

Equivalences of 5-dimensional CR-manifolds IV: Six ambiguity matrix groups (Initial G -structures)

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Abstract. Class I CR manifolds have initial G -structure a certain 4-dimensional subgroup of $GL_3(\mathbb{C})$. Class II CR manifolds have initial G -structure a certain 10-dimensional subgroup of $GL_4(\mathbb{C})$. Class III₁ CR manifolds have initial G -structure a certain 10-dimensional subgroup of $GL_5(\mathbb{C})$. Class III₂ CR manifolds have initial G -structure a certain 18-dimensional subgroup of $GL_5(\mathbb{C})$. Class IV₁ CR manifolds have initial G -structure a certain 13-dimensional subgroup of $GL_5(\mathbb{C})$. Class IV₂ CR manifolds have initial G -structure a certain 10-dimensional subgroup of $GL_5(\mathbb{C})$.

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1. General class I: 4-dimensional real subgroup of $GL_3(\mathbb{C})$

Equip \mathbb{C}^2 with coordinates:

$$(z, w) \in \mathbb{C}^2.$$

Let a connected real hypersurface:

$$M^3 \subset \mathbb{C}^2$$

be of smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^2.$$

By definition ([2, 3]):

$$\left(M^3 \subset \mathbb{C}^2 \right) \in \text{General Class I,}$$

belongs to the first class if:

$$\mathbb{C} \otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M].$$

This means that for any local vector field generator:

$$\mathcal{L} = \text{section of } T^{1,0}(M \cap U_p),$$

one has at every point $q \in M \cap U_p$:

$$\mathfrak{3} = \text{rank}_{\mathbb{C}} \left(\mathcal{L}|_q, \overline{\mathcal{L}}|_q, [\mathcal{L}, \overline{\mathcal{L}}]|_q \right).$$

Next, take any (local) biholomorphism:

$$h: U_p \xrightarrow{\sim} U'_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain hypersurface:

$$M'^3 := h(M \cap U_p) \subset \mathbb{C}^2,$$

having the same smoothness (mental exercise):

$$\mathcal{C}^{\kappa} \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^{\infty}, \quad \text{or} \quad \mathcal{C}^{\omega}.$$

Take also any local vector field generator:

$$\mathcal{L}' = \text{section of } T^{1,0}M'.$$

Then necessarily at every point $q' \in M'$, one also has ([2], Section 4):

$$\mathfrak{3} = \text{rank}_{\mathbb{C}} \left(\mathcal{L}'|_{q'}, \overline{\mathcal{L}'}|_{q'}, [\mathcal{L}', \overline{\mathcal{L}'}]|_{q'} \right).$$

Write the components of h and the target coordinates as:

$$\begin{aligned} (z, w) &\longmapsto (z'(z, w), w'(z, w)) \\ &=: (z', w'). \end{aligned}$$

To lower the number of primes in equations, it proves to be better to work instead with the *inverse* of h , denoted here not with an exponent $^{-1}$, but with a *prime*:

$$h': \quad \begin{aligned} U'_{p'} &\xrightarrow{\sim} U_p && (p = h'(p')), \\ (z', w') &\longmapsto (z(z', w'), w(z', w')) = (z, w). \end{aligned}$$

Then from [2], there exists a nowhere vanishing function:

$$a: M \cap U_p \longrightarrow \mathbb{C} \setminus \{0\},$$

such that:

$$h'_*(\mathcal{L}') = a \mathcal{L}.$$

Of course simultaneously ([2]):

$$h'_*(\overline{\mathcal{L}'}) = \overline{a} \overline{\mathcal{L}}.$$

Now set:

$$\begin{aligned}\mathcal{I} &:= \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}], \\ \mathcal{I}' &:= \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}'].\end{aligned}$$

The advantage of the $\sqrt{-1}$ -factor is *reality*:

$$\begin{aligned}\overline{\mathcal{I}} &= \mathcal{I}, \\ \overline{\mathcal{I}'} &= \mathcal{I}',\end{aligned}$$

so that one has two (local) frames:

$$\begin{aligned}\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{I}\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} T(M \cap U_p), \\ \{\mathcal{L}', \overline{\mathcal{L}'}, \mathcal{I}'\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM'.\end{aligned}$$

The Lie bracket \mathcal{I}' transfers back to $M \cap U_p$ through h'_* as:

$$\begin{aligned}h'_*(\mathcal{I}') &= h'_*(\sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}']) \\ &= \sqrt{-1} [h'_*(\mathcal{L}'), h'_*(\overline{\mathcal{L}}')] \\ &= \sqrt{-1} [a \mathcal{L}, \overline{a \mathcal{L}}] \\ &= a\overline{a} \underbrace{\sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]}_{=: \mathcal{I}} + \underbrace{\sqrt{-1} a \mathcal{L}(\overline{a}) \cdot \overline{\mathcal{L}}}_{=: \overline{b}} - \underbrace{\sqrt{-1} \overline{a} \mathcal{L}(a) \cdot \mathcal{L}}_{=: b},\end{aligned}$$

so that, if one decides to set:

$$b := -\sqrt{-1} \overline{a} \mathcal{L}(a),$$

forgetting how this coefficient-function is related to a , one obtains:

$$h'_*(\mathcal{I}') = a\overline{a} \mathcal{I} + \overline{b} \overline{\mathcal{L}} + b \mathcal{L}.$$

Summary. *Through any local biholomorphic equivalences between hypersurfaces of \mathbb{C}^2 belonging to the General Class I:*

$$M^3 \xrightarrow{\sim} M'^3,$$

for any two choices of local vector field generators:

$$\begin{aligned}\mathcal{L} &\quad \text{for } T^{1,0}M, \\ \mathcal{L}' &\quad \text{for } T^{1,0}M',\end{aligned}$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{L}' \\ \overline{\mathcal{L}'} \\ \mathcal{I}' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & \overline{a} & 0 \\ b & \overline{b} & a\overline{a} \end{pmatrix} \begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{I} \end{pmatrix},$$

for some two local functions:

$$a: M \longrightarrow \mathbb{C} \setminus \{0\}, \quad b: M \longrightarrow \mathbb{C}.$$

This means that the *ambiguity* in the choice of a local frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}\} \quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM$$

which comes — *naturally from the point of view of CR geometry* — from a choice of a local frame:

$$\mathcal{L} \quad \text{for } T^{1,0}M$$

is represented by general changes of frames whose matrices are of the form:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ b & \bar{b} & a\bar{a} \end{pmatrix},$$

the entries being coefficient-functions depending on existing equivalences $M \xrightarrow{\sim} M'$, and/or on the choice of local coordinates.

Within Élie Cartan's theory of equivalence between differential-geometric structures, this means that the *initial G-structure* for the biholomorphic equivalence problem between hypersurface $M^3 \subset \mathbb{C}^2$ belonging to the General Class I is a reduction of the full linear group $\mathrm{GL}_3(\mathbb{C})$ to the mentioned subgroup in which the — possibly unknown — functions a, b are *replaced* by independent complex variables.

