{t} \mapsto 2\mathsf{t}, \quad \mathsf{s}_1 \mapsto 3\mathsf{s}_1, \quad \mathsf{s}_2 \mapsto 3\mathsf{s}_2,$$

$$\mathsf{r}: \quad \mathsf{l}_1 \mapsto -\mathsf{l}_2, \quad \mathsf{l}_2 \mapsto \mathsf{l}_1, \quad \mathsf{t} \mapsto 0, \quad \mathsf{s}_1 \mapsto -\mathsf{s}_2, \quad \mathsf{s}_2 \mapsto \mathsf{s}_1.$$

Next component $\mathfrak{g}_1$ of $\mathfrak{g}$ is the set of all liner maps $\mathsf{d} = (\mathsf{d}_1, \mathsf{d}_2, \mathsf{d}_3)$ with $\mathsf{d}_i \in \mathsf{Lin}\big(\mathfrak{g}_{-i}, \mathfrak{g}_{1-i}\big)$, $i = 1, 2, 3$ satisfying the fundamental equation introduced in (67). One can write the values of the map d as follows:

$$\mathsf{d}_1(\mathsf{l}_1) = r_1\mathsf{d} + r_2\mathsf{r}, \quad \mathsf{d}_1(\mathsf{l}_2) = r_3\mathsf{d} + r_4\mathsf{r}, \quad \mathsf{d}_2(\mathsf{t}) = k_1\mathsf{l}_1 + k_2\mathsf{l}_2,$$
$$\mathsf{d}_3(\mathsf{s}_1) = m_1\mathsf{t}, \quad \mathsf{d}_3(\mathsf{s}_2) = m_2\mathsf{t}.$$

Applying the equality (67) for $\mathsf{y} = \mathsf{l}_1$ and $\mathsf{z} = \mathsf{l}_2$ gives:

$$\mathsf{d}([\mathsf{l}_1, \mathsf{l}_2]) = [r_1\mathsf{d} + r_2\mathsf{r}, \mathsf{l}_2] - [r_3\mathsf{d} + r_4\mathsf{r}, \mathsf{l}_1],$$

which can be read as:

$$k_1\mathsf{l}_1 + k_2\mathsf{l}_2 = r_1\mathsf{l}_2 + r_2\mathsf{l}_1 - r_3\mathsf{l}_1 + r_4\mathsf{l}_2$$

which immediately implies that:

$$k_1 = r_2 - r_3, \quad k_2 = r_1 + r_4.$$

Similar inspecting of the other values $\mathsf{d}\big([\mathsf{y}, \mathsf{z}]\big)$ for $\mathsf{y}, \mathsf{z} = \mathsf{l}_1, \mathsf{l}_2, \mathsf{t}, \mathsf{s}_1$ and s_2 gives, moreover, the following relations between the coefficients (here we present only the useful equalities):

$$k_1 = r_2 - r_3, \;\; k_2 = r_1 + r_4, \;\; m_1 = k_2 + 2r_1, \;\; r_2 = 0, \;\; m_1 + 3r_1 = 0,$$

$$r_1 = 0, \;\; m_2 + r_2 = 0, \;\; m_2 = 2r_3 - k_1, \;\; r_3 = 0, \;\; m_1 - r_4 = 0, \;\; \ldots.$$

It follows from the above relations that $r_i = k_j = m_t = 0$ for $i = 1, \ldots 4$ and $j, t = 1, 2$ which means that the subalgebra $\mathfrak{g}_1$ is trivial, *i.e.* $\mathfrak{g}_1 = 0$. Accordingly, the transitivity of Tanaka algebras ([74, 77]) and also fundamentality of the Levi-Tanaka subalgebra $\mathfrak{g}_-$ ([6]) imply that all components $\mathfrak{g}_l$, $l > 0$ vanish identically (*see* [68], Proposition 4.6). So we have:

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

This prolonged Lie algebra is seven-dimensional with the basis elements $\mathsf{d}, \mathsf{r}, \mathsf{l}_1, \mathsf{l}_2, \mathsf{t}, \mathsf{s}_1$ and s_2 and according to the definition it has the following table of commutators:

	s_2	s_1	t	l_2	l_1	d	r
s_2	0	0	0	0	0	$3\mathsf{s}_2$	$-\mathsf{s}_1$
s_1	$*$	0	0	0	0	$3\mathsf{s}_1$	s_2
t	$*$	$*$	0	$4\mathsf{s}_2$	$4\mathsf{s}_1$	$2\mathsf{t}$	0
l_2	$*$	$*$	$*$	0	$-4\mathsf{t}$	l_2	$-\mathsf{l}_1$
l_1	$*$	$*$	$*$	$*$	0	l_1	l_2
d	$*$	$*$	$*$	$*$	$*$	0	0
r	$*$	$*$	$*$	$*$	$*$	$*$	0

By the correspondence $X \mapsto \mathsf{x}$, for $X = D, R, \dots, S_2$, one easily verifies that this Lie algebra is isomorphic to the algebra of infinitesimal automorphisms $\mathfrak{aut}(M)$ computed in the former subsection[3]. That is why we chose the same letters for generators of the Lie algebra as was done in Section 10.

12. Equivalence computations for the cubic model

12.1. Initial Lie bracket structure. For the 3-codimensional *cubic model* CR-generic submanifold $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ represented as a graph:

$$\begin{cases} v_1 = 2\,i\,z\,\overline{z}, \\ v_2 = 2\,i\,\big(z^2\,\overline{z} + z\,\overline{z}^2\big), \\ v_3 = 2\,\big(z^2\,\overline{z} - z\,\overline{z}^2\big), \end{cases} \tag{69}$$

the $(0,1)$-complex tangent bundle $T^{0,1}M^5_{\mathsf{c}}$ is spanned by the single $(0,1)$-vector field:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \underline{z}} - 2\,i\,z\,\frac{\partial}{\partial \overline{w}_1} - \big(2\,i\,z^2 + 4\,i\,z\,\overline{z}\big)\,\frac{\partial}{\partial \overline{w}_2} - \big(2\,z^2 - 4\,z\,\overline{z}\big)\,\frac{\partial}{\partial \overline{w}_3}.$$

Here, the expression of $\overline{\mathscr{L}}$ is presented as a vector field which lives in a neighborhood of M^5_{c} in $\mathbb{C}^4$, while M^5_{c} itself, is a real five-dimensional hypersurface equipped with the five real coordinates x, y, u_1, u_2, u_3. But, in order to express $\overline{\mathscr{L}}$ *intrinsically*, one must drop $\frac{\partial}{\partial v_1}$, $\frac{\partial}{\partial v_2}$ and $\frac{\partial}{\partial v_3}$ and also simultaneously replace the v_i by their expressions in (69) for $i = 1, 2, 3$ in its coefficients. Then, after expanding $\overline{\mathscr{L}}$ in real and imaginary parts:

$$\overline{\mathscr{L}}\big|_M = \frac{\partial}{\partial \overline{z}} - \,i\,z\,\frac{\partial}{\partial u_1} - \big(i\,z^2 + 2\,i\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_2} - \big(z^2 - 2\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_3},$$

one gains a result that can now be summarized as follows.

[3] Indeed, it is proved (*see* [6, 24, 77]) that the Tanaka prolongation of the Levi-Tanaka algebra associated to M *contains* the algebra of infinitesimal automorphisms $\mathfrak{aut}(M)$.

Proposition 12.1. *For the cubic 5-dimensional real algebraic CR-generic model submanifold* $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ *represented near the origin as a graph:*

$$\begin{cases} v_1 := 2\,i\,z\,\overline{z}, \\ v_2 := 2\,i\,\big(z^2\,\overline{z} + z\,\overline{z}^2\big), \\ v_3 := 2\,\big(z^2\,\overline{z} - z\,\overline{z}^2\big), \end{cases}$$

in coordinates:

$$\big(z, w_1, w_2, w_3\big) = \big(x + iy, u_1 + iv_1, u_2 + iv_2, u_3 + iv_3\big),$$

its complex bundles $T^{0,1}M^5_{\mathsf{c}}$ *and* $T^{1,0}M^5_{\mathsf{c}}$ *are generated by:*

$$\boxed{\begin{aligned} \overline{\mathscr{L}} &= \frac{\partial}{\partial \overline{z}} - i\,z\,\frac{\partial}{\partial u_1} - \big(i\,z^2 + 2\,i\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_2} - \big(z^2 - 2\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_3} \qquad \text{and} \\ \mathscr{L} &= \frac{\partial}{\partial z} + i\,\overline{z}\,\frac{\partial}{\partial u_1} + \big(i\,\overline{z}^2 + 2\,i\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_2} - \big(\overline{z}^2 - 2\,z\,\overline{z}\big)\,\frac{\partial}{\partial u_3}. \end{aligned}}$$

12.2. **Length-two Lie bracket.** Between these two complex vector fields $\mathscr{L}$ and $\overline{\mathscr{L}}$, there is of course only one Lie bracket $\big[\mathscr{L}, \overline{\mathscr{L}}\big]$ of length two. This vector field is in fact *imaginary*. In order to get a *real* vector field, we multiply it by i:

$$\mathscr{T} := i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big].$$

A direct computation yields its expression:

$$\boxed{\mathscr{T} = 2\,\frac{\partial}{\partial u_1} + 4\,\big(z + \overline{z}\big)\,\frac{\partial}{\partial u_2} - 4\,i\,\big(z - \overline{z}\big)\,\frac{\partial}{\partial u_3}.}$$

12.3. **Length-three Lie brackets.** In this length, we have two Lie brackets:

$$\mathscr{S} := \big[\mathscr{L}, \mathscr{T}\big], \qquad \overline{\mathscr{S}} := \big[\overline{\mathscr{L}}, \mathscr{T}\big].$$

Again, direct easy computations provide the following expressions for them:

$$\boxed{\begin{aligned} \mathscr{S} &= 4\,\frac{\partial}{\partial u_2} - 4\,i\,\frac{\partial}{\partial u_3} \qquad \text{and} \\ \overline{\mathscr{S}} &= 4\,\frac{\partial}{\partial u_2} + 4\,i\,\frac{\partial}{\partial u_3}. \end{aligned}}$$

Lemma 12.2. *The five vector fields* $\overline{\mathscr{L}}, \mathscr{L}, \mathscr{T}, \overline{\mathscr{S}}, \mathscr{S}$ *constitute a (complex) frame for* $TM^5_{\mathsf{c}} \otimes_{\mathbb{R}} \mathbb{C}$.

Proof. It is sufficient to see from their expressions that these vector fields are linearly independent and hence they constitute a frame. □

12.4. **Other iterated Lie brackets.** We saw that the collection of five vector fields:

$$\big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T}, \overline{\mathscr{L}},\, \mathscr{L}\big\},$$

where:

$$\begin{aligned}\mathscr{T} &:= i\big[\mathscr{L}, \overline{\mathscr{L}}\big],\\ \mathscr{S} &:= \big[\mathscr{L},\, \mathscr{T}\big],\\ \overline{\mathscr{S}} &:= \big[\overline{\mathscr{L}},\, \mathscr{T}\big],\end{aligned}$$

makes up a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM^5_{\mathsf{c}}$. Having five vector fields implies that there are in sum $\binom{5}{2} = 10$ Lie brackets between them. Thus, there remain seven such brackets to be looked at. However, simple computations show that all of these remaining vector fields vanish identically, namely we have:

$$\begin{aligned}&\big[\mathscr{L},\, \mathscr{S}\big] = 0, \quad &&\big[\overline{\mathscr{L}},\, \mathscr{S}\big] = 0, \quad &&\big[\mathscr{L},\, \overline{\mathscr{S}}\big] = 0,\\ &\big[\overline{\mathscr{L}},\, \overline{\mathscr{S}}\big] = 0, \quad &&\big[\mathscr{T},\, \mathscr{S}\big] = 0, \quad &&\big[\mathscr{T},\, \overline{\mathscr{S}}\big] = 0,\\ &\big[\mathscr{S},\, \overline{\mathscr{S}}\big] = 0.\end{aligned}$$

When we will study arbitrary geometry-preserving deformations of this cubic model, the corresponding seven supplementary Lie brackets will be highly more complicated.

12.5. **Passage to a dual coframe and its Darboux-Cartan structure.** On the natural agreement that the coframe:

$$\big\{du_3, du_2, du_1,\, dz,\, d\overline{z}\big\}$$

is dual to the frame:

$$\big\{\tfrac{\partial}{\partial u_3},\, \tfrac{\partial}{\partial u_2},\, \tfrac{\partial}{\partial u_1},\, \tfrac{\partial}{\partial z},\, \tfrac{\partial}{\partial \overline{z}}\big\},$$

let us introduce the coframe:

$$\big\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\big\} \quad \text{which is dual to the frame} \quad \big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T}, \overline{\mathscr{L}},\, \mathscr{L}\big\},$$

that is to say which satisfies by definition:

$$\begin{array}{lllll}
\overline{\sigma_0}(\overline{\mathscr{S}}) = 1 & \overline{\sigma_0}(\mathscr{S}) = 0 & \overline{\sigma_0}(\mathscr{T}) = 0 & \overline{\sigma_0}(\overline{\mathscr{L}}) = 0 & \overline{\sigma_0}\big(\mathscr{L}\big) = 0,\\
\sigma_0(\overline{\mathscr{S}}) = 0 & \sigma_0(\mathscr{S}) = 1 & \sigma_0(\mathscr{T}) = 0 & \sigma_0(\overline{\mathscr{L}}) = 0 & \sigma_0\big(\mathscr{L}\big) = 0,\\
\rho_0(\overline{\mathscr{S}}) = 0 & \rho_0(\mathscr{S}) = 0 & \rho_0(\mathscr{T}) = 1 & \rho_0(\overline{\mathscr{L}}) = 0 & \rho_0\big(\mathscr{L}\big) = 0,\\
\overline{\zeta_0}(\overline{\mathscr{S}}) = 0 & \overline{\zeta_0}(\mathscr{S}) = 0 & \overline{\zeta_0}(\mathscr{T}) = 0 & \overline{\zeta_0}(\overline{\mathscr{L}}) = 1 & \overline{\zeta_0}\big(\mathscr{L}\big) = 0,\\
\zeta_0(\overline{\mathscr{S}}) = 0 & \zeta_0(\mathscr{S}) = 0 & \zeta_0(\mathscr{T}) = 0 & \zeta_0(\overline{\mathscr{L}}) = 0 & \zeta_0\big(\mathscr{L}\big) = 1.
\end{array}$$

Since neither $\mathscr{T}$, nor $\mathscr{S}$, nor $\overline{\mathscr{S}}$ incorporates any $\frac{\partial}{\partial u_j}$, $j = 1, 2, 3$, we have:

$$\zeta_0 = dz \qquad \text{and} \qquad \overline{\zeta_0} = d\overline{z}.$$

In order to launch the Cartan algorithm of equivalence for the cubic model, initially we need the expressions of the five 2-forms $d\overline{\sigma}_0, d\sigma_0, d\rho_0, d\overline{\zeta}_0, d\zeta_0$

in terms of the wedge products of $\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0$. To find them, we use the following well known duality correspondence.

Lemma 12.3. *Given a frame $\big\{\mathscr{L}_1, \dots, \mathscr{L}_n\big\}$ on an open subset of $\mathbb{R}^n$ enjoying the Lie structure:*

$$\big[\mathscr{L}_{i_1},\, \mathscr{L}_{i_2}\big] = \sum_{k=1}^{n} a_{i_1,i_2}^k\, \mathscr{L}_k \qquad\qquad {\scriptstyle (1 \leqslant i_1 < i_2 \leqslant n)},$$

where the a_{i_1,i_2}^k are certain functions on $\mathbb{R}^n$, the dual coframe $\{\omega^1, \dots, \omega^n\}$ satisfying by definition:

$$\omega^k\big(\mathscr{L}_i\big) = \delta_i^k$$

enjoys a quite similar Darboux-Cartan structure, up to an overall minus sign:

$$d\omega^k = -\sum_{1\leqslant i_1<i_2\leqslant n} a_{i_1,i_2}^k\, \omega^{i_1} \wedge \omega^{i_2} \qquad\qquad {\scriptstyle (k = 1 \cdots n)}.$$

Proof. Just apply the so-called Cartan formula $d\omega(\mathscr{X},\mathscr{Y}) = \mathscr{X}\big(\omega(\mathscr{Y})\big) - \mathscr{Y}\big(\omega(\mathscr{X})\big) - \omega\big([\mathscr{X},\mathscr{Y}]\big)$. □

Thank to this Lemma, minding the overall minus sign, we can readily find the expressions of the exterior derivatives of our five 1-forms that provide the associated Darboux-Cartan structure:

$$\boxed{\begin{array}{ll} d\overline{\sigma}_0 = \rho_0 \wedge \overline{\zeta}_0, & d\sigma_0 = \rho_0 \wedge \zeta_0, \\ d\rho_0 = i\,\zeta_0 \wedge \overline{\zeta_0}, & \\ d\overline{\zeta_0} = 0, & d\zeta_0 = 0. \end{array}} \tag{70}$$

12.6. **Ambiguity matrix.** Our next goal is to set up, in a coordinate-free manner, the *Cartan ambiguity matrix* associated to the *problem of local biholomorphic equivalences*:

$$h\colon \quad (z,w) \longmapsto \big(f(z,w),\, g(z,w)\big) =: (z',w')$$

between our cubic 5-dimensional CR-generic cubic model M_{c}^5 and another arbitrary local real analytic CR-generic maximally minimal real submanifold $M'^5 \subset \mathbb{C}^4$ in coordinates $\big(z', w_1', w_2', w_3'\big)$. Naturally, we assume that M'^5 is also equipped with a collection of five vector fields:

$$\Big\{\mathscr{L}',\, \overline{\mathscr{L}}',\, \mathscr{T}',\, \mathscr{S}',\, \overline{\mathscr{S}}'\Big\}$$

where $\mathscr{L}'$ is a local generator of $T^{1,0}M'$ and where:

$$\mathscr{T}' := i\,\big[\mathscr{L}', \overline{\mathscr{L}}'\big], \qquad \mathscr{S}' := \big[\mathscr{L}',\, \mathscr{T}'\big], \qquad \overline{\mathscr{S}}' := \big[\overline{\mathscr{L}}',\, \mathscr{T}'\big],$$

which so that they also make up a frame for its rank-five complexified tangent bundle $\mathbb{C} \otimes_{\mathbb{R}} TM'^5$, in the sense that, at every point $p' \in M'^5$ one has:

$$\mathbb{C} \otimes T_{p'} M'^5 = \mathbb{C}\,\mathscr{L}'\big|_{p'} \oplus \mathbb{C}\,\overline{\mathscr{L}}'\big|_{p'} \oplus \mathbb{C}\,\mathscr{T}'\big|_{p'} \oplus \mathbb{C}\,\mathscr{S}'\big|_{p'} \oplus \mathbb{C}\,\overline{\mathscr{L}}'\big|_{p'}.$$

The fact that M'^5 is maximally minimal in the sense of Proposition 5.3 if and only these fields make up such a frame is essentially tautological, but it will be made more explicit in Proposition 13.2 below.

Thus, suppose that two given hypersurfaces M^5_{c} and M'^5 are CR-equivalent under some (possibly unknown) local equivalence:

$$h\colon \quad M^5_{\mathsf{c}} \longrightarrow M'^5$$

which is a biholomorphism from some neighborhood of M^5_{c} onto some neighborhood of M'^5. Then, the associated differential of h:

$$h_*\colon \quad TM^5_{\mathsf{c}} \longrightarrow TM'^5$$

induces a push-forward complexified map, still denoted with the same symbol:

$$h_*\colon \quad \mathbb{C} \otimes TM^5_{\mathsf{c}} \longrightarrow \mathbb{C} \otimes TM'^5,$$

which naturally defined by (*see* [11], Subsection 3.1):

$$h_*\big(\mathsf{z} \otimes_{\mathbb{R}} \mathscr{X}\big) := \mathsf{z} \otimes_{\mathbb{R}} h_*(\mathscr{X}), \qquad \mathsf{z} \in \mathbb{C}, \quad \mathscr{X} \in T_p M^5_{\mathsf{c}}, \quad p \in M^5_{\mathsf{c}}.$$

Proposition 12.4. *The initial ambiguity matrix associated to the local biholomorphic equivalence problem between the cubic* 5*-dimensional model CR-generic submanifold* M^5_{c} *and any other maximally minimal CR-generic* 5*-dimensional submanifolds* $M'^5 \subset \mathbb{C}^4$ *under local biholomorphic transformations is of the general form:*

$$\begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & \overline{\mathsf{c}} & \overline{\mathsf{e}} & \overline{\mathsf{d}} \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & \mathsf{c} & \mathsf{d} & \mathsf{e} \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & \overline{\mathsf{b}} & \mathsf{b} \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix},$$

where $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{e}, \mathsf{d}$ *are complex numbers. Moreover, the collection of all these matrices makes up a real* 10*-dimensional matrix Lie subgroup of* $\mathsf{GL}_5(\mathbb{C})$.

Proof. As a CR mapping, $h\colon M^5_{\mathsf{c}} \longrightarrow M'^5$ respects the complex structure (*see* [11], page 149, Definition 1), and hence we necessarily have:

$$h_*(\mathscr{L}) := \mathsf{a}'\mathscr{L}',$$

for some nonzero complex function a' defined on M'^5. By conjugation, it obviously comes that one also has:

$$h_*\big(\overline{\mathscr{L}}\big) := \overline{\mathsf{a}}'\,\overline{\mathscr{L}}'.$$

Next, let us look at what happens with Lie brackets. Since the differential commutes with brackets, we have:

$$h_*(\mathscr{T}) = h_*\big(i[\mathscr{L},\overline{\mathscr{L}}]\big) = i\,h_*\big([\mathscr{L},\overline{\mathscr{L}}]\big) = i\,\big[h_*(\mathscr{L}),h_*(\overline{\mathscr{L}})\big] = i\big[\mathsf{a}'\mathscr{L}',\overline{\mathsf{a}}'\overline{\mathscr{L}}'\big],$$

and if we expand this last bracket, we obtain:

$$\begin{aligned} h_*(\mathscr{T}) &= \mathsf{a}'\overline{\mathsf{a}}'\cdot i\,\big[\mathscr{L}',\overline{\mathscr{L}'}\big]\underbrace{-i\,\overline{\mathsf{a}}'\overline{\mathscr{L}}'(\mathsf{a}')}_{=:\,\mathsf{b}'}\cdot\mathscr{L}' + i\,\mathsf{a}'\,\mathscr{L}'\big(\overline{\mathsf{a}}'\big)\cdot\overline{\mathscr{L}}' \\ &=: \mathsf{a}'\overline{\mathsf{a}}'\,\mathscr{T}' + \mathsf{b}'\mathscr{L}' + \overline{\mathsf{b}}'\overline{\mathscr{L}}', \end{aligned}$$

by introducing — in accordance with the general principles of Cartan's approach — a name b' for a certain complicated function that might remain unknown as long the problem of equivalence is not settled.

Now, a quite similar computation for the next Lie bracket:

$$\begin{aligned} h_*(\mathscr{S}) &= h_*\big([\mathscr{L},\mathscr{T}]\big) = \big[h_*(\mathscr{L}),h_*(\mathscr{T})\big] = \\ &= \big[\mathsf{a}'\mathscr{L}',\mathsf{a}'\overline{\mathsf{a}}'\mathscr{T}' + \mathsf{b}'\mathscr{L}' + \overline{\mathsf{b}}'\overline{\mathscr{L}}'\big] = \\ &= \mathsf{a}'\mathsf{a}'\overline{\mathsf{a}}'\mathscr{S}' + 0\mathscr{S}' + \underbrace{\big(\mathsf{a}'\mathscr{L}'\big(\mathsf{a}'\overline{\mathsf{a}}'\big) - i\,\mathsf{a}'\overline{\mathsf{b}}'\big)}_{=:\mathsf{c}'}\mathscr{T}' + \\ &\quad + \underbrace{\big(-\mathsf{a}'\overline{\mathsf{a}}'\mathscr{T}'(\mathsf{a}') + \mathsf{a}'\overline{\mathscr{L}}'(\mathsf{b}') - \mathsf{b}'\mathscr{L}'(\mathsf{a}') - \overline{\mathsf{b}}'\overline{\mathscr{L}}'(\mathsf{a}')\big)}_{=:\mathsf{e}'}\mathscr{L}' + \\ &\quad + \underbrace{\mathsf{a}'\mathscr{L}'\big(\overline{\mathsf{b}}'\big)}_{=:\mathsf{d}'}\overline{\mathscr{L}}', \end{aligned}$$

shows us that:

$$h_*(\mathscr{S}) = \mathsf{a}'\mathsf{a}'\overline{\mathsf{a}}'\mathscr{S}' + 0\mathscr{S}' + \mathsf{c}'\mathscr{T}' + \mathsf{e}'\mathscr{L}' + \mathsf{d}'\overline{\mathscr{L}}',$$

for some three complex-valued functions c', d' and e' defined on M'^5. This means that the initial ambiguity matrix associated to the equivalence problem under local biholomorphic maps for maximally minimal CR-generic submanifolds $M'^5 \subset \mathbb{C}^4$ is of the general form — we drop the primes in the group variables $\mathsf{a}', \mathsf{b}', \mathsf{c}', \mathsf{d}', \mathsf{e}'$ —:

$$\begin{pmatrix} \overline{\mathscr{S}} \\ \mathscr{S} \\ \mathscr{T} \\ \overline{\mathscr{L}} \\ \mathscr{L} \end{pmatrix} = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & \overline{\mathsf{c}} & \overline{\mathsf{e}} & \overline{\mathsf{d}} \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & \mathsf{c} & \mathsf{d} & \mathsf{e} \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & \overline{\mathsf{b}} & \mathsf{b} \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix} \begin{pmatrix} \overline{\mathscr{S}}' \\ \mathscr{S}' \\ \mathscr{T}' \\ \overline{\mathscr{L}}' \\ \mathscr{L}' \end{pmatrix},$$

as announced. The group property follows by verifying that the product of any two such matrices is again a matrix of this general form. □

12.7. **Setting up the equivalence problem.** According to the general principles ([25, 62]), the so-called *lifted coframe* in terms of the dual basis of 1-form then becomes, after a plain matrix transposition:

$$\begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} := \underbrace{\begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a} \end{pmatrix}}_{=:g} \begin{pmatrix} \overline{\sigma_0} \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta_0} \\ \zeta_0 \end{pmatrix}, \tag{71}$$

that is to say:

$$\begin{aligned} \overline{\sigma} &= \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}}\,\overline{\sigma_0}, \\ \sigma &= \mathsf{a}\mathsf{a}\overline{\mathsf{a}}\,\sigma_0, \\ \rho &= \overline{\mathsf{c}}\,\overline{\sigma_0} + \mathsf{c}\,\sigma_0 + \mathsf{a}\overline{\mathsf{a}}\,\rho_0, \\ \overline{\zeta} &= \overline{\mathsf{e}}\,\overline{\sigma_0} + \mathsf{d}\,\sigma_0 + \overline{b}\,\rho_0 + \overline{\mathsf{a}}\,\overline{\zeta_0}, \\ \zeta &= \overline{\mathsf{d}}\,\overline{\sigma_0} + \mathsf{e}\,\sigma_0 + \mathsf{b}\,\rho_0 + \mathsf{a}\,\zeta_0. \end{aligned}$$

Of course, the 1-form ρ is real and the 1-forms $\overline{\sigma}$ and $\overline{\zeta}$ are the conjugate of σ and ζ.

The main aim of the next sections will be to construct one (or more) absolute parallelism(s) on a certain principal bundle(s) over a general maximally minimal $M^5 \subset \mathbb{C}^4$ by performing the Cartan equivalence method with initial data a coframe $\{\overline{\sigma}, \sigma, \rho, \overline{\zeta}, \zeta\}$ related to M^5 — to be computed explicitly later — and the structure group:

$$G := \left\{ g = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a} \end{pmatrix}, \quad \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} \in \mathbb{C} \right\}.$$

In this way, we will encounter invariants of the desired holomorphic equivalence. *But our tricky strategy of approach is to perform this algorithm beforehand in the case where the five* 1*-form* $\{\overline{\sigma}, \sigma, \rho, \overline{\zeta}, \zeta\}$ *come from the model cubic* $M^5_{\mathsf{c}} \subset \mathbb{C}^4$*, because the much simpler computations will serve as a guide before treating the more delicate general case of an arbitrary maximally minimal* $M^5 \subset \mathbb{C}^4$. We essentially admit that the reader knows the so-called *Cartan algorithm* which consists of three major parts *absorbtion, normalization* and *prolongation*, but we briefly provide a summary (*cf.* [62], pp. 305–310).

12.8. **Absorption and normalization: general features.** Suppose that on an n-dimensional (local) manifold M, one has an initial coframe:

$$\theta_0 = \big(\theta_0^1, \dots, \theta_0^n\big)^t$$

of 1-forms written as a column vector. Suppose that a certain closed matrix structure subgroup $G \subset \mathsf{GL}(n, \mathbb{R})$ specifies a certain geometric equivalence problem, and introduce the lifted coframe:

$$\theta = g \cdot \theta_0.$$

Differentiating both sides of this equality gives in concise notation:

$$d\theta = dg \wedge \theta_0 + g \cdot d\theta_0. \tag{72}$$

Assume that the expressions of $d\theta_0^1, \dots, d\theta_0^n$ in terms of the initial 2-forms $\theta_0^j \wedge \theta_0^k$ with $1 \leqslant j < k \leqslant n$, are at hand thanks to some preliminary computations related to the specific geometric features of the problem under study, say of the form:

$$d\theta_0^i = \sum_{1 \leqslant j < k \leqslant n} T_{0jk}^i \cdot \theta_0^j \wedge \theta_0^k \qquad {\scriptstyle (i = 1 \cdots n)},$$

for some explicitly known functions $T_{0\bullet\bullet}^{\bullet}$. For $i = 1, \dots, n$, let $g^{(i)}$ denote the i-th row of the matrix g. Passing through the inverted lifting:

$$\theta_0 = g^{-1} \cdot \theta,$$

it is generally possible — after computations that are almost always delicate (even on a computer) in nontrivial applications — to re-express each scalar expression $g^{(i)} \cdot d\theta_0$ in terms of the lifted basis $\theta^j \wedge \theta^k$ of 2-forms, and one gets expressions of the quite similar general form:

$$g^{(i)} \cdot d\theta_0 = \sum_{1 \leqslant j < k \leqslant n} T_{jk}^i \cdot \theta^j \wedge \theta^k,$$

in which there appear certain functions $T_{\bullet\bullet}^{\bullet}$, called *torsion coefficients*, which express explicitly in terms of the initial structure functions T_{0jk}^i and in terms of the group parameters. This treats the second term of the right-hand side of (72).

For the first term, one rewrites:

$$dg \wedge \theta_0 = \underbrace{dg \cdot g^{-1}}_{\omega_{\mathsf{MC}}} \wedge \underbrace{g \cdot \theta_0}_{\theta},$$

and there naturally appears an $n \times n$ matrix:

$$\begin{aligned}\omega_{\mathsf{MC}} &= \big((\omega_{\mathsf{MC}})_j^i\big)_{1 \leqslant j \leqslant n}^{1 \leqslant i \leqslant n} \\ &:= \sum_{k=1}^n dg_k^i \, \big(g^{-1}\big)_j^k\end{aligned}$$

(where i is the index of rows) which is called the *Maurer-Cartan matrix* associated to G.

As a result, the expression of $d\theta$ rewritten in terms of the lifted coframe of 2-forms $\theta^j \wedge \theta^k$ can be read as follows:

$$d\theta^i = \sum_{j=1}^{n} \big(\omega_{\mathsf{MC}}\big)_j^i \wedge \theta^j + \sum_{1 \leqslant j < k \leqslant n} T_{jk}^i \cdot \theta^j \wedge \theta^k \qquad (i = 1 \cdots n). \tag{73}$$

The obtained equations are called *structure equations.*

To pursue further, it is necessary to express each entry of the Maurer-Cartan matrix in terms of a basis of 1-forms living on the abstract Lie group corresponding to G. Thus, with $r := \dim_{\mathbb{R}} G$, and with a basis of left-invariant 1-forms on the group in question, say $\alpha^1, \dots, \alpha^r$, one can decompose:

$$\big(\omega_{\mathsf{MC}}\big)_j^i = \sum_{s=1}^{r} a_{js}^i \, \alpha^s \qquad (i, \, j = 1 \dots n),$$

in terms of certain *constants* $a_{\bullet\bullet}^{\bullet}$. Consequently, the equations (73) can be brought into the form:

$$d\theta^i = \sum_{k=1}^{n} \bigg(\sum_{s=1}^{r} a_{ks}^i \, \alpha^s + \sum_{j=1}^{k-1} T_{jk}^i \, \theta^j \bigg) \wedge \theta^k \qquad (i = 1 \cdots n). \tag{74}$$

Since the constant coefficients $a_{\bullet\bullet}^{\bullet}$ depend merely on the structure Lie group G, for another similar lifted coframe $\widetilde{\theta} = \widetilde{g} \cdot \widetilde{\theta}_0$ with the same group $\widetilde{G} = G$ but on another manifold $\widetilde{M}$ with another initial coframe $\big(\widetilde{\theta}_0^1, \dots, \widetilde{\theta}_0^n\big)$, one has in a completely similar way:

$$d\widetilde{\theta}^i = \sum_{k=1}^{n} \bigg(\sum_{s=1}^{r} a_{ks}^i \, \widetilde{\alpha}^s + \sum_{j=1}^{k-1} \widetilde{T}_{jk}^i \, \widetilde{\theta}^j \bigg) \wedge \widetilde{\theta}^k \qquad (i = 1 \cdots n), \tag{75}$$

with *unchanged* constants a_{ks}^i. So, if an equivalence holds between the two initial coframes, according to the fundamental result of the theory ([25]), this equivalence lifts up and it provides an equality — through an unwritten pull-back — between the two lifted coframes:

$$\theta = \widetilde{\theta}.$$

Applying exterior differentiation, it follows at once that:

$$d\theta = d\widetilde{\theta}.$$

But then we may subtract the two representation (74) and (75) of the $d\theta^i$, and this gives us:

(76)

$$0 \equiv \sum_{k=1}^{n} \Bigg(\underbrace{\sum_{s=1}^{r} a_{ks}^i \left(\widetilde{\alpha}^s - \alpha^s\right) + \sum_{j=1}^{k-1} \left(\widetilde{T}_{jk}^i - T_{jk}^i\right)\theta^j}_{=:\eta_k^i} \Bigg) \wedge \theta^k \qquad {\scriptstyle (i=1\cdots n)}.$$

Lemma 12.5. (CARTAN) *Let* $\{\vartheta^1, \dots, \vartheta^n\}$ *be a set of locally defined linearly independent 1-forms. Then some* n *arbitrary* 1*-forms* $\eta_1, \dots, \eta_n$ *satisfy* $\sum_{k=1}^n \eta_k \wedge \vartheta^k = 0$ *if and only if they write as* $\eta_k = \sum_{l=1}^n A_{kl}\, \vartheta^l$ *for some symmetric matrix of functions* $A_{kl} = A_{lk}$. □

The Cartan's Lemma plays an essential role in the theory of equivalence problems; here, by applying it to the equality (76), we obtain that, for each $i = 1, \dots, n$, there exist functions A_{kl}^i with $A_{kl}^i = A_{lk}^i$ such that:

$$\sum_{s=1}^{r} a_{ks}^i \left(\widetilde{\alpha}^s - \alpha^s\right) + \sum_{j=1}^{k-1} \left(\widetilde{T}_{jk}^i - T_{jk}^i\right)\theta^j = \sum_{l=1}^{n} A_{kl}^i\, \theta^l \qquad {\scriptstyle (i,\ k=1\cdots n)}. \tag{77}$$

Next, multiplying both sides of each of these n equations by $\theta^1 \wedge \cdots \wedge \theta^n$ brings the equalities:

$$\sum_{s=1}^{r} a_{ks}^i \left(\widetilde{\alpha}^s - \alpha^s\right) \wedge \theta^1 \wedge \cdots \wedge \theta^n = 0 \qquad {\scriptstyle (i=1\cdots n)}.$$

If we apply once again Cartan's Lemma to each of these n relations, we conclude that for every $s = 1, \dots, r$, there exist n functions $z_1^s, \dots, z_n^s$ defined on the base manifold M such that:

$$\widetilde{\alpha}^s = \alpha^s + \sum_{j=1}^{n} z_j^s\, \theta^j \qquad {\scriptstyle (s=1\cdots r)}.$$

Substituting this expression into (76) gives:

$$0 \equiv \sum_{k=1}^{n} \Bigg(\sum_{s=1}^{r} \sum_{j=1}^{n} a_{ks}^i\, z_j^s\, \theta^j + \sum_{j=1}^{k-1} \left(\widetilde{T}_{jk}^i - T_{jk}^i\right)\theta^j \Bigg) \wedge \theta^k.$$

Then all the $\frac{n(n-1)}{2}$ coefficients of the basis 2-forms $\theta^j \wedge \theta^k$ for $1 \leqslant j < k \leqslant n$ must vanish. Extracting these coefficients and equating them to zero yields:

$$\widetilde{T}_{jk}^i = T_{jk}^i + \sum_{s=1}^{r} \left(a_{js}^i \, z_k^s - a_{ks}^i \, z_j^s \right) \qquad {\scriptstyle (i=1\cdots n\,;\ 1\leqslant j<k\leqslant n)}.$$

Proposition 12.6. *In the structure equations* (74):

$$d\theta^i = \sum_{k=1}^{n} \left(\sum_{s=1}^{r} a_{ks}^i \, \alpha^s + \sum_{j=1}^{k-1} T_{jk}^i \, \theta^j \right) \wedge \theta^k \qquad {\scriptstyle (i=1\cdots n)},$$

one can replace each Maurer-Cartan form α^s and each torsion coefficient T_{jk}^i with:

$$\boxed{\begin{aligned} \alpha^s &\longmapsto \alpha^s + \sum_{j=1}^{n} z_j^s \, \theta^j && {\scriptstyle (s=1\cdots r)}, \\ T_{jk}^i &\longmapsto T_{jk}^i + \sum_{s=1}^{r} \left(a_{js}^i \, z_k^s - a_{ks}^i \, z_j^s \right) && {\scriptstyle (i=1\cdots n\,;\ 1\leqslant j<k\leqslant n)}, \end{aligned}} \tag{78}$$

for some arbitrary functions $z_{\bullet}^{\bullet}$ on the base manifold M. □

12.9. **Absorbtion and normalization for the model.** Now, differentiating both sides of (71) gives:

$$d \begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} = dg \, \wedge \begin{pmatrix} \overline{\sigma}_0 \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta}_0 \\ \zeta_0 \end{pmatrix} + g \cdot \begin{pmatrix} d\overline{\sigma}_0 \\ d\sigma_0 \\ d\rho_0 \\ d\overline{\zeta}_0 \\ d\zeta_0 \end{pmatrix}.$$

Also, differentiating the entries of the matrix g and putting the expressions of $d\overline{\sigma}_0$, $d\sigma_0$, $d\rho_0$, $d\overline{\zeta}_0$, $d\zeta_0$ in the second term of the above equation gives the structure equations as follows:

(79)

$$d \begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} = \underbrace{\begin{pmatrix} \overline{\mathsf{a}}^2 \, d\mathsf{a} + 2\mathsf{a}\overline{\mathsf{a}} \, d\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & 2\mathsf{a}\overline{\mathsf{a}} \, d\mathsf{a} + \mathsf{a}^2\overline{\mathsf{a}} \, d\overline{\mathsf{a}} & 0 & 0 & 0 \\ d\overline{\mathsf{c}} & d\mathsf{c} & \mathsf{a} \, d\overline{\mathsf{a}} + \overline{\mathsf{a}} \, d\mathsf{a} & 0 & 0 \\ d\overline{\mathsf{e}} & d\mathsf{d} & d\overline{\mathsf{b}} & d\overline{\mathsf{a}} & 0 \\ d\overline{\mathsf{d}} & d\mathsf{e} & d\mathsf{b} & 0 & d\mathsf{a} \end{pmatrix}}_{dg} \wedge \begin{pmatrix} \overline{\sigma}_0 \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta}_0 \\ \zeta_0 \end{pmatrix} +$$

$$+ \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} \, d\overline{\sigma}_0 \\ \mathsf{a}\mathsf{a}\overline{\mathsf{a}} \, d\sigma_0 \\ \overline{\mathsf{c}} \, d\overline{\sigma}_0 + \mathsf{c} \, d\sigma_0 + \mathsf{a}\overline{\mathsf{a}} \, d\rho_0 \\ \overline{\mathsf{e}} \, d\overline{\sigma}_0 + \mathsf{d} \, d\sigma_0 + \overline{\mathsf{b}} \, d\rho_0 + \overline{\mathsf{a}} \, d\overline{\zeta}_0 \\ \overline{\mathsf{d}} \, d\overline{\sigma}_0 + \mathsf{e} \, d\sigma_0 + \mathsf{b} \, d\rho_0 + \mathsf{a} \, d\zeta_0 \end{pmatrix}.$$

Moreover, since the determinant of the matrix g is $(\mathsf{a}\overline{\mathsf{a}})^5$ and since the variable a, lying on the diagonal of the matrix group, must necessarily be nonzero, g is invertible with inverse:

$$g^{-1} = \begin{pmatrix} \frac{1}{\mathsf{a}\overline{\mathsf{a}}^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}} & 0 & 0 & 0 \\ -\frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3} & -\frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2} & \frac{1}{\mathsf{a}\overline{\mathsf{a}}} & 0 & 0 \\ \frac{\overline{\mathsf{b}}\,\overline{\mathsf{c}}-\overline{\mathsf{e}}\mathsf{a}\overline{\mathsf{a}}}{\mathsf{a}^2\overline{\mathsf{a}}^4} & \frac{\overline{\mathsf{b}}\mathsf{c}-\mathsf{a}\overline{\mathsf{a}}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^3} & -\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2} & \frac{1}{\overline{\mathsf{a}}} & 0 \\ \frac{\mathsf{b}\overline{\mathsf{c}}-\mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} & \frac{\mathsf{b}\mathsf{c}-\mathsf{e}\mathsf{a}\overline{\mathsf{a}}}{\mathsf{a}^4\overline{\mathsf{a}}^2} & -\frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}} & 0 & \frac{1}{\mathsf{a}} \end{pmatrix}.$$

Multiplying this matrix by the transpose of $\big(\overline{\sigma}, \sigma, \rho, \overline{\zeta}, \zeta\big)$ gives the inverse expressions:

$$\begin{aligned} \sigma_0 &= \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,\sigma, \\ \rho_0 &= -\frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\sigma} - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\sigma + \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\rho, \\ \zeta_0 &= \frac{\mathsf{b}\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{\sigma} + \frac{\mathsf{b}\mathsf{c} - \mathsf{a}\overline{\mathsf{a}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,\sigma - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\,\rho + \frac{1}{\mathsf{a}}\,\zeta, \end{aligned} \tag{80}$$

with plain conjugations to obtain $\overline{\sigma}_0$ and $\overline{\zeta}_0$ — remembering that both ρ_0 and ρ are real.

