MULTIZETA CALCULUS (I)

Joël MERKER

Table of contents

1. Introduction .... **1.**
2. Weight, Depth, Height .... **11.**
3. Dualities .... **13.**
4. Detachable and Flexible Notations .... **15.**
5. Total Ordering $\prec$ Between Polyzetas of Fixed Weight .... **16.**
6. Intermediate Countings .... **20.**
7. Shuffle Minus Stuffle .... **22.**
8. Computing $-(1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 1 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$ .... **23.**
9. Computing $-(2) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 01 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$ .... **31.**
10. Computing $-(3) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 001 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$ ..... **44.**
11. Computing $-(2,1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 011 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$ .. **44.**

The Holy Grail in the field of MZV's [Multiple Zeta Values] *is an algorithm to express each MZV into an unique basis in a constructive way. That way we would have a (hopefully) small procedure rather than giant tables. Thus far this has not been found.* J.A.M. VERMASEREN.

1. INTRODUCTION

As is known ([14, 8, 10, 4], convergent polyzetas are infinite numerical sums:

$$\zeta\big(s_1, s_2, \dots, s_{\mathsf{d}}\big) = \sum_{n_1 > n_2 > \cdots > n_{\mathsf{d}} > 0} \frac{1}{(n_1)^{s_1}\,(n_2)^{s_2}\,\cdots\,(n_{\mathsf{d}})^{s_{\mathsf{d}}}},$$

for certain integers $s_1, s_2, \dots, s_{\mathsf{d}} \geqslant 1$, all positive, with the additional requirement that $s_1 \geqslant 2$ in order to insure (absolute) convergence of this multiple series in which $n_1 > n_2 > \cdots > n_{\mathsf{d}} > 0$ are integers. The number d of the appearing entries s_i is called the *depth* of the polyzeta, while the sum of entries:

$$s_1 + s_2 + \cdots + s_{\mathsf{d}} =: \mathsf{w}$$

is said to be its *weight*.

It is expected — or conjectured — that all the so-called *double shuffle relations*:

$$0 = -\,\zeta\big(s_1, s_2, \dots, s_{\mathsf{d}}\big) * \zeta\big(t_1, t_2, \dots, t_{\mathsf{e}}\big) + \zeta\big(s_1, s_2, \dots, s_{\mathsf{d}}\big) \sqcup\!\sqcup\, \zeta\big(t_1, t_2, \dots, t_{\mathsf{e}}\big),$$

including the ones for which $s_1 = 1$ — a single non-convergent polyzeta in the two products is then erased by the subtraction in this case — provide all $\mathbb{Q}$-linear relations between the convergent polyzetas. Since double shuffle relations are $\mathbb{Q}$-*linear*

Date: 2012-8-29.

between polyzetas of any fixed weight, such a conjecture amounts, metaphysically speaking, to believe that easy-to-discover structures exhaust *all* polyzeta relations.

Intentionally, we will not produce here any 'sexy' basic presentation of conjectures and structures (fundamental references are listed in the bibliography), because our goal is instead to set up 'handy' formulas towards a (new?) *Polyzeta Calculus*.

A first immediate aspect of the problems that are open in the field would be to understand *in a closed way* all what is contained in these double shuffle relations.

A second, much more delicate, aspect would be to hope for irrationality and even for transcendence concerning polyzetas *modulo* these double shuffle relations, but this question will not at all be dealt with here, because, due to its complexity, a satisfactory settlement of it could well require yet several decades of intensive mathematical research ([5]).

Concerning the first aspect, the so-called *Hoffman conjecture* expects that the polyzetas whose entries s_i are equal all to either 2 or 3 make up a basis of the vector space of all polyzetas of any fixed weight w modulo all existing double shuffle ($\mathbb{Q}$-linear) relations within a same fixed weight.

The correctness of this quite appealing conjecture has been verified on powerful computer machines up to weight $\mathsf{w} = 22$ ([11, 1]) and a striking proof that every polyzeta is a certain $\mathbb{Q}$-linear combination of the $\zeta(s_1, \dots, s_{\mathsf{d}})$ with the $s_i \in \{2, 3\}$ was obtained recently by Francis Brown ([3]).

Still, experts agree that a (wide?) gap exists between, on one hand, abstract, indirect or non computationally complete arguments, and, on the other hand, what our sparkling computer machines really handle within their most intimate core.

Therefore, our present goal here will be to offer a contribution that, hopefully, might be new, in the sense that it would launch the construction of some archway for a bridge between brain and machine towards satisfactory constructiveness in the $(2, 3)$-conjecture. A strong result of Jean Écalle ([7]) already showed in an effective way that all entries $s_i = 1$ may be eliminated.

In 2008, Masanobu Kaneko, Masayuki Noro and Ken'ichi Tsurumaki ([11]) showed that up to weight $\mathsf{w} = 20$, it suffices in fact to look only at all double shuffle relations whose first polyzeta $\zeta(s_1, \dots, s_{\mathsf{d}})$ is either $\zeta(1)$, $\zeta(2)$, $\zeta(3)$ or $\zeta(2, 1)$ — the consideration of duality relations is also advantageous. This improves the $(2, 3)$-conjecture in a highly natural manner, because the number of relations obtained in such a way for polyzetas of weight w:

$$2^{\mathsf{w}-3} + 2^{\mathsf{w}-4} + 2^{\mathsf{w}-5} + 2^{\mathsf{w}-5}$$

is *exactly equal* to the number:

$$2^{\mathsf{w}-2}$$

of polyzetas of weight w, hence its exceeds in the right way the expected number $2^{\mathsf{w}-2} - \delta_{\mathsf{w}}$ of *dependent* polyzetas, where $\delta_2 = \delta_3 = \delta_4 = 1$, $\delta_{\mathsf{w}} = \delta_{\mathsf{w}-2} + \delta_{\mathsf{w}-3}$ is the number of polyzetas of weight w having entries belonging to the set $\{2, 3\}$.

In this paper, we develop techniques and we establish theorems in order to explicitly write down these four families of *conjecturably relevant* double shuffle relations:

$$0 = -\,\zeta(1) * \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \zeta(1) \sqcup\!\sqcup\, \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \qquad \text{[w-1]},$$

$$0 = -\,\zeta(2) * \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \zeta(2) \sqcup\!\sqcup\, \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \qquad \text{[w-2]},$$

$$0 = -\,\zeta(3) * \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \zeta(3) \sqcup\!\sqcup\, \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \qquad \text{[w-3]},$$

$$0 = -\,\zeta(2,1) * \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \zeta(2,1) \sqcup\!\sqcup\, \zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \qquad \text{[w-3]}.$$

By definition, the *height* h of a polyzeta is the number of its entries s_i that are $\geqslant 2$. It follows that a general, arbitrary convergent polyzeta can always be written under the specific form:

$$\zeta\big(a_1, 1^{b_1}, a_2, 1^{b_2}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),$$

where the integer $\mathsf{h} \geqslant 1$ is its height, where all the a-integers are $\geqslant 2$:

$$a_1 \geqslant 2, \qquad a_2 \geqslant 2, \qquad \dots\dots, \qquad a_{\mathsf{h}} \geqslant 2,$$

and where all the b-integers are $\geqslant 0$:

$$b_1 \geqslant 0, \qquad b_2 \geqslant 0, \qquad \dots\dots, \qquad b_{\mathsf{h}} \geqslant 0.$$

Theorem 1.1. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} =: \mathsf{w} - 1,$$
$$1 + b_1 + \cdots + 1 + b_{\mathsf{h}} =: \mathsf{d},$$

then the so-called regularized *double shuffle relation:*

$$0 = -\,(1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 1 \sqcup\!\sqcup\, 0^{a_1-1}1\,1^{b_1} \,\cdots\, 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$$

between $\zeta(1)$ *and any* $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 1 \geqslant 2$*, of depth* d *and of height* h *writes out after complete finalization:*

$$\boxed{\begin{aligned}
0 = &-\sum_{1 \leqslant i \leqslant \mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 1, \bullet\bullet\bullet\bullet\bullet\big) - \\
&- \sum_{\substack{1 \leqslant j \leqslant \mathsf{h} \\ b_j \geqslant 1}} \Bigg(\sum_{b_j' + b_j'' = b_j - 1} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
&+ \sum_{1 \leqslant j \leqslant \mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\
&+ \sum_{\substack{1 \leqslant i \leqslant \mathsf{h} \\ a_i \geqslant 3}} \Bigg(\sum_{\substack{a_i' + a_i'' = a_i + 1 \\ a_i' \geqslant 2,\ a_i'' \geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg).
\end{aligned}}$$

Here by convention, only the terms of the initial polyzeta $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}$ that are *changed* are written down, so that the symbol:

$$\bullet\bullet\bullet\bullet\bullet$$

means that all other entries are unchanged, as one might have guessed at once.

Theorem 1.2. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$\begin{aligned} a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 2, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d}, \end{aligned}$$

then the so-called double shuffle relation:

$$0 = -(2) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 01 \sqcup 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$$

between $\zeta(2)$ *and any* $\zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 2 \geqslant 2$*, of depth* d *and of height* h *writes out after complete finalization:*

$$\boxed{\begin{aligned} 0 = &- \sum_{1 \leqslant i \leqslant \mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 2, \bullet\bullet\bullet\bullet\bullet\big) - \\ &- \sum_{\substack{1 \leqslant j \leqslant \mathsf{h} \\ b_j \geqslant 1}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 3, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ &+ \sum_{1 \leqslant i \leqslant j \leqslant \mathsf{h}} a_i\,(b_j + 2) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 1, \bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\ &+ \sum_{\substack{1 \leqslant j_1 < j_2 \leqslant \mathsf{h} \\ b_{j_1} \geqslant 1}} \Bigg(\sum_{\substack{b_{j_1}' + b_{j_1}'' = b_{j_1} - 1 \\ b_{j_1}' \geqslant 0,\ b_{j_1}'' \geqslant 0}} (b_{j_2} + 2) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_{j_1}'}, 2, 1^{b_{j_1}''}, \bullet\bullet\bullet\bullet\bullet, 1^{b_{j_2}+1}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ &+ \sum_{1 \leqslant j \leqslant \mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} b_j'' \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ &+ \sum_{\substack{1 \leqslant i \leqslant \mathsf{h} \\ a_i \geqslant 5}} \Bigg(\sum_{\substack{a_i' + a_i'' = a_i + 2 \\ a_i' \geqslant 3,\ a_i'' \geqslant 2}} (a_i' - 1) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ &+ \sum_{\substack{1 \leqslant i_1 < i_2 \leqslant \mathsf{h} \\ a_{i_2} \geqslant 3}} \Bigg(\sum_{\substack{a_{i_2}' + a_{i_2}'' = a_{i_2} + 1 \\ a_{i_2}' \geqslant 2,\ a_{i_2''} \geqslant 2}} a_{i_1} \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1} + 1, \bullet\bullet\bullet\bullet\bullet, a_{i_2}', 1^0, a_{i_2}'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ &+ \sum_{\substack{1 \leqslant j < i \leqslant \mathsf{h} \\ b_j \geqslant 1,\ a_i \geqslant 3}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \sum_{\substack{a_i' + a_i'' = a_i + 1 \\ a_i' \geqslant 2,\ a_i'' \geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg). \end{aligned}}$$

Theorem 1.3. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$\begin{aligned} a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 3, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d}, \end{aligned}$$

then the so-called $*$-stuffle product between $\zeta(3)$ and any $\zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ of weight $\mathsf{w}-3 \geqslant 2$, of depth d and of height h writes out after complete finalization:

$$
\begin{aligned}
\zeta(3) * \zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) = \\
= \sum_{1 \leqslant i \leqslant \mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 3, \bullet\bullet\bullet\bullet\bullet\big) + \\
+ \sum_{\substack{1 \leqslant j \leqslant \mathsf{h} \\ b_j \geqslant 1}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 4, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
+ \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(3, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \\
\sum_{1 \leqslant j \leqslant \mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 3, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg).
\end{aligned}
$$

Theorem 1.4. *For every height $\mathsf{h} \geqslant 1$, every entries $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$, every entries $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$, if one sets:*

$$
\begin{aligned}
a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 3, \\
1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d},
\end{aligned}
$$

then the so-called $\sqcup\!\sqcup$-stuffle product between $\zeta(3)$ and any $\zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ of weight $\mathsf{w} - 3 \geqslant 2$, of depth d and of height h writes in full:

$$
\begin{aligned}
& 001 \sqcup\!\sqcup\, 0^{a_1 - 1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}} - 1} 1\, 1^{b_{\mathsf{h}}} = \\
& = \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(3, a_1, 1^{b_1}, \dots, a_k, 1^{b_k}\big) + \\
& + \sum_{\substack{1 \leqslant i \leqslant \mathsf{h} \\ a_i \geqslant 3}} \Bigg(\sum_{\substack{a_i' + a_i'' = a_i + 3 \\ a_i' \geqslant 4,\ a_i'' \geqslant 2}} \frac{(a_i' - 1)(a_i' - 2)}{2} \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
& + \sum_{1 \leqslant j \leqslant \mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} (b_j'' + 1) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 3, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
& + \sum_{\substack{1 \leqslant j \leqslant \mathsf{h} \\ b_j \geqslant 1}} \Bigg(\sum_{\substack{b_j' + b_j'' + b_j''' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0,\ b_j''' \geqslant 0}} (b_j''' + 1) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, 2, 1^{b_j'''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
& + \sum_{\substack{1 \leqslant i_1 < i_2 \leqslant \mathsf{h} \\ a_{i_2} \geqslant 3}} \Bigg(\sum_{\substack{a_{i_2}' + a_{i_2}'' = a_{i_2} + 1 \\ a_{i_2}' \geqslant 2,\ a_{i_2}'' \geqslant 2}} \frac{(a_{i_1} + 1)\, a_{i_1}}{2} \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1} + 2, \bullet\bullet\bullet\bullet\bullet, a_{i_2}', a_{i_2}'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
& + \sum_{1 \leqslant i \leqslant j \leqslant \mathsf{h}} \Bigg(\frac{(a_i + 1)\, a_i}{2} (b_j + 2) \cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 2, \bullet\bullet\bullet\bullet\bullet, 1^{b_j + 1}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) +
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}\ \sum_{\substack{a_i'+a_i''=a_i+1\\ a_i'\geqslant 2,\ a_i''\geqslant 2}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},3,1^{b_j''},\bullet\bullet\bullet\bullet\bullet,a_i',a_i'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 2,\ a_i\geqslant 3}}\Bigg(\sum_{\substack{b_j'+b_j''+b_j'''=b_j-2\\ b_j'\geqslant,\ b_j''\geqslant 0,\ b_j'''\geqslant 0}}\ \sum_{\substack{a_i'+a_i''=a_i+1\\ a_i'\geqslant 2,\ a_i''\geqslant 2}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+3]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},2,1^{b_j'''},\bullet\bullet\bullet\bullet\bullet,a_i',a_i'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}-1\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}(b_{j_2}+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}{}'},3,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 2}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''+b_{j_1}'''=b_{j_1}-2\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0,\ b_{j_1}'''\geqslant 0}}(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}'},2,1^{b_{j_1}''},2,1^{b_{j_1}'''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}}\Bigg(\sum_{\substack{a_{i_2}'+a_{i_2}''=a_{i_2}+2\\ ai_2{}'\geqslant 3,\ a_{i_2}''\geqslant 2}}a_{i_1}\,(a_{i_2}'-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a_{i_2}',a_{i_2}'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}a_i\,(b_j''+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}\ \sum_{\substack{a_i'+a_i''=a_i+2\\ a_i'\geqslant 3,\ a_i''\geqslant 2}}(a_i'-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},\bullet\bullet\bullet\bullet\bullet,a_i',a_i'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}-1\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}\ \sum_{\substack{b_{j_2}'+b_{j_2}''=b_{j_2}\\ b_{j_2}'\geqslant 0,\ b_{j_2}''\geqslant 0}}(b_{j_2}''+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}'},2,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}'},2,1^{b_{j_2}''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant i_1<i_2<i_3\leqslant \mathsf{h}\\ a_{i_3}\geqslant 3}}\Bigg(\sum_{\substack{a_{i_3}'+a_{i_3}''=a_{i_3}+1\\ a_{i_3}'\geqslant 2,\ a_{i_3}''\geqslant 2}}a_{i_1}a_{i_2}\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a_{i_2}+1,\bullet\bullet\bullet\bullet\bullet,a_{i_3}',a_{i_3}'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant i_1<i_2\leqslant j\leqslant \mathsf{h}}a_{i_1}a_{i_2}(b_j+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a_{i_2}+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet\big)+\\
&+\sum_{\substack{1\leqslant i_1\leqslant j<i_2\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_{i_2}\geqslant 3}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}\ \sum_{\substack{a_{i_2}'+a_{i_2}''=a_{i_2}+1\\ a_{i_2}'\geqslant 2,\ a_{i_2}''\geqslant 2}}a_{i_1}\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},\bullet\bullet\bullet\bullet\bullet,a_{i_2}',a_{i_2}'',\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant i\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}-1\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}a_i(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+
\end{aligned}
$$

$$+\sum_{\substack{1\leqslant j<i_1<i_2\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_{i_2}\geqslant 3}}\left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}\ \sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_j},2,1^{b''_j},\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a'_{i_2},a''_{i_2},\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{\substack{1\leqslant j_1<i\leqslant j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\left(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b_{j'_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}a_i(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{\substack{1\leqslant j_1<j_2<i\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1,\, b_{j_2}\geqslant 1,\, a_i\geqslant 3}}\left(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\ \sum_{\substack{b'_{j_2}+b''_{j_2}=b_{j_2}-1\\ b'_{j_2}\geqslant 0,\ b''_{j_2}\geqslant 0}}\ \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}}\right.$$
$$\left.\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+3]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_2}},2,1^{b''_{j_2}},\bullet\bullet\bullet\bullet\bullet,a'_i,a''_i,\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{\substack{1\leqslant j_1<j_2<j_3\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1,\ b_{j_2}\geqslant 1}}\left(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\ \sum_{\substack{b'_{j_2}+b''_{j_2}=b_{j_2}-1\\ b'_{j_2}\geqslant 0,\ b''_{j_2}\geqslant 0}}\right.$$
$$\left.(b_{j_3}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_2}},2,1^{b''_{j_2}},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_3}+1},\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