Indeed, one has a very elementary:

Lemma. *The set of matrices:*

$$\mathbb{G}_I^{\text{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ b & \bar{b} & a\bar{a} \end{pmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \right\}$$

is a (closed) 4-dimensional real matrix subgroup of the full:

$$\mathrm{GL}_3(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \end{pmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proof. Well, closedness under multiplication (composition):

$$\begin{aligned} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \bar{a}_1 & 0 \\ b_1 & \bar{b}_1 & a_1\bar{a}_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 & 0 \\ 0 & \bar{a}_2 & 0 \\ b_2 & \bar{b}_2 & a_2\bar{a}_2 \end{pmatrix} &= \begin{pmatrix} a_1a_2 & 0 & 0 \\ 0 & \bar{a}_1\bar{a}_2 & 0 \\ b_1a_2 + a_1\bar{a}_1b_2 & \bar{b}_1\bar{a}_2 + a_1\bar{a}_1\bar{b}_2 & a_1\bar{a}_1a_2\bar{a}_2 \end{pmatrix} \\ &=: \begin{pmatrix} a_3 & 0 & 0 \\ 0 & \bar{a}_3 & 0 \\ b_3 & \bar{b}_3 & a_3\bar{a}_3 \end{pmatrix} \end{aligned}$$

is visibly clear, after setting:

$$\begin{aligned} a_3 &:= a_1 a_2 & (\neq 0, \text{ again}) \\ b_3 &:= b_1 a_2 + a_1 \bar{a}_1 b_2. \end{aligned}$$

Quite similarly, the inverse:

$$\begin{aligned} \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ b & \bar{b} & a\bar{a} \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{\bar{a}} & 0 \\ \frac{-b}{a\bar{a}\bar{a}} & \frac{-\bar{b}}{a\bar{a}\bar{a}} & \frac{1}{a\bar{a}} \end{pmatrix} \\ &=: \begin{pmatrix} a^\sim & 0 & 0 \\ 0 & \bar{a}^\sim & 0 \\ b^\sim & \bar{b}^\sim & a^\sim \bar{a}^\sim \end{pmatrix}, \end{aligned}$$

also belongs to the subgroup, with:

$$\begin{aligned} a^\sim &:= \frac{1}{a} & (\neq 0), \\ b^\sim &:= -\frac{b}{a\bar{a}}, \end{aligned}$$

which concludes. □

Proposition. *On a 3-dimensional hypersurface:*

$$\left(M^3 \subset \mathbb{C}^2 \right) \in \text{General Class I,}$$

having biholomorphically invariant $(1, 0)$ CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of local vector field generator:

$$\mathcal{L} \quad \text{for } T^{1,0}M,$$

the associated frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]\} =: \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{I}\}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $GL_3(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 4-dimensional subgroup:

$$G_1^{\text{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ b & \bar{b} & a\bar{a} \end{pmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \right\},$$

which is the initial ambiguity group when launching the Cartan equivalence method. □

2. Scholium: The multiplier function a

Although Cartan's method equivalence classically *decides* not to enter the computations of coefficient-functions like a and b in the preceding section, one may wonder what is the expression of a , at least.

Through a (local) biholomorphism:

$$h': (z', w') \longmapsto (z(z', w'), w(z', w')),$$

basic vector fields transfer as (without writing h'_*):

$$\begin{aligned} \frac{\partial}{\partial z'} &= z_{z'} \frac{\partial}{\partial z} + w_{z'} \frac{\partial}{\partial w}, \\ \frac{\partial}{\partial w'} &= z_{w'} \frac{\partial}{\partial z} + w_{w'} \frac{\partial}{\partial w}. \end{aligned}$$

If M and M' have local graphing equations:

$$\begin{aligned} v &= \varphi(x, y, u), \\ v' &= \varphi'(x', y', u'), \end{aligned}$$

two generators for $T^{1,0}M$ and $T^{1,0}M'$ are ([2]):

$$\begin{aligned} \mathcal{L} &= \frac{\partial}{\partial z} + 2A \frac{\partial}{\partial w}, \\ \mathcal{L}' &= \frac{\partial}{\partial z'} + 2A' \frac{\partial}{\partial w'}, \end{aligned}$$

where:

$$\begin{aligned} A &:= \frac{\varphi_z}{\sqrt{-1} + \varphi_u}, \\ A' &:= \frac{\varphi'_{z'}}{\sqrt{-1} + \varphi'_{u'}}. \end{aligned}$$

Consequently:

$$\begin{aligned} \mathcal{L}' &= \frac{\partial}{\partial z'} + 2A' \frac{\partial}{\partial w'} \\ &= (z_{z'} + 2A' z_{w'}) \frac{\partial}{\partial z} + (w_{z'} + 2A' w_{w'}) \frac{\partial}{\partial w} \\ &= (z_{z'} + 2A' z_{w'}) \underbrace{\left[\frac{\partial}{\partial z} + \frac{w_{z'} + 2A' w_{w'}}{z_{z'} + 2A' z_{w'}} \frac{\partial}{\partial w} \right]}_{=\mathcal{L}}, \end{aligned}$$

so that necessarily:

$$\begin{aligned} a &= z_{z'} + 2A' z_{w'}, \\ A &= \frac{w_{z'} + 2A' w_{w'}}{z_{z'} + 2A' z_{w'}}, \end{aligned}$$

or more precisely writing all arguments:

$$\begin{aligned} a(z, w) &= (z_{z'} + 2A' z_{w'}) (z'(z, w), w'(z, w)), \\ A(z, w) &= \frac{w_{z'} + 2A' w_{w'}}{z_{z'} + 2A' z_{w'}} (z'(z, w), w'(z, w)), \end{aligned}$$

so that:

$$\begin{aligned} h'_*(\mathcal{L}') &= h'_* \left(\frac{\partial}{\partial z'} + A' \frac{\partial}{\partial w'} \right) \\ &= a \left(\frac{\partial}{\partial z} + A \frac{\partial}{\partial w} \right) \\ &= a \mathcal{L}. \end{aligned}$$

Here, because h' is (local) biholomorphism, its Jacobian matrix h'_* is invertible, hence on restriction to any complex 1-dimensional line of the form:

$$T_{q'}^{1,0} M' = \mathbb{C} \cdot \mathcal{L}'|_{q'} \quad (q' \in M'),$$

it is of rank 1 (also invertible):

$$h'_* : T_{q'}^{1,0} M' \xrightarrow{\sim} T_{h'(q')}^{1,0} M,$$

from which it follows that the function a , appearing also in denominator place in the above expression of A , vanishes *nowhere*.

3. General class II

Equip \mathbb{C}^3 with coordinates:

$$(z, w_1, w_2) \in \mathbb{C}^3.$$

Let a connected CR-generic submanifold:

$$M^4 \subset \mathbb{C}^3$$

be of smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^3.$$

By definition ([2, 3]):

$$(M^4 \subset \mathbb{C}^3) \in \text{General Class II},$$

if:

$$\mathbb{C} \otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]].$$

This means that for any local vector field generator:

$$\mathcal{L} = \text{section of } T^{1,0}(M \cap U_p),$$

one has at every point $q \in M \cap U_p$:

$$4 = \text{rank}_{\mathbb{C}} \left(\mathcal{L}|_q, \overline{\mathcal{L}}|_q, [\mathcal{L}, \overline{\mathcal{L}}]|_q, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]|_q \right).$$

Next, take any (local) biholomorphism:

$$h: U_p \xrightarrow{\sim} U_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain CR-generic submanifold:

$$M'^4 := h(M \cap U_p) \subset \mathbb{C}^3,$$

having of course the same smoothness:

$$\mathcal{C}^{\kappa} \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^{\infty}, \quad \text{or} \quad \mathcal{C}^{\omega}.$$

Take also any local vector field generator:

$$\mathcal{L}' = \text{section of } T^{1,0}M'.$$

Then necessarily at every point $q' \in M'$, one also has (exercise, or *see* below):

$$4 = \text{rank}_{\mathbb{C}} \left(\mathcal{L}'|_{q'}, \overline{\mathcal{L}'}|_{q'}, [\mathcal{L}', \overline{\mathcal{L}'}]|_{q'}, [\mathcal{L}', [\mathcal{L}', \overline{\mathcal{L}'}]]|_{q'} \right).$$

Write the components of h and the target coordinates as:

$$\begin{aligned} (z, w_1, w_2) &\longmapsto (z'(z, w_1, w_2), w'_1(z, w_1, w_2), w'_2(z, w_1, w_2)) \\ &=: (z', w'_1, w'_2). \end{aligned}$$

Consider the *inverse* of h :

$$\begin{aligned} &U'_{p'} \xrightarrow{\sim} U_p \quad (p = h'(p')), \\ h': (z', w'_1, w'_2) &\longmapsto (z(z', w'_1, w'_2), w_1(z', w'_1, w'_2), w_2(z', w'_1, w'_2)) \\ &= (z, w_1, w_2). \end{aligned}$$