Thus, one can then replace the first term $dg \wedge (\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0)^t$ of (79) by:

$$(dg \cdot g^{-1}) \wedge \big[g \cdot \big(\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\big)^t\big],$$

and this, according to (71), gives:

$$dg \wedge \begin{pmatrix} \overline{\sigma}_0 \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta}_0 \\ \zeta_0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2\,\overline{\alpha}_1 + \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 2\,\alpha_1 + \overline{\alpha}_1 & 0 & 0 & 0 \\ \overline{\alpha}_2 & \alpha_2 & \alpha_1 + \overline{\alpha}_1 & 0 & 0 \\ \overline{\alpha}_3 & \overline{\alpha}_4 & \overline{\alpha}_5 & \overline{\alpha}_1 & 0 \\ \alpha_4 & \alpha_3 & \alpha_5 & 0 & \alpha_1 \end{pmatrix}}_{\omega_{\mathsf{MC}} := dg\cdot g^{-1}} \wedge \underbrace{\begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix}}_{g\cdot(\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0)^t},$$

where the *Maurer-Cartan 1-forms* — minding the typographical distinction between the group variable d and the standard symbol d for exterior differentiation — are:

$$\begin{aligned}
\alpha_1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\alpha_2 &:= \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} - \frac{\mathsf{c}\,d\mathsf{a}}{\mathsf{a}^3\overline{\mathsf{a}}} - \frac{\mathsf{c}\,d\overline{\mathsf{a}}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
\alpha_3 &:= -\frac{\mathsf{c}\,d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \left(\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\right) d\mathsf{a} + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,d\mathsf{e},\\
\alpha_4 &:= \frac{d\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2} - \frac{\overline{\mathsf{c}}\,d\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^3} + \left(\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right) d\mathsf{a},\\
\alpha_5 &:= \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}} - \frac{\mathsf{b}\,d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}}.
\end{aligned}$$

Here, the so-defined 5×5 matrix ω_{MC} of 1-forms is of course the *Maurer-Cartan form* associated to our 10-dimensional structure group G.

Next, the above expressions of σ_0, ρ_0, ζ_0 in terms of the 1-forms σ, ρ, ζ and the conjugate expressions as well enable us to express the five 2-forms $d\overline{\sigma}_0$, $d\sigma_0$, $d\rho_0$, $d\overline{\zeta}_0$, $d\zeta_0$ of (70) in terms of exterior products by pairs of the 1-forms $\overline{\sigma}$, σ, ρ, $\overline{\zeta}$, $d\zeta$. After non-painful computations, we obtain:

$$\begin{aligned}
d\sigma_0 = \sigma\wedge\overline{\sigma}\left(\frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^4} - \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\right) + \sigma\wedge\rho\left(\frac{\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\right) + \sigma\wedge\zeta\left(\frac{\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\right) +\\
+ \overline{\sigma}\wedge\rho\left(\frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\right) + \overline{\sigma}\wedge\zeta\left(-\frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\right) + \rho\wedge\zeta\left(\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\right).
\end{aligned}$$

and:

$$\begin{aligned}
d\rho_0 = \sigma\wedge\overline{\sigma}\bigg(-i\,\frac{\mathsf{b}\mathsf{c}\overline{\mathsf{e}}}{\mathsf{a}^5\overline{\mathsf{a}}^5} - i\,\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^5} + i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^5\overline{\mathsf{a}}^5} + i\,\frac{\overline{\mathsf{b}}\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^5} - i\,\frac{\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\bigg) +\\
+ \sigma\wedge\rho\left(i\,\frac{\overline{\mathsf{b}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^3} - i\,\frac{\mathsf{b}\mathsf{d}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\right) + \sigma\wedge\zeta\left(-i\,\frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\right)\\
+ \sigma\wedge\overline{\zeta}\left(i\,\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3} - i\,\frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\right) + \overline{\sigma}\wedge\rho\left(-i\,\frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\overline{\mathsf{b}\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\right) +\\
+ \overline{\sigma}\wedge\zeta\left(-i\,\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\right) + \overline{\sigma}\wedge\overline{\zeta}\left(i\,\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} - i\,\frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\right) +\\
+ \rho\wedge\zeta\left(i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right) + \rho\wedge\overline{\zeta}\left(-i\,\frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right) + \frac{i}{\mathsf{a}\overline{\mathsf{a}}}\,\zeta\wedge\overline{\mathsf{a}},
\end{aligned}$$

while trivially:

$$d\zeta_0 = 0.$$

After that, we can express $d\overline{\sigma}$, $d\sigma$, $d\rho$, $d\overline{\zeta}$ in the structure equation (79) in terms of the one forms $\overline{\sigma}$, σ, ρ, $\overline{\zeta}$ instead of $\overline{\sigma}_0$, σ_0, ρ_0, $\overline{\zeta}_0$ and we obtain:

$$
\begin{aligned}
d\sigma = \big(2\,\alpha_1 + \overline{\alpha}_1\big) \wedge \sigma + & \\
+ \left(\frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\right)\sigma\wedge\overline{\sigma} + \left(\frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}}\right)\sigma\wedge\rho + \left(-\,\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}}\right)\sigma\wedge\zeta + 0 + & \\
+ \left(\frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2}\right)\overline{\sigma}\wedge\rho + \left(-\,\frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2}\right)\overline{\sigma}\wedge\zeta + 0 + & \\
+ \rho\wedge\zeta + 0 + & \\
+ 0. &
\end{aligned}
$$

For the order between the 10 two-forms, we choose:

$$
\begin{array}{cccc}
\sigma\wedge\overline{\sigma}, & \sigma\wedge\rho, & \sigma\wedge\zeta, & \sigma\wedge\overline{\zeta}, \\
 & \overline{\sigma}\wedge\rho, & \overline{\sigma}\wedge\zeta, & \overline{\sigma}\wedge\overline{\zeta}, \\
 & & \rho\wedge\zeta, & \rho\wedge\overline{\zeta}, \\
 & & & \zeta\wedge\overline{\zeta}.
\end{array}
$$

Let us abbreviate this first structure equation by introducing specific numbered letters $U_\bullet$ for the torsion coefficients:

$$
\begin{aligned}
d\sigma &= \big(2\,\alpha_1 + \overline{\alpha}_1\big)\wedge\sigma + \\
&\quad + U_1\,\sigma\wedge\overline{\sigma} + U_2\,\sigma\wedge\rho + U_3\,\sigma\wedge\zeta + \\
&\quad + U_5\,\overline{\sigma}\wedge\rho + U_6\,\overline{\sigma}\wedge\zeta + \rho\wedge\zeta.
\end{aligned}
\tag{81}
$$

Similarly, one may compute elementarily the torsion coefficients V_1, V_2, V_3, V_4, V_8 which appear in the finalized expression of:

$$
\begin{aligned}
d\rho = \alpha_2\wedge\sigma + \overline{\alpha}_2\wedge\overline{\sigma} + \alpha_1\wedge\rho + \overline{\alpha}_1\wedge\rho + & \\
+ V_1\,\sigma\wedge\overline{\sigma} + V_2\,\sigma\wedge\rho + V_3\,\sigma\wedge\zeta + V_4\,\sigma\wedge\overline{\zeta} + & \\
+ \overline{V_2}\,\overline{\sigma}\wedge\rho + \overline{V_4}\,\overline{\sigma}\wedge\zeta + \overline{V_3}\,\overline{\sigma}\wedge\overline{\zeta} + & \\
+ V_8\,\rho\wedge\zeta + \overline{V_8}\,\rho\wedge\overline{\zeta} + & \\
+ i\,\zeta\wedge\overline{\zeta}, &
\end{aligned}
\tag{82}
$$

and their explicit expressions are:

$$
\begin{aligned}
V_1 &= \frac{\mathsf{c}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + \frac{\mathsf{c}\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^4} - \frac{\mathsf{e}\mathsf{c}\overline{\mathsf{c}}}{\mathsf{a}^5\overline{\mathsf{a}}^4} - \frac{\overline{\mathsf{c}}\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^4\overline{\mathsf{a}}^5} + \frac{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^5} - \\
&\quad - i\,\frac{\mathsf{b}\mathsf{c}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^4} - i\,\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{b}\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^4\overline{\mathsf{a}}^4} + i\,\frac{\overline{\mathsf{b}}\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^4} - i\,\frac{\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}, \\
V_2 &= \frac{\mathsf{c}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^2} + \frac{\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + i\,\frac{\overline{\mathsf{b}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2} - i\,\frac{\mathsf{b}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^2},
\end{aligned}
$$

$$
\begin{aligned}
V_3 &= -\,\frac{\mathsf{c}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^4} - i\,\frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + i\,\frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
V_4 &= -\,\frac{\mathsf{c}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2} - i\,\frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
V_8 &= \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} + i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}}.
\end{aligned}
$$

Lastly, the 10 torsion coefficients $W_\bullet$ in:

$$
\begin{aligned}
d\zeta = \alpha_3\wedge\sigma + \alpha_4\wedge\overline{\sigma} + \alpha_5\wedge\rho + \alpha_1\wedge\zeta &+ \\
+ W_1\,\sigma\wedge\overline{\sigma} + W_2\,\sigma\wedge\rho + W_3\,\sigma\wedge\zeta + W_4\,\sigma\wedge\overline{\zeta} &+ \\
+ W_5\,\overline{\sigma}\wedge\rho + W_6\,\overline{\sigma}\wedge\zeta + W_7\,\overline{\sigma}\wedge\overline{\zeta} &+ \\
+ W_8\,\rho\wedge\zeta + W_9\,\rho\wedge\overline{\zeta} &+ \\
+ W_{10}\,\zeta\wedge\overline{\zeta}&,
\end{aligned}
\tag{83}
$$

are equal to:

$$
\begin{aligned}
W_1 &= \frac{\mathsf{d}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + \frac{\mathsf{c}\overline{\mathsf{d}}\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^4} - \frac{\overline{\mathsf{c}}\mathsf{e}\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^4} - \frac{\overline{\mathsf{c}}\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^5} + \frac{\mathsf{c}\overline{\mathsf{d}}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^5} - \\
&\quad - i\,\frac{\mathsf{b}\mathsf{b}\mathsf{c}\overline{\mathsf{e}}}{\mathsf{a}^5\overline{\mathsf{a}}^5} - i\,\frac{\mathsf{b}\overline{\mathsf{b}}\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^5} + i\,\frac{\mathsf{b}\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\mathsf{b}\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^5\overline{\mathsf{a}}^5} + i\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^5} - i\,\frac{\mathsf{b}\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^4},\\
W_2 &= \frac{\mathsf{e}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^4} + \frac{\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^3} - i\,\frac{\mathsf{b}\mathsf{b}\mathsf{d}}{\mathsf{a}^4\overline{\mathsf{a}}^3},\\
W_3 &= -\,\frac{\mathsf{c}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - i\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{b}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^2},\\
W_4 &= -\frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{b}\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^4} - i\,\frac{\mathsf{b}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2},\\
W_5 &= \frac{\overline{\mathsf{d}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + \frac{\overline{\mathsf{e}}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^4} - i\,\frac{\mathsf{b}\mathsf{b}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\overline{\mathsf{b}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^4},\\
W_6 &= -\,\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - i\,\frac{\mathsf{b}\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^3},\\
W_7 &= -\,\frac{\overline{\mathsf{c}}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} - i\,\frac{\mathsf{b}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3},\\
W_8 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}} + i\,\frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
W_9 &= \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2} - i\,\frac{\mathsf{b}\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
W_{10} &= i\,\frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}}.
\end{aligned}
$$

12.10. **First-loop absorbtion.** Now, we are ready to realize which of the above torsion coefficients are *normalizable*. According to Proposition 12.6, we must modify the five 1-forms α_1, α_2, α_3, α_4, α_5 by adding to them general linear combinations of the 1-forms σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$:

$$\begin{aligned}
\alpha_1 &\longmapsto \alpha_1 + p_1\,\sigma + q_1\,\overline{\sigma} + r_1\,\rho + s_1\,\zeta + t_1\,\overline{\zeta},\\
\alpha_2 &\longmapsto \alpha_2 + p_2\,\sigma + q_2\,\overline{\sigma} + r_2\,\rho + s_2\,\zeta + t_2\,\overline{\zeta},\\
\alpha_3 &\longmapsto \alpha_3 + p_3\,\sigma + q_3\,\overline{\sigma} + r_3\,\rho + s_3\,\zeta + t_3\,\overline{\zeta},\\
\alpha_4 &\longmapsto \alpha_4 + p_4\,\sigma + q_4\,\overline{\sigma} + r_4\,\rho + s_4\,\zeta + t_4\,\overline{\zeta},\\
\alpha_5 &\longmapsto \alpha_5 + p_5\,\sigma + q_5\,\overline{\sigma} + r_5\,\rho + s_5\,\zeta + t_5\,\overline{\zeta},
\end{aligned}$$

with 25 arbitrary real analytic functions p_i, q_i, r_i, s_i, t_i, hence by conjugation, we also have the two useful replacements:

$$\begin{aligned}
\overline{\alpha}_1 &\longmapsto \overline{\alpha}_1 + \overline{q}_1\,\sigma + \overline{p}_1\,\overline{\sigma} + \overline{r}_1\,\rho + \overline{t}_1\,\zeta + t_1\,\overline{\zeta},\\
\overline{\alpha}_2 &\longmapsto \overline{\alpha}_2 + \overline{q}_2\,\sigma + \overline{p}_2\,\overline{\sigma} + \overline{r}_2\,\rho + \overline{t}_2\,\zeta + t_2\,\overline{\zeta}.
\end{aligned}$$

Performing the replacement with these modified Maurer-Cartan forms, we obtain the three new structure equations which change the expressions of $d\sigma, d\rho$ and $d\zeta$ in (81), (82) and (83) as follows:

$$\begin{aligned}
d\sigma = &\big(2\alpha_1 + \overline{\alpha}_1\big)\wedge\sigma +\\
&+ \sigma\wedge\overline{\sigma}\big[U_1 - 2q_1 - \overline{p}_1\big] + \sigma\wedge\rho\big[U_2 - 2r_1 - \overline{r}_1\big] + \sigma\wedge\zeta\big[U_3 - 2s_1 - \overline{t}_1\big] + \sigma\wedge\overline{\zeta}\big[0 - 2t_1 - \overline{s}_1\big]\\
&+ \overline{\sigma}\wedge\rho\big[U_5\big] + \overline{\sigma}\wedge\zeta\big[U_6\big] + 0 +\\
&+ \rho\wedge\zeta,
\end{aligned}$$

$$\begin{aligned}
d\rho = &\,\alpha_2\wedge\sigma + \overline{\alpha}_2\wedge\overline{\sigma} + \alpha_1\wedge\rho + \overline{\alpha}_1\wedge\rho +\\
&+ \sigma\wedge\overline{\sigma}\big[V_1 - q_2 - \overline{q}_2\big] + \sigma\wedge\rho\big[V_2 - r_2 + p_1 + \overline{q}_1\big] + \sigma\wedge\zeta\big[V_3 - s_2\big] + \sigma\wedge\overline{\zeta}\big[V_4 - t_2\big]\\
&+ \overline{\sigma}\wedge\rho\big[\overline{V}_2 - \overline{r}_2 + q_1 + \overline{p}_1\big] + \overline{\sigma}\wedge\zeta\big[\overline{V}_4 - \overline{t}_2\big] + \overline{\sigma}\wedge\overline{\zeta}\big[\overline{V}_3 - \overline{s}_2\big] +\\
&+ \rho\wedge\zeta\big[V_8 - s_1 - \overline{t}_1\big] + \rho\wedge\overline{\zeta}\big[\overline{V}_8 - t_1 - \overline{s}_1\big] +\\
&+ i\,\zeta\wedge\overline{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\zeta = &\,\alpha_3\wedge\sigma + \alpha_4\wedge\overline{\sigma} + \alpha_5\wedge\rho + \alpha_1\wedge\zeta +\\
&+ \sigma\wedge\overline{\sigma}\big[W_1 - q_3 + p_4\big] + \sigma\wedge\rho\big[W_2 - r_3 + p_5\big] + \sigma\wedge\zeta\big[W_3 - s_3 + p_1\big] + \sigma\wedge\overline{\zeta}\big[W_4 - t_3\big]\\
&+ \overline{\sigma}\wedge\rho\big[W_5 - r_4 + q_5\big] + \overline{\sigma}\wedge\zeta\big[W_6 - s_4 + q_1\big] + \overline{\sigma}\wedge\overline{\zeta}\big[W_7 - t_4\big] +\\
&+ \rho\wedge\zeta\big[W_8 - s_5 + r_1\big] + \rho\wedge\overline{\zeta}\big[W_9 - t_5\big] +\\
&+ \zeta\wedge\overline{\zeta}\big[W_{10} - t_1\big].
\end{aligned}$$

12.11. **First loop normalization.** Now, according to the general principles ([62], Chapter 10), in order to know what are the precise linear combinations of the 20 torsion coefficients:

$$
\begin{aligned}
&U_1, \quad U_2, \quad U_3, \quad U_5, \quad U_6,\\
&V_1, \quad V_2, \quad V_3, \quad V_4, \quad V_8,\\
&W_1, \quad W_2, \quad W_3, \quad W_4, \quad W_5, \quad W_6, \quad W_7, \quad W_8, \quad W_9, \quad W_{10}
\end{aligned}
$$

that are *necessarily normalizable*, one must determine all possible linear combinations of the following $6+5+10=21$ equations — including their (unwritten) conjugates —:
(84)

$$
\left[
\begin{aligned}
U_1 &= 2q_1+\overline{p}_1,\\
U_2 &= 2r_1+\overline{r}_1,\\
U_3 &= 2s_1+\overline{t}_1,\\
0 &= 2t_1+\overline{s}_1,\\
U_5 &= 0,\\
U_6 &= 0,
\end{aligned}
\right.
\qquad
\left[
\begin{aligned}
V_1 &= q_2-\overline{q}_2,\\
V_2 &= r_2-p_1-\overline{q}_1,\\
V_3 &= s_2,\\
V_4 &= t_2,\\
V_8 &= s_1+\overline{t}_1,
\end{aligned}
\right.
\qquad
\left[
\begin{aligned}
W_1 &= q_3-p_4,\\
W_2 &= r_3-p_5,\\
W_3 &= s_3-p_1,\\
W_4 &= t_3,\\
W_5 &= r_4-q_5,\\
W_6 &= s_4-q_1,\\
W_7 &= t_4,\\
W_8 &= s_5-r_1,\\
W_9 &= t_5,\\
W_{10} &= t_1.
\end{aligned}
\right.
$$

so as to obtain null right-hand sides, though without exchanging any left-hand side term with any right-hand side term.

One then easily convinces oneself just visually that some complete appropriate linear combinations are:

$$
\begin{aligned}
0 &= U_5,\\
0 &= U_6,\\
0 &= U_3-3\,V_8,\\
0 &= V_8+\overline{W}_{10},
\end{aligned}
$$

and that is all. This means that these four right linear combinations are normalizable — minding that this synoptic way of finding normalizable linear combinations with the notation:

‘$0=$ combination of torsion coefficients’

does not necessarily mean that one will assign the value 0 to them, and in fact, some normalizable right-hand sides could well be assigned the value 1 in certain circumstances.

Now, if one just replaces the appearing torsion coefficients with the values computed a while ago, one plainly obtains:

$$\begin{aligned} U_5 &= \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\ U_6 &= -\frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\ U_3 - 3\,V_8 &= -\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} - 3\,\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}}, \\ V_8 + \overline{W}_{10} &= \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}}. \end{aligned}$$

Since none of the 3 group parameters b, c, d appearing in denominator place here does belong to the diagonal of our initial lower triangular matrix subgroup of $\mathsf{GL}_5(\mathbb{C})$, it is clear that we can normalize all of them to zero, namely we can set:

$$\boxed{\mathsf{b} := 0, \qquad \mathsf{c} := 0, \qquad \mathsf{d} := 0.}$$

12.12. **Second-loop absorbtion and normalization.** This assignment then considerably simplifies a lot of the expressions of the three differentials $d\sigma$, $d\rho$, $d\zeta$ in (81), (82), (83)).

Also, the Maurer-Cartan 1-forms α_2, α_4, α_5 then vanish identically and the new Maurer-Cartan matrix ω_{MC} changes into the form — we employ a new letter β instead of α in this second loop and we re-number them —:

$$\omega_{\mathsf{MC}} = \begin{pmatrix} 2\,\overline{\beta}_1 + \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 2\,\beta_1 + \overline{\beta}_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 + \overline{\beta}_1 & 0 & 0 \\ \overline{\beta}_2 & 0 & 0 & \overline{\beta}_1 & 0 \\ 0 & \beta_2 & 0 & 0 & \beta_1 \end{pmatrix},$$

with two non-vanishing 1-forms:

$$\begin{aligned} \beta_1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\ \beta_2 &:= -\frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\,d\mathsf{a} + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,d\mathsf{e}. \end{aligned}$$

Moreover, among the 20 torsion coefficients $U_\bullet$, $V_\bullet$, $W_\bullet$, only the following five simplified ones remain non-vanishing:

$$
\begin{aligned}
U_2 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
V_1 &= \frac{i\,\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^3},\\
V_4 &= -\frac{i\,\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
W_2 &= \frac{\mathsf{e}^2}{\mathsf{a}^4\overline{\mathsf{a}}^2},\\
W_8 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}}.
\end{aligned}
$$

Hence, one has the following reduced expressions for $d\sigma$, $d\rho$, $d\zeta$:

$$
\begin{aligned}
d\sigma &= \big(2\,\beta_1 + \overline{\beta}_1\big)\wedge\sigma + U_2\,\sigma\wedge\rho + \rho\wedge\zeta,\\
d\rho &= (\beta_1+\overline{\beta}_1)\wedge\rho + V_1\,\sigma\wedge\overline{\sigma} + V_4\,\sigma\wedge\overline{\zeta} + \overline{V}_4\,\overline{\sigma}\wedge\zeta + i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \beta_2\wedge\sigma + \beta_1\wedge\zeta + W_2\,\sigma\wedge\rho + W_8\,\rho\wedge\zeta.
\end{aligned}
\tag{85}
$$

Again we must modify the two 1-forms β_1, β_2 by adding to them general linear combinations of the 1-forms σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$:

$$
\begin{aligned}
\beta_1 &\longmapsto \beta_1 + p_1\sigma + q_1\overline{\sigma} + r_1\rho + s_1\zeta + t_1\overline{\zeta},\\
\beta_2 &\longmapsto \beta_2 + p_2\sigma + q_2\overline{\sigma} + r_2\rho + s_2\zeta + t_2\overline{\zeta},
\end{aligned}
$$

with 10 arbitrary real analytic functions p_i, q_i, r_i, s_i, t_i. To get the new absorption equations, it just suffices to set $\mathsf{b} := 0$, $\mathsf{c} := 0$, $\mathsf{d} := 0$ in the ones obtained a moment ago, remembering that some torsion coefficients vanish (are normalized) and that there is a change of numbering due to $\beta_2 := \alpha_3$:

$$
\left[
\begin{aligned}
U_1 &= 2q_1 + \overline{p}_1,\\
U_2 &= 2r_1 + \overline{r}_1,\\
0 &= 2s_1 + \overline{t}_1,\\
0 &= 2t_1 + \overline{s}_1,\\
0 &= 0,\\
0 &= 0,
\end{aligned}
\right.
\qquad
\left[
\begin{aligned}
V_1 &= 0,\\
0 &= -\,p_1 - \overline{q}_1,\\
0 &= 0,\\
V_4 &= 0,\\
0 &= s_1 + \overline{t}_1,
\end{aligned}
\right.
\qquad
\left[
\begin{aligned}
0 &= q_2,\\
W_2 &= r_2,\\
0 &= s_2,\\
0 &= t_2,\\
0 &= 0,\\
0 &= -\,q_1,\\
0 &= 0,\\
W_8 &= -\,r_1,\\
0 &= 0,\\
0 &= t_1.
\end{aligned}
\right.
$$

Visually, one sees that three linear combinations of torsion coefficients are normalizable at this step, which are represented as the equations:

$$\begin{aligned} V_1 &= 0, \\ V_4 &= 0, \\ 2\,W_8 + \overline{W}_8 + U_2 &= 0. \end{aligned}$$

Thus, we can normalize:

$$\begin{aligned} V_1 &= \frac{i\,\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}, \\ V_4 &= -\,\frac{i\,\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}}, \\ 2\,W_8 + \overline{W}_8 + U_2 &= \frac{3\,\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \end{aligned}$$

and immediately, this amounts to just annihilating:

$$\boxed{\mathsf{e} := 0}.$$

After really setting this group parameter null, all the torsion coefficients vanish identically, the Maurer-Cartan 1-form β_2 also annihilates, and the expression of β_1 changes into the simpler form:

$$\beta_1 = \frac{d\mathsf{a}}{\mathsf{a}}.$$

Finally, one has the following greatly simplified expressions for $d\sigma$, $d\rho$, $d\zeta$:

$$\begin{aligned} d\sigma &= \big(2\,\beta_1 + \overline{\beta}_1\big) \wedge \sigma + \rho \wedge \zeta, \\ d\rho &= \big(\beta_1 + \overline{\beta}_1\big) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \beta_1 \wedge \zeta, \end{aligned} \tag{86}$$

in which no more nonconstant torsion coefficient appears. Therefore, executing once more an absorbtion-normalization loop is not valuable and we should start the *prolongation step*. But beforehand, let us present an important application of Cartan's Lemma.

Lemma 12.7. *The Maurer-Cartan form $\beta_1 = \frac{d\mathsf{a}}{\mathsf{a}}$ is the only 1-form which enjoys the structure equations* (86)*.*

Proof. Let β_1' be another 1-form, enjoying the structure equations (86). Then subtracting by pairs the expressions of $d\sigma$, $d\rho$, $d\zeta$ with β_1 and with β_1' immediately gives:

$$\begin{aligned} 0 &\equiv \big(2\,\beta_1 + \overline{\beta}_1 - 2\,\beta_1' - \overline{\beta}_1'\big) \wedge \sigma, \\ 0 &\equiv \big(\beta_1 + \overline{\beta}_1 - \beta_1' - \overline{\beta}_1'\big) \wedge \rho, \\ 0 &\equiv \big(\beta_1 - \beta_1'\big) \wedge \zeta. \end{aligned}$$

Applying the Cartan's Lemma 12.5 on the second and on the third equations yields that one has:

$$\begin{aligned} \beta_1 - \beta_1' + \overline{\beta}_1 - \overline{\beta}_1' &= A\,\rho, \\ \beta_1 - \beta_1' &= B\,\zeta, \end{aligned} \tag{87}$$

for some certain functions A, B. Now, substituting the expression $\beta_1' = \beta_1 - B\,\zeta$ obtained from the second equality into the first one gives:

$$B\,\zeta + \overline{B}\,\overline{\zeta} - A\,\rho = 0,$$

and consequently, we have $A = B = 0$, due to the fact that σ, ρ, ζ are linearly independent. Now, the second equation of (87) immediately implies that:

$$\beta_1' = \beta_1,$$

and hence the Maurer-Cartan 1-form β_1 is unique, as was claimed. □

12.13. **Prolongation: checking firstly non-involutiveness.** At this stage, we shall be interested in testing whether the obtained coframe $\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}\}$ is involutive. If this holds, the structure group is infinite-dimensional, else we have to prolong the lifted coframe. We stick to Chapter 11 of Olvers' book [62].

Assume that after performing all loops of the absorbtion-normalization procedure, the structure equations for the lifted coframe have the form:

$$d\theta^i = \sum_{k=1}^{n} \bigg(\sum_{s=1}^{r} a_{ks}^i\,\alpha^s + \sum_{j=1}^{k-1} T_{jk}^i\,\theta^j \bigg) \wedge \theta^k \qquad {\scriptstyle (i = 1 \cdots n)},$$

where $\alpha^1, \dots, \alpha^r$ are a basis of Maurer-Cartan 1-forms on the group, that is to say, assume that none of the essential — *i.e.* unabsorbable — torsion coefficients depend explicitly on the group parameters, since otherwise, some parameters would be normalizable. Modifying each Maurer-Cartan form by adding to it a linear combination of coframe elements:

$$\alpha^s \longmapsto \alpha^s + \sum_{j=1}^{n} z_j^s\,\theta^j \qquad {\scriptstyle (s = 1 \cdots r)},$$

results in the *linear absorption equations*:

$$\sum_{s=1}^{r} \big(a_{ks}^i\,z_j^s - a_{js}^i\,z_k^s \big) = T_{jk}^i \qquad {\scriptstyle (i = 1 \cdots n;\ 1 \leqslant j < k \leqslant n)}.$$

Definition 12.8. The *degree of indeterminacy* of a lifted coframe is the number of free variables in the associated homogeneous linear equations:

$$\sum_{s=1}^{r} \left(a_{ks}^{i}\, z_j^s - a_{js}^{i}\, z_k^s\right) = 0 \qquad\qquad (i=1\cdots n;\ 1\leqslant j<k\leqslant n).$$

Next, for a given vector $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, introduce $I(v)$ to be the $n \times r$ matrix with the entries:

$$I(v)_s^i := \sum_{l=1}^{n} a_{sl}^{i}\, v^l \qquad\qquad (i=1\cdots n;\ s=1\cdots r).$$

Definition 12.9. The *first* $n-1$ *reduced characters* $s_1', s_2', \dots, s_{n-1}'$ are defined, for $k = 1, 2, \dots, n-1$, recursively by:

$$s_1' + s_2' + \ldots + s_k' = \max\left\{ \text{rank} \begin{pmatrix} I(v_1) \\ I(v_2) \\ \vdots \\ I(v_k) \end{pmatrix} : \quad v_1, \dots, v_k \in \mathbb{R}^n \right\},$$

the ranks being always $\leqslant r$. Moreover, the final n-th reduced character is defined by:

$$s_1' + s_2' + \ldots + s_{n-1}' + s_n' = r.$$

Definition 12.10. Lastly, the coframe θ is said to be *involutive* if the value of the sum:

$$\sum_{k=1}^{n} k\, s_k'$$

is equal to the degree of indeterminancy.

Now, Lemma 12.7 showed that if we execute again the absorbtion procedure on the structure equation (86) by replacing β_1 with:

$$\beta_1 + p_1\sigma + q_1\overline{\sigma} + r_1\rho + s_1\zeta + t_1\overline{\zeta},$$

then in order to annihilate all the coefficients in the new expressions of $d\sigma$, $d\rho$, $d\zeta$, the only solution is:

$$p_1 = q_1 = r_1 = s_1 = t_1 := 0.$$

In other words, the number of free variables is null in our case.

On the other hand, we claim that the reduced characters cannot all be null, and hence the lifted coframe is certainly non-involutive. Let us check this fact by constructing the 5×2 matrix:

$$I(v) := \begin{pmatrix} 2\,v^{\sigma} & v^{\sigma} \\ v^{\overline{\sigma}} & 2\,v^{\overline{\sigma}} \\ v^{\rho} & v^{\rho} \\ v^{\zeta} & 0 \\ 0 & v^{\overline{\zeta}} \end{pmatrix} \begin{matrix} d\sigma \\ d\overline{\sigma} \\ d\rho \\ d\zeta \\ d\overline{\zeta} \end{matrix} \qquad (\text{columns } \alpha,\ \overline{\alpha})$$

according to the structure equations (86) and their conjugates. Then, by the above definition we have:

$$s_1' = \mathsf{rank}\big(I(v)\big) = 2,$$

which is the maximum rank of the above matrix for all arbitrary vectors $\big(v^{\sigma}, v^{\overline{\sigma}}, v^{\rho}, v^{\zeta}, v^{\overline{\zeta}}\big) \in \mathbb{R}^5$. Consequently, the sum $\sum_{k=1}^{5} k\,s_k' \geqslant 2$ is *larger* than the number, 0, of free variables.

In conclusion, the associated lifted coframe σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$ is certainly non-involutive and we have to start the *prolongation procedure*.

12.14. **Prolongation.** Now, we are ready to prolong the structure equation (86) of the non-involutive coframe $\big\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}\big\}$ — now, we rename β_1 as α —:

$$\begin{aligned} d\sigma &= \big(2\,\alpha + \overline{\alpha}\big) \wedge \sigma + \rho \wedge \zeta, \\ d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \alpha \wedge \zeta. \end{aligned}$$

We prolong the base manifold M^5_{c} to the prolonged space $M^5_{\mathsf{c}} \times G^{\mathsf{red}}$, where G^{red} is the reduced structure group:

$$G^{\mathsf{red}} := \left\{ g = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\,\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix} : \quad \mathsf{a} \in \mathbb{C} \right\}.$$

The prolonged space $M^5_{\mathsf{c}} \times G^{\mathsf{red}}$ is in fact a submanifold of the complex space $\mathbb{C}^5 := \mathbb{C}_{(z, w_1, w_2, w_3, \mathsf{a})}$. The idea behind the prolongation procedure is based upon the following fundamental proposition (*see* Proposition 12.1 page 375 of [62] for the general assertion):

Proposition 12.11. *Let θ and θ' be two lifted coframes on two manifolds M and M' having the same structure group G, let α and α' be the modified Maurer-Cartan forms obtained by solving the absorbtion equations and assume that neither group-dependent essential torsion coefficients exist nor free absorption variables remain. Then there exists a diffeomorphism $\Phi : M \to M'$ mapping θ to θ' for some choice of group parameters if and* *only if there is a diffeomorphism* $\Psi : M \times G \longrightarrow M' \times G$ *mapping the prolonged coframe* $\{\theta, \alpha\}$ *to* $\{\theta', \alpha'\}$. □

Accordingly, one transforms the equivalence problem for the 5-dimensional cubic M^5_{c} into an equivalence problem on the prolonged submanifold $M^5_{\mathsf{c}} \times G^{\mathsf{red}}$ equipped with the coframe $\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}, \alpha, \overline{\alpha}\}$, enlarged with the additional 1-form $\alpha = \frac{d\mathsf{a}}{\mathsf{a}}$. Since $d\alpha = d \log \mathsf{a} = 0$, we obtain gratuitously the following — fully expressed — structure equations on this space:

$$\begin{aligned} d\sigma &= \big(2\,\alpha + \overline{\alpha}\big) \wedge \sigma + \rho \wedge \zeta, \\ d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \alpha \wedge \zeta, \\ d\alpha &= 0. \end{aligned} \tag{88}$$

and this is nothing but the final, desired $\{e\}$-*structure*.

Now, recall that if $\{\mathsf{v}_1, \dots, \mathsf{v}_r\}$ is a basis of a real Lie algebra $\mathfrak{g}$ of left-invariant vector fields on an r-dimensional Lie group G which have Lie brackets:

$$\big[\mathsf{v}_i, \mathsf{v}_j\big] = \sum_{k=1}^{r} c_{ij}^k\, \mathsf{v}_k \qquad\qquad {\scriptstyle (1 \leqslant i < j \leqslant r)},$$

with certain *structure constants* $c_{\bullet\bullet}^{\bullet}$, and if $\alpha^1, \dots, \alpha^r$ is the dual Maurer-Cartan basis of left-invariant 1-forms, then their structure equations:

$$d\alpha^k = - \sum_{1 \leqslant i < j \leqslant r} c_{ij}^k\, \alpha^i \wedge \alpha^j \qquad\qquad {\scriptstyle (k = 1 \cdots r)}$$

have the same structure coefficients $c_{\bullet\bullet}^{\bullet}$ up to an overall minus sign.

As in the case of a Lie group, in our case (88), the structure functions are constant, hence the associated coframe is of *rank zero* (*see* [62], pages 266–268). To proceed the problem, now we need the following result.

Theorem 12.1. *(see* [62]*, page 268, Theorem 8.16). Let* θ *be a rank zero coframe on an* m*-dimensional manifold* M*, with constant structure functions* $T_{ij}^k = -c_{ij}^k$*. Let* G *be an* m*-dimensional Lie group whose Lie algebra* $\mathfrak{g}$ *has structure constants* c_{ij}^k *relative to a basis* $\{\mathsf{v}_1, \dots, \mathsf{v}_m\}$*, and let* $\{\alpha^1, \dots, \alpha^m\}$ *denote the dual basis of Maurer-Cartan forms. Then, there exists a local diffeomorphism* $\Phi : M \to G$ *mapping the given coframe to the Maurer-Cartan coframe on* G.

Accordingly, every generic submanifold $M'^5 \subset \mathbb{C}^4$ which is equivalent to M^5_{c} corresponds, after prolongation, to a 7-dimensional Lie group G whose Lie algebra $\mathfrak{g}$ has the same structure constants as the structure coefficients of (88). Let us examine this Lie algebra.

According to the reminder, taking account of the overall minus sign, we find the following complete table of commutators:

	v^σ	$\mathsf{v}^{\overline{\sigma}}$	v^ρ	v^ζ	$\mathsf{v}^{\overline{\zeta}}$	v^α	$\mathsf{v}^{\overline{\alpha}}$
v^σ	0	0	0	0	0	$2\,\mathsf{v}^\sigma$	v^σ
$\mathsf{v}^{\overline{\sigma}}$	$*$	0	0	0	0	$\mathsf{v}^{\overline{\sigma}}$	$2\,\mathsf{v}^{\overline{\sigma}}$
v^ρ	$*$	$*$	0	$-\mathsf{v}^\sigma$	$-\mathsf{v}^{\overline{\sigma}}$	v^ρ	v^ρ
v^ζ	$*$	$*$	$*$	0	$-i\,\mathsf{v}^\rho$	v^ζ	0
$\mathsf{v}^{\overline{\zeta}}$	$*$	$*$	$*$	$*$	0	0	$\mathsf{v}^{\overline{\zeta}}$
v^α	$*$	$*$	$*$	$*$	$*$	0	0
$\mathsf{v}^{\overline{\alpha}}$	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

A visual inspection of this table shows that this Lie algebra $\mathfrak{g}$ is a 3-*graded algebra* in the sense of Tanaka, of the form:

$$\begin{aligned}\mathfrak{g} &:= \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0, \\ [\mathfrak{g}_i, \mathfrak{g}_j] &\subset \mathfrak{g}_{i+j} \qquad\qquad {\scriptstyle (i,j=-3,-2-1,0)},\end{aligned}$$

where:

$$\begin{aligned}\mathfrak{g}_{-3} &:= \langle \mathsf{v}^\sigma, \mathsf{v}^{\overline{\sigma}} \rangle, \\ \mathfrak{g}_{-2} &:= \langle \mathsf{v}^\rho \rangle, \\ \mathfrak{g}_{-1} &:= \langle \mathsf{v}^\zeta, \mathsf{v}^{\overline{\zeta}} \rangle, \\ \mathfrak{g}_0 &:= \langle \mathsf{v}^\alpha, \mathsf{v}^{\overline{\alpha}} \rangle.\end{aligned}$$

But remember that in Section 10, we computed the Lie algebra $\mathfrak{aut}_{CR}(M^5_{\mathsf{c}})$ of infinitesimal CR-automorphisms of our cubic model M^5_{c} and there, we showed that it is 7-dimensional of the form:

$$\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big) := \mathfrak{a}_{-3} \oplus \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0,$$

where:

$$\begin{aligned}\mathfrak{a}_{-3} &:= \langle S_1, S_2 \rangle, \\ \mathfrak{a}_{-2} &:= \langle T \rangle, \\ \mathfrak{a}_{-1} &:= \langle L_1, L_2 \rangle, \\ \mathfrak{a}_0 &:= \langle D, R \rangle,\end{aligned}$$

and that it is equipped with the following table of commutators:

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	$*$	0	0	0	0	$3S_1$	S_2
T	$*$	$*$	0	$4S_2$	$4S_1$	$2T$	0
L_2	$*$	$*$	$*$	0	$-4T$	L_2	$-L_1$
L_1	$*$	$*$	$*$	$*$	0	L_1	L_2
D	$*$	$*$	$*$	$*$	$*$	0	0
R	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

Introduce now the linear map:

$$\Psi\colon\quad \mathfrak{aut}_{CR}(M^5_{\mathsf{c}}) \longrightarrow \mathfrak{g},$$

having the following values on the basis elements of $\mathfrak{aut}_{CR}(M^5_{\mathsf{c}})$:

Ψ	$\longmapsto$
S_1	$-\frac{i}{2}\,v^\sigma - \frac{i}{4}\,v^{\overline{\sigma}}$
S_2	$\frac{1}{2}\,v^\sigma - \frac{1}{4}\,v^{\overline{\sigma}}$
T	v^ρ
L_1	$2\,i\,v^\zeta + i\,v^{\overline{\zeta}}$
L_2	$-2\,v^\zeta + v^{\overline{\zeta}}$
D	$v^\alpha + v^{\overline{\alpha}}$
R	$i\,v^\alpha - i\,v^{\overline{\alpha}}.$

One checks that this linear map Ψ is an isomorphism. Consequently, the Lie algebra $\mathfrak{aut}_{CR}(M^5_{\mathsf{c}})$ of our cubic model M^5_{c} is isomorphic to the Lie algebra $\mathfrak{g}$ obtained after prolongation.

Theorem 12.2. *A 5-dimensional maximally minimal real analytic CR-generic local submanifold $M'^5 \subset \mathbb{C}^4$ is equivalent, through some local biholomorphic transformation, to the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$, if and only if the structure equations of the lifted coframe $\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}, \alpha, \overline{\alpha}\}$ associated to the 7-dimensional prolonged space $M^5 \times G^{\mathsf{red}}$ are in the form* (88), *if and only if they have structure constants of a Lie algebra isomorphic to $\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big)$ where G^{red} is the 2-dimensional matrix group:*

$$G^{\mathsf{red}} := \left\{ g = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\,\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix} : \quad \mathsf{a} \in \mathbb{C} \right\},$$

and, if and only if the prolonged space $M^5 \times G^{\mathsf{red}}$ is equivalent, through an $\{e\}$-structure to $\mathsf{Aut}_{CR}(M^5_{\mathsf{c}})$, the symmetry group of the model M^5_{c} with the associated Lie algebra $\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big)$.