Theorem 1.5. *For every height* $\mathsf{h}\geqslant 1$*, every entries* $a_1\geqslant 2,\ \dots,\ a_{\mathsf{h}}\geqslant 2$*, every entries* $b_1\geqslant 0,\dots,b_{\mathsf{h}}\geqslant 0$*, if one sets:*

$$a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}=:\mathsf{w}-3,$$
$$1+b_1+\cdots+1+b_{\mathsf{h}}=:\mathsf{d},$$

then the so-called $*$*-stuffle product between* $\zeta(2,1)$ *and any* $\zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w}-3\geqslant 2$*, of depth* d *and of height* h *writes in full:*

$$\zeta(2,1)*\zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)=$$
$$=\sum_{1\leqslant i_1<i_2\leqslant \mathsf{h}}\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+2,\bullet\bullet\bullet\bullet\bullet,a_{i_2}+1,\bullet\bullet\bullet\bullet\bullet\big)+$$

$$+\sum_{\substack{1\leqslant i\leqslant j\leqslant \mathsf{h}\\ b_j\geqslant 1}}\left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+2,\bullet\bullet\bullet\bullet\bullet,1^{b'_j},2,1^{b''_j},\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1}}\left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_j},3,1^{b''_j},\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}}\left(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\ \sum_{\substack{b'_{j_2}+b''_{j_2}=b_{j_2}-1\\ b'_{j_2}\geqslant 0,\ b''_{j_2}\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},3,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_2}},2,1^{b''_{j_2}},\bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+\sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}}(b_j+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+2,\bullet\bullet\bullet\bullet\bullet,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet\big)+$$

$$+\sum_{\substack{1\leqslant j\leqslant \mathsf{h}\\ b_j\geqslant 1}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}(b_j''+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},3,1^{b_j''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}-1\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}(b_{j_2}+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}'},3,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{\substack{1\leqslant j\leqslant \mathsf{h}\\ b_j\geqslant 1}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(2,\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{1\leqslant j\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b_j'+b_j''+b_j'''=b_j\\ b_j'\geqslant 0,\ b_j''\geqslant 0,\ b_j'''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},2,1^{b_j'''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_2}\geqslant 1}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}\ \sum_{\substack{b_{j_2}'+b_{j_2}''=b_{j_2}-1\\ b_{j_2}'\geqslant 0,\ b_{j_2}''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}'},2,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}'},2,1^{b_{j_2}''},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{1\leqslant i\leqslant \mathsf{h}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(2,\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{1\leqslant j<i\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b_j'+b_j''=b_j\\ b_j'\geqslant 0,\ b_j''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{1\leqslant j\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b_j'+b_j''+b_j'''=b_j\\ b_j'\geqslant 0,\ b_j''\geqslant 0,\ b_j'''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_j'},2,1^{b_j''},1^{b_j'''+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+$$

$$+\sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}}(b_{j_2}+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}'},2,1^{b_{j_1}''},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)$$

$$+\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(2,1,\bullet\bullet\bullet\bullet\bullet\big).$$

Theorem 1.6. *For every height* $\mathsf{h}\geqslant 1$*, every entries* $a_1\geqslant 2,\dots,a_{\mathsf{h}}\geqslant 2$*, every entries* $b_1\geqslant 0,\dots,b_{\mathsf{h}}\geqslant 0$*, if one sets:*

$$\begin{aligned}a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}&=:\mathsf{w}-3,\\ 1+b_1+\cdots+1+b_{\mathsf{h}}&=:\mathsf{d},\end{aligned}$$

then the so-called $\sqcup\!\sqcup$*-shuffle product between* $\zeta(2,1)$ *and any* $\zeta_{[\mathsf{w}-3,\mathsf{d},\mathsf{h}]}\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w}-3\geqslant 2$*, of depth* d *and of height* h *writes out in full:*

$$011\sqcup\!\sqcup\,0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}=$$

$$= \sum_{\substack{1\leqslant i\leqslant \mathsf{h}\\ a_i\geqslant 2}} \left(\sum_{\substack{a'_i+a''_i=a_i+2\\ a'_i\geqslant 2,\ a''_i\geqslant 2}} (a'_i-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a'_i, 1, a''_i, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant i\leqslant \mathsf{h}\\ a_i\geqslant 3}} \left(\sum_{\substack{a'_i+a''_i+a'''_i=a_i+3\\ a'_i\geqslant 2,\ a''_i\geqslant 2,\ a'''_i\geqslant 2}} (a'_i-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}2]}\big(\bullet\bullet\bullet\bullet\bullet, a'_i, a''_i, a'''_i, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant j\leqslant \mathsf{h}\\ b_j\geqslant 1}} \left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}} \frac{(b''_j+3)(b''_j+2)}{2}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b'_j}, 2, 1^{b''_j+2}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}} \left(\sum_{a'_{i_1}+a''_{i_1}=a_{i_1}+2} (a'_{i_1}-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, a'_{i_1}, a''_{i_1}, \bullet\bullet\bullet\bullet\bullet, a'_{i_2}, a''_{i_2}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} \left(\sum_{\substack{a'_i+a''_i=a_i+2\\ a'_i\geqslant 2,\ a''_i\geqslant 2}} (a'_i-1)(b_j+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a'_i, a''_i, \bullet\bullet\bullet\bullet\bullet, 1^{b_{j_2}+1}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}} \left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}} \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}} (b''_j+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b'_j}, 2, 1^{b''_j+1}, \bullet\bullet\bullet\bullet\bullet, a'_i, a''_i, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}} \left(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}} (b''_{j_1}+2)(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b'_{j_1}}, 2, 1^{b''_{j_1}+1}, \bullet\bullet\bullet\bullet\bullet, 1^{b_{j_2}+1}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}} \left(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}} a_{i_1}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1}+1, \bullet\bullet\bullet\bullet\bullet, a'_{i_2}, 1, a''_{i_2}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 4}} \left(\sum_{\substack{a'_{i_2}+a''_{i_2}+a'''_{i_2}=a_{i_2}+2\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2,\ a'''_{i_2}\geqslant 2}} a_{i_1}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1}+1, \bullet\bullet\bullet\bullet\bullet, a'_{i_2}, a''_{i_2}, a'''_{i_2}, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} a_i\,\frac{(b_j+3)(b_j+2)}{2}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i+1, \bullet\bullet\bullet\bullet\bullet, 1^{b_j+2}, \bullet\bullet\bullet\bullet\bullet\big)+$$

$$+ \sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}} \left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}} \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b'_j}, 2, 1^{b''_j}, \bullet\bullet\bullet\bullet\bullet, a'_i, 1, a''_i, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$+ \sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 4}} \left(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}} \sum_{\substack{a'_i+a''_i+a'''_i=a_i+2\\ a'_i\geqslant 2,\ a''_i\geqslant 2,\ a'''_i\geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+3]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b'_j}, 2, 1^{b''_j}, \bullet\bullet\bullet\bullet\bullet, a'_i, a''_i, a'''_i, \bullet\bullet\bullet\bullet\bullet\big)\right)+$$

$$
\begin{aligned}
&+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\frac{(b_{j_2}+3)(b_{j_2}+2)}{2}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+2},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant i_1<i_2<i_3\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3,\ a_{i_3}\geqslant 3}}\Bigg(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}\ \sum_{\substack{a'_{i_3}+a''_{i_3}=a_{i_3}+1\\ a'_{i_3}\geqslant 2,\ a''_{i_3}\geqslant 2}}\\
&\qquad\qquad a_{i_1}\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a'_{i_2},a''_{i_2},\bullet\bullet\bullet\bullet\bullet,a'_{i_3},a''_{i_3},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{\substack{1\leqslant i_1<i_2\leqslant j\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}}\Bigg(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}a_{i_1}(b_j+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a'_{i_2},a''_{i_2},\bullet\bullet\bullet\bullet\bullet,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant i_1\leqslant j<i_2\leqslant \mathsf{h}}\Bigg(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}a_{i_1}(b_j+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet,a'_{i_2},a''_{i_2},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant i\leqslant j_1<j_2\leqslant \mathsf{h}}a_i(b_{j_1}+2)(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet,1^{b_{j_1}+1},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)+\\
&+\sum_{\substack{1\leqslant j<i_1<i_2\leqslant \mathsf{h}\\ a_{i_1}\geqslant 3,\ a_{i_2}\geqslant 3}}\Bigg(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}\ \sum_{\substack{a'_{i_1}+a''_{i_1}=a_{i_1}+1\\ a'_{i_1}\geqslant 2,\ a''_{i_1}\geqslant 2}}\ \sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}\\
&\qquad\qquad \zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+3]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_j},2,1^{b''_j},\bullet\bullet\bullet\bullet\bullet,a'_{i_1},a''_{i_1},\bullet\bullet\bullet\bullet\bullet,a'_{i_2},a''_{i_2},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant j_1<i\leqslant j_2\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\ \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}}\\
&\qquad\qquad (b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,a'_i,a''_i,\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant j_1<j_2<i\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\ \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}}\\
&\qquad\qquad (b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet,a'_i,a''_i,\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\sum_{1\leqslant j_1<j_2<j_3\leqslant \mathsf{h}}\Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}\\
&\qquad\qquad (b_{j_2}+2)(b_{j_3}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_3}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+\\
&+\zeta_{[\mathsf{w},\mathsf{d}+2,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,2,1\big).
\end{aligned}
$$

1.1. **Acknowledgments and research ‘trail’.** Once in January 2011, at the *Séminaire de Philosophie des Mathématiques* (École Normale Supérieure, Paris), Pierre Cartier ([5]) expressed the natural desire that a constructive approach existed towards Hoffman’s $(2,3)$-conjecture, and this motivated us to invest energy in order to set up usable general formulas for double shuffle relations. Quickly, we realized that presumably, the four families on p. 3 of double shuffle relations could suffice, we checked this manually up to weight $\mathsf{w} = 7$, and we also spent enough time in Spring 2011 to finalize all the formulas presented here. However, after a while of further experiments and explorations (the content of which will appear in some forthcoming arxiv.org prepublications), a prudent mathematical wise fear appeared in our thoughts that possibly, some hidden unpleasant complexities could probably contradict the sufficiency of the four families of relations on p. 3. It was only in July 2012 that the arxiv.org preprint [6] of German Combariza made us aware that the improved $(2,3)$-conjecture we devised intuitively without conjecturable certainty had already been verified by Masanobu Kaneko, Masayuki Noro and Ken’ichi Tsurumaki to be correct on powerful computer machines up to weight $\mathsf{w} = 20$! This wonderful confirmation decided us to typeset our manuscript.

Special thanks are addressed to Professor Masanobu Kaneko for sending us a pdf file of the publication [11] and for transmitting data that are unreachable by hand. Further interactions between mind and machine are expectable, hopefully towards a full constructive resolution of various polyzeta conjectures in arbitrary weight.

In July 2012, Olivier Bouillot provided us with precious Maple files which automatize the production of double-shuffle relations, a tool we are very grateful for.

Of course, the (extensive) formulas provided in this article are only a preliminary — in a sense easy and elementary — step towards constructiveness in the $(2,3)$-conjecture, for experts know well that double shuffle computations just lie on the threshold of higher level linear algebra computations with matrix minors whose size, unfortunately, increases exponentially with the weights of the polyzetas.

2. Weight, depth, height

2.1. **Principles of ζ-notations.** Thus, any convergent polyzeta writes out:

$$\zeta\big(s_1, s_2, \dots, s_{\mathsf{d}}\big) = \sum_{n_1 > n_2 > \cdots > n_{\mathsf{d}} > 0} \frac{1}{(n_1)^{s_1}\,(n_2)^{s_2}\,\cdots\,(n_{\mathsf{d}})^{s_{\mathsf{d}}}},$$

for certain integers $s_1, s_2, \dots, s_{\mathsf{d}} \geqslant 1$, all positive, with the additional requirement that $s_1 \geqslant 2$ in order to insure (absolute) convergence of this multiple series in which $n_1 > n_2 > \cdots > n_{\mathsf{d}} > 0$ are integers.

Definition 2.1. Classically, the integer $\mathsf{d} \geqslant 1$ is called the *depth* $\mathsf{d}(\zeta)$ of the polyzeta $\zeta\big(s_1, s_2, \dots, s_{\mathsf{d}}\big)$, while the total sum of the s_i:

$$\mathsf{w} := s_1 + s_2 + \cdots + s_{\mathsf{d}}$$

is said to be its *weight* $\mathsf{w}(\zeta)$.

Here are a few simple instances:

$$\zeta(2,1), \qquad \zeta(3,2), \qquad \zeta(6,2), \qquad \zeta(2,1,2,1,1), \qquad \zeta(5,3,1),$$

the weights and the depths of which may be at once computed mentally.

Although it appears in the literature to be a less widespread feature, the concept of *height* is also intrinsically necessary to deal with when one enters a *more-in-depth*, general, systematic *Polyzeta Calculus.*

Definition 2.2. The *height* of a convergent polyzeta $\zeta(s_1,\dots,s_{\mathsf{d}})$ is the number of entries s_i that are $\geqslant 2$:

$$\mathsf{h}(\zeta) := \mathsf{Card}\,\big\{i \in \{1,2,\dots,\mathsf{d}(\zeta)\} :\; s_i \geqslant 2\big\}.$$

Of course, the height is always $\geqslant 1$. Equivalently, one could just count the number of entries that are equal to 1. It is appropriate, then, to adapt the notation in order to specify how many entries 1 are repeated one after the other. Here are a few examples:

$$\begin{aligned}\zeta(3,1,2,1,1) &= \zeta(3,1,2,1^2),\\ \zeta(4,1,1,1) &= \zeta(4,1^3),\\ \zeta(5,3,2) &= \zeta(5,1^0,3,1^0,2,1^0),\end{aligned}$$

and such *coincidences of notations* will be admitted throughout the present paper. With the intention of lightening the writing, we want to denote the number of 1 that are present in some continuous succession, say $b \geqslant 0$ times, using just by a plain exponent, generally as:

$$1^b.$$

However, we will not use any further specific symbol like $1^{\#b}$, 1^{*2}, $1^{\sim 1}$. With such a convention, it must then be clear that 1^0 means that no 1 is present at all. Furthermore, between any two successive entries that are $\geqslant 2$, as *e.g.* in $\zeta(6,2)$, an 1^0 is always implicitly present, and a last 1^0 is also present when the very last s_{d} happens to be $\geqslant 2$, as for instance in:

$$\zeta(6,2) = \zeta(6,1^0,2,1^0).$$

Notation 2.3. A general, arbitrary convergent polyzeta will always be written under the specific form:

$$\boxed{\zeta\big(a_1,1^{b_1},a_2,1^{b_2},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)},$$

where the integer $\mathsf{h} \geqslant 1$ is its height, where all the a-integers are $\geqslant 2$:

$$a_1 \geqslant 2, \qquad a_2 \geqslant 2, \qquad \dots\dots, \qquad a_{\mathsf{h}} \geqslant 2,$$

and where all the b-integers are $\geqslant 0$:

$$b_1 \geqslant 0, \qquad b_2 \geqslant 0, \qquad \dots\dots, \qquad b_{\mathsf{h}} \geqslant 0.$$

In summary, *three fundamental numerical quantities are attached to any convergent polyzeta* ζ:

□ its weight $\mathsf{w}(\zeta) = a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}}$;

☐ its depth $\mathsf{d}(\zeta) = 1 + b_1 + \cdots + 1 + b_{\mathsf{h}}$;

☐ and its height $\mathsf{h}(\zeta)$.

Also, it will be admitted throughout the present text that *any polyzeta can be written either with or without pointing out the implicit* 1^0 *between any two subsequent* $a_i \geqslant 2$, $a_{i+1} \geqslant 2$ *in it.*

3. Dualities

Before proceeding further, it is appropriate, at this point, to speak of the important and very useful relations of duality (to be studied and exploited in a forthcoming publication). Here are a few examples:

$$\zeta(3) = \zeta(2,1) \qquad \text{that is to say:} \qquad \zeta(3,1^0) = \zeta(2,1),$$
$$\zeta(6,2) = \zeta(2,2,1,1,1,1) \qquad \text{that is to say:} \qquad \zeta(6,1^0,2,1^0) = \zeta(2,1^4).$$

Sometimes, the *dual* to a polyzeta ζ will be denoted using the sign $\circ$:

$$\zeta^\circ := \text{dual of the polyzeta } \zeta,$$

because in the existing theory, the sign $*$ is already used in order to denote the stuffle product (*see* below), and because we want to reserve the prime symbol $'$ for other technical purposes.

