Equivalences of 5-dimensional CR-manifolds III: Six models and (very) elementary normalizations

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Abstract. The six nondegeneracy conditions of geometric nature that are satisfied by the only six possibly existing nondegenerate general classes I, II, III_1, III_2, IV_1, IV_2 of 5-dimensional CR manifolds are shown to be readable instantaneously from their elementarily normalized respective defining graphed equations, without advanced Moser theory.

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1. Introduction

Consider a CR-generic submanifold:

$$M^{2n+c} \subset \mathbb{C}^{n+c}$$
$$(c = \mathsf{codim}\, M,\ n = \mathsf{CRdim}\, M),$$

of smoothness:

$$\mathscr{C}^{\kappa}\,(\kappa \geqslant 1), \qquad \text{or} \quad \mathscr{C}^{\infty}, \qquad \text{or} \quad \mathscr{C}^{\omega}.$$

According to [3], there are precisely *six* general classes of nondegenerate $M^{2n+c} \subset \mathbb{C}^{n+c}$ having dimension:

$$2n + c \leqslant \mathbf{5},$$

hence having CR dimension:

$$n = \begin{cases} \mathbf{1}, \\ \mathbf{2}. \end{cases}$$

namely if one denotes by:

$$\{\mathscr{L}\},$$
$$\{\mathscr{L}_1, \mathscr{L}_2\},$$

any local frame for $T^{1,0}M$, firstly the well known:

General Class I:

$$M^3 \subset \mathbb{C}^2 \ \text{ with } \Big\{\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}]\Big\} \\ \text{constituting a frame for } \ \mathbb{C} \otimes_{\mathbb{R}} TM,$$

secondly the known:

General Class II:

$$M^4 \subset \mathbb{C}^3 \ \text{ with } \Big\{\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\} \\ \text{constituting a frame for } \ \mathbb{C} \otimes_{\mathbb{R}} TM,$$

thirdly the known:

General Class III_1:

$$M^5 \subset \mathbb{C}^4 \ \text{ with } \Big\{\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \ [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\} \\ \text{constituting a frame for } \ \mathbb{C} \otimes_{\mathbb{R}} TM,$$

fourthly the *new:*

General Class III_2:

$$M^5 \subset \mathbb{C}^4 \ \text{ with } \Big\{\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \ [\mathscr{L}, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]]\Big\} \\ \text{constituting a frame for } \ \mathbb{C} \otimes_{\mathbb{R}} TM, \\ \text{while } \ \mathbf{4} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \ [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\Big),$$

fifthly the well known:

General Class IV_1:

$$M^5 \subset \mathbb{C}^3 \ \text{ with } \Big\{\mathscr{L}_1, \mathscr{L}_2, \overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2, \ [\mathscr{L}_1, \overline{\mathscr{L}}_1]\Big\} \\ \text{constituting a frame for } \ \mathbb{C} \otimes_{\mathbb{R}} TM, \\ \text{and with the Levi-Form:} \\ \mathsf{Levi\text{-}Form}^M(p) \\ \text{being of rank } \mathbf{2} \text{ at every point } p \in M,$$

sixthly and lastly the known:

General Class IV_2:

$M^5 \subset \mathbb{C}^3$ with $\Big\{\mathscr{L}_1,\, \mathscr{L}_2,\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{L}}_2,\; \big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\Big\}$
constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$,
with the Levi-Form:

$$\text{Levi-Form}^M(p)$$

being of rank $\mathbf{1}$ at every point $p \in M$,
while the Freeman-Form:

$$\text{Freeman-Form}^M(p)$$

is nondegenerate at every point.

The objective is to find elementary, preliminary normalizing coordinates in which the graphing equations would immediately and visibly show that these six nondegeneracy conditions are satisfied.

Remarks. For background foundational material, the reader is referred to Part II of this memoir ([3]), considered to be conceptually synthetized.

Part I, scheduled at the end, will include introduction, references, credit.

Part IV (forthcoming, brief) will set up the six related matrix ambiguity groups (initial G-structures).

Part V (forthcoming, *door to the core*), will set up the six initial coframes and their Darboux structures.

These thankless systematic re-foundations being achieved, the final launch of central computations will be firmly grounded.

2. Removing pluriharmonic monomials

Pick two integers:

$$n \geqslant 1, \qquad\qquad c \geqslant 1,$$

denote coordinates on $\mathbb{C}^{n+c}$ by:

$$(z_1, \dots, z_n, w_1, \dots, w_c) \;=\; \big(x_1 + \sqrt{-1}\, y_1, \dots, x_n + \sqrt{-1}\, y_n,\; u_1 + \sqrt{-1}\, v_1, \dots, u_c + \sqrt{-1}\, v_c\big),$$

and notationally contract them as:

$$\big(z_\bullet, w_\bullet\big) = \big(x_\bullet + \sqrt{-1}\, y_\bullet,\; u_\bullet + \sqrt{-1}\, v_\bullet\big).$$

Given therefore a *local* CR-generic *real analytic* ($\mathscr{C}^\omega$) submanifold:

$$M^{2n+c} \subset \mathbb{C}^{n+c}$$

passing through the origin:

$$0 \in M,$$

with of course:

$$c = \mathsf{codim}_{\mathbb{R}}\, M,$$
$$n = \mathsf{CRdim}\, M,$$

then provided only that:

$$\frac{\partial}{\partial z_1}\bigg|_0 \not\in T_0^{1,0}M, \ \ldots\ldots, \ \frac{\partial}{\partial z_n}\bigg|_0 \not\in T_0^{1,0}M,$$

one easily convinces oneself that the analytic implicit function theorem shows the existence of c real analytic graphing functions:

$$\varphi_1\big(z_\bullet, \overline{z}_\bullet, u_\bullet\big), \ \ldots\ldots\ldots, \ \varphi_c\big(z_\bullet, \overline{z}_\bullet, u_\bullet\big),$$

having convergent power series in some neighborhood of:

$$\big(0_\bullet, 0_\bullet, 0_\bullet\big) \ \in \ \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}^c,$$

and vanishing at the origin:

$$\begin{aligned} 0 &= \varphi_1\big(0_\bullet, 0_\bullet, 0_\bullet\big), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 &= \varphi_c\big(0_\bullet, 0_\bullet, 0_\bullet\big), \end{aligned}$$

such that M is locally represented by:

$$\left[\begin{aligned} v_1 &= \varphi_1\big(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, u_1, \ldots, u_c\big), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_c &= \varphi_c\big(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, u_1, \ldots, u_c\big). \end{aligned}\right.$$

After a $\mathbb{C}$-linear biholomorphism of $\mathbb{C}^{n+c}$, one can even assume that:

$$T_0 M = \big\{0 = v_1 = \cdots = v_c\big\},$$

so that the c graphing functions satisfy in addition:

$$\begin{aligned} 0 = \varphi_{1,z_k}\big(0_\bullet, 0_\bullet, 0_\bullet\big) &= \varphi_{1,\overline{z}_k}\big(0_\bullet, 0_\bullet, 0_\bullet\big) = \varphi_{1,u_l}\big(0_\bullet, 0_\bullet, 0_\bullet\big) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = \varphi_{c,z_k}\big(0_\bullet, 0_\bullet, 0_\bullet\big) &= \varphi_{c,\overline{z}_k}\big(0_\bullet, 0_\bullet, 0_\bullet\big) = \varphi_{c,u_l}\big(0_\bullet, 0_\bullet, 0_\bullet\big) \\ &{\scriptstyle (1 \leqslant k \leqslant n;\ 1 \leqslant l \leqslant c).} \end{aligned}$$

Precisely, the assumption of *local real analyticity* of the φ_j means that there exists a positive number:

$$\rho_0 > 0$$

for which the power series expansion:

$$\varphi_j\big(z_\bullet, \overline{z}_\bullet, u_\bullet\big) = \sum_{\alpha_\bullet \in \mathbb{N}^n} \sum_{\beta_\bullet \in \mathbb{N}^n} \sum_{\gamma_\bullet \in \mathbb{N}^c} \underbrace{\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}}_{\in \mathbb{R}} \big(z_\bullet\big)^{\alpha_\bullet} \big(\overline{z}_\bullet\big)^{\beta_\bullet} \big(u_\bullet\big)^{\gamma_\bullet} \qquad (1 \leqslant j \leqslant c),$$

converges when:

$$|z_\bullet| < \rho_0, \qquad\qquad |u_\bullet| < \rho_0,$$

in the sense that its coefficients satisfy a Cauchy-type estimate:

$$\big|\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\big| \leqslant \text{constant} \left(\frac{1}{\rho_0}\right)^{|\alpha_\bullet|+|\beta_\bullet|+|\gamma_\bullet|} \qquad (1 \leqslant j \leqslant c),$$

for some positive $\text{constant} > 0$.

Convention. Given a (formal or convergent) power series in N variables $(\mathsf{t}_1, \dots, \mathsf{t}_\mathrm{N})$ having complex coefficients:

$$\Phi\big(\mathsf{t}_1, \dots, \mathsf{t}_\mathrm{N}\big) = \sum_{\gamma_1=0}^{\infty} \cdots \sum_{\gamma_\mathrm{N}=0}^{\infty} \underbrace{\Phi_{\gamma_1,\dots,\gamma_\mathrm{N}}}_{\in \mathbb{C}} \big(\mathsf{t}_1\big)^{\gamma_1} \cdots \big(\mathsf{t}_\mathrm{N}\big)^{\gamma_\mathrm{N}},$$

its *conjugate series*:

$$\overline{\Phi}\big(\mathsf{t}_1, \dots, \mathsf{t}_\mathrm{N}\big) := \sum_{\gamma_1=0}^{\infty} \cdots \sum_{\gamma_\mathrm{N}=0}^{\infty} \overline{\Phi_{\gamma_1,\dots,\gamma_\mathrm{N}}} \big(\mathsf{t}_1\big)^{\gamma_1} \cdots \big(\mathsf{t}_\mathrm{N}\big)^{\gamma_\mathrm{N}}$$

is defined by conjugating only its coefficients, so that an overall conjugation operator:

$$\overline{\Phi(\mathsf{t}_1, \dots, \mathsf{t}_\mathrm{N})} \equiv \overline{\Phi}\big(\overline{\mathsf{t}}_1, \dots, \overline{\mathsf{t}}_\mathrm{N}\big)$$

distributes *simultaneously on the series symbol and on the variables*.

Notation. The *complexifications* of the antiholomorphic variables:

$$\big(\overline{z}_\bullet, \overline{w}_\bullet\big)$$

will be denoted using just some underlining:

$$\big(\underline{z}_\bullet, \underline{w}_\bullet\big),$$

and these are $(n + c)$ new variables totally independent of $(z_\bullet, w_\bullet)$.

If one also introduce new complex variables:

$$\nu_1, \dots, \nu_c \in \mathbb{C},$$

with:

$$\text{Re}\, \nu_1 = u_1, \; \dots\dots, \; \text{Re}\, \nu_c = u_c,$$

the *complexification* of $\varphi_\bullet$ is the power series:

$$\varphi_\bullet\big(z_\bullet,\underline{z}_\bullet,\nu_\bullet\big):=\sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^n}\sum_{\gamma_\bullet\in\mathbb{N}^c}\underbrace{\varphi_{\bullet,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}}_{\in\mathbb{R}}\big(z_\bullet\big)^{\alpha_\bullet}\big(\underline{z}_\bullet\big)^{\beta_\bullet}\big(\nu_\bullet\big)^{\gamma_\bullet}$$

in the $2n+c$ *complex* variables:

$$\big(z_\bullet,\underline{z}_\bullet,\nu_\bullet\big),$$

and it yet converges when

$$|z_\bullet|<\rho_0,\qquad\qquad |\underline{z}_\bullet|<\rho_0,\qquad\qquad |\nu_\bullet|<\rho_0,$$

because:

$$\big|\varphi_{\bullet,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\big|\leqslant\text{constant}\left(\frac{1}{\rho_0}\right)^{|\alpha_\bullet|+|\beta_\bullet|+|\gamma_\bullet|}.$$

Complexification-Identity Principle. *Two converging power series:*

$$G\big(z_\bullet,w_\bullet,\underline{z}_\bullet,\underline{w}_\bullet\big)=\sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^c}\sum_{\gamma_\bullet\in\mathbb{N}^n}\sum_{\delta_\bullet\in\mathbb{N}^c}\underbrace{G_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet}}_{\in\mathbb{C}}\big(z_\bullet\big)^{\alpha_\bullet}\big(w_\bullet\big)^{\beta_\bullet}\big(\underline{z}_\bullet\big)^{\gamma_\bullet}\big(\underline{w}_\bullet\big)^{\delta_\bullet}$$

$$H\big(z_\bullet,w_\bullet,\underline{z}_\bullet,\underline{w}_\bullet\big)=\sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^c}\sum_{\gamma_\bullet\in\mathbb{N}^n}\sum_{\delta_\bullet\in\mathbb{N}^c}\underbrace{H_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet}}_{\in\mathbb{C}}\big(z_\bullet\big)^{\alpha_\bullet}\big(w_\bullet\big)^{\beta_\bullet}\big(\underline{z}_\bullet\big)^{\gamma_\bullet}\big(\underline{w}_\bullet\big)^{\delta_\bullet}$$

in the $2n+2c$ *independent complex variables:*

$$\big(z_\bullet,w_\bullet,\underline{z}_\bullet\underline{w}_\bullet\big)\in\mathbb{C}^{n+c+n+c}$$

are identically equal:

$$G_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet}=H_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet}\qquad\qquad{\scriptstyle(\forall\,\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet)}$$

if and only if their restrictions to the antiholomorphic diagonal:

$$\overline{\Lambda}:=\big\{\big(z_\bullet,w_\bullet,\underline{z}_\bullet,\underline{w}_\bullet\big)\in\mathbb{C}^{n+c+n+c}:\ \underline{z}_\bullet=\overline{z}_\bullet,\ \underline{w}_\bullet=\overline{w}_\bullet\big\},$$

coincide as functions:

$$G\big(z_\bullet,w_\bullet,\overline{z}_\bullet,\overline{w}_\bullet\big)\equiv H\big(z_\bullet,w_\bullet,\overline{z}_\bullet,\overline{w}_\bullet\big),$$

that is more precisely, if and only if:

$$G\Big(x_\bullet+\sqrt{-1}\,y_\bullet,\ u_\bullet+\sqrt{-1}\,v_\bullet,\ x_\bullet-\sqrt{-1}\,y_\bullet,\ u_\bullet-\sqrt{-1}\,v_\bullet\Big)\equiv$$
$$\equiv H\Big(x_\bullet+\sqrt{-1}\,y_\bullet,\ u_\bullet+\sqrt{-1}\,v_\bullet,\ x_\bullet-\sqrt{-1}\,y_\bullet,\ u_\bullet-\sqrt{-1}\,v_\bullet\Big)$$

identically in:

$$\mathbb{C}\big\{x_\bullet,y_\bullet,u_\bullet,v_\bullet\big\},$$

i.e. *as functions of:*

$$\big(x_\bullet,y_\bullet,u_\bullet,v_\bullet\big)\in\mathbb{R}^{n+n+c+c}.$$

Proof. One direction being trivial, starting then after subtraction from:

$$0 \equiv \sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^c}\sum_{\gamma_\bullet\in\mathbb{N}^n}\sum_{\delta_\bullet\in\mathbb{N}^c}\Big[G_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet} - H_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet}\Big]\,(z_\bullet)^{\alpha_\bullet}\,(w_\bullet)^{\beta_\bullet}\,(\overline{z}_\bullet)^{\gamma_\bullet}\,(\overline{w}_\bullet)^{\delta_\bullet},$$

one easily convinces oneself by suitable induction on:

$$\big(\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet\big)\in\mathbb{N}^{n+c+n+c}$$

that all Taylor coefficients match up:

$$G_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet} = H_{\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet} \qquad (\forall\,\alpha_\bullet,\beta_\bullet,\gamma_\bullet,\delta_\bullet),$$

by successively differentiating this identity and by setting in it:

$$z_\bullet = w_\bullet = 0,$$

taking account of the elementary relations:

$$\begin{aligned}
\delta_{k_1,k_2} &= \frac{\partial}{\partial z_{k_1}}\big(z_{k_2}\big) = \frac{\partial}{\partial \overline{z}_{k_1}}\big(\overline{z}_{k_2}\big),\\
\delta_{l_1,l_2} &= \frac{\partial}{\partial w_{l_1}}\big(w_{l_2}\big) = \frac{\partial}{\partial \overline{w}_{l_1}}\big(\overline{w}_{l_2}\big),\\
0 &= \frac{\partial}{\partial z_k}\big(w_l\big) = \frac{\partial}{\partial \overline{z}_k}\big(w_l\big),\\
0 &= \frac{\partial}{\partial w_l}\big(z_k\big) = \frac{\partial}{\partial \overline{w}_l}\big(z_k\big),
\end{aligned}$$

term by term differentiation being justified by standard normal convergence arguments. □

Reality feature. *The c scalar functions φ_j being real-valued:*

$$\overline{\varphi_j\big(z_\bullet,\overline{z}_\bullet,u_\bullet\big)} \equiv \varphi_j\big(z_\bullet,\overline{z}_\bullet,u_\bullet\big) \qquad (j=1\cdots c),$$

their power series complex coefficients satisfy:

$$\overline{\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}} = \varphi_{j,\beta_\bullet,\alpha_\bullet,\gamma_\bullet},$$

and conversely.

Proof. Conjugating term by term the power series expansion is justified by normal convergence:

$$\begin{aligned}\overline{\varphi_j\big(z_\bullet,\overline{z}_\bullet,u_\bullet\big)} &= \overline{\sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^n}\sum_{\gamma_\bullet\in\mathbb{N}^c}\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\,\big(z_\bullet\big)^{\alpha_\bullet}\,\big(\overline{z}_\bullet\big)^{\beta_\bullet}\,\big(u_\bullet\big)^{\gamma_\bullet}} \\ &= \sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^n}\sum_{\gamma_\bullet\in\mathbb{N}^c}\overline{\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}}\,\big(\overline{z}_\bullet\big)^{\alpha_\bullet}\,\big(z_\bullet\big)^{\beta_\bullet}\,\big(u_\bullet\big)^{\gamma_\bullet} \\ &= \varphi_j\big(z_\bullet,\overline{z}_\bullet,u_\bullet\big) \\ &= \sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^n}\sum_{\gamma_\bullet\in\mathbb{N}^c}\varphi_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\,\big(z_\bullet\big)^{\alpha_\bullet}\,\big(\overline{z}_\bullet\big)^{\beta_\bullet}\,\big(u_\bullet\big)^{\gamma_\bullet},\end{aligned}$$

and the assumed reality yields the stated coefficient equalities, after reorganizing the index summation:

$$\alpha_\bullet \longleftrightarrow \beta_\bullet$$

for comparison-identification thanks to the complexification-identity principle. □

Scholium. *For* $\nu_\bullet\in\mathbb{C}^c$*, on has:*

$$\overline{\varphi_j\big(z_\bullet,\overline{z}_\bullet,\nu_\bullet\big)} \;\equiv\; \varphi_j\big(z_\bullet,\overline{z}_\bullet,\overline{\nu}_\bullet\big).$$

Proof. At the level of converging power series, this follows (exercise) from the above symmetry relations between coefficients. □

Now, writing the c scalar equations of M as:

$$\frac{w_\bullet-\overline{w}_\bullet}{2\sqrt{-1}} = \varphi_\bullet\bigg(z_\bullet,\,\overline{z}_\bullet,\,\frac{w_\bullet+\overline{w}_\bullet}{2}\bigg),$$

and reminding:

$$\varphi_\bullet = \mathrm{O}(2),$$

the implicit function theorem enables one to solve for either $w_\bullet$ or $\overline{w}_\bullet$, but to be appropriately rigorous, it is better in fact to *complexify* the equation that one wants to solve, namely to look at the c scalar equations:

$$\frac{w_\bullet-\underline{w}_\bullet}{2\sqrt{-1}} = \varphi_\bullet\bigg(z_\bullet,\,\underline{z}_\bullet,\,\frac{w_\bullet+\underline{w}_\bullet}{2}\bigg),$$

that are now *holomorphic* with respect to the $2n+2c$ variables:

$$\big(z_\bullet,w_\bullet,\underline{z}_\bullet,\underline{w}_\bullet\big)\;\in\;\mathbb{C}^{n+c+n+c}.$$

Using the analytic implicit function theorem, two options indeed present themselves:

□ either solve with respect to $w_\bullet$;

□ or solve with respect to $\underline{w}_\bullet$.