Proof. As we already observed, the structure constants of the structure equations (88) are precisely equal to those of the Lie algebra $\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big)$. Furthermore, the Lie algebra $\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big)$ associates to the Lie group $\mathsf{Aut}_{CR}\big(M^5_{\mathsf{c}}\big)$. Now, the result comes immediately from Theorem 12.1. □

One should notice that actually the problem of equivalency to our cubic model M^5_{c} is not fully-completed, yet. Being more precise and according to the first part of the above theorem, to realize the equivalency of an arbitrary 5-dimensional CR-manifold M'^5 to this model, we need to have the structure equations of the lifted coframe, associated to it. In Corollary 17.4 we will finalize the solution of the current equivalence problem.

Remark 12.12. Let us denot by G_- the Lie subgroup of $\mathsf{Aut}_{CR}(M^5_{\mathsf{c}})$ associated to the Levi-Tanaka subalgebra $\mathfrak{g}_-$ of $\mathfrak{aut}_{CR}\big(M^5_{\mathsf{c}}\big)$. It is known (*see* [6], Proposition 3) that this group is diffeomorphic to the model M^5_{c}. Then, the above theorem asserts that the *prolonged* space $M^5_{\mathsf{c}} \times G^{\mathsf{red}}$ is equivalent to the Tanaka *prolongation* of the Lie subgroup G_-, namely $\mathsf{Aut}_{CR}(M^5_{\mathsf{c}})$. This may show the close coherency between two concepts of prolongation in the senses of Cartan and Tanaka.

Corollary 12.13. *Let M^5 be an under consideration 5-dimensional CR-manifold which is equivalent, through some local biholomorphism to the model M^5_{c}. Then, the symmetry group of the corresponding 7-dimensional prolonged space $M^5 \times G^{\mathsf{red}}$ is $\mathsf{Aut}_{CR}(M^5_{\mathsf{c}})$.*

Proof. The assertion is a direct consequence of Corollary 14.20 page 435 of [62]. □

13. Initial complex frame for geometry-preserving deformations of the model

13.1. **Initial Lie bracket structure.** Given a 3-codimensional CR-generic submanifold $M^5 \subset \mathbb{C}^4$ represented as a graph:

$$
\begin{aligned}
v_1 &= \varphi_1\big(x, y, u_1, u_2, u_3\big),\\
v_2 &= \varphi_2\big(x, y, u_1, u_2, u_3\big),\\
v_3 &= \varphi_3\big(x, y, u_1, u_2, u_3\big),
\end{aligned}
$$

with $\varphi_j(0) = 0$ and $d\varphi_j(0) = 0$, the $(0,1)$-complex tangent bundle $T^{0,1}M^5$ is spanned by a certain single $(0,1)$-vector field of the form:

$$
\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \mathsf{A}_1 \frac{\partial}{\partial \overline{w}_1} + \mathsf{A}_2 \frac{\partial}{\partial \overline{w}_2} + \mathsf{A}_3 \frac{\partial}{\partial \overline{w}_3},
$$

where the coefficient-functions A_1, A_2, A_3 may be expressed explicitly in terms of the first-order jets $J^1_{x,y,u_1,u_2,u_3}\varphi_j$ of the functions φ_j. Writing the tangency equations gives that its three (unknown) coefficients A_1, A_2, A_3 should satisfy the three equations:

$$\begin{aligned}
\frac{i}{2}\mathsf{A}_1 &= \varphi_{1\overline{z}} + \frac{1}{2}\mathsf{A}_1\varphi_{1u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{1u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{1u_3},\\
\frac{i}{2}\mathsf{A}_2 &= \varphi_{2\overline{z}} + \frac{1}{2}\mathsf{A}_1\varphi_{2u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{2u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{2u_3},\\
\frac{i}{2}\mathsf{A}_3 &= \varphi_{3\overline{z}} + \frac{1}{2}\mathsf{A}_1\varphi_{3u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{3u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{3u_3}.
\end{aligned}$$

Equivalently, these equations can be read as a non-homogeneous system with right hand-side vanishing at the origin:

$$\begin{aligned}
-\frac{1}{2i}\mathsf{A}_1 &= \frac{1}{2}\varphi_{1x} - \frac{1}{2i}\varphi_{1y} + \frac{1}{2}\mathsf{A}_1\varphi_{1u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{1u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{1u_3},\\
-\frac{1}{2i}\mathsf{A}_2 &= \frac{1}{2}\varphi_{2x} - \frac{1}{2i}\varphi_{2y} + \frac{1}{2}\mathsf{A}_1\varphi_{2u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{2u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{2u_3},\\
-\frac{1}{2i}\mathsf{A}_3 &= \frac{1}{2}\varphi_{3x} - \frac{1}{2i}\varphi_{2y} + \frac{1}{2}\mathsf{A}_1\varphi_{3u_1} + \frac{1}{2}\mathsf{A}_2\varphi_{3u_2} + \frac{1}{2}\mathsf{A}_3\varphi_{3u_3}.
\end{aligned}$$

Solving this linear system of three equations and three unknowns $\mathsf{A}_1, \mathsf{A}_2$, A_3 and expanding the solutions according to their real and imaginary parts provides the rational solutions:

$$\begin{aligned}
\mathsf{A}_1 &= \frac{\Lambda_1^1}{\Delta} + i\frac{\Lambda_2^1}{\Delta},\\
\mathsf{A}_2 &= \frac{\Lambda_1^2}{\Delta} + i\frac{\Lambda_2^2}{\Delta},\\
\mathsf{A}_3 &= \frac{\Lambda_1^3}{\Delta} + i\frac{\Lambda_2^3}{\Delta},
\end{aligned}$$

in which the denominator Δ, in terms of the first-order derivatives of φ_1, φ_2, φ_3, is:

$$\Delta = \sigma^2 + \tau^2,$$

with the squared functions of:

$$\begin{aligned}
\sigma &= \varphi_{3u_3} + \varphi_{1u_1} + \varphi_{2u_2} - \varphi_{1u_2}\varphi_{3u_1}\varphi_{2u_3} - \varphi_{1u_3}\varphi_{2u_1}\varphi_{3u_2} + \varphi_{1u_2}\varphi_{2u_1}\varphi_{3u_3} -\\
&\quad - \varphi_{1u_1}\varphi_{2u_2}\varphi_{3u_3} + \varphi_{1u_1}\varphi_{2u_3}\varphi_{3u_2} + \varphi_{1u_3}\varphi_{3u_1}\varphi_{2u_2},\\
\tau &= -1 + \varphi_{1u_1}\varphi_{2u_2} - \varphi_{2u_3}\varphi_{3u_2} - \varphi_{1u_3}\varphi_{3u_1} + \varphi_{2u_2}\varphi_{3u_3} - \varphi_{1u_2}\varphi_{2u_1} + \varphi_{1u_1}\varphi_{3u_3},
\end{aligned}$$

and in which the numerators $\Lambda_{\bullet}^{\bullet}$ of $\mathsf{A}_1, \mathsf{A}_2$ and A_3 are in the (longer) forms:

$$\begin{aligned}
\Lambda_1^1 &= \big(- \varphi_{3u_3}\varphi_{2x}\varphi_{1u_2} - \varphi_{1u_3}\varphi_{3y} + \varphi_{2u_2}\varphi_{1x}\varphi_{3u_3} + \varphi_{3u_3}\varphi_{1y} - \varphi_{1x} - \varphi_{2y}\varphi_{1u_2} +\\
&\quad + \varphi_{2u_3}\varphi_{3x}\varphi_{1u_2} + \varphi_{2u_2}\varphi_{1y} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3x} + \varphi_{2x}\varphi_{1u_3}\varphi_{3u_2}\big)\sigma +\\
&\quad + \big(\varphi_{1u_3}\varphi_{3x} - \varphi_{1y} + \varphi_{2x}\varphi_{1u_2} + \varphi_{2u_3}\varphi_{1u_2}\varphi_{3y} - \varphi_{2u_2}\varphi_{1x} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1y} -\\
&\quad - \varphi_{3u_3}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3y} - \varphi_{3u_3}\varphi_{1u_2}\varphi_{2y} + \varphi_{1u_3}\varphi_{3u_2}\varphi_{2y} + \varphi_{2u_2}\varphi_{3u_3}\varphi_{1y}\big)\tau,\\
\Lambda_2^1 &= \big(\varphi_{1u_3}\varphi_{3x} - \varphi_{1y} + \varphi_{2x}\varphi_{1u_2} + \varphi_{2u_3}\varphi_{1u_2}\varphi_{3y} - \varphi_{2u_2}\varphi_{1x} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1y} -\\
&\quad - \varphi_{3u_3}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3y} - \varphi_{3u_3}\varphi_{1u_2}\varphi_{2y} + \varphi_{1u_3}\varphi_{3u_2}\varphi_{2y} + \varphi_{2u_2}\varphi_{3u_3}\varphi_{1y}\big)\sigma -\\
&\quad - \big(- \varphi_{3u_3}\varphi_{2x}\varphi_{1u_2} - \varphi_{1u_3}\varphi_{3y} + \varphi_{2u_2}\varphi_{1x}\varphi_{3u_3} + \varphi_{3u_3}\varphi_{1y} - \varphi_{1x} - \varphi_{2y}\varphi_{1u_2} +\\
&\quad + \varphi_{2u_3}\varphi_{3x}\varphi_{1u_2} + \varphi_{2u_2}\varphi_{1y} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3x} + \varphi_{2x}\varphi_{1u_3}\varphi_{3u_2}\big)\tau,
\end{aligned}$$

$$\Lambda_1^2 = \big(- \varphi_{2x} + \varphi_{3u_3}\varphi_{2y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3x} - \varphi_{2u_3}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2x} - \varphi_{2u_1}\varphi_{1y} - \\ - \varphi_{2u_1}\varphi_{3u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{2y} - \varphi_{1u_1}\varphi_{2u_3}\varphi_{3x} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2x}\big)\sigma + \\ + \big(- \varphi_{1u_1}\varphi_{2u_3}\varphi_{3y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2y} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1y} + \varphi_{2u_3}\varphi_{3x} - \\ - \varphi_{2u_1}\varphi_{3u_3}\varphi_{1y} - \varphi_{3u_3}\varphi_{2x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2y} + \varphi_{2u_1}\varphi_{1x} - \varphi_{1u_1}\varphi_{2x} - \varphi_{2y}\big)\tau,$$

$$\Lambda_2^2 = \big(- \varphi_{1u_1}\varphi_{2u_3}\varphi_{3y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2y} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1y} + \varphi_{2u_3}\varphi_{3x} - \\ - \varphi_{2u_1}\varphi_{3u_3}\varphi_{1y} - \varphi_{3u_3}\varphi_{2x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2y} + \varphi_{2u_1}\varphi_{1x} - \varphi_{1u_1}\varphi_{2x} - \varphi_{2y}\big)\sigma - \\ - \big(- \varphi_{2x} + \varphi_{3u_3}\varphi_{2y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3x} - \varphi_{2u_3}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2x} - \varphi_{2u_1}\varphi_{1y} - \\ - \varphi_{2u_1}\varphi_{3u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{2y} - \varphi_{1u_1}\varphi_{2u_3}\varphi_{3x} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2x}\big)\tau,$$

$$\Lambda_1^3 = \big(- \varphi_{2u_1}\varphi_{1u_2}\varphi_{3x} - \varphi_{3u_1}\varphi_{1y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1x} + \varphi_{1u_1}\varphi_{3y} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1x} - \\ - \varphi_{3u_2}\varphi_{2y} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2x} + \varphi_{2u_2}\varphi_{3y} - \varphi_{3u_2}\varphi_{1u_1}\varphi_{2x} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3x} - \varphi_{3x}\big)\sigma + \\ + \big(- \varphi_{3u_2}\varphi_{1u_1}\varphi_{2y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1y} + \varphi_{3u_1}\varphi_{1x} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2y} - \varphi_{2u_1}\varphi_{1u_2}\varphi_{3y} + \\ + \varphi_{3u_2}\varphi_{2x} - \varphi_{1u_1}\varphi_{3x} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1y} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3y} - \varphi_{3y} - \varphi_{2u_2}\varphi_{3x}\big)\tau,$$

$$\Lambda_2^3 = \big(- \varphi_{3u_2}\varphi_{1u_1}\varphi_{2y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1y} + \varphi_{3u_1}\varphi_{1x} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2y} - \varphi_{2u_1}\varphi_{1u_2}\varphi_{3y} + \\ + \varphi_{3u_2}\varphi_{2x} - \varphi_{1u_1}\varphi_{3x} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1y} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3y} - \varphi_{3y} - \varphi_{2u_2}\varphi_{3x}\big)\sigma - \\ - \big(- \varphi_{2u_1}\varphi_{1u_2}\varphi_{3x} - \varphi_{3u_1}\varphi_{1y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1x} + \varphi_{1u_1}\varphi_{3y} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1x} - \\ - \varphi_{3u_2}\varphi_{2y} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2x} + \varphi_{2u_2}\varphi_{3y} - \varphi_{3u_2}\varphi_{1u_1}\varphi_{2x} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3x} - \varphi_{3x}\big)\tau,$$

again in terms of the first-order derivatives of φ_1, φ_2, φ_3.

Here, the expression of $\overline{\mathscr{L}}$ is presented as a vector field which lives in a neighborhood of M^5 in $\mathbb{C}^4$, while M^5 itself, is a real five-dimensional hypersurface equipped with the five real coordinates x, y, u_1, u_2, u_3. But, in order to express $\overline{\mathscr{L}}$ *intrinsically*, one must drop $\frac{\partial}{\partial v_1}$, $\frac{\partial}{\partial v_2}$ and $\frac{\partial}{\partial v_3}$ and also replace v_i by $\varphi_i(x, y, u_1, u_2, u_3)$ for $i = 1, 2, 3$, simultaneously in its expression. Then, after expanding $\overline{\mathscr{L}}$ in real and imaginary parts:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \frac{\mathsf{A}_1}{2}\frac{\partial}{\partial u_1} + \frac{\mathsf{A}_2}{2}\frac{\partial}{\partial u_2} + \frac{\mathsf{A}_3}{2}\frac{\partial}{\partial u_3},$$

one gains a result that can now be summarized as follows.

Proposition 13.1. *For any local real analytic CR-generic submanifold $M^5 \subset \mathbb{C}^4$ which is represented near the origin as a graph:*

$$\begin{cases} v_1 := \varphi_1(x, y, u_1, u_2, u_3), \\ v_2 := \varphi_2(x, y, u_1, u_2, u_3), \\ v_3 := \varphi_3(x, y, u_1, u_2, u_3), \end{cases}$$

in coordinates:

$$\big(z, w_1, w_2, w_3\big) = \big(x + iy, u_1 + iv_1, u_2 + iv_2, u_3 + iv_3\big),$$

its complex bundle $T^{1,0}M$ is generated by:

$$\boxed{\overline{\mathscr{L}} = \frac{\partial}{\partial\overline{z}} + \frac{\mathsf{A}_1}{2}\frac{\partial}{\partial u_1} + \frac{\mathsf{A}_2}{2}\frac{\partial}{\partial u_2} + \frac{\mathsf{A}_3}{2}\frac{\partial}{\partial u_3},}$$

whose numerators and denominators have the explicit expressions shown above in terms of the first order jets $J^1_{x,y,u_1,u_2,u_3}\varphi_j$, $j = 1, 2, 3$.

In particular, for the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$, represented as the graph:

$$\left[\begin{aligned} v_1 &:= x^2 + y^2, \\ v_2 &:= 2\,x^3 + 2\,xy^2, \\ v_3 &:= 2\,x^2y + 2\,y^3, \end{aligned}\right.$$

we have:

$$\begin{aligned} \overline{\mathscr{L}}_{\mathsf{c}} &= \frac{\partial}{\partial\overline{z}} + \big(y - ix\big)\frac{\partial}{\partial u_1} + \big(2xy - i(3x^2 + y^2)\big)\frac{\partial}{\partial u_2} + \big(x^2 + 3y^2 - 2ixy\big)\frac{\partial}{\partial u_3} \\ \mathscr{L}_{\mathsf{c}} &= \frac{\partial}{\partial z} + \big(y + ix\big)\frac{\partial}{\partial u_1} + \big(2xy + i(3x^2 + y^2)\big)\frac{\partial}{\partial u_2} + \big(x^2 + 3y^2 + 2ixy\big)\frac{\partial}{\partial u_3}. \end{aligned}$$

13.2. **Length-two Lie bracket.** Between the two already presented complex vector fields $\mathscr{L}$ and $\overline{\mathscr{L}}$, of course there is only one Lie bracket $[\mathscr{L}, \overline{\mathscr{L}}]$ of length two. This vector field is in fact *imaginary*. Being more precise, expressing $\overline{\mathscr{L}} := L_1 + iL_2$ in real and imaginary parts, then the commutator:

$$\begin{aligned} \big[\mathscr{L}, \overline{\mathscr{L}}\big] &= \big[L_1 - iL_2, L_1 + iL_2\big], \\ &= 2i\,\big[L_1, L_2\big] \end{aligned}$$

is imaginary. Hence let us denote the third (*real*) vector field by:

$$\mathscr{T} := i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big].$$

Performing direct but painful computations provides the expression of:

$$\boxed{\mathscr{T} = \frac{\Upsilon_1}{\Delta^3}\frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^3}\frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^3}\frac{\partial}{\partial u_3},}$$

in which the three new numerators Υ_i are given as follows:

$$\begin{aligned} \Upsilon_1 = -\big(&\Delta^2\Lambda^1_{2x} - \Delta\Delta_x\Lambda^1_2 - \Delta^2\Lambda^1_{1y} + \Delta\Delta_y\Lambda^1_1 + \Delta\Lambda^1_1\Lambda^1_{2u_1} - \Delta\Lambda^1_2\Lambda^1_{1u_1} - \Delta\Lambda^2_2\Lambda^1_{1u_2} + \\ &+ \Delta_{u_2}\Lambda^1_1\Lambda^2_2 - \Delta\Lambda^3_2\Lambda^1_{1u_3} + \Delta_{u_3}\Lambda^3_2\Lambda^1_1 + \Delta\Lambda^2_1\Lambda^1_{2u_2} - \Delta_{u_2}\Lambda^2_1\Lambda^1_2 + \Delta\Lambda^3_1\Lambda^1_{2u_3} - \Delta_{u_3}\Lambda^3_1\Lambda^1_2\big), \\ \Upsilon_2 = -\big(&\Delta^2\Lambda^2_{2x} - \Delta\Delta_x\Lambda^2_2 + \Delta\Lambda^1_1\Lambda^2_{2u_1} - \Delta_{u_1}\Lambda^1_1\Lambda^2_2 - \Delta^2\Lambda^2_{1y} + \Delta\Delta_y\Lambda^2_1 - \Delta\Lambda^1_2\Lambda^2_{1u_1} + \\ &+ \Delta_{u_1}\Lambda^1_2\Lambda^2_1 + \Delta\Lambda^2_1\Lambda^2_{2u_2} - \Delta\Lambda^2_2\Lambda^2_{1u_2} + \Delta\Lambda^3_1\Lambda^2_{2u_3} - \Delta_{u_3}\Lambda^3_1\Lambda^2_2 - \Delta\Lambda^3_2\Lambda^2_{1u_3} + \Delta_{u_3}\Lambda^3_2\Lambda^2_1\big), \\ \Upsilon_3 = -\big(&\Delta^2\Lambda^3_{2x} - \Delta\Delta_x\Lambda^3_2 + \Delta\Lambda^1_1\Lambda^3_{2u_1} - \Delta_{u_1}\Lambda^1_1\Lambda^3_2 - \Delta^2\Lambda^3_{1y} + \Delta\Delta_y\Lambda^3_1 - \Delta\Lambda^1_2\Lambda^3_{1u_1} + \\ &+ \Delta_{u_1}\Lambda^1_2\Lambda^3_1 - \Delta\Lambda^2_2\Lambda^3_{1u_2} + \Delta_{u_2}\Lambda^2_2\Lambda^3_1 + \Delta\Lambda^3_1\Lambda^3_{2u_3} - \Delta\Lambda^3_2\Lambda^3_{1u_3} + \Delta\Lambda^2_1\Lambda^3_{2u_2} - \Delta_{u_2}\Lambda^2_1\Lambda^3_2\big). \end{aligned}$$

In particular, for the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^3$, we have:

$$\mathscr{T}_{\mathsf{c}} = 4\,\frac{\partial}{\partial u_1} + 16x\,\frac{\partial}{\partial u_2} + 16y\,\frac{\partial}{\partial u_3}.$$

13.3. **Length-three Lie brackets.** In this length, we have two Lie brackets:

$$\mathscr{S} := [\mathscr{L}, \mathscr{T}] \qquad \text{and} \qquad \overline{\mathscr{S}} := [\overline{\mathscr{L}}, \mathscr{T}].$$

According to the performed computations, we have the explicit expressions of these two *complex* vector fields in terms of the defining functions $\varphi_1, \varphi_2, \varphi_3$ as:

$$\mathscr{S} = \frac{\Gamma^1_1 - i\Gamma^1_2}{\Delta^5}\frac{\partial}{\partial u_1} + \frac{\Gamma^2_1 - i\Gamma^2_2}{\Delta^5}\frac{\partial}{\partial u_2} + \frac{\Gamma^3_1 - i\Gamma^3_2}{\Delta^5}\frac{\partial}{\partial u_3},$$

where, by allowing the two notational coincidences $x \equiv x_1$ and $y \equiv x_2$, the numerators are (for $i = 1, 2$):

$$\begin{aligned}
\Gamma^1_i &= -2\big(\tfrac{1}{4}\Delta^2\Upsilon_{1x_i} - 3\Delta\Delta_{x_i}\Upsilon_1 + \Delta\Lambda^1_i\Upsilon_{1u_1} - 2\Delta_{u_1}\Lambda^1_i\Upsilon_1 - \Delta\Lambda^1_{iu_1}\Upsilon_1 - \Delta\Lambda^1_{iu_2}\Upsilon_2 + \\
&\quad + \Delta_{u_2}\Lambda^1_i\Upsilon_2 - \Delta\Lambda^1_{iu_3}\Upsilon_3 + \Delta_{u_3}\Lambda^1_i\Upsilon_3 + \Delta\Lambda^2_i\Upsilon_{1u_2} - 3\Delta_{u_2}\Lambda^2_i\Upsilon_1 + \Delta\Lambda^3_i\Upsilon_{1u_3} - 3\Delta_{u_3}\Lambda^3_i\Upsilon_1\big),\\
\Gamma^2_i &= -2\big(\Delta^2\Upsilon_{2x_i} - 3\Delta\Delta_{x_i}\Upsilon_2 + \Delta\Lambda^1_i\Upsilon_{2u_1} - 3\Delta_{u_1}\Lambda^1_i\Upsilon_2 - \Delta\Lambda^2_{iu_1}\Upsilon_1 + \Delta_{u_1}\Lambda^2_i\Upsilon_1 + \\
&\quad + \Delta\Lambda^2_i\Upsilon_{2u_2} - 2\Delta_{u_2}\Lambda^2_i\Upsilon_2 - \Delta\Lambda^2_{iu_2}\Upsilon_2 - \Delta\Lambda^2_{iu_3}\Upsilon_3 + \Delta_{u_3}\Lambda^2_i\Upsilon_3 + \Delta\Lambda^3_i\Upsilon_{2u_3} - 3\Delta_{u_3}\Lambda^3_i\Upsilon_2\big),\\
\Gamma^3_i &= -2\big(\Delta^2\Upsilon_{3x_i} - 3\Delta\Delta_{x_i}\Upsilon_3 + \Delta\Lambda^1_i\Upsilon_{3u_1} - 3\Delta_{u_1}\Lambda^1_i\Upsilon_3 + \Delta\Lambda^2_i\Upsilon_{3u_2} - 3\Delta_{u_2}\Lambda^2_i\Upsilon_3 - \\
&\quad - \Delta\Lambda^3_{iu_1}\Upsilon_1 + \Delta_{u_1}\Lambda^3_i\Upsilon_1 - \Delta\Lambda^3_{iu_2}\Upsilon_2 + \Delta_{u_2}\Lambda^3_i\Upsilon_2 + \Delta\Lambda^3_i\Upsilon_{3u_3} - 2\Delta_{u_3}\Lambda^3_i\Upsilon_3 - \Delta\Lambda^3_{iu_3}\Upsilon_3\big).
\end{aligned}$$

In particular for the cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$, the above expressions give:

$$\begin{aligned}
\mathscr{S}_{\mathsf{c}} &= 8\frac{\partial}{\partial u_2} - 8i\frac{\partial}{\partial u_3},\\
\overline{\mathscr{S}}_{\mathsf{c}} &= 8\frac{\partial}{\partial u_2} + 8i\frac{\partial}{\partial u_3}.
\end{aligned}$$

Proposition 13.2. *The five vector fields* $\overline{\mathscr{L}}, \mathscr{L}, \mathscr{T}, \overline{\mathscr{S}}, \mathscr{S}$ *constitute a (complex) frame for* $TM^5 \otimes_{\mathbb{R}} \mathbb{C}$*.*

Proof. Thanks to assumption that remainders in the graphing functions φ_1, φ_2 and φ_3 are all an $\mathsf{OW}(4)$, the values of these vector fields at the origin are the same as for the corresponding ones of the model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$, namely:

$$\begin{aligned}
&\overline{\mathscr{L}}\big|_0 = \overline{\mathscr{L}}_{\mathsf{c}}\big|_0 = \frac{\partial}{\partial \overline{z}}, \qquad \mathscr{L}\big|_0 = \mathscr{L}_{\mathsf{c}}\big|_0 = \frac{\partial}{\partial z},\\
&\mathscr{T}\big|_0 = \mathscr{T}_{\mathsf{c}}\big|_0 = 4\,\frac{\partial}{\partial u_1},\\
&\overline{\mathscr{S}}|_0 = \overline{\mathscr{S}}_{\mathsf{c}}\big|_0 = 8\frac{\partial}{\partial u_2} + 8i\frac{\partial}{\partial u_3}, \qquad \mathscr{S}\big|_0 = \mathscr{S}_{\mathsf{c}}\big|_0 = 8\frac{\partial}{\partial u_2} - 8i\frac{\partial}{\partial u_3},
\end{aligned}$$

whence the five vectors $\overline{\mathscr{L}}, \mathscr{L}, \mathscr{T}, \overline{\mathscr{S}}, \mathscr{S}$ are linearly independent in a neighborhood of the origin and they constitute a local frame for M^5. □

13.4. **Other iterated Lie brackets.** We saw that the collection of five vector fields:

$$\Big\{\overline{\mathscr{S}},\,\mathscr{S},\,\mathscr{T},\,\overline{\mathscr{L}},\,\mathscr{L}\Big\},$$

where:

$$\begin{aligned}\mathscr{T}&:=i\big[\mathscr{L},\overline{\mathscr{L}}\big],\\ \mathscr{S}&:=\big[\mathscr{L},\,\mathscr{T}\big],\\ \overline{\mathscr{S}}&:=\big[\overline{\mathscr{L}},\,\mathscr{T}\big],\end{aligned}$$

makes up a frame for $\mathbb{C}\otimes TM^5$. Having five fields implies that there are in sum ten Lie brackets between them. Thus, there remain seven such brackets to be looked at.

Let us start with the following group of four Lie brackets:

$$\big[\mathscr{L},\,\mathscr{S}\big],\qquad\big[\overline{\mathscr{L}},\,\mathscr{S}\big],\qquad\big[\mathscr{L},\,\overline{\mathscr{S}}\big],\qquad\big[\overline{\mathscr{L}},\,\overline{\mathscr{S}}\big].$$

Because of the coefficient $\frac{\partial}{\partial z}$ in $\mathscr{L}$ is 1, we observe that each one of these four Lie brackets is a linear combination of just $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_3}$. For the same reason, we point out that $\mathscr{T}$, $\mathscr{S}$ and $\overline{\mathscr{S}}$ were also already a linear combination of just $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_3}$. Thus, we are sure that there are six complex-valued functions P, Q, R, A, B, C defined on M^5 such that:

$$\begin{aligned}\big[\mathscr{L},\,\mathscr{S}\big]&=P\,\mathscr{T}+Q\,\mathscr{S}+R\,\overline{\mathscr{S}},\\ \big[\overline{\mathscr{L}},\,\mathscr{S}\big]&=A\,\mathscr{T}+B\,\mathscr{S}+C\,\overline{\mathscr{S}},\\ \big[\mathscr{L},\,\overline{\mathscr{S}}\big]&=\overline{A}\,\mathscr{T}+\overline{C}\,\mathscr{S}+\overline{B}\,\overline{\mathscr{S}},\\ \big[\overline{\mathscr{L}},\,\overline{\mathscr{S}}\big]&=\overline{P}\,\mathscr{T}+\overline{R}\,\mathscr{S}+\overline{Q}\,\overline{\mathscr{S}}.\end{aligned}$$

Lemma 13.3. *In fact, the above vector fields $[\overline{\mathscr{L}},\mathscr{S}]$ and $[\mathscr{L},\overline{\mathscr{S}}]$ are real and equal. In particular, A is a real-valued function and $C=\overline{B}$.*

Proof. Indeed, given any two real or complex vector fields H_1 and H_2 on any manifold, one always has the following consequence of the Jacobi identity (*cf.* [58], eq. (15), p. 1817):

$$\big[H_2,\,\big[H_1,\,[H_1,\,H_2]\big]\big]=\big[H_1,\,\big[H_2,\,[H_1,\,H_2]\big]\big],$$

an identity which is true in free Lie algebras. Applied to $H_1:=\mathscr{L}$ and $H_2:=\overline{\mathscr{L}}$, this identity gives that the following two vector fields which are visibly conjugate to each other:

$$\begin{aligned}\big[\overline{\mathscr{L}},\,\mathscr{S}\big]&=\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,\mathscr{T}\big]\big]=\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,i[\mathscr{L},\,\overline{\mathscr{L}}]\big]\big]=i\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,[\mathscr{L},\,\overline{\mathscr{L}}]\big]\big],\\ \big[\mathscr{L},\,\overline{\mathscr{S}}\big]&=\big[\mathscr{L},\,\big[\overline{\mathscr{L}},\,\mathscr{T}\big]\big]=\big[\mathscr{L},\,\big[\overline{\mathscr{L}},\,i[\mathscr{L},\,\overline{\mathscr{L}}]\big]\big]=i\big[\mathscr{L},\,\big[\overline{\mathscr{L}},\,[\mathscr{L},\,\overline{\mathscr{L}}]\big]\big],\end{aligned}$$

are also in fact *equal*. This immediately implies that A is a real function and $C=\overline{B}$. □

Consequently, the expression of the third of the above four Lie brackets simplifies as:

$$\tag{89} \boxed{\begin{aligned} \big[\mathscr{L},\, \mathscr{S}\big] &= P\,\mathscr{T} + Q\,\mathscr{S} + R\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\, \mathscr{S}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}, \\ \big[\mathscr{L},\, \overline{\mathscr{S}}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\, \overline{\mathscr{S}}\big] &= \overline{P}\,\mathscr{T} + \overline{R}\,\mathscr{S} + \overline{Q}\,\overline{\mathscr{S}}. \end{aligned}}$$

Expressing these five functions A, B, P, Q and R explicitly in terms of the 4-th order jets $J^4_{x,y,u_1,u_2,u_3}\varphi_j$ of the three graphing functions φ_j brings formulas that are of an already quite impressive size. Nonetheless, we aim at presenting them, in semi-expanded form. At first, we need to have the expressions of the three *coordinate fields* $\frac{\partial}{\partial u_i}, i = 1, 2, 3$ in terms of the five complex vector fields $\overline{\mathscr{L}}, \mathscr{L}, \mathscr{T}, \overline{\mathscr{S}}, \mathscr{S}$. According to our computations we have:

$$\tag{90} \begin{aligned} \frac{\partial}{\partial u_1} &= (\Pi^1_t)\mathscr{T} + (\Pi^1_{s_1} - i\Pi^1_{s_2})\mathscr{S} + (\Pi^1_{s_1} + i\Pi^1_{s_2})\overline{\mathscr{S}}, \\ \frac{\partial}{\partial u_2} &= (\Pi^2_t)\mathscr{T} + (\Pi^2_{s_1} - i\Pi^2_{s_2})\mathscr{S} + (\Pi^2_{s_1} + i\Pi^2_{s_2})\overline{\mathscr{S}}, \\ \frac{\partial}{\partial u_3} &= (\Pi^3_t)\mathscr{T} + (\Pi^3_{s_1} - i\Pi^3_{s_2})\mathscr{S} + (\Pi^3_{s_1} + i\Pi^3_{s_2})\overline{\mathscr{S}}, \end{aligned}$$

where the coefficients are the real functions:

$$\begin{aligned} \Pi^1_t &= \Delta^3\big(\Gamma^2_2\Gamma^3_1 - \Gamma^3_2\Gamma^2_1\big)/4\Pi, & \Pi^1_{s_1} &= \Delta^5\big(\Upsilon_3\Gamma^2_2 - \Gamma^3_2\Upsilon_2\big)/16\Pi, \\ \Pi^1_{s_2} &= \Delta^5\big(\Upsilon_2\Gamma^3_1 - \Upsilon_3\Gamma^2_1\big)/16\Pi, & \Pi^2_t &= \Delta^3\big(\Gamma^3_1\Gamma^1_2 - \Gamma^1_1\Gamma^3_2\big)/4\Pi, \\ \Pi^2_{s_1} &= \Delta^5\big(\Upsilon_3\Gamma^1_2 - \Upsilon_1\Gamma^3_2\big)/16\Pi, & \Pi^2_{s_2} &= \Delta^5\big(\Upsilon_1\Gamma^3_1 - \Gamma^1_1\Upsilon_3\big)/16\Pi, \\ \Pi^3_t &= \Delta^3\big(\Gamma^2_1\Gamma^1_2 - \Gamma^1_1\Gamma^2_2\big)/16\Pi, & \Pi^3_{s_1} &= \Delta^5\big(\Upsilon_2\Gamma^1_2 - \Upsilon_1\Gamma^2_2\big)/16\Pi, \\ \Pi^3_{s_3} &= \Delta^5\big(\Upsilon_1\Gamma^2_1 - \Upsilon_2\Gamma^1_1\big)/16\Pi, \end{aligned}$$

and where the denominator Π explicitly is:

$$\Pi = -\Upsilon_1\Gamma^2_2\Gamma^3_1 - \Upsilon_2\Gamma^1_1\Gamma^3_2 + \Upsilon_2\Gamma^1_2\Gamma^3_1 + \Upsilon_1\Gamma^3_2\Gamma^2_1 + \Upsilon_3\Gamma^1_1\Gamma^2_2 - \Upsilon_3\Gamma^1_2\Gamma^2_1.$$

Computing directly the Lie brackets $[\mathscr{L}, \mathscr{S}]$ and $[\overline{\mathscr{L}}, \mathscr{S}]$ and putting the above expressions of $\frac{\partial}{\partial u_i}, i = 1, 2, 3$, the following expressions bring for A, B, C, P, Q and R (as mentioned before, the expressions of these functions are too much extensive. That is why we divide them into several sub-terms):

$$\begin{aligned}
P &= \Phi_3^1 - \Phi_1^1 + 2\,i\,\Phi_2^1,\\
Q &= \tfrac{1}{4}\big(\Phi_1^2 - \Phi_3^2 - 2\,\Phi_2^3\big) + \tfrac{i}{4}\big(\Phi_3^3 - \Phi_1^3 - 2\,\Phi_2^2\big),\\
R &= \tfrac{1}{4}\big(\Phi_1^2 - \Phi_3^2 + 2\,\Phi_2^3\big) + \tfrac{i}{4}\big(\Phi_1^3 - \Phi_3^3 - 2\,\Phi_2^2\big),\\
A &= -\Phi_1^1 - \Phi_3^1,\\
B &= \tfrac{1}{4}\big(\Phi_1^2 + \Phi_3^2\big) + \tfrac{i}{4}\big(\Phi_1^3 + \Phi_3^3\big),
\end{aligned}$$

where the terms $\Phi_\bullet^\bullet$ are:

$$\begin{aligned}
\Phi_1^1 &= \frac{-\Gamma_1^1\Gamma_2^3\Omega_1^2 - \Gamma_2^1\Gamma_1^2\Omega_1^3 + \Gamma_1^3\Gamma_2^1\Omega_1^2 + \Gamma_1^1\Omega_1^3\Gamma_2^2 + \Omega_1^1\Gamma_1^2\Gamma_2^3 - \Omega_1^1\Gamma_1^3\Gamma_2^2}{\Delta^2\Sigma},\\
\Phi_1^2 &= -\frac{\Upsilon_3\Gamma_2^1\Omega_1^2 - \Upsilon_3\Omega_1^1\Gamma_2^2 - \Omega_1^2\Upsilon_1\Gamma_2^3 + \Upsilon_2\Omega_1^1\Gamma_2^3 - \Omega_1^3\Gamma_2^1\Upsilon_2 + \Omega_1^3\Upsilon_1\Gamma_2^2}{\Sigma},\\
\Phi_1^3 &= \frac{\Gamma_1^2\Omega_1^3\Upsilon_1 - \Gamma_1^2\Upsilon_3\Omega_1^1 + \Upsilon_3\Gamma_1^1\Omega_1^2 - \Omega_1^3\Gamma_1^1\Upsilon_2 - \Omega_1^2\Gamma_1^3\Upsilon_1 + \Gamma_1^3\Upsilon_2\Omega_1^1}{\Sigma},\\
\Phi_2^1 &= \frac{-\Gamma_1^1\Gamma_2^3\Omega_2^2 - \Gamma_2^1\Gamma_1^2\Omega_2^3 + \Gamma_1^3\Gamma_2^1\Omega_2^2 + \Gamma_1^1\Omega_2^3\Gamma_2^2 + \Omega_2^1\Gamma_1^2\Gamma_2^3 - \Omega_2^1\Gamma_1^3\Gamma_2^2}{\Delta^2\Sigma},\\
\Phi_2^2 &= -\frac{\Upsilon_3\Gamma_2^1\Omega_2^2 - \Upsilon_3\Omega_2^1\Gamma_2^2 - \Omega_2^2\Upsilon_1\Gamma_2^3 + \Upsilon_2\Omega_2^1\Gamma_2^3 - \Omega_2^3\Gamma_2^1\Upsilon_2 + \Omega_2^3\Upsilon_1\Gamma_2^2}{\Sigma},\\
\Phi_2^3 &= \frac{\Gamma_1^2\Omega_2^3\Upsilon_1 - \Gamma_1^2\Upsilon_3\Omega_2^1 + \Upsilon_3\Gamma_1^1\Omega_2^2 - \Omega_2^3\Gamma_1^1\Upsilon_2 - \Omega_2^2\Gamma_1^3\Upsilon_1 + \Gamma_1^3\Upsilon_2\Omega_2^1}{\Sigma},\\
\Phi_3^1 &= \frac{-\Gamma_1^1\Gamma_2^3\Omega_3^2 - \Gamma_2^1\Gamma_1^2\Omega_3^3 + \Gamma_1^3\Gamma_2^1\Omega_3^2 + \Gamma_1^1\Omega_3^3\Gamma_2^2 + \Omega_3^1\Gamma_1^2\Gamma_2^3 - \Omega_3^1\Gamma_1^3\Gamma_2^2}{\Delta^2\Sigma},\\
\Phi_3^2 &= -\frac{\Upsilon_3\Gamma_2^1\Omega_3^2 - \Upsilon_3\Omega_3^1\Gamma_2^2 - \Omega_3^2\Upsilon_1\Gamma_2^3 + \Upsilon_2\Omega_3^1\Gamma_2^3 - \Omega_3^3\Gamma_2^1\Upsilon_2 + \Omega_3^3\Upsilon_1\Gamma_2^2}{\Sigma},\\
\Phi_3^3 &= \frac{\Gamma_1^2\Omega_3^3\Upsilon_1 - \Gamma_1^2\Upsilon_3\Omega_3^1 + \Upsilon_3\Gamma_1^1\Omega_3^2 - \Omega_3^3\Gamma_1^1\Upsilon_2 - \Omega_3^2\Gamma_1^3\Upsilon_1 + \Gamma_1^3\Upsilon_2\Omega_3^1}{\Sigma},
\end{aligned}$$

where Σ in the denominator is given explicitly by:

$$\Sigma = \Delta^2\big(\Gamma_1^3\Gamma_2^1\Upsilon_2 - \Gamma_1^1\Gamma_2^3\Upsilon_2 - \Gamma_2^1\Gamma_1^2\Upsilon_3 + \Gamma_1^1\Upsilon_3\Gamma_2^2 + \Upsilon_1\Gamma_1^2\Gamma_2^3 - \Upsilon_1\Gamma_1^3\Gamma_2^2\big),$$

and where — again admitting the notational coincidences $x \equiv x_1$ and $y \equiv x_2$ — the functions $\Omega_\bullet^\bullet$ are:

$$\begin{aligned}
\Omega_i^1 &= \Delta^2\Gamma_{ix}^1 - 5\Delta\Delta_{x_i}\Gamma_1^1 + \Delta\Lambda_1^1\Gamma_{iu_1}^1 - 4\Delta_{u_1}\Lambda_1^1\Gamma_i^1 - \Delta\Lambda_{1u_1}^1\Gamma_i^1 - \Delta\lambda_{u_2}^1\Gamma_1^2 +\\
&\quad + \Delta_{u_2}\Lambda_1^1\Gamma_{21}^1 - \Delta\Lambda_{1u_3}^1\Gamma_i^3 + \Delta_{u_3}\Lambda_1^1\Gamma_i^3 + \Delta\Lambda_1^2\Gamma_{iu_2}^1 - 5\Delta_{u_2}\Lambda_1^2\Gamma_i^1 + \Delta\Lambda_1^3\Gamma_{iu_3}^1 - 5\Delta_{u_3}\Lambda_1^3\Gamma_i^1,\\
\Omega_i^2 &= \Delta^2\Gamma_{ix}^2 - 5\Delta\Delta_{x_i}\Gamma_1^2 + \Delta\Lambda_1^1\Gamma_{iu_1}^2 - 5\Delta_{u_1}\Lambda_1^1\Gamma_i^2 - \Delta\Lambda_{1u_1}^2\Gamma_i^1 + \Delta_{u_1}\Lambda_1^2\Gamma_i^1 +\\
&\quad + \Delta\Lambda_1^2\Gamma_{iu_2}^2 - 4\Delta_{u_2}\Lambda_1^2\Gamma_i^2 - \Delta\Lambda_{1u_2}^2\Gamma_i^2 - \Delta\Lambda_{1u_3}^2\Gamma_i^3 + \Delta_{u_3}\Lambda_1^2\Gamma_i^3 + \Delta\Lambda_1^3\Gamma_{iu_3}^2 - 5\Delta_{u_3}\Lambda_1^3\Gamma_i^2,\\
\Omega_i^3 &= \Delta^2\Gamma_{ix}^3 - 5\Delta\Delta_{x_i}\Gamma_1^3 + \Delta\Lambda_1^1\Gamma_{iu_1}^3 - 5\Delta_{u_1}\Lambda_1^1\Gamma_i^3 + \Delta\Lambda_1^2\Gamma_{iu_2}^3 - 5\Delta_{u_2}\Lambda_1^2\Gamma_i^3 -\\
&\quad - \Delta\Lambda_{1u_1}^3\Gamma_i^1 + \Delta_{u_1}\Lambda_1^3\Gamma_i^1 - \Delta\Lambda_{1u_2}^3\Gamma_i^2 + \Delta_{u_2}\Lambda_1^3\Gamma_i^2 + \Delta\Lambda_1^3\Gamma_{iu_3}^3 - 4\Delta_{u_3}\Lambda_1^3\Gamma_i^3 - \Delta\Lambda_{1u_3}^3\Gamma_i^3,\\
\Omega_3^1 &= \Delta^2\Gamma_{2x}^1 - 5\Delta\Delta_y\Gamma_1^1 + \Delta\Lambda_1^1\Gamma_{2u_1}^1 - 4\Delta_{u_1}\Lambda_1^1\Gamma_3^1 - \Delta\Lambda_{2u_1}^1\Gamma_3^1 - \Delta\lambda_{u_2}^1\Gamma_1^2 +\\
&\quad + \Delta_{u_2}\Lambda_1^1\Gamma_{21}^1 - \Delta\Lambda_{2u_3}^1\Gamma_3^3 + \Delta_{u_3}\Lambda_1^1\Gamma_3^3 + \Delta\Lambda_1^2\Gamma_{2u_2}^1 - 5\Delta_{u_2}\Lambda_1^2\Gamma_3^1 + \Delta\Lambda_1^3\Gamma_{2u_3}^1 - 5\Delta_{u_3}\Lambda_1^3\Gamma_3^1,
\end{aligned}$$

$$\Omega_3^2 = \Delta^2\Gamma_{2x}^2 - 5\Delta\Delta_y\Gamma_1^2 + \Delta\Lambda_1^1\Gamma_{2u_1}^2 - 5\Delta_{u_1}\Lambda_1^1\Gamma_3^2 - \Delta\Lambda_{2u_1}^2\Gamma_3^1 + \Delta_{u_1}\Lambda_1^2\Gamma_3^1 +$$
$$+ \Delta\Lambda_2^2\Gamma_{2u_2}^2 - 4\Delta_{u_2}\Lambda_1^2\Gamma_3^2 - \Delta\Lambda_{2u_2}^2\Gamma_3^2 - \Delta\Lambda_{2u_3}^2\Gamma_3^3 + \Delta_{u_3}\Lambda_1^2\Gamma_3^3 + \Delta\Lambda_1^3\Gamma_{2u_3}^2 - 5\Delta_{u_3}\Lambda_1^3\Gamma_3^2,$$
$$\Omega_3^3 = \Delta^2\Gamma_{2x}^3 - 5\Delta\Delta_y\Gamma_1^3 + \Delta\Lambda_1^1\Gamma_{2u_1}^3 - 5\Delta_{u_1}\Lambda_1^1\Gamma_3^3 + \Delta\Lambda_2^2\Gamma_{2u_2}^3 - 5\Delta_{u_2}\Lambda_1^2\Gamma_3^3 -$$
$$- \Delta\Lambda_{2u_1}^3\Gamma_3^1 + \Delta_{u_1}\Lambda_1^3\Gamma_3^1 - \Delta\Lambda_{2u_2}^3\Gamma_3^2 + \Delta_{u_2}\Lambda_2^3\Gamma_3^2 + \Delta\Lambda_1^3\Gamma_{2u_3}^3 - 4\Delta_{u_3}\Lambda_1^3\Gamma_3^3 - \Delta\Lambda_{2u_3}^3\Gamma_3^3.$$

It yet remains to compute the 3 among 10 structure Lie brackets:

$$\big[\mathscr{T},\,\mathscr{S}\big], \qquad \big[\mathscr{T},\,\overline{\mathscr{S}}\big], \qquad \big[\mathscr{S},\,\overline{\mathscr{S}}\big].$$

13.5. **Two structure brackets of length 5.** Next, for the two iterated Lie brackets:

$$\big[\mathscr{T},\,\mathscr{S}\big] = \Big[i[\mathscr{L},\overline{\mathscr{L}}],\; [\mathscr{L},\,i[\mathscr{L},\overline{\mathscr{L}}]]\Big],$$
$$\big[\mathscr{T},\,\overline{\mathscr{S}}\big] = \Big[i[\mathscr{L},\overline{\mathscr{L}}],\; [\overline{\mathscr{L}},\,i[\mathscr{L},\overline{\mathscr{L}}]]\Big],$$

that are visibly of length 5. Again the Jacobi identity helps us to specify their expressions.