Then there exists a nowhere vanishing function:

$$a: M \cap U_p \longrightarrow \mathbb{C} \setminus \{0\},$$

such that:

$$h'_*(\mathcal{L}') = a \mathcal{L}.$$

Simultaneously:

$$h'_*(\overline{\mathcal{L}'}) = \overline{a} \overline{\mathcal{L}}.$$

Now setting:

$$\begin{aligned}\mathcal{T} &:= \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}], \\ \mathcal{T}' &:= \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}'],\end{aligned}$$

and setting:

$$\begin{aligned}\mathcal{S} &:= [\mathcal{L}, \mathcal{T}], \\ \mathcal{S}' &:= [\mathcal{L}', \mathcal{T}'],\end{aligned}$$

one has two (local) frames:

$$\begin{aligned}\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} T(M \cap U_p), \\ \{\mathcal{L}', \overline{\mathcal{L}}', \mathcal{T}', \mathcal{S}'\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM'.\end{aligned}$$

The Lie bracket \mathcal{T}' transfers back to $M \cap U_p$ through h'_* as:

$$\begin{aligned}h'_*(\mathcal{T}') &= h'_*(\sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}']) \\ &= \sqrt{-1} [h'_*(\mathcal{L}'), h'_*(\overline{\mathcal{L}}')] \\ &= \sqrt{-1} [a \mathcal{L}, \overline{a} \overline{\mathcal{L}}] \\ &= a\overline{a} \underbrace{\sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]}_{=\mathcal{T}} + \underbrace{\sqrt{-1} a \mathcal{L}(\overline{a}) \cdot \overline{\mathcal{L}}}_{=:\overline{b}} - \underbrace{\sqrt{-1} \overline{a} \overline{\mathcal{L}}(a) \cdot \mathcal{L}}_{=:b},\end{aligned}$$

so that, if one decides to set:

$$b := -\sqrt{-1} \overline{a} \overline{\mathcal{L}}(a),$$

forgetting how this coefficient-function is related to a , one obtains, exactly as for the General Class I:

$$h'_*(\mathcal{T}') = a\overline{a} \mathcal{T} + \overline{b} \overline{\mathcal{L}} + b \mathcal{L}.$$

Next:

$$\begin{aligned}h'_*(\mathcal{S}') &= h'_*([\mathcal{L}', \mathcal{T}']) \\ &= [h'_*(\mathcal{L}'), h'_*(\mathcal{T}')] \\ &= [a \mathcal{L}, a\overline{a} \mathcal{T} + \overline{b} \overline{\mathcal{L}} + b \mathcal{L}] \\ &= aa\overline{a} \underbrace{[\mathcal{L}, \mathcal{T}]}_{=\mathcal{S}} + a\overline{b} \underbrace{[\mathcal{L}, \overline{\mathcal{L}}]}_{=-\sqrt{-1} \mathcal{T}} + ab \underbrace{[\mathcal{L}, \mathcal{L}]}_{\circ} + \\ &\quad + a \mathcal{L}(a\overline{a}) \cdot \mathcal{T} + a \mathcal{L}(\overline{b}) \cdot \overline{\mathcal{L}} + a \mathcal{L}(b) \cdot \mathcal{L} - \\ &\quad - a\overline{a} \mathcal{T}(a) \cdot \mathcal{L} - \overline{b} \overline{\mathcal{L}}(a) \cdot \mathcal{L} - b \mathcal{L}(a) \cdot \mathcal{L},\end{aligned}$$

which is:

$$\begin{aligned}h'_*(\mathcal{S}') &= aa\overline{a} \cdot \mathcal{S} + (-\sqrt{-1} a\overline{b} + a \mathcal{L}(a\overline{a})) \cdot \mathcal{T} + (a \mathcal{L}(\overline{b})) \cdot \overline{\mathcal{L}} + \\ &\quad + (a \mathcal{L}(b) - a\overline{a} \mathcal{T}(a) - \overline{b} \overline{\mathcal{L}}(a) - b \mathcal{L}(a)) \cdot \mathcal{L},\end{aligned}$$

so that if one decides to set:

$$\begin{aligned} c &:= -\sqrt{-1}a\bar{b} + a\mathcal{L}(a\bar{a}), \\ d &:= a\mathcal{L}(\bar{b}), \\ e &:= a\mathcal{L}(b) - a\bar{a}\mathcal{T}(a) - \bar{b}\overline{\mathcal{L}}(a) - b\mathcal{L}(a), \end{aligned}$$

one has in:

Summary. *Through any local biholomorphic equivalences between CR-generic submanifolds of \mathbb{C}^3 belonging to the General Class II:*

$$M^4 \xrightarrow{\sim} M'^4,$$

for any two choices of local vector field generators:

$$\begin{aligned} \mathcal{L} &\text{ for } T^{1,0}M, \\ \mathcal{L}' &\text{ for } T^{1,0}M', \end{aligned}$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{L}' \\ \overline{\mathcal{L}'} \\ \mathcal{T}' \\ \mathcal{S}' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 \\ e & d & c & a\bar{a} \end{pmatrix} \begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{T} \\ \mathcal{S} \end{pmatrix},$$

for some five local functions:

$$\begin{aligned} a &: M \longrightarrow \mathbb{C} \setminus \{0\}, \\ b, c, d, e &: M \longrightarrow \mathbb{C}. \end{aligned}$$

This means that the *ambiguity* in the choice of a local frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}\} \text{ for } \mathbb{C} \otimes_{\mathbb{R}} TM$$

which comes — *naturally from the point of view of CR geometry* — from a choice of a local frame:

$$\mathcal{L} \text{ for } T^{1,0}M$$

is represented by general changes of frames whose matrices are of the form:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 \\ e & d & c & a\bar{a} \end{pmatrix},$$

the entries being coefficient-functions depending on existing equivalences $M \xrightarrow{\sim} M'$, and/or on the choice of local coordinates.

Thus, the *initial G-structure* for the biholomorphic equivalence problem between CR-generic submanifolds $M^4 \subset \mathbb{C}^3$ belonging to the General Class II is a reduction of the full linear group $\mathrm{GL}_4(\mathbb{C})$ to the mentioned subgroup in which the — possibly unknown — functions a, b, c, d, e are replaced by independent complex variables.

Lemma. *The set of matrices:*

$$G_{\mathrm{II}}^{\mathrm{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 \\ e & d & c & a\bar{a}\bar{a} \end{pmatrix} \in \mathcal{M}_{4 \times 4}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e \in \mathbb{C} \right\}$$

is a (closed) 10-dimensional real matrix subgroup of the full:

$$\mathrm{GL}_4(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} & \pi_{2,4} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} \end{pmatrix} \in \mathcal{M}_{4 \times 4}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proof. Closedness under multiplication (composition):

$$\begin{aligned} & \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & \bar{a}_1 & 0 & 0 \\ b_1 & \bar{b}_1 & a_1\bar{a}_1 & 0 \\ e_1 & d_1 & c_1 & a_1a_1\bar{a}_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & \bar{a}_2 & 0 & 0 \\ b_2 & \bar{b}_2 & a_2\bar{a}_2 & 0 \\ e_2 & d_2 & c_2 & a_2a_2\bar{a}_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2 & 0 & 0 & 0 \\ 0 & \bar{a}_1\bar{a}_2 & 0 & 0 \\ b_1a_2 + a_1\bar{a}_1b_2 & \bar{b}_1\bar{a}_2 + a_1\bar{a}_1\bar{b}_2 & a_1\bar{a}_1a_2\bar{a}_2 & 0 \\ e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2 & d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2 & c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2 & a_1a_1\bar{a}_1a_2a_2\bar{a}_2 \end{pmatrix} \\ &=: \begin{pmatrix} a_3 & 0 & 0 & 0 \\ 0 & \bar{a}_3 & 0 & 0 \\ b_3 & \bar{b}_3 & a_3\bar{a}_3 & 0 \\ e_3 & d_3 & c_3 & a_3a_3\bar{a}_3 \end{pmatrix} \end{aligned}$$

is visibly clear, after setting:

$$\begin{aligned} a_3 &:= a_1a_2 && (\neq 0, \text{ again}), \\ b_3 &:= b_1a_2 + a_1\bar{a}_1b_2, \\ c_3 &:= c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2, \\ d_3 &:= d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2, \\ e_3 &:= e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2. \end{aligned}$$