The first solving provides:

$$w_\bullet = \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big),$$

for some certain c *complex-analytic* (holomorphic) functions:

$$\Theta_j\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) = \sum_{\alpha_\bullet\in\mathbb{N}^n}\sum_{\beta_\bullet\in\mathbb{N}^n}\sum_{\gamma_\bullet\in\mathbb{N}^c} \Theta_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\,\big(z_\bullet\big)^{\alpha_\bullet}\,\big(\underline{z}_\bullet\big)^{\beta_\bullet}\,\big(\underline{w}_\bullet\big)^{\gamma_\bullet} \qquad {\scriptstyle (1\leqslant j\leqslant c)},$$

these c power series being also convergent, namely having coefficients which also satisfy — shrinking $\rho_0 > 0$ if necessary — a Cauchy-type estimate:

$$\big|\Theta_{j,\alpha_\bullet,\beta_\bullet,\gamma_\bullet}\big| \leqslant \text{constant}\left(\frac{1}{\rho_0}\right)^{|\alpha_\bullet|+|\beta_\bullet|+|\gamma_\bullet|} \qquad {\scriptstyle (1\leqslant j\leqslant c)}.$$

By definition, the solution $\Theta_\bullet$ satisfies:

$$\frac{\Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) - \underline{w}_\bullet}{2\sqrt{-1}} \equiv \varphi_\bullet\left(z_\bullet,\, \underline{z}_\bullet,\, \frac{\Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) + \underline{w}_\bullet}{2}\right),$$

identically in the ring of c converging power series:

$$\mathbb{C}\big\{z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}^c.$$

On restriction to the antiholomorphic diagonal:

$$\big\{\underline{z}_\bullet = \overline{z}_\bullet,\, \underline{w}_\bullet = \overline{w}_\bullet\big\},$$

one then obtains:

$$\frac{\Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big) - \overline{w}_\bullet}{2\sqrt{-1}} \equiv \varphi_\bullet\left(z_\bullet,\, \overline{z}_\bullet,\, \frac{\Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big) + \overline{w}_\bullet}{2}\right),$$

so that the c complex scalar equations:

$$w_\bullet = \Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big)$$

also constitute defining equations for $M^{2n+c}\subset\mathbb{C}^{n+c}$, *the c real part equations and the c imaginary part equations being equivalent* (exercise of understanding, or *see* below).

Furthermore, using:

$$\overline{\varphi_\bullet\big(z_\bullet, \overline{z}_\bullet, \nu_\bullet\big)} \equiv \varphi_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{\nu}_\bullet\big),$$

a plain conjugation of what precedes yields:

$$\frac{-\overline{\Theta}_\bullet\big(\overline{z}_\bullet, z_\bullet, w_\bullet\big) + w_\bullet}{2\sqrt{-1}} \equiv \varphi_\bullet\left(z_\bullet,\, \overline{z}_\bullet,\, \frac{\overline{\Theta}_\bullet\big(\overline{z}_\bullet, z_\bullet, w_\bullet\big) + w_\bullet}{2}\right),$$

and a last complexification yields:

$$\frac{-\overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big)+w_\bullet}{2\sqrt{-1}} \equiv \varphi_\bullet\bigg(z_\bullet,\, \underline{z}_\bullet,\, \frac{\overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big)+w_\bullet}{2}\bigg).$$

On the other hand, if one solves secondly for $\underline{w}_\bullet$ the same c equations:

$$\frac{w_\bullet-\underline{w}_\bullet}{2\sqrt{-1}} = \varphi_\bullet\bigg(z_\bullet, \underline{z}_\bullet, \frac{w_\bullet+\underline{w}_\bullet}{2}\bigg),$$

getting:

$$\underline{w}_\bullet = \widetilde{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big),$$

for a certain $\mathbb{C}^c$-valued holomorphic map $\widetilde{\Theta}_\bullet$ satisfying by definition:

$$\frac{w_\bullet-\widetilde{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big)}{2\sqrt{-1}} \equiv \varphi_\bullet\bigg(z_\bullet, \underline{z}_\bullet,\, \frac{w_\bullet+\widetilde{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big)}{2}\bigg),$$

then because this $\widetilde{\Theta}_\bullet$ provided by the analytic implicit function theorem is *unique*, a comparison with what has been written at the moment yields the coincidence:

$$\widetilde{\Theta}_\bullet = \overline{\Theta}_\bullet.$$

In summary:

Proposition. *Solving either with respect to $w_\bullet$ or to $\underline{w}_\bullet$ some c real analytic local defining equations:*

$$\frac{w_\bullet-\overline{w}_\bullet}{2\sqrt{-1}} = v_\bullet = \varphi_\bullet\big(z_\bullet, \overline{z}_\bullet, u_\bullet\big) = \varphi_\bullet\bigg(z_\bullet, \overline{z}_\bullet,\, \frac{w_\bullet+\overline{w}_\bullet}{2}\bigg)$$

for a $\mathscr{C}^\omega$ CR-generic submanifold $M^{2n+c} \subset \mathbb{C}^{n+c}$, there exists a single $\mathbb{C}^c$-valued local holomorphic function:

$$\Theta_\bullet = \Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big)$$

such that the two solutions are conjugate to each other:

$$\begin{aligned} w_\bullet &= \Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big), \\ \overline{w}_\bullet &= \overline{\Theta}_\bullet\big(\overline{z}_\bullet, z_\bullet, w_\bullet\big), \end{aligned}$$

and a point of coordinates:

$$\big(z_\bullet, w_\bullet\big) \in M$$

belongs to M if and only if it satisfies either one of these three systems of c equations (hence the other two as well):

$$\begin{aligned} v_\bullet &= \varphi_\bullet\big(z_\bullet, \overline{z}_\bullet, u_\bullet\big), \\ w_\bullet &= \Theta_\bullet\big(z_\bullet, \overline{z}_\bullet, \overline{w}_\bullet\big), \\ \overline{w}_\bullet &= \overline{\Theta}_\bullet\big(\overline{z}_\bullet, z_\bullet, w_\bullet\big). \quad \square \end{aligned}$$

Passing to complexified variables:

$$\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \in \mathbb{C}^{n+c+n+c},$$

the three systems of c scalar equations:

$$\begin{aligned}
\tfrac{w_\bullet - \underline{w}_\bullet}{2\sqrt{-1}} &= \varphi_\bullet\big(z_\bullet, \underline{z}_\bullet, \tfrac{w_\bullet + \underline{w}_\bullet}{2}\big),\\
w_\bullet &= \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big),\\
\underline{w}_\bullet &= \overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big)
\end{aligned}$$

are therefore all equivalent by pairs, and since their differentials at the origin are all of maximal rank equal to c, if follows from the so-called *Hadamard lemma* that there exist two $c \times c$ invertible matrices of converging power series:

$$\begin{aligned}
\mathsf{b}_{\bullet,\bullet}\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) &\in \mathsf{GL}_{c\times c}\Big(\mathbb{C}\big\{z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}\Big),\\
\mathsf{a}_{\bullet,\bullet}\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) &\in \mathsf{GL}_{c\times c}\Big(\mathbb{C}\big\{z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}\Big),
\end{aligned}$$

such that:

$$\begin{aligned}
\tfrac{w_\bullet - \underline{w}_\bullet}{2\sqrt{-1}} - \varphi_\bullet\big(z_\bullet, \underline{z}_\bullet, \tfrac{w_\bullet + \underline{w}_\bullet}{2}\big) &\equiv \mathsf{b}_{\bullet,\bullet}\,\big[w_\bullet - \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\big],\\
\underline{w}_\bullet - \overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big) &\equiv \mathsf{a}_{\bullet,\bullet}\,\big[w_\bullet - \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\big].
\end{aligned}$$

Next, the *extrinsic complexification* of M is the complex submanifold:

$$M^{e_c} \subset \mathbb{C}^{n+c+n+c}$$

of complex codimension c which is defined by either one of the two (equivalent (mental exercise) systems of c equations:

$$\begin{aligned}
w_\bullet &= \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big),\\
\underline{w}_\bullet &= \overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, w_\bullet\big). \quad \square
\end{aligned}$$

It follows (exercise of understanding) that the two functional equations:

$$\boxed{\begin{aligned}
w_\bullet &\equiv \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \overline{\Theta}_\bullet(\underline{z}_\bullet, z_\bullet, w_\bullet)\big),\\
\underline{w}_\bullet &\equiv \overline{\Theta}_\bullet\big(\underline{z}_\bullet, z_\bullet, \Theta_\bullet(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet)\big),
\end{aligned}}$$

are identically satisfied, respectively, in:

$$\begin{aligned}
&\mathbb{C}\big\{\underline{z}_\bullet, z_\bullet, w_\bullet\big\}^c,\\
&\mathbb{C}\big\{z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}^c.
\end{aligned}$$

Consider now a local biholomorphic map:

$$\begin{aligned}
\big(z_\bullet, w_\bullet\big) \longmapsto &\ \big(z'_\bullet(z_\bullet, w_\bullet),\, w'_\bullet(z_\bullet, w_\bullet)\big)\\
=: &\ \big(z'_\bullet, w'_\bullet\big),
\end{aligned}$$

which, to fix ideas, sends $(0_\bullet, 0_\bullet)$ to $(0'_\bullet, 0'_\bullet)$, and moreover — after a possible renumbering of coordinates —, which sends M onto another CR-generic:

$$M'^{2n+c} \subset \mathbb{C}'^{n+c}$$

that is also representable under similar equivalent graphed forms:

$$\begin{aligned} v'_\bullet &= \varphi'_\bullet\big(z'_\bullet, \overline{z}'_\bullet, u'_\bullet\big), \\ w'_\bullet &= \Theta'_\bullet\big(z'_\bullet, \overline{z}'_\bullet, \overline{w}'_\bullet\big), \\ \overline{w}'_\bullet &= \overline{\Theta}'_\bullet\big(\overline{z}'_\bullet, z'_\bullet, w'_\bullet\big). \end{aligned}$$

Since the differentials of each among these three collections of c equations are of rank c at the origin (hence at every nearby point), the so-called Hadamard lemma provides a $c \times c$ invertible matrix of converging power series:

$$\mathsf{c}_{\bullet,\bullet}\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \in \mathsf{GL}_{c\times c}\Big(\mathbb{C}\big\{z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}\Big),$$

such that one has:

$$\begin{aligned} w'_\bullet\big(z_\bullet, w_\bullet\big) - \Theta'_\bullet\big(z'_\bullet(z_\bullet, w_\bullet),\, \overline{z}'_\bullet(\underline{z}_\bullet, \underline{w}_\bullet),\, \overline{w}'_\bullet(\underline{z}_\bullet, \underline{w}_\bullet)\big) &\equiv \\ \equiv \mathsf{c}_{\bullet,\bullet}\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\, \big[w_\bullet - \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\big]&, \end{aligned}$$

identically in:

$$\mathbb{C}\big\{z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}^c.$$

In other (interpretational) words, the *complexified mapping:*

$$\begin{aligned} \big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \longmapsto\ & \big(z'_\bullet(z_\bullet, w_\bullet),\, w'_\bullet(z_\bullet, w_\bullet),\, \overline{z}'_\bullet(\underline{z}_\bullet, \underline{w}_\bullet),\, \overline{w}'_\bullet(\underline{z}_\bullet, \underline{w}_\bullet)\big) \\ & =: \big(z'_\bullet, w'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big), \end{aligned}$$

sends the complexification:

$$M^{e_c} = \big\{(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet) \colon\ w_\bullet = \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\big\}$$

to the complexification:

$$M'^{e_c} = \big\{(z'_\bullet, w'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet) \colon\ w'_\bullet = \Theta'_\bullet\big(z'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big)\big\}.$$

Proposition. *After a local biholomorphism fixing the origin of the form:*

$$\begin{aligned} z_k &\longmapsto z_k && (1 \leqslant k \leqslant n), \\ w_l &\longmapsto w_l\big(z_\bullet, w_\bullet\big) && (1 \leqslant l \leqslant c), \end{aligned}$$

the new real analytic local graphing functions satisfy:

$$\begin{aligned} 0 &\equiv \varphi_1\big(0, \overline{z}_\bullet, u_\bullet\big) \equiv \varphi_1\big(z_\bullet, 0, u_\bullet\big), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &\equiv \varphi_c\big(0, \overline{z}_\bullet, u_\bullet\big) \equiv \varphi_c\big(z_\bullet, 0, u_\bullet\big), \end{aligned}$$

that is to say after notational contraction:

$$0 \equiv \varphi_\bullet\big(z_\bullet, 0_\bullet, u_\bullet\big) \equiv \varphi_\bullet\big(0_\bullet, \overline{z}_\bullet, u_\bullet\big),$$

and simultaneously also:

$$\begin{aligned} \underline{w}_\bullet &\equiv \Theta_\bullet\big(0_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \equiv \ \Theta_\bullet\big(z_\bullet, 0_\bullet, \underline{w}_\bullet\big), \\ w_\bullet &\equiv \overline{\Theta}\big(0_\bullet, z_\bullet, w_\bullet\big) \ \ \equiv \ \overline{\Theta}_\bullet\big(\underline{z}_\bullet, 0_\bullet, w_\bullet\big). \end{aligned}$$

Proof. A composition of *two* appropriate local biholomorphisms will do the job.

As a *Step I*, introduce the local holomorphic transformation defined by:

$$\begin{aligned} z_\bullet &= z'_\bullet, \\ w_\bullet &= w'_\bullet + \sqrt{-1}\,\varphi_\bullet\big(0_\bullet, 0_\bullet, w'_\bullet\big). \end{aligned}$$

It has an $(n+c)\times(n+c)$ Jacobian matrix at the origin equal to the identity, since:

$$\varphi_\bullet = \mathrm{O}(2)$$

after the (assumed) preliminary affine normalization:

$$T_0M = \big\{v_\bullet = 0_\bullet\big\},$$

hence this transformation is a local *biholomorphism*.

Through it, M is tranformed to a certain CR-generic submanifold:

$$M'^{2n+c} \subset \mathbb{C}'^{n+c}$$

still satisfying:

$$T_{0'}M' = \big\{v'_\bullet = 0'_\bullet\big\},$$

so that the analytic implicit function theorem solves:

$$v'_\bullet = \varphi'_\bullet\big(z'_\bullet, \overline{z}'_\bullet, u'_\bullet\big).$$

Visibly:

$$\big\{z_\bullet = 0_\bullet\big\} \ = \ \big\{z'_\bullet = 0'_\bullet\big\},$$

whence:

$$\big\{z_\bullet = 0_\bullet\big\} \cap M \ = \ \big\{z'_\bullet = 0'_\bullet\big\} \cap M',$$

that is to say:

$$\big\{\big(0_\bullet,\ u_\bullet + \sqrt{-1}\,\varphi_\bullet(0_\bullet, 0_\bullet, u_\bullet)\big)\big\} = \big\{\big(0'_\bullet,\ u'_\bullet + \sqrt{-1}\,\varphi'_\bullet(0'_\bullet, 0'_\bullet, u'_\bullet)\big)\big\},$$

this coincidence being through the related restriction of the biholomorphism.