Lemma 13.4. *The coefficients of the two Lie brackets:*

$$\big[\mathscr{T},\,\mathscr{S}\big] = E\,\mathscr{T} + F\,\mathscr{S} + G\,\overline{\mathscr{S}},$$
$$\big[\mathscr{T},\,\overline{\mathscr{S}}\big] = \overline{E}\,\mathscr{T} + \overline{G}\,\mathscr{S} + \overline{F}\,\overline{\mathscr{S}},$$

are three complex-valued functions E, F, G which can be expressed as follows in terms of P, Q, R, A, B and their first-order frame derivatives:

$$E = -\,i\,\overline{\mathscr{L}}(P) - i\,A\,Q - i\,\overline{P}\,R + i\,\mathscr{L}(A) + i\,B\,P + i\,A\,\overline{B},$$
$$F = -\,i\,\overline{\mathscr{L}}(Q) - i\,R\,\overline{R} + i\,A + i\,\mathscr{L}(B) + i\,B\,\overline{B},$$
$$G = -\,i\,P - i\,\overline{B}\,Q - i\,R\,\overline{Q} - i\,\overline{\mathscr{L}}(R) + i\,B\,R + i\,\overline{B}\overline{B} + i\,\mathscr{L}(\overline{B}).$$

Proof. A glance at the explicit expressions of the three complex fields $\mathscr{T}, \mathscr{S}, \overline{\mathscr{S}}$ shows that they are some combinations of the three coordinate fields $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_3}$, *without* any $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}}$. Therefore, the two considered Lie brackets must merely be some combinations of these three coordinate fields $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_3}$, too. Then by inversion, they must become linear combinations of the three fields $\mathscr{T}$, $\mathscr{S}$, $\overline{\mathscr{S}}$. In fact, using (90), one can expand the following Jacobi identity:

$$\begin{aligned}
-i\,[\mathscr{T},\mathscr{S}] &= \big[[\mathscr{L},\overline{\mathscr{L}}],\mathscr{S}\big] \\
&= \big[\overline{\mathscr{L}},[\mathscr{S},\mathscr{L}]\big] + \big[\mathscr{L},[\overline{\mathscr{L}},\mathscr{S}]\big] \\
&= [\overline{\mathscr{L}}, -P\,\mathscr{T} - Q\,\mathscr{S} - R\,\overline{\mathscr{S}}] + [\mathscr{L}, A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}] \\
&= -\overline{\mathscr{L}}(P)\,\mathscr{T} - P\,\overline{\mathscr{S}} - \overline{\mathscr{L}}(Q)\,\mathscr{S} - Q\,(A\,\mathscr{T} + \underline{B\,\mathscr{S}}_{\circ} + \overline{B}\,\overline{\mathscr{S}}) - \overline{\mathscr{L}}(R)\,\overline{\mathscr{S}} - \\
&\quad - R(\overline{P}\,\mathscr{T} + \overline{R}\,\mathscr{S} + \overline{Q}\,\overline{\mathscr{S}}) + A\,\mathscr{S} + \mathscr{L}(A)\,\mathscr{T} + \\
&\quad + B(P\,\mathscr{T} + \underline{Q\,\mathscr{S}}_{\circ} + R\,\overline{\mathscr{S}}) + \mathscr{L}(B)\,\mathscr{S} + \overline{B}\,(A\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}) + \mathscr{L}(\overline{B})\,\overline{\mathscr{S}}.
\end{aligned}$$

Here the underlined terms vanish by pair. Now, extracting the coefficients of $\mathscr{T}$, $\mathscr{S}$, $\overline{\mathscr{S}}$ gives the desired expressions of E, F and G, respectively. □

13.6. **The last structure bracket of length 6.** Now, the last remaining iterated bracket:

$$[\mathscr{S}, \overline{\mathscr{S}}] = \Big[\big[\mathscr{L}, i[\mathscr{L}, \overline{\mathscr{L}}]\big],\ \big[\overline{\mathscr{L}}, i[\mathscr{L}, \overline{\mathscr{L}}]\big]\Big]$$

is of length 6, and because Jacobi identities at this level start to become more complex, we must take care of how to re-express it. In a preceding publication (*see* equations $\overset{1}{=}$ and $\overset{2}{=}$ page 1818 of [58]), we showed that, in a free Lie algebra generated by two vectors h_1 and h_2, the following two relations hold true:

$$\begin{aligned}
0 = \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] - \big[h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]\big]+ \\
+ \big[h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]\big], \\
0 = \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] + \big[h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]\big]- \\
- \big[h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]\big].
\end{aligned}$$

Then by a plain addition, replacing $h_1 := \mathscr{L}$ and $h_2 := \overline{\mathscr{L}}$, we may express our length-six Lie bracket in terms of simple-words Lie brackets:

$$\begin{aligned}
-2\Big[\big[\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]\big],\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\Big] = -\Big[\overline{\mathscr{L}},\Big[\overline{\mathscr{L}},\big[\mathscr{L},\big[\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big] + \Big[\overline{\mathscr{L}},\Big[\mathscr{L},\big[\mathscr{L},\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]+ \\
+ \Big[\mathscr{L},\Big[\overline{\mathscr{L}},\big[\mathscr{L},\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big] - \Big[\mathscr{L},\Big[\mathscr{L},\big[\overline{\mathscr{L}},\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big].
\end{aligned}$$

Notably, this expression points out the purely imaginary character *of both sides*, hence it is appropriate for what will follow. By expanding the four simple-words Lie brackets, we obtain:

Lemma 13.5. *The coefficients of the last, tenth structure bracket:*

$$[\mathscr{S}, \overline{\mathscr{S}}] = i\,J\,\mathscr{T} + K\,\mathscr{S} - \overline{K}\,\overline{\mathscr{S}},$$

are one complex-valued function K and one real-valued function J which can be expressed as follows in terms of P, Q, R, A, B and their frame derivatives up to order 2:

$$\begin{aligned}
-2\,J = & -\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(P)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(A)\big) - \mathscr{L}\big(\mathscr{L}(\overline{P})\big) \\
& - Q\,\overline{\mathscr{L}}(A) - 2\,A\,\overline{\mathscr{L}}(Q) - R\,\overline{\mathscr{L}}(\overline{P}) - 2\,\overline{P}\,\overline{\mathscr{L}}(R) - 2\,AR\overline{R} - 2\,P\overline{P} - \overline{BP}Q - \overline{PQ}R- \\
& - \overline{R}\mathscr{L}(P) - 2\,P\,\mathscr{L}(\overline{R}) - \overline{Q}\,\mathscr{L}(A) - 2\,A\mathscr{L}(\overline{Q}) - PQ\overline{R} - BP\overline{Q}+ \\
& + 2\,P\overline{\mathscr{L}}(B) + B\overline{\mathscr{L}}(P) + 2\,A\overline{\mathscr{L}}(\overline{B}) + \overline{B}\,\overline{\mathscr{L}}(A) + 2\,A\,\mathscr{L}(B) + 2\,AA + 2\,AB\overline{B} + 2\,\overline{P}\,\mathscr{L}(\overline{B})+ \\
& + B\overline{P}R + \overline{BBP} + B\,\mathscr{L}(A) + \overline{B}\,\mathscr{L}(\overline{P}) + BBP + \overline{B}P\overline{R},
\end{aligned}$$

$$
\begin{aligned}
2i\,K = &-\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(Q)\big)+\overline{\mathscr{L}}\big(\mathscr{L}(B)\big)+\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)-\mathscr{L}\big(\mathscr{L}(\overline{R})\big)-\\
&-2\,\overline{R}\,\overline{\mathscr{L}}(R)-R\,\overline{\mathscr{L}}(\overline{R})-B\,\overline{\mathscr{L}}(Q)-BR\overline{R}-2\,P\overline{R}-\overline{Q}R\overline{R}-2\,\mathscr{L}(\overline{P})-\overline{R}\,\mathscr{L}(Q)-\\
&-\,2\,Q\,\mathscr{L}(\overline{R})-\overline{Q}\,\mathscr{L}(B)-2\,B\,\mathscr{L}(\overline{Q})-A\overline{Q}-\overline{P}Q-QQ\overline{R}-BQ\overline{Q}+\\
&+2\,\overline{\mathscr{L}}(A)+\overline{B}\,\overline{\mathscr{L}}(B)+2\,B\,\overline{\mathscr{L}}(\overline{B})+3\,B\,\mathscr{L}(B)+3\,AB+BBQ+2\,BB\overline{B}+2\,\overline{R}\,\mathscr{L}(\overline{B})+\\
&+\overline{BBR}+\overline{B}\,\mathscr{L}(\overline{R})+\overline{BP}+Q\,\overline{\mathscr{L}}(B). \quad \square
\end{aligned}
$$

13.7. **Further relations.** Moreover, we also showed in [58] (*see* equations $\overset{9}{=}$, $\overset{10}{=}$, $\overset{11}{=}$ pp. 1818–1819 of this paper) that exactly *three* independent linear relations hold between simple-word Lie brackets, which, applied to $h_1 := \mathscr{L}$ and $h_2 := \overline{\mathscr{L}}$, are:

$$
\begin{aligned}
0 &\overset{1}{=} \Big[\mathscr{L},\,\Big[\mathscr{L},\,\big[\mathscr{L},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]-2\Big[\mathscr{L},\,\Big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,\big[\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]+\Big[\overline{\mathscr{L}},\,\Big[\mathscr{L},\,\big[\mathscr{L},\,\big[\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big],\\
0 &\overset{2}{=} \Big[\overline{\mathscr{L}},\,\Big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]-2\Big[\overline{\mathscr{L}},\,\Big[\mathscr{L},\,\big[\overline{\mathscr{L}},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]+\Big[\mathscr{L},\,\Big[\overline{\mathscr{L}},\,\big[\overline{\mathscr{L}},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big],\\
0 &\overset{3}{=} \Big[\mathscr{L},\,\Big[\mathscr{L},\,\big[\overline{\mathscr{L}},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]-3\Big[\mathscr{L},\,\Big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]+3\Big[\overline{\mathscr{L}},\,\Big[\mathscr{L},\,\big[\mathscr{L},\,\big[\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big]-\\
&\qquad-\Big[\overline{\mathscr{L}},\,\Big[\overline{\mathscr{L}},\,\big[\mathscr{L},\,\big[\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]\big]\big]\Big]\Big].
\end{aligned}
$$

Applying the computations relevant to the above three equations, one sees that the two equations 1 and 2 bring two same outcomes, since each of them is the conjugation of each other with a negative sign. Also for the equation 3, the coefficient of $\overline{\mathscr{S}}$ is the conjugation of that of $\mathscr{S}$, again with a negative sign. Overall, these equations give the following five independent relationships between the fundamental functions A, B, P, Q, R, their conjugations and derivations:

(91)

$$
\begin{aligned}
0 &\overset{1}{\equiv} 2\,\mathscr{L}(\overline{\mathscr{L}}(P))-\mathscr{L}(\mathscr{L}(A))-\overline{\mathscr{L}}(\mathscr{L}(P))-\\
&\;2\,P\mathscr{L}(B)-B\mathscr{L}(P)-2\,A\mathscr{L}(\overline{B})-\overline{B}\mathscr{L}(A)+P\overline{\mathscr{L}}(Q)+A\mathscr{L}(Q)+\\
&+2\,Q\mathscr{L}(A)-Q\overline{\mathscr{L}}(P)+A\overline{\mathscr{L}}(R)+2\,R\mathscr{L}(\overline{P})+\overline{P}\mathscr{L}(R)-R\overline{\mathscr{L}}(A)-\\
&-PB\overline{B}-A\overline{B}^2+PBQ+2\,AQ\overline{B}-AQ^2-2\,ABR+2\,RP\overline{R}+2\,AR\overline{Q}-QR\overline{P}-R\overline{B}\,\overline{P},\\
0 &\overset{2}{\equiv} 2\,\mathscr{L}(\overline{\mathscr{L}}(Q))-\mathscr{L}(\mathscr{L}(B))-\overline{\mathscr{L}}(\mathscr{L}(Q))-\\
&-2\,\mathscr{L}(A)-2\,B\mathscr{L}(\overline{B})-\overline{B}\mathscr{L}(B)+B\overline{\mathscr{L}}(R)+2\,R\mathscr{L}(\overline{R})+\overline{R}\mathscr{L}(R)-R\overline{\mathscr{L}}(B)+\overline{\mathscr{L}}(P)+\\
&+2\,R\overline{P}+BQ\overline{B}-A\overline{B}-B\overline{B}^2+AQ+QR\overline{R}+2\,BR\overline{Q}-2\,B^2R-R\overline{B}\,\overline{R},\\
0 &\overset{3}{\equiv} 2\,\mathscr{L}(\overline{\mathscr{L}}(R))-\mathscr{L}(\mathscr{L}(\overline{B}))-\overline{\mathscr{L}}\big(\mathscr{L}(R)\big)-\\
&-3\,\overline{B}\mathscr{L}(\overline{B})+\overline{B}\mathscr{L}(Q)+2\,Q\mathscr{L}(\overline{B})-2\,R\mathscr{L}(B)-B\mathscr{L}(R)+R\overline{\mathscr{L}}(Q)+\overline{B}\,\overline{\mathscr{L}}(R)+\\
&+2\,R\mathscr{L}(\overline{Q})+\overline{Q}\mathscr{L}(R)-Q\overline{\mathscr{L}}(R)-\overline{\mathscr{L}}\mathscr{L}(R)-R\overline{\mathscr{L}}(\overline{B})+\mathscr{L}(P)+\\
&+2\,Q\overline{B}^2-QP-Q^2\overline{B}-\overline{B}^3+P\overline{B}-2\,AR-2\,BR\overline{B}+BQR+2\,R^2\overline{R}+R\overline{B}\,\overline{Q}-QR\overline{Q},\\
0 &\overset{4}{\equiv} -3\overline{\mathscr{L}}(\mathscr{L}(A))-\mathscr{L}(\mathscr{L}(\overline{P}))+3\,\mathscr{L}(\overline{\mathscr{L}}(A))+\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))-\\
&-2\,A\mathscr{L}(\overline{Q})-\overline{Q}\mathscr{L}(A)+3\,B\mathscr{L}(A)+3\,\overline{B}\mathscr{L}(\overline{P})-3\,B\overline{\mathscr{L}}(P)-3\,\overline{B}\,\overline{\mathscr{L}}(A)+2\,A\overline{\mathscr{L}}(Q)-
\end{aligned}
$$

$$
\begin{aligned}
&+Q\overline{\mathscr{L}}(A)-2\,P\mathscr{L}(\overline{R})-\overline{R}\mathscr{L}(P)+2\,\overline{P}\,\overline{\mathscr{L}}(R)+R\overline{\mathscr{L}}(\overline{P})-\\
&-BP\overline{Q}+3\,B^2P+2\,A\overline{B}\,\overline{Q}-2\,BQA-3\,\overline{B}^2\overline{P}+Q\overline{B}\,\overline{P}-PQ\overline{R}+3\,P\overline{B}\,\overline{R}-3\,BR\overline{P}+R\overline{P}\,\overline{Q},\\
0\overset{5}{\equiv}&-3\,\overline{\mathscr{L}}(\mathscr{L}(B))+3\,\mathscr{L}(\overline{\mathscr{L}}(B))+\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q))-\mathscr{L}(\mathscr{L}(\overline{R}))+\\
&+3\,B\mathscr{L}(B)-3\,\overline{B}\,\overline{\mathscr{L}}(B)+Q\overline{\mathscr{L}}(B)-B\overline{\mathscr{L}}(Q)-2\,Q\mathscr{L}(\overline{R})-\overline{R}\mathscr{L}(Q)-\\
&-2\,B\mathscr{L}(\overline{Q})-\overline{Q}\mathscr{L}(B)+3\,\overline{B}\mathscr{L}(\overline{R})+2\,\overline{R}\,\overline{\mathscr{L}}(R)+R\overline{\mathscr{L}}(\overline{R})-2\,\mathscr{L}(\overline{P})-\\
&-Q\overline{P}-A\overline{Q}-BQ\overline{Q}+3\,AB+3\,\overline{B}\,\overline{P}+2\,B\overline{B}\,\overline{Q}+B^2Q-Q^2\overline{R}+4\,Q\overline{B}\,\overline{R}-\\
&-3\,BR\overline{R}-3\,\overline{B}^2\overline{R}+R\overline{Q}\,\overline{R}.
\end{aligned}
$$

14. PASSAGE TO A DUAL COFRAME AND ITS DARBOUX-CARTAN STRUCTURE

On the natural agreement that:

the coframe $\big\{du_3, du_2, du_1,\, dz,\, d\overline{z}\big\}$ is dual to the frame $\big\{\frac{\partial}{\partial u_3},\, \frac{\partial}{\partial u_2},\, \frac{\partial}{\partial u_1},\, \frac{\partial}{\partial z},\, \frac{\partial}{\partial \overline{z}}\big\}$,

let us introduce the coframe:

$\big\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\big\}$ which is dual to the frame $\big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\}$,

that is to say which satisfies by definition:

$$
\begin{array}{lllll}
\overline{\sigma_0}(\overline{\mathscr{S}})=1 & \overline{\sigma_0}(\mathscr{S})=0 & \overline{\sigma_0}(\mathscr{T})=0 & \overline{\sigma_0}(\overline{\mathscr{L}})=0 & \overline{\sigma_0}\big(\mathscr{L}\big)=0,\\
\sigma_0(\overline{\mathscr{S}})=0 & \sigma_0(\mathscr{S})=1 & \sigma_0(\mathscr{T})=0 & \sigma_0(\overline{\mathscr{L}})=0 & \sigma_0\big(\mathscr{L}\big)=0,\\
\rho_0(\overline{\mathscr{S}})=0 & \rho_0(\mathscr{S})=0 & \rho_0(\mathscr{T})=1 & \rho_0(\overline{\mathscr{L}})=0 & \rho_0\big(\mathscr{L}\big)=0,\\
\overline{\zeta_0}(\overline{\mathscr{S}})=0 & \overline{\zeta_0}(\mathscr{S})=0 & \overline{\zeta_0}(\mathscr{T})=0 & \overline{\zeta_0}(\overline{\mathscr{L}})=1 & \overline{\zeta_0}\big(\mathscr{L}\big)=0,\\
\zeta_0(\overline{\mathscr{S}})=0 & \zeta_0(\mathscr{S})=0 & \zeta_0(\mathscr{T})=0 & \zeta_0(\overline{\mathscr{L}})=0 & \zeta_0\big(\mathscr{L}\big)=1.
\end{array}
$$

Since neither $\mathscr{T}$, nor $\mathscr{S}$, nor $\overline{\mathscr{S}}$ incorporate any $\frac{\partial}{\partial u_j}$, $j=1,2,3$, we have:

$$
\zeta_0=dz \qquad\text{and}\qquad \overline{\zeta_0}=d\overline{z}.
$$

In order to launch the Cartan algorithm, initially we need the expressions of the five 2-forms $d\overline{\sigma}_0, d\sigma_0, d\rho_0, d\overline{\zeta}_0, d\zeta_0$ in terms of the wedge products of $\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0$.

To find them, we remember that if a frame $\big\{\mathscr{L}_1,\dots,\mathscr{L}_n\big\}$ on an open subset of $\mathbb{R}^n$ enjoys the Lie structure:

$$
\big[\mathscr{L}_{i_1},\, \mathscr{L}_{i_2}\big]=\sum_{k=1}^n a_{i_1,i_2}^k\,\mathscr{L}_k \qquad\qquad {\scriptstyle (1\leqslant i_1<i_2\leqslant n)},
$$

then its dual coframe $\{\omega^1,\dots,\omega^n\}$ enjoys the Darboux-Cartan structure:

$$
d\omega^k=-\sum_{1\leqslant i_1<i_2\leqslant n} a_{i_1,i_2}^k\,\omega^{i_1}\wedge\omega^{i_2} \qquad\qquad {\scriptstyle (k=1\cdots n)}.
$$

Granted this reminder, it is convenient to rewrite the ten Lie brackets under the form of a convenient auxiliary array:

$$
\begin{array}{ccccccccccccc}
 & & \overline{\mathscr{S}} & & \mathscr{S} & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & & \\
 & & \boxed{d\overline{\sigma_0}} & & \boxed{d\sigma_0} & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & & \\
\big[\overline{\mathscr{S}},\mathscr{S}\big] & = & \overline{K}\cdot\overline{\mathscr{S}} & + & -K\cdot\mathscr{S} & + & -i\,J\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\sigma_0} \\
\big[\overline{\mathscr{S}},\mathscr{T}\big] & = & -\overline{F}\cdot\overline{\mathscr{S}} & + & -\overline{G}\cdot\mathscr{S} & + & -\overline{E}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\rho_0} \\
\big[\overline{\mathscr{S}},\overline{\mathscr{L}}\big] & = & -\overline{Q}\cdot\overline{\mathscr{S}} & + & -\overline{R}\cdot\mathscr{S} & + & -\overline{P}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\overline{\zeta_0}} \\
\big[\overline{\mathscr{S}},\mathscr{L}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\zeta_0} \\
\big[\mathscr{S},\mathscr{T}\big] & = & -G\cdot\overline{\mathscr{S}} & + & -F\cdot\mathscr{S} & + & -E\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\rho_0} \\
\big[\mathscr{S},\overline{\mathscr{L}}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{S},\mathscr{L}\big] & = & -R\cdot\overline{\mathscr{S}} & + & -Q\cdot\mathscr{S} & + & -P\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\zeta_0} \\
\big[\mathscr{T},\overline{\mathscr{L}}\big] & = & -\overline{\mathscr{S}} & + & 0 & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{T},\mathscr{L}\big] & = & 0 & + & -\mathscr{S} & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\zeta_0} \\
\big[\overline{\mathscr{L}},\mathscr{L}\big] & = & 0 & + & 0 & + & i\,\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\zeta_0}\wedge\zeta_0}
\end{array}
$$

Thank to this array, we can *vertically read* the expressions of the 10 forms of degree 2 that provides the associated Darboux-Cartan structure, putting an overall minus sign:

$$
\tag{92}
\boxed{
\begin{aligned}
d\overline{\sigma}_0 &= -\overline{K}\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{F}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{Q}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+\overline{B}\cdot\overline{\sigma}_0\wedge\zeta_0+ \\
&\qquad +G\cdot\sigma_0\wedge\rho_0+\overline{B}\cdot\sigma_0\wedge\overline{\zeta}_0+R\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\overline{\zeta}_0, \\
d\sigma_0 &= K\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{G}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{R}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+B\cdot\overline{\sigma}_0\wedge\zeta_0+ \\
&\qquad +F\cdot\sigma_0\wedge\rho_0+B\cdot\sigma_0\wedge\overline{\zeta}_0+Q\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\zeta_0, \\
d\rho_0 &= i\,J\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{E}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{P}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+A\cdot\overline{\sigma}_0\wedge\zeta_0+ \\
&\qquad +E\cdot\sigma_0\wedge\rho_0+A\cdot\sigma_0\wedge\overline{\zeta}_0+P\cdot\sigma_0\wedge\zeta_0-i\,\overline{\zeta}_0\wedge\zeta_0, \\
d\overline{\zeta}_0 &= 0, \\
d\zeta_0 &= 0.
\end{aligned}
}
$$

14.1. **Ambiguity matrix.** Consider now a local biholomorphic equivalence:

$$
h\colon\quad (z,w)\longmapsto\big(f(z,w),\,g(z,w)\big)=:(z',w')
$$

between any two real analytic local CR-generic maximally minimal real submanifolds:

$$
M^5\subset\mathbb{C}^4_{(z,w)}\qquad\text{and}\qquad M'^5\subset\mathbb{C}^4_{(z',w')}.
$$

As we saw in what precedes, the assumption that both M^5 and M'^5 are maximally minimal means that the two sets of five vector fields:

$$\Big\{\mathscr{L},\,\overline{\mathscr{L}},\,\mathscr{T},\,\mathscr{S},\,\overline{\mathscr{S}}\Big\}\qquad\text{and}\qquad\Big\{\mathscr{L}',\,\overline{\mathscr{L}}',\,\mathscr{T}',\,\mathscr{S}',\,\overline{\mathscr{S}}'\Big\}$$

make up frames for $TM^5\otimes_{\mathbb{R}}\mathbb{C}$ and for $TM'^5\otimes_R\mathbb{C}$, respectively.

By a quick inspection of the proof of Proposition 12.4, one easily convinces oneself that for it to hold true, we in fact did not at all use the assumption that M^5 was the model cubic $M^5_{\sf c}$. So in the general case, we also have:

Proposition 14.1. *The initial ambiguity matrix associated to the equivalence problem under local biholomorphic transformations for maximally minimal CR-generic* 3*-codimensional submanifolds* $M^5\subset\mathbb{C}^4$ *is of the general form:*

$$\begin{pmatrix}{\sf a}\overline{\sf a}\overline{\sf a} & 0 & \overline{\sf c} & \overline{\sf e} & \overline{\sf d}\\ 0 & {\sf a}{\sf a}\overline{\sf a} & {\sf c} & {\sf d} & {\sf e}\\ 0 & 0 & {\sf a}\overline{\sf a} & \overline{\sf b} & {\sf b}\\ 0 & 0 & 0 & \overline{\sf a} & 0\\ 0 & 0 & 0 & 0 & {\sf a}\end{pmatrix},$$

where ${\sf a}, {\sf b}, {\sf c}, {\sf e}, {\sf d}$ *are complex numbers. Moreover, the collection of all these matrices makes up a real 10-dimensional matrix Lie subgroup of* ${\sf GL}_5(\mathbb{C})$.

□

14.2. **Setting up the equivalence problem.** Quite similarly as in the case where M^5 was the cubic model $M^5_{\sf c}$ — though the levels of complexity will rapidly diverge —, with the initial coframe $\big\{\overline{\sigma}_0,\sigma_0,\rho_0,\overline{\zeta}_0,\zeta_0\big\}$ dual to the explicitly computed frame $\big\{\overline{\mathscr{S}},\mathscr{S},\mathscr{T},\overline{\mathscr{L}},\mathscr{L}\big\}$, the lifted coframe is then (one must transpose the ambiguity matrix):

$$\begin{pmatrix}\overline{\sigma}\\ \sigma\\ \rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} := \underbrace{\begin{pmatrix}{\sf a}\overline{\sf a}\overline{\sf a} & 0 & 0 & 0 & 0\\ 0 & {\sf a}{\sf a}\overline{\sf a} & 0 & 0 & 0\\ \overline{\sf c} & {\sf c} & {\sf a}\overline{\sf a} & 0 & 0\\ \overline{\sf e} & {\sf d} & \overline{\sf b} & \overline{\sf a} & 0\\ \overline{\sf d} & {\sf e} & {\sf b} & 0 & {\sf a}\end{pmatrix}}_{=:g}\begin{pmatrix}\overline{\sigma}_0\\ \sigma_0\\ \rho_0\\ \overline{\zeta}_0\\ \zeta_0\end{pmatrix}, \tag{93}$$

that is to say:

$$\begin{aligned}
\overline{\sigma} &= {\sf a}\overline{\sf a}\overline{\sf a}\,\overline{\sigma_0},\\
\sigma &= {\sf a}{\sf a}\overline{\sf a}\,\sigma_0,\\
\rho &= \overline{\sf c}\,\overline{\sigma_0}+{\sf c}\,\sigma_0+{\sf a}\overline{\sf a}\,\rho_0,\\
\overline{\zeta} &= \overline{\sf e}\,\overline{\sigma_0}+{\sf d}\,\sigma_0+\overline{b}\,\rho_0+\overline{\sf a}\,\overline{\zeta_0},\\
\zeta &= \overline{\sf d}\,\overline{\sigma_0}+{\sf e}\,\sigma_0+{\sf b}\,\rho_0+{\sf a}\,\zeta_0.
\end{aligned}$$

Again, the 1-form ρ is real and the 1-forms $\overline{\sigma}$ and $\overline{\zeta}$ are the conjugate of σ and ζ.

Now, our objective is to perform the equivalence method with these general data, taking advantage of what has been already finalized in the simpler case of the model in Section 12.

15. Absorption and normalization

Proceeding exactly as in the beginning of Subsection 12.9, differentiating both sides of (93) yields in matrix notation:

$$
d\begin{pmatrix}\overline{\sigma}\\ \sigma\\ \rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} = dg\wedge\begin{pmatrix}\overline{\sigma}_0\\ \sigma_0\\ \rho_0\\ \overline{\zeta}_0\\ \zeta_0\end{pmatrix} + g\cdot\begin{pmatrix}d\overline{\sigma}_0\\ d\sigma_0\\ d\rho_0\\ d\overline{\zeta}_0\\ d\zeta_0\end{pmatrix}
$$

$$
= \underbrace{\begin{pmatrix} 2\,\overline{\alpha}_1+\alpha_1 & 0 & 0 & 0 & 0\\ 0 & 2\,\alpha_1+\overline{\alpha}_1 & 0 & 0 & 0\\ \overline{\alpha}_2 & \alpha_2 & \alpha_1+\overline{\alpha}_1 & 0 & 0\\ \overline{\alpha}_3 & \overline{\alpha}_4 & \overline{\alpha}_5 & \overline{\alpha}_1 & 0\\ \alpha_4 & \alpha_3 & \alpha_5 & 0 & \alpha_1\end{pmatrix}}_{\omega_{\mathsf{MC}}:=dg\cdot g^{-1}} \wedge \underbrace{\begin{pmatrix}\overline{\sigma}\\ \sigma\\ \rho\\ \overline{\zeta}\\ \zeta\end{pmatrix}}_{g\cdot(\overline{\sigma}_0,\sigma_0,\rho_0,\overline{\zeta}_0,\zeta_0)^t} +
$$

$$
+\begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}}\,d\overline{\sigma}_0\\ \mathsf{a}\mathsf{a}\overline{\mathsf{a}}\,d\sigma_0\\ \overline{\mathsf{c}}\,d\overline{\sigma}_0+\mathsf{c}\,d\sigma_0+\mathsf{a}\overline{\mathsf{a}}\,d\rho_0\\ \overline{\mathsf{e}}\,d\overline{\sigma}_0+\mathsf{d}\,d\sigma_0+\overline{\mathsf{b}}\,d\rho_0+\overline{\mathsf{a}}\,d\overline{\zeta}_0\\ \overline{\mathsf{d}}\,d\overline{\sigma}_0+\mathsf{e}\,d\sigma_0+\mathsf{b}\,d\rho_0+\mathsf{a}\,d\zeta_0\end{pmatrix}.
$$

Of course here, the Maurer-Cartan forms are the same as the ones computed before when dealing with the model:

$$
\begin{aligned}
\alpha_1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\alpha_2 &:= \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} - \frac{\mathsf{c}\,d\mathsf{a}}{\mathsf{a}^3\overline{\mathsf{a}}} - \frac{\mathsf{c}\,d\overline{\mathsf{a}}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
\alpha_3 &:= -\frac{\mathsf{c}\,d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \left(\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\right)d\mathsf{a} + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,d\mathsf{e},\\
\alpha_4 &:= \frac{d\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2} - \frac{\overline{\mathsf{c}}\,d\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^3} + \left(\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right)d\mathsf{a},\\
\alpha_5 &:= \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}} - \frac{\mathsf{b}\,d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}}.
\end{aligned}
$$

Also, from Subsection 12.9 again, we know that the inverse of the general matrix g of our ambiguity group is:

$$g^{-1} = \begin{pmatrix} \frac{1}{a\overline{a}^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2\overline{a}} & 0 & 0 & 0 \\ -\frac{\overline{c}}{a^2\overline{a}^3} & -\frac{c}{a^3\overline{a}^2} & \frac{1}{a\overline{a}} & 0 & 0 \\ \frac{\overline{b}\,\overline{c}-\overline{e}a\overline{a}}{a^2\overline{a}^4} & \frac{\overline{b}c-a\overline{a}d}{a^3\overline{a}^3} & -\frac{\overline{b}}{a\overline{a}^2} & \frac{1}{\overline{a}} & 0 \\ \frac{b\overline{c}-a\overline{a}\overline{d}}{a^3\overline{a}^3} & \frac{bc-ea\overline{a}}{a^4\overline{a}^2} & -\frac{b}{a^2\overline{a}} & 0 & \frac{1}{a} \end{pmatrix},$$

hence we have the same inversion formulas as we had when dealing with the model:

$$\begin{aligned} \sigma_0 &= \frac{1}{a^2\overline{a}}\,\sigma, \\ \rho_0 &= -\frac{\overline{c}}{a^2\overline{a}^3}\,\overline{\sigma} - \frac{c}{a^3\overline{a}^2}\,\sigma + \frac{1}{a\overline{a}}\,\rho, \\ \zeta_0 &= \frac{b\overline{c} - a\overline{a}\overline{d}}{a^3\overline{a}^3}\,\overline{\sigma} + \frac{bc - a\overline{a}e}{a^4\overline{a}^2}\,\sigma - \frac{b}{a^2\overline{a}}\,\rho + \frac{1}{a}\,\zeta. \end{aligned} \tag{94}$$

However, at this precise point, computations start to be differ and to become substantially harder.

Indeed, leaving aside the trivial:

$$d\zeta_0 = 0,$$

coming back to the two initial structure equations (92) that were set up in the preceding section —, we modify appropriately the order of appearance of terms —:

$$\begin{aligned} d\sigma_0 = &- K \cdot \sigma_0 \wedge \overline{\sigma}_0 + F \cdot \sigma_0 \wedge \rho_0 + Q \cdot \sigma_0 \wedge \zeta_0 + B \cdot \sigma_0 \wedge \overline{\zeta}_0 + \\ &+ \overline{G} \cdot \overline{\sigma}_0 \wedge \rho_0 + B \cdot \overline{\sigma}_0 \wedge \zeta_0 + \overline{R} \cdot \overline{\sigma}_0 \wedge \overline{\zeta}_0 + \\ &+ \rho_0 \wedge \zeta_0, \end{aligned} \tag{95}$$

$$\begin{aligned} d\rho_0 = &- i\, J \cdot \sigma_0 \wedge \overline{\sigma}_0 + E \cdot \sigma_0 \wedge \rho_0 + P \cdot \sigma_0 \wedge \zeta_0 + A \cdot \sigma_0 \wedge \overline{\zeta}_0 + \\ &+ \overline{E} \cdot \overline{\sigma}_0 \wedge \rho_0 + \overline{P} \cdot \overline{\sigma}_0 \wedge \overline{\zeta}_0 + A \cdot \overline{\sigma}_0 \wedge \zeta_0 + \\ &+ i\, \zeta_0 \wedge \overline{\zeta}_0, \end{aligned}$$

we must replace the so obtained values of σ_0, $\overline{\sigma}_0$, ρ_0, ζ_0, $\overline{\zeta}_0$ in terms of σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$. In the process, we organize the computations so that group variables appear first, so that monomials $a^\mu\overline{a}^\nu$ always land at denominator place, so that the alphabetical order is respected in all numerators, and so that the initial structure functions always appear after group variables. Of course, we remember that the 5 coefficient-functions:

$$E, \qquad F, \qquad G, \qquad J, \qquad K$$

express in fact in terms of the 5 functions:

$$P, \qquad Q, \qquad R, \qquad A, \qquad B$$

and their frame derivatives up to order 2, but we prevent from inserting these expressions at this stage, planning to perform the replacements later.