Quite similarly, the inverse:

$$\begin{aligned} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 \\ e & d & c & aa\bar{a} \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{\bar{a}} & 0 & 0 \\ \frac{-b}{aa\bar{a}} & \frac{-\bar{b}}{a\bar{a}\bar{a}} & \frac{1}{a\bar{a}} & 0 \\ \frac{bc}{a^4\bar{a}^2} - \frac{e}{a^3\bar{a}} & \frac{c\bar{b}}{a^3\bar{a}^3} - \frac{d}{a^2\bar{a}^2} & \frac{-c}{a^3\bar{a}^2} & \frac{1}{a^2\bar{a}} \end{pmatrix} \\ &=: \begin{pmatrix} a^\sim & 0 & 0 & 0 \\ 0 & \bar{a}^\sim & 0 & 0 \\ b^\sim & \bar{b}^\sim & a^\sim\bar{a}^\sim & 0 \\ e^\sim & d^\sim & c^\sim & a^\sim a^\sim \bar{a}^\sim \end{pmatrix}, \end{aligned}$$

also belongs to the subgroup, with:

$$\begin{aligned} a^\sim &:= \frac{1}{a} & (\neq 0), \\ b^\sim &:= -\frac{b}{aa\bar{a}}, \\ c^\sim &:= -\frac{c}{aaaa\bar{a}}, \\ d^\sim &:= \frac{c\bar{b}}{aaaa\bar{a}} - \frac{d}{aaa\bar{a}}, \\ e^\sim &:= \frac{bc}{aaaa\bar{a}} - \frac{e}{aaa\bar{a}}, \end{aligned}$$

which concludes. □

Proposition. *On a 4-dimensional CR-generic submanifold:*

$$\left(M^4 \subset \mathbb{C}^3 \right) \in \text{General Class II},$$

having biholomorphically invariant (1, 0) CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of local vector field generator:

$$\mathcal{L} \quad \text{for } T^{1,0}M,$$

the associated frame:

$$\{\mathcal{L}, \bar{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \bar{\mathcal{L}}], [\mathcal{L}, \sqrt{-1}[\mathcal{L}, \bar{\mathcal{L}}]]\} =: \{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{I}, \mathcal{J}\}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $\mathrm{GL}_4(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 10-dimensional subgroup:

$$G_{\mathrm{II}}^{\mathrm{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 \\ e & d & c & a\bar{a}\bar{a} \end{pmatrix} \in \mathcal{M}_{4 \times 4}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e \in \mathbb{C} \right\}. \quad \square$$

4. General class III₁

Equip \mathbb{C}^4 with coordinates:

$$(z, w_1, w_2, w_3) \in \mathbb{C}^4.$$

Let a connected CR-generic submanifold:

$$M^5 \subset \mathbb{C}^4$$

be of smoothness:

$$\mathcal{C}^{\kappa} \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^{\infty}, \quad \text{or} \quad \mathcal{C}^{\omega}.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^4.$$

By definition ([2, 3]):

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class III}_1,$$

if:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] \\ + [T^{0,1}M, [T^{1,0}M, T^{0,1}M]]. \end{aligned}$$

This means that for any local vector field generator:

$$\mathcal{L} = \text{section of } T^{1,0}(M \cap U_p),$$

one has at every point $q \in M \cap U_p$:

$$5 = \mathrm{rank}_{\mathbb{C}} \left(\mathcal{L}|_q, \overline{\mathcal{L}}|_q, [\mathcal{L}, \overline{\mathcal{L}}]|_q, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]|_q, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]|_q \right).$$

Next, take any (local) biholomorphism:

$$h: U_p \xrightarrow{\sim} U'_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain CR-generic submanifold:

$$M'^5 := h(M \cap U_p) \subset \mathbb{C}^4,$$

having of course the same smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Take also any local vector field generator:

$$\mathcal{L}' = \text{section of } T^{1,0}M'.$$

Then necessarily at every point $q' \in M'$, one also has (exercise, or *see* below):

$$5 = \text{rank}_{\mathbb{C}} \left(\mathcal{L}'|_{q'}, \overline{\mathcal{L}'}|_{q'}, [\mathcal{L}', \overline{\mathcal{L}'}]|_{q'}, [\mathcal{L}', [\mathcal{L}', \overline{\mathcal{L}'}]]|_{q'}, [\overline{\mathcal{L}'}', [\mathcal{L}', \overline{\mathcal{L}'}]]|_{q'} \right).$$

Write the components of h and the target coordinates as:

$$\begin{aligned} (z, w_1, w_2, w_3) &\longmapsto (z'(z, w_1, w_2, w_3), w'_1(z, w_1, w_2, w_3), w'_2(z, w_1, w_2, w_3), w'_3(z, w_1, w_2, w_3)) \\ &=: (z', w'_1, w'_2, w'_3). \end{aligned}$$

Consider the *inverse* h' of h :

$$\begin{aligned} U'_{p'} &\xrightarrow{\sim} U_p && (p = h'(p')), \\ (z', w'_1, w'_2, w'_3) &\longmapsto (z(z', w'_1, w'_2, w'_3), w_1(z', w'_1, w'_2, w'_3), w_2(z', w'_1, w'_2, w'_3), w_3(z', w'_1, w'_2, w'_3)) \\ &= (z, w_1, w_2, w_3). \end{aligned}$$

Then there exists a nowhere vanishing function:

$$a: M \cap U_p \longrightarrow \mathbb{C} \setminus \{0\},$$

such that:

$$h'_*(\mathcal{L}') = a \mathcal{L}.$$

Simultaneously:

$$h'_*(\overline{\mathcal{L}'}) = \overline{a} \overline{\mathcal{L}}.$$

Now setting:

$$\begin{aligned} \mathcal{F} &:= \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}], \\ \mathcal{F}' &:= \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}'}], \end{aligned}$$

setting:

$$\begin{aligned} \mathcal{S} &:= [\mathcal{L}, \mathcal{F}], \\ \mathcal{S}' &:= [\mathcal{L}', \mathcal{F}'], \end{aligned}$$

whence:

$$\begin{aligned} \overline{\mathcal{F}} &= [\overline{\mathcal{L}}, \mathcal{F}], \\ \overline{\mathcal{F}'} &= [\overline{\mathcal{L}'}, \mathcal{F}'], \end{aligned}$$

one has two (local) frames:

$$\begin{aligned} \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \overline{\mathcal{T}}\} & \quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} T(M \cap \mathbf{U}_p), \\ \{\mathcal{L}', \overline{\mathcal{L}'}, \mathcal{T}', \mathcal{S}', \overline{\mathcal{T}'}\} & \quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM'. \end{aligned}$$

As for the General Class II:

$$\begin{aligned} h'_*(\mathcal{T}') &= a\bar{a}\mathcal{T} + \bar{b}\overline{\mathcal{L}} + b\mathcal{L}, \\ h'_*(\mathcal{S}') &= aa\bar{a}\mathcal{S} + c\mathcal{T} + d\overline{\mathcal{L}} + e\mathcal{L}, \end{aligned}$$

whence by plain conjugation:

$$h'_*(\overline{\mathcal{T}'}) = a\bar{a}\bar{a}\overline{\mathcal{T}} + \bar{c}\mathcal{T} + \bar{e}\overline{\mathcal{L}} + \bar{d}\mathcal{L}.$$

Summary. *Through any local biholomorphic equivalences between CR-generic submanifolds of \mathbb{C}^4 belonging to the General Class III₁:*

$$M^5 \xrightarrow{\sim} M'^5,$$

for any two choices of local vector field generators:

$$\begin{aligned} \mathcal{L} & \quad \text{for } T^{1,0}M, \\ \mathcal{L}' & \quad \text{for } T^{1,0}M', \end{aligned}$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{L}' \\ \overline{\mathcal{L}'} \\ \mathcal{T}' \\ \mathcal{S}' \\ \overline{\mathcal{T}' } \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ \bar{d} & \bar{e} & \bar{c} & 0 & a\bar{a}\bar{a} \end{pmatrix} \begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{T} \\ \mathcal{S} \\ \overline{\mathcal{T}} \end{pmatrix},$$

for some five local functions:

$$\begin{aligned} a: & \quad M \longrightarrow \mathbb{C} \setminus \{0\}, \\ b, c, d, e: & \quad M \longrightarrow \mathbb{C}. \end{aligned}$$

This means that the *ambiguity* in the choice of a local frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \overline{\mathcal{T}}\} \quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM$$

which comes — *naturally from the point of view of CR geometry* — from a choice of a local frame:

$$\mathcal{L} \quad \text{for } T^{1,0}M$$

is represented by general changes of frames whose matrices are of the form:

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ \bar{d} & \bar{e} & \bar{c} & 0 & a\bar{a}\bar{a} \end{pmatrix},$$

the entries being coefficient-functions depending on existing equivalences $M \xrightarrow{\sim} M'$, and/or on the choice of local coordinates.