But through the biholomorphism in question, when:

$$\begin{aligned} w'_\bullet &= u'_\bullet + \sqrt{-1}\,0'_\bullet \\ &= u'_\bullet, \end{aligned}$$

and when:

$$z'_\bullet = 0'_\bullet,$$

one visibly covers arbitrary points:

$$\begin{aligned}\big(z_\bullet, w_\bullet\big) &= \big(0_\bullet,\, u'_\bullet + \sqrt{-1}\,\varphi_\bullet\big(0_\bullet, 0_\bullet, u'_\bullet\big)\big)\\ &\in \big\{z_\bullet = 0_\bullet\big\} \cap M,\end{aligned}$$

so that necessarily:

$$\big\{z'_\bullet = 0'_\bullet\big\} \cap M' = \big\{\big(0'_\bullet, u'_\bullet\big)\big\},$$

and one concludes that:

$$\varphi'_\bullet\big(0'_\bullet, 0'_\bullet, u'_\bullet\big) \;\equiv\; 0'_\bullet.$$

Dropping the primes, one may therefore now assume in order to pursue the proof that:

$$\varphi_\bullet\big(0_\bullet, 0_\bullet, u_\bullet\big) \equiv 0_\bullet.$$

It follows (mental exercise) that:

$$\begin{aligned}\Theta_\bullet\big(0_\bullet, 0_\bullet, \overline{w}_\bullet\big) &\;\equiv\; \overline{w}_\bullet,\\ \overline{\Theta}_\bullet\big(0_\bullet, 0_\bullet, w_\bullet\big) &\;\equiv\; w_\bullet.\end{aligned}$$

As a *Step II,* introduce the (second) local biholomorphism:

$$\begin{aligned}z'_\bullet &= z_\bullet,\\ w'_\bullet &= \overline{\Theta}_\bullet\big(0_\bullet, z_\bullet, w_\bullet\big).\end{aligned}$$

It transforms M to a certain generic submanifold:

$$M'^{2n+c} \subset \mathbb{C}'^{n+c},$$

still satisfying:

$$T_{0'}M' = \big\{v'_\bullet = 0'_\bullet\big\},$$

whence M' is defined by:

$$\begin{aligned}v'_\bullet &= \varphi'_\bullet\big(z'_\bullet, \overline{z}'_\bullet, u'_\bullet\big),\\ w'_\bullet &= \Theta'_\bullet\big(z'_\bullet, \overline{z}'_\bullet, \overline{w}'_\bullet\big),\\ \overline{w}'_\bullet &= \overline{\Theta}'_\bullet\big(\overline{z}'_\bullet, z'_\bullet, w'_\bullet\big).\end{aligned}$$

So the *complexification* is the second biholomorphism:

$$\begin{aligned}z'_\bullet &= z_\bullet, & \underline{z}'_\bullet &= \underline{z}_\bullet,\\ w'_\bullet &= \overline{\Theta}_\bullet\big(0_\bullet, z_\bullet, w_\bullet\big), & \underline{w}'_\bullet &= \Theta\big(0_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big),\end{aligned}$$

sends biholomorphically the extrinsic complexification M^{e_c} onto the extrinsic complexification M'^{e_c}, namely:

$$\begin{aligned}\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \,\in\, M^{e_c} \quad &\Longleftrightarrow \quad \big(z'_\bullet, w'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big) \,\in\, M'^{e_c} \\ &\Longleftrightarrow \quad \big(z_\bullet,\, \overline{\Theta}_\bullet(0_\bullet, z_\bullet, w_\bullet),\, \underline{z}_\bullet,\, \Theta_\bullet(0_\bullet, \underline{z}_\bullet, \underline{w}_\bullet)\big) \,\in\, M',\end{aligned}$$

that is to say, if one takes as equations for M'^{e_c}:

$$\underline{w}'_\bullet = \overline{\Theta}'\big(\underline{z}'_\bullet, z'_\bullet, w'_\bullet\big),$$

one has:

$$\Theta_\bullet\big(0_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) = \overline{\Theta}'_\bullet\big(\underline{z}_\bullet, z_\bullet, \overline{\Theta}_\bullet(0_\bullet, z_\bullet, w_\bullet)\big),$$

still for:

$$\big(z_\bullet, w_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \,\in\, M^{e_c}.$$

But this means that after replacing $w_\bullet$ occuring at the very last place by:

$$w_\bullet = \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big),$$

one has the c equations:

$$\Theta_\bullet\big(0_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big) \,\equiv\, \overline{\Theta}'_\bullet\Big(\underline{z}_\bullet, z_\bullet, \overline{\Theta}_\bullet\big(0_\bullet, z_\bullet, \Theta_\bullet\big(z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big)\big)\Big),$$

identically satisfied in:

$$\mathbb{C}\big\{z_\bullet, \underline{z}_\bullet, \underline{w}_\bullet\big\}^c.$$

Set in these:

$$\underline{z}_\bullet := 0_\bullet,$$

and get:

$$\underbrace{\Theta_\bullet\big(0_\bullet, 0_\bullet, \underline{w}_\bullet\big)}_{\substack{\equiv\, \underline{w}_\bullet \\ \text{after Step I}}} \,\equiv\, \overline{\Theta}'_\bullet\Big(0'_\bullet, z_\bullet, \underbrace{\overline{\Theta}_\bullet\big(0_\bullet, z_\bullet, \Theta_\bullet\big(z_\bullet, 0_\bullet, \underline{w}_\bullet\big)\big)}_{\equiv\, \underline{w}_\bullet \text{ by one of the functional equations}}\Big),$$

that is to say:

$$\underline{w}_\bullet \equiv \overline{\Theta}'_\bullet\big(0'_\bullet, z_\bullet, \underline{w}_\bullet\big).$$

Changing the name of variables to improve notational harmony, one therefore arrives at one of the announced identities:

$$w'_\bullet \equiv \overline{\Theta}'_\bullet\big(0'_\bullet, z'_\bullet, w'_\bullet\big).$$

Conjugating and renaming again variables, one also obtains:

$$\underline{w}'_\bullet \equiv \Theta'_\bullet\big(0'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big).$$

In the functional equation:

$$w'_\bullet \equiv \Theta'\big(z'_\bullet, \overline{z}'_\bullet,\, \overline{\Theta}'(\underline{z}'_\bullet, z'_\bullet, w'_\bullet)\big),$$

setting:

$$\underline{z}'_\bullet := 0'_\bullet,$$

taking account of what has been just obtained, one gets:

$$w'_\bullet \equiv \Theta'_\bullet\big(z'_\bullet, 0'_\bullet, \underbrace{\overline{\Theta}'_\bullet(0'_\bullet, z'_\bullet, w'_\bullet)}_{\equiv w'_\bullet}\big),$$

that is to say after renaming variables:

$$\underline{w}'_\bullet \equiv \Theta'_\bullet\big(z'_\bullet, 0'_\bullet, \underline{w}'_\bullet\big).$$

Conjugating and complexifying:

$$w'_\bullet \equiv \overline{\Theta}'_\bullet\big(\underline{z}'_\bullet, 0'_\bullet, w'_\bullet\big).$$

It only remains to treat $\varphi'_\bullet$. But from:

$$\frac{\Theta'_\bullet\big(z'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big) - \underline{w}'_\bullet}{2\sqrt{-1}} \equiv \varphi'_\bullet\bigg(z'_\bullet, \underline{z}'_\bullet, \frac{\Theta'_\bullet\big(z'_\bullet, \underline{z}'_\bullet, \underline{w}'_\bullet\big) + \underline{w}'_\bullet}{2}\bigg),$$
$$\frac{w'_\bullet - \overline{\Theta}'_\bullet\big(\underline{z}'_\bullet, z'_\bullet, w'_\bullet\big)}{2\sqrt{-1}} \equiv \varphi'_\bullet\bigg(z'_\bullet, \underline{z}'_\bullet, \frac{w'_\bullet + \overline{\Theta}'_\bullet\big(\underline{z}'_\bullet, z'_\bullet, w'_\bullet\big)}{2}\bigg),$$

setting either $z'_\bullet := 0'_\bullet$ or (not simultaneously) $\underline{z}'_\bullet := 0'_\bullet$, one obtains:

$$0'_\bullet \equiv \varphi'_\bullet\big(0'_\bullet, \overline{z}'_\bullet, \nu'_\bullet\big) \equiv \varphi'_\bullet\big(z'_\bullet, 0'_\bullet, \nu'_\bullet\big),$$

which finishes. □

3. General class I

Proposition. *A local real analytic hypersurface passing through the origin:*

$$0 \in M^3 \subset \mathbb{C}^2$$

which belongs to the general Class I*, namely such that, for any local vector field generator $\mathscr{L}$ of $T^{1,0}M$:*

$$\Big\{\mathscr{L}, \overline{\mathscr{L}}, \ \big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big\}$$

constitute a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, may always be represented, in suitable local holomorphic coordinates:

$$(z, w) \in \mathbb{C}^2$$

by a specific real analytic equation of the form:

$$\boxed{\underline{(\mathsf{I})}: \qquad v = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u).}$$

Proof. Choose first coordinates (z, w) so that M is:

$$v = \varphi\big(z, \overline{z}, u\big),$$

with:

$$0 = \varphi_z(0) = \varphi_{\overline{z}}(0) = \varphi_u(0),$$

and with:

$$0 \equiv \varphi\big(0, \overline{z}, u\big) \equiv \varphi(z, 0, u).$$

Equivalently, the *converging* power series expansion:

$$\varphi = \sum_{\substack{j,k,l \in \mathbb{N} \\ j+k+l \geqslant 2}} \underbrace{\varphi_{j,k,l}}_{\in \mathbb{C}} z^j \, \overline{z}^k \, u^l \qquad\qquad \big(\overline{\varphi_{j,k,l}} = \varphi_{k,j,l}\big),$$

has vanishing coefficients:

$$0 = \varphi_{0,k,l} = \varphi_{j,0,l},$$

so that:

$$v = a\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\, u\, \mathrm{O}_0\big(z, \overline{z}, u\big),$$

or shortly:

$$v = a\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\, \mathrm{O}_1\big(u\big),$$

with of course:

$$a \,\in\, \mathbb{R}.$$

The standard intrinsic generator for $T^{1,0}M$ (*see* [3]):

$$\mathscr{L} = \frac{\partial}{\partial z} \underbrace{- \frac{\varphi_z}{\sqrt{-1} + \varphi_u}}_{=: A} \frac{\partial}{\partial u}$$

has coefficient:

$$\begin{aligned} A &= - \frac{a\, \overline{z} + \mathrm{O}(2)}{\sqrt{-1} + \mathrm{O}(2)} \\ &= \sqrt{-1}\, a\, \overline{z} + \mathrm{O}(2). \end{aligned}$$

Thus starting from:

$$\begin{aligned} \mathscr{L} &= \frac{\partial}{\partial z} + \big(\sqrt{-1}\, a\, \overline{z} + \mathrm{O}(2)\big) \frac{\partial}{\partial u}, \\ \overline{\mathscr{L}} &= \frac{\partial}{\partial \overline{z}} - \big(\sqrt{-1}\, a\, z + \mathrm{O}(2)\big) \frac{\partial}{\partial u}, \end{aligned}$$

one computes the Lie bracket:

$$\big[\mathscr{L}, \overline{\mathscr{L}}\big] = \big(2\, a + \mathrm{O}(1)\big) \frac{\partial}{\partial u}.$$

By hypothesis, the three vectors at the origin:

$$\mathscr{L}\big|_0 = \frac{\partial}{\partial z}\bigg|_0,$$
$$\overline{\mathscr{L}}\big|_0 = \frac{\partial}{\partial \overline{z}}\bigg|_0,$$
$$\sqrt{-1}\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 = 2a\,\frac{\partial}{\partial u},$$

should make a basis for $\mathbb{C}\otimes_{\mathbb{R}} T_0M$ having:

$$\mathbf{3} = \mathsf{dim}_{\mathbb{C}}\big(\mathbb{C}\otimes_{\mathbb{R}} T_0M\big),$$

and this forces:

$$a \neq 0.$$

If $a < 0$, changing:

$$w \longmapsto -w,$$

one makes:

$$a > 0,$$

and lastly changing:

$$z \longmapsto \sqrt{a}\,z$$

one makes $a = 1$, which finishes. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalization at the origin:*

$$\mathscr{L}\big|_0 = \frac{\partial}{\partial z}\bigg|_0,$$
$$\overline{\mathscr{L}}\big|_0 = \frac{\partial}{\partial \overline{z}}\bigg|_0,$$
$$\big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 = -\,2\sqrt{-1}\,\frac{\partial}{\partial u}\bigg|_0,$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure [4]. □

Taking zero remainders, one obtains the:

Model (I):	$v = z\overline{z}.$

4. General class II

Proposition. *A local real analytic* 4*-dimensional CR-generic submanifold passing through the origin:*

$$0 \in M^4 \subset \mathbb{C}^3$$

which belongs to the general Class II*, namely such that, for any local vector field generator* $\mathscr{L}$ *of* $T^{1,0}M$*:*

$$\Big\{ \mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]] \Big\}$$

constitute a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$*, may always be represented, in suitable local holomorphic coordinates:*

$$(z, w_1, w_2) \in \mathbb{C}^3$$

by two specific real analytic equation of the form:

$$\boxed{\underline{\text{(II)}}: \quad \begin{aligned} v_1 &= z\overline{z} && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2). \end{aligned}}$$

Proof. Choose first coordinates (z, w_1, w_2) so that M is:

$$\begin{aligned} v_1 &= \varphi_1\big(z, \overline{z}, u_1, u_2\big), \\ v_2 &= \varphi_2\big(z, \overline{z}, u_1, u_2\big), \end{aligned}$$

with:

$$\begin{aligned} 0 &= \varphi_{1,z}(0) = \varphi_{1,\overline{z}}(0) = \varphi_{1,u_1}(0) = \varphi_{1,u_2}(0), \\ 0 &= \varphi_{2,z}(0) = \varphi_{2,\overline{z}}(0) = \varphi_{2,u_1}(0) = \varphi_{2,u_2}(0), \end{aligned}$$

and with:

$$\begin{aligned} 0 &\equiv \varphi_1\big(0, \overline{z}, u_1, u_2\big) \equiv \varphi_1\big(z, 0, u_1, u_2\big), \\ 0 &\equiv \varphi_2\big(0, \overline{z}, u_1, u_2\big) \equiv \varphi_2\big(z, 0, u_1, u_2\big). \end{aligned}$$

Equivalently, the *converging* power series expansions:

$$\varphi_1 = \sum_{\substack{j,k,l_1,l_2 \in \mathbb{N} \\ j+k+l_1+l_2 \geqslant 2}} \underbrace{\varphi_{1,j,k,l_1,l_2}}_{\in \mathbb{C}} z^j \, \overline{z}^k \, u_1^{l_1} \, u_2^{l_2} \qquad \big(\overline{\varphi_{1,j,k,l_1,l_2}} = \varphi_{1,k,j,l_1,l_2}\big),$$

$$\varphi_2 = \sum_{\substack{j,k,l_1,l_2 \in \mathbb{N} \\ j+k+l_1+l_2 \geqslant 2}} \underbrace{\varphi_{2,j,k,l_1,l_2}}_{\in \mathbb{C}} z^j \, \overline{z}^k \, u_1^{l_1} \, u_2^{l_2} \qquad \big(\overline{\varphi_{2,j,k,l_1,l_2}} = \varphi_{2,k,j,l_1,l_2}\big),$$

have vanishing coefficients:

$$\begin{aligned} 0 &= \varphi_{1,0,k,l_1,l_2} = \varphi_{1,j,0,l_1,l_2}, \\ 0 &= \varphi_{2,0,k,l_1,l_2} = \varphi_{2,j,0,l_1,l_2}, \end{aligned}$$

so that:

$$v_1 = a_1\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\, u_1\, \mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\, u_2\, \mathrm{O}_0\big(z,\overline{z},u_1,u_2\big),$$
$$v_2 = a_2\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\, u_1\, \mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\, u_2\, \mathrm{O}_0\big(z,\overline{z},u_1,u_2\big),$$

or shortly:

$$\begin{aligned} v_1 &= a_1\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\, \mathrm{O}_1(u_1) + z\overline{z}\, \mathrm{O}_1(u_2) \\ v_2 &= a_2\, z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\, \mathrm{O}_1(u_1) + z\overline{z}\, \mathrm{O}_1(u_2), \end{aligned}$$

with of course:

$$\begin{aligned} a_1 &\in \mathbb{R}, \\ a_2 &\in \mathbb{R}. \end{aligned}$$

The standard intrinsic generator for $T^{1,0}M$ (*see* [3]):

$$\mathscr{L} = \frac{\partial}{\partial z} + A_1\,\frac{\partial}{\partial u_1} + A_2\,\frac{\partial}{\partial u_2}$$

has its two coefficients given by the formulas:

$$A_1 = \left|\begin{matrix} -\varphi_{1,z} & \varphi_{1,u_2} \\ -\varphi_{2,z} & \sqrt{-1}+\varphi_{2,u_2} \end{matrix}\right| \Big/ \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} \end{matrix}\right|,$$
$$A_2 = \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & -\varphi_{1,z} \\ \varphi_{2,u_1} & -\varphi_{2,z} \end{matrix}\right| \Big/ \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} \end{matrix}\right|.$$

An approximant for A_1 is:

$$\begin{aligned} A_1 &= \left|\begin{matrix} -a_1\overline{z}+\mathrm{O}(2) & \mathrm{O}(2) \\ -a_2\overline{z}+\mathrm{O}(2) & \sqrt{-1}+\mathrm{O}(2) \end{matrix}\right| \Big/ \left|\begin{matrix} \sqrt{-1}+\mathrm{O}(2) & \mathrm{O}(2) \\ \mathrm{O}(2) & \sqrt{-1}+\mathrm{O}(2) \end{matrix}\right| \\ &= \sqrt{-1}\, a_1\,\overline{z} + \mathrm{O}(2), \end{aligned}$$

and similarly:

$$A_2 = \sqrt{-1}\, a_2\,\overline{z} + \mathrm{O}(2),$$

so that:

$$\begin{aligned} \mathscr{L} &= \frac{\partial}{\partial z} + \sqrt{-1}\,\big(a_1\,\overline{z}+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1} + \sqrt{-1}\,\big(a_2\,\overline{z}+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2}, \\ \overline{\mathscr{L}} &= \frac{\partial}{\partial \overline{z}} - \sqrt{-1}\,\big(a_1\,z+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1} - \sqrt{-1}\,\big(a_2\,z+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2}, \end{aligned}$$

whence:

$$\sqrt{-1}\,\big[\mathscr{L},\overline{\mathscr{L}}\big] = \big(2\,a_1+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_1} + \big(2\,a_2+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_2},$$

and at the origin:

$$\sqrt{-1}\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 = 2a_1\,\frac{\partial}{\partial u_1}\Big|_0 + 2a_2\,\frac{\partial}{\partial u_2}\Big|_0.$$

Since the three vectors:

$$\mathscr{L}\big|_0, \qquad \overline{\mathscr{L}}\big|_0, \qquad \big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big|_0$$

must in particular by hypothesis necessarily be $\mathbb{C}$-linearly independent at the origin, one has:

$$a_1 \neq 0 \qquad \text{or} \qquad a_2 \neq 0.$$

Say:

$$a_1 \neq 0.$$

Then do if necessary:

$$w_1 \longmapsto -w_1$$

to make:

$$a_1 > 0,$$

and do:

$$z \longmapsto \sqrt{a_1}\, z,$$

to make:

$$a_1 = 1.$$

Writing out now the third-order $(z, \overline{z})$-terms, one therefore comes to:

$$v_1 = z\overline{z} + \alpha_1\, z^2\overline{z} + \overline{\alpha}_1\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2),$$
$$v_2 = a_2\, z\overline{z} + \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2),$$

with of course:

$$\alpha_1 \in \mathbb{C} \qquad \text{and} \qquad \alpha_2 \in \mathbb{C}.$$

Changing:

$$w_2 \longmapsto w_2 - a_2\, w_1,$$

one makes:

$$a_2 = 0.$$

In the process, α_2 is modified.