We obtain:

$$\begin{aligned} d\sigma_0 = \sigma\wedge\overline{\sigma}\bigg[& -\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,K-\frac{\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,F+\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\,Q-\frac{\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,Q+\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^5}\,B-\frac{\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,B+\frac{\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\overline{G}- \\ & -\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\,B+\frac{\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,B-\frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^5}\,\overline{R}+\frac{\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\overline{R}+\frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^4}-\frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\bigg]+ \\ & +\sigma\wedge\rho\left[\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,F-\frac{\mathsf{b}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,Q-\frac{\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,B+\frac{\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\right]+ \\ & +\sigma\wedge\zeta\left[\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}}\,Q-\frac{\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\right]+\sigma\wedge\overline{\zeta}\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,B\right]+ \\ & +\overline{\sigma}\wedge\rho\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{G}-\frac{\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,B-\frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{R}+\frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\right]+ \\ & +\overline{\sigma}\wedge\zeta\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,B-\frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\right]+\overline{\sigma}\wedge\overline{\zeta}\left[\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{R}\right]+\rho\wedge\zeta\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\right], \end{aligned}$$

and:

$$\begin{aligned} d\rho_0 = \sigma\wedge\overline{\sigma}\bigg[& -i\,\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,J-\frac{\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,E+\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\,P-\frac{\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,P+\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^5}\,A-\frac{\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,A+ \\ & +\frac{\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\overline{E}-\frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^5}\,\overline{P}+\frac{\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\overline{P}-\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^5\overline{\mathsf{a}}^4}\,A+\frac{\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,A-i\,\frac{\mathsf{b}\mathsf{c}\overline{\mathsf{e}}}{\mathsf{a}^5\overline{\mathsf{a}}^5}- \\ & -i\,\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^5}+i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}+i\,\frac{\mathsf{b}\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^5\overline{\mathsf{a}}^5}+i\,\frac{\overline{\mathsf{b}}\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^5\overline{\mathsf{a}}^5}-i\,\frac{\mathsf{d}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\bigg]+ \\ & +\sigma\wedge\rho\left[\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,E-\frac{\mathsf{b}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,P-\frac{\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,A+i\,\frac{\overline{\mathsf{b}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^3}-i\,\frac{\mathsf{b}\mathsf{d}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\right]+ \\ & +\sigma\wedge\zeta\left[\frac{1}{\mathsf{a}^3\overline{\mathsf{a}}}\,P-i\,\frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3}+i\,\frac{\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\right]+ \\ & +\sigma\wedge\overline{\zeta}\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,A+i\,\frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3}-i\,\frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\right]+ \\ & +\overline{\sigma}\wedge\rho\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\overline{E}-\frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{P}-\frac{\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{A}-i\,\frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}+i\,\frac{\overline{\mathsf{b}\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\right]+ \\ & +\overline{\sigma}\wedge\zeta\left[\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,A-i\,\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}+i\,\frac{\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\right]+ \\ & +\overline{\sigma}\wedge\overline{\zeta}\left[\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{P}+i\,\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}-i\,\frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\right]+ \\ & +\rho\wedge\zeta\left[i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right]+\rho\wedge\overline{\zeta}\left[-i\,\frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\right]+\frac{i}{\mathsf{a}\overline{\mathsf{a}}}\,\zeta\wedge\overline{\mathsf{a}}. \end{aligned}$$

It yet remains to compute the three last terms:

$$\begin{aligned}
&\mathsf{a}^2\overline{\mathsf{a}}\, d\sigma_0,\\
&\mathsf{c}\, d\sigma_0 + \overline{\mathsf{c}}\, d\overline{\sigma}_0 + \mathsf{a}\overline{\mathsf{a}}\, d\rho_0,\\
&\mathsf{e}\, d\sigma_0 + \overline{\mathsf{d}}\, d\overline{\sigma}_0 + \mathsf{b}\, d\sigma_0 + \underline{\mathsf{a}\, d\zeta_{0}}_{\circ},
\end{aligned}$$

which happen to be complicated and to give rise to torsion coefficients. We do this, and this provides the complete structure equations:

$$\begin{aligned}
d\sigma &= \big(2\,\alpha_1 + \overline{\alpha}_1\big) \wedge \sigma + \\
&\quad + U_1\, \sigma \wedge \overline{\sigma} + U_2\, \sigma \wedge \rho + U_3\, \sigma \wedge \zeta + U_4\, \sigma \wedge \overline{\zeta} + \\
&\quad + U_5\, \overline{\sigma} \wedge \rho + U_6\, \overline{\sigma} \wedge \zeta + U_7\, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + \rho \wedge \zeta,
\end{aligned}\tag{96}$$

$$\begin{aligned}
d\rho &= \alpha_2 \wedge \sigma + \overline{\alpha}_2 \wedge \overline{\sigma} + \alpha_1 \wedge \rho + \overline{\alpha}_1 \wedge \rho + \\
&\quad + V_1\, \sigma \wedge \overline{\sigma} + V_2\, \sigma \wedge \rho + V_3\, \sigma \wedge \zeta + V_4\, \sigma \wedge \overline{\zeta} + \\
&\quad + \overline{V_2}\, \overline{\sigma} \wedge \rho + \overline{V_4}\, \overline{\sigma} \wedge \zeta + \overline{V_3}\, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + V_8\, \rho \wedge \zeta + \overline{V_8}\, \rho \wedge \overline{\zeta} + \\
&\quad + i\, \zeta \wedge \overline{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\zeta &= \alpha_3 \wedge \sigma + \alpha_4 \wedge \overline{\sigma} + \alpha_5 \wedge \rho + \alpha_1 \wedge \zeta + \\
&\quad + W_1\, \sigma \wedge \overline{\sigma} + W_2\, \sigma \wedge \rho + W_3\, \sigma \wedge \zeta + W_4\, \sigma \wedge \overline{\zeta} + \\
&\quad + W_5\, \overline{\sigma} \wedge \rho + W_6\, \overline{\sigma} \wedge \zeta + W_7\, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + W_8\, \rho \wedge \zeta + W_9\, \rho \wedge \overline{\zeta} + \\
&\quad + W_{10}\, \zeta \wedge \overline{\zeta},
\end{aligned}$$

expressed in terms of the lifted coframe, where the appearing torsion coefficients are as follows. For $d\sigma$:

$$\begin{aligned}
U_1 &= -\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\, K - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\, F + \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\, Q - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\, Q + \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\, B - \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^3}\, B + \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\, \overline{G} - \\
&\quad - \frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\, B + \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\, B - \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\, \overline{R} + \frac{\mathsf{d}}{\mathsf{a}\overline{\mathsf{a}}^3}\, \overline{R} + \frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3},\\
U_2 &= \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\, F - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\, Q - \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2}\, B + \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
U_3 &= \frac{1}{\mathsf{a}}\, Q - \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
U_4 &= \frac{1}{\overline{\mathsf{a}}}\, B,\\
U_5 &= \frac{1}{\overline{\mathsf{a}}^2}\, \overline{G} - \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^2}\, B - \frac{\overline{\mathsf{b}}}{\overline{\mathsf{a}}^3}\, \overline{R} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2},
\end{aligned}$$

$$U_6 = \frac{1}{\bar{a}}\,B - \frac{\bar{c}}{a\bar{a}^2},$$

$$U_7 = \frac{a}{\bar{a}^2}\,\overline{R}.$$

For $d\rho$:

$$\begin{aligned}
V_1 = &-\frac{c}{a^3\bar{a}^3}\,K - \frac{c\bar{c}}{a^4\bar{a}^4}\,F + \frac{bc\bar{c}}{a^5\bar{a}^4}\,Q - \frac{c\bar{d}}{a^4\bar{a}^3}\,Q + \frac{\bar{b}c\bar{c}}{a^4\bar{a}^5}\,B - \frac{c\bar{e}}{a^3\bar{a}^4}\,B + \frac{cc}{a^4\bar{a}^4}\,\overline{G} - \\
&- \frac{bcc}{a^5\bar{a}^4}\,B + \frac{ce}{a^4\bar{a}^3}\,B - \frac{\bar{b}cc}{a^4\bar{a}^5}\,\overline{R} + \frac{cd}{a^3\bar{a}^4}\,\overline{R} + \frac{cc\bar{d}}{a^5\bar{a}^4} - \frac{ec\bar{c}}{a^5\bar{a}^4} + \\
&+ \frac{\bar{c}}{a^3\bar{a}^3}\,\overline{K} + \frac{c\bar{c}}{a^4\bar{a}^4}\,\overline{F} - \frac{\bar{b}c\bar{c}}{a^4\bar{a}^5}\,\overline{Q} + \frac{\bar{c}d}{a^3\bar{a}^4}\,\overline{Q} - \frac{bc\bar{c}}{a^5\bar{a}^4}\,\overline{B} + \frac{\bar{c}e}{a^4\bar{a}^3}\,\overline{B} - \frac{\bar{c}\bar{c}}{a^4\bar{a}^4}\,G + \\
&+ \frac{\bar{b}\bar{c}\bar{c}}{a^4\bar{a}^5}\,\overline{B} - \frac{\bar{c}\bar{e}}{a^3\bar{a}^4}\,\overline{B} + \frac{b\bar{c}\bar{c}}{a^5\bar{a}^4}\,R - \frac{\bar{c}\bar{d}}{a^4\bar{a}^3}\,R - \frac{\bar{c}\bar{c}d}{a^4\bar{a}^5} + \frac{c\bar{c}\bar{e}}{a^4\bar{a}^5} - \\
&- i\,\frac{1}{a^2\bar{a}^2}\,J - \frac{\bar{c}}{a^3\bar{a}^3}\,E + \frac{b\bar{c}}{a^4\bar{a}^3}\,P - \frac{\bar{d}}{a^3\bar{a}^2}\,P + \frac{\bar{b}\bar{c}}{a^3\bar{a}^4}\,A - \frac{\bar{e}}{a^2\bar{a}^3}\,A + \frac{c}{a^3\bar{a}^3}\,\overline{E} - \frac{\bar{b}c}{a^3\bar{a}^4}\,\overline{P} + \frac{d}{a^2\bar{a}^3}\,\overline{P} - \\
&- \frac{bc}{a^4\bar{a}^3}\,A + \frac{e}{a^3\bar{a}^2}\,A - i\,\frac{bc\bar{e}}{a^4\bar{a}^4} - i\,\frac{\bar{b}\bar{c}e}{a^4\bar{a}^4} + i\,\frac{e\bar{e}}{a^3\bar{a}^3} + i\,\frac{b\bar{c}d}{a^4\bar{a}^4} + i\,\frac{\bar{b}c\bar{d}}{a^4\bar{a}^4} - i\,\frac{d\bar{d}}{a^3\bar{a}^3},
\end{aligned}$$

$$\begin{aligned}
V_2 = &\frac{c}{a^3\bar{a}^2}\,F - \frac{bc}{a^4\bar{a}^2}\,Q - \frac{\bar{b}c}{a^3\bar{a}^3}\,B + \frac{ce}{a^4\bar{a}^2} + \frac{\bar{c}}{a^3\bar{a}^2}\,G - \frac{\bar{b}\bar{c}}{a^3\bar{a}^3}\,\overline{B} - \frac{b\bar{c}}{a^4\bar{a}^2}\,R + \frac{\bar{c}d}{a^3\bar{a}^3} + \\
&+ \frac{1}{a^2\bar{a}}\,E - \frac{b}{a^3\bar{a}}\,P - \frac{\bar{b}}{a^2\bar{a}^2}\,A + i\,\frac{\bar{b}e}{a^3\bar{a}^2} - i\,\frac{bd}{a^3\bar{a}^2},
\end{aligned}$$

$$V_3 = \frac{c}{a^3\bar{a}}\,Q - \frac{cc}{a^4\bar{a}^2} + \frac{\bar{c}}{a^3\bar{a}}\,R + \frac{1}{a^2}\,P - i\,\frac{\bar{b}c}{a^3\bar{a}^2} + i\,\frac{d}{a^2\bar{a}},$$

$$V_4 = \frac{c}{a^2\bar{a}^2}\,B + \frac{\bar{c}}{a^2\bar{a}^2}\,\overline{B} - \frac{c\bar{c}}{a^3\bar{a}^3} + \frac{1}{a\bar{a}}\,A + i\,\frac{bc}{a^3\bar{a}^2} - i\,\frac{e}{a^2\bar{a}},$$

$$V_8 = \frac{c}{a^2\bar{a}} + i\,\frac{\bar{b}}{a\bar{a}}.$$

Lastly, for $d\zeta$:

$$\begin{aligned}
W_1 = &-\frac{e}{a^3\bar{a}^3}\,K - \frac{\bar{c}e}{a^4\bar{a}^4}\,F + \frac{b\bar{c}e}{a^5\bar{a}^4}\,Q - \frac{\bar{d}e}{a^4\bar{a}^3}\,Q + \frac{\bar{b}\bar{c}e}{a^4\bar{a}^5}\,B - \frac{e\bar{e}}{a^3\bar{a}^4}\,B + \frac{ce}{a^4\bar{a}^4}\,\overline{G} - \\
&- \frac{bce}{a^5\bar{a}^4}\,B + \frac{ee}{a^4\bar{a}^3}\,B - \frac{\bar{b}ce}{a^4\bar{a}^5}\,\overline{R} + \frac{de}{a^3\bar{a}^4}\,\overline{R} + \frac{c\bar{d}e}{a^5\bar{a}^4} - \frac{\bar{c}ee}{a^5\bar{a}^4} + \\
&+ \frac{\bar{d}}{a^3\bar{a}^3}\,\overline{K} + \frac{c\bar{d}}{a^4\bar{a}^4}\,\overline{F} - \frac{\bar{b}c\bar{d}}{a^4\bar{a}^5}\,\overline{Q} + \frac{d\bar{d}}{a^3\bar{a}^4}\,\overline{Q} - \frac{bc\bar{d}}{a^5\bar{a}^4}\,\overline{B} + \frac{\bar{d}e}{a^4\bar{a}^3}\,\overline{B} - \frac{\bar{c}\bar{d}}{a^4\bar{a}^4}\,G + \\
&+ \frac{\bar{b}\bar{c}\bar{d}}{a^4\bar{a}^5}\,\overline{B} - \frac{\bar{d}\bar{e}}{a^3\bar{a}^4}\,\overline{B} + \frac{b\bar{c}\bar{d}}{a^5\bar{a}^4}\,R - \frac{\bar{d}\bar{d}}{a^4\bar{a}^3}\,R - \frac{\bar{c}d\bar{d}}{a^4\bar{a}^5} + \frac{c\bar{d}\bar{e}}{a^4\bar{a}^5} - \\
&- i\,\frac{b}{a^3\bar{a}^3}\,J - \frac{b\bar{c}}{a^4\bar{a}^4}\,E + \frac{bb\bar{c}}{a^5\bar{a}^4}\,P - \frac{b\bar{d}}{a^4\bar{a}^3}\,P + \frac{b\bar{b}\bar{c}}{a^4\bar{a}^5}\,A - \frac{b\bar{e}}{a^3\bar{a}^4}\,A + \frac{bc}{a^4\bar{a}^4}\,\overline{E} - \frac{b\bar{b}c}{a^4\bar{a}^5}\,\overline{P} + \frac{bd}{a^3\bar{a}^4}\,\overline{P} - \\
&- \frac{bbc}{a^5\bar{a}^4}\,A + \frac{be}{a^4\bar{a}^3}\,A - i\,\frac{bbc\bar{e}}{a^5\bar{a}^5} - i\,\frac{b\bar{b}\bar{c}e}{a^5\bar{a}^5} + i\,\frac{be\bar{e}}{a^4\bar{a}^4} + i\,\frac{bb\bar{c}d}{a^5\bar{a}^5} + i\,\frac{b\bar{b}c\bar{d}}{a^5\bar{a}^5} - i\,\frac{bd\bar{d}}{a^4\bar{a}^4},
\end{aligned}$$

$$\begin{aligned}
W_2 = &\frac{e}{a^3\bar{a}^2}\,F - \frac{be}{a^4\bar{a}^2}\,Q - \frac{\bar{b}e}{a^3\bar{a}^3}\,B + \frac{ee}{a^4\bar{a}^2} + \frac{\bar{d}}{a^3\bar{a}^2}\,G - \frac{\bar{b}\bar{d}}{a^3\bar{a}^3}\,\overline{B} - \frac{b\bar{d}}{a^4\bar{a}^2}\,R + \frac{d\bar{d}}{a^3\bar{a}^3} + \\
&+ \frac{b}{a^3\bar{a}^2}\,E - \frac{bb}{a^4\bar{a}^2}\,P - \frac{b\bar{b}}{a^3\bar{a}^3}\,A + i\,\frac{b\bar{b}e}{a^4\bar{a}^3} - i\,\frac{bbd}{a^4\bar{a}^3},
\end{aligned}$$

$$
\begin{aligned}
W_3 &= \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\,Q - \frac{\mathsf{ce}}{\mathsf{a}^4\overline{\mathsf{a}}^2} + \frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}}\,R + \frac{\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}}\,P - i\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3} + i\,\frac{\mathsf{bd}}{\mathsf{a}^3\overline{\mathsf{a}}^2},\\
W_4 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,B + \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{B} - \frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,A + i\,\frac{\mathsf{bbc}}{\mathsf{a}^4\overline{\mathsf{a}}^3} - i\,\frac{\mathsf{be}}{\mathsf{a}^3\overline{\mathsf{a}}^2},\\
W_5 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{G} - \frac{\mathsf{be}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,B - \frac{\overline{\mathsf{b}}\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{R} + \frac{\overline{\mathsf{d}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{F} - \frac{\overline{\mathsf{b}}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{Q} - \frac{\mathsf{b}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{B} + \frac{\overline{\mathsf{e}}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^4} +\\
&\quad + \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{E} - \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{P} - \frac{\mathsf{bb}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,A - i\,\frac{\mathsf{bb}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\overline{\mathsf{bd}}}{\mathsf{a}^3\overline{\mathsf{a}}^4},\\
W_6 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,B - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3} + \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{B} + \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,A - i\,\frac{\mathsf{b}\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} + i\,\frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^3},\\
W_7 &= \frac{\mathsf{e}}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{R} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{Q} - \frac{\overline{\mathsf{c}}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^4} + \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{P} + i\,\frac{\mathsf{bb}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4} - i\,\frac{\mathsf{b}\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3},\\
W_8 &= \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}} + i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
W_9 &= \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2} - i\,\frac{\mathsf{bb}}{\mathsf{a}^2\overline{\mathsf{a}}^2},\\
W_{10} &= i\,\frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}}.
\end{aligned}
$$

15.1. **First loop absorbtion.** Similarly as when we treated the model in Subsection 12.10, according to Proposition 12.6, we must modify the five 1-forms α_1, α_2, α_3, α_4, α_5 by adding to them general linear combinations of the 1-forms σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$:

$$
\begin{aligned}
\alpha_1 &\longmapsto \alpha_1 + p_1\,\sigma + q_1\,\overline{\sigma} + r_1\,\rho + s_1\,\zeta + t_1\,\overline{\zeta},\\
\alpha_2 &\longmapsto \alpha_2 + p_2\,\sigma + q_2\,\overline{\sigma} + r_2\,\rho + s_2\,\zeta + t_2\,\overline{\zeta},\\
\alpha_3 &\longmapsto \alpha_3 + p_3\,\sigma + q_3\,\overline{\sigma} + r_3\,\rho + s_3\,\zeta + t_3\,\overline{\zeta},\\
\alpha_4 &\longmapsto \alpha_4 + p_4\,\sigma + q_4\,\overline{\sigma} + r_4\,\rho + s_4\,\zeta + t_4\,\overline{\zeta},\\
\alpha_5 &\longmapsto \alpha_5 + p_5\,\sigma + q_5\,\overline{\sigma} + r_5\,\rho + s_5\,\zeta + t_5\,\overline{\zeta},
\end{aligned}
$$

with 25 arbitrary real analytic functions p_i, q_i, r_i, s_i, t_i. Then the expressions of $d\sigma$, $d\rho$, $d\zeta$ become — mind now that, contrary to the model case, the two torsion coefficients U_4 and U_7 do not vanish —:

$$
\begin{aligned}
d\sigma &= \big(2\alpha_1 + \overline{\alpha}_1\big)\wedge\sigma +\\
&\quad + \sigma\wedge\overline{\sigma}\big[U_1 - 2q_1 - \overline{p}_1\big] + \sigma\wedge\rho\big[U_2 - 2r_1 - \overline{r}_1\big] + \sigma\wedge\zeta\big[U_3 - 2s_1 - \overline{t}_1\big] + \sigma\wedge\overline{\zeta}\big[U_4 - 2t_1 - \overline{s}_1\big]\\
&\quad + \overline{\sigma}\wedge\rho\big[U_5\big] + \overline{\sigma}\wedge\zeta\big[U_6\big] + \overline{\sigma}\wedge\overline{\zeta}\big[U_7\big] +\\
&\quad + \rho\wedge\zeta,\\
d\rho &= \alpha_2\wedge\sigma + \overline{\alpha}_2\wedge\overline{\sigma} + \alpha_1\wedge\rho + \overline{\alpha}_1\wedge\rho +\\
&\quad + \sigma\wedge\overline{\sigma}\big[V_1 - q_2 - \overline{q}_2\big] + \sigma\wedge\rho\big[V_2 - r_2 + p_1 + \overline{q}_1\big] + \sigma\wedge\zeta\big[V_3 - s_2\big] + \sigma\wedge\overline{\zeta}\big[V_4 - t_2\big]\\
&\quad + \overline{\sigma}\wedge\rho\big[\overline{V}_2 - \overline{r}_2 + q_1 + \overline{p}_1\big] + \overline{\sigma}\wedge\zeta\big[\overline{V}_4 - \overline{t}_2\big] + \overline{\sigma}\wedge\overline{\zeta}\big[\overline{V}_3 - \overline{s}_2\big] +
\end{aligned}
$$

$$
\begin{aligned}
&+\rho\wedge\zeta\big[V_8-s_1-\overline{t}_1\big]+\rho\wedge\overline{\zeta}\big[\overline{V}_8-t_1-\overline{s}_1\big]+\\
&+i\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta=\alpha_3\wedge\sigma+\alpha_4\wedge\overline{\sigma}+\alpha_5\wedge\rho+\alpha_1\wedge\zeta+\\
+\sigma\wedge\overline{\sigma}\big[W_1-q_3+p_4\big]+\sigma\wedge\rho\big[W_2-r_3+p_5\big]+\sigma\wedge\zeta\big[W_3-s_3+p_1\big]+\sigma\wedge\overline{\zeta}\big[W_4-t_3\big]\\
+\overline{\sigma}\wedge\rho\big[W_5-r_4+q_5\big]+\overline{\sigma}\wedge\zeta\big[W_6-s_4+q_1\big]+\overline{\sigma}\wedge\overline{\zeta}\big[W_7-t_4\big]+\\
+\rho\wedge\zeta\big[W_8-s_5+r_1\big]+\rho\wedge\overline{\zeta}\big[W_9-t_5\big]+\\
+\zeta\wedge\overline{\zeta}\big[W_{10}-t_1\big].
\end{aligned}
$$

15.2. **First loop normalization.** In order to know what are the precise linear combinations of the 22 torsion coefficients:

$$
\begin{array}{llllllllll}
U_1, & U_2, & U_3, & U_4, & U_5, & U_6, & U_7, \\
V_1, & V_2, & V_3, & V_4, & V_8, \\
W_1, & W_2, & W_3, & W_4, & W_5, & W_6, & W_7, & W_8, & W_9, & W_{10}
\end{array}
$$

that are *necessarily normalizable*, one must determine all possible linear combinations of the following $7+5+10=22$ equations — including their (unwritten) conjugates —:

$$
\left[\begin{aligned}
U_1&=2q_1+\overline{p}_1,\\
U_2&=2r_1+\overline{r}_1,\\
U_3&=2s_1+\overline{t}_1,\\
U_4&=2t_1+\overline{s}_1,\\
U_5&=0,\\
U_6&=0,\\
U_7&=0,
\end{aligned}\right.
\qquad
\left[\begin{aligned}
V_1&=q_2-\overline{q}_2,\\
V_2&=r_2-p_1-\overline{q}_1,\\
V_3&=s_2,\\
V_4&=t_2,\\
V_8&=s_1+\overline{t}_1,
\end{aligned}\right.
\qquad
\left[\begin{aligned}
W_1&=q_3-p_4,\\
W_2&=r_3-p_5,\\
W_3&=s_3-p_1,\\
W_4&=t_3,\\
W_5&=r_4-q_5,\\
W_6&=s_4-q_1,\\
W_7&=t_4,\\
W_8&=s_5-r_1,\\
W_9&=t_5,\\
W_{10}&=t_1,
\end{aligned}\right.
$$

so as to obtain null right-hand sides. Visually, some complete appropriate linear combinations are:

$$
\begin{aligned}
0&=U_5,\\
0&=U_6,\\
0&=U_7,\\
0&=U_3+\overline{U}_4-3\,V_8,\\
0&=\overline{U}_4-V_8-\overline{W}_{10},
\end{aligned}
\tag{97}
$$

and that is all (exercise). Now, if one just replaces the appearing torsion coefficients, one plainly obtains *five* normalizable linear combinations:

$$
\boxed{
\begin{aligned}
U_5 &= \frac{1}{\overline{\mathsf{a}}^2}\,\overline{G} - \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^2}\,B - \frac{\overline{\mathsf{b}}}{\overline{\mathsf{a}}^3}\,\overline{R} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2},\\
U_6 &= \frac{1}{\overline{\mathsf{a}}}\,B - \frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2},\\
U_7 &= \frac{\mathsf{a}}{\overline{\mathsf{a}}^2}\,\overline{R},\\
U_3 + \overline{U}_4 - 3\,V_8 &= \frac{1}{\mathsf{a}}\,Q - 4\,\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{1}{\mathsf{a}}\,\overline{B} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}},\\
\overline{U}_4 - V_8 - \overline{W}_{10} &= \frac{1}{\mathsf{a}}\,\overline{B} - \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}}.
\end{aligned}
}
\tag{98}
$$

Visibly, the second and the fifth combinations are conjugate of each other, *hence the fifth can be removed*. A confirmation of computational correctness is yielded by the following obvious

Assertion 15.1. One recovers all equations along with the equivalence process applied to the model in Section 12 just by assigning the value zero to all of the functions:

$$
\begin{array}{ccc}
P, & Q, & R,\\
 & A, & B,\\
E, & F, & G,\\
 & J, & K,
\end{array}
$$

appearing in the structure equations (95) of the initial coframe $\{\sigma_0, \overline{\sigma}_0, \rho_0, \zeta_0, \overline{\zeta}_0\}$. □

But then, in the model case, instead of the third normalizable expression above:

$$
\frac{\mathsf{a}}{\overline{\mathsf{a}}^2}\,\overline{R},
$$

we had only the trivial combination '0', just because $R = 0$ in the model case. It is thus necessary and unavoidable to *distinguish two branches* in the future issue of the equivalence procedure:

□ either $R \equiv 0$ as a function on our CR-generic maximally minimal submanifold $M^5 \subset \mathbb{C}^4$;

□ or else, $R \not\equiv 0$, so that after relocalizing if necessary the consideration at another Zariski-generic central point (shifting the origin of the coordinate system), we may assume that $R \not\equiv 0$ vanishes at no point.

Of course, in this second case, we will be able to normalize the complex parameter a, but let us continue at first the procedure of equivalence under the assumption that $R \equiv 0$, a 'branch' which is closer to the model case.

16. The branch $R \equiv 0$

16.1. **Normalization of three group parameters.** Assuming therefore that $R \equiv 0$ vanishes identically, the above five normalizable expressions (98) reduce to exactly three:

$$
\begin{aligned}
U_5 &= \frac{1}{\overline{\mathsf{a}}^2}\,\overline{G} - \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^2}\,B - \underbrace{\frac{\overline{\mathsf{b}}}{\overline{\mathsf{a}}^3}\,\overline{R}}_{\circ} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\
U_6 &= \frac{1}{\overline{\mathsf{a}}}\,B - \frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\
U_3 + \overline{U}_4 - 3\,V_8 &= \frac{1}{\mathsf{a}}\,Q - 4\,\frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{1}{\mathsf{a}}\,\overline{B} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}}.
\end{aligned}
\tag{99}
$$

Equating the second expression to zero then specifies the expression of the group parameter c as:

$$
\boxed{\mathsf{c} := \mathsf{a}\overline{\mathsf{a}}\,\overline{B}.}
$$

This changes the third expression into a simplified form:

$$
\frac{1}{\mathsf{a}}\,Q - 3\,\frac{1}{\mathsf{a}}\,\overline{B} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}},
$$

and then, equating it similarly to zero normalizes the expression of b as:

$$
\boxed{\mathsf{b} := \mathsf{a}\left(-\,i\,B + \tfrac{i}{3}\,\overline{Q}\right).}
$$

Lastly, putting this into the first expression of (99) and equating it to zero, we also determine the expression of d:

$$
\mathsf{d} = \overline{\mathsf{a}}\left(-\;G + i\,\overline{B}\,\overline{B} - \tfrac{i}{3}\,\overline{B}\,Q\right).
$$

But if we remember at this stage from Lemma 13.4 that the function G has an expression in terms of P, Q, R, A, B, then still under the assumption that $R = 0$, the normalization of d is in fact more precisely:

$$
\boxed{\mathsf{d} = \overline{\mathsf{a}}\left(-\;i\,\mathscr{L}\big(\overline{B}\big) + i\,P + \tfrac{2i}{3}\,\overline{B}\,Q\right).}
$$

16.2. **Changing up the initial coframe.** Now, if we insert these normalized values of b, c, d into the expressions of the 1-forms of our lifted coframe:

$$
\begin{aligned}
\sigma &= \mathsf{a}^2\overline{\mathsf{a}}\,\sigma_0, \\
\rho &= \mathsf{c}\cdot\sigma_0 + \overline{\mathsf{c}}\cdot\overline{\sigma}_0 + \mathsf{a}\overline{\mathsf{a}}\cdot\rho_0, \\
\zeta &= \mathsf{e}\cdot\sigma_0 + \overline{\mathsf{d}}\cdot\overline{\sigma}_0 + \mathsf{b}\cdot\rho_0 + \mathsf{a}\cdot\zeta_0,
\end{aligned}
$$

we realize after reorganization:

$$\sigma = \mathsf{a}^2\overline{\mathsf{a}}\cdot\sigma_0,$$
$$\rho = \mathsf{a}\overline{\mathsf{a}}\cdot\big(\underbrace{\overline{B}\,\sigma_0 + B\,\overline{\sigma}_0 + \rho_0}_{=:\,\rho_0^\sim}\big),$$
$$\zeta = \mathsf{e}\cdot\sigma_0 + \mathsf{a}\cdot\big[\underbrace{\big(-\overline{G} - i\,BB + \tfrac{i}{3}\,B\overline{Q}\big)\,\overline{\sigma_0} + \big(-i\,B + \tfrac{i}{3}\,\overline{Q}\big)\cdot\rho_0 + \zeta_0}_{=:\,\zeta_0^\sim}\big]$$

that it is natural to introduce the two new *modified initial* 1*-forms*:

$$\rho_0^\sim \qquad\qquad \text{and} \qquad\qquad \zeta_0^\sim$$

in terms of which the *reduced* lifted coframe rewrites:

$$\left[\begin{aligned}&\sigma = \mathsf{a}^2\overline{\mathsf{a}}\cdot\sigma_0,\\ &\rho = \mathsf{a}\overline{\mathsf{a}}\cdot\rho_0^\sim,\\ &\zeta = \mathsf{e}\cdot\sigma_0 + \mathsf{a}\cdot\zeta_0^\sim,\end{aligned}\right.$$

so that the corresponding *reduced* matrix group becomes:

$$G^\sim := \left\{ g^\sim = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}^2 & 0 & 0 & 0 & 0\\ 0 & \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0\\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{e}} & 0 & 0 & \overline{\mathsf{a}} & 0\\ 0 & \mathsf{e} & 0 & 0 & \mathsf{a}\end{pmatrix} \;:\; \mathsf{a},\,\mathsf{e}\in\mathbb{C}\right\};$$

of course, it is immediate that $(\overline{\sigma}_0, \sigma_0, \rho_0^\sim, \overline{\zeta}_0^\sim, \zeta_0^\sim)$ still constitutes a coframe on our generic submanifold M. Furthermore, a simple computation shows that:

$$dg^\sim\cdot g^{\sim -1} = \begin{pmatrix} 2\overline{\beta}_1+\beta_1 & 0 & 0 & 0 & 0\\ 0 & 2\beta_1+\overline{\beta}_1 & 0 & 0 & 0\\ 0 & 0 & \beta_1+\overline{\beta}_1 & 0 & 0\\ \overline{\beta}_2 & 0 & 0 & \overline{\beta}_1 & 0\\ 0 & \beta_2 & 0 & 0 & \beta_1\end{pmatrix},$$

where:

$$\beta_1 := \frac{d\mathsf{a}}{\mathsf{a}} \qquad\qquad \text{and} \qquad\qquad \beta_2 := \frac{d\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}} - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\,d\mathsf{a}.$$

16.3. **Second-loop absorbtion and normalization.** Now, let us pursue the computations with our initial coframe $\{\sigma_0, \overline{\sigma}_0, \rho_0, \zeta_0, \overline{\zeta}_0\}$. We have to replace the normalized values of b, c, d obtained above in the definition of the lifted coframe. Let us abbreviate these three normalizations as:

$$\mathsf{b} := \mathsf{a}\,\mathbf{B}_0, \qquad \text{where:} \qquad \boxed{\mathbf{B}_0 := -\,i\,B + \tfrac{i}{3}\,\overline{Q},}$$

$$\mathsf{c} := \mathsf{a}\overline{\mathsf{a}}\,\mathbf{C}_0, \qquad \text{where:} \qquad \boxed{\mathbf{C}_0 := \overline{B},}$$

$$\mathsf{d} := \overline{\mathsf{a}}\,\mathbf{D}_0, \qquad \text{where:} \qquad \boxed{\mathbf{D}_0 := -\,i\,\mathscr{L}\big(\overline{B}\big) + i\,P + \tfrac{2i}{3}\,\overline{B}\,Q,}$$

in terms of three new functions $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$ defined on the base manifold $M^5 \subset \mathbb{C}^4$. Then the new lifted coframe becomes:

$$\begin{aligned}
\sigma &= \mathsf{a}^2\overline{\mathsf{a}}\cdot\sigma_0,\\
\rho &= \mathsf{a}\overline{\mathsf{a}}\cdot\big(\mathbf{C}_0\,\sigma_0 + \overline{\mathbf{C}}_0\,\overline{\sigma}_0 + \rho_0\big),\\
\zeta &= \mathsf{e}\cdot\sigma_0 + \mathsf{a}\cdot\big(\overline{\mathbf{D}}_0\,\overline{\sigma}_0 + \mathbf{B}_0\,\rho_0 + \zeta_0\big).
\end{aligned}$$

Now, we apply the exterior differentiation operator and we obtain, in terms of the Maurer-Cartan forms β_1 and β_2:

$$\begin{aligned}
d\sigma &= \big(2\beta_1 + \overline{\beta}_1\big)\wedge\sigma +\\
&\quad + \mathsf{a}^2\overline{\mathsf{a}}\cdot d\sigma_0,\\
d\rho &= \big(\beta_1 + \overline{\beta}_1\big)\wedge\rho +\\
&\quad + \mathsf{a}\overline{\mathsf{a}}\cdot\big(\mathbf{C}_0\,d\sigma_0 + \overline{\mathbf{C}}_0\,d\overline{\sigma}_0 + d\rho_0 +\\
&\qquad\qquad + d\mathbf{C}_0\wedge\sigma_0 + d\overline{\mathbf{C}}_0\wedge\overline{\sigma}_0\big),\\
d\zeta &= \beta_2\wedge\sigma + \beta_1\wedge\zeta +\\
&\quad + \mathsf{e}\cdot d\sigma_0 + \mathsf{a}\cdot\big(\overline{\mathbf{D}}_0\,d\overline{\sigma}_0 + \mathbf{B}_0\,d\rho_0 + d\zeta_0 +\\
&\qquad\qquad\qquad + d\overline{\mathbf{D}}_0\wedge\overline{\sigma}_0 + d\mathbf{B}_0\wedge\rho_0\big).
\end{aligned} \tag{100}$$

One readily observes there is no change in the structure equation of $d\sigma$, hence the torsion coefficients $U_\bullet$ will be essentially unchanged. Intentionally here, each right-hand side is organized in three lines having distinct meanings. On the first line, only Maurer-Cartan forms appear. On the second line, only exterior derivatives of the base 1-forms appear. One easily convinces oneself that the contribution of these second-line terms to the new torsion coefficients U_i', V_j', W_k':

$$\begin{aligned}
d\sigma = \big(2\,\beta_1 + \overline{\beta}_1\big)\wedge\sigma &+\\
+ U_1'\,\sigma\wedge\overline{\sigma} + U_2'\,\sigma\wedge\rho &+ U_3'\,\sigma\wedge\zeta + U_4'\,\sigma\wedge\overline{\zeta} +\\
&+ U_5'\,\overline{\sigma}\wedge\rho + U_6'\,\overline{\sigma}\wedge\zeta + U_7'\,\overline{\sigma}\wedge\overline{\zeta} +\\
&\qquad\qquad + \rho\wedge\zeta,
\end{aligned}$$

$$
\begin{aligned}
d\rho = \big(\beta_1 + \overline{\beta}_1\big) \wedge \rho + \\
+ V_1'\,\sigma\wedge\overline{\sigma} + V_2'\,\sigma\wedge\rho + V_3'\,\sigma\wedge\zeta + V_4'\,\sigma\wedge\overline{\zeta} + \\
+ \overline{V_2}'\,\overline{\sigma}\wedge\rho + \overline{V_4}'\,\overline{\sigma}\wedge\zeta + \overline{V_3}'\,\overline{\sigma}\wedge\overline{\zeta} + \\
+ V_8'\,\rho\wedge\zeta + \overline{V_8}'\,\rho\wedge\overline{\zeta} + \\
+ i\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = \beta_2\wedge\sigma + \beta_1\wedge\zeta + \\
+ W_1'\,\sigma\wedge\overline{\sigma} + W_2'\,\sigma\wedge\rho + W_3'\,\sigma\wedge\zeta + W_4'\,\sigma\wedge\overline{\zeta} + \\
+ W_5'\,\overline{\sigma}\wedge\rho + W_6'\,\overline{\sigma}\wedge\zeta + W_7'\,\overline{\sigma}\wedge\overline{\zeta} + \\
+ W_8'\,\rho\wedge\zeta + W_9'\,\rho\wedge\overline{\zeta} + \\
+ W_{10}'\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

just consists in taking back the previous torsion coefficients U_i, V_j, W_k shown above and replacing the values of b, c, d by their normalized values:

$$
\begin{aligned}
U_i'\big|_{\mathsf{replace(b,c,d)}} &= \text{second line of } \big(d\sigma\big) && {\scriptstyle (i=1,2,3,4,5,6,7)}, \\
V_j\big|_{\mathsf{replace(b,c,d)}} &= \text{second line of } \big(d\rho\big) && {\scriptstyle (j=1,2,3,4,8)}, \\
W_k\big|_{\mathsf{replace(b,c,d)}} &= \text{second line of } \big(d\zeta\big) && {\scriptstyle (k=1,2,3,4,5,6,7,8,9,10)}.
\end{aligned}
$$

Now, to compute the third lines in the new torsion coefficients, we apply a lemma, the proof of which is a direct consequence of a formula shown above for the inverse g^{-1} of a general element g in our ambiguity matrix group.

Lemma 16.1. *The exterior differential:*

$$
d\mathbf{G}_0 = \mathscr{S}(\mathbf{G}_0)\cdot\sigma_0 + \overline{\mathscr{S}}(\mathbf{G}_0)\cdot\overline{\sigma}_0 + \mathscr{T}(\mathbf{G}_0)\cdot\rho_0 + \mathscr{L}(\mathbf{G}_0)\cdot\zeta_0 + \overline{\mathscr{L}}(\mathbf{G}_0)\cdot\overline{\zeta}_0
$$

of any function $\mathbf{G}_0$ on the base manifold $M^5 \subset \mathbb{C}^4$ re-expresses, in terms of the lifted coframe, as:

$$
\begin{aligned}
d\mathbf{G}_0 = \sigma\cdot\bigg(&\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{S}(\mathbf{G}_0) - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{T}(\mathbf{G}_0) + \frac{\mathsf{bc}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{G}_0) - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{G}_0) + \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{G}_0) - \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
+\overline{\sigma}\cdot\bigg(&\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{S}}(\mathbf{G}_0) - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{T}(\mathbf{G}_0) + \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{G}_0) - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{G}_0) + \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{G}_0) - \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
+\rho\cdot\bigg(&\frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\mathscr{T}(\mathbf{G}_0) - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{G}_0) - \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
+\zeta\cdot\bigg(&\frac{1}{\mathsf{a}}\,\mathscr{L}(\mathbf{G}_0)\bigg) + \\
+\overline{\zeta}\cdot\bigg(&\frac{1}{\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg),
\end{aligned}
$$

before any normalization of the coefficients. □

Then all the new torsion coefficients may be computed completely. Although the computations start to become quite substantial, an accessible focused computation, left to the reader, yields the expression of the new fourth torsion coefficient in $d\rho$:

$$V_4' = -\frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\overline{B}) + \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,A + \frac{2}{\mathsf{a}\overline{\mathsf{a}}}\,B\overline{B} - \frac{1}{3}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{B}\,\overline{Q} - i\,\frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}}.$$

In just a while, we will show that this torsion coefficient is essential, hence normalizable. Assigning to it the value 0, we deduce that the group parameter e can always be normalized as:

$$\boxed{\mathsf{e} := \mathsf{a}\cdot\big(i\,\overline{\mathscr{L}}(\overline{B}) - i\,A - 2i\,B\overline{B} + \tfrac{i}{3}\,\overline{B}\,\overline{Q}\big).}$$

We may also abbreviate the appearing auxiliary-normalizing function defined on the base manifold $M^5 \subset \mathbb{C}^4$ as:

$$\mathsf{e} := \mathsf{a}\cdot\mathbf{E}_0 \qquad \text{where:} \qquad \boxed{\mathbf{E}_0 := i\,\overline{\mathscr{L}}(\overline{B}) - i\,A - 2i\,B\overline{B} + \tfrac{i}{3}\,\overline{B}\,\overline{Q}.}$$

16.4. **New torsion coefficients.** In general, adding second lines to third lines by means of Lemma 16.1 just above, we obtain temporary (unclosed, unfinished) formulas for the new torsion coefficients.

At first, in the U_i, plain replacements of the values of b, c, d must be done:

$$U_i' := U_i\Big|_{\mathsf{replace(b,c,d)}} \qquad\qquad (i=1,2,3,4,5,6,7).$$

But for the V_i and for the W_k, supplementary terms coming from the exterior differentiations of the three auxiliary-normalizing functions $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$ appear in the concerned third lines. Applying the lemma, we obtain formulas in which one must replace b, c, d afterwards:

(101)

$$\begin{aligned}
V_1' := V_1 &- \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{\mathscr{S}}(\mathbf{C}_0) + \frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{T}(\mathbf{C}_0) - \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{C}_0) + \frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{C}_0) - \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{C}_0) + \frac{\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{C}_0) +\\
&+ \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\mathscr{S}(\overline{\mathbf{C}}_0) - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{T}(\overline{\mathbf{C}}_0) + \frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^3}\,\mathscr{L}(\overline{\mathbf{C}}_0) - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{L}(\overline{\mathbf{C}}_0) + \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\overline{\mathbf{C}}_0) - \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{C}}_0),\\
V_2' &:= V_2 - \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{T}(\mathbf{C}_0) + \frac{\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{C}_0) + \frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{C}_0),\\
V_3' &:= V_3 - \frac{1}{\mathsf{a}^2}\,\mathscr{L}(\mathbf{C}_0),\\
V_4' &:= V_4 - \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\mathbf{C}_0),\\
V_8' &:= V_8,\\
W_1' := W_1 &+ \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{S}(\overline{\mathbf{D}}_0) - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\mathscr{T}(\overline{\mathbf{D}}_0) + \frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\mathscr{L}(\overline{\mathbf{D}}_0) - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\overline{\mathbf{D}}_0) + \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^5}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0) - \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0) -\\
&- \frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\mathscr{S}(\mathbf{B}_0) + \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\overline{\mathscr{S}}(\mathbf{B}_0) + \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\mathscr{L}(\mathbf{B}_0) - \frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\mathscr{L}(\mathbf{B}_0) + \frac{\overline{\mathsf{c}}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^5}\,\overline{\mathscr{L}}(\mathbf{B}_0) - \frac{\mathsf{c}\overline{\mathsf{e}}}{\mathsf{a}^3\overline{\mathsf{a}}^5}\,\overline{\mathscr{L}}(\mathbf{B}_0),
\end{aligned}$$

$$W_2' := W_2 + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\mathscr{S}(\mathbf{B}_0) - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{B}_0) - \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{B}_0),$$
$$W_3' := W_3 + \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{B}_0),$$
$$W_4' := W_4 + \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{B}_0),$$
$$W_5' := W_5 - \frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\,\mathscr{T}(\overline{\mathbf{D}}_0) + \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{L}(\overline{\mathbf{D}}_0) + \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0) + \frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{\mathscr{S}}(\mathbf{B}_0) - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{B}_0) - \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{B}_0),$$
$$W_6' := W_6 - \frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\mathscr{L}(\overline{\mathbf{D}}_0) + \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{B}_0),$$
$$W_7' := W_7 - \frac{1}{\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0) + \frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{B}_0),$$
$$W_8' := W_8 - \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{B}_0),$$
$$W_9' := W_9 - \frac{1}{\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{B}_0),$$
$$W_{10}' := W_{10}.$$

Now, a direct computation shows that among the above torsion coefficients, V_3' vanishes identically as soon as we substitute the obtained expressions of $\mathbf{B_0}, \mathbf{C_0}$ and $\mathbf{D_0}$.