Thus, the *initial G-structure* for the biholomorphic equivalence problem between CR-generic submanifolds $M^5 \subset \mathbb{C}^4$ belonging to the General Class III₁ is a reduction of the full linear group $\mathrm{GL}_5(\mathbb{C})$ to the mentioned subgroup in which the — possibly unknown — functions a, b, c, d, e are *replaced* by independent complex variables.

Lemma. *The set of matrices:*

$$G_{\mathrm{III}_1}^{\mathrm{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ \bar{d} & \bar{e} & \bar{c} & 0 & a\bar{a}\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e \in \mathbb{C} \right\}$$

is a (closed) 10-dimensional real matrix subgroup of the full:

$$\mathrm{GL}_5(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proof. Closedness under multiplication (composition):

$$\begin{aligned}
& \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_1 & 0 & 0 & 0 \\ b_1 & \bar{b}_1 & a_1\bar{a}_1 & 0 & 0 \\ e_1 & d_1 & c_1 & a_1a_1\bar{a}_1 & 0 \\ \bar{d}_1 & \bar{e}_1 & \bar{c}_1 & 0 & a_1a_1\bar{a}_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_2 & 0 & 0 & 0 \\ b_2 & \bar{b}_2 & a_2\bar{a}_2 & 0 & 0 \\ e_2 & d_2 & c_2 & a_2a_2\bar{a}_2 & 0 \\ \bar{d}_2 & \bar{e}_2 & \bar{c}_2 & 0 & a_2a_2\bar{a}_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1a_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_1\bar{a}_2 & 0 & 0 & 0 \\ b_1a_2 + a_1\bar{a}_1b_2 & \bar{b}_1\bar{a}_2 + a_1\bar{a}_1\bar{b}_2 & a_1\bar{a}_1a_2\bar{a}_2 & 0 & 0 \\ e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2 & d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2 & c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2 & a_1a_1\bar{a}_1a_2a_2\bar{a}_2 & 0 \\ \bar{d}_1a_2 + \bar{c}_1b_2 + a_1\bar{a}_1\bar{a}_1\bar{d}_2 & \bar{e}_1\bar{a}_2 + \bar{c}_1\bar{b}_2 + a_1\bar{a}_1\bar{a}_1\bar{e}_2 & \bar{c}_1a_2\bar{a}_2 + a_1\bar{a}_1\bar{a}_1\bar{c}_2 & 0 & a_1\bar{a}_1\bar{a}_1a_2\bar{a}_2\bar{a}_2 \end{pmatrix} \\
&=: \begin{pmatrix} a_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_3 & 0 & 0 & 0 \\ b_3 & \bar{b}_3 & a_3\bar{a}_3 & 0 & 0 \\ e_3 & d_3 & c_3 & a_3a_3\bar{a}_3 & 0 \\ \bar{d}_3 & \bar{e}_3 & \bar{c}_3 & 0 & a_3\bar{a}_3\bar{a}_3 \end{pmatrix}
\end{aligned}$$

is visibly clear, after setting:

$$\begin{aligned}
a_3 &:= a_1a_2 & (\neq 0, \text{ again}), \\
b_3 &:= b_1a_2 + a_1\bar{a}_1b_2, \\
c_3 &:= c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2, \\
d_3 &:= d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2, \\
e_3 &:= e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2.
\end{aligned}$$

Quite similarly, the inverse:

$$\begin{aligned}
& \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & a\bar{a}\bar{a} & 0 \\ \bar{d} & \bar{e} & \bar{c} & 0 & a\bar{a}\bar{a} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\bar{a}} & 0 & 0 & 0 \\ \frac{-b}{a\bar{a}} & \frac{-\bar{b}}{a\bar{a}} & \frac{1}{a\bar{a}} & 0 & 0 \\ \frac{bc}{a^4\bar{a}^2} - \frac{e}{a^3\bar{a}} & \frac{c\bar{b}}{a^3\bar{a}^3} - \frac{d}{a^2\bar{a}^2} & \frac{-c}{a^3\bar{a}^2} & \frac{1}{a^2\bar{a}} & 0 \\ \frac{bc}{a^3\bar{a}^3} - \frac{d}{a^2\bar{a}^2} & \frac{bc}{a^2\bar{a}^4} - \frac{e}{a\bar{a}^3} & \frac{-c}{a^2\bar{a}^3} & 0 & \frac{1}{a\bar{a}^2} \end{pmatrix} \\
&=: \begin{pmatrix} \tilde{a} & 0 & 0 & 0 & 0 \\ 0 & \tilde{\bar{a}} & 0 & 0 & 0 \\ \tilde{b} & \tilde{\bar{b}} & \tilde{a}\tilde{\bar{a}} & 0 & 0 \\ \tilde{e} & \tilde{d} & \tilde{c} & \tilde{a}\tilde{\bar{a}}\tilde{\bar{a}} & 0 \\ \tilde{\bar{d}} & \tilde{\bar{e}} & \tilde{\bar{c}} & 0 & \tilde{a}\tilde{\bar{a}}\tilde{\bar{a}} \end{pmatrix},
\end{aligned}$$

also belongs to the subgroup, with:

$$\begin{aligned} a^\sim &:= \frac{1}{a} && (\neq 0), \\ b^\sim &:= -\frac{b}{aa\bar{a}}, \\ c^\sim &:= -\frac{c}{aaaa\bar{a}}, \\ d^\sim &:= \frac{c\bar{b}}{aaaa\bar{a}\bar{a}} - \frac{d}{aa\bar{a}}, \\ e^\sim &:= \frac{bc}{aaaa\bar{a}\bar{a}} - \frac{e}{aaa\bar{a}}, \end{aligned}$$

which concludes. \square

Proposition. *On a 5-dimensional CR-generic submanifold:*

$$\left(M^5 \subset \mathbb{C}^4\right) \in \text{General Class III}_1,$$

having biholomorphically invariant $(1, 0)$ CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of local vector field generator:

$$\mathcal{L} \quad \text{for } T^{1,0}M,$$

the associated frame:

$$\begin{aligned} &\{\mathcal{L}, \overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]]\} =: \\ &=: \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \overline{\mathcal{P}}\} \end{aligned}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $\text{GL}_5(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 10-dimensional subgroup:

$$G_{\text{III}_1}^{\text{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ \bar{d} & \bar{e} & \bar{c} & 0 & a\bar{a}\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e \in \mathbb{C} \right\}. \quad \square$$

5. General class III₂

Equip \mathbb{C}^4 with coordinates:

$$(z, w_1, w_2, w_3) \in \mathbb{C}^4.$$

Let a connected CR-generic submanifold:

$$M^5 \subset \mathbb{C}^4$$

be of smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 4), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^4.$$

Recall ([2, 3]) that if:

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class III}_2,$$

then:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] + \\ + [T^{1,0}M, [T^{1,0}M, [T^{1,0}M, T^{0,1}M]]]. \end{aligned}$$