Changing:

$$z \longmapsto z + \alpha_1\, z^2,$$

one makes:

$$\alpha_1 = 0.$$

One thus arrives at:

$$v_1 = z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2),$$
$$v_2 = \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2).$$

Next, a deeper approximant for A_1 is:

$$A_1 = \left| \begin{array}{cc} -\overline{z} + \mathrm{O}_3(z,\overline{z}) + \overline{z}\mathrm{O}_1(u_1,u_2) & z\overline{z}\mathrm{O}(0) \\ -2\alpha_2 z\overline{z} - \overline{\alpha}_2\overline{z}^2 + \mathrm{O}_3(z,\overline{z}) + \overline{z}\mathrm{O}_1(u_1,u_2) & \sqrt{-1} + z\overline{z}\mathrm{O}(0) \end{array} \right| \Big/ \left| \begin{array}{cc} \sqrt{-1} + z\overline{z}\mathrm{O}(0) & z\overline{z}\mathrm{O}(0) \\ z\overline{z}\mathrm{O}(0) & \sqrt{-1} + z\overline{z}\mathrm{O}(0) \end{array} \right|$$
$$= \sqrt{-1}\,\overline{z} + \overline{z}\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3),$$

while an approximant for A_2 is:

$$A_2 = \left| \begin{array}{cc} \sqrt{-1} + z\overline{z}\mathrm{O}(0) & -\overline{z} + \mathrm{O}_3(z,\overline{z}) + \overline{z}\mathrm{O}_1(u_1,u_2) \\ z\overline{z}\mathrm{O}(0) & -2\alpha_2 z\overline{z} - \overline{\alpha}_2\overline{z}^2 + \mathrm{O}_3(z,\overline{z}) + \overline{z}\mathrm{O}_1(u_1,u_2) \end{array} \right| \Big/ \left| \begin{array}{cc} \sqrt{-1} + z\overline{z}\mathrm{O}(0) & z\overline{z}\mathrm{O}(0) \\ z\overline{z}\mathrm{O}(0) & \sqrt{-1} + z\overline{z}\mathrm{O}(0) \end{array} \right|$$
$$= 2\sqrt{-1}\,\alpha_2\, z\overline{z} + \sqrt{-1}\,\overline{\alpha}_2\,\overline{z}^2 + \overline{z}\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3),$$

so that:

$$\mathscr{L} = \frac{\partial}{\partial z} + \sqrt{-1}\,\big(\overline{z} + \overline{z}\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3)\big)\,\frac{\partial}{\partial u_1} + \sqrt{-1}\,\big(2\alpha_2\,z\overline{z} + \overline{\alpha}_2\,\overline{z}^2 + \overline{z}\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3)\big)\,\frac{\partial}{\partial u_2},$$
$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} - \sqrt{-1}\,\big(z + z\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3)\big)\,\frac{\partial}{\partial u_1} - \sqrt{-1}\,\big(2\overline{\alpha}_2\,z\overline{z} + \alpha_2\,z^2 + z\,\mathrm{O}_1(u_1,u_2) + \mathrm{O}(3)\big)\,\frac{\partial}{\partial u_2},$$

whence:

$$\big[\mathscr{L},\overline{\mathscr{L}}\big] = -\sqrt{-1}\,\big(2+\mathrm{O}_1(u_1,u_2)+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1} - \sqrt{-1}\,\big(4\alpha_2\,z + 4\overline{\alpha}_2\,\overline{z} + \mathrm{O}_1(u_1,u_2) + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2},$$

whence further:

$$\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big] = \mathrm{O}(1)\,\frac{\partial}{\partial u_1} - \sqrt{-1}\,\big(4\alpha_2 + \mathrm{O}(1)\big)\,\frac{\partial}{\partial u_2},$$

and lastly, at the origin:

$$\mathscr{L}\big|_0 = \frac{\partial}{\partial z}\bigg|_0,$$
$$\overline{\mathscr{L}}\big|_0 = \frac{\partial}{\partial \overline{z}}\bigg|_0,$$
$$\big[\mathscr{L},\overline{\mathscr{L}}\big]\big|_0 = -2\sqrt{-1}\,\frac{\partial}{\partial u_1}\bigg|_0,$$
$$\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\big|_0 = -4\sqrt{-1}\,\alpha_2\,\frac{\partial}{\partial u_2}\bigg|_0.$$

Since these four vectors should by hypothesis constitute a basis for:

$$\mathbb{C}\otimes_{\mathbb{R}} T_0M \;=\; \mathbb{C}\frac{\partial}{\partial z}\bigg|_0 \oplus \mathbb{C}\frac{\partial}{\partial \overline{z}}\bigg|_0 \oplus \mathbb{C}\frac{\partial}{\partial u_1}\bigg|_0 \oplus \mathbb{C}\frac{\partial}{\partial u_2}\bigg|_0,$$

one necessarily has:

$$\alpha_2 \neq 0.$$

Using a complex dilation:

$$z \longmapsto \lambda\,z,$$

with λ solution of (exercise):

$$1 = \alpha_2 \, \lambda^2 \, \overline{\lambda},$$

one makes:

$$\alpha_2 = 1,$$

so that the second equation receives the form announced:

$$v_2 = z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2).$$

In the process, the first equation is changed to:

$$v_1 = \lambda\overline{\lambda}\,z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2),$$

but one can yet replace:

$$w_1 \longmapsto \frac{w_1}{\lambda\overline{\lambda}},$$

to keep $1 \cdot z\overline{z}$, which concludes. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalization at the origin:*

$$\begin{aligned}
\mathscr{L}\big|_0 &= \frac{\partial}{\partial z}\bigg|_0, \\
\overline{\mathscr{L}}\big|_0 &= \frac{\partial}{\partial \overline{z}}\bigg|_0, \\
\big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 &= -\,2\,\sqrt{-1}\,\frac{\partial}{\partial u_1}\bigg|_0, \\
\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big|_0 &= -\,4\,\sqrt{-1}\,\frac{\partial}{\partial u_2}\bigg|_0,
\end{aligned}$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure. □

Taking zero remainders, one obtains Beloshapka's cubic:

Model (II):	$v_1 = z\overline{z},$ $v_2 = z^2\overline{z} + z\overline{z}^2.$

5. General class III_1

Proposition. *A local real analytic* 5*-dimensional CR-generic submanifold passing through the origin:*

$$0 \in M^5 \subset \mathbb{C}^4$$

which belongs to the general Class III_1*, namely such that, for any local vector field generator* $\mathscr{L}$ *of* $T^{1,0}M$*:*

$$\Big\{\mathscr{L}, \overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}], \ [\mathscr{L}, \ [\mathscr{L}, \overline{\mathscr{L}}]], \ [\overline{\mathscr{L}}, \ [\mathscr{L}, \overline{\mathscr{L}}]]\Big\}$$

constitute a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$*, may always be represented, in suitable local holomorphic coordinates:*

$$(z, w_1, w_2, w_3) \in \mathbb{C}^3$$

by three specific real analytic equation of the form:

$$(\mathrm{III})_1: \quad \begin{aligned} v_1 &= z\overline{z} && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\ v_3 &= \sqrt{-1}\,\big(z^2\overline{z} - z\overline{z}^2\big) && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3). \end{aligned}$$

Proof. Choose first coordinates (z, w_1, w_2, w_3) so that M is:

$$\begin{aligned} v_1 &= \varphi_1\big(z, \overline{z}, u_1, u_2, u_3\big), \\ v_2 &= \varphi_2\big(z, \overline{z}, u_1, u_2, u_3\big), \\ v_3 &= \varphi_3\big(z, \overline{z}, u_1, u_2, u_3\big), \end{aligned}$$

with:

$$\begin{aligned} 0 &= \varphi_{1,z}(0) = \varphi_{1,\overline{z}}(0) = \varphi_{1,u_1}(0) = \varphi_{1,u_2}(0) = \varphi_{1,u_3}(0), \\ 0 &= \varphi_{2,z}(0) = \varphi_{2,\overline{z}}(0) = \varphi_{2,u_1}(0) = \varphi_{2,u_2}(0) = \varphi_{2,u_3}(0), \\ 0 &= \varphi_{3,z}(0) = \varphi_{3,\overline{z}}(0) = \varphi_{3,u_1}(0) = \varphi_{3,u_2}(0) = \varphi_{3,u_3}(0), \end{aligned}$$

and with:

$$\begin{aligned} 0 &\equiv \varphi_1\big(0, \overline{z}, u_1, u_2, u_3\big) \equiv \varphi_1\big(z, 0, u_1, u_2, u_3\big), \\ 0 &\equiv \varphi_2\big(0, \overline{z}, u_1, u_2, u_3\big) \equiv \varphi_2\big(z, 0, u_1, u_2, u_3\big), \\ 0 &\equiv \varphi_3\big(0, \overline{z}, u_1, u_2, u_3\big) \equiv \varphi_3\big(z, 0, u_1, u_2, u_3\big). \end{aligned}$$

Equivalently, the *converging* power series expansions:

$$\varphi_1 = \sum_{\substack{j,k,l_1,l_2,l_3\in\mathbb{N}\\ j+k+l_1+l_2+l_3\geqslant 2}} \underbrace{\varphi_{1,j,k,l_1,l_2,l_3}}_{\in\mathbb{C}} z^j\,\overline{z}^k\,u_1^{l_1}\,u_2^{l_2}\,u_3^{l_3} \qquad (\overline{\varphi_{1,j,k,l_1,l_2,l_3}} = \varphi_{1,k,j,l_1,l_2,l_3}),$$

$$\varphi_2 = \sum_{\substack{j,k,l_1,l_2,l_3\in\mathbb{N}\\ j+k+l_1+l_2+l_3\geqslant 2}} \underbrace{\varphi_{2,j,k,l_1,l_2,l_3}}_{\in\mathbb{C}} z^j\,\overline{z}^k\,u_1^{l_1}\,u_2^{l_2}\,u_3^{l_3} \qquad (\overline{\varphi_{2,j,k,l_1,l_2,l_3}} = \varphi_{2,k,j,l_1,l_2,l_3}),$$

$$\varphi_3 = \sum_{\substack{j,k,l_1,l_2,l_3\in\mathbb{N}\\ j+k+l_1+l_2+l_3\geqslant 2}} \underbrace{\varphi_{3,j,k,l_1,l_2,l_3}}_{\in\mathbb{C}} z^j\,\overline{z}^k\,u_1^{l_1}\,u_2^{l_2}\,u_3^{l_3} \qquad (\overline{\varphi_{3,j,k,l_1,l_2,l_3}} = \varphi_{3,k,j,l_1,l_2,l_3}),$$

have vanishing coefficients:

$$\begin{aligned}
0 &= \varphi_{1,0,k,l_1,l_2,l_3} = \varphi_{1,j,0,l_1,l_2,l_3},\\
0 &= \varphi_{2,0,k,l_1,l_2,l_3} = \varphi_{2,j,0,l_1,l_2,l_3},\\
0 &= \varphi_{3,0,k,l_1,l_2,l_3} = \varphi_{3,j,0,l_1,l_2,l_3},
\end{aligned}$$

so that:

$$\begin{aligned}
v_1 &= a_1\,z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_2 &= a_2\,z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_3 &= a_3\,z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),
\end{aligned}$$

with of course:

$$\begin{aligned}
a_1 &\in \mathbb{R},\\
a_2 &\in \mathbb{R},\\
a_3 &\in \mathbb{R}.
\end{aligned}$$

The standard intrinsic generator for $T^{1,0}M$ (*see* [3]):

$$\mathscr{L} = \frac{\partial}{\partial z} + A_1\,\frac{\partial}{\partial u_1} + A_2\,\frac{\partial}{\partial u_2} + A_3\,\frac{\partial}{\partial u_3}$$

has its three coefficients given by the formulas:

$$A_1 = \left|\begin{matrix} -\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3}\\ -\varphi_{2,z} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\ -\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right| \Bigg/ \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3}\\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right|,$$

$$A_2 = \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & -\varphi_{1,z} & \varphi_{1,u_3}\\ \varphi_{2,u_1} & -\varphi_{2,z} & \varphi_{2,u_3}\\ \varphi_{3,u_1} & -\varphi_{3,z} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right| \Bigg/ \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3}\\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right|,$$

$$A_3 = \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & -\varphi_{1,z}\\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & -\varphi_{2,z}\\ \varphi_{3,u_1} & \varphi_{3,u_2} & -\varphi_{3,z}\end{matrix}\right| \Bigg/ \left|\begin{matrix} \sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3}\\ \varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right|.$$

An approximant for A_1 is:

$$A_1 = \begin{vmatrix} -a_1\overline{z} + \mathrm{O}(2) & \mathrm{O}(2) & \mathrm{O}(2) \\ -a_2\overline{z} + \mathrm{O}(2) & \sqrt{-1} + \mathrm{O}(2) & \mathrm{O}(2) \\ -a_3\overline{z} + \mathrm{O}(2) & \mathrm{O}(2) & \sqrt{-1} + \mathrm{O}(2) \end{vmatrix} \Bigg/ \begin{vmatrix} \sqrt{-1} + \mathrm{O}(2) & \mathrm{O}(2) & \mathrm{O}(2) \\ \mathrm{O}(2) & \sqrt{-1} + \mathrm{O}(2) & \mathrm{O}(2) \\ \mathrm{O}(2) & \mathrm{O}(2) & \sqrt{-1} + \mathrm{O}(2) \end{vmatrix}$$
$$= \sqrt{-1}\, a_1\, \overline{z} + \mathrm{O}(2),$$

and similarly:

$$A_2 = \sqrt{-1}\, a_2\, \overline{z} + \mathrm{O}(2),$$
$$A_3 = \sqrt{-1}\, a_3\, \overline{z} + \mathrm{O}(2),$$

so that:

$$\mathscr{L} = \frac{\partial}{\partial z} + \sqrt{-1}\,\big(a_1\,\overline{z} + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1} + \sqrt{-1}\,\big(a_2\,\overline{z} + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2} + \sqrt{-1}\,\big(a_3\,\overline{z} + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_3},$$
$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} - \sqrt{-1}\,\big(a_1\,z + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1} - \sqrt{-1}\,\big(a_2\,z + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2} - \sqrt{-1}\,\big(a_3\,z + \mathrm{O}(2)\big)\,\frac{\partial}{\partial u_3},$$

whence:

$$\sqrt{-1}\,\big[\mathscr{L}, \overline{\mathscr{L}}\big] = \big(2\,a_1 + \mathrm{O}(1)\big)\,\frac{\partial}{\partial u_1} + \big(2\,a_2 + \mathrm{O}(1)\big)\,\frac{\partial}{\partial u_2} + \big(2\,a_2 + \mathrm{O}(1)\big)\,\frac{\partial}{\partial u_3},$$

and at the origin:

$$\sqrt{-1}\,\big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big|_0 = 2a_1\,\frac{\partial}{\partial u_1}\Big|_0 + 2a_2\,\frac{\partial}{\partial u_2}\Big|_0 + 2a_3\,\frac{\partial}{\partial u_2}\Big|_0.$$

Since the three vectors:

$$\mathscr{L}\big|_0, \qquad \overline{\mathscr{L}}\big|_0, \qquad \big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big|_0$$

must in particular by hypothesis necessarily be $\mathbb{C}$-linearly independent at the origin, one has:

$a_1 \neq 0,$ or $a_2 \neq 0,$ or $a_3 \neq 0.$

Say:

$$a_1 \neq 0.$$

Then do if necessary:

$$w_1 \longmapsto -w_1$$

to make:

$$a_1 > 0,$$

and do:

$$z \longmapsto \sqrt{a_1}\, z,$$

to make:

$$a_1 = 1.$$

Writing out now the third-order $(z,\overline{z})$-terms, one therefore comes to:

$$\begin{aligned}
v_1 &= z\overline{z} + \alpha_1\, z^2\overline{z} + \overline{\alpha}_1\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_2 &= a_2\, z\overline{z} + \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_3 &= a_3\, z\overline{z} + \alpha_3\, z^2\overline{z} + \overline{\alpha}_3\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),
\end{aligned}$$

with of course:

$$\alpha_1 \in \mathbb{C}, \qquad\qquad \alpha_2 \in \mathbb{C}, \qquad\qquad \alpha_3 \in \mathbb{C}.$$

Changing:

$$\begin{aligned}
w_2 &\longmapsto w_2 - a_2\, w_1,\\
w_3 &\longmapsto w_3 - a_3\, w_1,
\end{aligned}$$

one makes:

$$\begin{aligned}
a_2 &= 0,\\
a_3 &= 0.
\end{aligned}$$

In the process, α_2, α_3 are modified.

Changing:

$$z \longmapsto z + \alpha_1\, z^2,$$

one makes:

$$\alpha_1 = 0.$$

One thus arrives at:

$$\begin{aligned}
v_1 &= z\overline{z} && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_2 &= \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),\\
v_3 &= \alpha_3\, z^2\overline{z} + \overline{\alpha}_3\, z\overline{z}^2 && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3).
\end{aligned}$$

Take again the approximant for the denominator common to A_1, A_2, A_3:

$$\begin{aligned}
\left|\begin{array}{ccc}
\sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3}\\
\varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\
\varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}
\end{array}\right|
&= \left|\begin{array}{ccc}
\sqrt{-1}+\mathrm{O}(2) & \mathrm{O}(2) & \mathrm{O}(2)\\
\mathrm{O}(2) & \sqrt{-1}+\mathrm{O}(2) & \mathrm{O}(2)\\
\mathrm{O}(2) & \mathrm{O}(2) & \sqrt{-1}+\mathrm{O}(2)
\end{array}\right|\\
&= \sqrt{-1}^{\,3} + \mathrm{O}(2).
\end{aligned}$$

Next, a deeper approximant for the numerator of A_1 is:

$$\left|\begin{array}{ccc}
-\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3}\\
-\varphi_{2,z} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3}\\
-\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}
\end{array}\right|
= \left|\begin{array}{ccc}
\begin{array}{c}-\overline{z}+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\\ +\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\end{array} & z\overline{z}(0) & z\overline{z}(0)\\
\begin{array}{c}-2\alpha_2 z\overline{z}-\overline{\alpha}_2\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\\ +\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\end{array} & \sqrt{-1}+z\overline{z}(0) & z\overline{z}(0)\\
\begin{array}{c}-2\alpha_3 z\overline{z}-\overline{\alpha}_3\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\\ +\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\end{array} & z\overline{z}(0) & \sqrt{-1}+z\overline{z}(0)
\end{array}\right|$$

that is:

$$\sqrt{-1}^{\,2}\big(-\ \overline{z} + \mathrm{O}_3(z,\overline{z}) + \overline{z}\,\mathrm{O}_1(u_1) + \overline{z}\,\mathrm{O}_1(u_3) + \overline{z}\,\mathrm{O}_1(u_3)\big),$$

so that:

$$\begin{aligned}A_1 &= \frac{\sqrt{-1}^2\big(-\,\overline{z}+\mathrm{O}_3(z,\overline{z})+\overline{z}\,\mathrm{O}_1(u_1)+\overline{z}\,\mathrm{O}_1(u_3)+\overline{z}\,\mathrm{O}_1(u_3)\big)}{\sqrt{-1}^3+\mathrm{O}(2)}\\ &= \sqrt{-1}\,\big(\overline{z}+\mathrm{O}_3(z,\overline{z})+\overline{z}\,\mathrm{O}_1(u_1)+\overline{z}\,\mathrm{O}_1(u_3)+\overline{z}\,\mathrm{O}_1(u_3)\big).\end{aligned}$$

Next, a deeper approximant for the numerator of A_2 is:

$$\left|\begin{matrix}\sqrt{-1}+\varphi_{1,u_2} & -\varphi_{1,z} & \varphi_{1,u_3}\\ \varphi_{2,u_1} & -\varphi_{2,z} & \varphi_{2,u_3}\\ \varphi_{3,u_1} & -\varphi_{3,z} & \sqrt{-1}+\varphi_{3,u_3}\end{matrix}\right| = \left|\begin{matrix}\sqrt{-1}+z\overline{z}(0) & \substack{-\overline{z}+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\\ +\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)} & z\overline{z}(0)\\ z\overline{z}(0) & \substack{-2\alpha_2z\overline{z}-\overline{\alpha}_2\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\\ +\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)} & z\overline{z}(0)\\ z\overline{z}(0) & \substack{-2\alpha_3z\overline{z}-\overline{\alpha}_3\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\\ +\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)} & \sqrt{-1}+z\overline{z}(0)\end{matrix}\right|,$$

that is:

$$\sqrt{-1}^2\big(-\,2\alpha_2\,z\overline{z}-\overline{\alpha}_2\,\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\big),$$

so that:

$$\begin{aligned}A_2 &= \frac{\sqrt{-1}^2\big(-\,2\alpha_2\,z\overline{z}-\overline{\alpha}_2\,\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\big)}{\sqrt{-1}^3+\mathrm{O}(2)}\\ &= \sqrt{-1}\,\big(2\alpha_2\,z\overline{z}+\overline{\alpha}_2\,\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\big).\end{aligned}$$