In order to find the *dual* of any polyzeta:

$$\zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}-1}, 1^{b_{\mathsf{h}-1}}, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),$$

the recipe is simple as soon as one adopts the general notation of the previous section:

☐ read the entries backwards, starting from the last entry;

☐ replace any incoming 1^{b_j} by a new single entry $b_j + 2$ which is $\geqslant 2$;

☐ replace any incoming $a_i \geqslant 2$ by a new 1^{a_i-2} with a number of repetitions of 1's which is $\geqslant 0$.

Duality Theorem 3.1. *Any convergent polyzeta is equal to its dual polyzeta:*

$$\boxed{\begin{aligned}\zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}-1}, 1^{b_{\mathsf{h}-1}}, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) &= \zeta\big(b_{\mathsf{h}}+2, 1^{a_{\mathsf{h}}-2}, b_{\mathsf{h}-1}+2, 1^{a_{\mathsf{h}-1}-2}, \dots\dots, b_1+2, 1^{a_1-2}\big) \\ &=: \Big[\zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)\Big]^\circ,\end{aligned}}$$

for every height $\mathsf{h} \geqslant 1$, *every* $a_1, \dots, a_{\mathsf{h}} \geqslant 2$ *and every* $b_1, \dots, b_{\mathsf{h}} \geqslant 0$. *Furthermore, duality is an involution:*

$$(\zeta^\circ)^\circ = \zeta$$

between convergent polyzetas.

The reader is referred to [15], p.11, for an elementary proof of these relations of duality which will not be copied here; it comes from a simple and natural change of variable in a Chen integral representing the polyzeta.

In weight $\mathsf{w} = 2$, there is only one duality: $\zeta(2) = \zeta(2)$, but it gives no information. Polyzetas equal to their dual will be said to be *self-dual*, they exist only when the weight is even, and they can well be counted, as a forthcoming work will show.

In weight $\mathsf{w} = 3$, there is only one duality:

$$\zeta(3)^\circ = \zeta(2,1),$$

namely Euler's relation.

In weight $\mathsf{w} = 4$, there are two dualities:

$$\zeta(4)^\circ = \zeta(2,1,1) \qquad \text{and} \qquad \zeta(2,2)^\circ = \zeta(2,2),$$

the latter being self-dual.

In weight $\mathsf{w} = 5$, there are four dualities:

$$\zeta(5)^\circ = \zeta(2,1,1,1), \quad \zeta(4,1)^\circ = \zeta(3,1,1), \quad \zeta(3,2)^\circ = \zeta(2,2,1), \quad \zeta(2,3)^\circ = \zeta(2,1,2).$$

In weight $\mathsf{w} = 6$, there are six dualities:

$$\begin{aligned} &\zeta(6)^\circ = \zeta(2,1,1,1,1), && \zeta(5,1)^\circ = \zeta(3,1,1,1), && \zeta(4,2)^\circ = \zeta(2,2,1,1), \\ &\zeta(3,3)^\circ = \zeta(2,1,2,1), && \zeta(2,4)^\circ = \zeta(2,1,1,2), && \zeta(3,1,2)^\circ = \zeta(2,3,1). \end{aligned}$$

Lemma 3.1. *Weight and height remain unchanged through duality:*

$$\mathsf{w}\big(\zeta^\circ\big) = \mathsf{w}(\zeta) \qquad \text{and} \qquad \mathsf{h}\big(\zeta^\circ\big) = \mathsf{h}(\zeta),$$

while depth is symmetrized across the medium weight $\frac{\mathsf{w}}{2}$*:*

$$\mathsf{d}\big(\zeta^\circ\big) = \mathsf{w}(\zeta) - \mathsf{d}(\zeta).$$

Proof. We use the general writing $\zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$. Firstly, invariance of weight amounts to the trivial arithmetical identity:

$$a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} = b_{\mathsf{h}} + 2 + a_{\mathsf{h}} - 2 + \cdots + b_1 + 2 + a_1 - 2.$$

Secondly, invariance of height is immediately visible in the duality theorem. Thirdly and lastly, one has by definition:

$$\mathsf{d}(\zeta) = 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} \qquad \text{and} \qquad \mathsf{w}(\zeta) = a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}},$$

while on the dual side:

$$\begin{aligned} d\big(\zeta^\circ\big) = 1 + a_{\mathsf{h}-2} + \cdots + 1 + a_1 - 2 &= a_1 + b_1 - 1 - b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} - 1 - b_{\mathsf{h}} \\ &= \mathsf{w}(\zeta) - \mathsf{d}(\zeta), \end{aligned}$$

which completes this quite elementary proof. □

4. Detachable and Flexible Notations

The previous section showed that the notation:

$$\zeta\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{\mathsf{b}_h}\big)$$

is, in particular, well adapted to the expression of the dualities. But in several circumstances, it is advisable, and useful, to also make visible the weight w, the depth d and the height h of any written polyzeta, since only the height is visible in such a writing. This is why we shall regularly employ the more precise notation:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{\mathsf{b}_h}\big),$$

in which w, d, h appear as lower case indices, just before the entries of the polyzeta. Sometimes, when it is understood from the context what the weight is, the mention of w will be allowed to be dropped:

$$\zeta_{[\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{\mathsf{b}_h}\big).$$

What matters most is that the notations be *flexible*, with the lower case indices $_{[\mathsf{w},\mathsf{d},\mathsf{h}]}$ or $_{[\mathsf{d},\mathsf{h}]}$ being detachable, either present or absent.

In concrete examples for instance, it is rarely advisable to specify weights, depths and heights, because they may be reconstituted by a glance. In fact, writing $\zeta_{[8,2,2]}(6,2)$ for $\zeta(6,2)$ just makes the reading harder.

Beyond this, we shall also even adopt the convention of *sometimes not writing the letter ζ at all*, writing for instance Euler's relation simply under the form:

$$(3) = (2,1).$$

Since only polyzetas will be dealt with in this paper, no risk of ambiguity, of confusion, or of misunderstanding exists. We would like to mention that in all our hand manuscripts, we found it more economical and expeditious not to write the ζ letters, exactly as one would do in computer programming.

Summary about notations 4.1. *In various places, four flexible, detachable, interchangeable notations will be employed to denote a general, arbitrary polyzeta:*

$$\boxed{\begin{aligned}&\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),\\ &\zeta\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),\\ &\zeta_{[\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),\\ &\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),\end{aligned}}$$

and these four notations will be considered to be completely equivalent.

Lemma 4.2. *For any weight $\mathsf{w} \geqslant 2$, the total number of convergent polyzetas having weight w is equal to:*

$$\mathbf{n}(\mathsf{w}) := 2^{\mathsf{w}-2}.$$

Proof. As we saw implicitly above when listing the dualities in weights $\mathsf{w} = 2, 3, 4, 5, 6$, this is already known to be true for small w.

By induction on w, suppose now that the number of convergent polyzetas $\zeta(s_1, \dots, s_\mathsf{d})$ of weight w is indeed equal to $2^{\mathsf{w}-2}$. Then we claim that any convergent polyzeta of weight $\mathsf{w}+1$ may be obtained from:

$$\zeta(s_1, \dots, s_\mathsf{d})$$

in exactly two non-overlapping ways:

□ either by adding $+1$ to the last entry, getting:

$$\zeta(s_1, \dots, s_\mathsf{d} + 1);$$

□ or by concatenating 1 after the last entry, getting:

$$\zeta(s_1, \dots, s_\mathsf{d}, 1).$$

One easily convinces oneself of this fact by observing that any polyzeta of weight $\mathsf{w}+1$ either has 1 as its last entry, or has a last entry which is $\geqslant 2$. Consequently, the number of convergent polyzetas of weight $\mathsf{w}+1$ equals twice that of weight w, namely $2 \cdot 2^{\mathsf{w}-2} = 2^{\mathsf{w}+1-2}$, completing the induction. □

Since the number of polyzetas grows exponentially with their weights, one must set up a fine ordering between all of them.

5. TOTAL ORDERING $\prec$ BETWEEN POLYZETAS OF FIXED WEIGHT

To launch the main computations of the paper, the next initial goal is to set up an appropriate total ordering between all polyzetas having equal weight. Several mental speculations that are uneasy to reproduce imposed the specific choice presented here.

In accordance to experimental and speculative conjectures about the diophantine properties of polyzetas which conducted experts to believe that no algebraic relation exists between polyzetas of different weights, only polyzetas of equal weight will be compared.

Thus, let us take any two *distinct* polyzetas of equal weight w:

$$\begin{aligned} & \zeta_{[\mathsf{w},\mathsf{d}',\mathsf{h}']}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}'}', 1^{b_{\mathsf{h}'}'}\big) \\ \text{and:} \qquad & \zeta_{[\mathsf{w},\mathsf{d}'',\mathsf{h}'']}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}''}'', 1^{b_{\mathsf{h}''}''}\big), \end{aligned}$$

but of possibly unequal depths d', d'' and heights h', h'', with entries that are arbitrary.

First of all, increasing depth will be the dominant criterion of ordering, and we declare that

$$\zeta_{[\mathsf{w},\mathsf{d}',\mathsf{h}']}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}'}', 1^{b_{\mathsf{h}'}'}\big) \prec \zeta_{[\mathsf{w},\mathsf{d}'',\mathsf{h}'']}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}''}'', 1^{b_{\mathsf{h}''}''}\big),$$

(strictly) for any entries in both sides whenever:

$$\mathsf{d}' < \mathsf{d}''.$$

Next, when the two depths are equal while heights are unequal, namely when:

$$\mathsf{d}' = \mathsf{d}'' =: \mathsf{d} \qquad \text{but} \qquad \mathsf{h}' < \mathsf{h}'',$$

the height will be the sub-dominant criterion, and we also declare in this case that:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}']}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}'}', 1^{b_{\mathsf{h}'}'}\big) \prec \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}'']}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}''}'', 1^{b_{\mathsf{h}''}''}\big),$$

for any entries in both sides.

When the two depths and the two heights are equal:

$$\mathsf{d}' = \mathsf{d}'' =: \mathsf{d} \qquad \text{and} \qquad \mathsf{h}' = \mathsf{h}'' =: \mathsf{h},$$

the third criterion for ordering will be the place where the 1 lie. Thus, consider two distinct polyzetas of same weight:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}}', 1^{b_{\mathsf{h}}'}\big) \qquad \text{and} \qquad \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}}'', 1^{b_{\mathsf{h}}''}\big)$$

having equal height h and equal depth:

$$a_1' + b_1' + \cdots + a_{\mathsf{h}}' + b_{\mathsf{h}}' = a_1'' + b_1'' + \cdots + a_{\mathsf{h}}'' + b_{\mathsf{h}}'' =: \mathsf{d}.$$

It follows that the number of 1 present in each of them is the same, for this number is, as known, equal to the difference between the (common) height and the (common) depth:

$$b_1' + \cdots + b_{\mathsf{h}}' = b_1'' + \cdots + b_{\mathsf{h}}'' = \mathsf{d} - \mathsf{h}.$$

Consequently, in any such two polyzetas to be compared and ordered, the places of the 1 are encoded by two multiindices:

$$\mathbb{N}^{\mathsf{h}} \ni \big(b_1', \dots, b_{\mathsf{h}}'\big) \qquad \text{and} \qquad \big(b_1'', \dots, b_{\mathsf{h}}''\big) \in \mathbb{N}^{\mathsf{h}}$$

of equal length:

$$|b'| = b_1' + \cdots + b_{\mathsf{h}}' = b_1'' + \cdots + b_{\mathsf{h}}'' = |b_{\mathsf{h}}''|.$$

Then by definition, whenever these two multiindices are distinct:

$$b' \neq b'',$$

we declare that:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}}', 1^{b_{\mathsf{h}}'}\big) \prec \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}}'', 1^{\mathsf{h}''}\big)$$

if the first multiindex encoding the 1 is *larger* than the second one, with respect to the *reverse lexicographic ordering*:

$$\big(b_1', \dots, b_{\mathsf{h}}'\big) >_{\mathsf{revlex}} \big(b_1'', \dots, b_{\mathsf{h}}''\big).$$

We recall that the so-called *reverse lexicographic ordering* is the standard lexicographic ordering, though read backwards from the last term, namely for any two distinct multiindices $b' \neq b''$ in $\mathbb{N}^h$, one defines:

$$\big(b_1', \dots, b_{\mathsf{h}}'\big) >_{\mathsf{revlex}} \big(b_1'', \dots, b_{\mathsf{h}}''\big)$$

in all of the following mutually exclusive and exhaustive circumstances:

□ when $b_{\mathsf{h}}' > b_{\mathsf{h}}''$;

□ or when $b_{\mathsf{h}'} = b''_{\mathsf{h}}$ but $b'_{\mathsf{h}-1} > b''_{\mathsf{h}-1}$;

□ ....

□ or when $b_{\mathsf{h}'} = b''_{\mathsf{h}}$, $b'_{\mathsf{h}-1} > b''_{\mathsf{h}-1}, \dots, b'_2 = b''_2$ but $b'_1 > b''_1$.

For instance, convergent polyzetas of depth $\mathsf{d} = 5$ and height $\mathsf{h} = 3$ are ordered in the following schematic way (entries are drawn vertically without parentheses):

$$\begin{array}{ccccccccccc}
a & & a & & a & & a & & a & & a \\
a & & a & & 1 & & a & & 1 & & 1 \\
a & \prec & 1 & \prec & a & \prec & 1 & \prec & a & \prec & 1 \\
1 & & a & & a & & 1 & & 1 & & a \\
1 & & 1 & & 1 & & a & & a & & a,
\end{array}$$

whatever the entries $a \geqslant 2$ are.

It therefore only remains to compare and to order any two distinct convergent polyzetas having same weight (as agreed), same depth, same height and in which the places of the 1 are exactly the same, so that:

$$b'_1 = b''_1 =: b_1, \qquad b'_2 = b''_2 =: b_2, \qquad \cdots\cdots, \qquad b'_{\mathsf{h}} = b''_{\mathsf{h}} =: b_{\mathsf{h}},$$

that is to say, to compare any two:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a'_1, 1^{b_1}, \dots, a'_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \qquad \text{and} \qquad \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a''_1, 1^{b_1}, \dots, a''_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big).$$

In such a circumstance, one observes that equality of weights:

$$a'_1 + b_1 + \cdots + a'_{\mathsf{h}} + b_{\mathsf{h}} = a''_1 + b_1 + \cdots + a''_{\mathsf{h}} + b_{\mathsf{h}}$$

and equality of the number of 1 immediately gives that the two multiindices:

$$\mathbb{N}^{\mathsf{h}} \ni \big(a'_1, \dots, a'_{\mathsf{h}}\big) \qquad \text{and} \qquad \big(a''_1, \dots, a''_{\mathsf{h}}\big) \in \mathbb{N}^{\mathsf{h}}$$

which encode all the entries that are $\geqslant 2$ in the two polyzetas have equal length:

$$|a'| = a'_1 + \cdots + a'_{\mathsf{h}} = a''_1 + \cdots + a''_{\mathsf{h}} = |a''|.$$

Most simply then, to compare such two distinct multiindices a' and a'' with $|a'| = |a''|$, we will declare that:

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a'_1, 1^{b_1}, \dots, a'_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \prec \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a''_1, 1^{b_1}, \dots, a''_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),$$

whenever the first multiindex is *larger* than the second one:

$$\big(a'_1, \dots, a'_{\mathsf{h}}\big) \prec_{\mathsf{lex}} \big(a''_1, \dots, a''_{\mathsf{h}}\big),$$

with respect to the standard lexicographic ordering.