This means that for any local vector field generator:

$$\mathcal{L} = \text{section of } T^{1,0}(M \cap U_p),$$

one has at every point $q \in M \cap U_p$:

$$5 = \text{rank}_{\mathbb{C}} \left(\mathcal{L}|_q, \overline{\mathcal{L}}|_q, [\mathcal{L}, \overline{\mathcal{L}}]|_q, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]|_q, [\mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]]|_q \right).$$

Next, take any (local) biholomorphism:

$$h: U_p \xrightarrow{\sim} U_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain CR-generic submanifold:

$$M'^5 := h(M \cap U_p) \subset \mathbb{C}^4,$$

having of course the same smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 4), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Take also any local vector field generator:

$$\mathcal{L}' = \text{section of } T^{1,0}M'.$$

Then necessarily at every point $q' \in M'$, one also has (exercise, or *see* below):

$$5 = \text{rank}_{\mathbb{C}} \left(\mathcal{L}'|_{q'}, \overline{\mathcal{L}'}|_{q'}, [\mathcal{L}', \overline{\mathcal{L}'}]|_{q'}, [\mathcal{L}', [\mathcal{L}', \overline{\mathcal{L}'}]]|_{q'}, [\mathcal{L}', [\mathcal{L}', [\mathcal{L}', \overline{\mathcal{L}'}]]]|_{q'} \right).$$

Write the components of h and the target coordinates as:

$$(z, w_1, w_2, w_3) \mapsto (z'(z, w_1, w_2, w_3), w'_1(z, w_1, w_2, w_3), w'_2(z, w_1, w_2, w_3), w'_3(z, w_1, w_2, w_3)) \\ =: (z', w'_1, w'_2, w'_3).$$

Consider the *inverse* h' of h :

$$\begin{aligned} \mathbb{U}'_{p'} &\xrightarrow{\sim} \mathbb{U}_p && (p = h'(p')), \\ (z', w'_1, w'_2, w'_3) &\mapsto (z(z', w'_1, w'_2, w'_3), w_1(z', w'_1, w'_2, w'_3), w_2(z', w'_1, w'_2, w'_3), w_3(z', w'_1, w'_2, w'_3)) \\ &= (z, w_1, w_2, w_3). \end{aligned}$$

Then there exists a nowhere vanishing function:

$$a: M \cap \mathbb{U}_p \longrightarrow \mathbb{C} \setminus \{0\},$$

such that:

$$h'_*(\mathcal{L}') = a \mathcal{L}.$$

Simultaneously:

$$h'_*(\overline{\mathcal{L}'}) = \bar{a} \overline{\mathcal{L}}.$$

Now setting:

$$\begin{aligned} \mathcal{T} &:= \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}], \\ \mathcal{T}' &:= \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}'}], \end{aligned}$$

setting:

$$\begin{aligned} \mathcal{S} &:= [\mathcal{L}, \mathcal{T}], \\ \mathcal{S}' &:= [\mathcal{L}', \mathcal{T}'], \end{aligned}$$

and setting

$$\begin{aligned} \mathcal{R} &= [\mathcal{L}, [\mathcal{L}, \mathcal{T}]], \\ \mathcal{R}' &= [\mathcal{L}', [\mathcal{L}', \mathcal{T}']], \end{aligned}$$

one has two (local) frames:

$$\begin{aligned} \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \mathcal{R}\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} T(M \cap \mathbb{U}_p), \\ \{\mathcal{L}', \overline{\mathcal{L}'}, \mathcal{T}', \mathcal{S}', \mathcal{R}'\} &\quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM'. \end{aligned}$$

As for the General Class III₁:

$$\begin{aligned} h'_*(\mathcal{T}') &= a\bar{a} \mathcal{T} + \bar{b} \overline{\mathcal{L}} + b \mathcal{L}, \\ h'_*(\mathcal{S}') &= aa\bar{a} \mathcal{S} + c \mathcal{T} + d \overline{\mathcal{L}} + e \mathcal{L}. \end{aligned}$$

Next, compute:

$$\begin{aligned}
h'_*(\mathcal{R}') &= h'_*([\mathcal{L}', \mathcal{S}']) \\
&= [h'_*(\mathcal{L}'), h'_*(\mathcal{S}')] \\
&= [a\mathcal{L}, aa\bar{a}\mathcal{S} + c\mathcal{T} + d\bar{\mathcal{L}} + e\mathcal{L}] \\
&= aaa\bar{a} \underbrace{[\mathcal{L}, \mathcal{S}]}_{=\mathcal{R}} + ac \underbrace{[\mathcal{L}, \mathcal{T}]}_{=\mathcal{S}} + ad \underbrace{[\mathcal{L}, \bar{\mathcal{L}}]}_{=-\sqrt{-1}\mathcal{T}} + ae \underbrace{[\mathcal{L}, \mathcal{L}]}_{\circ} + \\
&\quad + a\mathcal{L}(aa\bar{a}) \cdot \mathcal{S} + a\mathcal{L}(c) \cdot \mathcal{T} + a\mathcal{L}(d) \cdot \bar{\mathcal{L}} + a\mathcal{L}(e) \cdot \mathcal{L} - \\
&\quad - aa\bar{a}\mathcal{S}(a) \cdot \mathcal{L} - c\mathcal{T}(a) \cdot \mathcal{L} - d\bar{\mathcal{L}}(a) \cdot \mathcal{L} - e\mathcal{L}(a) \cdot \mathcal{L},
\end{aligned}$$

that is to say:

$$\begin{aligned}
h'_*(\mathcal{R}') &= aaa\bar{a} \cdot \mathcal{R} + (ac + a\mathcal{L}(aa\bar{a})) \cdot \mathcal{S} + \\
&\quad + (-\sqrt{-1}ad + a\mathcal{L}(c)) \cdot \mathcal{T} + (a\mathcal{L}(d)) \cdot \bar{\mathcal{L}} + \\
&\quad + (a\mathcal{L}(e) - aa\bar{a}\mathcal{S}(a) - c\mathcal{T}(a) - d\bar{\mathcal{L}}(a) - e\mathcal{L}(a)) \cdot \mathcal{L} \\
&=: aaa\bar{a} \cdot \mathcal{R} + f \cdot \mathcal{S} + g \cdot \mathcal{T} + h \cdot \bar{\mathcal{L}} + k \cdot \mathcal{L}.
\end{aligned}$$

Summary. Through any local biholomorphic equivalences between CR-generic submanifolds of \mathbb{C}^4 belonging to the General Class III₂:

$$M^5 \xrightarrow{\sim} M'^5,$$

for any two choices of local vector field generators:

$$\begin{aligned}
\mathcal{L} &\quad \text{for } T^{1,0}M, \\
\mathcal{L}' &\quad \text{for } T^{1,0}M',
\end{aligned}$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{L}' \\ \bar{\mathcal{L}}' \\ \mathcal{T}' \\ \mathcal{S}' \\ \mathcal{R}' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ k & h & g & f & aaa\bar{a} \end{pmatrix} \begin{pmatrix} \mathcal{L} \\ \bar{\mathcal{L}} \\ \mathcal{T} \\ \mathcal{S} \\ \mathcal{R} \end{pmatrix},$$

for some nine local functions:

$$\begin{aligned}
a &: M \longrightarrow \mathbb{C} \setminus \{0\}, \\
b, c, d, e, f, g, h, k &: M \longrightarrow \mathbb{C}.
\end{aligned}$$

This means that the *ambiguity* in the choice of a local frame:

$$\{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \mathcal{R}\} \quad \text{for } \mathbb{C} \otimes_{\mathbb{R}} TM$$

which comes — *naturally from the point of view of CR geometry* — from a choice of a local frame:

$$\mathcal{L} \quad \text{for } T^{1,0}M$$

is represented by general changes of frames whose matrices are of the form:

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ k & h & g & f & aaa\bar{a} \end{pmatrix},$$

the entries being coefficient-functions depending on existing equivalences $M \xrightarrow{\sim} M'$, and/or on the choice of local coordinates.

Thus, the *initial G-structure* for the biholomorphic equivalence problem between CR-generic submanifolds $M^5 \subset \mathbb{C}^4$ belonging to the General Class III₂ is a reduction of the full linear group $\text{GL}_5(\mathbb{C})$ to the mentioned subgroup in which the — possibly unknown — functions a, b, c, d, e are *replaced* by independent complex variables.