Quite similarly:

$$A_3 = \sqrt{-1}\,\big(2\alpha_3\,z\overline{z}+\overline{\alpha}_3\,\overline{z}^2+\mathrm{O}_3(z,\overline{z})+\overline{z}\mathrm{O}_1(u_1)+\overline{z}\mathrm{O}_1(u_2)+\overline{z}\mathrm{O}_1(u_3)\big).$$

It follows that:

$$\begin{aligned}\mathscr{L} = \frac{\partial}{\partial z} &+ \sqrt{-1}\,\big(\overline{z}+\overline{z}\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_1}+\\ &+ \sqrt{-1}\,\big(2\alpha_2\,z\overline{z}+\overline{\alpha}_2\,\overline{z}^2+\overline{z}\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_2},\\ &+ \sqrt{-1}\,\big(2\alpha_3\,z\overline{z}+\overline{\alpha}_3\,\overline{z}^2+\overline{z}\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

$$\begin{aligned}\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} &- \sqrt{-1}\,\big(z+z\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_1}-\\ &- \sqrt{-1}\,\big(2\overline{\alpha}_2\,z\overline{z}+\alpha_2\,z^2+z\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_2},\\ &- \sqrt{-1}\,\big(2\overline{\alpha}_3\,z\overline{z}+\alpha_3\,z^2+z\,\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(3)\big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

whence:

$$\begin{aligned}\big[\mathscr{L},\overline{\mathscr{L}}\big] = &-\sqrt{-1}\,\big(2+\mathrm{O}_1(u_1,u_2)+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_1}-\\ &-\sqrt{-1}\,\big(4\alpha_2\,z+4\overline{\alpha}_2\,\overline{z}+\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_2},\\ &-\sqrt{-1}\,\big(4\alpha_3\,z+4\overline{\alpha}_3\,\overline{z}+\mathrm{O}_1(u_1,u_2,u_3)+\mathrm{O}(2)\big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

whence further:

$$\begin{aligned}\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big] &= \mathrm{O}(1)\,\frac{\partial}{\partial u_1}-\sqrt{-1}\,\big(4\alpha_2+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_2}-\sqrt{-1}\,\big(4\alpha_3+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_3},\\ \big[\overline{\mathscr{L}},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big] &= \mathrm{O}(1)\,\frac{\partial}{\partial u_1}-\sqrt{-1}\,\big(4\overline{\alpha}_2+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_2}-\sqrt{-1}\,\big(4\overline{\alpha}_3+\mathrm{O}(1)\big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

and lastly, at the origin:

$$\begin{aligned}\mathscr{L}\big|_0 &= \frac{\partial}{\partial z}\bigg|_0,\\ \overline{\mathscr{L}}\big|_0 &= \frac{\partial}{\partial \overline{z}}\bigg|_0,\\ \big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 &= -\,2\,\sqrt{-1}\,\frac{\partial}{\partial u_1}\bigg|_0,\\ \big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big|_0 &= -\,4\,\sqrt{-1}\,\alpha_2\,\frac{\partial}{\partial u_2}\bigg|_0-4\,\sqrt{-1}\,\alpha_3\,\frac{\partial}{\partial u_3}\bigg|_0,\\ \big[\overline{\mathscr{L}},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big|_0 &= -\,4\,\sqrt{-1}\,\overline{\alpha}_2\,\frac{\partial}{\partial u_2}\bigg|_0-4\,\sqrt{-1}\,\overline{\alpha}_3\,\frac{\partial}{\partial u_3}\bigg|_0.\end{aligned}$$

These four vectors should by hypothesis constitute a basis for:

$$\mathbb{C}\otimes_{\mathbb{R}}T_0M \;=\; \mathbb{C}\frac{\partial}{\partial z}\bigg|_0\oplus\mathbb{C}\frac{\partial}{\partial \overline{z}}\bigg|_0\oplus\mathbb{C}\frac{\partial}{\partial u_1}\bigg|_0\oplus\mathbb{C}\frac{\partial}{\partial u_2}\bigg|_0\oplus\mathbb{C}\frac{\partial}{\partial u_3}\bigg|_0,$$

having dimension $\mathbf{5}$, which necessitates the nonzeroness:

$$0\neq\begin{vmatrix}2 & 0 & 0\\ 0 & 4\alpha_2 & 4\alpha_3\\ 0 & 4\overline{\alpha}_2 & 4\overline{\alpha}_3\end{vmatrix}.$$

At least:

$$\alpha_2\neq 0.$$

Using a complex dilation:

$$z\longmapsto\lambda\,z,$$

with λ solution of:

$$1=\alpha_2\,\lambda^2\,\overline{\lambda},$$

one makes:

$$\alpha_2 = 1,$$

so that the second equation receives the form announced:

$$v_2 = z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3).$$

In the process, the first equation is changed to:

$$v_1 = \lambda\overline{\lambda}\,z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3),$$

but one can yet replace:

$$w_1 \longmapsto \frac{w_1}{\lambda\overline{\lambda}},$$

to keep $1 \cdot z\overline{z}$.

Writing:

$$\alpha_3 = a_3 + \sqrt{-1}\,b_3,$$

replacing:

$$w_3 \longmapsto w_3 - a_3\,w_2,$$

one makes:

$$a_3 = 0.$$

Lastly:

$$0 \neq \left| \begin{array}{ccc} 2 & 0 & 0 \\ 0 & 4 & 4\sqrt{-1}\,b_3 \\ 0 & 4 & -4\sqrt{-1}\,b_3 \end{array} \right|.$$

means:

$$b_3 \neq 0,$$

and doing:

$$w_3 \longmapsto \frac{w_3}{b_3},$$

one arrives at the announced form for the third graphing equation. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalization at the origin:*

$$\begin{aligned}
\mathscr{L}\big|_0 &= \frac{\partial}{\partial z}\bigg|_0, \\
\overline{\mathscr{L}}\big|_0 &= \frac{\partial}{\partial \overline{z}}\bigg|_0, \\
\big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big|_0 &= -2\sqrt{-1}\,\frac{\partial}{\partial u_1}\bigg|_0, \\
\big[\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]\big]\Big|_0 &= -4\sqrt{-1}\,\frac{\partial}{\partial u_2}\bigg|_0 + 4\,\frac{\partial}{\partial u_3}\bigg|_0, \\
\big[\overline{\mathscr{L}},\, [\mathscr{L}, \overline{\mathscr{L}}]\big]\Big|_0 &= -4\sqrt{-1}\,\frac{\partial}{\partial u_2}\bigg|_0 - 4\,\frac{\partial}{\partial u_3}\bigg|_0,
\end{aligned}$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure. □

Taking zero remainders, one obtains Beloshapka's second cubic:

$$\boxed{\text{Model (III)}_1: \qquad \begin{aligned} v_1 &= z\overline{z}, \\ v_2 &= z^2\overline{z} + z\overline{z}^2, \\ v_3 &= \sqrt{-1}\,\big(z^2\overline{z} - z\overline{z}^2\big). \end{aligned}}$$

6. General class III_2

Proposition. *A local real analytic 5-dimensional CR-generic submanifold passing through the origin:*

$$0 \in M^5 \subset \mathbb{C}^4$$

which belongs to the general Class III_1, namely such that, for any local vector field generator $\mathscr{L}$ of $T^{1,0}M$, one has at every point:

$$\begin{aligned}
\mathbf{3} &= \mathrm{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}]\Big\}\Big), \\
\mathbf{4} &= \mathrm{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}],\ [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\}\Big), \\
\mathbf{4} &= \mathrm{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}],\ [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]],\ [\overline{\mathscr{L}},\, [\mathscr{L}, \overline{\mathscr{L}}]]\Big\}\Big), \\
\mathbf{5} &= \mathrm{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L}, \overline{\mathscr{L}},\ [\mathscr{L}, \overline{\mathscr{L}}],\ [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]],\ [\mathscr{L},\, [\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]]]\Big\}\Big),
\end{aligned}$$

may always be represented, in suitable local holomorphic coordinates:

$$(z, w_1, w_2, w_3) \in \mathbb{C}^4$$

by three specific real analytic equation of the form:

$$
\underline{(\mathsf{III})_2:}\quad
\begin{aligned}
v_1 &= z\overline{z} + c_1\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_2 &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_3 &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big).
\end{aligned}
$$

Exercise. Show that one can make $c_1 = 0$, somewhere at the right place in the reasoning conducted below.

Proof. Choose first coordinates (z, w_1, w_2, w_3) so that M is:

$$
\begin{aligned}
v_1 &= \varphi_1\big(z,\overline{z},u_1,u_2,u_3\big) = a_1\, z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_2 &= \varphi_2\big(z,\overline{z},u_1,u_2,u_3\big) = a_2\, z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_3 &= \varphi_3\big(z,\overline{z},u_1,u_2,u_3\big) = a_3\, z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big),
\end{aligned}
$$

with:

$$
a_1,\ a_2,\ a_3\ \in \mathbb{R}.
$$

As for the general class III_1, one sees (mental exercise) that at least one of a_1, a_2, a_3 must be nonzero, say:

$$
a_1 \neq 0.
$$

Elementary linear transformations along coordinate axes yield (exercise):

$$
\begin{aligned}
v_1 &= z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_2 &= \qquad\ z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\
v_3 &= \qquad\ z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big).
\end{aligned}
$$

To better organize these general remainders:

$$
z\overline{z}\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big),
$$

observe that every local converging power series function:

$$
F\big(\mathsf{x}_1,\mathsf{x}_2,\dots,\mathsf{x}_{\rm N}\big) = \sum_{\alpha_1,\alpha_2,\dots,\alpha_{\rm N}\in\mathbb{N}} F_{\alpha_1,\alpha_2,\dots,\alpha_{\rm N}}\,\big(\mathsf{x}_1\big)^{\alpha_1}\big(\mathsf{x}_2\big)^{\alpha_2}\cdots\big(\mathsf{x}_{\rm N}\big)^{\alpha_{\rm N}}
$$

can always be written under the progressively factorized form:

$$
F\big(\mathsf{x}_1,\mathsf{x}_2,\dots,\mathsf{x}_{\rm N}\big) = F_0 + \mathsf{x}_1\,F_1\big(\mathsf{x}_1\big) + \mathsf{x}_2\,F\big(\mathsf{x}_1,\mathsf{x}_2\big) + \cdots + \mathsf{x}_{\rm N}\,F_{\rm N}\big(\mathsf{x}_1,\mathsf{x}_2,\dots,\mathsf{x}_{\rm N}\big).
$$

Hence:

$$v_1 = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_0\big(z,\overline{z},u_1\big) + z\overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big),$$
$$v_2 = \qquad z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_0\big(z,\overline{z},u_1\big) + z\overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big),$$
$$v_3 = \qquad z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_0\big(z,\overline{z},u_1\big) + z\overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big).$$

In the right-hand sides, let us emphasize three monomials in each line:

$$\begin{aligned}
&d_1\,z\overline{z}\,u_1 + e_1\,z\overline{z}\,u_2 + f_1\,z\overline{z}\,u_3,\\
&d_2\,z\overline{z}\,u_1 + e_2\,z\overline{z}\,u_2 + f_2\,z\overline{z}\,u_3,\\
&d_3\,z\overline{z}\,u_1 + e_3\,z\overline{z}\,u_2 + f_3\,z\overline{z}\,u_3,
\end{aligned}$$

so that:

$$\begin{aligned}
v_1 = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) &+ d_1\,z\overline{z}\,u_1 + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) +\\
&+ e_1\,z\overline{z}\,u_2 + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) +\\
&+ f_1\,z\overline{z}\,u_3 + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big),\\
v_2 = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) &+ d_2\,z\overline{z}\,u_1 + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) +\\
&+ e_2\,z\overline{z}\,u_2 + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) +\\
&+ f_2\,z\overline{z}\,u_3 + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big),\\
v_3 = z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z,\overline{z}\big) &+ d_3\,z\overline{z}\,u_1 + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) +\\
&+ e_3\,z\overline{z}\,u_2 + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) +\\
&+ f_3\,z\overline{z}\,u_3 + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big).
\end{aligned}$$

Assertion. *An appropriate choice of real constants in the biholomorphic change of coordinates:*

$$\begin{aligned}
z &= z',\\
w_1 &= w_1' + l_1'\,w_1'w_1' + m_1'\,w_1'w_2' + n_1'\,w_1'w_3',\\
w_2 &= w_2' + l_2'\,w_1'w_1' + m_2'\,w_1'w_2' + n_2'\,w_1'w_3',\\
w_3 &= w_3' + l_3'\,w_1'w_1' + m_3'\,w_1'w_2' + n_3'\,w_1'w_3',
\end{aligned}$$

annihilates the 9 new constants:

$$\begin{aligned}
0 &= d_1' = e_1' = f_1',\\
0 &= d_2' = e_2' = f_2',\\
0 &= d_3' = e_3' = f_3'.
\end{aligned}$$

Proof. Indeed, the real and imaginary parts are:

$$\begin{aligned}
u_1 &= u_1' + l_1'\big(u_1'u_1' - v_1'v_1'\big) + m_1'\big(u_1'u_2' - v_1'v_2'\big) + n_1'\big(u_1'u_3' - v_1'v_3'\big),\\
v_1 &= v_1' + l_1'\big(u_1'v_1' + u_1'v_1'\big) + m_1'\big(u_1'v_2' + u_2'v_1'\big) + n_1'\big(u_1'v_3' + u_3'v_1'\big),
\end{aligned}$$

$$
\begin{aligned}
u_2 &= u_2' + l_2'\big(u_1'u_1' - v_1'v_1'\big) + m_2'\big(u_1'u_2' - v_1'v_2'\big) + n_2'\big(u_1'u_3' - v_1'v_3'\big),\\
v_2 &= v_2' + l_2'\big(u_1'v_1' + u_1'v_1'\big) + m_2'\big(u_1'v_2' + u_2'v_1'\big) + n_2'\big(u_1'v_3' + u_3'v_1'\big),\\
u_3 &= u_3' + l_3'\big(u_1'u_1' - v_1'v_1'\big) + m_3'\big(u_1'u_2' - v_1'v_2'\big) + n_3'\big(u_1'u_3' - v_1'v_3'\big),\\
v_3 &= v_3' + l_3'\big(u_1'v_1' + u_1'v_1'\big) + m_3'\big(u_1'v_2' + u_2'v_1'\big) + n_3'\big(u_1'v_3' + u_3'v_1'\big).
\end{aligned}
$$

Disgregarding terms of order 4 and higher, a replacement in the three equations of M gives:

$$
\begin{aligned}
&v_1' + l_1'\big(u_1'v_1' + u_1'v_1'\big) + m_1'\big(u_1'v_2' + u_2'v_1'\big) + n_1'\big(u_1'v_3' + u_3'v_1'\big) \equiv\\
&\qquad \equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_1\big(z',\overline{z}'\big) + d_1\,z'\overline{z}'\,u_1' + e_1\,z'\overline{z}'\,u_2' + f_1\,z'\overline{z}'\,u_3' \ \ \mathsf{mod}\,\mathrm{O}(4),\\
&v_2' + l_2'\big(u_1'v_1' + u_1'v_1'\big) + m_2'\big(u_1'v_2' + u_2'v_1'\big) + n_2'\big(u_1'v_3' + u_3'v_1'\big) \equiv\\
&\qquad \equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_1\big(z',\overline{z}'\big) + d_2\,z'\overline{z}'\,u_1' + e_2\,z'\overline{z}'\,u_2' + f_2\,z'\overline{z}'\,u_3' \ \ \mathsf{mod}\,\mathrm{O}(4),\\
&v_3' + l_3'\big(u_1'v_1' + u_1'v_1'\big) + m_3'\big(u_1'v_2' + u_2'v_1'\big) + n_3'\big(u_1'v_3' + u_3'v_1'\big) \equiv\\
&\qquad \equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_1\big(z',\overline{z}'\big) + d_3\,z'\overline{z}'\,u_1' + e_3\,z'\overline{z}'\,u_2' + f_3\,z'\overline{z}'\,u_3' \ \ \mathsf{mod}\,\mathrm{O}(4).
\end{aligned}
$$

The equations:

$$
\begin{aligned}
v_1' &= \varphi_1'\big(z',\overline{z}',u_1',u_2',u_3'\big),\\
v_2' &= \varphi_2'\big(z',\overline{z}',u_1',u_2',u_3'\big),\\
v_3' &= \varphi_3'\big(z',\overline{z}',u_1',u_2',u_3'\big),
\end{aligned}
$$

of the transformed CR-generic submanifold:

$$
M'^5 \subset \mathbb{C}'^4
$$

are obtained, up to order 4, by solving the above three equations with respect to v_1', v_2', v_3', and one gets, modulo $\mathrm{O}(4)$:

$$
\begin{aligned}
v_1' &\equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_1\big(z',\overline{z}'\big) + \underbrace{\big(d_1 - 2l_1'\big)}_{=:\,d_1'}\,z'\overline{z}'\,u_1' + \underbrace{\big(e_1 - m_1'\big)}_{=:\,e_1'}\,z'\overline{z}'\,u_2' + \underbrace{\big(f_1 - n_1'\big)}_{=:\,f_1'}\,z'\overline{z}'\,u_3',\\
v_2' &\equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_1\big(z',\overline{z}'\big) + \underbrace{\big(d_2 - 2l_2'\big)}_{=:\,d_2'}\,z'\overline{z}'\,u_1' + \underbrace{\big(e_2 - m_2'\big)}_{=:\,e_2'}\,z'\overline{z}'\,u_2' + \underbrace{\big(f_2 - n_2'\big)}_{=:\,f_2'}\,z'\overline{z}'\,u_3',\\
v_1' &\equiv z'\overline{z}' + z'\overline{z}'\,\mathrm{O}_3\big(z',\overline{z}'\big) + \underbrace{\big(d_3 - 2l_3'\big)}_{=:\,d_3'}\,z'\overline{z}'\,u_1' + \underbrace{\big(e_3 - m_3'\big)}_{=:\,e_3'}\,z'\overline{z}'\,u_2' + \underbrace{\big(f_3 - n_3'\big)}_{=:\,f_3'}\,z'\overline{z}'\,u_3'.
\end{aligned}
$$