For instance, the ordering of all polyzetas of weight 10, of depth 6, of height 3, and of 1-type $\zeta\big(a, 1, a, 1, 1, a\big)$ is the following — again, entries are drawn vertically without parentheses —:

$$
\begin{array}{ccccccccccc}
4 & & 3 & & 3 & & 2 & & 2 & & 2 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 3 & & 2 & & 4 & & 3 & & 2 \\
1 & \prec & 1 & \prec & 1 & \prec & 1 & \prec & 1 & \prec & 1 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 2 & & 3 & & 2 & & 3 & & 4.
\end{array}
$$

5.1. **Total ordering in small weights** $\mathsf{w} = 3, 4, 5, 6$**.** To illustrate this, let us list convergent polyzetas in accordance to this ordering (entries are drawn vertically without parentheses):

$$
\begin{array}{lcc}
 & & \mathbf{2} \\
\underline{\mathsf{w} = 3:} & \mathbf{3} & \mathbf{1}
\end{array}
\qquad
\begin{array}{lcccc}
 & & & & \mathbf{2} \\
 & & \mathbf{3} & \mathbf{2} & \mathbf{1} \\
\underline{\mathsf{w} = 4:} & \mathbf{4} & \mathbf{1} & \mathbf{2} & \mathbf{1}
\end{array}
$$

$$
\begin{array}{lcccccccc}
 & & & & & & & & \mathbf{2} \\
 & & & & & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} \\
 & & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\
\underline{\mathsf{w} = 5:} & \mathbf{5} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1}
\end{array}
$$

$$
\begin{array}{lcccccccccccccccc}
 & & & & & & & & & & & & & & & & \mathbf{2} \\
 & & & & & & & & & & & & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{1} \\
 & & & & & & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
 & & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\
\underline{\mathsf{w} = 6:} & \mathbf{6} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1}
\end{array}
$$

5.2. **Summarized presentation of the total ordering between convergent polyzetas of fixed weight.** *By definition, two arbitrary distinct convergent polyzetas of equal weight are smaller one to another:*

$$
\zeta_{[\mathsf{w},\mathsf{d}',\mathsf{h}']}\big(a_1', 1^{b_1'}, \dots, a_{\mathsf{h}'}', 1^{b_{\mathsf{h}'}'}\big) \prec \zeta_{[\mathsf{w},\mathsf{d}',\mathsf{h}']}\big(a_1'', 1^{b_1''}, \dots, a_{\mathsf{h}''}'', 1^{b_{\mathsf{h}''}''}\big)
$$

whenever firstly, either the depth of the first is smaller than that of the second:

$$
1 + b_1' + \cdots + 1 + b_{\mathsf{h}'}' = \mathsf{d}' < \mathsf{d}'' = 1 + b_1'' + \cdots + 1 + b_{\mathsf{h}''}'',
$$

or secondly, whenever their depths coincide, but the height of the first is smaller than that of the second:

$$
\mathsf{d}' = \mathsf{d}'' \qquad \text{but} \qquad \mathsf{h}' < \mathsf{h}'',
$$

or thirdly, whenever both their depths and their heights $\mathsf{h}' = \mathsf{h}'' =: \mathsf{h}$ *coincide, but their* b*-integers encoding their places of the* 1 *satisfy:*

$$
\big(b_1', \dots, b_{\mathsf{h}}'\big) >_{\mathsf{revlex}} \big(b_1'', \dots, b_{\mathsf{h}}''\big),
$$

or fourthly and lastly, whenever depths, heights and places of the 1 *coincide, but:*

$$
\big(a_1', \dots, a_{\mathsf{h}}'\big) >_{\mathsf{lex}} \big(a_1'', \dots, a_{\mathsf{h}}''\big).
$$

6. INTERMEDIATE COUNTINGS

We remember that the total number of convergent polyzetas of weight w is equal to:

$$\mathbf{n}(\mathsf{w}) := 2^{\mathsf{w}-2}.$$

More precise intermediate countings can now be provided.

Lemma 6.1. *The total number of convergent polyzetas:*

$$\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$$

of weight w*, of depth* d*, of height* h *and whose entries* 1 *are placed according to a fixed multiindex* $(b_1, \dots, b_{\mathsf{h}}) \in \mathbb{N}^{\mathsf{h}}$ *is equal to:*

$$\mathbf{n}\big(\mathsf{w}, \mathsf{d}, \mathsf{h}, (b_1, \dots, b_{\mathsf{h}})\big) := \binom{\mathsf{w}-\mathsf{d}-1}{\mathsf{h}-1}.$$

Observe that this number does not depend on the assignment of places to the 1, *i.e.* $\mathbf{n}\big(\mathsf{w}, \mathsf{d}, \mathsf{h}, (b_1, \dots, b_{\mathsf{h}})\big)$ is — visibly — independent of $(b_1, \dots, b_{\mathsf{h}}) \in \mathbb{N}^{\mathsf{h}}$.

Proof. Indeed, in such convergent polyzetas, only the integers $a_i \geqslant 2$ can still vary, but with the only constraint that their sum equals:

$$\begin{aligned} a_1 + \cdots + a_{\mathsf{h}} &= \mathsf{w} - b_1 - \cdots - b_{\mathsf{h}} \\ &= \mathsf{w} + \mathsf{h} - \mathsf{d}. \end{aligned}$$

But it is either elementary to check or already known that:

$$\mathsf{Card}\big\{\big(\widetilde{a}_1, \dots, \widetilde{a}_{\mathsf{h}}\big) \in \mathbb{N}^{\mathsf{h}} \colon\; \widetilde{a}_1 + \cdots + \widetilde{a}_{\mathsf{h}} = \mathsf{m}\big\} = \binom{\mathsf{m}+\mathsf{h}-1}{\mathsf{h}-1}.$$

Since the a_i must be $\geqslant 2$, in order to come to some $\widetilde{a}_i$ that are $\geqslant 0$, one has to subtract 2 each time:

$$\underbrace{a_1 - 2}_{=:\,\widetilde{a}_1} + \cdots + \underbrace{a_{\mathsf{h}} - 2}_{=:\,\widetilde{a}_{\mathsf{h}}} = \mathsf{w} + \mathsf{h} - \mathsf{d} - 2\,\mathsf{h} = \underbrace{\mathsf{w} - \mathsf{d} - \mathsf{h}}_{=:\,\mathsf{m}}$$

so that substituting $\mathsf{m} := \mathsf{w} + \mathsf{h} - \mathsf{d}$ in the binomial completes the proof. □

Lemma 6.2. *The total number of convergent polyzetas* $\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}$ *of weight* w*, of depth* d *and of height* h *is equal to:*

$$\mathbf{n}(\mathsf{w}, \mathsf{d}, \mathsf{h}) := \binom{\mathsf{d}-1}{\mathsf{d}-\mathsf{h}}\binom{\mathsf{w}-\mathsf{d}-1}{\mathsf{h}-1}.$$

Proof. Of course, the first entry of a convergent polyzeta $\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}$ must be $\geqslant 2$, namely it must always be an $a_1 \geqslant 2$. But then, the $\mathsf{d} - \mathsf{h}$ existing 1 can take any place in the $\mathsf{d} - 1$ entries of a convergent $\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}$ which lie after a_1. As is known, the number of places that the 1 can be so given in a convergent $\zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}$ is equal to the plain binomial:

$$\binom{\mathsf{d}-1}{\mathsf{d}-\mathsf{h}}.$$

Once the places of the 1 are fixed, the number $\binom{\mathsf{w}-\mathsf{d}-1}{\mathsf{h}-1}$ of variations of the a_i was counted just above, and lastly, the total number of possibilities is multiplicative. □

Corollary 6.3. *The total number of convergent polyzetas of weight* w *and of depth* d *is equal to:*

$$\mathbf{n}(\mathsf{w},\mathsf{d}) := \sum_{\mathsf{h}=1}^{\mathsf{h}=2} \binom{\mathsf{d}-1}{\mathsf{d}-\mathsf{h}} \binom{\mathsf{w}-\mathsf{d}-1}{\mathsf{h}-1}. \quad \square$$

For fun and for later use, let us list these dimensions $\mathbf{n}(\mathsf{w},\mathsf{d})$ in weights 6, 7, 8, 9, 10.

$$\underline{\mathsf{w}=6:} \qquad \left.\begin{aligned} 1 &= \mathbf{1} \\ 1\cdot 1 + 1\cdot 3 &= \mathbf{4} \\ 1\cdot 1 + 2\cdot 2 + 1\cdot 1 &= \mathbf{6} \\ 1\cdot 1 + 1\cdot 3 &= \mathbf{4} \\ 1\cdot 1 &= \mathbf{1} \end{aligned}\right] \quad \mathbf{16}.$$

$$\underline{\mathsf{w}=7:} \qquad \left.\begin{aligned} 1 &= \mathbf{1} \\ 1\cdot 1 + 1\cdot 4 &= \mathbf{5} \\ 1\cdot 1 + 2\cdot 3 + 1\cdot 3 &= \mathbf{10} \\ 1\cdot 1 + 2\cdot 3 + 1\cdot 3 &= \mathbf{10} \\ 1\cdot 1 + 1\cdot 4 &= \mathbf{5} \\ 1\cdot 1 &= \mathbf{1} \end{aligned}\right] \quad \mathbf{32}.$$

$$\underline{\mathsf{w}=8:} \qquad \left.\begin{aligned} 1 &= \mathbf{1} \\ 1\cdot 1 + 1\cdot 5 &= \mathbf{6} \\ 1\cdot 1 + 2\cdot 4 + 1\cdot 6 &= \mathbf{15} \\ 1\cdot 1 + 3\cdot 3 + 3\cdot 3 + 1\cdot 1 &= \mathbf{20} \\ 1\cdot 1 + 2\cdot 4 + 1\cdot 6 &= \mathbf{15} \\ 1\cdot 1 + 1\cdot 5 &= \mathbf{6} \\ 1 &= \mathbf{1} \end{aligned}\right] \quad \mathbf{64}.$$

$$\underline{\mathsf{w}=9:} \qquad \left.\begin{aligned} 1 &= \mathbf{1} \\ 1\cdot 1 + 1\cdot 6 &= \mathbf{7} \\ 1\cdot 1 + 2\cdot 5 + 1\cdot 10 &= \mathbf{21} \\ 1\cdot 1 + 3\cdot 4 + 3\cdot 6 + 1\cdot 4 &= \mathbf{35} \\ 1\cdot 1 + 3\cdot 4 + 3\cdot 6 + 1\cdot 4 &= \mathbf{35} \\ 1\cdot 1 + 2\cdot 5 + 1\cdot 10 &= \mathbf{21} \\ 1\cdot 1 + 1\cdot 6 &= \mathbf{7} \\ 1 &= \mathbf{1} \end{aligned}\right] \quad \mathbf{128}.$$

$$
\underline{\mathsf{w}=10:}\qquad
\left.\begin{array}{r}
1=\mathbf{1}\\
1\cdot 1+1\cdot 7=\mathbf{8}\\
1\cdot 1+2\cdot 6+1\cdot 15=\mathbf{28}\\
1\cdot 1+3\cdot 5+3\cdot 10+1\cdot 10=\mathbf{56}\\
1\cdot 1+4\cdot 4+6\cdot 6+4\cdot 4+1\cdot 1=\mathbf{70}\\
1\cdot 1+3\cdot 5+3\cdot 10+1\cdot 10=\mathbf{56}\\
1\cdot 1+2\cdot 6+1\cdot 15=\mathbf{28}\\
1\cdot 1+1\cdot 7=\mathbf{8}\\
1=\mathbf{1}
\end{array}\right]\ \mathbf{256}.
$$

7. SHUFFLE MINUS STUFFLE

The result of the usual multiplication between any two polyzetas is known to be re-expressible in *two* ways, either as a polyzeta whose entries come from the combinatorial $*$-stuffle product between the two concerned entries, or as a polyzeta whose entries come from the combinatorial $\sqcup\!\sqcup$-shuffle product between the two concernend entries:

$$
\begin{aligned}
\underbrace{\zeta\big(\alpha_1,1^{\beta_1},\dots,\alpha_{\mathsf{g}},1^{b_{\mathsf{g}}}\big)\cdot\zeta\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)}_{\text{usual multiplication}} &= \\
&=\zeta\Big(\underbrace{\big(\alpha_1,1^{\beta_1},\dots,\alpha_{\mathsf{g}},1^{b_{\mathsf{g}}}\big)*\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)}_{\text{combinatorial s\underline{t}uffle product between the two entries}}\Big)\\
&=\zeta\Big(\underbrace{\big(0^{\alpha_1-1}1\,1^{\beta_1}\cdots 0^{\alpha_{\mathsf{g}}-1}1\,1^{\beta_{\mathsf{g}}}\big)\sqcup\!\sqcup\big(0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}\big)}_{\text{combinatorial s\underline{h}uffle product between the two entries}}\Big).
\end{aligned}
$$

Here, we use the simplest binary symbols 0 and 1 — instead of x_0 and x_1, or instead of x and y as is usually done — to translate any polyzeta in its second encoding:

$$
\boxed{\underbrace{\big(a_1,1^{b_1},\dots,a_{\mathsf{h}},1^{b_{\mathsf{h}}}\big)}_{\substack{\text{encoding used}\\ \text{to compute }*\text{-products}}}\overset{\longleftrightarrow}{=}\underbrace{\big(0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}\big)}_{\substack{\text{encoding used}\\ \text{to compute }\sqcup\!\sqcup\text{-products}}}.}
$$

Since the introductory literature is rich enough, we will not make any further reminder here. Observably, we will not copy the classical inductive rules for the computation of stuffle and shuffle products, *because the non-closed character of these inductive rules is a landmark of a hidden defective knowledge concerning what they really give.*

Just let us restrict ourselves with expressing in full generality the known-to-hold double shuffle relations that we want, in the four special cases mentioned in the introduction, to close up, namely in the particular circumstances when:

$$\big(\alpha_1, 1^{\beta_1}, \ldots, \alpha_{\mathsf{g}}, 1^{\beta_{\mathsf{h}}}\big) = \begin{cases} (1); \\ (2); \\ (3); \\ (2,1). \end{cases}$$

Theorem 7.1. *For every two heights* $\mathsf{h} \geqslant 1$, $\mathsf{g} \geqslant 1$, *every two collections of entries* $\alpha_1 \geqslant 2, \ldots, \alpha_{\mathsf{g}} \geqslant 2$, $a_1 \geqslant 2, \ldots, a_{\mathsf{h}} \geqslant 2$ *and for every two collections of entries* $\beta_1 \geqslant 0, \ldots, \beta_{\mathsf{g}} \geqslant 0$, $b_1 \geqslant 0, \ldots, b_{\mathsf{h}} \geqslant 0$, *one has the so-called double-shuffle relations:*

$$\begin{aligned} 0 = &- \big(\alpha_1, 1^{\beta_1}, \ldots, \alpha_{\mathsf{g}}, 1^{b_{\mathsf{g}}}\big) * \big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \\ &+ \big(0^{\alpha_1-1} 1\, 1^{\beta_1} \cdots 0^{\alpha_{\mathsf{g}}-1} 1\, 1^{\beta_{\mathsf{g}}}\big) \sqcup\!\sqcup \big(0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}\big). \end{aligned}$$

By automatic cancelling out of two nonconvergent polyzetas both equal to $\big(1, a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *through the subtraction, these double shuffle relations also hold true when:*

$$\big(\alpha_1, 1^{\beta_1}, \ldots, \alpha_{\mathsf{g}}, 1^{\beta_{\mathsf{g}}}\big) = (1).$$

8. Computing $-(1) * \big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 1 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$

8.1. **Computing firstly the stuffle product** $(1) * \big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$**.** Granted that we want a result of weight w, we aim to compute in general the stuffle product of $\zeta(1)$ with an arbitrary convergent polyzeta of weight $\mathsf{w} - 1$:

$$\zeta(1) * \zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big),$$

that it to say in greater length, we aim to compute:

$$(1) * \big(a_1, \overbrace{1, \ldots, 1}^{b_1}, a_2, \overbrace{1, \ldots, 1}^{b_2}, \bullet\bullet\bullet\bullet\bullet, a_{\mathsf{h}}, \overbrace{1, \ldots, 1}^{b_{\mathsf{h}}}\big).$$

According to the general rule, one must either add 1 to each one of the entries of $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}$, or insert 1 between each two of its entries, including the extremes. For instance (remember that we allow dropping the letter ζ):

$$\begin{aligned} (1) * (4,1,1) = &(5,1,1) + (4,2,1) + (4,1,2) + && \text{[additions]} \\ &+ (1,4,1,1) + (4,1,1,1) + (4,1,1,1) + (4,1,1,1) && \text{[insertions],} \end{aligned}$$

and this gives, after gathering equal terms:

$$(1) * (4,1,1) = (5,1,1) + (4,2,1) + (4,1,2) + (1,4,1,1) + 3\,(4,1,1,1).$$

Drawing on this example, let us perform the general computation in the simpler case where the height h is equal to 1:

$$\begin{aligned}(1)*\big(a_1,\overbrace{1,\dots,1}^{b_1}\big) &= \big(a_1+1,1,\dots,1\big)+\big(a_1,2,1,\dots,1\big)+\big(a_1,1,2,1,\dots,1\big)+\\ &+\cdots+\big(a_1,1,\dots,2,1\big)+\big(a_1,1,\dots,1,2\big)+ \qquad \text{[additions]}\\ &+\big(1,a_1,1,\dots,1\big)+\big(a_1,1,1,\dots,1\big)+\big(a_1,1,1,1,\dots,1\big)+\\ &+\cdots+\big(a_1,1,\dots,1,1,1\big)+\big(a_1,1,\dots,1,1\big) \qquad \text{[insertions].}\end{aligned}$$

In the last two lines, all terms except the first one are equal so that after gathering:

$$\begin{aligned}\text{insertions} &= \big(1,a_1,\overbrace{1,\dots,1}^{b_1}\big)+(b_1+1)\big(a_1,\overbrace{1,1,\dots,1}^{b_1+1}\big)\\ &= \zeta\big(1,a_1,1^{b_1}\big)+(b_1+1)\,\zeta\big(a_1,1^{b_1+1}\big).\end{aligned}$$

Of course, the first $\zeta\big(1,a_1,1^{b_1}\big)$ is non-convergent, but as is known, it will simply disappear in the double shuffle-stuffle subtraction:

$$-(1)*\big(a_1,1^{b_1}\big)+(1)\sqcup\!\sqcup\big(a_1,1^{b_1}\big),$$

since it will also appear in $1\sqcup\!\sqcup\big(a_1,1^{b_1}\big)$. Here and everywhere below, we intentionally write the minus sign *in the first position*, because the terms involving the stuffle $*$ are of smallest depth, so that they should come *first* if one respects the chosen total order between polyzetas of fixed weight.