Lemma 16.2. *After determining the group parameters* $\mathsf{b}, \mathsf{c}, \mathsf{d}$, *the torsion coefficient* V_3' *vanishes identically.*

Proof. Putting the obtained expressions of $\mathsf{b}, \mathsf{c}, \mathsf{d}$ in the expression of V_3 gives:

$$V_3 = \frac{1}{\mathsf{a}^2}\mathscr{L}(\overline{B}).$$

Now, it remains only to subtract $\frac{1}{\mathsf{a}^2}\mathscr{L}(\mathbf{C}_0)$ and easily see the vanishing of V_3'. □

16.5. **Normalizable essential torsion combinations.** As is known from the general Cartan equivalence procedure, we must next modify the two remaining Maurer-Cartan 1-forms $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_3$ by adding to them general linear combinations of the 1-forms σ, $\overline{\sigma}$, ρ, ζ, $\overline{\zeta}$:

$$\beta_1 \longmapsto \beta_1 + p_1\,\sigma + q_1\,\overline{\sigma} + r_1\,\rho + s_1\,\zeta + t_1\,\overline{\zeta},$$
$$\beta_2 \longmapsto \beta_2 + p_2\,\sigma + q_2\,\overline{\sigma} + r_2\,\rho + s_2\,\zeta + t_2\,\overline{\zeta},$$

which then modifies the torsion coefficients in some way. Of course, from the first loop we already know (still in the branch $R = 0$) from (97) that:

$$U_5' = 0, \qquad U_6' = 0, \qquad U_7' = 0,$$
$$V_8' = \tfrac{1}{3}\,U_3' + \tfrac{1}{3}\,\overline{U}_4', \qquad W_{10}' = \tfrac{2}{3}\,U_4' - \tfrac{1}{3}\,\overline{U}_3',$$

and we must take account of these normalizations. In fact, we may skip computations and just use the previously obtained formulas (84), setting simply in them:

$$p_2 = q_2 = r_2 = s_2 = t_2 = 0, \qquad p_3 = p_2, \;\; q_3 = q_2, \;\; r_3 = r_2, \;\; s_3 = s_2, \;\; t_3 = t_2,$$
$$p_4 = q_4 = r_4 = s_4 = t_4 = 0, \qquad p_5 = q_5 = r_5 = s_5 = t_5 = 0,$$

which yields the following equations — including their (unwritten) conjugates — :

$$\left[\begin{aligned} U_1' &= 2q_1 + \overline{p}_1, \\ U_2' &= 2r_1 + \overline{r}_1, \\ U_3' &= 2s_1 + \overline{t}_1, \\ U_4' &= 2t_1 + \overline{s}_1, \\ 0 &= 0, \\ 0 &= 0, \\ 0 &= 0, \end{aligned}\right. \qquad \left[\begin{aligned} V_1' &= 0, \\ V_2' &= -p_1 - \overline{q}_1, \\ V_3' &= 0, \\ V_4' &= 0, \\ \tfrac{1}{3}\,U_3' + \tfrac{1}{3}\,\overline{U}_4' &= s_1 + \overline{t}_1, \end{aligned}\right. \qquad \left[\begin{aligned} W_1' &= q_2, \\ W_2' &= r_2, \\ W_3' &= s_2 - p_1, \\ W_4' &= t_2, \\ W_5' &= 0, \\ W_6' &= -q_1, \\ W_7' &= 0, \\ W_8' &= -r_1, \\ W_9' &= 0, \\ \tfrac{2}{3}\,U_4' - \tfrac{1}{3}\,\overline{U}_3' &= t_1, \end{aligned}\right.$$

so as to obtain null right-hand sides. Then visually, we realize that 8 new appropriate linear combinations potentially provide normalizations:

$$\begin{aligned} 0 &= V_1', \\ 0 &= V_3', \\ 0 &= V_4', \\ 0 &= W_5', \\ 0 &= W_7', \\ 0 &= W_9', \\ 0 &= U_2' + 2\,W_8' + \overline{W}_8', \\ 0 &= \overline{U}_1' + V_2' + \overline{W}_6'. \end{aligned} \tag{102}$$

As we already pointed out V_4' is indeed normalizable.

Abbreviate the last two essential torsions as:

$$\begin{aligned} X_1' &:= U_2' + 2\,W_8' + \overline{W}_8', \\ X_2' &:= \overline{U}_1' + V_2' + \overline{W}_6'. \end{aligned}$$

16.6. **Principles for the systematic computation of new torsion coefficients.** Now, we remember that our initial frame:

$$\{\mathscr{L}, \;\overline{\mathscr{L}}, \;\mathscr{T}, \;\mathscr{S}, \;\overline{\mathscr{S}}\}$$

on the base manifold $M^5 \subset \mathbb{C}^4$ had its last three elements given by:

$$
\begin{aligned}
\mathscr{T} &= i\,\big[\mathscr{L},\,\overline{\mathscr{L}}\big],\\
\mathscr{S} &= \big[\mathscr{L},\,\mathscr{T}\big],\\
\overline{\mathscr{S}} &= \big[\overline{\mathscr{L}},\,\mathscr{T}\big],
\end{aligned}
$$

complemented by the following Lie bracket structure:

$$
\begin{aligned}
\big[\mathscr{L},\,\mathscr{S}\big] &= P\cdot\mathscr{T} + Q\cdot\mathscr{S} + R\cdot\overline{\mathscr{S}},\\
\big[\overline{\mathscr{L}},\,\mathscr{S}\big] &= A\cdot\mathscr{T} + B\cdot\mathscr{S} + \overline{B}\cdot\overline{\mathscr{S}},\\
\big[\mathscr{L},\,\overline{\mathscr{S}}\big] &= A\cdot\mathscr{T} + B\cdot\mathscr{S} + \overline{B}\cdot\overline{\mathscr{S}},\\
\big[\overline{\mathscr{L}},\,\overline{\mathscr{S}}\big] &= \overline{P}\cdot\mathscr{T} + \overline{R}\cdot\mathscr{S} + \overline{Q}\cdot\overline{\mathscr{S}},\\
\big[\mathscr{T},\,\mathscr{S}\big] &= E\cdot\mathscr{T} + F\cdot\mathscr{S} + G\cdot\overline{\mathscr{S}},\\
\big[\mathscr{T},\,\overline{\mathscr{S}}\big] &= \overline{E}\cdot\mathscr{T} + \overline{G}\cdot\mathscr{S} + \overline{F}\cdot\overline{\mathscr{S}},\\
\big[\mathscr{S},\,\overline{\mathscr{S}}\big] &= i\,J\cdot\mathscr{T} + K\cdot\mathscr{S} - \overline{K}\cdot\overline{\mathscr{S}}.
\end{aligned}
$$

In fact, the 5 functions E, F, G, J, K express themselves in terms of the 5 functions P, Q, R, A, B and their coframes derivatives. Consequently, an important observation is in order:

Principle. *All the subsequent computations must necessarily be achieved only in terms of the* 5 *functions* P*,* Q*,* R*,* A*,* B *that are really fundamental and independent.* □

We must therefore replace in the torsion coefficients (after setting $R = 0$):

$$
\begin{aligned}
E &= -i\,\overline{\mathscr{L}}(P) - i\,A\,Q - i\,\overline{P}\,R + i\,\mathscr{L}(A) + i\,B\,P + i\,A\,\overline{B},\\
F &= -i\,\overline{\mathscr{L}}(Q) - i\,R\,\overline{R} + i\,A + i\,\mathscr{L}(B) + i\,B\,\overline{B},\\
G &= -i\,P - i\,\overline{B}\,Q - i\,R\,\overline{Q} - i\,\overline{\mathscr{L}}(R) + i\,B\,R + i\,\overline{BB} + i\,\mathscr{L}(\overline{B}).
\end{aligned}
$$

$$
\begin{aligned}
-\,2\,J = {} & -\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(P)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(A)\big) - \mathscr{L}\big(\mathscr{L}(\overline{P})\big)\\
& -\,Q\,\overline{\mathscr{L}}(A) - 2\,A\,\overline{\mathscr{L}}(Q) - R\,\overline{\mathscr{L}}(\overline{P}) - 2\,\overline{P}\,\overline{\mathscr{L}}(R) - 2\,AR\overline{R} - 2\,P\overline{P} - \overline{BP}Q - \overline{PQ}R -\\
& -\,\overline{R}\mathscr{L}(P) - 2\,P\,\mathscr{L}(\overline{R}) - \overline{Q}\,\mathscr{L}(A) - 2\,A\mathscr{L}(\overline{Q}) - PQ\overline{R} - BP\overline{Q} +\\
& +\,2\,P\overline{\mathscr{L}}(B) + B\overline{\mathscr{L}}(P) + 2\,A\overline{\mathscr{L}}(\overline{B}) + \overline{B}\,\overline{\mathscr{L}}(A) + 2\,A\,\mathscr{L}(B) + 2\,AA + 2\,AB\overline{B} + 2\,\overline{P}\,\mathscr{L}(\overline{B}) +\\
& +\,B\overline{P}R + \overline{BBP} + B\,\mathscr{L}(A) + \overline{B}\,\mathscr{L}(\overline{P}) + BBP + \overline{B}P\overline{R},
\end{aligned}
$$

$$
\begin{aligned}
2i\,K = {} & -\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(Q)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(B)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(B)\big) - \mathscr{L}\big(\mathscr{L}(\overline{R})\big) -\\
& -\,2\,\overline{R}\,\overline{\mathscr{L}}(R) - R\,\overline{\mathscr{L}}(\overline{R}) - B\,\overline{\mathscr{L}}(Q) - BR\overline{R} - 2\,P\overline{R} - \overline{Q}R\overline{R} - 2\,\mathscr{L}(\overline{P}) - \overline{R}\,\mathscr{L}(Q) -\\
& -\,2\,Q\,\mathscr{L}(\overline{R}) - \overline{Q}\,\mathscr{L}(B) - 2\,B\,\mathscr{L}(\overline{Q}) - A\overline{Q} - \overline{P}Q - QQ\overline{R} - BQ\overline{Q} +\\
& +\,2\,\overline{\mathscr{L}}(A) + \overline{B}\,\overline{\mathscr{L}}(B) + 2\,B\,\overline{\mathscr{L}}(\overline{B}) + 3\,B\,\mathscr{L}(B) + 3\,AB + BBQ + 2\,BB\overline{B} + 2\,\overline{R}\,\mathscr{L}(\overline{B}) +\\
& +\,\overline{BBR} + \overline{B}\,\mathscr{L}(\overline{R}) + \overline{BP} + Q\,\overline{\mathscr{L}}(B).
\end{aligned}
$$

Moreover, for any real analytic function $\mathbf{G}_0$ on the base manifold $M^5 \subset \mathbb{C}^4$, we must expand its last three frame derivatives precisely as:

$$\begin{aligned}
\mathscr{T}(\mathbf{G}_0) &= i\,\mathscr{L}\big(\overline{\mathscr{L}}(\mathbf{G}_0)\big) - i\,\overline{\mathscr{L}}\big(\mathscr{L}(\mathbf{G}_0)\big),\\
\mathscr{S}(\mathbf{G}_0) &= i\,\mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\mathbf{G}_0)\big)\big) - 2i\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\mathbf{G}_0)\big)\big) + i\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\mathscr{L}(\mathbf{G}_0)\big)\big),\\
\overline{\mathscr{S}}(\mathbf{G}_0) &= -\,i\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\mathbf{G}_0)\big)\big) + 2i\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\mathbf{G}_0)\big)\big) - i\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(\mathbf{G}_0)\big)\big).
\end{aligned}$$

Doing so, we obtain the following formulas that are useful to expand in an systematic way the new torsion coefficients on a computer machine:

$$\begin{aligned}
\mathscr{S}(\overline{\mathbf{D}}_0) = &-\mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(B)\big)\big)\big) + 2\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big)\big) - \overline{\mathscr{L}}\big(\mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big)\big) +\\
&+ \mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\overline{P})\big)\big) - 2\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\overline{P})\big)\big) + \overline{\mathscr{L}}\big(\mathscr{L}\big(\mathscr{L}(\overline{P})\big)\big)\\
&+ \tfrac{2}{3}\,\overline{Q}\,\mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big) - \tfrac{4}{3}\,\overline{Q}\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\mathscr{L}(B)\big)\big) + \tfrac{2}{3}\,\overline{Q}\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\mathscr{L}(B)\big)\big)\\
&+ \tfrac{2}{3}\,B\,\mathscr{L}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\overline{Q})\big)\big) - \tfrac{4}{3}\,B\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\overline{Q})\big)\big) + \tfrac{2}{3}\,B\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\mathscr{L}(\overline{Q})\big)\big),
\end{aligned}$$

$$\begin{aligned}
\overline{\mathscr{S}}(\overline{\mathbf{D}}_0) = &\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big)\big) - 2\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(B)\big)\big)\big) + \mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(B)\big)\big)\big) -\\
&- \overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\overline{P})\big)\big) + 2\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\overline{P})\big)\big) - \mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(\overline{P})\big)\big) -\\
&- \tfrac{2}{3}\,\overline{Q}\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}(B)\big)\big) + \tfrac{4}{3}\,\overline{Q}\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big) - \tfrac{2}{3}\,\overline{Q}\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(B)\big)\big) -\\
&- \tfrac{2}{3}\,B\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}(\overline{Q})\big)\big) + \tfrac{4}{3}\,B\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(\overline{Q})\big)\big) - \tfrac{2}{3}\,B\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(\overline{Q})\big)\big),
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}(\overline{\mathbf{D}}_0) = &-\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(B)\big)\big) + \overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)\big) + \mathscr{L}\big(\overline{\mathscr{L}}(\overline{P})\big) - \overline{\mathscr{L}}\big(\mathscr{L}(\overline{P})\big) +\\
&+ \tfrac{2}{3}\,\overline{Q}\,\mathscr{L}\big(\overline{\mathscr{L}}(B)\big) - \tfrac{2}{3}\,\overline{Q}\,\overline{\mathscr{L}}\big(\mathscr{L}(B)\big) + \tfrac{2}{3}\,B\,\mathscr{L}\big(\overline{\mathscr{L}}(\overline{Q})\big) - \tfrac{2}{3}\,B\,\overline{\mathscr{L}}\big(\mathscr{L}(\overline{Q})\big),
\end{aligned}$$

with quite similar formulas for $\mathbf{B}_0$ and $\mathbf{E}_0$.

In subsection 13.7, we obtained five relations between the fundamental functions A, B, P, Q, R, extracted from the iterated Lie brackets of the length six. By the assumption $R \equiv 0$, they become:

(103)

$$\begin{aligned}
0 \overset{1}{\equiv}\, & 2\,\mathscr{L}(\overline{\mathscr{L}}(P)) - \mathscr{L}(\mathscr{L}(A)) - \overline{\mathscr{L}}(\mathscr{L}(P)) - 2\,P\mathscr{L}(B) - B\mathscr{L}(P) - 2\,A\mathscr{L}(\overline{B}) - \overline{B}\mathscr{L}(A) + P\overline{\mathscr{L}}(Q) +\\
&+ A\mathscr{L}(Q) + 2\,Q\mathscr{L}(A) - Q\overline{\mathscr{L}}(P) - PB\overline{B} - A\overline{B}^2 + PBQ + 2\,AQ\overline{B} - AQ^2,\\
0 \overset{2}{\equiv}\, & 2\,\mathscr{L}(\overline{\mathscr{L}}(Q)) - \mathscr{L}(\mathscr{L}(B)) - \overline{\mathscr{L}}(\mathscr{L}(Q)) -\\
&- 2\,\mathscr{L}(A) - 2\,B\mathscr{L}(\overline{B}) - \overline{B}\mathscr{L}(B) + \overline{\mathscr{L}}(P) + BQ\overline{B} - A\overline{B} - B\overline{B}^2 + AQ,\\
0 \overset{3}{\equiv}\, & -\mathscr{L}(\mathscr{L}(\overline{B})) -\\
&- 3\,\overline{B}\mathscr{L}(\overline{B}) + \overline{B}\mathscr{L}(Q) + 2\,Q\mathscr{L}(\overline{B}) + \mathscr{L}(P) + 2\,Q\overline{B}^2 - QP - Q^2\overline{B} - \overline{B}^3 + P\overline{B},\\
0 \overset{4}{\equiv}\, & -3\overline{\mathscr{L}}(\mathscr{L}(A)) - \mathscr{L}(\mathscr{L}(\overline{P})) + 3\,\mathscr{L}(\overline{\mathscr{L}}(A)) + \overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) -\\
&- 2\,A\mathscr{L}(\overline{Q}) - \overline{Q}\mathscr{L}(A) + 3\,B\mathscr{L}(A) + 3\,\overline{B}\mathscr{L}(\overline{P}) - 3\,B\overline{\mathscr{L}}(P) - 3\,\overline{B}\,\overline{\mathscr{L}}(A) + 2\,A\overline{\mathscr{L}}(Q) + Q\overline{\mathscr{L}}(A) -\\
&- BP\overline{Q} + 3\,B^2P + 2\,A\overline{B}\,\overline{Q} - 2\,BQA - 3\,\overline{B}^2\overline{P} + Q\overline{B}\,\overline{P},\\
0 \overset{5}{\equiv}\, & -3\,\overline{\mathscr{L}}(\mathscr{L}(B)) + 3\,\mathscr{L}(\overline{\mathscr{L}}(B)) + \overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) +\\
&+ 3\,B\mathscr{L}(B) - 3\,\overline{B}\,\overline{\mathscr{L}}(B) + Q\overline{\mathscr{L}}(B) - B\overline{\mathscr{L}}(Q) - 2\,B\mathscr{L}(\overline{Q}) - \overline{Q}\mathscr{L}(B) - 2\,\mathscr{L}(\overline{P}) -\\
&- Q\overline{P} - A\overline{Q} - BQ\overline{Q} + 3\,AB + 3\,\overline{B}\,\overline{P} + 2\,B\overline{B}\,\overline{Q} + B^2Q.
\end{aligned}$$

These five equations enable us to simplify the results obtained during the computations. In particular, we have:

Lemma 16.3. *After determining the group parameter* e*, the two (normalizable) torsion coefficients* W_7' *and* X_2' *vanish identically.*

Proof. After determining e, the expressions of W_7' and X_2' take the forms:

$$
\begin{aligned}
W_7' = -\frac{i}{\overline{\mathsf{a}}^3}\Big(&\overline{\mathscr{L}}(\overline{\mathscr{L}}(B)) - \overline{\mathscr{L}}(\overline{P}) - B\overline{\mathscr{L}}(\overline{Q}) + 3\,B\overline{\mathscr{L}}(B) - 2\,\overline{\mathscr{L}}(B)\overline{Q} + \\
&+ \overline{Q}\,\overline{P} + B\overline{Q}^2 - 2\,B^2\overline{Q} + B^3 - B\overline{P}\Big),
\end{aligned}
$$

$$
\begin{aligned}
X_2' = -\frac{i}{6\mathsf{a}\overline{\mathsf{a}}^2}\Big(&\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) - 3\,\overline{\mathscr{L}}(\mathscr{L}(B)) + 3\,\mathscr{L}(\overline{\mathscr{L}}(B)) - 2\,\mathscr{L}(\overline{P}) - B\overline{\mathscr{L}}(Q) + \\
&+ 3\,AB + 3\,B\mathscr{L}(B) - \overline{Q}\mathscr{L}(B) - A\overline{Q} - Q\overline{P} - 2\,B\mathscr{L}(\overline{Q}) - 3\,\overline{B\mathscr{L}}(B) + B^2Q + \\
&+ Q\overline{\mathscr{L}}(B) + 3\,\overline{B}\,\overline{P} - BQ\overline{Q} + 2\overline{B}B\overline{Q}\Big).
\end{aligned}
$$

Now, it suffices to use the already presented equation $\overset{5}{=}$ and the conjugation of $\overset{3}{=}$ to extract the expressions of $\overline{\mathscr{L}}(\mathscr{L}(B))$ and of $\mathscr{L}(\mathscr{L}(\overline{B}))$ in W_7', respectively and subsequently to insert them in X_2' and in W_7'. □

17. FOUR GROUP PARAMETER GENERAL NORMALIZATIONS

17.1. **Setting up the torsion coefficients.** After normalizing the group parameter e and inserting it into the expressions of the lifted coframe, one obtains:

$$
\begin{aligned}
\sigma &= \mathsf{a}^2\overline{\mathsf{a}}\cdot\,\sigma_0,\\
\rho &= \mathsf{a}\overline{\mathsf{a}}\cdot\,\big(\underbrace{\mathbf{C}_0\,\sigma_0 + \overline{\mathbf{C}}_0\overline{\sigma}_0 + \rho_0}_{\rho_0^{\sim}}\big),\\
\zeta &= \mathsf{a}\cdot\Big[\underbrace{\mathbf{E}_0\,\sigma_0 + \underbrace{\overline{\mathbf{D}}_0\,\overline{\sigma}_0 + \mathbf{B}_0\cdot\rho_0 + \zeta_0}_{\zeta_0^{\sim}}}_{=:\zeta_0^{\approx}}\Big].
\end{aligned}
$$

In other words, the lifted coframe converts into the form:

$$
\left[
\begin{aligned}
\sigma &= \mathsf{a}^2\overline{\mathsf{a}}\cdot\sigma_0,\\
\rho &= \mathsf{a}\overline{\mathsf{a}}.\rho_0^{\sim},\\
\zeta &= \mathsf{a}\cdot\zeta_0^{\approx},
\end{aligned}
\right.
$$

with the reduced matrix group:

$$
G^{\approx} := \left\{ g^{\approx} := \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}^2 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix} : \mathsf{a}\in\mathbb{C} \right\},
$$

and with the modified Maurer-Cartan matrix:

$$dg^{\approx}\cdot g^{\approx -1}=\begin{pmatrix} 2\,\overline{\beta}+\beta & 0 & 0 & 0 & 0\\ 0 & 2\,\beta+\overline{\beta} & 0 & 0 & 0\\ 0 & 0 & \beta+\overline{\beta} & 0 & 0\\ 0 & 0 & 0 & \overline{\beta} & 0\\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}.$$

Here, there remains just one Maurer-Cartan 1-form:

$$\beta:=\beta_1=\frac{d\mathsf{a}}{\mathsf{a}}.$$

Now, applying the exterior differentiation operator on the already obtained lifted coframe gives:

$$\begin{aligned} d\sigma &= (2\beta+\overline{\beta})\wedge\sigma+\\ &\quad+\mathsf{a}^2\overline{\mathsf{a}}\cdot d\sigma_0,\\ d\rho &= (\beta+\overline{\beta})\wedge\rho+\\ &\quad+\mathsf{a}\overline{\mathsf{a}}\cdot(\mathbf{C}_0\,d\sigma_0+\overline{\mathbf{C}}_0\,d\overline{\sigma}_0+d\rho_0+\\ &\qquad+d\mathbf{C}_0\wedge\sigma_0+d\overline{\mathbf{C}}_0\wedge\overline{\sigma}_0),\\ d\zeta &= \beta\wedge\zeta+\\ &\quad+\mathsf{a}\cdot(\mathbf{E}_0\,d\sigma_0+\overline{\mathbf{D}}_0\,d\overline{\sigma}_0+\mathbf{B}_0 d\rho_0+d\zeta_0+\\ &\qquad+d\mathbf{E}_0\wedge\sigma_0+d\overline{\mathbf{D}}_0\wedge\overline{\sigma}_0+d\mathbf{B}_0\wedge\rho_0). \end{aligned}$$

Comparing with (100), one easily verifies that the expressions of $d\sigma$ and $d\rho$ are unchanged. Hence except possible replacements of $\mathsf{e}=\mathsf{a}\,\mathbf{E}_0$, we have no essential change in the expressions of the new torsion coefficients U_i'' and V_j''. Nevertheless, some of the torsion coefficients W_k'' change after determining e. More precisely, our computations show that just the four torsion coefficients W_1'', W_2'', W_3'', W_4'' convert into the following modified forms:

(104)

$$\begin{aligned} W_1'' &:= W_1'-\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{S}}(\mathbf{E}_0)+\frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\mathscr{T}(\mathbf{E}_0)-\frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\mathscr{L}(\mathbf{E}_0)+\frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{E}_0)-\\ &\quad-\frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^5}\,\overline{\mathscr{L}}(\mathbf{E}_0)+\frac{\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{E}_0),\\ W_2'' &:= W_2'-\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\mathscr{T}(\mathbf{E}_0)+\frac{\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{E}_0)+\frac{\overline{\mathsf{b}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{E}_0),\\ W_3'' &:= W_3'-\frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{E}_0),\\ W_4'' &:= W_4'-\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{E}_0). \end{aligned}$$

17.2. **The remaining normalizable expressions in the second loop.** At the second loop absorbtion, we found eight normalizable expressions (102). Among them, V_3' vanished automatically and also $V_4' \equiv 0$ after determining e. Subsequently, W_7' and X_2' vanished identically, as soon as we took account of the above relations $\overset{1}{=}$, $\overset{2}{=}$, $\overset{3}{=}$, $\overset{4}{=}$, $\overset{5}{=}$ coming from a study of Jacobi identities between iterated Lie brackets of length six. Then, for the moment, a computer-assisted calculation provides the following expressions for all the essential torsions (102):

(105)

$$
\begin{aligned}
V_1'' := -\frac{i}{3\,\mathsf{a}^2\overline{\mathsf{a}}^2}\Big(& 3\,\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(B))) + 3\,\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{Q}))) - 6\,\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))) + 2\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{Q}))) + \\
& + 3\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{B}))) + 3\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(Q))) - 6\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(Q))) + 2\,\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q))) - \\
& - 4\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) + 9\,\mathscr{L}(\overline{\mathscr{L}}(A)) - 7\,\mathscr{L}(\mathscr{L}(\overline{P})) + 2\,\overline{Q}\,\overline{\mathscr{L}}(\mathscr{L}(\overline{B})) - 6\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) + 10\,\overline{B}\,\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) + \\
& + 2\,Q\mathscr{L}(\overline{\mathscr{L}}(B)) + 10\,B\mathscr{L}(\overline{\mathscr{L}}(Q)) - 6\,B\overline{\mathscr{L}}(\mathscr{L}(Q)) - 4\,\overline{Q}\mathscr{L}(\overline{\mathscr{L}}(Q)) + 2\,\overline{Q}\,\overline{\mathscr{L}}(\mathscr{L}(Q)) - 3\,B\mathscr{L}(\mathscr{L}(\overline{Q})) - \\
& - 3\,\overline{B}\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) - 4\,Q\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) + 2\,Q\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) - 2\,\overline{P}\mathscr{L}(Q) + 6\,\overline{P}\mathscr{L}(\overline{B}) - 2\,P\overline{\mathscr{L}}(\overline{Q}) + 6\,P\overline{\mathscr{L}}(B) + \\
& + 3\,A\mathscr{L}(B) - 5\,A\mathscr{L}(\overline{Q}) + 3\,A\overline{\mathscr{L}}(\overline{B}) + A\overline{\mathscr{L}}(Q) + 2\,Q\overline{\mathscr{L}}(A) - \overline{Q}\mathscr{L}(A) + 7\,B\overline{\mathscr{L}}(P) - 9\,\overline{B}\,\overline{\mathscr{L}}(A) + \\
& + 16\,\overline{B}\mathscr{L}(\overline{P}) - 2\,\overline{B}Q\overline{\mathscr{L}}(\overline{Q}) - 2\,\mathscr{L}(B)\overline{\mathscr{L}}(Q) + 2\,B^2\mathscr{L}(Q) + 2\,\mathscr{L}(Q)\overline{\mathscr{L}}(B) + 6\,\mathscr{L}(B)^2 + 9\,\overline{\mathscr{L}}(\overline{B})\mathscr{L}(B) - \\
& - 6\,B^2\overline{B}^2 - 9\,B\overline{B}\,\overline{\mathscr{L}}(\overline{B}) + 5\,BQ\overline{\mathscr{L}}(\overline{B}) - 9\,B\overline{B}\mathscr{L}(B) + 5\,B^2\overline{B}Q + 5\,\overline{B}\,\overline{Q}\mathscr{L}(B) + 6\,\overline{B}Q\overline{\mathscr{L}}(B) + \\
& + 4\,B\overline{B}\,\overline{\mathscr{L}}(Q) + 4\,B\overline{B}\mathscr{L}(\overline{Q}) + 6\,B\overline{Q}\mathscr{L}(\overline{B}) - 9\,\overline{B}^2\overline{\mathscr{L}}(B) - 9\,B^2\mathscr{L}(\overline{B}) - 3\,\mathscr{L}(\overline{B})\overline{\mathscr{L}}(B) - 6\,\overline{\mathscr{L}}(\overline{B})\overline{\mathscr{L}}(Q) - \\
& - 2\,\overline{B}\,\overline{Q}\mathscr{L}(Q) + 6\,\overline{\mathscr{L}}(\overline{B})^2 + 4\,\overline{B}\,\overline{Q}\,\overline{\mathscr{L}}(\overline{B}) - 2\,Q\overline{Q}\,\overline{\mathscr{L}}(\overline{B}) - 4\,\overline{Q}\,\overline{\mathscr{L}}(P) - 2\,\overline{\mathscr{L}}(\overline{B})\mathscr{L}(\overline{Q}) + 2\,\mathscr{L}(\overline{B})\overline{\mathscr{L}}(\overline{Q}) - \\
& - 4\,Q\mathscr{L}(\overline{P}) - 6\,\mathscr{L}(B)\mathscr{L}(\overline{Q}) + 2\,\overline{B}^2\overline{\mathscr{L}}(\overline{Q}) - 2\,B\mathscr{L}(Q)\overline{Q} - 2\,Q\overline{Q}\mathscr{L}(B) + 4\,BQ\mathscr{L}(B) - \\
& - 2\,BQ\mathscr{L}(\overline{Q}) - 6\,\overline{B}^2\,\overline{P} + 10\,\overline{B}\,\overline{Q}A - BP\overline{Q} + 2\,\overline{B}\,\overline{P}Q + 4\,ABQ - 9\,AB\overline{B} - 4\,QA\overline{Q} - \\
& - 3\,BQ\overline{B}\,\overline{Q} + 5\,\overline{B}^2\overline{Q}B + 3\,B^2P\Big)
\end{aligned}
$$

$$V_3'' = 0,$$

$$V_4'' = 0,$$

$$
\begin{aligned}
W_5'' = -\frac{1}{3\,\mathsf{a}\overline{\mathsf{a}}^3}\Big(& 2\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{Q}))) - 3\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(B))) + 3\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(B))) - \overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))) - \mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{Q}))) + \\
& + 3\mathscr{L}(\overline{\mathscr{L}}(\overline{P})) - 3\,\overline{\mathscr{L}}(\mathscr{L}(\overline{P})) + 3\,\overline{Q}\mathscr{L}(\overline{\mathscr{L}}(B)) - 2\,\overline{Q}\,\overline{\mathscr{L}}(\mathscr{L}(B)) + 2\,B\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) - 2\,B\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) - \\
& - 3\,B\mathscr{L}(\overline{\mathscr{L}}(B)) - Q\overline{\mathscr{L}}(\overline{\mathscr{L}}(B)) + 3\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(B)) + +QB\overline{\mathscr{L}}(\overline{Q}) - 4\,B\overline{B}\,\overline{\mathscr{L}}(\overline{Q}) + 3\,\overline{P}\,\overline{\mathscr{L}}(\overline{B}) \\
& - 3\,\overline{\mathscr{L}}(\overline{B})\overline{\mathscr{L}}(B) + 3\,A\overline{\mathscr{L}}(B) + 2\,B\overline{Q}\,\overline{\mathscr{L}}(\overline{B}) + 15\,B\overline{B}\,\overline{\mathscr{L}}(B) - 7\,\overline{B}\,\overline{Q}\,\overline{\mathscr{L}}(B) - \overline{Q}^2\mathscr{L}(B) - \\
& - 3\,B^2\mathscr{L}(B) - 2\,B\overline{Q}\mathscr{L}(\overline{Q}) + 2\,Q\overline{Q}\,\overline{\mathscr{L}}(B) - 3\,BQ\overline{\mathscr{L}}(B) + 2\,\overline{\mathscr{L}}(B)\mathscr{L}(\overline{Q}) - 2\,\overline{P}\mathscr{L}(\overline{Q}) - \\
& - 3\,\mathscr{L}(B)\overline{\mathscr{L}}(B) + 3\,B\overline{\mathscr{L}}(A) - \overline{Q}\,\overline{\mathscr{L}}(A) - A\overline{\mathscr{L}}(\overline{Q}) + 2\,B^2\mathscr{L}(\overline{Q}) - 3\,\overline{B}\,\overline{\mathscr{L}}(\overline{P}) + \\
& + 4\,B\overline{Q}\mathscr{L}(B) + \mathscr{L}(B)\overline{\mathscr{L}}(\overline{Q}) + Q\overline{\mathscr{L}}(\overline{P}) + 3\,AB^2 + 6\,B^3\overline{B} - 4\,AB\overline{Q} - 3\,B\overline{B}\,\overline{P} - 10\,B^2\overline{B}\,\overline{Q} + \\
& + 3\,\overline{B}\,\overline{Q}\,\overline{P} + 4\,B\overline{B}\,\overline{Q}^2 - B^3Q + A\overline{Q}^2 + BQ\overline{P} - BQ\overline{Q}^2 - Q\overline{Q}\,\overline{P} + 2\,B^2Q\overline{Q}\Big),
\end{aligned}
$$

$$W_7'' = 0,$$

$$W_9'' = \frac{i}{9\overline{\mathsf{a}}^2}\Big(18\,\overline{\mathscr{L}}(B) - 3\,\overline{\mathscr{L}}(\overline{Q}) - 9\,\overline{P} - 12\,B\overline{Q} + 9\,B^2 + \overline{Q}^2\Big),$$
$$X_1'' = -\frac{i}{9\mathsf{a}\overline{\mathsf{a}}}\Big(6\,\overline{\mathscr{L}}(Q) + 6\,\mathscr{L}(\overline{Q}) - 18\,\mathscr{L}(B) - 18\,\overline{\mathscr{L}}(\overline{B}) + 27\,B\overline{B} - 6\,BQ - 6\,\overline{B}\,\overline{Q} + 2\,Q\overline{Q} + 9\,A\Big),$$
$$X_2'' = 0.$$

Theorem 17.1. *When at least one among the above four independent essential torsions V_1'', W_5'', W_9'', X_1'' does not vanish identically, one can normalize either* a *or* $\mathsf{a}\overline{\mathsf{a}}$ *after relocalization to a neighborhood of a generic point, and more precisely:*

- **(i)** *when $X_1'' \not\equiv 0$, setting $X_1'' := -\frac{i}{9}$ (noticing that X_1'' is a purely imaginary valued function), one normalizes* $\mathsf{a}\overline{\mathsf{a}}$*, and in this case, the three remaining expressions W_9'', W_5'', V_1'' are the invariants of the equivalence problem;*
- **(ii)** *when $W_9'' \not\equiv 0$ but $X_1'' \equiv 0$, setting $W_9'' := \frac{i}{9}$, one normalizes* a*, and in this case the two remaining expressions W_5'' and V_1'' are the invariants of the equivalence problem;*
- **(iii)** *when $W_5'' \neq 0$ but $X_1'' \equiv W_9'' \equiv 0$, setting $W_5'' := \frac{1}{3}$, one normalizes* a*, and in this case, V_1'' is the single invariant of the problem.*
- **(iv)** *when, if $V_1'' \neq 0$ but $X_1'' \equiv W_9'' \equiv W_5'' \equiv 0$, setting $V_1'' := -\frac{i}{3}$ (noticing that V_1'' is a purely imaginary valued function), one normalizes* $\mathsf{a}\overline{\mathsf{a}}$.
- **(v)** *when $X_1'' \equiv W_9'' \equiv W_5'' \equiv V_1'' \equiv 0$, one has to start the third loop of the Cartan equivalence procedure under the assumption:*

$$V_1'' = W_5'' = W_9'' = X_1'' \equiv 0. \tag{106}$$

Proof. The reason of choosing first the expressions with the lowest degree of derivations — namely X_1'', W_9'' of $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivation order 1, and afterwards W_5'', V_1'' of $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivation order 3 — is that the vanishing of the less complex ones may possibly cause the vanishing of the ones having greater complexity.

To check the independency of these expressions, we use the MAPLE command IDEALMEMBERSHIP. The procedure will be described in the next subsection 17.3. □

17.3. **Third loop normalizable essential torsion combinations.** Now, the assumptions (106) lead us to start the third-loop normalization. At this time, our structure equations have the following form:

$$
\begin{aligned}
d\sigma &= \big(2\,\beta + \overline{\beta}\big) \wedge \sigma + \\
&\quad + U_1''' \, \sigma \wedge \overline{\sigma} + U_2''' \, \sigma \wedge \rho + U_3''' \, \sigma \wedge \zeta + U_4''' \, \sigma \wedge \overline{\zeta} + \\
&\quad + \underline{U_5'''}_{\circ} \, \overline{\sigma} \wedge \rho + \underline{U_6'''}_{\circ} \, \overline{\sigma} \wedge \zeta + \underline{U_7'''}_{\circ} \, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + \rho \wedge \zeta, \\
d\rho &= \big(\beta + \overline{\beta}\big) \wedge \rho + \\
&\quad + \underline{V_1'''}_{\circ} \, \sigma \wedge \overline{\sigma} + V_2''' \, \sigma \wedge \rho + \underline{V_3'''}_{\circ} \, \sigma \wedge \zeta + \underline{V_4'''}_{\circ} \, \sigma \wedge \overline{\zeta} + \\
&\quad + \overline{V_2}''' \, \overline{\sigma} \wedge \rho + \underline{\overline{V_4}'''}_{\circ} \, \overline{\sigma} \wedge \zeta + \underline{\overline{V_3}'''}_{\circ} \, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + V_8''' \, \rho \wedge \zeta + \overline{V_8}''' \, \rho \wedge \overline{\zeta} + \\
&\quad + i \, \zeta \wedge \overline{\zeta}, \\
d\zeta &= \beta \wedge \zeta + \\
&\quad + W_1''' \, \sigma \wedge \overline{\sigma} + W_2''' \, \sigma \wedge \rho + W_3''' \, \sigma \wedge \zeta + W_4''' \, \sigma \wedge \overline{\zeta} + \\
&\quad + \underline{W_5'''}_{\circ} \, \overline{\sigma} \wedge \rho + W_6''' \, \overline{\sigma} \wedge \zeta + \underline{W_7'''}_{\circ} \, \overline{\sigma} \wedge \overline{\zeta} + \\
&\quad + W_8''' \, \rho \wedge \zeta + \underline{W_9'''}_{\circ} \, \rho \wedge \overline{\zeta} + \\
&\quad + W_{10}''' \, \zeta \wedge \overline{\zeta},
\end{aligned}
\tag{107}
$$

where we underline the torsions that vanish identically.

Starting this step of normalization, one has to replace the single remaining Maurer-Cartan 1-form β as:

$$\beta \mapsto \beta + p\,\sigma + q\,\overline{\sigma} + r\,\rho + s\,\zeta + t\,\overline{\zeta}$$

and to proceed with the same line of computations as in the former steps. Then, by anticipation, one obtains 4 new potentially normalizable expressions:

$$
\left[
\begin{aligned}
0 &= W_1''', \\
0 &= W_2''', \\
0 &= W_4''', \\
0 &= V_2''' - W_3''' - \overline{W}_6''' =: Y'''.
\end{aligned}
\right.
$$

Putting the lastly obtained expressions of V_2''', W_3''', W_6''' in Y''' immediately implies that:

Lemma 17.1. *The last normalizable expression Y''' vanishes identically.* □

Similarly to the second step of normalization, if at least one of the above expressions does not vanish, then it can be employed to determine the last group parameter a and next, the remaining nonzero normalizable expressions will be the invariants of the equivalence problem. Otherwise, one has to start the prolongation procedure.