Setting:

$$
\begin{aligned}
2\,l_1' &:= d_1, & m_1' &:= e_1, & n_1' &:= f_1,\\
2\,l_2' &:= d_2, & m_2' &:= e_2, & n_2' &:= f_2,\\
2\,l_3' &:= d_3, & m_3' &:= e_3, & n_3' &:= f_3,
\end{aligned}
$$

one concludes. □

Emphasizing order 3 terms in $(z,\overline{z})$, plus order 4 terms in the first line, one therefore arrives at:

$$
\begin{aligned}
v_1 &= z\overline{z} + \alpha_1\, z^2\overline{z} + \overline{\alpha}_1\, z\overline{z}^2 + \beta_1\, z^3\overline{z} + \overline{\beta}_1\, z\overline{z}^3 + c_1\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big) \\
v_2 &= \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big) \\
v_3 &= \alpha_3\, z^2\overline{z} + \overline{\alpha}_3\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big).
\end{aligned}
$$

In the first line, replacing:

$$
z \longmapsto z + \alpha_1\, z^2 + \beta_1\, z^3,
$$

one makes:

$$
\alpha_1 = 0 = \beta_1,
$$

without modifying the general form of the remainders. So:

$$
\begin{aligned}
v_1 &= z\overline{z} + c_1\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big) \\
v_2 &= \alpha_2\, z^2\overline{z} + \overline{\alpha}_2\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big) \\
v_3 &= \alpha_3\, z^2\overline{z} + \overline{\alpha}_3\, z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\
&\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big).
\end{aligned}
$$

It is now time to recall ([3]) that the coefficients of the natural local generator for $T^{1,0}M$:

$$
\mathscr{L} = \frac{\partial}{\partial z} + A_1\,\frac{\partial}{\partial u_1} + A_2\,\frac{\partial}{\partial u_2} + A_3\,\frac{\partial}{\partial u_3}
$$

are given by the formulas:

$$
A_1 = \frac{1}{\Delta}
\begin{vmatrix}
-\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3} \\
-\varphi_{2,z} & \sqrt{-1}+\varphi_{2,u_2} & \varphi_{2,u_3} \\
-\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1}+\varphi_{3,u_3}
\end{vmatrix},
$$

$$
A_2 = \frac{1}{\Delta}
\begin{vmatrix}
\sqrt{-1}+\varphi_{1,u_1} & -\varphi_{1,z} & \varphi_{1,u_3} \\
\varphi_{2,u_1} & -\varphi_{2,z} & \varphi_{2,u_3} \\
\varphi_{3,u_1} & -\varphi_{3,z} & \sqrt{-1}+\varphi_{3,u_3}
\end{vmatrix},
$$

$$
A_3 = \frac{1}{\Delta}
\begin{vmatrix}
\sqrt{-1}+\varphi_{1,u_1} & \varphi_{1,u_2} & -\varphi_{1,z} \\
\varphi_{2,u_1} & \sqrt{-1}+\varphi_{2,u_2} & -\varphi_{2,z} \\
\varphi_{3,u_1} & \varphi_{3,u_2} & -\varphi_{3,z}
\end{vmatrix},
$$

with common denominator:

$$\Delta := \left| \begin{matrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{matrix} \right|.$$

One notices:

$$\varphi_{j,u_l} = \mathrm{O}(3) \qquad\qquad {\scriptstyle (1 \leqslant j, l \leqslant 3)},$$

whence:

$$\begin{aligned} \Delta &= \left| \begin{matrix} \sqrt{-1} + \mathrm{O}(3) & \mathrm{O}(3) & \mathrm{O}(3) \\ \mathrm{O}(3) & \sqrt{-1} + \mathrm{O}(3) & \mathrm{O}(3) \\ \mathrm{O}(3) & \mathrm{O}(3) & \sqrt{-1} + \mathrm{O}(3) \end{matrix} \right| \\ &= \sqrt{-1}^3 + \mathrm{O}(3). \end{aligned}$$

Concerning numerators, that of A_1 is:

$$A_1^{\mathsf{num}} = \left| \begin{matrix} -\overline{z} + \mathrm{O}(3) & \mathrm{O}(3) & \mathrm{O}(3) \\ -2\alpha_2 z\overline{z} - \overline{\alpha}_2\overline{z}^2 + \mathrm{O}(3) & \sqrt{-1} + \mathrm{O}(3) & \mathrm{O}(3) \\ -2\alpha_3 z\overline{z} - \overline{\alpha}_3\overline{z}^2 + \mathrm{O}(3) & \mathrm{O}(3) & \sqrt{-1} + \mathrm{O}(3) \end{matrix} \right|,$$

so that:

$$\begin{aligned} A_1 &= \frac{-\sqrt{-1}^2\,\overline{z} + \mathrm{O}(3)}{\sqrt{-1}^3 + \mathrm{O}(3)} \\ &= \sqrt{-1}\,\overline{z} + \mathrm{O}(3). \end{aligned}$$

Next:

$$A_2^{\mathsf{num}} = \left| \begin{matrix} \sqrt{-1} + \mathrm{O}(3) & -\overline{z} + \mathrm{O}(3) & \mathrm{O}(3) \\ \mathrm{O}(3) & -2\alpha_2 z\overline{z} - \overline{\alpha}_2\overline{z}^2 + \mathrm{O}(3) & \mathrm{O}(3) \\ \mathrm{O}(3) & -2\alpha_3 z\overline{z} - \overline{\alpha}_3\overline{z}^2 + \mathrm{O}(3) & \sqrt{-1} + \mathrm{O}(3) \end{matrix} \right|,$$

so that:

$$\begin{aligned} A_1 &= \frac{-\sqrt{-1}^2\,2\alpha_2\,z\overline{z} - \sqrt{-1}^2\,\overline{\alpha}_2\,\overline{z}^2 + \mathrm{O}(3)}{\sqrt{-1}^3 + \mathrm{O}(3)} \\ &= \sqrt{-1}\,2\alpha_2\,z\overline{z} + \sqrt{-1}\,\overline{\alpha}_2\,\overline{z}^2 + \mathrm{O}(3). \end{aligned}$$

Quite similarly (mental exercise):

$$A_3 = \sqrt{-1}\,2\alpha_3\,z\overline{z} + \sqrt{-1}\,\overline{\alpha}_3\,\overline{z}^2 + \mathrm{O}(3).$$

Thus:

$$\begin{aligned} \mathscr{L} = \frac{\partial}{\partial z} + \sqrt{-1}\Big(\overline{z} + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_1} + \sqrt{-1}\Big(2\alpha_2\,z\overline{z} + \overline{\alpha}_2\,\overline{z}^2 + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_2} + \\ + \sqrt{-1}\Big(2\alpha_3\,z\overline{z} + \overline{\alpha}_3\,\overline{z}^2 + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_3}, \end{aligned}$$

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} - \sqrt{-1}\Big(z + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_1} - \sqrt{-1}\Big(2\overline{\alpha}_2\, z\overline{z} + \alpha_2\, z^2 + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_2} + \\ + \sqrt{-1}\Big(2\overline{\alpha}_3\, z\overline{z} + \alpha_3\, z^2 + \mathrm{O}(3)\Big)\frac{\partial}{\partial u_3},$$

whence:

$$\big[\mathscr{L}, \overline{\mathscr{L}}\big] = -\sqrt{-1}\Big(2 + \mathrm{O}(2)\Big)\frac{\partial}{\partial u_1} - \sqrt{-1}\Big(4\alpha_2\, z + 4\overline{\alpha}_2\,\overline{z} + \mathrm{O}(2)\Big)\frac{\partial}{\partial u_2} - \\ - \sqrt{-1}\Big(4\alpha_3\, z + 4\overline{\alpha}_3\,\overline{z} + \mathrm{O}(2)\Big)\frac{\partial}{\partial u_3},$$

and lastly:

$$\big[\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]\big] = \mathrm{O}(1)\frac{\partial}{\partial u_1} - \sqrt{-1}\Big(4\alpha_2 + \mathrm{O}(1)\Big)\frac{\partial}{\partial u_2} - \sqrt{-1}\Big(4\alpha_3 + \mathrm{O}(1)\Big)\frac{\partial}{\partial u_3},$$

$$\big[\overline{\mathscr{L}},\, [\mathscr{L}, \overline{\mathscr{L}}]\big] = \mathrm{O}(1)\frac{\partial}{\partial u_1} - \sqrt{-1}\Big(4\overline{\alpha}_2 + \mathrm{O}(1)\Big)\frac{\partial}{\partial u_2} - \sqrt{-1}\Big(4\overline{\alpha}_3 + \mathrm{O}(1)\Big)\frac{\partial}{\partial u_3}.$$

At the origin, visibly:

$$\begin{aligned}
\mathscr{L}\big|_0 &= \frac{\partial}{\partial z}\Big|_0, \\
\overline{\mathscr{L}}\big|_0 &= \frac{\partial}{\partial \overline{z}}\Big|_0, \\
\big[\mathscr{L}, \overline{\mathscr{L}}\big]\Big|_0 &= -2\sqrt{-1}\,\frac{\partial}{\partial u_1}\Big|_0, \\
\big[\mathscr{L},\, [\mathscr{L}, \overline{\mathscr{L}}]\big]\Big|_0 &= -4\sqrt{-1}\,\alpha_2\,\frac{\partial}{\partial u_2}\Big|_0 - 4\sqrt{-1}\,\alpha_3\,\frac{\partial}{\partial u_3}\Big|_0, \\
\big[\overline{\mathscr{L}},\, [\mathscr{L}, \overline{\mathscr{L}}]\big]\Big|_0 &= -4\sqrt{-1}\,\overline{\alpha}_2\,\frac{\partial}{\partial u_2}\Big|_0 - 4\sqrt{-1}\,\overline{\alpha}_3\,\frac{\partial}{\partial u_3}\Big|_0,
\end{aligned}$$

By hypothesis, these five vectors should *not* constitute a basis for:

$$\mathbb{C}\otimes_{\mathbb{R}} T_0M \;=\; \mathbb{C}\frac{\partial}{\partial z}\Big|_0 \oplus \mathbb{C}\frac{\partial}{\partial \overline{z}}\Big|_0 \oplus \mathbb{C}\frac{\partial}{\partial u_1}\Big|_0 \oplus \mathbb{C}\frac{\partial}{\partial u_2}\Big|_0 \oplus \mathbb{C}\frac{\partial}{\partial u_3}\Big|_0,$$

of dimension **5**, while the first four should be independent, hence:

$$\mathbf{4} = \mathsf{rank}_{\mathbb{C}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & -4\sqrt{-1}\alpha_2 & -4\sqrt{-1}\alpha_3 \\ 0 & 0 & 0 & -4\overline{\sqrt{-1}\alpha_2} & -4\overline{\sqrt{-1}\alpha_3} \end{pmatrix}.$$

Without loss of generality:

$$\alpha_2 \neq 0,$$

and doing:

$$z \longmapsto \lambda\, z,$$

with:

$$1 = \lambda^2 \overline{\lambda}\, \alpha_2,$$

one makes:

$$\alpha_2 = 1.$$

Thus:

$$0 = \begin{vmatrix} 1 & \alpha_3 \\ 1 & \overline{\alpha}_3 \end{vmatrix},$$

that is to say:

$$\alpha_3 =: a_3 \,\in\, R.$$

Lastly, doing:

$$w_3 \longmapsto w_3 - a_3\, w_2,$$

one makes:

$$\alpha_3 = 0.$$

Emphasize now fourth order terms in all lines:

$$\begin{aligned} v_1 &= z\overline{z} + c_1\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 + \beta_2\, z^3\overline{z} + \overline{\beta}_2\, z\overline{z}^3 + c_2\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_3 &= \beta_3\, z^3\overline{z} + \overline{\beta}_3\, z\overline{z}^3 + c_3\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big). \end{aligned}$$

Again:

$$\Delta = \sqrt{-1}^3 + \mathrm{O}(3).$$

Next:

$$A_1^{\mathsf{num}} = \begin{vmatrix} \begin{smallmatrix} -\overline{z} - 2c_1 z\overline{z}^2 + \mathrm{O}_4(z,\overline{z}) + \overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1) + \\ + z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2) + z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2) + \\ + z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} \\ \begin{smallmatrix} -2z\overline{z} - \overline{z}^2 - 3\beta_2 z^2\overline{z} - \overline{\beta}_2\overline{z}^3 - 2c_2 z\overline{z}^2 + \\ + \mathrm{O}_4(z,\overline{z}) + \overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1) + \\ + z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2) + z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} \sqrt{-1} + z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2) + \\ + z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} \\ \begin{smallmatrix} -3\beta_3 z^2\overline{z} - \overline{\beta}_3\overline{z}^3 - 2c_3 z\overline{z}^2 + \\ + \mathrm{O}_4(z,\overline{z}) + \overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1) + \\ + z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2) + z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2) + \\ + z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} & \begin{smallmatrix} \sqrt{-1} + z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3) \end{smallmatrix} \end{vmatrix}$$

$$= -\sqrt{-1}^2 \Big(\overline{z} + 2c_2 z\overline{z}^2 + \mathrm{O}_4\big(z,\overline{z}\big) + \overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + \overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big) \Big),$$

so that:

$$\begin{aligned} A_1 &= \frac{-\sqrt{-1}^2 \Big(\overline{z} + 2c_2 z\overline{z}^2 + \mathrm{O}_4\big(z,\overline{z}\big) + \overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + \overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big) \Big)}{\sqrt{-1}^3 + \mathrm{O}(3)} \\ &= \sqrt{-1} \Big(\overline{z} + 2c_2 z\overline{z}^2 + \mathrm{O}_4\big(z,\overline{z}\big) + \overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \overline{z}\,u_2\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2\big) + \overline{z}\,u_3\,\mathrm{O}_0\big(z,\overline{z},u_1,u_2,u_3\big) \Big). \end{aligned}$$

Secondly:

$$
\begin{aligned}
A_2^{\mathsf{num}} &= \left|\begin{matrix}
\begin{smallmatrix}\sqrt{-1}+z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2)+\\ +z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}-\overline{z}-2c_1z\overline{z}^2+\mathrm{O}_4(z,\overline{z})+\overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1)+\\ +z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} \\
\begin{smallmatrix}z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2)+\\ +z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}-2z\overline{z}-\overline{z}^2-3\beta_2z^2\overline{z}-\overline{\beta}_2\overline{z}^3-2c_2z\overline{z}^2+\\ +\mathrm{O}_4(z,\overline{z})+\overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1)+\\ +z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} \\
\begin{smallmatrix}z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2)+\\ +z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}-3\beta_3z^2\overline{z}-\overline{\beta}_3\overline{z}^3-2c_3z\overline{z}^2+\\ +\mathrm{O}_4(z,\overline{z})+\overline{z}u_1\mathrm{O}_1(z,\overline{z},u_1)+\\ +z\overline{z}u_2\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}u_3\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix} &
\begin{smallmatrix}\sqrt{-1}+z\overline{z}\mathrm{O}_1(z,\overline{z},u_1,u_2,u_3)\end{smallmatrix}
\end{matrix}\right| \\
&= -\sqrt{-1}^2\Big(2\,z\overline{z}+\overline{z}^2+3\beta_2\,z^2\overline{z}+\overline{\beta}_2\,\overline{z}^3+2c_2\,z\overline{z}^2+ \\
&\qquad\qquad + \mathrm{O}_4(z,\overline{z})+\overline{z}\,u_1\,\mathrm{O}_1(z,\overline{z},u_1)+z\overline{z}\,u_2\,\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}\,u_3\,\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\Big),
\end{aligned}
$$

so that:

$$
\begin{aligned}
A_2 &= \frac{A_2^{\mathsf{num}}}{\sqrt{-1}^3+\mathrm{O}(3)} \\
&= \sqrt{-1}\Big(2\,z\overline{z}+\overline{z}^2+3\beta_2\,z^2\overline{z}+\overline{\beta}_2\,\overline{z}^3+2c_2\,z\overline{z}^2+ \\
&\qquad\qquad + \mathrm{O}_4(z,\overline{z})+\overline{z}\,u_1\,\mathrm{O}_1(z,\overline{z},u_1)+z\overline{z}\,u_2\,\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}\,u_3\,\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\Big).
\end{aligned}
$$

Quite similarly:

$$
\begin{aligned}
A_3 &= \sqrt{-1}\Big(3\beta_3\,z^2\overline{z}+\overline{\beta}_3\,\overline{z}^3+2c_3\,z\overline{z}^2+ \\
&\qquad\qquad + \mathrm{O}_4(z,\overline{z})+\overline{z}\,u_1\,\mathrm{O}_1(z,\overline{z},u_1)+z\overline{z}\,u_2\,\mathrm{O}_0(z,\overline{z},u_1,u_2)+z\overline{z}\,u_3\,\mathrm{O}_0(z,\overline{z},u_1,u_2,u_3)\Big).
\end{aligned}
$$

To compactify remainders, attribute weights:

$$
\begin{aligned}
\mathsf{weight}(z) &:= 1, \\
\mathsf{weight}(w_1) &:= 2, \\
\mathsf{weight}(w_2) &:= 3, \\
\mathsf{weight}(w_3) &:\geqslant 4.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathscr{L} &= \frac{\partial}{\partial z}+\sqrt{-1}\left(\overline{z}+2c_1\,z\overline{z}^2+\mathrm{O}_{\mathsf{weighted}}(4)\right)\frac{\partial}{\partial u_1}+ \\
&\quad+\sqrt{-1}\Big(2\,z\overline{z}+\overline{z}^2+3\beta_2\,z^2\overline{z}+\overline{\beta}_2\,\overline{z}^3+2c_2\,z\overline{z}^2+\mathrm{O}_{\mathsf{weighted}}(4)\Big)\frac{\partial}{\partial u_2}+ \\
&\quad+\sqrt{-1}\Big(3\beta_3\,z^2\overline{z}+\overline{\beta}_3\,\overline{z}^3+2c_3\,z\overline{z}^2+\mathrm{O}_{\mathsf{weighted}}(4)\Big)\frac{\partial}{\partial u_3}, \\
\overline{\mathscr{L}} &= \frac{\partial}{\partial \overline{z}}-\sqrt{-1}\left(z+2c_1\,z^2\overline{z}+\mathrm{O}_{\mathsf{weighted}}(4)\right)\frac{\partial}{\partial u_1}- \\
&\quad-\sqrt{-1}\Big(2\,z\overline{z}+z^2+3\overline{\beta}_2\,z\overline{z}^2+\beta_2\,z^3+2c_2\,z^2\overline{z}+\mathrm{O}_{\mathsf{weighted}}(4)\Big)\frac{\partial}{\partial u_2}- \\
&\quad-\sqrt{-1}\Big(3\overline{\beta}_3\,z\overline{z}^2+\beta_3\,z^3+2c_3\,z^2\overline{z}+\mathrm{O}_{\mathsf{weighted}}(4)\Big)\frac{\partial}{\partial u_3}.
\end{aligned}
$$