Let us also rewrite the result of additions under a concise sum-like form:

$$\text{additions} = \big(a_1+1,1^{b_1}\big)+\sum_{\substack{b_1'+b_1''=b_1-1\\ b_1'\geqslant 0,\ b_1''\geqslant 0}}\big(a_1,\dots,1^{b_1'},2,1^{b_1''},\dots\big).$$

An inspection of depths and heights of appearing terms enables us to summarize the result in the special case $\mathsf{h}=1$.

Lemma 8.1. *For every $a_1\geqslant 2$ and every $b_1\geqslant 0$, if one sets:*

$$a_1+b_1=:\mathsf{w}-1 \qquad \text{and} \qquad 1+b_1=:\mathsf{d},$$

then the $*$-stuffle product of $\zeta(1)$ with any $\zeta_{[\mathsf{w}-1,\mathsf{d},1]}\big(a_1,1^{b_1}\big)$ of weight $\mathsf{w}-1\geqslant 2$, of height d and of height 1 writes out, after complete finalization:

$$\begin{aligned}\zeta(1)*\zeta_{[\mathsf{w}-1,\mathsf{d},1]}\big(a_1,1^{b_1}\big) &= \zeta_{[\mathsf{w},\mathsf{d},1]}\big(a_1+1,1^{b_1}\big)+\\ &+\sum_{\substack{b_1'+b_1''=b_1-1\\ b_1'\geqslant 0,\ b_1''\geqslant 0}}\zeta_{[\mathsf{w},\mathsf{d},2]}\big(a_1,1^{b_1'},2,1^{b_1''}\big)+\\ &+\underbrace{\zeta_{[\mathsf{w},\mathsf{d}+1,1]}\big(1,a_1,1^{b_1}\big)}_{\substack{\text{non-convergent}\\ \text{but will disappear}}}+(b_1+1)\,\zeta_{[\mathsf{w},\mathsf{d}+1,1]}\big(a_1,1^{b_1+1}\big).\end{aligned}$$

Notice that, except for the single appearing non-convergent polyzeta, the terms are ordered in accordance with the total (sub-)ordering:

$$\zeta_{[\mathsf{w},\mathsf{d}',\mathsf{h}']} \prec \zeta_{[\mathsf{w},\mathsf{d}'',\mathsf{h}'']} \quad \text{if} \quad \begin{cases} \text{either } \ \mathsf{d}' < \mathsf{d}'' \\ \text{or} \qquad \mathsf{d}' = \mathsf{d}'' \quad \text{but } \ \mathsf{h}' < \mathsf{h}''. \end{cases}$$

Now, in the general case $\mathsf{h} \geqslant 1$, the procedure will be quite similar, with no obstacle of combinatorics understanding.

One one hand, additions of 1 will concern firstly the a_i, $i = 1, \dots, \mathsf{h}$, and secondly, the 1 of each 1^{b_j}, $j = 1, \dots, \mathsf{h}$, hence we will have similar concise sum-like expressions for addition terms.

One the second hand, insertions of the additional 1 will be achieved in such a way that, if 1 is inserted *just after* an a_i, one will consider that *equivalently*, the 1 is inserted just before the first 1 of 1^{b_i} which sits just after the a_i in question, so that one gets a group of $(b_i + 1)$ entries 1, namely 1^{b_i+1} (the 1^{b_i} '*absorbs*' the inserted 1). Completely similarly, if the additional 1 is inserted *just before* an a_i, and if i is $\geqslant 2$, one will consider that the inserted 1 is *absorbed* by the group of 1 of the $1^{b_{i-1}}$ which sits just before the a_i in question, so that one gets an $1^{b_{i-1}+1}$ in this way. The only exception is when 1 is inserted just before a_1, and this produces the only non-convergent polyzeta $\big(1, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$, which will anyway disappear at the end.

Granted these explanations, we may state the general lemma without further proof, but with a few comments just after. Later, subsequent statements that are less direct and more complex will be established with all details.

Lemma 8.2. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$\begin{aligned} a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 1, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d}, \end{aligned}$$

then the $*$*-stuffle product of* $\zeta(1)$ *with any* $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 1 \geqslant 2$*, of depth* d *and of height* h *writes out, after complete finalization:*

$$\begin{aligned} \zeta(1) * \zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) &= \sum_{i=1}^{\mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 1, \bullet\bullet\bullet\bullet\bullet\big) + \\ &\quad + \sum_{\substack{j=1 \\ b_j \geqslant 1}}^{\mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \end{aligned}$$

$$+ \underbrace{\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(1, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)}_{\substack{\text{non-convergent} \\ \text{but will disappear}}} +$$

$$+ \sum_{j=1}^{\mathsf{h}} (b_j + 1)\, \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big).$$

Here by convention, in the right-hand side, only the terms of the initial polyzeta $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}$ that are *changed* are written down, so that the symbol:

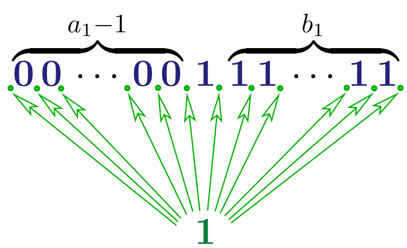
$$\bullet\bullet\bullet\bullet\bullet$$

means that all other entries are unchanged.

With constancy and regularity, the letter 'i' will always be used in relation with the entries $a_i \geqslant 2$, especially in summation symbols. Later below, i', i'', i''' and i_1, i_2, i_3 will also be used.

Similarly, the letter j will always be used in link with the entries $b_j \geqslant 0$, specially in summation symbols.

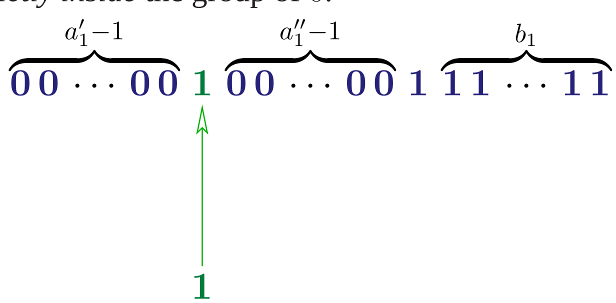
8.2. **Computing secondly the shuffle product** $(1) \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$**.** To begin with, let us study the case of smallest height $\mathsf{h} = 1$, so that one has to insert 1 in any place between two successive entries of the polyzeta $\big(a_1, 1^{b_1}\big)$, after re-coding it in terms of 0 and 1:



Since its depth equals $a_1 + b_1$, there visibly are $a_1 + b_1 + 1$ possible insertions of 1. Let us set apart the case where 1 is inserted in the front place:

$$\mathbf{1}\, \overbrace{\mathbf{0\,0} \cdots \mathbf{0\,0}}^{a_1-1}\, \mathbf{1}\, \overbrace{\mathbf{1\,1} \cdots \mathbf{1\,1}}^{b_1},$$

since it corresponds to the only obtained non-convergent polyzeta $\zeta\big(1, a_1, 1^{b_1}\big)$. Suppose now that the 1 is inserted *strictly inside* the group of 0:



with $a_1' - 1 \geqslant 1$ in the diagram, with $a_1'' - 1 \geqslant 1$ and with:

$$a_1' - 1 + a_1'' - 1 = a_1 - 1 = \text{unchanged total number of } 0,$$

so that one has:

$$a_1' + a_1'' = a_1 + 1.$$

In other words, we have required that the inserted 1 does not enter in contact neither with the 1 which terminates the series $00 \cdots 001$ which encodes a_1, nor with the 1 which constitute 1^{b_1}, and one obtains in sum:

$$\sum_{\substack{a_1' + a_1'' = a_1 + 1 \\ a_1' \geqslant 2,\ a_1'' \geqslant 2}} \zeta\big(a_1', a_1'', 1^{b_1}\big).$$

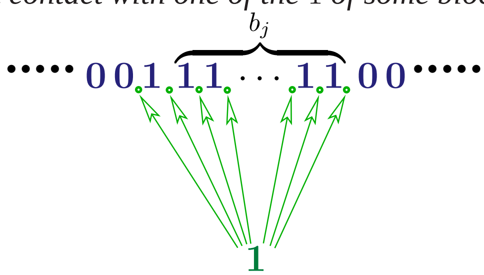
Principle of agglutination for shuffle-insertions of 1. *Any additional entry:*

$$\mathbf{1}$$

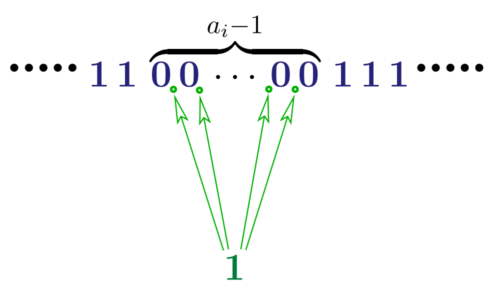
which is shuffle-inserted in some general polyzeta:

$$\overbrace{0\,0 \cdots 0\,0}^{a_1-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_1}\,\overbrace{0\,0 \cdots 0\,0}^{a_2-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_2} \,\cdots\cdots\, \overbrace{0\,0 \cdots 0\,0}^{a_{\mathsf{h}}-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_{\mathsf{h}}}$$

at some place lying in direct contact with one of the 1 of some block $1\,1^{b_j}$:

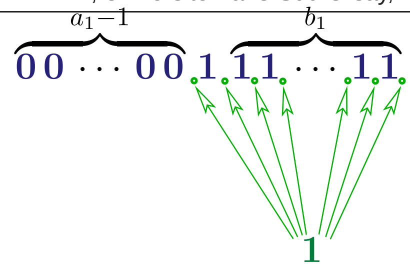


should be thought of as being automatically agglutinated to the group of 1 of the $1\,1^{b_j}$ in question, and therefore, other insertions of such an additional 1 in the 0 of some block 0^{a_i-1}:



should always be done strictly inside these 0, namely never in any two extremities of a block 0^{a_i-1}.

In accordance with such a rule and coming back to our sample $(1) \sqcup \big(a_1, 1^{b_1}\big)$ studied in the simple case of height $\mathsf{h} = 1$, it therefore only remains to look at all insertions of 1 which enter in contact with the 1^{b_1}:



But evidently, in each one of these b_1+2 circumstances, one gets:

$$\overbrace{0\,0\cdots 0\,0}^{a_1-1}\,1\,\overbrace{1\,1\cdots 1\cdots 1\,1}^{b_1+1}=\big(a_1,1^{b_1+1}\big),$$

so that one obtains in sum:

$$(b_1+2)\,\zeta\big(a_1,1^{b_1+1}\big).$$

Lemma 8.3. *For every $a_1\geqslant 2$ and every $b_1\geqslant 0$, if one sets:*

$$a_1+b_1=:\mathsf{w}-1 \qquad\text{and}\qquad 1+b_1=:\mathsf{d},$$

then the $ш$-shuffle product of $\zeta(1)$ with any $\zeta_{[\mathsf{w}-1,\mathsf{d},1]}\big(a_1,1^{b_1}\big)$ of weight $\mathsf{w}-1\geqslant 2$, of depth d and of height 1 writes out, after complete finalization:

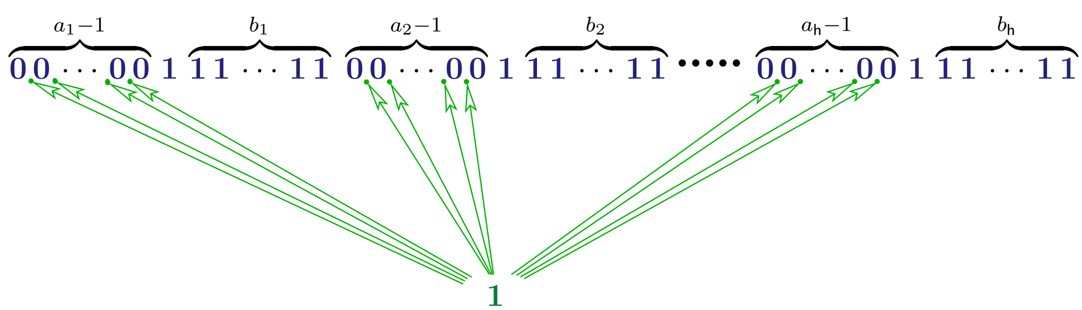
$$\zeta(1)\,ш\,\zeta_{[\mathsf{w}-1,\mathsf{d},1]}\big(a_1,1^{b_1}\big)=(b_1+2)\,\zeta_{[\mathsf{w},\mathsf{d}+1,1]}\big(a_1,1^{b_1+1}\big)+\underbrace{\zeta_{[\mathsf{w},\mathsf{d}+1,1]}\big(1,a_1,1^{b_1}\big)}_{\text{non-convergent}}+$$

$$+\sum_{\substack{a_1'+a_1''=a_1+1\\ a_1'\geqslant 2,\ a_1''\geqslant 2}}\zeta_{[\mathsf{w},\mathsf{d}+1,2]}\big(a_1',a_1'',1^{b_1}\big).\qquad\square$$

In the general case of arbitrary height $\mathsf{h}\geqslant 1$:

$$\mathbf{1}\,ш\,\overbrace{0\,0\cdots 0\,0}^{a_1-1}\,1\,\overbrace{1\,1\cdots 1\,1}^{b_1}\,\bullet\bullet\bullet\bullet\bullet\,\overbrace{0\,0\cdots 0\,0}^{a_{\mathsf{h}}-1}\,1\,\overbrace{1\,1\cdots 1\,1}^{b_{\mathsf{h}}},$$

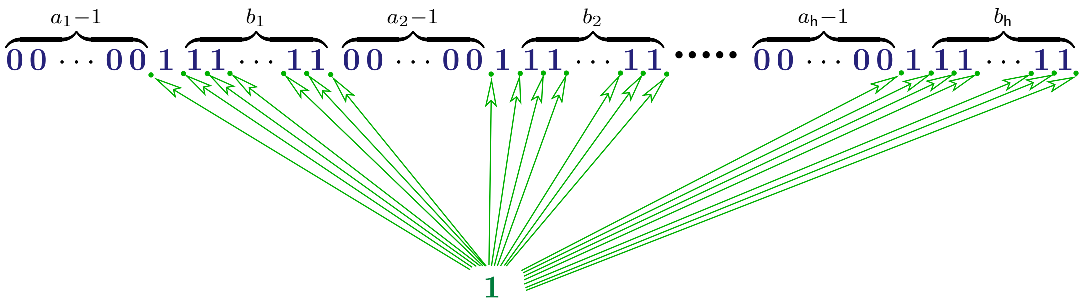
the reasonings are quite similar. Setting apart the insertion of 1 in the front place which provides the only non-convergent polyzeta, one firstly considers all possible insertions of 1 strictly inside some group of 0 of some 0^{a_i-1}:



and this gives without delay:

$$\sum_{\substack{i=1 \\ a_i \geqslant 3}}^{\mathsf{h}} \Bigg(\sum_{\substack{a_i'+a_i''=a_i+1 \\ a_i' \geqslant 2,\ a_i'' \geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg).$$

(Let us remind that implicitly, there is an 1^0 between a_i' and a_i'' just here.) Secondly, in accordance with the principle of agglutination stated above, one considers all possible insertions of 1 which happen to lie in direct contact with some 1 of some block $1\, 1^{b_j}$:



and naturally, one obtains a sum, for $j = 1, \ldots, \mathsf{h}$, of terms like the first one of the preceding lemma:

$$\sum_{j=1}^{\mathsf{h}} (b_j + 2)\, \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big).$$

All the explanations provided so far offer us the following result, fully proved now.

Lemma 8.4. *For every height* $\mathsf{h} \geqslant 1$, *every entries* $a_1 \geqslant 2, \ldots, a_{\mathsf{h}} \geqslant 2$, *every entries* $b_1 \geqslant 0, \ldots, b_{\mathsf{h}} \geqslant 0$, *if one sets:*

$$\begin{aligned} a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 1, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d}, \end{aligned}$$

then the $⧢$*-shuffle product of* $\zeta(1)$ *with any* $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 1 \geqslant 2$, *of depth* d *and of height* h *writes out, after complete finalization:*

$$\begin{aligned} \zeta(1) ⧢ \zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) = & \sum_{j=1}^{\mathsf{h}} (b_j + 2)\, \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\ & + \underbrace{\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(1, a_1, 1^{b_1}, \ldots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)}_{\substack{\text{non-convergent} \\ \text{but will disappear}}} + \\ & + \sum_{\substack{i=1 \\ a_i \geqslant 3}}^{\mathsf{h}} \Bigg(\sum_{\substack{a_i'+a_i''=a_i+1 \\ a_i' \geqslant 2,\ a_i'' \geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg). \end{aligned}$$

A concrete example — respecting the above three-lines writing — is:

$$\begin{aligned}(1) \sqcup\!\sqcup \big(3,1,4,1\big) = 3\,\big(3,1,1,4,1\big) + 3\,\big(3,1,4,1,1\big) &+ \\ + \big(1,3,1,4,1\big) &+ \\ + \big(2,2,1,4,1\big) + \big(3,1,3,2,1\big) + \big(3,1,2,3,1\big)&.\end{aligned}$$

8.3. **The subtraction.** Now that we have computed both the stuffle and the shuffle production of $\zeta(1)$ with any $\zeta\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$, it only remains to perform the final subtraction of the two results provided by the lemmas above:

$$\begin{aligned}0 &= -\,(1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 1 \sqcup\!\sqcup \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) \\ &= -\sum_{i=1}^{\mathsf{h}} \big(\bullet\bullet\bullet\bullet\bullet, a_i+1, \bullet\bullet\bullet\bullet\bullet\big) - \\ &\quad - \sum_{\substack{j=1\\ b_j \geqslant 1}}^{\mathsf{h}} \Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}} \big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg) - \\ &\quad - \underline{\big(1, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)}_{\circ} - \sum_{j=1}^{\mathsf{h}} (b_j+1)\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\ &\quad + \sum_{j=1}^{\mathsf{h}} (b_j+2)\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\ &\quad + \underline{\big(1, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)}_{\circ} + \\ &\quad + \sum_{\substack{i=1\\ a_i\geqslant 3}}^{\mathsf{h}} \Bigg(\sum_{\substack{a_i'+a_i''=a_i+1\\ a_i'\geqslant 2,\ a_i''\geqslant 2}} \big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg).\end{aligned}$$

As is known and as was expected, the two (underlined) non-convergent polyzetas annihilate. Also, the terms involving the multiplicities $-(b_j+1)$ in the fourth line and the ones involving the multiplicities (b_j+2) in the fifth line immediately collapse, while no other terms simplify.