Lemma. *The set of matrices:*

$$G_{\text{III}_2}^{\text{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ k & h & g & f & aaa\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e, f, g, h, k \in \mathbb{C} \right\}$$

is a (closed) 18-dimensional real matrix subgroup of the full:

$$\text{GL}_5(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proof. Closedness under multiplication (composition):

$$\begin{aligned}
& \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_1 & 0 & 0 & 0 \\ b_1 & \bar{b}_1 & a_1\bar{a}_1 & 0 & 0 \\ e_1 & d_1 & c_1 & a_1a_1\bar{a}_1 & 0 \\ k_1 & h_1 & g_1 & f_1 & a_1a_1a_1\bar{a}_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_2 & 0 & 0 & 0 \\ b_2 & \bar{b}_2 & a_2\bar{a}_2 & 0 & 0 \\ e_2 & d_2 & c_2 & a_2a_2\bar{a}_2 & 0 \\ k_2 & h_2 & g_2 & f_2 & a_2a_2a_2\bar{a}_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1a_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_1\bar{a}_2 & 0 & 0 & 0 \\ b_1a_2 + a_1\bar{a}_1b_2 & \bar{b}_1\bar{a}_2 + a_1\bar{a}_1\bar{b}_2 & a_1\bar{a}_1a_2\bar{a}_2 & 0 & 0 \\ e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2 & d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2 & c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2 & a_1a_1\bar{a}_1a_2a_2\bar{a}_2 & 0 \\ k_1a_2 + g_1b_2 + f_1e_2 + a_1a_1a_1\bar{a}_1k_2 & h_1\bar{a}_2 + g_1\bar{b}_2 + f_1d_2 + a_1a_1a_1\bar{a}_1h_2 & g_1a_2\bar{a}_2 + f_1c_2 + a_1a_1a_1\bar{a}_1g_2 & f_1a_2a_2\bar{a}_2 + a_1a_1a_1\bar{a}_1f_2 & a_1a_1\bar{a}_1\bar{a}_1 \cdot a_2a_2\bar{a}_2\bar{a}_2 \end{pmatrix} \\
&=: \begin{pmatrix} a_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_3 & 0 & 0 & 0 \\ b_3 & \bar{b}_3 & a_3\bar{a}_3 & 0 & 0 \\ e_3 & d_3 & c_3 & a_3a_3\bar{a}_3 & 0 \\ k_3 & h_3 & g_3 & f_3 & a_3a_3a_3\bar{a}_3 \end{pmatrix}
\end{aligned}$$

is visibly clear, after setting:

$$\begin{aligned}
a_3 &:= a_1a_2 && (\neq 0, \text{ again}), \\
b_3 &:= b_1a_2 + a_1\bar{a}_1b_2, \\
c_3 &:= c_1a_2\bar{a}_2 + a_1a_1\bar{a}_1c_2, \\
d_3 &:= d_1\bar{a}_2 + c_1\bar{b}_2 + a_1a_1\bar{a}_1d_2, \\
e_3 &:= e_1a_2 + c_1b_2 + a_1a_1\bar{a}_1e_2, \\
f_3 &:= f_1a_2a_2\bar{a}_2 + a_1a_1a_1\bar{a}_1f_2, \\
g_3 &:= g_1a_2\bar{a}_2 + f_1c_2 + a_1a_1a_1\bar{a}_1g_2, \\
h_3 &:= h_1\bar{a}_2 + g_1\bar{b}_2 + f_1d_2 + a_1a_1a_1\bar{a}_1h_2, \\
k_3 &:= k_1a_2 + g_1b_2 + f_1e_2 + a_1a_1a_1\bar{a}_1k_2.
\end{aligned}$$

Quite similarly, the inverse:

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ k & h & g & f & aaa\bar{a} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\bar{a}} & 0 & 0 & 0 \\ \frac{-b}{aa\bar{a}} & \frac{-\bar{b}}{aa\bar{a}} & \frac{1}{a\bar{a}} & 0 & 0 \\ \frac{bc}{a^4\bar{a}^2} - \frac{e}{a^3\bar{a}} & \frac{c\bar{b}}{a^3\bar{a}^3} - \frac{d}{a^2\bar{a}^2} & \frac{-c}{a^3\bar{a}^2} & \frac{1}{a^2\bar{a}} & 0 \\ \frac{-bcf}{a^7\bar{a}^3} + \frac{bg}{a^5\bar{a}^2} + \frac{ef}{a^6\bar{a}^2} - \frac{k}{a^4\bar{a}} & \frac{-fcb}{a^6\bar{a}^4} + \frac{g\bar{b}}{a^4\bar{a}^3} + \frac{fd}{a^5\bar{a}^3} - \frac{h}{a^3\bar{a}^2} & \frac{fc}{a^6\bar{a}^3} - \frac{g}{a^4\bar{a}^2} & \frac{-f}{a^5\bar{a}^2} & \frac{1}{a^3\bar{a}} \end{pmatrix} =: \begin{pmatrix} a^\sim & 0 & 0 & 0 & 0 \\ 0 & \bar{a}^\sim & 0 & 0 & 0 \\ b^\sim & \bar{b}^\sim & a^\sim\bar{a}^\sim & 0 & 0 \\ e^\sim & d^\sim & c^\sim & a^\sim a^\sim\bar{a}^\sim & 0 \\ k^\sim & h^\sim & g^\sim & f^\sim & a^\sim a^\sim\bar{a}^\sim\bar{a}^\sim \end{pmatrix},$$

also belongs to the subgroup, with:

$$\begin{aligned} a^\sim &:= \frac{1}{a} & (\neq 0), \\ b^\sim &:= -\frac{b}{aa\bar{a}}, \\ c^\sim &:= -\frac{c}{aaa\bar{a}\bar{a}}, \\ d^\sim &:= \frac{c\bar{b}}{aaa\bar{a}\bar{a}\bar{a}} - \frac{d}{aa\bar{a}}, \\ e^\sim &:= \frac{bc}{aaa\bar{a}\bar{a}\bar{a}} - \frac{e}{aaa\bar{a}}, \\ f^\sim &:= -\frac{f}{a^5\bar{a}^2}, \\ g^\sim &:= \frac{cf}{a^6\bar{a}^3} - \frac{g}{a^4\bar{a}^2}, \\ h^\sim &:= -\frac{fcb}{a^6\bar{a}^4} + \frac{g\bar{b}}{a^4\bar{a}^3} + \frac{fd}{a^5\bar{a}^3} - \frac{h}{a^3\bar{a}^2}, \\ k^\sim &:= -\frac{bcf}{a^7\bar{a}^3} + \frac{bg}{a^5\bar{a}^2} + \frac{ef}{a^6\bar{a}^2} - \frac{k}{a^4\bar{a}}, \end{aligned}$$

which concludes. \square

Proposition. *On a 5-dimensional CR-generic submanifold:*

$$\left(M^5 \subset \mathbb{C}^4\right) \in \text{General Class III}_2,$$

having biholomorphically invariant (1, 0) CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of local vector field generator:

$$\mathcal{L} \quad \text{for } T^{1,0}M,$$

the associated frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]], [\mathcal{L}, [\mathcal{L}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]]]\} =:$$

$$=: \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \mathcal{R}\}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $\mathrm{GL}_5(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 18-dimensional subgroup:

$$\mathbf{G}_{\mathrm{III}_2}^{\mathrm{initial}} := \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & 0 & 0 & 0 \\ b & \bar{b} & a\bar{a} & 0 & 0 \\ e & d & c & aa\bar{a} & 0 \\ k & h & g & f & aaa\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a \in \mathbb{C} \setminus \{0\}, b, c, d, e, f, g, h, k \in \mathbb{C} \right\}. \quad \square$$

6. General class IV₁

Equip \mathbb{C}^3 with coordinates:

$$(z_1, z_2, w) \in \mathbb{C}^3.$$

Let a connected hypersurface:

$$M^5 \subset \mathbb{C}^3$$

be of smoothness:

$$\mathcal{C}^{\kappa} \quad (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^{\infty}, \quad \text{or} \quad \mathcal{C}^{\omega}.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^3.$$

Take a local frame:

$$\{\mathcal{L}_1, \mathcal{L}_2\} \quad \text{for } T^{1,0}(M \cap U_p).$$

Recall ([2, 3]) that:

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class IV}_1,$$

if firstly, after a possible constant change of frame:

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in \mathrm{GL}_2(\mathbb{C})} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix},$$

setting:

$$\mathcal{I} := \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1],$$

the five field:

$$\{\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \mathcal{I}\}$$

make up a (local) frame for:

$$\mathbb{C} \otimes_{\mathbb{R}} TM,$$

and if, secondly, the Levi form of M is nondegenerate (of rank 2) at every point, a second condition which reads as follows.