When computing brackets up to length three, remainders of weighted order $\geqslant 4$ do not contribute (exercise):

$$
\begin{aligned}
\big[\mathscr{L},\overline{\mathscr{L}}\big] = & -2\sqrt{-1}\Big(1+4c_1\,z\overline{z}+\mathrm{O}_{\mathsf{weighted}}(3)\Big)\frac{\partial}{\partial u_1}+\\
& -\,2\sqrt{-1}\Big(2\,z+2\,\overline{z}+3\beta_2\,z^2+3\overline{\beta}_2\,\overline{z}^2+4c_2\,z\overline{z}+\mathrm{O}_{\mathsf{weighted}}(3)\Big)\frac{\partial}{\partial u_2}+\\
& -\,2\sqrt{-1}\Big(3\beta_3\,z^2+3\overline{\beta}_3\,\overline{z}^2+4c_3\,z\overline{z}+\mathrm{O}_{\mathsf{weighted}}(3)\Big)\frac{\partial}{\partial u_3},\\
\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big] = & -2\sqrt{-1}\Big(4c_1\,\overline{z}+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_1}+\\
& -\,2\sqrt{-1}\Big(2+6\beta_2\,z+4c_2\,\overline{z}+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_2}+\\
& -\,2\sqrt{-1}\Big(6\beta_3\,z+4c_3\,\overline{z}+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_3},\\
\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big] = & -2\sqrt{-1}\Big(4c_1 z+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_1}+\\
& -\,2\sqrt{-1}\Big(2+6\overline{\beta}_2\,\overline{z}+4c_2\,z+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_2}+\\
& -\,2\sqrt{-1}\Big(6\overline{\beta}_3\,\overline{z}+4c_3\,z+\mathrm{O}_{\mathsf{weighted}}(2)\Big)\frac{\partial}{\partial u_3}.
\end{aligned}
$$

The hypothesis:

$$
\mathbf{4}\;=\;\mathsf{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L},\,\overline{\mathscr{L}},\;\big[\mathscr{L},\overline{\mathscr{L}}\big],\;\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big],\;\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big\}\Big)
$$

reads:

$$
0\equiv\left|\begin{array}{ccc}
1+4c_1z\overline{z}+\mathrm{O}_{\mathsf{weighted}}(3) & 4c_1\overline{z}+\mathrm{O}_{\mathsf{weighted}}(2) & 4c_1z+\mathrm{O}_{\mathsf{weighted}}(2)\\[2ex]
\begin{smallmatrix}2z+2\overline{z}+3\beta_2z^2+\overline{\beta}_2\overline{z}^2+\\ +4c_2z\overline{z}+\mathrm{O}_{\mathsf{weighted}}(3)\end{smallmatrix} & \begin{smallmatrix}2+6\beta_2z+4c_2\overline{z}+\\ +\mathrm{O}_{\mathsf{weighted}}(2)\end{smallmatrix} & \begin{smallmatrix}2+6\overline{\beta}_2\overline{z}+4c_2z+\\ +\mathrm{O}_{\mathsf{weighted}}(2)\end{smallmatrix}\\[2ex]
\begin{smallmatrix}3\beta_3z^2+3\overline{\beta}_3\overline{z}^2+4c_3z\overline{z}+\\ +\mathrm{O}_{\mathsf{weighted}}(3)\end{smallmatrix} & \begin{smallmatrix}6\beta_3z+4c_3\overline{z}+\\ +\mathrm{O}_{\mathsf{weighted}}(2)\end{smallmatrix} & \begin{smallmatrix}6\overline{\beta}_3\overline{z}+4c_3z+\\ +\mathrm{O}_{\mathsf{weighted}}(2)\end{smallmatrix}
\end{array}\right|,
$$

and picking only order 1 terms:

$$
0\equiv 12\,\overline{\beta}_3\,\overline{z}-12\,\beta_3\,z+8\,c_3\,z-8\,c_3\,\overline{z},
$$

which is:

$$
3\,\beta_3=2\,c_3,
$$

so that:

$$
\boxed{\beta_3\text{ is real.}}
$$

Further, the hypothesis:

$$
\mathbf{5}\;=\;\mathsf{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L},\,\overline{\mathscr{L}},\;\big[\mathscr{L},\overline{\mathscr{L}}\big],\;\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big],\;\big[\mathscr{L},\,\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\big]\Big\}\Big),
$$

requires (exercise):

$$c_3 \neq 0.$$

Naturally, a real dilation of z makes:

$$\beta_3 := 2, \qquad\qquad c_3 := 3,$$

so that order 4 terms in the third line are the ones announced:

$$2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2,$$

while a simultaneous correcting dilation along the w_1-axis and along the w_2-axis keeps unchanged the preceding normalizations.

Furthermore, an inspection of the last two remainders in the bottom line of the above 3×3 determinant shows (exercise) that they both are of the form:

$$\overline{z}\,\mathrm{O}_{\mathsf{weighted}}(1),$$

so that when one picks only:

$$\mathsf{coefficient}\big(z^2\big),$$

one obtains:

$$\begin{aligned} 0 &= \mathsf{coefficient}\big(z^2\big) \\ &= 6 \cdot 12\,\beta_2 - 12 \cdot 4\,c_2, \end{aligned}$$

which similarly implies that:

$$\boxed{\beta_2 \text{ is real,}}$$

and furthermore, the quartic terms in the second line:

$$\tfrac{2\,c_2}{3}\,z^3\overline{z} + \tfrac{2\,c_2}{3}\,z\overline{z}^3 + c_2\,z^2\overline{z}^2$$

are real multiples of those just normalized on the third line:

$$2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2.$$

The final transformation:

$$w_2 \longmapsto w_2 - \tfrac{c_2}{3}\,w_3$$

makes:

$$\beta_2 = c_2 = 0,$$

which concludes. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalization at the origin:*

$$
\begin{aligned}
\mathscr{L}\big|_0 &= \frac{\partial}{\partial z}\bigg|_0, \\
\overline{\mathscr{L}}\big|_0 &= \frac{\partial}{\partial \overline{z}}\bigg|_0, \\
\big[\mathscr{L},\overline{\mathscr{L}}\big]\Big|_0 &= -\,2\,\sqrt{-1}\,\frac{\partial}{\partial u_1}\bigg|_0, \\
\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big|_0 &= -\,4\,\sqrt{-1}\,\frac{\partial}{\partial u_2}\bigg|_0, \\
\big[\mathscr{L},\,\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\big]\Big|_0 &= -\,24\,\sqrt{-1}\,\frac{\partial}{\partial u_3}\bigg|_0,
\end{aligned}
$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure. □

Taking zero remainders, one obtains:

$$
\boxed{\text{Model (III)}_2: \qquad
\begin{aligned}
v_1 &= z\overline{z}, \\
v_2 &= z^2\overline{z} + z\overline{z}^2, \\
v_3 &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2.
\end{aligned}}
$$

To conclude, one must express the constraint:

$$
\mathbf{4} \;=\; \text{rank}_{\mathbb{C}}\Big(\Big\{\mathscr{L},\,\overline{\mathscr{L}},\;\big[\mathscr{L},\overline{\mathscr{L}}\big],\;\big[\mathscr{L},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big],\;\big[\overline{\mathscr{L}},\,\big[\mathscr{L},\overline{\mathscr{L}}\big]\big]\Big\}\Big).
$$

Starting from:

$$
\begin{aligned}
\mathscr{L} &= \frac{\partial}{\partial z} + A_1\,\frac{\partial}{\partial u_1} + A_2\,\frac{\partial}{\partial u_2} + A_3\,\frac{\partial}{\partial u_3}, \\
\overline{\mathscr{L}} &= \frac{\partial}{\partial \overline{z}} + \overline{A}_1\,\frac{\partial}{\partial u_1} + \overline{A}_2\,\frac{\partial}{\partial u_2} + \overline{A}_3\,\frac{\partial}{\partial u_3},
\end{aligned}
$$

firstly, secondly, one has thirdly:

$$
\big[\mathscr{L},\overline{\mathscr{L}}\big] = \Big(\mathscr{L}\big(\overline{A}_1\big) - \overline{\mathscr{L}}\big(A_1\big)\Big)\,\frac{\partial}{\partial u_1} + \Big(\mathscr{L}\big(\overline{A}_2\big) - \overline{\mathscr{L}}\big(A_2\big)\Big)\,\frac{\partial}{\partial u_2} + \Big(\mathscr{L}\big(\overline{A}_3\big) - \overline{\mathscr{L}}\big(A_3\big)\Big)\,\frac{\partial}{\partial u_3},
$$

fourthly:

$$\begin{aligned}\big[\mathscr{L},\,[\mathscr{L},\overline{\mathscr{L}}]\big] = \Big(\mathscr{L}\big(\mathscr{L}(\overline{A}_1)\big) - 2\,\mathscr{L}\big(\overline{\mathscr{L}}(A_1)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A_1)\big)\Big)\,\frac{\partial}{\partial u_1} + \\ + \Big(\mathscr{L}\big(\mathscr{L}(\overline{A}_2)\big) - 2\,\mathscr{L}\big(\overline{\mathscr{L}}(A_2)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A_2)\big)\Big)\,\frac{\partial}{\partial u_2} + \\ + \Big(\mathscr{L}\big(\mathscr{L}(\overline{A}_3)\big) - 2\,\mathscr{L}\big(\overline{\mathscr{L}}(A_3)\big) + \overline{\mathscr{L}}\big(\mathscr{L}(A_3)\big)\Big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

and fifthly:

$$\begin{aligned}\big[\overline{\mathscr{L}},\,[\mathscr{L},\overline{\mathscr{L}}]\big] = \Big(-\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(A_1)\big) + 2\,\overline{\mathscr{L}}\big(\mathscr{L}(\overline{A}_1)\big) - \mathscr{L}\big(\overline{\mathscr{L}}(\overline{A}_1)\big)\Big)\,\frac{\partial}{\partial u_1} + \\ + \Big(-\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(A_2)\big) + 2\,\overline{\mathscr{L}}\big(\mathscr{L}(\overline{A}_2)\big) - \mathscr{L}\big(\overline{\mathscr{L}}(\overline{A}_2)\big)\Big)\,\frac{\partial}{\partial u_2} + \\ + \Big(-\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(A_3)\big) + 2\,\overline{\mathscr{L}}\big(\mathscr{L}(\overline{A}_3)\big) - \mathscr{L}\big(\overline{\mathscr{L}}(\overline{A}_3)\big)\Big)\,\frac{\partial}{\partial u_3},\end{aligned}$$

So the hypothesis means the identical vanishing of the 3×3 determinant:

$$0 \equiv \left|\begin{array}{ccc} \mathscr{L}(\overline{A}_1)-\overline{\mathscr{L}}(A_1) & \mathscr{L}(\overline{A}_2)-\overline{\mathscr{L}}(A_2) & \mathscr{L}(\overline{A}_3)-\overline{\mathscr{L}}(A_3) \\ \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_1))-2\mathscr{L}(\overline{\mathscr{L}}(A_1))+\\+\overline{\mathscr{L}}(\mathscr{L}(A_1))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_2))-2\mathscr{L}(\overline{\mathscr{L}}(A_2))+\\+\overline{\mathscr{L}}(\mathscr{L}(A_2))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_3))-2\mathscr{L}(\overline{\mathscr{L}}(A_3))+\\+\overline{\mathscr{L}}(\mathscr{L}(A_3))\end{array} \\ \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_1))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_1))-\\-\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_1))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_2))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_2))-\\-\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_2))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_3))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_3))-\\-\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_3))\end{array}\end{array}\right|.$$

When expressed back in terms of the three graphing functions:

$$\begin{aligned}&\varphi_1\big(z,\overline{z},u_1,u_2,u_3\big),\\&\varphi_2\big(z,\overline{z},u_1,u_2,u_3\big),\\&\varphi_3\big(z,\overline{z},u_1,u_2,u_3\big),\end{aligned}$$

one obtains a rational expression in the third-order jet of the three graphing functions φ_1, φ_2, φ_3 whose numerator contains hundreds of lines.

7. General class IV_1

Proposition. *A local real analytic hypersurface passing through the origin:*

$$0 \in M^5 \subset \mathbb{C}^3$$

which belongs to the general Class IV_1*, namely such that, for any two local generators* $\{\mathscr{L}_1, \mathscr{L}_2\}$ *for* $T^{1,0}M$*:*

$$\Big\{\mathscr{L}_1,\,\mathscr{L}_2,\,\overline{\mathscr{L}}_1,\,\overline{\mathscr{L}}_2,\;\big[\mathscr{L}_1,\overline{\mathscr{L}}_1\big]\Big\}$$

constitute a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$*, and such that in addition, the Levi form:*

$$\text{Levi-Form}^M(p)$$

if of rank 2 *at every point, may always be represented, in suitable local holomorphic coordinates:*

$$(z_1, z_2, w) \in \mathbb{C}^3$$

by a specific real analytic equation of the form:

$$\boxed{\underline{\text{(IV)}_1}: \qquad v = z_1\overline{z}_1 \pm z_2\overline{z}_2 + \mathrm{O}_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big),}$$

with remainder satisfying:

$$0 \equiv \text{remainder}\big(0, 0, \overline{z}_1, \overline{z}_2, u\big) \equiv \text{remainder}\big(z_1, z_2, 0, 0, u\big).$$

Proof. One starts with:

$$\begin{aligned} v &= \varphi\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big) \\ &= \mathrm{O}_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big), \end{aligned}$$

assuming:

$$0 \equiv \varphi\big(0, 0, \overline{z}_1, \overline{z}_2, u\big) \equiv \varphi\big(z_1, z_2, 0, 0, u\big).$$

Hence:

$$\begin{aligned} v &= P_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + \mathrm{O}_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big) \\ &= a\, z_1\overline{z}_1 + \beta\, z_2\overline{z}_1 + \overline{\beta}\, z_1\overline{z}_2 + c\, z_2\overline{z}_2 + \mathrm{O}_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big), \end{aligned}$$

with:

$$a \in \mathbb{R}, \qquad\qquad \beta \in \mathbb{C}, \qquad\qquad c \in \mathbb{R}.$$

The first natural local generator of $T^{1,0}M$ is:

$$\begin{aligned} \mathscr{L}_1 &= \frac{\partial}{\partial z_1} - \frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u}\,\frac{\partial}{\partial u} \\ &= \frac{\partial}{\partial z_1} - \bigg(\frac{a\,\overline{z}_1 + \overline{\beta}\,\overline{z}_2 + \mathrm{O}(2)}{\sqrt{-1} + \mathrm{O}(2)}\bigg)\,\frac{\partial}{\partial u}, \\ &= \frac{\partial}{\partial z_1} + \sqrt{-1}\,\Big(a\,\overline{z}_1 + \overline{\beta}\,\overline{z}_2 + \mathrm{O}(2)\Big)\,\frac{\partial}{\partial u}, \end{aligned}$$

and similarly, the second is:

$$\mathscr{L}_2 = \frac{\partial}{\partial z_2} + \sqrt{-1}\,\Big(\beta\,\overline{z}_1 + c\,\overline{z}_2 + \mathrm{O}(2)\Big)\,\frac{\partial}{\partial u}.$$

Conjugates are:

$$\overline{\mathscr{L}}_1 = \frac{\partial}{\partial \overline{z}_1} - \sqrt{-1}\,\Big(a\,z_1 + \beta\,z_2 + \mathrm{O}(2)\Big)\,\frac{\partial}{\partial u},$$
$$\overline{\mathscr{L}}_2 = \frac{\partial}{\partial \overline{z}_2} - \sqrt{-1}\,\Big(\overline{\beta}\,z_1 + c\,z_2 + \mathrm{O}(2)\Big)\,\frac{\partial}{\partial u}.$$

Taking:

$$\rho_0 = du - A_1\,dz_1 - A_2\,dz_2 - \overline{A}_1\,d\overline{z}_1 - \overline{A}_2\,d\overline{z}_2,$$

the Levi matrix at the origin is:

$$\begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_2\big]\big) \end{pmatrix}(0) = \sqrt{-1}\,\begin{pmatrix} \mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_1\big) & \mathscr{L}_2\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_2\big) \\ \mathscr{L}_1\big(\overline{A}_2\big) - \overline{\mathscr{L}}_2\big(A_1\big) & \mathscr{L}_2\big(\overline{A}_2\big) - \overline{\mathscr{L}}_2\big(A_2\big) \end{pmatrix}(0)$$
$$= \begin{pmatrix} 2a & 2\beta \\ 2\overline{\beta} & 2c \end{pmatrix},$$

and its determinant should be nonzero.

A linear transformation:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} \lambda & \mu \\ \nu & \chi \end{pmatrix}}_{\in \mathsf{GL}_2(\mathbb{C})} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

normalizes this 2×2 Hermitian matrix to:

$$2\,\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

which concludes. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalizations at the origin of the fields:*

$$\mathscr{L}_1\big|_0 = \frac{\partial}{\partial z_1}\bigg|_0,$$
$$\mathscr{L}_2\big|_0 = \frac{\partial}{\partial z_2}\bigg|_0,$$
$$\overline{\mathscr{L}}_1\big|_0 = \frac{\partial}{\partial \overline{z}_1}\bigg|_0,$$
$$\overline{\mathscr{L}}_2\big|_0 = \frac{\partial}{\partial \overline{z}_2}\bigg|_0,$$
$$\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\Big|_0 = -\,2\sqrt{-1}\,\frac{\partial}{\partial u}\bigg|_0,$$

and of the Levi Matrix:

$$\begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1,\overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2,\overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1,\overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2,\overline{\mathscr{L}}_2\big]\big) \end{pmatrix}(0) = 2\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure. □

Taking zero remainders, one obtains (two):

$$\boxed{\text{Model(s) (IV)}_1: \qquad v = z_1\overline{z}_1 \pm z_1\overline{z}_2.}$$

8. General class IV$_2$

Proposition. *A local real analytic hypersurface passing through the origin:*

$$0 \in M^5 \subset \mathbb{C}^3$$

which belongs to the general Class IV$_2$*, namely such that, for any two local generators* $\{\mathscr{L}_1,\mathscr{L}_2\}$ *for* $T^{1,0}M$*:*

$$\Big\{\mathscr{L}_1,\,\mathscr{L}_2,\,\overline{\mathscr{L}}_1,\,\overline{\mathscr{L}}_2,\,\,\big[\mathscr{L}_1,\overline{\mathscr{L}}_1\big]\Big\}$$

constitute a frame for $\mathbb{C}\otimes_{\mathbb{R}} TM$*, such that in addition, the Levi form:*

$$\text{Levi-Form}^M(p)$$

if of rank 1 *at every point, and such that lastly, the Freeman form:*

$$\text{Freeman-Form}^M(p)$$

is nondegenerate at every point, may always be represented, in suitable local holomorphic coordinates:

$$(z_1,z_2,w)\in\mathbb{C}^3$$

by a specific real analytic equation of the form:

$$\boxed{\underline{(\text{IV})_2}: \quad v = z_1\overline{z}_1 + \tfrac{1}{2}\,z_1z_1\overline{z}_2 + \tfrac{1}{2}\,z_2\overline{z}_1\overline{z}_1 + \mathrm{O}_4\big(z_1,z_2,\overline{z}_1,\overline{z}_2\big) + u\,\mathrm{O}_2\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big),}$$

with remainders both satisfying:

$$0 \equiv \text{remainder}\big(0,0,\overline{z}_1,\overline{z}_2,u\big) \equiv \text{remainder}\big(z_1,z_2,0,0,u\big).$$

Proof. One starts with:

$$\begin{aligned} v &= \varphi\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big) \\ &= \mathrm{O}_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big), \end{aligned}$$

assuming:

$$0 \equiv \varphi\big(0, 0, \overline{z}_1, \overline{z}_2, u\big) \equiv \varphi\big(z_1, z_2, 0, 0, u\big).$$

Emphasize second and third order terms:

$$\begin{aligned} v = P_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + P_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + \\ + u\, Q_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + \mathrm{O}_4\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big). \end{aligned}$$

Of course:

$$\begin{aligned} 0 &\equiv P_3\big(0, 0, \overline{z}_1, \overline{z}_2\big) \equiv P_3\big(z_1, z_2, 0, 0\big), \\ 0 &\equiv Q_2\big(0, 0, \overline{z}_1, \overline{z}_2\big) \equiv Q_2\big(z_1, z_2, 0, 0\big), \\ 0 &\equiv \mathrm{O}_4\big(0, 0, \overline{z}_1, \overline{z}_2, u\big) \equiv \mathrm{O}_4\big(z_1, z_2, 0, 0, u\big). \end{aligned}$$

As in what precedes:

$$P_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) = a\, z_1\overline{z}_1 + \beta\, z_2\overline{z}_1 + \overline{\beta}\, z_1\overline{z}_2 + c\, z_2\overline{z}_2,$$

with:

$$a \in \mathbb{R}, \qquad \beta \in \mathbb{C}, \qquad c \in \mathbb{R}.$$

Taking:

$$\rho_0 = du - A_1\, dz_1 - A_2\, dz_2 - \overline{A}_1\, d\overline{z}_1 - \overline{A}_2\, d\overline{z}_2,$$

what was done for the General Class $(\mathsf{IV})_1$ yields that the Levi matrix at the origin is:

$$\begin{aligned} \begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_2\big]\big) \end{pmatrix}(0) &= \sqrt{-1} \begin{pmatrix} \mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_1\big) & \mathscr{L}_2\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_2\big) \\ \mathscr{L}_1\big(\overline{A}_2\big) - \overline{\mathscr{L}}_2\big(A_1\big) & \mathscr{L}_2\big(\overline{A}_2\big) - \overline{\mathscr{L}}_2\big(A_2\big) \end{pmatrix}(0) \\ &= \begin{pmatrix} 2a & 2\beta \\ 2\overline{\beta} & 2c \end{pmatrix}, \end{aligned}$$

and it should be of rank 1.