With the mentions of weights, of depths and of heights, we therefore have gained the first fundamental theorem of the present article, classically attributed to Hoffman, but exhibited here in a more detailed way, including rigorous explicitation of quantifiers.

Theorem 8.1. *For every height* $\mathsf{h} \geqslant 1$, *every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$, *every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$, *if one sets:*

$$\begin{aligned}a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 1, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d},\end{aligned}$$

then the so-called double shuffle relation:

$$0 = -\,(1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 1 \sqcup\!\sqcup\, 0^{a_1-1}1\,1^{b_1}\ \cdots\ 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$$

between $\zeta(1)$ *and any* $\zeta_{[\mathsf{w}-1,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 1 \geqslant 2$*, of depth* d *and of height* h *writes out after complete finalization:*

$$
\boxed{
\begin{aligned}
0 = & -\sum_{i=1}^{\mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 1, \bullet\bullet\bullet\bullet\bullet\big) - \\
& - \sum_{\substack{j=1 \\ b_j \geqslant 1}}^{\mathsf{h}} \Bigg(\sum_{b_j' + b_j'' = b_j - 1} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\
& + \sum_{j=1}^{\mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\
& + \sum_{\substack{i=1 \\ a_i \geqslant 3}}^{\mathsf{h}} \Bigg(\sum_{\substack{a_i' + a_i'' = a_i + 1 \\ a_i' \geqslant 2,\ a_i'' \geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', a_i'', \bullet\bullet\bullet\bullet\bullet\big) \Bigg).
\end{aligned}
}
$$

In fact, the standard statement of Hoffman's relations (*cf. e.g.* [15], p. 3):

$$
\begin{aligned}
0 = & -\sum_{k=1}^{\mathsf{d}} \zeta\big(s_1, \dots, s_{k-1}, s_k + 1, s_{k+1}, \dots, s_{\mathsf{d}}\big) + \\
& + \sum_{\substack{k=1 \\ s_k \geqslant 2}}^{\mathsf{d}} \sum_{j=0}^{s_k - 2} \zeta\big(s_1, \dots, s_{k-1}, s_k - j, j + 1, s_{k+1}, \dots, s_{\mathsf{d}}\big),
\end{aligned}
$$

usually makes no mention of the variable depths and of the variable heights of the appearing polyzetas, while the summations are not organized according to any ordering between all polyzetas of fixed weight.

So is our first — by far the easiest amongst six — theorem.

9. Computing $-(2) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 01 \sqcup\!\sqcup\, 0^{a_1-1}1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1}1\, 1^{b_{\mathsf{h}}}$

9.1. **Computing firstly the stuffle product** $(2) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$**.** What next goal should be is clear: write out the general double shuffle relations with $\zeta(2)$ as a first member.

We start by treating the $*$-stuffle product, in the (simpler) sample case of height $\mathsf{h} = 1$. Exactly as in the case of $\zeta(1) * \big(a_1, 1^{b_1}\big)$, the computation of $\zeta(2) * \big(a_1, 1^{b_1}\big)$ involves addition terms and insertion terms:

$$
\begin{aligned}
(1) * \big(a_1, 1^{b_1}\big) = & \big(a_1 + 2, 1^{b_1}\big) + \big(a_1, 3, 1^{b_1-1}\big) + \cdots + \big(a_1, 1^{b_1-1}, 3\big) + && \text{[additions]} \\
& + \big(2, a_1, 1^{b_1}\big) + \big(a_1, 2, 1^{b_1}\big) + \cdots + \big(a_1, 1^{b_1}, 2\big), && \text{[insertions]}
\end{aligned}
$$

the only difference being that no non-convergent polyzeta appears, now. Using summation symbols, the result may then be expressed as:

$$(2) * \big(a_1, 1^{b_1}\big) = \big(a_1 + 2, 1^{b_1}\big) + \sum_{\substack{b_1' + b_1'' = b_1 - 1 \\ b_1' \geqslant 0,\ b_1'' \geqslant 0}} \big(a_1, 1^{b_1'}, 3, 1^{b_1''}\big) + \\ + \big(2, a_1, 1^{b_1}\big) + \sum_{\substack{b_1' + b_1'' = b_1 \\ b_1' \geqslant 0,\ b_1'' \geqslant 0}} \big(a_1, 1^{b_1'}, 2, 1^{b_1''}\big).$$

The general case of arbitrary height $\mathsf{h} \geqslant 1$ is quite similar, hence we state it directly without additional words.

Lemma 9.1. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} =: \mathsf{w} - 2, \\ 1 + b_1 + \cdots + 1 + b_{\mathsf{h}} =: \mathsf{d},$$

then the $*$*-stuffle product of* $\zeta(2)$ *with any* $\zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w} - 2 \geqslant 2$*, of depth* d *and of height* h *writes out, after complete finalization:*

$$\zeta(2) * \zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) = \sum_{i=1}^{\mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i + 2, \bullet\bullet\bullet\bullet\bullet\big) + \\ + \sum_{\substack{j=1 \\ b_j \geqslant 1}}^{\mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j - 1 \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 3, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg) + \\ + \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(2, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \\ + \sum_{j=1}^{\mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg).$$

Notice that in the right-hand side the polyzetas appear firstly according to increasing depth, secondly according to increasing height.

9.2. **Computing secondly the shuffle product** $01 \sqcup\!\sqcup\, 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1\, 1^{b_{\mathsf{h}}}$**.** Now, the computational task begins to be more substantial. One has to shuffle-insert 01:

$$\mathbf{01} \sqcup\!\sqcup\, \overbrace{\mathbf{0\,0} \cdots \mathbf{0\,0}}^{a_1-1}\, \mathbf{1}\, \overbrace{\mathbf{1\,1} \cdots \mathbf{1\,1}}^{b_1} \bullet\bullet\bullet\bullet\bullet \overbrace{\mathbf{0\,0} \cdots \mathbf{0\,0}}^{a_{\mathsf{h}}-1}\, \mathbf{1}\, \overbrace{\mathbf{1\,1} \cdots \mathbf{1\,1}}^{b_{\mathsf{h}}}$$

in a general polyzeta of weight, say, equal to:

$$a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} =: \mathsf{w} - 2,$$

of depth, say, equal to:

$$1 + b_1 + \cdots + 1 + b_{\mathsf{h}} =: \mathsf{d},$$

and of height visibly equal to h. Of course, the total number of obtained polyzetas must be equal to the number of choices of two (binary) digits among $\mathsf{w} - 2 + 2 = \mathsf{w}$ digits, namely:

$$\begin{aligned}\text{number of terms} &= \binom{\mathsf{w}}{2} = \frac{\mathsf{w}(\mathsf{w}-1)}{2}\\ &= \frac{\big(a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} + 2\big)\big(a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} + 1\big)}{2}.\end{aligned}$$

Another simple initial observation is:

Lemma 9.2. *All terms of* $01 \sqcup 0^{a_1-1}1\,1^{b_1} \cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$ *are of depth* $\mathsf{d} + 1$.

Proof. Remembering that the depth of a general polyzeta is:

$$\begin{aligned}\mathsf{d} &= 1 + b_1 + \cdots + 1 + b_{\mathsf{h}}\\ &= \text{number of } 1 \text{ in its encoding } 0^{a_1-1}1\,1^{b_1} \cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}},\end{aligned}$$

it is then clear that any shuffle-insertion of 01 always increases the number of 1 by one unit, exactly. □

How will we, then, insert the 0 and the 1 of 01 inside a general polyzeta $0^{a_1-1}1\,1^{b_1} \cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$? The principle of agglutination stated in the preceding section for insertions of 1 also has a mirror-companion concerning insertions of 0.

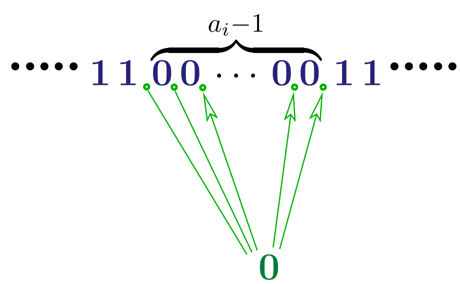
Principle of agglutination for shuffle-insertions of 0**.** *Any additional entry:*

$$\mathbf{0}$$

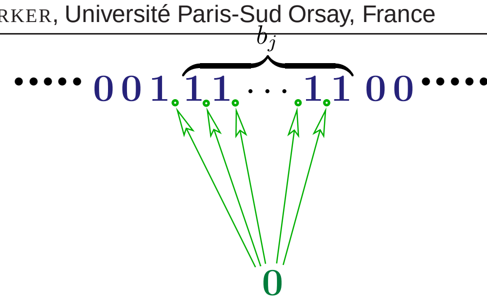
which is shuffle-inserted in some general polyzeta:

$$\overbrace{0\,0 \cdots 0\,0}^{a_1-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_1}\,\overbrace{0\,0 \cdots 0\,0}^{a_2-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_2}\;\cdots\cdots\;\overbrace{0\,0 \cdots 0\,0}^{a_{\mathsf{h}}-1}\,1\,\overbrace{1\,1 \cdots 1\,1}^{b_{\mathsf{h}}}$$

at some place lying in direct contact with one of the 0 *of some block* 0^{a_i-1}*:*



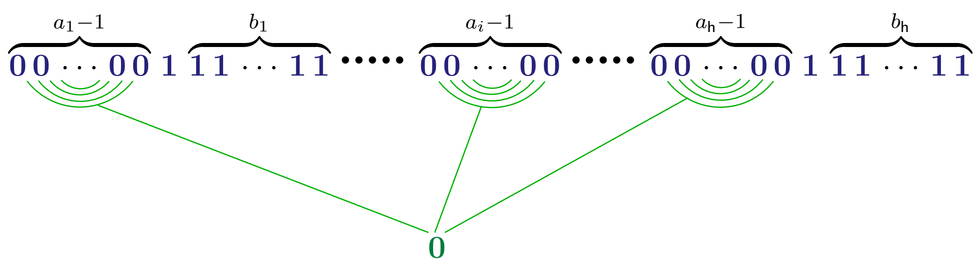
should be thought of as being automatically agglutinated *to the group of* 0 *of the block* 0^{a_i-1} *in question, and therefore, other insertions of such an additional* 0 *in the* 1 *of some block* $1\,1^{b_j}$*:*



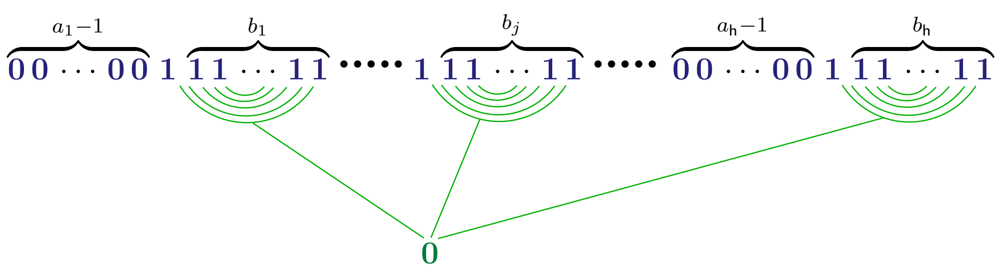
should always be done strictly inside these 1, *namely never in any two extremities of a block* $1\,1^{b_j}$.

So we have to insert the 0 and the 1 of 01 one after the other in a general polyzeta. The possibilities are:

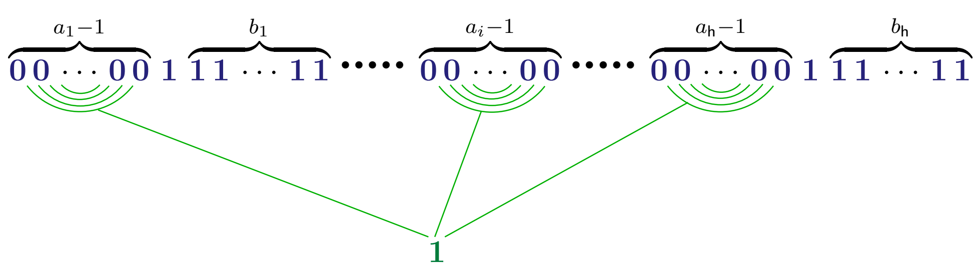
☐ the 0 goes into the 0:



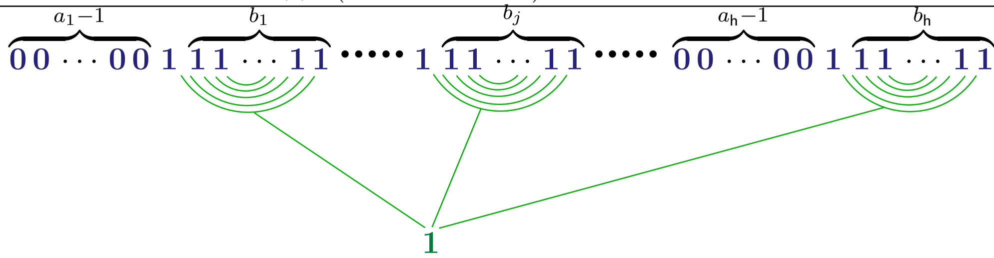
☐ the 0 goes into the 1:



☐ the 1 goes into the 0:



☐ the 1 goes into the 1:



Since one has to insert the 0 and the 1 of 01 *simultaneously one after the other* in a general polyzeta, four *common* occurrences hold:

$\boxed{i_1|i_2}$: the 0 goes into the 0 of some $0^{a_{i_1}-1}$ and the 1 also goes into the 0 of some $0^{a_{i_2}-1}$;

$\boxed{i|j}$: the 0 goes into the 0 of some 0^{a_i-1} while the 1 goes into the 1 of some $1\, 1^{b_j}$;

$\boxed{j|i}$: the 0 goes into the 1 of some $1\, 1^{b_j}$ while the 1 goes into the 0 of some 0^{a_i-1};

$\boxed{j_1|j_2}$: the 0 goes into the 1 of some $1\, 1^{b_{j_1}}$ and the 1 also goes into the 1 of some $1\, 1^{b_{j_2}}$.

Of course there also are the two somehow special subcases of the first one and of the fourth one, respectively, namely when $i_1 = i_2$ and when $j_1 = j_2$:

$\boxed{i\, i}$: the 0 goes into the 0 of some 0^{a_i-1} and the 1 also goes into the 0 of the *same* 0^{a_i-1};

$\boxed{j\, j}$: the 0 goes into the 1 of some $1\, 1^{b_j}$ and the 1 also goes into the 1 of the *same* $1\, 1^{b_j}$.

These two last subcases will be treated separately, hence precisely six cases will be dealt with, in fact.

Although the depth of any polyzeta appearing in $01 ⧢ 0^{a_1-1} 1\, 1^{b_1} \cdots 0^{a_{\mathsf{h}}-1} 1 1^{b_{\mathsf{h}}}$ is always equal to $\mathsf{d}+1$, the heights of such appearing polyzetas can vary, as examples show. *A precise control of the heights in question — to be made below — happens to be available only when one applies the two principles agglutinations, for the insertion of the* 0 *of* 01 *and as well, for the insertion of the* 1 *of* 01. Complete explanations being provided in just a while, let us list in advance the obtained heights — we intentionally change the order of appearance of the six cases, so as to fit with the ordering we introduced in Section 5 — :

☐ First family $\boxed{i|j}$ with $1 \leqslant i \leqslant j \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height h.