Before proceeding, let us check whether one of the above normalizable expressions can be expressed as a combination of the remaining ones or those in (106). For this aim, first we extract the zero-order terms of each expression. Since these terms do not admit any derivations of $\mathscr{L}$ or $\overline{\mathscr{L}}$, then it is not required to consider also the derivations of the mentioned expressions. Now, we check the independency of these extracted zero-order expressions. For this aim, we use the MAPLE command IDEALMEMBERSHIP[4] and realize that the first order terms of W_4''' can be eliminated by the first-order terms of W_9''' and X_1'''. Then, we *surmise* that W_4''' may be expressed as a combination of — assumed to be vanishing — W_9''' and X_1'''. The maximum $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivation order in W_4''' equals 2, while it equals 1 in the expressions of W_9''' and X_1'''. Hence, we guess that W_4''' can be expressed as a combination of W_9''', X_1''' and $\mathscr{L}(X_1'''), \overline{\mathscr{L}}(X_1'''), \mathscr{L}(W_9'''), \ldots, \overline{\mathscr{L}}(\overline{W}_9''')$. After somehow tremendous computations, we realized that:

$$W_4''' = -\frac{6\,\overline{B} - 2\,Q}{5\,\mathsf{a}}\,W_9''' + \frac{\overline{Q}}{5\overline{\mathsf{a}}}X_1''' + \frac{3}{5\,\mathsf{a}}\mathscr{L}(W_9''') - \frac{3}{5\overline{\mathsf{a}}}\overline{\mathscr{L}}(X_1''').$$

Lemma 17.2. *Under the assumptions* (106), *the normalizable expression* W_4''' *vanishes identically.* □

Our inspections show that it is not possible to express one of the two remaining expressions W_1''', W_2''' in terms of the other one or in terms of the expressions V_1'', W_5''', W_9''', X_1'''. Hence, in this step we encounter two essential torsion coefficients having the following expressions[5]:

(108)

$$\begin{aligned}
W_1''' &= W_1''\Big|_{\text{replace } \mathsf{e}=\mathsf{a}\mathbf{E}_0} - \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\overline{\mathscr{S}}(\mathbf{E}_0) + \frac{\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^4}\,\mathscr{T}(\mathbf{E}_0) - \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^4\overline{\mathsf{a}}^4}\,\mathscr{L}(\mathbf{E}_0) + \frac{\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}((\mathbf{E}_0) - \\
&\quad - \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^5}\,\overline{\mathscr{L}}(\mathbf{E}_0) + \frac{\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{E}_0) \\
&= \frac{1}{9\,\mathsf{a}^2\overline{\mathsf{a}}^3}\Big(\text{long expression}\Big), \\
W_2''' &= -\frac{1}{405\mathsf{a}^2\overline{\mathsf{a}}^2}\Big[1485\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(Q))) - 2610\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(Q))) + 270\,\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q))) - 1080\,\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))) + \\
&\quad + 270\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{Q}))) + 540\,\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{Q}))) + 405\,\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(B))) - 675\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{B}))) + 4320\,\mathscr{L}(\overline{\mathscr{L}}(A)) + \\
&\quad + 111\,P\overline{Q}^2 + 3150\,B\mathscr{L}(\overline{\mathscr{L}}(Q)) + 156\,Q^2\overline{P} - 198Q\mathscr{L}(\overline{\mathscr{L}}(B)) - 1980\,\overline{B}\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) + 999\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) - \\
&\quad - 918\,\overline{Q}\overline{\mathscr{L}}(\mathscr{L}(\overline{B})) + 6345\,B\overline{\mathscr{L}}(\mathscr{L}(\overline{B})) + 2241\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(B)) - 1980\,\mathscr{L}(\mathscr{L}(\overline{P})) - 2250\,B\overline{\mathscr{L}}(\mathscr{L}(Q)) -
\end{aligned}$$

[4] It is in fact a time and memory consuming command of MAPLE and that it is why that we use first just the zero-order terms of the expressions instead of their corresponding lengthy ones.

[5] The full expression of W_1''' is much too long — it involves $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivations of order 4 —, hence we do not typeset its expanded expression. Nevertheless, it is available in the MAPLE worksheet [59]

$$
\begin{aligned}
&- 1320\,\overline{Q}\mathscr{L}(\overline{\mathscr{L}}(Q)) - 270B\mathscr{L}(\mathscr{L}(\overline{Q})) + 1215\,\overline{B\mathscr{L}}(\overline{\mathscr{L}}(Q)) + 18\,Q\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) + 120\,Q\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))+\\
&+ 120\,\mathscr{L}(Q)\overline{\mathscr{L}}(\overline{Q}) + 858\,\overline{\mathscr{L}}(\mathscr{L}(Q))\overline{Q} - 120\,Q\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) + 1305\,A\overline{\mathscr{L}}(Q) - 2610\,A\mathscr{L}(\overline{Q})+\\
&+ 1350\,A\overline{\mathscr{L}}(\overline{B}) - 270\,P\overline{\mathscr{L}}(B) + 90\,P\overline{\mathscr{L}}(\overline{Q}) + 810\,\overline{P}\mathscr{L}(\overline{B}) - 270\,\mathscr{L}(Q)\overline{P} + 52\,\mathscr{L}(Q)\overline{Q}^2+\\
&+ 1827\,\overline{B}\mathscr{L}(\overline{P}) - 2745B\overline{\mathscr{L}}(P) + 2808\,\overline{B}^2\overline{\mathscr{L}}(B) - 3780\,\big(\overline{\mathscr{L}}(\overline{B})\big)^2 - 120\,\big(\overline{\mathscr{L}}(Q)^2\big) - 180\,\big(\mathscr{L}(\overline{Q})^2\big)-\\
&- 68\,Q\overline{Q}\mathscr{L}(\overline{Q}) + 156\,BQ\mathscr{L}(\overline{Q}) + 11394\,\overline{\mathscr{L}}(\overline{B})B\overline{B} - 3042\,\overline{\mathscr{L}}(\overline{B})BQ + 456\,\overline{BQ}\mathscr{L}(\overline{Q}) - 2052\,B\overline{B}\mathscr{L}(\overline{Q})-\\
&- 3978\,B\overline{B\mathscr{L}}(Q) + 3366\,B\overline{Q}\mathscr{L}(\overline{B}) - 540\,\overline{B}Q\overline{\mathscr{L}}(B) + 150\,\overline{B}Q\overline{\mathscr{L}}(\overline{Q}) + 894\,\overline{BQ\mathscr{L}}(Q) - 5670\,B^2\mathscr{L}(\overline{B})-\\
&- 210\,B^2Q^2 + 4860\mathscr{L}(\overline{B})\overline{\mathscr{L}}(B) + 1800\,\overline{\mathscr{L}}(\overline{B})\overline{\mathscr{L}}(Q) - 336\,\overline{Q\mathscr{L}}(P) + 1440\,\overline{\mathscr{L}}(\overline{B})\mathscr{L}(\overline{Q})-\\
&- 450\,\mathscr{L}(\overline{B})\overline{\mathscr{L}}(\overline{Q}) - 378\,\overline{B}^2\overline{\mathscr{L}}(\overline{Q}) - 267\,\mathscr{L}(\overline{B})\overline{Q}^2 + 24\,Q\mathscr{L}(\overline{P}) - 90\,\mathscr{L}(Q)\overline{\mathscr{L}}(B) + 630\,B^2\mathscr{L}(Q)-\\
&- 192\,\overline{\mathscr{L}}(B)Q^2 + 12\,Q^2\overline{\mathscr{L}}(\overline{Q}) - 180\,\mathscr{L}(\overline{Q})\overline{\mathscr{L}}(Q) - 2934\,\overline{\mathscr{L}}(\overline{B})\overline{BQ} + 780\,\overline{Q}Q\overline{\mathscr{L}}(\overline{B}) + 294\,QB\overline{\mathscr{L}}(Q)-\\
&- 546\,B\mathscr{L}(Q)\overline{Q} - 88\,Q\overline{\mathscr{L}}(Q)\overline{Q} + 3780\,B^2P - 1674\,\overline{B}^2\overline{P} - 432\,\overline{B}^2\overline{Q}^2 + 3753\,AB\overline{B} - 444\,QA\overline{Q}-\\
&- 1584\,ABQ + 873\,\overline{BQ}A - 1728\,BP\overline{Q} + 90\,\overline{BP}Q + 405\,A^2 + 2583\,\overline{B}^2\,\overline{Q}B + 3096\,B^2\overline{B}Q-\\
&- 20\,Q^2\overline{Q}^2 - 1242\,B^2\overline{B}^2 + 278\,\overline{Q}^2Q\overline{B} + 234\,\overline{Q}BQ^2 - 2868\,\overline{B}\,\overline{Q}BQ\Big].
\end{aligned}
$$

Similarly to what we did for the second loop, non-vanishing of either of the above two essential torsion coefficients enables us to determine either a or $\mathsf{a}\overline{\mathsf{a}}$, and to yet consider the remaining ones as the invariants of the problem. Hence, pursuant to Theorem 17.1, we have:

Theorem 17.2. *Assume that the four expressions V_1''', W_5''', W_9''', X_1''' vanish identically.*

- **(vi)** *When $W_2''' \neq 0$, one can normalize $\mathsf{a}\overline{\mathsf{a}}$ and W_1''' is the single invariant of the equivalence problem.*
- **(vii)** *When $W_1''' \neq 0$ but $W_2''' \equiv 0$, one can normalize a from $\mathsf{a}^2\overline{\mathsf{a}}^3$ and the equivalence problem has no invariant.*
- **(viii)** *Otherwise, when both these two normalizable expressions vanish identically, one has to start the prolongation procedure under the assumptions:*

$$
\begin{aligned}
&0 \equiv V_1''' \equiv W_5''' \equiv W_9''' \equiv X_1''',\\
&0 \equiv W_1''' \equiv W_2''' \equiv 0.
\end{aligned}
\tag{109}
$$

17.4. **Prolongation.** After determining the four group parameters $\mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}$ and in the case that the equations (109) hold, we have to prolong the equivalence problem with one undetermined group parameter a and with one Maurer-Cartan 1-form:

$$
\beta = \frac{d\mathsf{a}}{\mathsf{a}}.
$$

Here the structure equations have the form (*cf.* (107)):

(110)
$$\begin{aligned}
d\sigma &= \big(2\,\beta+\overline{\beta}\big)\wedge\sigma+ \\
&\quad + U_1'''\,\sigma\wedge\overline{\sigma}+U_2'''\,\sigma\wedge\rho+U_3'''\,\sigma\wedge\zeta+U_4'''\,\sigma\wedge\overline{\zeta}+\rho\wedge\zeta, \\
d\rho &= \big(\beta+\overline{\beta}\big)\wedge\rho+ \\
&\quad + V_2'''\,\sigma\wedge\rho+\overline{V_2}'''\,\overline{\sigma}\wedge\rho+V_8'''\,\rho\wedge\zeta+\overline{V_8}'''\,\rho\wedge\overline{\zeta}+i\,\zeta\wedge\overline{\zeta}, \\
d\zeta &= \beta\wedge\zeta+ \\
&\quad + W_3'''\,\sigma\wedge\zeta+W_6'''\,\overline{\sigma}\wedge\zeta+W_8'''\,\rho\wedge\zeta+W_{10}'''\,\zeta\wedge\overline{\zeta},
\end{aligned}$$

which, after the replacement:

$$\beta\longmapsto\beta+p\,\sigma+q\,\overline{\sigma}+r\,\rho+s\,\zeta+t\,\overline{\zeta}$$

take the form:

$$\begin{aligned}
d\sigma &= \big(2\,\beta+\overline{\beta}\big)\wedge\sigma+ \\
&\quad + \big(U_1'''-\overline{p}-2\,q\big)\,\sigma\wedge\overline{\sigma}+\big(U_2'''-2\,r-\overline{r}\big)\,\sigma\wedge\rho+\big(U_3'''-2\,s-\overline{t}\big)\,\sigma\wedge\zeta+\big(U_4'''-2\,t-\overline{s}\big)\,\sigma\wedge\overline{\zeta}+\rho\wedge\zeta, \\
d\rho &= \big(\beta+\overline{\beta}\big)\wedge\rho+ \\
&\quad + \big(V_2'''+p+\overline{q}\big)\,\sigma\wedge\rho+\big(\overline{V_2}'''+\overline{p}+q\big)\,\overline{\sigma}\wedge\rho+\big(V_8'''-s-\overline{t}\big)\,\rho\wedge\zeta+\big(\overline{V_8}'''-\overline{s}-t\big)\,\rho\wedge\overline{\zeta}+i\,\zeta\wedge\overline{\zeta}, \\
d\zeta &= \beta\wedge\zeta+ \\
&\quad + \big(W_3'''+p\big)\,\sigma\wedge\zeta+\big(W_6'''+q\big)\,\overline{\sigma}\wedge\zeta+\big(W_8'''+r\big)\,\rho\wedge\zeta+\big(W_{10}'''-t\big)\,\zeta\wedge\overline{\zeta}.
\end{aligned}$$

Taking account of the vanishing expressions of the previous subsections, one verifies that it is possible to annihilate all the above coefficients by determining:

$$p:=-W_3''',\quad q:=-W_6''',\quad r:=-W_8''',\quad s:=V_8'''-\overline{W}_{10}''',\quad t:=W_{10}'''.$$

Then, putting:

$$\beta:=\frac{d\mathsf{a}}{\mathsf{a}}+W_3'''\,\sigma+W_6'''\,\overline{\sigma}+W_8'''\,\rho+\big(\overline{W}_{10}'''-V_8'''\big)\,\zeta-W_{10}'''\,\overline{\zeta}, \tag{111}$$

the structure equations convert into the form:

$$\begin{aligned}
d\sigma &= \big(2\,\beta+\overline{\beta}\big)\wedge\sigma+\rho\wedge\zeta, \\
d\rho &= \big(\beta+\overline{\beta}\big)\wedge\rho+i\,\zeta\wedge\overline{\zeta}, \\
d\zeta &= \beta\wedge\zeta.
\end{aligned} \tag{112}$$

Before starting the prolongation procedure similarly to the procedure we performed in subsection 12.13, one should observe the non-involutiveness of the above structure equations. Since all of the coefficients p,q,r,s,t were determined, there remains no free variable, hence the above modified β is the *unique* 1-form enjoying the structure equations. According to Proposition 12.11, we can therefore transform the $G^{\approx}$-structure equivalence problem of the 5-dimensional base manifolds $M^5 \subset \mathbb{C}^4$ to the $\{e\}$-*structure* problem on the 7-dimensional prolonged spaces $M^5 \times G^{\approx} \subset \mathbb{C}^5 := \mathbb{C}\{z, w_1, w_2, w_3, \mathsf{a}\}$ with $G^{\approx}$ as follows:

$$G^{\approx} := \left\{ g^{\approx} := \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}^2 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathsf{a}} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix} : \mathsf{a} \in \mathbb{C} \right\}.$$

In this case, the lifted coframe $\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}\}$ will be extended by the two 1-forms β and $\overline{\beta}$. Hence, to find the structure equations of the new equivalence problem, it suffices to extend the structure equations (112) by computing the exterior derivation $d\beta$ of β in (111) in terms of these 7 lifted 1-forms.

To do this, first let us compute $d\beta$ directly, taking account of Lemma 16.1:

(113)

$$\begin{aligned} d\beta = & \underline{d\Big(\frac{d\mathsf{a}}{\mathsf{a}}\Big)}_{\circ} + W_3''' \, d\sigma + W_6''' \, d\overline{\sigma} + W_8''' \, d\rho + \big(\overline{W}_{10}''' - V_8'''\big) \, d\zeta - W_{10}''' \, d\overline{\zeta} + \\ & + \Big(\mathscr{S}(W_3''') \, \sigma_0 + \overline{\mathscr{S}}(W_3''') \, \overline{\sigma}_0 + \mathscr{T}(W_3''') \, \rho_0 + \mathscr{L}(W_3''') \, \zeta_0 + \overline{\mathscr{L}}(W_3''') \, \overline{\zeta}_0 \Big) \wedge \sigma + \\ & + \Big(\mathscr{S}(W_6''') \, \sigma_0 + \overline{\mathscr{S}}(W_6''') \, \overline{\sigma}_0 + \mathscr{T}(W_6''') \, \rho_0 + \mathscr{L}(W_6''') \, \zeta_0 + \overline{\mathscr{L}}(W_6''') \, \overline{\zeta}_0 \Big) \wedge \overline{\sigma} + \\ & + \Big(\mathscr{S}(W_8''') \, \sigma_0 + \overline{\mathscr{S}}(W_8''') \, \overline{\sigma}_0 + \mathscr{T}(W_8''') \, \rho_0 + \mathscr{L}(W_8''') \, \zeta_0 + \overline{\mathscr{L}}(W_8''') \, \overline{\zeta}_0 \Big) \wedge \rho + \\ & + \Big(\mathscr{S}(\overline{W}_{10}''' - V_8''') \, \sigma_0 + \overline{\mathscr{S}}(\overline{W}_{10}''' - V_8''') \, \overline{\sigma}_0 + \mathscr{T}(\overline{W}_{10}''' - V_8''') \, \rho_0 + \mathscr{L}(\overline{W}_{10}''' - V_8''') \, \zeta_0 + \overline{\mathscr{L}}(\overline{W}_{10}''' - V_8''') \, \overline{\zeta}_0 \Big) \wedge \zeta - \\ & - \Big(\mathscr{S}(W_{10}''') \, \sigma_0 + \overline{\mathscr{S}}(W_{10}''') \, \overline{\sigma}_0 + \mathscr{T}(W_{10}''') \, \rho_0 + \mathscr{L}(W_{10}''') \, \zeta_0 + \overline{\mathscr{L}}(W_{10}''') \, \overline{\zeta}_0 \Big) \wedge \overline{\zeta}. \end{aligned}$$

We need the following useful lemma:

Lemma 17.3. *The exterior derivation $d\beta$ of the 1-form β has the form:*

$$d\beta := T_1 \, \sigma \wedge \zeta + T_2 \, \overline{\sigma} \wedge \zeta + T_3 \, \rho \wedge \zeta + T_4 \, \zeta \wedge \overline{\zeta},$$

for some certain functions $T_1, \ldots, T_4$.

Proof. According to (113), the expression of $d\beta$ admits no variable a or 1-form $d\mathsf{a}$ in its expression. Hence, it will be independent of any wedge product of the form $\bullet \wedge \beta$ and $\bullet \wedge \overline{\beta}$. On the other hand, differentiating the expression of $d\zeta$ in (112) gives:

$$0 \equiv d\beta \wedge \zeta - \beta \wedge d\zeta = d\beta \wedge \zeta - \underline{\beta \wedge \beta \wedge \zeta}_{\circ},$$

which, according to Cartan's Lemma 12.5, implies that:

$$d\beta := \mathscr{F} \wedge \zeta$$

for some certain 1-form $\mathscr{F}$. Now, since seven 1-forms $\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}, \beta, \overline{\beta}$ constitute a basis for the set of all 1-forms on the prolonged space, $d\beta$ may be expressed as follows:

$$d\beta := T_1\, \sigma \wedge \zeta + T_2\, \overline{\sigma} \wedge \zeta + T_3\, \rho \wedge \zeta + T_4\, \zeta \wedge \overline{\zeta},$$

for some certain functions $T_1, \ldots, T_4$, as claimed. □

According to this lemma, to compute the exterior derivation $d\beta$, it suffices to compute only four coefficients $T_1, \ldots, T_4$ instead of computing all 21 coefficients of the possible wedge products between seven 1-forms $\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}, \beta, \overline{\beta}$ in (113). Extracting the coefficients of $\sigma \wedge \zeta, \overline{\sigma} \wedge \zeta, \rho \wedge \zeta, \overline{\zeta} \wedge \zeta$ in the expression (113) gives:

(114)

$$T_1 = -\frac{1}{\mathsf{a}}\mathscr{L}(W_3''') + \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\mathscr{S}(\overline{W}_{10}''' - V_8''') - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\mathscr{T}(\overline{W}_{10}''' - V_8''') + \frac{\mathsf{b}\mathsf{c} - \mathsf{a}\overline{\mathsf{a}}\mathsf{e}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,\mathscr{L}(\overline{W}_{10}''' - V_8''') + \frac{\overline{\mathsf{b}}\mathsf{c} - \mathsf{a}\overline{\mathsf{a}}\mathsf{d}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\overline{W}_{10}''' - V_8'''),$$
$$T_2 = -\frac{1}{\mathsf{a}}\mathscr{L}(W_6''') + \frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\overline{\mathscr{S}}(\overline{W}_{10}''' - V_8''') - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\mathscr{T}(\overline{W}_{10}''' - V_8''') + \frac{\mathsf{b}\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\overline{W}_{10}''' - V_8''') + \frac{\overline{\mathsf{b}}\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\,\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\overline{W}_{10}''' - V_8'''),$$
$$T_3 = W_3''' - \frac{1}{\mathsf{a}}\,\mathscr{L}(W_8''') + \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\mathscr{T}(\overline{W}_{10}''' - V_8''') - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\overline{W}_{10}''' - V_8''') - \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\overline{W}_{10}''' - V_8'''),$$
$$T_4 = i\,W_8''' - \frac{1}{\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\overline{W}_{10}''' - V_8''') - \frac{1}{\mathsf{a}}\,\mathscr{L}(W_{10}''').$$

Therefore, the $\{e\}$-structure equivalence problem on the prolonged space $M \times G^{\approx}$ enjoys the structure equations of the form:

$$(115)\qquad \begin{aligned} d\sigma &= \big(2\,\beta + \overline{\beta}\big) \wedge \sigma + \rho \wedge \zeta, \\ d\rho &= \big(\beta + \overline{\beta}\big) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \beta \wedge \zeta, \\ d\beta &= T_1\, \sigma \wedge \zeta + T_2\, \overline{\sigma} \wedge \zeta + T_3\, \rho \wedge \zeta + T_4\, \zeta \wedge \overline{\zeta}, \end{aligned}$$

with four essential invariants T_1, T_2, T_3, T_4 as above.

Theorem 17.3. *In the case that six functions V_1''', W_5''', W_9''', X_1''', W_1''' and W_2''' vanish identically (see* (101) *and* (104) *for their expressions), then the under consideration equivalence problem has the four essential invariants T_1, T_2, T_3 and T_4 as* (114)*, in terms of the five complex variables $z, w_1, w_2, w_3, \mathsf{a}$.*

In Section 12, we considered the equivalence problem of an arbitrary 5-dimensional CR-manifold M^5 to the cubic model M^5_{c}. In Theorem 12.2, we observed that such equivalency holds if and only if the structure equations associated to the lifted coframe $\{\sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}, \alpha, \overline{\alpha}\}$ of M^5 takes the structure equations as (88). Now, by a careful comparison between two structure equations (88) and (115) and thanks to the above theorem, one realizes that these two structure equations take the same form whenever the appearing invariants vanish, identically. Then we have:

Corollary 17.4. *An arbitrary 5-dimensional CR-manifold M^5 is equivalence, through some biholomorphism, to the cubic model M^5_{c} if and only if we have:*

$$
\begin{aligned}
&0 \equiv V_1''' = W_5''' = W_9''' = X_1''' = W_1''',\\
&0 \equiv T_1 = T_2 = T_3 = T_4.
\end{aligned}
$$

□

This corollary completes the procedure at the end of Section 12. On the other hand, in [6, Proposition 12], Beloshapka proved that the cubic model M^5_{c} is the most symmetric nondegenerate surface, *i.e.* the dimension of the symmetry group of each M^5 is not greater than that of M^5_{c}. Now, granted the above result one finds out that — *see* also Corollary 12.13 — :

Corollary 17.5. *If an arbitrary CR-manifold M^5 satisfies the assumptions of the above Corollary 17.4, then its symmetry group has the maximum dimension, equal to the dimension of the symmetry group of M^5_{c}.* □

18. THE BRANCH $R \not\equiv 0$

18.1. **Normalization of four group parameters.** After performing the long and complicated computations of the under consideration equivalence problem with the assumption $R \equiv 0$ in Sections 16 and 17, now we have somehow simpler computations to conclude the memoir by inspecting the equivalence problem in the case $R \not\equiv 0$. For this aim, we have to return to the subsection 15.2 of the first loop normalization of the structure equations (96):

$$
\begin{aligned}
d\sigma = &\big(2\,\alpha_1 + \overline{\alpha}_1\big)\wedge\sigma + \\
&+ U_1\,\sigma\wedge\overline{\sigma} + U_2\,\sigma\wedge\rho + U_3\,\sigma\wedge\zeta + U_4\,\sigma\wedge\overline{\zeta} + \\
&+ U_5\,\overline{\sigma}\wedge\rho + U_6\,\overline{\sigma}\wedge\zeta + U_7\,\overline{\sigma}\wedge\overline{\zeta} + \\
&+ \rho\wedge\zeta,
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
d\rho = &\,\alpha_2\wedge\sigma + \overline{\alpha}_2\wedge\overline{\sigma} + \alpha_1\wedge\rho + \overline{\alpha}_1\wedge\rho + \\
&+ V_1\,\sigma\wedge\overline{\sigma} + V_2\,\sigma\wedge\rho + V_3\,\sigma\wedge\zeta + V_4\,\sigma\wedge\overline{\zeta} + \\
&+ \overline{V_2}\,\overline{\sigma}\wedge\rho + \overline{V_4}\,\overline{\sigma}\wedge\zeta + \overline{V_3}\,\overline{\sigma}\wedge\overline{\zeta} + \\
&+ V_8\,\rho\wedge\zeta + \overline{V_8}\,\rho\wedge\overline{\zeta} + \\
&+ i\,\zeta\wedge\overline{\zeta},
\end{aligned}
$$

$$\begin{aligned} d\zeta = \alpha_3 \wedge \sigma + \alpha_4 \wedge \overline{\sigma} + \alpha_5 \wedge \rho + \alpha_1 \wedge \zeta + \\ + W_1\, \sigma \wedge \overline{\sigma} + W_2\, \sigma \wedge \rho + W_3\, \sigma \wedge \zeta + W_4\, \sigma \wedge \overline{\zeta} + \\ + W_5\, \overline{\sigma} \wedge \rho + W_6\, \overline{\sigma} \wedge \zeta + W_7\, \overline{\sigma} \wedge \overline{\zeta} + \\ + W_8\, \rho \wedge \zeta + W_9\, \rho \wedge \overline{\zeta} + \\ + W_{10}\, \zeta \wedge \overline{\zeta}. \end{aligned}$$

In this early normalization step, we found the following four[6] normalizable expressions in (98):

$$\begin{aligned} U_5 &= \frac{1}{\overline{\mathsf{a}}^2}\, \overline{G} - \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}^2}\, B - \frac{\overline{\mathsf{b}}}{\overline{\mathsf{a}}^3}\, \overline{R} + \frac{\overline{\mathsf{d}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\ U_6 &= \frac{1}{\overline{\mathsf{a}}}\, B - \frac{\overline{\mathsf{c}}}{\mathsf{a}\overline{\mathsf{a}}^2}, \\ U_7 &= \frac{\mathsf{a}}{\overline{\mathsf{a}}^2}\, \overline{R}, \\ U_3 + \overline{U}_4 - 3\, V_8 &= \frac{1}{\mathsf{a}}\, Q - 4\, \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{1}{\mathsf{a}}\, \overline{B} - 3i\, \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}}. \end{aligned} \tag{117}$$

Normalizing these expressions, this time with the assumption $R \not\equiv 0$, enables us to determine the first group parameter a besides b, c and d (*cf.* subsection 16.1). Normalizing the third expression to 1 and the remaining ones to 0, and taking account of the expression of G in Lemma 13.4, one obtains:

$$\boxed{\begin{aligned} &\mathsf{a} := \mathbf{A}_0, \\ &\mathsf{b} := \mathbf{A}_0 \left(- i\, B + \tfrac{i}{3}\, \overline{Q} \right), \\ &\mathsf{c} := \mathbf{A}_0\, \overline{\mathbf{A}}_0\, B, \\ &\mathsf{d} := \overline{\mathbf{A}}_0 \left(i\, \overline{\mathscr{L}}(R) - i\, \mathscr{L}(\overline{B}) + \tfrac{4i}{3}\, \overline{Q} R + iP + \tfrac{2i}{3}\, \overline{B} Q - 2i\, BR \right), \end{aligned}} \tag{118}$$

where $\mathbf{A}_0$ is a nonzero complex function satisfying $\frac{\mathbf{A}_0^2}{\overline{\mathbf{A}}_0} = R$.

18.2. **Changing up the initial coframe.** After determining four of the five group parameters in the previous section, now the equivalence problem takes the form:

$$\left[\begin{aligned} \sigma &= \mathbf{A}_0^2 \overline{\mathbf{A}} \cdot \sigma_0, \\ \rho &= \rho_0^\dagger, \\ \zeta &= \mathsf{e} \cdot \sigma_0 + \zeta_0^\dagger, \end{aligned} \right.$$

[6]In fact they were five but the second and the fifth ones were conjugate of each other.

where the two new modified initial 1-forms are:

$$\rho_0^\dagger = \mathbf{A}_0\overline{\mathbf{A}}_0\big(\overline{B}\sigma_0 + B\overline{\sigma}_0 + \rho_0\big),$$
$$\zeta_0^\dagger = \mathbf{A}_0\Big[\big(-i\,\mathscr{L}(\overline{R}) + i\,\overline{\mathscr{L}}(B) - \tfrac{4i}{3}\,Q\overline{R} - i\overline{P} - \tfrac{2i}{3}\,B\overline{Q} + 2i\,\overline{B}\,\overline{R}\big)\overline{\sigma}_0 + $$
$$+\big(-i\,B + \tfrac{i}{3}\,\overline{Q}\big)\sigma_0 + \zeta_0\Big].$$

Under this new initial coframe $(\overline{\sigma}_0, \sigma_0, \rho_0^\dagger, \overline{\zeta}_0^\dagger, \zeta_0^\dagger)^t$, the ambiguity group reduces to the form:

$$G^\dagger := \left\{ g^\dagger = \begin{pmatrix} \mathbf{A}_0\overline{\mathbf{A}}_0^2 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}_0^2\overline{\mathbf{A}}_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \overline{\mathsf{e}} & 0 & 0 & 1 & 0 \\ 0 & \mathsf{e} & 0 & 0 & 1 \end{pmatrix} : \mathsf{e} \in \mathbb{C} \right\},$$

and the new Maurer-Cartan form is:

$$dg^\dagger \cdot {g^\dagger}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \overline{\gamma} & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 \end{pmatrix},$$

where:

$$\gamma := \frac{d\mathsf{e}}{\mathbf{A}_0^2\overline{\mathbf{A}}_0}.$$

18.3. **Second-loop absorbtion and normalization.** Similar to what we did in subsection 16.3, let us introduce the notations:

$$\begin{aligned} &\mathsf{a} := \mathbf{A}_0, && \text{where:} \quad \frac{\mathbf{A}_0^2}{\overline{\mathbf{A}}_0} = R, \\ &\mathsf{b} := \mathbf{A}_0\,\mathbf{B}_0', && \text{where:} \quad \mathbf{B}_0' := -i\,B + \tfrac{i}{3}\,\overline{Q}, \\ &\mathsf{c} := \mathbf{A}_0\overline{\mathbf{A}}_0\,\mathbf{C}_0', && \text{where:} \quad \mathbf{C}_0' := \overline{B}, \\ &\mathsf{d} := \overline{\mathbf{A}}_0\,\mathbf{D}_0', && \text{where:} \quad \mathbf{D}_0' := i\,\overline{\mathscr{L}}\big(R\big) - i\,\mathscr{L}\big(\overline{B}\big) + \\ & && \qquad\qquad + \tfrac{4i}{3}\,\overline{Q}R + i\,P + \tfrac{2i}{3}\,\overline{B}\,Q - 2i\,BR. \end{aligned}$$

In this case, the new lifted coframe takes the form:

$$\begin{aligned} \sigma &= \mathbf{A}_0^2\overline{\mathbf{A}}_0 \cdot \sigma_0, \\ \rho &= \mathbf{A}_0\overline{\mathbf{A}}_0 \cdot \big(\mathbf{C}_0'\,\sigma_0 + \overline{\mathbf{C}}_0'\,\overline{\sigma}_0 + \rho_0\big), \\ \zeta &= \mathsf{e}\cdot\sigma_0 + \mathbf{A}_0 \cdot \big(\overline{\mathbf{D}}_0'\,\overline{\sigma}_0 + \mathbf{B}_0'\,\rho_0 + \zeta_0\big). \end{aligned}$$

To proceed with the Cartan's method, one has to consider the exterior differentiation of these expressions:

$$\begin{aligned} d\sigma &= \mathbf{A}_0^2\overline{\mathbf{A}}_0 \cdot d\overline{\sigma}_0 + \\ &\quad + 2\,\mathbf{A}_0\overline{\mathbf{A}}_0\,d\mathbf{A}_0 \wedge \sigma_0 + \mathbf{A}_0^2\,d\overline{\mathbf{A}}_0 \wedge \sigma_0, \\ d\rho &= \mathbf{A}_0\overline{\mathbf{A}}_0 \cdot \big(\mathbf{C}_0'\,d\sigma_0 + \overline{\mathbf{C}}_0'\,d\overline{\sigma}_0 + d\rho_0\big) + \\ &\quad + \big(\mathbf{A}_0\overline{\mathbf{A}}_0\,d\mathbf{C}_0' + \mathbf{A}_0\mathbf{C}_0'\,d\overline{\mathbf{A}}_0 + \overline{\mathbf{A}}_0\mathbf{C}_0'\,d\mathbf{A}_0\big) \wedge \sigma_0 + \\ &\quad + \big(\mathbf{A}_0\overline{\mathbf{A}}_0\,d\overline{\mathbf{C}}_0' + \mathbf{A}_0\overline{\mathbf{C}}_0'\,d\overline{\mathbf{A}}_0 + \overline{\mathbf{A}}_0\,\overline{\mathbf{C}}_0'\,d\mathbf{A}_0\big) \wedge \overline{\sigma}_0 + \\ &\quad + \big(\overline{\mathbf{A}}_0\,d\mathbf{A}_0 + \mathbf{A}_0\,d\overline{\mathbf{A}}_0\big) \wedge \rho_0, \\ d\zeta &= \gamma \wedge \sigma + \\ &\quad + \mathsf{e} \cdot d\sigma_0 + \mathbf{A}_0 \cdot \big(\overline{\mathbf{D}}_0'\,d\overline{\sigma}_0 + \mathbf{B}_0'\,d\rho_0 + d\zeta_0\big) + \\ &\quad + \big(\mathbf{A}_0\,d\overline{\mathbf{D}}_0' + \overline{\mathbf{D}}_0'\,d\mathbf{A}_0\big) \wedge \overline{\sigma}_0 + \big(\mathbf{A}_0\,d\mathbf{B}_0' + \mathbf{B}_0'\,d\mathbf{A}_0\big) \wedge \rho_0 + d\mathbf{A}_0 \wedge \zeta_0. \end{aligned}$$

Reading these new structure equations in terms of the lifted coframe (*cf.* subsection 16.3) gives the following modified structure equations:

(119)

$$\begin{aligned} d\sigma &= U_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + U_2^{\mathsf{new}}\,\sigma\wedge\rho + U_3^{\mathsf{new}}\,\sigma\wedge\zeta + U_4^{\mathsf{new}}\,\sigma\wedge\overline{\zeta} + \overline{\sigma}\wedge\overline{\zeta} + \rho\wedge\zeta, \\ d\rho &= V_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + V_2^{\mathsf{new}}\,\sigma\wedge\rho + V_3^{\mathsf{new}}\,\sigma\wedge\zeta + V_4^{\mathsf{new}}\,\sigma\wedge\overline{\zeta} + \\ &\qquad + \overline{V}_2^{\mathsf{new}}\,\overline{\sigma}\wedge\rho + \overline{V}_4^{\mathsf{new}}\,\overline{\sigma}\wedge\zeta + \overline{V}_3^{\mathsf{new}}\,\overline{\sigma}\wedge\overline{\zeta} + \\ &\qquad\quad + V_8^{\mathsf{new}}\,\rho\wedge\zeta + \overline{V}_8^{\mathsf{new}}\,\rho\wedge\overline{\zeta} + \\ &\qquad\qquad + i\,\zeta\wedge\overline{\zeta}, \\ d\zeta &= \gamma\wedge\sigma + \\ &\quad + W_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + W_2^{\mathsf{new}}\,\sigma\wedge\rho + W_3^{\mathsf{new}}\,\sigma\wedge\zeta + W_4^{\mathsf{new}}\,\sigma\wedge\overline{\zeta} + \\ &\qquad + W_5^{\mathsf{new}}\,\overline{\sigma}\wedge\rho + W_6^{\mathsf{new}}\,\overline{\sigma}\wedge\zeta + W_7^{\mathsf{new}}\,\overline{\sigma}\wedge\overline{\zeta} + \\ &\qquad\quad + W_8^{\mathsf{new}}\,\rho\wedge\zeta + W_9^{\mathsf{new}}\,\rho\wedge\overline{\zeta} + \\ &\qquad\qquad + W_{10}^{\mathsf{new}}\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

with the new extended torsion coefficients which can be presented as follows after a large amount of simplification[7]:

$$\begin{aligned} U_1^{\mathsf{new}} &:= U_1 - 2\,\frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{\mathsf{a}^2\overline{\mathsf{a}}}\left(\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{S}}(\mathbf{A}_0) - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{T}(\mathbf{A}_0) + \frac{\mathsf{b}\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{A}_0) + \frac{\overline{\mathsf{b}}\,\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\,\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right) - \\ &\quad - \frac{\mathbf{A}_0^2}{\mathsf{a}^2\overline{\mathsf{a}}}\left(\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{S}}(\overline{\mathbf{A}}_0) - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{T}(\overline{\mathbf{A}}_0) + \frac{\mathsf{b}\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\overline{\mathbf{A}}_0) + \frac{\overline{\mathsf{b}}\,\overline{\mathsf{c}} - \mathsf{a}\overline{\mathsf{a}}\,\overline{\mathsf{e}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right), \end{aligned}$$

[7] Of course here we have $\mathsf{a} = \mathbf{A}_0$, but in these expressions we still keep the notation a to better emphasize their appearance in denominators.

$$
\begin{aligned}
U_2^{\mathsf{new}} &:= U_2 - 2\,\frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^2\bar{a}}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\mathbf{A}_0) - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\mathbf{A}_0) - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right) - \\
&\quad - \frac{\mathbf{A}_0^2}{a^2\bar{a}}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\overline{\mathbf{A}}_0) - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\overline{\mathbf{A}}_0) - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)\right),
\end{aligned}
$$

$$
U_3^{\mathsf{new}} := U_3 - 2\,\frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^3\bar{a}}\,\mathscr{L}(\mathbf{A}_0) - \frac{\mathbf{A}_0^2}{a^3\bar{a}}\,\mathscr{L}(\overline{\mathbf{A}}_0),
$$

$$
U_4^{\mathsf{new}} := U_4 - 2\,\frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^2\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{A}_0) - \frac{\mathbf{A}_0^2}{a^2\bar{a}^2}\,\overline{\mathscr{L}}(\overline{\mathbf{A}}_0),
$$

$$
\begin{aligned}
V_1^{\mathsf{new}} &:= V_1 - \frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^2\bar{a}}\left(\frac{1}{a\bar{a}^2}\,\overline{\mathscr{S}}(\mathbf{C}_0') - \frac{\bar{c}}{a^2\bar{a}^3}\,\mathscr{T}(\mathbf{C}_0') + \frac{b\bar{c} - a\bar{a}\,\bar{d}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{C}_0') + \frac{\bar{b}\,\bar{c} - a\bar{a}\,\bar{e}}{a^2\bar{a}^4}\,\overline{\mathscr{L}}(\mathbf{C}_0')\right) + \\
&\quad + \frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a\bar{a}^2}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\overline{\mathbf{C}}_0') - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\overline{\mathbf{C}}_0') + \frac{bc - a\bar{a}\,e}{a^4\bar{a}^2}\,\mathscr{L}(\overline{\mathbf{C}}_0') + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{C}}_0')\right),
\end{aligned}
$$

$$
\begin{aligned}
V_2^{\mathsf{new}} &:= V_2 - \frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^2\bar{a}}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\mathbf{C}_0') - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\mathbf{C}_0') - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{C}_0')\right) + \\
&\quad + \frac{\overline{\mathbf{A}}_0}{a\bar{a}}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\mathbf{A}_0) - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\mathbf{A}_0) + \frac{bc - a\bar{a}e}{a^4\bar{a}^2}\,\mathscr{L}(\mathbf{A}_0) + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right) + \\
&\quad + \frac{\mathbf{A}_0}{a\bar{a}}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\overline{\mathbf{A}}_0) - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\overline{\mathbf{A}}_0) + \frac{bc - a\bar{a}e}{a^4\bar{a}^2}\,\mathscr{L}(\overline{\mathbf{A}}_0) + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)\right),
\end{aligned}
$$

$$
V_3^{\mathsf{new}} := V_3 - \frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^3\bar{a}}\,\mathscr{L}(\mathbf{C}_0'),
$$

$$
V_4^{\mathsf{new}} := V_4 - \frac{\mathbf{A}_0\overline{\mathbf{A}}_0}{a^2\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{C}_0'),
$$

$$
\begin{aligned}
W_1^{\mathsf{new}} &:= W_1 + \frac{\mathbf{A}_0}{a\bar{a}^2}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\overline{\mathbf{D}}_0') - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\overline{\mathbf{D}}_0') + \frac{bc - a\bar{a}e}{a^4\bar{a}^2}\,\mathscr{L}(\overline{\mathbf{D}}_0') + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0')\right) - \\
&\quad - \frac{\mathbf{A}_0\bar{c}}{a^2\bar{a}^3}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\mathbf{B}_0') - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\mathbf{B}_0') + \frac{bc - a\bar{a}e}{a^4\bar{a}^2}\,\mathscr{L}(\mathbf{B}_0') + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\mathbf{B}_0')\right) + \\
&\quad + \frac{\mathbf{A}_0 c}{a^3\bar{a}^2}\left(\frac{1}{a\bar{a}^2}\,\overline{\mathscr{S}}(\mathbf{B}_0') - \frac{\bar{c}}{a^2\bar{a}^3}\,\mathscr{T}(\mathbf{B}_0') + \frac{b\bar{c} - a\bar{a}\,\bar{d}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{B}_0') + \frac{\bar{b}\,\bar{c} - a\bar{a}\,\bar{e}}{a^2\bar{a}^4}\,\overline{\mathscr{L}}(\mathbf{B}_0')\right) + \\
&\quad + \frac{e}{a^3\bar{a}}\left(\frac{1}{a\bar{a}^2}\,\overline{\mathscr{S}}(\mathbf{A}_0) - \frac{\bar{c}}{a^2\bar{a}^3}\,\mathscr{T}(\mathbf{A}_0) + \frac{b\bar{c} - a\bar{a}\,\bar{d}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{A}_0) + \frac{\bar{b}\,\bar{c} - a\bar{a}\,\bar{e}}{a^2\bar{a}^4}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right),
\end{aligned}
$$

$$
\begin{aligned}
W_2^{\mathsf{new}} &:= W_2 + \frac{\mathbf{A}_0}{a\bar{a}}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\mathbf{B}_0') - \frac{e}{a^3\bar{a}}\,\mathscr{L}(\mathbf{B}_0') - \frac{d}{a^2\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{B}_0')\right) + \\
&\quad + \frac{e}{a^3\bar{a}}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\mathbf{A}_0) - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\mathbf{A}_0) - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right),
\end{aligned}
$$

$$
W_3^{\mathsf{new}} := W_3 + \frac{\mathbf{A}_0 c}{a^4\bar{a}^2}\,\mathscr{L}(\mathbf{B}_0') + \frac{1}{a}\left(\frac{1}{a^2\bar{a}}\,\mathscr{S}(\mathbf{A}_0) - \frac{c}{a^3\bar{a}^2}\,\mathscr{T}(\mathbf{A}_0) + \frac{bc}{a^4\bar{a}^2}\,\mathscr{L}(\mathbf{A}_0) + \frac{\bar{b}c - a\bar{a}d}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right),
$$

$$
W_4^{\mathsf{new}} := W_4 + \frac{\mathbf{A}_0 c}{a^3\bar{a}^3}\,\overline{\mathscr{L}}(\mathbf{B}_0') + \frac{e}{a^3\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{A}_0),
$$

$$
\begin{aligned}
W_5^{\mathsf{new}} &:= W_5 + \frac{\mathbf{A}_0}{a\bar{a}}\left(\frac{1}{a\bar{a}^2}\,\overline{\mathscr{S}}(\mathbf{B}_0') - \frac{\bar{c}}{a^2\bar{a}^3}\,\mathscr{T}(\mathbf{B}_0') + \frac{b\bar{c} - a\bar{a}\bar{d}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{B}_0') + \frac{\bar{b}\,\bar{c} - a\bar{a}\,\bar{e}}{a^2\bar{a}^4}\,\overline{\mathscr{L}}(\mathbf{B}_0')\right) - \\
&\quad - \frac{\mathbf{A}_0}{a\bar{a}^2}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\overline{\mathbf{D}}_0') - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\overline{\mathbf{D}}_0') - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0')\right) + \frac{\mathbf{A}_0\bar{c}}{a^2\bar{a}^3}\left(\frac{1}{a\bar{a}}\,\mathscr{T}(\mathbf{B}_0') - \frac{b}{a^2\bar{a}}\,\mathscr{L}(\mathbf{B}_0') - \frac{\bar{b}}{a\bar{a}^2}\,\overline{\mathscr{L}}(\mathbf{B}_0')\right),
\end{aligned}
$$

$$
\begin{aligned}
W_6^{\mathsf{new}} &:= W_6 + \frac{1}{a}\left(\frac{1}{a\bar{a}^2}\,\overline{\mathscr{S}}(\mathbf{A}_0) - \frac{\bar{c}}{a^2\bar{a}^3}\,\mathscr{T}(\mathbf{A}_0) + \frac{b\bar{c} - a\bar{a}\bar{d}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{A}_0) + \frac{\bar{b}\,\bar{c} - a\bar{a}\,\bar{e}}{a^2\bar{a}^4}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right) - \\
&\quad - \frac{\mathbf{A}_0}{a^2\bar{a}^2}\,\mathscr{L}(\overline{\mathbf{D}}_0') + \frac{\mathbf{A}_0\bar{c}}{a^3\bar{a}^3}\,\mathscr{L}(\mathbf{B}_0'),
\end{aligned}
$$

$$W_7^{\mathsf{new}} := W_7 - \frac{\mathbf{A}_0}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\overline{\mathbf{D}}_0') - \frac{\overline{\mathbf{D}}_0'}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{A}_0) + \frac{\mathbf{A}_0\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{B}_0'),$$

$$W_8^{\mathsf{new}} := W_8 - \frac{\mathbf{A}_0}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{B}_0') + \frac{1}{\mathsf{a}}\left(\frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\mathscr{T}(\mathbf{A}_0) - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{A}_0) - \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{A}_0)\right),$$

$$W_9^{\mathsf{new}} := W_9 - \frac{\mathbf{A}_0}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{B}_0'),$$

$$W_{10}^{\mathsf{new}} := W_{10} - \frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\mathbf{A}_0).$$