A linear transformation:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} \lambda & \mu \\ \nu & \chi \end{pmatrix}}_{\in \mathsf{GL}_2(\mathbb{C})} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

normalizes it to:

$$2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

whence:

$$v = z_1\overline{z}_1 + P_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + \\ + u\, Q_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + \mathrm{O}_4\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big).$$

To compute:

$$\overline{\mathscr{L}}_1\big(A_1\big),$$

observing:

$$\varphi_u = \mathrm{O}(2), \\ \varphi_{z_1} = \mathrm{O}(1),$$

start with:

$$\begin{aligned} A_1 &= -\frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u} \\ &= \sqrt{-1}\left(\frac{\varphi_{z_1}}{1 - \sqrt{-1}\,\varphi_u}\right) \\ &= \sqrt{-1}\,\varphi_{z_1}\left(1 + \sqrt{-1}\,\varphi_u + \big(\sqrt{-1}\,\varphi_u\big)^2 + \cdots\right) \\ &= \sqrt{-1}\,\varphi_{z_1} + \mathrm{O}(3) \\ &= \sqrt{-1}\big(\overline{z}_1 + P_{3,z_1} + u\,Q_{2,z_1} + \mathrm{O}(3)\big). \end{aligned}$$

Next:

$$\begin{aligned} \overline{\mathscr{L}}_1 &= \frac{\partial}{\partial \overline{z}_1} + \overline{A}_1\,\frac{\partial}{\partial u} \\ &= \frac{\partial}{\partial \overline{z}_1} + \mathrm{O}(1)\,\frac{\partial}{\partial u}. \end{aligned}$$

Observing that:

$$\begin{aligned} \varphi_{z_1 u} &= Q_{2,z_1} + \mathrm{O}(3) \\ &= \mathrm{O}(1), \end{aligned}$$

one computes:

$$\begin{aligned} \overline{\mathscr{L}}_1\big(A_1\big) &= \left(\frac{\partial}{\partial \overline{z}_1} + \mathrm{O}(1)\,\frac{\partial}{\partial u}\right)\big[\sqrt{-1}\,\varphi_{z_1} + \mathrm{O}(3)\big] \\ &= \sqrt{-1}\,\varphi_{z_1\overline{z}_1} + \mathrm{O}(1)\,\varphi_{z_1 u} + \mathrm{O}(2) \\ &= \sqrt{-1}\,\varphi_{z_1\overline{z}_1} + \mathrm{O}(2) \\ &= \sqrt{-1}\,\big(1 + P_{3,z_1\overline{z}_1} + u\,Q_{2,z_1\overline{z}_1} + \mathrm{O}(2)\big). \end{aligned}$$

One treats similarly the other

$$\overline{\mathscr{L}}_1\big(A_2\big), \qquad \overline{\mathscr{L}}_2\big(A_1\big), \qquad \overline{\mathscr{L}}_2\big(A_2\big),$$

plus all conjugates, and obtains:

$$\text{Levi-Matrix} \ = \ 2 \begin{pmatrix} {\scriptstyle 1+P_{3,z_1\overline{z}_1}+ \atop \scriptstyle +uQ_{2,z_1\overline{z}_1}+\mathrm{O}(2)} & {\scriptstyle P_{3,z_2\overline{z}_1}+ \atop \scriptstyle uQ_{2,z_2\overline{z}_1}+\mathrm{O}(2)} \\ {\scriptstyle P_{3,z_1\overline{z}_2}+ \atop \scriptstyle +uQ_{2,z_1\overline{z}_2}+\mathrm{O}(2)} & {\scriptstyle P_{3,z_2\overline{z}_2}+ \atop \scriptstyle uQ_{2,z_2\overline{z}_2}+\mathrm{O}(2)} \end{pmatrix}.$$

The vanishing of its determinant modulo $\mathrm{O}(2)$-terms:

$$0 \equiv P_{3,z_2\overline{z}_2} + u\,Q_{2,z_2\overline{z}_2} + \mathrm{O}(2)$$

yields:

$$\begin{aligned} 0 &\equiv P_{3,z_2\overline{z}_2} \\ 0 &\equiv Q_{2,z_2\overline{z}_2}. \end{aligned}$$

The graphing equation then has the form:

$$\begin{aligned} v = z_1\overline{z}_1 + \alpha\, z_1z_1\overline{z}_1 + \overline{\alpha}_1\, z_1\overline{z}_1\overline{z}_1 + \beta\, z_1z_2\overline{z}_2 + \overline{\beta}\, z_2\overline{z}_1\overline{z}_2 + \\ + \gamma\, z_1z_2\overline{z}_1 + \overline{\gamma}\, z_1\overline{z}_1\overline{z}_2 + \delta\, z_2z_2\overline{z}_1 + \overline{\delta}\, z_1\overline{z}_2\overline{z}_2 + \\ + a\,u\,z_1\overline{z}_1 + \varepsilon\,u\,z_1\overline{z}_2 + \overline{\varepsilon}\,u\,\overline{z}_1z_2 + \mathrm{O}_4\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big). \end{aligned}$$

Doing:

$$z_1 \longmapsto z_1 + \alpha\, z_1z_1 + \gamma\, z_1z_2 + \delta\, z_2z_2,$$

one makes:

$$\alpha = \gamma = \delta = 0.$$

It remains:

$$\begin{aligned} v = z_1\overline{z}_1 + \beta\, z_1z_2\overline{z}_2 + \overline{\beta}\, z_2\overline{z}_1\overline{z}_2 + \\ + a\,u\,z_1\overline{z}_1 + \varepsilon\,u\,z_1\overline{z}_2 + \overline{\varepsilon}\,u\,\overline{z}_1z_2 + \mathrm{O}_4\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big). \end{aligned}$$

The function k of Section 9, page 81 of [3] is:

$$\begin{aligned} k &= \frac{-\mathscr{L}_2\big(\overline{A}_1\big) + \overline{\mathscr{L}}_1\big(A_2\big)}{\mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_1\big)} \\ &= \frac{2\sqrt{-1}\,\varphi_{z_2\overline{z}_1} + \mathrm{O}(1)}{-2\sqrt{-1} + \mathrm{O}(1)} \\ &= -\varphi_{z_2\overline{z}_1} + \mathrm{O}(2) \\ &= -\overline{\beta}\,\overline{z}_1 - \overline{\varepsilon}\,u + \mathrm{O}(2). \end{aligned}$$

Finally, the nondegeneracy of the Freeman form:

$$0 \neq \overline{\mathscr{L}}_1(k)$$

reads:

$$\overline{\mathscr{L}}_1(k) = \left(\frac{\partial}{\partial \overline{z}_1} + \mathrm{O}(1)\,\frac{\partial}{\partial u}\right)\Big[-\,\overline{\beta}\,\overline{z}_1 - \overline{\varepsilon}\,u + \mathrm{O}(2)\Big] \\ = -\,\overline{\beta} + \mathrm{O}(1),$$

which necessitates:

$$\boxed{\beta \neq 0.}$$

A complex dilation in the z_2-axis then terminates. □

Scholium. *In such elementarily normalized coordinates, one has the diagonal normalizations at the origin of the fields:*

$$\mathscr{L}_1\big|_0 = \frac{\partial}{\partial z_1}\bigg|_0,$$
$$\mathscr{L}_2\big|_0 = \frac{\partial}{\partial z_2}\bigg|_0,$$
$$\overline{\mathscr{L}}_1\big|_0 = \frac{\partial}{\partial \overline{z}_1}\bigg|_0,$$
$$\overline{\mathscr{L}}_2\big|_0 = \frac{\partial}{\partial \overline{z}_2}\bigg|_0,$$
$$\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\Big|_0 = -\,2\,\sqrt{-1}\;\frac{\partial}{\partial u}\bigg|_0,$$

and of the Levi Matrix:

$$\begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_2\big]\big) \end{pmatrix}(0) = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and furthermore, one has the normalization of the function k:

$$\boxed{k = -\,\overline{z}_1 + \mathrm{O}(1),}$$

whence the nondegeneracy of the Freeman form reads:

$$\boxed{\overline{\mathscr{L}}_1(k)\big|_0 = -\,1,}$$

which conveniently fixes ideas when performing explicitly the Cartan equivalence procedure. □

Taking, not zero remainders, but adapted ones ([1]):

$$\boxed{\text{Model (III)}_1: \qquad v = \frac{z_1\overline{z}_1 + \frac{1}{2}\,z_1z_1\overline{z}_2 + \frac{1}{2}\,z_2\overline{z}_1\overline{z}_1}{1 - z_2\overline{z}_2}.}$$

9. Smooth categories

Since the power series expansions have been used only up to a certain order in all reasonings, the hypothesis of real analyticity is not necessary. For instance, existence of preliminary coordinates in which all pluriharmonic terms are removed are (well) known to also exist when $M^{2n+c} \subset \mathbb{C}^{n+c}$ is of class $\mathscr{C}^\kappa$, with $\kappa \in \mathbb{N}$, $\kappa \geqslant 2$, or $\kappa = \infty$, provided one truncates the requirements to a certain order governed by the size of κ. Adapting the elementary proofs provided here, one may obtain (exercise) easy generalizations of the six propositions.

10. Graphing ontologies

General class I. Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\big(M^3 \subset \mathbb{C}^2\big) \in \text{General Class I},$$

and graphing functions:

$$\begin{aligned} v &= \varphi(x, y, u) \\ &= z\overline{z} + z\overline{z}\,\mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u), \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 3$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, *without any further restriction*.

General class II. Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\big(M^4 \subset \mathbb{C}^3\big) \in \text{General Class II},$$

and pairs of graphing functions:

$$\begin{aligned} v_1 &= \varphi_1\big(x, y, u_1, u_2\big) \\ &= z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2), \\ v_2 &= \varphi_2\big(x, y, u_1, u_2\big) \\ &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2). \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 3$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, again *without any further restriction*.

General class III_1. Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\big(M^5 \subset \mathbb{C}^4\big) \in \text{General Class III}_1,$$

and triples of graphing functions:

$$\begin{aligned} v_1 &= \varphi_1\big(x, y, u_1, u_2, u_3\big) \\ &= z\overline{z} + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \end{aligned}$$

$$\begin{aligned} v_2 &= \varphi_2\big(x, y, u_1, u_2, u_3\big) \\ &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \end{aligned}$$

$$\begin{aligned} v_3 &= \varphi_3\big(x, y, u_1, u_2, u_3\big) \\ &= \sqrt{-1}\,\big(z^2\overline{z} - z\overline{z}^2\big) + z\overline{z}\,\mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 4$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, still *without any further restriction*.

General class III_2. Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\Big(M^5 \subset \mathbb{C}^4\Big) \in \text{General Class } \mathsf{III}_2,$$

and triples of graphing functions:

$$\begin{aligned} v_1 &= \varphi_1\big(x, y, u_1, u_2, u_3\big) \\ &= z\overline{z} + z\overline{z}\,\mathrm{O}_3\big(z, \overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z, \overline{z}, u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2, u_3\big), \end{aligned}$$

$$\begin{aligned} v_2 &= \varphi_2\big(x, y, u_1, u_2, u_3\big) \\ &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z, \overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z, \overline{z}, u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2, u_3\big), \end{aligned}$$

$$\begin{aligned} v_3 &= \varphi_3\big(x, y, u_1, u_2, u_3\big) \\ &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z, \overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z, \overline{z}, u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z, \overline{z}, u_1, u_2, u_3\big), \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 5$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, *however with the restriction that:*

$$0 \equiv \left| \begin{matrix} \mathscr{L}(\overline{A}_1) - \overline{\mathscr{L}}(A_1) & \mathscr{L}(\overline{A}_2) - \overline{\mathscr{L}}(A_2) & \mathscr{L}(\overline{A}_3) - \overline{\mathscr{L}}(A_3) \\ \begin{matrix} \mathscr{L}(\mathscr{L}(\overline{A}_1)) - 2\mathscr{L}(\overline{\mathscr{L}}(A_1)) + \\ + \overline{\mathscr{L}}(\mathscr{L}(A_1)) \end{matrix} & \begin{matrix} \mathscr{L}(\mathscr{L}(\overline{A}_2)) - 2\mathscr{L}(\overline{\mathscr{L}}(A_2)) + \\ + \overline{\mathscr{L}}(\mathscr{L}(A_2)) \end{matrix} & \begin{matrix} \mathscr{L}(\mathscr{L}(\overline{A}_3)) - 2\mathscr{L}(\overline{\mathscr{L}}(A_3)) + \\ + \overline{\mathscr{L}}(\mathscr{L}(A_3)) \end{matrix} \\ \begin{matrix} -\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_1)) + 2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_1)) - \\ - \mathscr{L}(\overline{\mathscr{L}}(\overline{A}_1)) \end{matrix} & \begin{matrix} -\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_2)) + 2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_2)) - \\ - \mathscr{L}(\overline{\mathscr{L}}(\overline{A}_2)) \end{matrix} & \begin{matrix} -\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_3)) + 2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_3)) - \\ - \mathscr{L}(\overline{\mathscr{L}}(\overline{A}_3)) \end{matrix} \end{matrix} \right| .$$

a possibly subtle and complicated differential relation/condition one should take account of when launching the Cartan equivalence method.

General class IV_1**.** Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\big(M^5 \subset \mathbb{C}^3\big) \ \in \ \text{General Class IV}_1,$$

and graphing functions:

$$\begin{aligned} v &= \varphi\big(x_1, x_2, y_1, y_2, u\big) \\ &= z_1\overline{z}_1 \pm z_2\overline{z}_2 + \mathrm{O}_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big), \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 3$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, *without any further restriction* for the fourth time.

General class IV_2**.** Before finer biholomorphic equivalence considerations, there is a one-to one correspondence between:

$$\big(M^5 \subset \mathbb{C}^3\big) \ \in \ \text{General Class IV}_1,$$

and graphing functions:

$$\begin{aligned} v &= \varphi\big(x_1, x_2, y_1, y_2, u\big) \\ &= z_1\overline{z}_1 + z_1z_1\overline{z}_2 + z_2\overline{z}_1\overline{z}_1 + \mathrm{O}_4\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + u\,\mathrm{O}_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big), \end{aligned}$$

of smoothness $\mathscr{C}^\kappa$ ($\kappa \geqslant 4$), or $\mathscr{C}^\infty$, or $\mathscr{C}^\omega$, *however with the restriction of identical vanishing:*

$$\begin{aligned} 0 \equiv \text{Levi-Determinant} = \frac{4}{(\sqrt{-1}+\varphi_u)^3(-\sqrt{-1}+\varphi_u)^3} \Big\{ & \varphi_{z_2\overline{z}_2}\varphi_{z_1\overline{z}_1} - \varphi_{z_2\overline{z}_1}\varphi_{z_1\overline{z}_2} + \\ & + \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1u}\varphi_u - \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1}\varphi_{uu} - \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_{z_1}\varphi_{\overline{z}_2u} + \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_u\varphi_{z_1\overline{z}_2} - \\ & - \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_{\overline{z}_2}\varphi_{z_1u} - \varphi_{z_2}\varphi_{\overline{z}_1}\varphi_{uu}\varphi_{z_1\overline{z}_2} + \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_u\varphi_{z_1\overline{z}_2} - \varphi_{z_2\overline{z}_2}\varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u + \\ & + \varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu} - \varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u + \varphi_{z_2\overline{z}_1}\varphi_{z_1}\varphi_{\overline{z}_2u}\varphi_u + \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{\overline{z}_1}\varphi_{z_1u} - \\ & - \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{z_1\overline{z}_1}\varphi_u + \varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_{z_1}\varphi_{\overline{z}_1u} - \varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_u\varphi_{z_1\overline{z}_1} + \varphi_{\overline{z}_2}\varphi_{z_2}\varphi_{uu}\varphi_{z_1\overline{z}_1} + \\ & + \sqrt{-1}\big(\varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1u} + \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_{z_1\overline{z}_2} + \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1u} + \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{z_1\overline{z}_1}\big) - \\ & - \sqrt{-1}\big(\varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_{z_1\overline{z}_1} + \varphi_{z_2\overline{z}_1}\varphi_{z_1}\varphi_{\overline{z}_2u} + \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_{z_1\overline{z}_2} + \varphi_{z_2\overline{z}_2}\varphi_{\overline{z}_1}\varphi_{z_1u}\big) - \\ & - \varphi_{z_2\overline{z}_1}\varphi_{z_1\overline{z}_2}\varphi_u\varphi_u + \varphi_{z_2\overline{z}_2}\varphi_{z_1\overline{z}_1}\varphi_u\varphi_u \Big\}, \end{aligned}$$

a possibly subtle and complicated differential relation/condition that Samuel Pocchiola took systematically account of while performing *completely explicitly* the Cartan equivalence method in [5].

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