☐ Second family $\boxed{j_1|j_2}$ with $1 \leqslant j_1 < j_2 \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height $\mathsf{h}+1$;

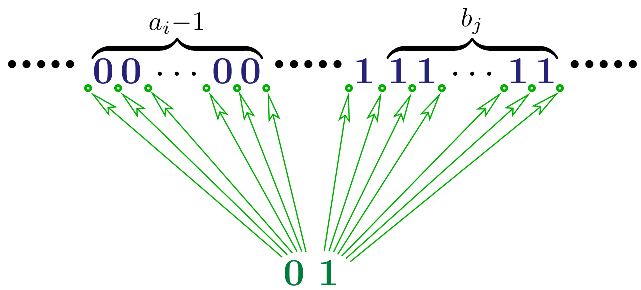
☐ Third family $\boxed{j\, j}$ with $1 \leqslant j \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height $\mathsf{h}+1$;

☐ Fourth family $\boxed{i\, i}$ with $1 \leqslant i \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height $\mathsf{h}+1$;

☐ Fifth family $\boxed{i_1|i_2}$ with $1 \leqslant i_1 < i_2 \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height $\mathsf{h}+1$;

☐ Sixth family $\boxed{j|i}$ with $1 \leqslant j < i \leqslant \mathsf{h}$: depth $\mathsf{d}+1$; height $\mathsf{h}+2$.

9.3. **First family** $\boxed{i|j}$**.** Thus, assume that 0 goes into a block of 0 while 1 goes into a block of 1:



Of course, one has $i \leqslant j$ here, for 0 must always be inserted left to the insertion of 1 (shuffle rule). For arbitrary fixed i and j with $1 \leqslant i \leqslant j \leqslant \mathsf{h}$, it is then diagrammatically visible and rigorously clear that for every such simultaneous insertion of 01, one obtains the same polyzeta, and the result is:

$$a_i\,(b_j+2)\cdot\zeta\Big(\bullet\bullet\bullet\bullet\bullet\,\overbrace{0\,0\cdots 0\cdots 0\,0}^{a_i}\,\bullet\bullet\bullet\bullet\bullet 1\,\overbrace{1\,1\cdots 1\cdots 1\,1}^{b_j+1}\bullet\bullet\bullet\bullet\bullet\Big).$$

One easily convinces oneself that the height is unchanged. As a result, the polyzetas generated by this first family — with the letter $\mathscr{F}$ — may be collected using a triangular summation symbol:

$$\boxed{\mathscr{F}_{i|j} := \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} a_i\,(b_j+2)\,\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,\,a_i+1,\bullet\bullet\bullet\bullet\bullet,\,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet\big).}$$

We recall that the bold dots mean that other entries of the polyzeta are unchanged. Lastly, we observe *passim* that the total number of terms in this family $\mathscr{F}_{i|j}$ equals:

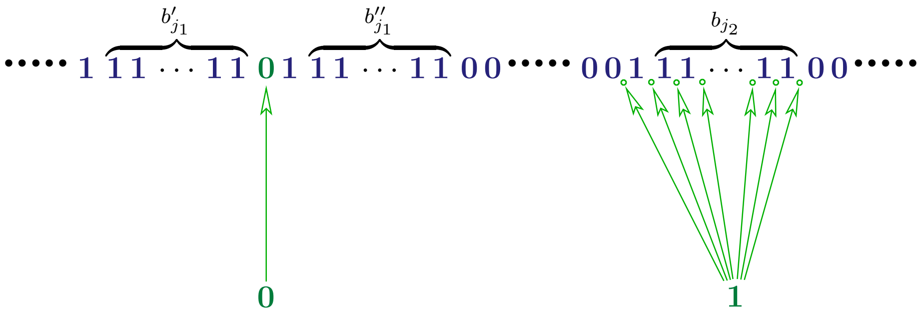
$$\mathbf{n}_{i|j} := \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} a_i\,b_j + \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} 2\,a_i.$$

In fact, in Subsection 9.9 below, we will check that the total sum of the numbers of polyzetas appearing in each one of our six families:

$$\begin{aligned}\mathbf{n}_{i|j}+\mathbf{n}_{j_1|j_2}+\mathbf{n}_{j\,j}+\mathbf{n}_{i\,i}+\mathbf{n}_{i_1|i_2}+\mathbf{n}_{j|i} &= \frac{\mathsf{w}(\mathsf{w}-1)}{2}\\ &= \frac{(a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}+2)\,(a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}+1)}{2}\end{aligned}$$

is indeed equal to the total expected number shown at the beginning of Subsection 9.2. In the future two Sections 10 and 11 below, the checkings will become a bit harder when dealing with $001 \sqcup\!\sqcup\, 0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$ and with $011 \sqcup\!\sqcup\, 0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$ (respectively), hence let us admit that the present computational level remains accessible.

9.4. **Second family** $\boxed{j_1|j_2}$ **with** $1 \leqslant j_1 < j_2 \leqslant \mathsf{h}$**.** Suppose now that the 0 and the 1 of 01 both go into a block of 1, the blocks being *distinct*, say into $1\,1^{b_{j_1}}$ and into $1\,1^{b_{j_2}}$ (respectively), for some $1 \leqslant j_1 < j_2 \leqslant \mathsf{h}$.



According to the agglutination principles, the 1 on the right can take $b_{j_2}+2$ positions, while the 0 on the left must be inserted *strictly* inside $1\,1^{b_{j_1}}$. Consequently, there are two integers $b'_{j_1} \geqslant 0$ and $b''_{j_1} \geqslant 0$ as in the diagram, which satisfy — notice that one more 1 is kept just after the inserted 0 —:

$$b'_{j_1} + b''_{j_1} = b_{j_1} - 1.$$

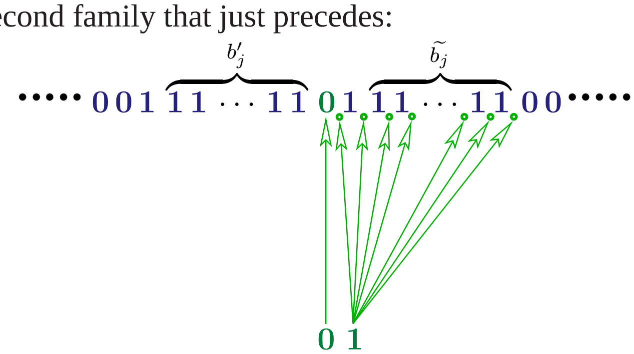
As a result, the original $1^{b_{j_1}}$ is replaced by $1^{b'_{j_1}}, 2, 1^{b''_{j_1}}$, while the $1^{b_{j_2}}$ is replaced by $1^{b_{j_2}+1}$, with multiplicity $(b_{j_2}+2)$. Thus, the polyzetas generated by this second family may be collected as follows using two summations symbols:

$$\boxed{\mathscr{F}_{j_1|j_2} := \sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}} \Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}} (b_{j_2}+2)\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_{j'_1}}, 2, 1^{b''_{j_1}}, \bullet\bullet\bullet\bullet\bullet, 1^{b_{j_2}+1}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg).}$$

Lastly, let us observe that the total number of terms in this second family $\mathscr{F}_{j_1|j_2}$ equals:

$$\mathbf{n}_{j_1|j_2} := \sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}} b_{j_1}\,(b_{j_2}+2).$$

9.5. **Third family** $\boxed{j\,j}$ **with** $1 \leqslant j \leqslant \mathsf{h}$**.** Next, assume that the 0 and the 1 of 01 go together into the *same* block of 1, a case which would correspond to the limit case $j_1 = j_2 =: j$ in the second family that just precedes:



Thus, there are two integers $b_j' \geqslant 0$ and $\widetilde{b}_j \geqslant 0$ as shown in the diagram satisfying:

$$b_j' + \widetilde{b}_j = b_j - 1.$$

As shuffle rules dictate, the insertion of the 1 must be done after that of the 0. Once this is performed, there is one more 1 in the right group:

$$\bullet\bullet\bullet\bullet\bullet\, 0\,0\,1\, \overbrace{1\,1\,\cdots\,1\,1}^{b_j'}\, 0\, \overbrace{1\,1\,1\,\cdots\,1\,\cdots\,1\,1}^{b_j''}\, 0\,0 \bullet\bullet\bullet\bullet\bullet,$$

namely we may set:

$$b_j'' := \widetilde{b}_j + 1,$$

so that b_j' and b_j'' now satisfy:

$$b_j' + b_j'' = b_j.$$

As a result, the polyzetas generated by this third family may be collected as follows using two summation symbols:

$$\boxed{\mathscr{F}_{j\,j} := \sum_{1 \leqslant j \leqslant \mathsf{h}} \Bigg(\sum_{\substack{b_j' + b_j'' = b_j \\ b_j' \geqslant 0,\ b_j'' \geqslant 0}} (b_j'' + 1) \cdot \zeta_{[\mathsf{w}, \mathsf{d}+1, \mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big) \Bigg).}$$

Clearly, the number of terms present here equals:

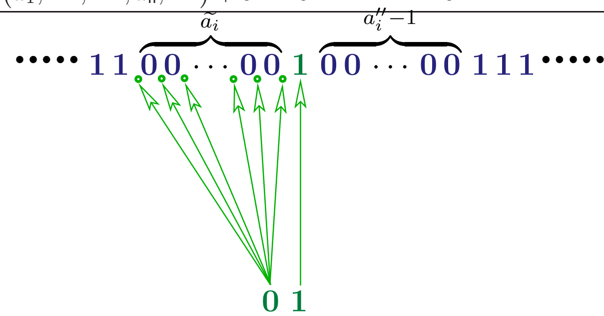
$$\mathbf{n}_{j\,j} := \sum_{1 \leqslant j \leqslant \mathsf{h}} \frac{(b_j + 2)(b_j + 1)}{2}.$$

But before passing to the fourth family, we must point out in advance that, for each such fixed block $1\,1^{b_j}$, the last obtained polyzeta, namely the one for $b_j' = b_j$ and $b_j'' = 0$, happens to be the following special polyzeta which might appear elsewhere because of some possible ambiguity:

$$\bullet\bullet\bullet\bullet\bullet\, 0\,0\,1\, \overbrace{1\,1\,\cdots\,1\,1}^{b_j}\, 0\,1\,0\,0 \bullet\bullet\bullet\bullet\bullet.$$

Indeed, there is no reason that the families we delineated above do not have small overlaps, and in fact, there will be some (restricted) overlaps (only) between the third and the fourth families — but we will avoid them.

9.6. **Fourth family** $\boxed{i\,i}$ **with** $1 \leqslant i \leqslant \mathsf{h}$**.** Now, the 0 and the 1 of the 01 are assumed to go into the *same* group of 0, say into some block $0^{a_i - 1}$:



Again, the principles of agglutination command to insert firstly the 1 strictly inside this block 0^{a_i-1}, so that in the exhibited diagram the following (in)equalities must hold:

$$\widetilde{a}_i \geqslant 1, \qquad a_i'' - 1 \geqslant 1, \qquad \text{and of course} \qquad \widetilde{a}_i + a_i'' - 1 = a_i - 1.$$

Then the 0 is inserted at any place before the insertion of the 1, and there are $\widetilde{a}_i + 1$ possibilities giving the same polyzeta:

$$\bullet\bullet\bullet\bullet\bullet\, 1\, 1\, \overbrace{0\, 0 \cdots 0 \cdots 0\, 0}^{a_i'-1}\, 1\, \overbrace{0\, 0 \cdots 0\, 0}^{a_i''-1}\, 1\, 1\, 1 \bullet\bullet\bullet\bullet\bullet,$$

in which it is natural to set:

$$a_i' - 1 := \widetilde{a}_i + 1, \qquad \text{that is to say:} \qquad a_i' = \widetilde{a}_i + 2.$$

Doing so, with $a_i' \geqslant 1$, one excludes the polyzeta:

$$\bullet\bullet\bullet\bullet\bullet\, 1\, 1\, 01\, \overbrace{0\, 0 \cdots 0\, 0}^{a_i-1}\, 1\, 1\, 1 \bullet\bullet\bullet\bullet\bullet$$

which appeared already in the third family, *except when* $i = 1$, *so that one must add:*

$$0\, 1\, \overbrace{0\, 0 \cdots 0\, 0}^{a_1-1}\, 1\, \overbrace{1\, 1 \cdots 1\, 1}^{b_1} \bullet\bullet\bullet\bullet\bullet\, \overbrace{0\, 0 \cdots 0\, 0}^{a_{\mathsf{h}}-1}\, 1\, \overbrace{1\, 1 \cdots 1\, 1}^{b_{\mathsf{h}}} = \big(2, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big).$$

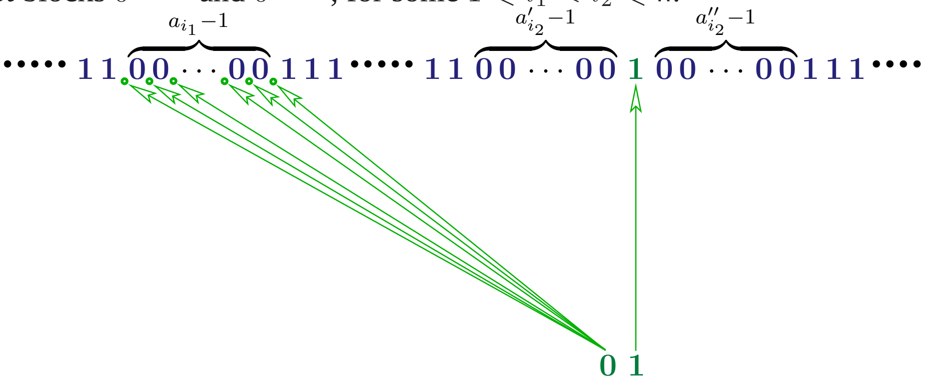
As a result, without any overlapping with what precedes, the polyzetas of the fourth family are:

$$\boxed{\begin{aligned} \mathscr{F}_{i\,i} &:= \big(2, a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + \\ &\quad + \sum_{\substack{1 \leqslant i \leqslant \mathsf{h} \\ a_i \geqslant 5}} \Bigg(\sum_{\substack{a_i' + a_i'' = a_i + 2 \\ a_i' \geqslant 3,\ a_i'' \geqslant 2}} (a_i' - 1) \cdot \zeta_{[\mathsf{w}, \mathsf{d}+1, \mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i', \bullet\bullet\bullet\bullet\bullet\big) \Bigg). \end{aligned}}$$

Lastly, the total number of terms in this fourth family may readily be checked to be equal to:

$$\mathbf{n}_{i\,i} := 1 + \sum_{1 \leqslant i \leqslant \mathsf{h}} \bigg(\frac{a_i(a_i - 1)}{2} - 1 \bigg).$$

9.7. **Fifth family** $\boxed{i_1|i_2}$ **with** $1 \leqslant i_1 < i_2 \leqslant \mathsf{h}$**.** Next, assume that the 0 and the 1 of 01 go into *distinct* blocks $0^{a_{i_1}-1}$ and $0^{a_{i_2}-1}$, for some $1 \leqslant i_1 < i_2 \leqslant \mathsf{h}$:



The placement of the 0 on the left always gives $a_{i_1}+1$ (with multiplicity a_{i_1}) instead of a_{i_1} in the original polyzeta, while the placement of 1 on the right imposes to replace the a_{i_2} in the original polyzeta by a'_{i_2}, a''_{i_2} with:

$$a'_{i_2} - 1 + a''_{i_2} - 1 = a_{i_2} - 1,$$

that is to say:

$$a'_{i_2} + a''_{i_2} = a_{i_2} + 1.$$

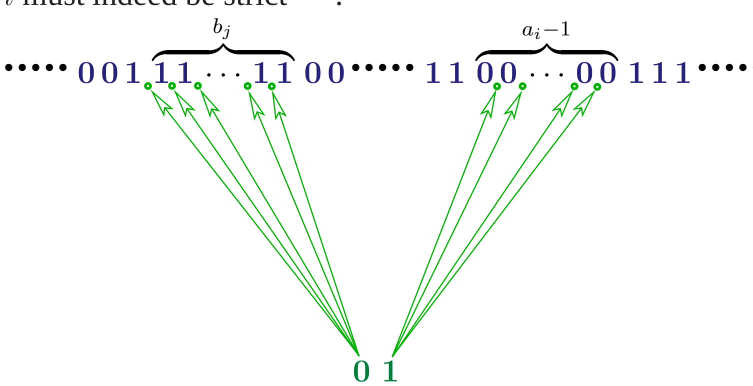
As a result, the polyzetas obtained in this fifth family are:

$$\boxed{\mathscr{F}_{i_1|i_2} := \sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}} \Bigg(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}} a_{i_1}\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1}+1, \bullet\bullet\bullet\bullet\bullet, a'_{i_2}, 1^0, a''_{i_2}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg).}$$

Visibly, their total number is:

$$\mathbf{n}_{i_1|i_2} := \sum_{1\leqslant i_2<i_2\leqslant \mathsf{h}} a_{i_1}\,(a_{i_2}-2).$$

9.8. **Sixth family** $\boxed{j|i}$ **with** $1 \leqslant j < i \leqslant \mathsf{h}$**.** Finally, assume that the 0 of 01 goes into a block $1\,1^{b_j}$ and that the 1 of 01 goes into a block 0^{a_i-1}, with $0 \leqslant j < i \leqslant \mathsf{h}$ — the inequality $j < i$ must indeed be strict —:



The agglutination principles require that all insertions are strict. No repetition appears, no multiplicity holds, all obtained polyzetas:

$$\bullet\bullet\bullet\bullet\bullet 0\,0\,1\,\overbrace{1\,1\cdots1\,1}^{b_j'}\,0\,1\,\overbrace{1\,1\cdots1\,1}^{b_j''}\,0\,0\bullet\bullet\bullet\bullet\bullet 1\,1\,\overbrace{0\,0\cdots0\,0}^{a_i'-1}\,1\,\overbrace{0\,0\cdots0\,0}^{a_i''-1}\,1\,1\,1\bullet\bullet\bullet\bullet\bullet.$$

are pairwise distinct, and the appearing new integers must satisfy:

$$b_j' + b_j'' = b_j - 1 \qquad\qquad \text{and} \qquad\qquad a_i' + a_i'' = a_i + 1.$$