After computing the above preparatory equations, we are now ready to apply the second normalization-absorbtion step. Replacing the single Maurer-Cartan form γ by:

$$\gamma \longmapsto \gamma + p\,\sigma + q\,\overline{\sigma} + r\,\rho + s\,\zeta + t\,\overline{\zeta} \tag{120}$$

has no effect on the two first expressions $d\sigma$ and $d\rho$ of (119), but it changes the third one as follows:

$$\begin{aligned}
d\zeta = \gamma\wedge\sigma + &\\
&+ \big(W_1^{\mathsf{new}} - q\big)\,\sigma\wedge\overline{\sigma} + \big(W_2^{\mathsf{new}} - r\big)\,\sigma\wedge\rho + \big(W_3^{\mathsf{new}} - s\big)\,\sigma\wedge\zeta + \big(W_4^{\mathsf{new}} - t\big)\,\sigma\wedge\overline{\zeta} + \\
&\qquad + W_5^{\mathsf{new}}\,\overline{\sigma}\wedge\rho + W_6^{\mathsf{new}}\,\overline{\sigma}\wedge\zeta + W_7^{\mathsf{new}}\,\overline{\sigma}\wedge\overline{\zeta} + \\
&\qquad\qquad + W_8^{\mathsf{new}}\,\rho\wedge\zeta + W_9^{\mathsf{new}}\,\rho\wedge\overline{\zeta} + \\
&\qquad\qquad\qquad + W_{10}^{\mathsf{new}}\,\zeta\wedge\overline{\zeta}.
\end{aligned}$$

Visibly, one can annihilate all the coefficients appearing at the second line of this expression by putting:

$$p = 0, \quad q = W_1^{\mathsf{new}}, \quad r = W_2^{\mathsf{new}}, \quad s = W_3^{\mathsf{new}}, \quad t = W_4^{\mathsf{new}}.$$

In other words, we can annihilate the second line of the above expression by modifying the single Maurer-Cartan 1-form into the form:

$$\gamma \mapsto \gamma - W_1^{\mathsf{new}}\,\overline{\sigma} - W_2^{\mathsf{new}}\,\rho - W_3^{\mathsf{new}}\,\zeta - W_4^{\mathsf{new}}\,\overline{\zeta}.$$

Therefore, we find 15 normalizable expressions:

$$\left[\begin{aligned}
&0 \equiv U_1^{\mathsf{new}} = U_2^{\mathsf{new}} = U_3^{\mathsf{new}} = U_4^{\mathsf{new}},\\
&0 \equiv V_1^{\mathsf{new}} = V_2^{\mathsf{new}} = V_3^{\mathsf{new}} = V_4^{\mathsf{new}} = V_8^{\mathsf{new}},\\
&0 \equiv W_5^{\mathsf{new}} = W_6^{\mathsf{new}} = W_7^{\mathsf{new}} = W_8^{\mathsf{new}} = W_9^{\mathsf{new}} = W_{10}^{\mathsf{new}}.
\end{aligned}\right. \tag{121}$$

Many of these torsion coefficients include the as yet undetermined parameter e, but as in the branch $R = 0$, let us plainly employ V_4^{new} to normalize e. Of course, we quite similarly have:

$$V_4^{\mathsf{new}} = -i\,\frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{1}{3\mathsf{a}\overline{\mathsf{a}}}\Big(6\,B\overline{B} + 3\,A - \overline{B}\,\overline{Q} - 3\,\overline{\mathscr{L}}(\overline{B})\Big).$$

Normalizing this expression to 0 determines the expression of the last group parameter e as:

$$\boxed{\mathsf{e} = -\tfrac{i}{3}\,\mathbf{A}_0\,\big(6\,B\overline{B} - 3\,\overline{\mathscr{L}}(\overline{B}) + 3\,A - \overline{B}\,\overline{Q}\big).}$$

After determining the last group parameter e in such a way, the remaining fourteen normalizable expressions change into the form:

$$
\begin{aligned}
U_1^{\text{new}} = {} & \tfrac{i}{6\,\mathbf{A}_0^2\overline{\mathbf{A}}^3}\big(12\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0))) - 24\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0))) + 12\,\overline{\mathbf{A}}_0\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathbf{A}_0))) + 6\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathbf{A}}_0)))- \\
& - 12\,\mathbf{A}_0\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0))) + 6\,\mathbf{A}_0\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0))) - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\mathscr{L}(\mathscr{L}(\overline{R})) - 2\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) + 6\,\mathbf{A}_0\overline{\mathbf{A}}_0\mathscr{L}(\overline{\mathscr{L}}(B))+ \\
& + 12\,\overline{\mathbf{A}}_0B\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) - 12\overline{\mathbf{A}}_0B\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0)) + 6\,\mathbf{A}_0B\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)) - 6\,\mathbf{A}_0B\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathbf{A}}_0)) - 8a\overline{\mathbf{A}}_0\mathscr{L}(\overline{P}) + 6\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{\mathscr{L}}(A)- \\
& - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0R\overline{\mathscr{L}}(\overline{R}) + 2\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{\mathscr{L}}(Q) - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{R}\mathscr{L}(Q) - 4\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{Q}\mathscr{L}(B) + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{B}\mathscr{L}(\overline{R}) - 2\,\mathbf{A}_0\overline{\mathbf{A}}_0Q\overline{\mathscr{L}}(B)- \\
& - 8\,\mathbf{A}_0\overline{\mathbf{A}}_0B\mathscr{L}(\overline{Q}) + 6\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{\mathscr{L}}(\overline{B}) + 12\,\mathbf{A}_0\overline{\mathbf{A}}_0B\mathscr{L}(B) - 12\overline{\mathbf{A}}_0B\overline{Q}\mathscr{L}(\mathbf{A})_0 - 16\overline{\mathbf{A}}_0Q\overline{R}\mathscr{L}(\mathbf{A}_0) + 24\,\overline{\mathbf{A}}_0\overline{B}R\mathscr{L}(\mathbf{A}_0)+ \\
& + 12\,B\overline{B}\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\mathbf{A}_0) - 6\,\mathbf{A}_0B\overline{Q}\mathscr{L}(\overline{\mathbf{A}}_0) - 8\,\mathbf{A}_0Q\overline{R}\mathscr{L}(\overline{\mathbf{A}}_0) + 12\,\mathbf{A}_0\overline{B}\,\overline{R}\mathscr{L}(\overline{\mathbf{A}}_0) + 6\,\mathbf{A}_0B\overline{B}\mathscr{L}(\overline{\mathbf{A}}_0) + 12\,\overline{\mathbf{A}}_0B^2\mathscr{L}(\mathbf{A}_0)- \\
& - 12\,\overline{\mathbf{A}}_0\mathscr{L}(\mathbf{A}_0)\mathscr{L}(\overline{R}) + 12\,\overline{\mathbf{A}}_0\mathscr{L}(\mathbf{A}_0)\overline{\mathscr{L}}(B) - 12\,\overline{\mathbf{A}}_0\overline{P}\mathscr{L}(\mathbf{A}_0) - 12\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\mathbf{A}_0)\mathscr{L}(B) + 12\,\overline{\mathbf{A}}_0A\overline{\mathscr{L}}(\mathbf{A}_0) + 6\,\mathbf{A}_0B^2\mathscr{L}(\overline{\mathbf{A}}_0)- \\
& - 6\,\mathbf{A}_0\mathscr{L}(\overline{\mathbf{A}}_0)\mathscr{L}(\overline{R}) + 6\,\mathbf{A}_0\mathscr{L}(\overline{\mathbf{A}}_0)\overline{\mathscr{L}}(B) - 6\,\mathbf{A}_0\overline{P}\mathscr{L}(\overline{\mathbf{A}}_0) - 6\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)\mathscr{L}(B) + 6\,\mathbf{A}_0A\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 2\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{B}\overline{Q}- \\
& - 4\,\mathbf{A}_0\overline{\mathbf{A}}_0A\overline{Q} + 2\,\mathbf{A}_0\overline{\mathbf{A}}_0Q\overline{P} + 5\,\mathbf{A}_0\overline{\mathbf{A}}_0Q^2\overline{R} - 2\,\mathbf{A}_0\overline{\mathbf{A}}_0B^2Q + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{B}^2R + 6\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{B}P - 6\,\mathbf{A}_0\overline{\mathbf{A}}_0B^2\overline{B}- \\
& - 9\,\mathbf{A}_0\overline{\mathbf{A}}_0BR\overline{R} + 5\,\mathbf{A}_0\overline{\mathbf{A}}_0R\overline{R}\overline{Q} + 2\,\mathbf{A}_0\overline{\mathbf{A}}_0BQ\overline{Q} - 8\,\mathbf{A}_0\overline{\mathbf{A}}_0Q\overline{B}\,\overline{R}\big),
\end{aligned}
$$

$$
\begin{aligned}
U_2^{\text{new}} = {} & \tfrac{i}{3\,\mathbf{A}_0^2\overline{\mathbf{A}}_0^2}\big(-6\,\overline{\mathbf{A}}_0\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) + 6\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0)) - 3\,\mathbf{A}_0\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)) + 3\,\mathbf{A}_0\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathbf{A}}_0)) - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{\mathscr{L}}(Q)- \\
& - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0R\overline{R} + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\mathscr{L}(B) + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\overline{B}) - 6\,\overline{\mathbf{A}}_0B\mathscr{L}(\mathbf{A}_0) + 2\,\overline{\mathbf{A}}_0\overline{Q}\mathscr{L}(\mathbf{A}_0) + 6\,\overline{\mathbf{A}}_0\overline{B}\,\mathscr{L}(\mathbf{A}_0) - 2\,\overline{\mathbf{A}}_0Q\overline{\mathscr{L}}(\mathbf{A}_0)- \\
& - 3\,\mathbf{A}_0B\mathscr{L}(\overline{\mathbf{A}}_0) + \mathbf{A}_0\overline{Q}\mathscr{L}(\overline{\mathbf{A}}_0) + 3\,\mathbf{A}_0\overline{B}\mathscr{L}(\overline{\mathbf{A}}_0) - \mathbf{A}_0Q\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 4\,\mathbf{A}_0\overline{\mathbf{A}}_0BQ - \mathbf{A}_0\overline{\mathbf{A}}_0Q\overline{Q} - 6\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{B} + \mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{B}\,\overline{Q}\big),
\end{aligned}
$$

$$
U_3^{\text{new}} = 2\,\overline{U}_4^{\text{new}} - 3\,\overline{W}_{10}^{\text{new}},
$$

$$
U_4^{\text{new}} = \tfrac{1}{\mathbf{A}_0\overline{\mathbf{A}}_0^2}\big(-2\,\overline{\mathbf{A}}_0\overline{\mathscr{L}}(\mathbf{A}_0) - \mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + \mathbf{A}_0\overline{\mathbf{A}}_0B\big),
$$

$$
\begin{aligned}
V_1^{\text{new}} = {} & -\tfrac{i}{18\,\mathbf{A}_0^2\overline{\mathbf{A}}_0^2}\big(-18\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(B))) + 18\,\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{Q}))) - 36\,\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))) + 12\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{Q}))) + 18\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{B})))+ \\
& + 6\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q))) - 18\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) - 6\,\mathscr{L}(\mathscr{L}(\overline{P})) - 12\,B\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) + 9R\overline{\mathscr{L}}(\overline{P}) + 18\,\mathscr{L}(\overline{\mathscr{L}}(A)) + 18\,\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))- \\
& - 12\,\overline{\mathscr{L}}(\mathscr{L}(\overline{P})) + 30\,\overline{Q}\mathscr{L}(\overline{\mathscr{L}}(B)) + 18\,B\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) + 27\,B^3R + 9\,\overline{B}\mathscr{L}(\mathscr{L}(\overline{R})) - 24\,\overline{B}\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) + 18\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(B))+ \\
& + 6B\mathscr{L}(\mathscr{L}(\overline{Q})) + 9\,B\overline{\mathscr{L}}(\overline{\mathscr{L}}(R)) - 36\,B\overline{\mathscr{L}}(\mathscr{L}(\overline{B})) - 2\,\overline{Q}\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) - 36\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) - 24\,Q\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))+ \\
& + 12\,Q\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) + 24\,B^2\mathscr{L}(\overline{Q}) - 36\,B\mathscr{L}(\overline{\mathscr{L}}(B)) + 60\,\overline{B}\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) + 12\,\overline{Q}\,\overline{\mathscr{L}}(\mathscr{L}(\overline{B})) - 27\,\overline{B}P\overline{R} + 24\,\overline{\mathscr{L}}(B)\mathscr{L}(\overline{Q})+ \\
& + 18\,A\overline{\mathscr{L}}(B) - 6\,B\overline{B}\,\overline{\mathscr{L}}(\overline{Q}) + 6\,B\mathscr{L}(\overline{P}) + 9\overline{R}\mathscr{L}(P) + 18\,\overline{P}\,\overline{\mathscr{L}}(\overline{B}) + 18\,AR\overline{R} + 6\,A\mathscr{L}(\overline{Q}) + 42\,B\overline{\mathscr{L}}(P) + 18\,Q\overline{\mathscr{L}}(A)- \\
& - 6\,\overline{Q}\mathscr{L}(A) - 36\,\overline{B}\overline{\mathscr{L}}(A) + 36\,\overline{B}\mathscr{L}(\overline{P}) - 18\,\overline{B}^2\,\overline{P} - 18\,B^2P + 36\,P\overline{\mathscr{L}}(B) + 18\,\overline{P}\,\overline{\mathscr{L}}(R) + 18\,P\mathscr{L}(\overline{R}) - 18\,A\mathscr{L}(B)- \\
& - 18\,\overline{P}\mathscr{L}(\overline{Q}) - 14\,\overline{Q}\mathscr{L}(\overline{P}) - 54\,\overline{B}^2\,\overline{\mathscr{L}}(B) - 6\,\overline{Q}\,\overline{\mathscr{L}}(A) + 18\,B^2\mathscr{L}(\overline{B}) + 18\,B\overline{\mathscr{L}}(A) - 27\,\overline{B}^2\mathscr{L}(\overline{R}) - 27\,B^2\overline{\mathscr{L}}(R)- \\
& - 12\,B\overline{Q}\mathscr{L}(\overline{B}) + 48\,\overline{B}Q\overline{\mathscr{L}}(B) + 12\,B\overline{B}\mathscr{L}(\overline{Q}) - 90\,B\overline{B}\overline{\mathscr{L}}(\overline{B}) + 48\,B\overline{B}\,\overline{\mathscr{L}}(Q) - 36\,B\overline{B}\mathscr{L}(B) + 9\,\overline{B}R\overline{\mathscr{L}}(\overline{R}) + 9\,\overline{B}\,\overline{R}\mathscr{L}(Q)+ \\
& + 9\,BR\overline{\mathscr{L}}(\overline{Q}) + 48\,BQ\overline{\mathscr{L}}(\overline{B}) + 9\,B\overline{R}\mathscr{L}(R) + 36\,\overline{\mathscr{L}}(\overline{B})\mathscr{L}(B) + 24\,Q\overline{R}\overline{\mathscr{L}}(R) + 18\,B\overline{Q}\,\overline{\mathscr{L}}(R) - 36\,\overline{B}\,\overline{R}\,\overline{\mathscr{L}}(R)+ \\
& + 24\,\overline{Q}R\mathscr{L}(\overline{R}) + 18\,\overline{B}Q\mathscr{L}(\overline{R}) - 36\,BR\mathscr{L}(\overline{R}) + 18\,\overline{\mathscr{L}}(R)\mathscr{L}(\overline{R}) - 18\,\mathscr{L}(\overline{B})\overline{\mathscr{L}}(B) - 18\,\overline{\mathscr{L}}(B)\overline{\mathscr{L}}(\overline{B}) + 12B\overline{Q}\overline{\mathscr{L}}(\overline{B})- \\
& - 32\,B\overline{Q}\mathscr{L}(\overline{Q}) - 18\,\overline{Q}\,\overline{B}\,\overline{\mathscr{L}}(B) + 4\overline{Q}Q\overline{\mathscr{L}}(B) - 24\,B^2\overline{B}\,\overline{Q} + 36\,B\overline{B}\,\overline{\mathscr{L}}(B) + 60\,B\overline{Q}\mathscr{L}(B) - 36\,\mathscr{L}(B)\overline{\mathscr{L}}(B) + 18\,\overline{P}\mathscr{L}(B)- \\
& - 36\,B^2\mathscr{L}(B) - 16\,\overline{Q}^2\mathscr{L}(B) + 12\,\mathscr{L}(B)\overline{\mathscr{L}}(\overline{Q}) - 6\,A\overline{\mathscr{L}}(\overline{Q}) - 12\,Q\mathscr{L}(\overline{P}) - 6\,\overline{P}\overline{\mathscr{L}}(Q) - 4B\overline{Q}\,\overline{\mathscr{L}}(Q) + 24\,\overline{Q}\,\overline{B}\,\overline{\mathscr{L}}(\overline{B})- \\
& - 12\,Q\overline{Q}\,\overline{\mathscr{L}}(\overline{B}) - 12\,\overline{B}\,\overline{Q}\,\overline{\mathscr{L}}(Q) - 12\,\overline{B}Q\overline{\mathscr{L}}(\overline{Q}) - 12\,P\overline{\mathscr{L}}(\overline{Q}) - 12\,\overline{Q}\,\overline{\mathscr{L}}(P) - 12\,\overline{\mathscr{L}}(\overline{B})\mathscr{L}(\overline{Q}) + 12\,\overline{\mathscr{L}}(\overline{Q})\mathscr{L}(\overline{B})+ \\
& + 12\,\overline{B}^2\overline{\mathscr{L}}(\overline{Q}) - 36\,\overline{\mathscr{L}}(\overline{B})\overline{\mathscr{L}}(Q) + 36\,\overline{\mathscr{L}}(\overline{B})^2 + 9\,\overline{P}\,\overline{Q}R + 9\,PQ\overline{R} + 6\,BP\overline{Q} - 27\,BR\overline{P} + 6\,\overline{B}\,\overline{P}Q - 36\,AB\overline{B} + 27\,\overline{B}^3\,\overline{R}+ \\
& + 2\,\overline{Q}^2A - 36\,B^2R\overline{Q} - 12\,\overline{B}B^2Q - 36\,\overline{B}^2Q\overline{R} + 6\,\overline{B}^2B\overline{Q} + 9\,B\overline{Q}^2R + 12\,ABQ + 9\,\overline{B}Q^2\overline{R} + 30\,\overline{B}A\overline{Q} - 63\,\overline{B}\overline{R}\,\overline{Q}R+
\end{aligned}
$$

$$+\,126\,\overline{B}\,\overline{R}BR - 63\,BRQ\overline{R} + 32\,\overline{Q}RQ\overline{R} + 18\,B^3\overline{B} + 18AB^2 + 12\,\overline{Q}\,\overline{B}\,\overline{P} + 14\,\overline{B}B\overline{Q}^2 - 4\,\overline{Q}Q\overline{P}-$$

$$-\,4\,QB\overline{Q}^2 - 12\,AB\overline{Q} + 4\,B^2Q\overline{Q} - 12\,QA\overline{Q}\big),$$

$$V_2^{\mathsf{new}} = \tfrac{i}{9\,\mathbf{A}_0^3\overline{\mathbf{A}}_0^2}\big(-9\,\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\mathbf{A}_0))) - 9\,\mathbf{A}_0\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0))) + 18\,\mathbf{A}_0\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathbf{A}}_0))) - 9\,\mathbf{A}_0\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{\mathbf{A}}_0)))+$$

$$+\,9\,\overline{\mathbf{A}}_0\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) - 9\,\overline{\mathbf{A}}_0\,\overline{B}\,\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0)) + 9\,\mathbf{A}_0\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)) - 9\,\overline{\mathbf{A}}_0\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) - 9\,\mathbf{A}_0\overline{B}\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathbf{A}}_0))+$$

$$+\,3\,\mathbf{A}_0\overline{\mathbf{A}}_0\mathscr{L}(\mathscr{L}(\overline{Q})) + 18\,\overline{\mathbf{A}}_0\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0)) + 9\,\mathbf{A}_0P\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 9\,\overline{\mathbf{A}}_0\mathscr{L}(\mathbf{A}_0)\overline{\mathscr{L}}(\overline{B}) + 9\mathbf{A}_0^2\overline{B}\,\overline{\mathscr{L}}(\overline{B}) - 9\,\mathbf{A}_0\overline{\mathbf{A}}_0\mathscr{L}(A)-$$

$$-\,9\,\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0)\mathscr{L}(\overline{B}) + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(P) - 9\,\mathbf{A}_0A\mathscr{L}(\overline{\mathbf{A}}_0) + 9\,\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0)\overline{\mathscr{L}}(R) - 9\,\overline{\mathbf{A}}_0\,\overline{B}^2\,\overline{\mathscr{L}}(\mathbf{A}_0) - 3\,\mathbf{A}_0^2Q\overline{\mathscr{L}}(\overline{B})-$$

$$-\,9\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}})_0\mathscr{L}(\overline{B}) - 9\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)\overline{B}^2 + 9\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}}_0)\overline{\mathscr{L}}(R) + 9\,P\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0) + 9\,\mathbf{A}_0\mathscr{L}(\overline{\mathbf{A}}_0)\overline{\mathscr{L}}(\overline{B}) - 9\,\overline{\mathbf{A}}_0A\mathscr{L}(\mathbf{A}_0)-$$

$$-\,18\,\mathbf{A}_0BR\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 12\,\mathbf{A}_0\overline{Q}R\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{B}\,\overline{\mathscr{L}}(Q) - 9\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{B}\mathscr{L}(B) - 9\,\mathbf{A}_0\overline{B}\,\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\overline{B}) - 4\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{Q}^2R-$$

$$-\,3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{Q}\,\overline{\mathscr{L}}(R) + 9\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{\mathscr{L}}(R) - 9\,B\overline{B}\,\overline{\mathbf{A}}_0\mathscr{L}(\mathbf{A}_0) + 9\,\overline{B}Q\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0) + 12\,\overline{Q}R\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0) - 18\,BR\overline{\mathbf{A}}_0\,\overline{\mathscr{L}}(\mathbf{A}_0)-$$

$$-\,9\,B\overline{B}\mathbf{A}_0\mathscr{L}(\overline{\mathbf{A}}_0) + 9\,\overline{B}Q\mathbf{A}_0\overline{\mathscr{L}}(\overline{\mathbf{A}}_0) + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{Q}\mathscr{L}(\overline{B}) - 9\,\mathbf{A}_0\overline{\mathbf{A}}_0B\mathscr{L}(\overline{B}) - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{B}\mathscr{L}(\overline{Q}) + 3\,\mathbf{A}_0^2AQ + 3\,\mathbf{A}_0^2\overline{B}^2\overline{Q}-$$

$$-\,9\mathbf{A}_0^2A\overline{B} - 18\,\mathbf{A}_0^2B\overline{B}^2 + 18\,\mathbf{A}_0\overline{B}^2\overline{\mathbf{A}}_0B - \mathbf{A}_0^2\overline{B}Q\overline{Q} + 6\,\mathbf{A}_0^2\overline{B}BQ + 3\,\mathbf{A}_0\overline{\mathbf{A}}_0AQ + 9\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{P}R - 18\,\mathbf{A}_0\overline{\mathbf{A}}_0B^2R+$$

$$+\,9\,\mathbf{A}_0\overline{\mathbf{A}}_0\overline{B}R\overline{R} - 2\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{B}Q\overline{Q} + 18\,\mathbf{A}_0\overline{\mathbf{A}}_0B\overline{Q}R - 3\,\mathbf{A}_0\overline{\mathbf{A}}_0\,\overline{Q}P + 9\,\mathbf{A}_0\overline{\mathbf{A}}_0A\overline{B}\big),$$

$$V_3^{\mathsf{new}} = \tfrac{1}{3\,\mathbf{A}_0^2}\big(-3\,\overline{\mathscr{L}}(R) + 9\,B\,R - 4\,\overline{Q}R\big),$$

$$V_8^{\mathsf{new}} = \overline{U}_4^{\mathsf{new}} - \overline{W}_{10}^{\mathsf{new}},$$

$$W_5^{\mathsf{new}} = -\tfrac{1}{27\,\mathbf{A}_0\overline{\mathbf{A}}_0^3}\big(27\,\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{R}))) - 27\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{R}))) + 18\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{Q}))) - 9\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{Q}))) - 9\,\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{Q})))+$$

$$+\,9\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q))) + 27\,\mathscr{L}(\overline{\mathscr{L}}(\overline{P})) - 45\,\overline{\mathscr{L}}(\mathscr{L}(\overline{P})) + 18\,B\mathscr{L}(\overline{\mathscr{L}}(\overline{Q})) - 36\,B\overline{\mathscr{L}}(\mathscr{L}(\overline{Q})) - 36\,\overline{R}\overline{\mathscr{L}}(\mathscr{L}(Q))-$$

$$-\,27\,Q\overline{\mathscr{L}}(\mathscr{L}(\overline{R})) + 27\,\overline{B}\,\overline{\mathscr{L}}(\mathscr{L}(\overline{R})) + 36\,\overline{R}\mathscr{L}(\overline{\mathscr{L}}(Q)) + 36\,Q\mathscr{L}(\overline{\mathscr{L}}(\overline{R})) - 54\,\overline{B}\mathscr{L}(\overline{\mathscr{L}}(\overline{R})) + 24\,BQ^2\overline{R} - 180\,B\overline{B}Q\overline{R}-$$

$$-\,9\,\overline{Q}\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(Q)) + 27\,B\mathscr{L}(\mathscr{L}(\overline{R})) - 9\,\overline{Q}\mathscr{L}(\mathscr{L}(\overline{R})) - 18\,\overline{R}\mathscr{L}(\mathscr{L}(\overline{Q}))54\,A\overline{\mathscr{L}}(B) - 18\,B\overline{B}\,\overline{\mathscr{L}}(\overline{Q}) - 18\,B\mathscr{L}(\overline{P})-$$

$$-\,12\,\overline{Q}\,\overline{R}\mathscr{L}(Q) + 54\,\overline{P}\,\overline{\mathscr{L}}(\overline{B}) - 18\,\overline{P}\mathscr{L}(\overline{Q}) + 18\,\overline{Q}\mathscr{L}(\overline{P}) - 18\,\overline{Q}\,\overline{\mathscr{L}}(A) + 54\,B\overline{\mathscr{L}}(A) + 36\,B\overline{R}\mathscr{L}(Q) + 12\,\overline{R}Q\overline{\mathscr{L}}(Q)-$$

$$-\,18\,\mathscr{L}(\overline{R})\mathscr{L}(\overline{Q}) - 54\,\overline{B}Q\overline{\mathscr{L}}(\overline{R}) + 27\,\mathscr{L}(\overline{R})\overline{\mathscr{L}}(\overline{B}) - 54\,\overline{\mathscr{L}}(B)\overline{\mathscr{L}}(\overline{B}) + 36\,B\overline{Q}\,\overline{\mathscr{L}}(\overline{B}) - 27\,\mathscr{L}(\overline{R})R\overline{R} - 24\,Q\overline{R}\mathscr{L}(\overline{Q})+$$

$$+\,36\,Q\overline{R}\,\overline{\mathscr{L}}(\overline{B}) + 27\,R\overline{R}\overline{\mathscr{L}}(B) + 54\,\overline{B}\,\overline{R}\mathscr{L}(\overline{Q}) - 54\,\overline{B}\,\overline{R}\,\overline{\mathscr{L}}(\overline{B}) + 63\,\overline{Q}\,\overline{B}\mathscr{L}(\overline{R}) - 18\,\overline{Q}\,\overline{B}\,\overline{\mathscr{L}}(B) - 24\,\overline{Q}Q\mathscr{L}(\overline{R})-$$

$$-\,135\,B\overline{B}\mathscr{L}(\overline{R}) + 108\,B\overline{B}\,\overline{\mathscr{L}}(B) - 72\,AB\overline{Q} + 54\,BQ\mathscr{L}(\overline{R}) - 18\,A\overline{\mathscr{L}}(\overline{Q}) + 12\,Q^2\overline{\mathscr{L}}(\overline{R}) + 54\,\overline{B}^2\,\overline{\mathscr{L}}(\overline{R}) + 36\,\overline{R}\,\overline{\mathscr{L}}(P)-$$

$$-\,9\,\overline{P}\overline{\mathscr{L}}(Q) + 18\,\overline{Q}^2A + 54\,B^3\overline{B} + 54\,AB^2 + 54\,\overline{B}\,\overline{R}^2R - 36\,Q\overline{R}^2R - 27\,\overline{P}R\overline{R} + 18\,\overline{B}B\overline{Q}^2 - 108\,\overline{Q}\,\overline{B}^2\,\overline{R}-$$

$$-\,16\,\overline{Q}Q^2\overline{R} - 81\,BP\overline{R} + 27\,\overline{Q}P\overline{R} - 18\,B\overline{Q}R\overline{R} + 102\,\overline{Q}\,\overline{B}Q\overline{R} - 72\,B^2\overline{B}\,\overline{Q} + 162\,B\overline{B}^2\overline{R} + 18\,Q\overline{R}A - 54\,\overline{B}\,\overline{R}A\big),$$

$$W_6^{\mathsf{new}} = -\tfrac{i}{3\,\mathbf{A}_0^2\overline{\mathbf{A}}_0^2}\big(3\,\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathbf{A}_0))) + 3\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0))) - 6\,\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0))) + 3\,B\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) - 3\,B\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0))+$$

$$+\,3\,\mathbf{A}_0\mathscr{L}(\overline{\mathscr{L}}(B)) - 3\,\mathbf{A}_0\mathscr{L}(\mathscr{L}(\overline{R})) - 4\,\mathbf{A}_0Q\mathscr{L}(\overline{R}) - 3\,\mathbf{A}_0\overline{B}\,\overline{\mathscr{L}}(B) + 3\,B\overline{B}\,\overline{\mathscr{L}}(\mathbf{A}_0) + 9\,\mathbf{A}_0\overline{B}\mathscr{L}(\overline{R})+$$

$$+\,6\,\overline{B}\,\overline{R}\mathscr{L}(\mathbf{A}_0) - 3\,B\overline{Q}\mathscr{L}(\mathbf{A}_0) - 4\,Q\overline{R}\mathscr{L}(\mathbf{A}_0) + 3\,B^2\mathscr{L}(\mathbf{A}_0) - 3\,\mathscr{L}(\mathbf{A}_0)\mathscr{L}(\overline{R}) + 3\,\mathscr{L}(\mathbf{A}_0)\overline{\mathscr{L}}(B) - 3\,\overline{P}\mathscr{L}(\mathbf{A}_0)-$$

$$-\,3\,\overline{\mathscr{L}}(\mathbf{A}_0)\mathscr{L}(B) + 3\,A\overline{\mathscr{L}}(\mathbf{A}_0) + 6\,\mathbf{A}_0\overline{R}\mathscr{L}(\overline{B}) - 3\,\mathbf{A}_0\overline{Q}\mathscr{L}(B) - 4\,\mathbf{A}_0\overline{R}\mathscr{L}(Q) + 6\,\mathbf{A}_0B\mathscr{L}(B)-$$

$$-\,3\,\mathbf{A}_0B\mathscr{L}(\overline{Q}) - 3\,\mathbf{A}_0\mathscr{L}(\overline{P}) + 3\,\mathbf{A}_0\overline{B}\,\overline{P} - 3\,\mathbf{A}_0B^2\overline{B} - 6\,\mathbf{A}_0\overline{B}^2\,\overline{R} + 4\,\mathbf{A}_0Q\overline{B}\,\overline{R} + 3\,\mathbf{A}_0\overline{B}B\overline{Q}\big),$$

$$W_7^{\mathsf{new}} = -\tfrac{i}{9\,\mathbf{A}_0\overline{\mathbf{A}}_0^3}\big(-9\,\mathbf{A}_0\overline{\mathscr{L}}(\mathscr{L}(\overline{R})) - 12\,Q\overline{R}\overline{\mathscr{L}}(\mathbf{A}_0) + 18\,\mathbf{A}_0\overline{B}\,\overline{\mathscr{L}}(\overline{R}) - 12\,\overline{R}\mathbf{A}_0\overline{\mathscr{L}}(Q) - 6\,B\overline{Q}\overline{\mathscr{L}}(\mathbf{A}_0)-$$

$$-\,12\,\mathbf{A}_0Q\overline{\mathscr{L}}(\overline{R}) + 9\,\overline{R}\mathbf{A}_0\overline{\mathscr{L}}(\overline{B}) - 18\,\mathbf{A}_0B\mathscr{L}(\overline{R}) + 12\,\mathbf{A}_0\overline{Q}\mathscr{L}(\overline{R}) + 18\,\overline{B}\,\overline{R}\,\overline{\mathscr{L}}(\mathbf{A}_0) - 9\,\overline{\mathscr{L}}(\mathbf{A}_0)\mathscr{L}(\overline{R})+$$

$$+\,9\,\overline{\mathscr{L}}(\mathbf{A}_0)\overline{\mathscr{L}}(B) - 9\,\overline{P}\overline{\mathscr{L}}(\mathbf{A}_0) + 54\,\mathbf{A}_0B\overline{B}\,\overline{R} - 24\,\mathbf{A}_0BQ\overline{R} - 27\,\mathbf{A}_0\overline{Q}\,\overline{B}\,\overline{R} + 16\,\mathbf{A}_0\overline{Q}Q\overline{R} + 9\,\mathbf{A}_0A\overline{R}\big),$$

$$W_8^{\mathsf{new}} = -\tfrac{i}{9\,\mathbf{A}_2\overline{\mathbf{A}}_0}\big(-9\,\mathscr{L}(\overline{\mathscr{L}}(\mathbf{A}_0)) + 9\,\overline{\mathscr{L}}(\mathscr{L}(\mathbf{A}_0)) - 9\,\mathbf{A}_0\overline{\mathscr{L}}(\overline{B}) - 9\,\mathbf{A}_0\mathscr{L}(B) + 3\,\mathbf{A}_0\mathscr{L}(\overline{Q}) - \\ -9\,B\mathscr{L}(\mathbf{A}_0) + 3\,\overline{Q}\mathscr{L}(\mathbf{A}_0) + 9\,\overline{B}\overline{\mathscr{L}}(\mathbf{A}_0) - 3\,Q\overline{\mathscr{L}}(\mathbf{A}_0) + 9\,\mathbf{A}_0B\overline{B} + 3\,\mathbf{A}_0BQ - \mathbf{A}_0Q\overline{Q} + 9\,\mathbf{A}_0A\big),$$

$$W_9^{\mathsf{new}} = \tfrac{i}{9\,\overline{\mathbf{A}}_0^2}\big(18\,\overline{\mathscr{L}}(B) - 9\,\mathscr{L}(\overline{R}) - 3\overline{\mathscr{L}}(\overline{Q}) - 12\,Q\overline{R} - 9\,\overline{P} - 12\,B\overline{Q} + 18\,\overline{B}\,\overline{R} + 9\,B^2 + \overline{Q}^2\big),$$

$$W_{10}^{\mathsf{new}} = \tfrac{1}{3\,\mathbf{A}_0\overline{\mathbf{A}}_0}\big(3\,\mathbf{A}_0B - \mathbf{A}_0\overline{Q} - 3\,\overline{\mathscr{L}}(\mathbf{A}_0)\big).$$

After determining the last group parameter e and also modifying the single Maurer-Cartan form as (120), now the last structure equations (119) are converted into the *final form*:

(122)

$$\begin{aligned}
d\sigma &= U_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + U_2^{\mathsf{new}}\,\sigma\wedge\rho + \big(2\,\overline{U}_4^{\mathsf{new}} - 3\,\overline{W}_{10}^{\mathsf{new}}\big)\,\sigma\wedge\zeta + U_4^{\mathsf{new}}\,\sigma\wedge\overline{\zeta} + \overline{\sigma}\wedge\overline{\zeta} + \rho\wedge\zeta,\\
d\rho &= V_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + V_2^{\mathsf{new}}\,\sigma\wedge\rho + V_3^{\mathsf{new}}\,\sigma\wedge\zeta + \overline{V}_2^{\mathsf{new}}\,\overline{\sigma}\wedge\rho + \overline{V}_3^{\mathsf{new}}\,\overline{\sigma}\wedge\overline{\zeta} + \\
&\quad + \big(\overline{U}_4^{\mathsf{new}} - \overline{W}_{10}^{\mathsf{new}}\big)\,\rho\wedge\zeta + \big(U_4^{\mathsf{new}} - W_{10}^{\mathsf{new}}\big)\,\rho\wedge\overline{\zeta} + i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \gamma\wedge\sigma + \\
&\quad + W_1^{\mathsf{new}}\,\sigma\wedge\overline{\sigma} + W_2^{\mathsf{new}}\,\sigma\wedge\rho + W_3^{\mathsf{new}}\,\sigma\wedge\zeta + W_4^{\mathsf{new}}\,\sigma\wedge\overline{\zeta} + \\
&\quad + W_5^{\mathsf{new}}\,\overline{\sigma}\wedge\rho + W_6^{\mathsf{new}}\,\overline{\sigma}\wedge\zeta + W_7^{\mathsf{new}}\,\overline{\sigma}\wedge\overline{\zeta} + \\
&\quad + W_8^{\mathsf{new}}\,\rho\wedge\zeta + W_9^{\mathsf{new}}\,\rho\wedge\overline{\zeta} + \\
&\quad + W_{10}^{\mathsf{new}}\,\zeta\wedge\overline{\zeta},
\end{aligned}$$

with twelve essential invariants:

$$(123)\qquad \left[\begin{array}{l} 0 \equiv U_1^{\mathsf{new}} = U_2^{\mathsf{new}} = U_4^{\mathsf{new}},\\ 0 \equiv V_1^{\mathsf{new}} = V_2^{\mathsf{new}} = V_3^{\mathsf{new}},\\ 0 \equiv W_5^{\mathsf{new}} = W_6^{\mathsf{new}} = W_7^{\mathsf{new}} = W_8^{\mathsf{new}} = W_9^{\mathsf{new}} = W_{10}^{\mathsf{new}}. \end{array}\right.$$

Theorem 18.1. *Two real analytic generic CR-submanifolds of $\mathbb{C}^4$ represented as the graph of three homogeneous defining equations of the form:*

$$\begin{aligned}
w_1 - \overline{w}_1 &= 2i\,z\overline{z} + \mathrm{O}(3),\\
w_2 - \overline{w}_2 &= 2i\,z\overline{z}\,(z + \overline{z}) + \mathrm{O}(4),\\
w_3 - \overline{w}_3 &= 2\,z\overline{z}\,(z - \overline{z}) + \mathrm{O}(4),
\end{aligned}$$

with nonzero corresponding essential functions R are equivalent if and only if the essential invariants (123) *associated to them are identically equal.*

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