As a result, the polyzetas of this sixth and last family are expressed by means of three summation symbols:

$$\boxed{\mathscr{F}_{j|i} := \sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}} \Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}} \ \sum_{\substack{a_i'+a_i''=a_i+1\\ a_i'\geqslant 2,\ a_i''\geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg).}$$

Mentally, one sees that their total number is:

$$\mathbf{n}_{j|i} := \sum_{1\leqslant j<i\leqslant \mathsf{h}} b_j\,(a_i-2).$$

9.9. **Final counting checking.** In order to convince ourselves that the formulas are completely free of errors, let us verify the promised equality concerning the number of obtained polyzetas. On one hand, we may firstly expand — disregarding temporarily the underlinings whose rôle will be explained in a moment —:

$$\begin{aligned}
&\frac{\big(a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}+2\big)\big(a_1+b_1+\cdots+a_{\mathsf{h}}+b_{\mathsf{h}}+1\big)}{2} = \\
&= \underline{\sum_{1\leqslant i\leqslant \mathsf{h}} \frac{a_i\,a_i}{2}}_{\,①} + \underline{\sum_{1\leqslant j\leqslant \mathsf{h}} \frac{b_j\,b_j}{2}}_{\,②} + \underline{\sum_{\substack{1\leqslant i\leqslant \mathsf{h}\\ 1\leqslant j\leqslant \mathsf{h}}} a_i\,b_j}_{\,③} + \underline{\sum_{1\leqslant i_1<i_2\leqslant \mathsf{h}} a_{i_1}\,a_{i_2}}_{\,④} + \\
&\quad + \underline{\sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}} b_{j_1}\,b_{j_2}}_{\,⑤} + \underline{\frac{3}{2}\sum_{1\leqslant i\leqslant \mathsf{h}} a_i}_{\,⑥} + \underline{\frac{3}{2}\sum_{1\leqslant j\leqslant \mathsf{h}} b_j}_{\,⑦} + \underline{1}_{\,⑧}.
\end{aligned}$$

On the other hand, let us collect the total number of polyzetas we obtained in our six families:

$$
\begin{aligned}
\mathbf{n}_{i|j} + \mathbf{n}_{j_1|j_2} + \mathbf{n}_{j\,j} + \mathbf{n}_{i\,i} + \mathbf{n}_{i_1|i_2} + \mathbf{n}_{j|i} = {} & \underline{\sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} a_i\,b_j}_{\textcircled{3}} + \underline{\sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} 2\,a_i}_{\textcircled{6}} + \\
& + \underline{\sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}} b_{j_1}\,b_{j_2}}_{\textcircled{5}} + \underline{\sum_{1\leqslant j_1<j_2\leqslant \mathsf{h}} 2\,b_{j_1}}_{\circ} + \\
& + \sum_{1\leqslant j\leqslant \mathsf{h}} \bigg(\underline{\frac{b_i\,b_j}{2}}_{\textcircled{2}} + \underline{\frac{3\,b_j}{2}}_{\textcircled{7}} + \underline{1}_{\circ\circ} \bigg) + \\
& + \sum_{1\leqslant i\leqslant \mathsf{h}} \bigg(\underline{\frac{a_i\,a_i}{2}}_{\textcircled{1}} - \underline{\frac{a_i}{2}}_{\textcircled{6}} - \underline{1}_{\circ\circ} \bigg) + \\
& + \underline{\sum_{1\leqslant i_1<i_2\leqslant \mathsf{h}} a_{i_1}\,a_{i_2}}_{\textcircled{4}} - \underline{\sum_{1\leqslant i_1<i_2\leqslant \mathsf{h}} 2\,a_{i_2}}_{\textcircled{6}} + \\
& + \underline{\sum_{1\leqslant j<i\leqslant \mathsf{h}} a_i\,b_j}_{\textcircled{3}} - \underline{\sum_{1\leqslant j<i\leqslant \mathsf{h}} 2\,b_j}_{\circ} .
\end{aligned}
$$

Thus, we have to compare these two series of sums that both yield a nonnegative integer and to verify that the two resulting integers are precisely equal. To do this in a way that does not ‘leave’ some non-immediate painful computations to a reader, let us borrow from [12] the *technique of numbered underlinings*, the purpose of which is to enable a *pure eye-checking of computations*, without the need of any pencil or of any extra sheet of paper.

To begin with, in the second series of sums just above, let us observe that two pairs of terms auto-annihilate, and they are denoted by means of the two pairs of underlinings $\underline{\ }_{\circ}$ and $\underline{\ }_{\circ\circ}$.

Next, with the symbol $\bigcirc$, we employ numbered underlinings to point out the coincidences of sum-terms between the two expressions under study. Notice that, especially for number $\textcircled{6}$, the summation reduction:

$$
\sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} - \sum_{1\leqslant i_1<i_2\leqslant \mathsf{h}} = \sum_{1\leqslant i\leqslant \mathsf{h}}
$$

must be used, while a last mental addition concludes the coincidence. Thus, number of terms match, as was required by this test of coherency. □

All the preceding reasonings enabled us to gain a complete expansion of the shuffle product of $\zeta(2)$ with an arbitrary polyzeta.

Proposition 9.3. *For every height* $\mathsf{h} \geqslant 1$*, every entries* $a_1 \geqslant 2, \dots, a_{\mathsf{h}} \geqslant 2$*, every entries* $b_1 \geqslant 0, \dots, b_{\mathsf{h}} \geqslant 0$*, if one sets:*

$$
\begin{aligned}
a_1 + b_1 + \cdots + a_{\mathsf{h}} + b_{\mathsf{h}} &=: \mathsf{w} - 2, \\
1 + b_1 + \cdots + 1 + b_{\mathsf{h}} &=: \mathsf{d},
\end{aligned}
$$

then the ⧢-shuffle product of $\zeta(2)$ with any $\zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1,1^{b_1},\dots,a_{\mathsf h},1^{b_{\mathsf h}}\big)$ of weight $\mathsf{w}-2\geqslant 2$, of depth $\mathsf d$ and of height $\mathsf h$ writes out, after complete finalization:

$$
\begin{aligned}
\zeta(2)⧢\zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1,1^{b_1},\dots,a_{\mathsf h},1^{b_{\mathsf h}}\big) &= \mathscr{F}_{i|j}+\mathscr{F}_{j_1|j_2}+\mathscr{F}_{j\,j}+\mathscr{F}_{i\,i}+\mathscr{F}_{i_1|i_2}+\mathscr{F}_{j|i}= \\
&= \sum_{1\leqslant i\leqslant j\leqslant \mathsf h} a_i\,(b_j+2)\,\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet,a_i+1,\bullet\bullet\bullet\bullet\bullet,1^{b_j+1},\bullet\bullet\bullet\bullet\bullet\big)+ \\
&\quad+\sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf h\\ b_{j_1}\geqslant 1}}\Bigg(\sum_{\substack{b'_{j_1}+b''_{j_1}=b_{j_1}-1\\ b'_{j_1}\geqslant 0,\ b''_{j_1}\geqslant 0}}(b_{j_2}+2)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_{j_1}},2,1^{b''_{j_1}},\bullet\bullet\bullet\bullet\bullet,1^{b_{j_2}+1},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+ \\
&\quad+\sum_{1\leqslant j\leqslant \mathsf h}\Bigg(\sum_{\substack{b'_j+b''_j=b_j\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}(b''_j+1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_j},2,1^{b''_j},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+ \\
&\quad+\big(2,a_1,1^{b_1},\dots,a_{\mathsf h},1^{b_{\mathsf h}}\big)+ \\
&\quad+\sum_{\substack{1\leqslant i\leqslant \mathsf h\\ a_i\geqslant 5}}\Bigg(\sum_{\substack{a'_i+a''_i=a_i+2\\ a'_i\geqslant 3,\ a''_i\geqslant 2}}(a'_i-1)\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a'_i,1^0,a'_i,\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+ \\
&\quad+\sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf h\\ a_{i_2}\geqslant 3}}\Bigg(\sum_{\substack{a'_{i_2}+a''_{i_2}=a_{i_2}+1\\ a'_{i_2}\geqslant 2,\ a''_{i_2}\geqslant 2}}a_{i_1}\cdot\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet,a_{i_1}+1,\bullet\bullet\bullet\bullet\bullet,a'_{i_2},1^0,a''_{i_2},\bullet\bullet\bullet\bullet\bullet\big)\Bigg)+ \\
&\quad+\sum_{\substack{1\leqslant j<i\leqslant \mathsf h\\ b_j\geqslant 1,\ a_i\geqslant 3}}\Bigg(\sum_{\substack{b'_j+b''_j=b_j-1\\ b'_j\geqslant 0,\ b''_j\geqslant 0}}\ \sum_{\substack{a'_i+a''_i=a_i+1\\ a'_i\geqslant 2,\ a''_i\geqslant 2}}\zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet,1^{b'_j},2,1^{b''_j},\bullet\bullet\bullet\bullet\bullet,a'_i,1^0,a''_i,\bullet\bullet\bullet\bullet\bullet\big)\Bigg).
\end{aligned}
$$

9.10. **The subtraction.** Now that we have gained the two fundamental expressions of the stuffle and of the shuffle product of $\zeta(2)$ with an arbitrary polyzeta of weight $\mathsf{w}-2$, we can execute the final subtraction. Not much terms collapse, no hint is really needed, hence we plainly (re)state the result.

Theorem 9.1. *For every height $\mathsf h\geqslant 1$, every entries $a_1\geqslant 2,\dots,a_{\mathsf h}\geqslant 2$, every entries $b_1\geqslant 0,\dots,b_{\mathsf h}\geqslant 0$, if one sets:*

$$
\begin{aligned}
a_1+b_1+\cdots+a_{\mathsf h}+b_{\mathsf h} &=: \mathsf{w}-2,\\
1+b_1+\cdots+1+b_{\mathsf h} &=: \mathsf{d},
\end{aligned}
$$

then the so-called double shuffle relation:

$$
0=-\,(2)*\big(a_1,1^{b_1},\dots,a_{\mathsf h},1^{b_{\mathsf h}}\big)+01 ⧢ 0^{a_1-1}1\,1^{b_1}\,\cdots\,0^{a_{\mathsf h}-1}1\,1^{b_{\mathsf h}}
$$

between $\zeta(2)$ *and any* $\zeta_{[\mathsf{w}-2,\mathsf{d},\mathsf{h}]}\big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big)$ *of weight* $\mathsf{w}-2\geqslant 2$*, of depth* d *and of height* h *writes out after complete finalization:*

$$
\begin{aligned}
0 = & - \sum_{1\leqslant i\leqslant \mathsf{h}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i+2, \bullet\bullet\bullet\bullet\bullet\big) - \\
& - \sum_{\substack{1\leqslant j\leqslant \mathsf{h}\\ b_j\geqslant 1}} \Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}} \zeta_{[\mathsf{w},\mathsf{d},\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 3, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
& + \sum_{1\leqslant i\leqslant j\leqslant \mathsf{h}} a_i\,(b_j+2)\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}]}\big(\bullet\bullet\bullet\bullet\bullet, a_i+1, \bullet\bullet\bullet\bullet\bullet, 1^{b_j+1}, \bullet\bullet\bullet\bullet\bullet\big) + \\
& + \sum_{\substack{1\leqslant j_1<j_2\leqslant \mathsf{h}\\ b_{j_1}\geqslant 1}} \Bigg(\sum_{\substack{b_{j_1}'+b_{j_1}''=b_{j_1}-1\\ b_{j_1}'\geqslant 0,\ b_{j_1}''\geqslant 0}} (b_{j_2}+2)\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_{j_1}'}, 2, 1^{b_{j_1}''}, \bullet\bullet\bullet\bullet\bullet, 1^{b_{j_2}+1}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
& + \sum_{1\leqslant j\leqslant \mathsf{h}} \Bigg(\sum_{\substack{b_j'+b_j''=b_j\\ b_j'\geqslant 0,\ b_j''\geqslant 0}} b_j''\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
& + \sum_{\substack{1\leqslant i\leqslant \mathsf{h}\\ a_i\geqslant 5}} \Bigg(\sum_{\substack{a_i'+a_i''=a_i+2\\ a_i'\geqslant 3,\ a_i''\geqslant 2}} (a_i'-1)\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
& + \sum_{\substack{1\leqslant i_1<i_2\leqslant \mathsf{h}\\ a_{i_2}\geqslant 3}} \Bigg(\sum_{\substack{a_{i_2}'+a_{i_2}''=a_{i_2}+1\\ a_{i_2}'\geqslant 2,\ a_{i_2''}\geqslant 2}} a_{i_1}\cdot \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+1]}\big(\bullet\bullet\bullet\bullet\bullet, a_{i_1}+1, \bullet\bullet\bullet\bullet\bullet, a_{i_2}', 1^0, a_{i_2}'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg) + \\
& + \sum_{\substack{1\leqslant j<i\leqslant \mathsf{h}\\ b_j\geqslant 1,\ a_i\geqslant 3}} \Bigg(\sum_{\substack{b_j'+b_j''=b_j-1\\ b_j'\geqslant 0,\ b_j''\geqslant 0}} \ \sum_{\substack{a_i'+a_i''=a_i+1\\ a_i'\geqslant 2,\ a_i''\geqslant 2}} \zeta_{[\mathsf{w},\mathsf{d}+1,\mathsf{h}+2]}\big(\bullet\bullet\bullet\bullet\bullet, 1^{b_j'}, 2, 1^{b_j''}, \bullet\bullet\bullet\bullet\bullet, a_i', 1^0, a_i'', \bullet\bullet\bullet\bullet\bullet\big)\Bigg).
\end{aligned}
$$

10. COMPUTING $-(3) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 001 \sqcup 0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$

The principles are similar. As an appendix to this prepublication, a detailed proof is expected to appear on arxiv.org.

11. COMPUTING $-(2,1) * \big(a_1, 1^{b_1}, \dots, a_{\mathsf{h}}, 1^{b_{\mathsf{h}}}\big) + 011 \sqcup 0^{a_1-1}1\,1^{b_1}\cdots 0^{a_{\mathsf{h}}-1}1\,1^{b_{\mathsf{h}}}$

See in the future the same appendix to this prepublication, which might be twenty pages long. In fact, the point is not just to set up these formulas (proofs may then be forgotten), but mainly to *apply* them, to *handle* them, to *compute* with them.

REFERENCES

[1] Blümlein, J.; Broadhurst, D.J.; Vermaseren, J.A.M.: *The multiple zeta values data mine*, Comput. Phys. Commun. **181** (2010), 582-625

[2] Bouillot, O.: *Invariants analytiques des difféomorphismes et multizêtas*, Ph. D., Orsay University, Paris, France, 19 October 2011, 290 pages.

[3] Brown, F.: *Mixed Tate motives over* $\mathbb{Z}$, arxiv.org/abs/1102.1312

[4] Cartier, P.: *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents*, Séminaire Bourbaki, 53ème année, 2000–2001, **885**.

[5] Cartier, P.: *Personal communication*.

[6] Combariza, G.: *A few conjectures about the multiple zeta values*, arxiv.org/1207.1735

[7] Écalle, J.: *Explicit-canonical decomposition of multizetas into irreducibles*, www.math.u-psud.fr/∼ecalle/publi.html

[8] Hoffman, M.: *The algebra of multiple harmonic series*, J. Algebra **194** (1997), no. 2, 477–495.

[9] Ihara, K.; Kaneko, M.; Zagier, D.: *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), no. 2, 307–338.

[10] Jacob, G.; Minh, H.M.; Oussous, N.E.; Petitot, M.: *Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier*, Sém. Lothar. Combin. **43**, 1999, art. B43e, 29 pp.

[11] Kaneko, M.; Noro, M.; Tsurumaki, K.: *On a conjecture for the dimension of the space of the multiple zeta values*, Software for Algebraic Geometry, IMA 148 (2008), 4758.

[12] Merker, J.: *Characterization of the Newtonian free particle system in* $m \geqslant 2$ *dependent variables*, Acta Applicandæ Mathematicæ, **92** (2006), no. 2, 125–207.

[13] Waldschmidt, M.: *Valeurs zêta multiples: une introduction*, J. Théor. Nombres Bordeaux, **12** (2000), 581–595.

[14] Zagier, D.: *Values of zeta functions and their applications*, First European Congress of Mathematics (Paris, 1992), vol. II, Progress in Mathematics, 120, Birkhäuser, Boston, 1994, pp. 497–512.

[15] Zudilin, V.V.: *Algebraic relations for multiple zeta values*, Russian Math. Surveys, **58** (2003), no. 1, 1–29.

DÉPARTMENT DE MATHÉMATIQUES D'ORSAY, BÂTIMENT 425, FACULTÉ DES SCIENCES, UNIVERSITÉ PARIS XI - ORSAY, F-91405 ORSAY CEDEX, FRANCE

E-mail address: `Joel.Merker@math.u-psud.fr`