Alltogether:

$$\begin{aligned} \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1] &= \mathcal{I}, \\ \sqrt{-1} [\mathcal{L}_2, \overline{\mathcal{L}}_1] &= A \cdot \mathcal{I} + B_1 \cdot \overline{\mathcal{L}}_1 + B_2 \cdot \overline{\mathcal{L}}_2 + D_1 \cdot \mathcal{L}_1 + D_2 \cdot \mathcal{L}_2, \\ \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_2] &= \overline{A} \cdot \mathcal{I} + \overline{D}_1 \cdot \overline{\mathcal{L}}_1 + \overline{D}_2 \cdot \overline{\mathcal{L}}_2 + \overline{B}_1 \cdot \mathcal{L}_1 + \overline{B}_2 \cdot \mathcal{L}_2, \\ \sqrt{-1} [\mathcal{L}_2, \overline{\mathcal{L}}_2] &= C \cdot \mathcal{I} + \overline{E}_1 \cdot \overline{\mathcal{L}}_1 + \overline{E}_2 \cdot \overline{\mathcal{L}}_2 + E_1 \cdot \mathcal{L}_1 + E_2 \cdot \mathcal{L}_2, \end{aligned}$$

for certain functions:

$$\begin{aligned} C &: M \cap U_p \longrightarrow \mathbb{R}, \\ A, B_1, B_2, D_1, D_2, E_1, E_2 &: M \cap U_p \longrightarrow \mathbb{C}. \end{aligned}$$

Then the Levi form of M is of rank 2 at every point $q \in M \cap U_p$ if and only if (mental exercise, cf. [2]):

$$0 \neq \det \begin{pmatrix} 1 & A \\ \overline{A} & C \end{pmatrix} (q) \quad (\forall q \in M \cap U_p).$$

Next, take any (local) biholomorphism:

$$h: U_p \xrightarrow{\sim} U'_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain hypersurface:

$$M'^5 := h(M \cap U_p) \subset \mathbb{C}^3,$$

having the same smoothness (mental exercise):

$$\mathcal{C}^\kappa \ (\kappa \geq 3), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Consider the inverse of h :

$$h': U'_{p'} \xrightarrow{\sim} U_p \quad (p = h'(p')).$$

Take also any two local vector field generators:

$$\{\mathcal{L}'_1, \mathcal{L}'_2\}$$

making a frame for $T^{1,0}M'$. After a possible constant $\mathrm{GL}_2(\mathbb{C})$ -multiplication, setting similarly:

$$\mathcal{F}' := \sqrt{-1} [\mathcal{L}'_1, \overline{\mathcal{L}'_1}],$$

the five fields:

$$\{\mathcal{L}'_1, \mathcal{L}'_2, \overline{\mathcal{L}'_1}, \overline{\mathcal{L}'_2}, \mathcal{F}'\}$$

also make up (local) frame for:

$$\mathbb{C} \otimes_{\mathbb{R}} TM'.$$

Then ([2]), there are certain multiplier functions with:

$$\begin{aligned} h'_*(\mathcal{L}'_1) &= a_{11} \mathcal{L}_1 + a_{21} \mathcal{L}_2, \\ h'_*(\mathcal{L}'_2) &= a_{12} \mathcal{L}_1 + a_{22} \mathcal{L}_2, \end{aligned}$$

with:

$$0 \neq \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} (q) \quad (\forall q \in M \cap U_p),$$

whence also:

$$\begin{aligned} h'_*(\overline{\mathcal{L}'_1}) &= \overline{a}_{11} \overline{\mathcal{L}}_1 + \overline{a}_{21} \overline{\mathcal{L}}_2, \\ h'_*(\overline{\mathcal{L}'_2}) &= \overline{a}_{12} \overline{\mathcal{L}}_1 + \overline{a}_{22} \overline{\mathcal{L}}_2. \end{aligned}$$

Now, compute:

$$\begin{aligned} h'_*(\mathcal{F}') &= h'_*(\sqrt{-1} [\mathcal{L}'_1, \overline{\mathcal{L}'_1}]) \\ &= \sqrt{-1} [h'_*(\mathcal{L}'_1), h'_*(\overline{\mathcal{L}'_1})] \\ &= [a_{11} \mathcal{L}_1 + a_{21} \mathcal{L}_2, \overline{a}_{11} \overline{\mathcal{L}}_1 + \overline{a}_{21} \overline{\mathcal{L}}_2] \\ &= a_{11} \overline{a}_{11} \underbrace{\sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1]}_{=\mathcal{F}} + a_{21} \overline{a}_{11} \sqrt{-1} [\mathcal{L}_2, \overline{\mathcal{L}}_1] + \\ &\quad + a_{11} \overline{a}_{21} \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_2] + a_{21} \overline{a}_{21} \sqrt{-1} [\mathcal{L}_2, \overline{\mathcal{L}}_2] + \\ &\quad + \sqrt{-1} (a_{11} \mathcal{L}_1 (\overline{a}_{11}) + a_{21} \mathcal{L}_2 (\overline{a}_{11})) \cdot \overline{\mathcal{L}}_1 + \\ &\quad + \sqrt{-1} (a_{11} \mathcal{L}_1 (\overline{a}_{21}) + a_{21} \mathcal{L}_2 (\overline{a}_{21})) \cdot \overline{\mathcal{L}}_2 - \\ &\quad - \sqrt{-1} (\overline{a}_{11} \overline{\mathcal{L}}_1 (a_{11}) + \overline{a}_{21} \overline{\mathcal{L}}_2 (a_{11})) \cdot \mathcal{L}_1 - \\ &\quad - \sqrt{-1} (\overline{a}_{11} \overline{\mathcal{L}}_1 (a_{21}) + \overline{a}_{21} \overline{\mathcal{L}}_2 (a_{21})) \cdot \mathcal{L}_2, \end{aligned}$$

which, taking account of what precedes, is:

$$\begin{aligned}
h'_*(\mathcal{F}') &= \left(a_{11}\bar{a}_{11} + a_{21}\bar{a}_{11}A + a_{11}\bar{a}_{21}\bar{A} + a_{21}\bar{a}_{21}C \right) \cdot \mathcal{F} + \\
&+ \left(a_{21}\bar{a}_{11}B_1 + a_{11}\bar{a}_{21}\bar{D}_1 + a_{21}\bar{a}_{21}\bar{E}_1 + \sqrt{-1}a_{11}\mathcal{L}_1(\bar{a}_{11}) + \sqrt{-1}a_{21}\mathcal{L}_2(\bar{a}_{11}) \right) \cdot \bar{\mathcal{L}}_1 + \\
&+ \left(a_{21}\bar{a}_{11}B_2 + a_{11}\bar{a}_{21}\bar{D}_2 + a_{21}\bar{a}_{21}\bar{E}_2 + \sqrt{-1}a_{11}\mathcal{L}_1(\bar{a}_{21}) + \sqrt{-1}a_{21}\mathcal{L}_2(\bar{a}_{21}) \right) \cdot \bar{\mathcal{L}}_2 + \\
&+ \left(a_{21}\bar{a}_{11}D_1 + a_{11}\bar{a}_{21}\bar{B}_1 + a_{21}\bar{a}_{21}E_1 - \sqrt{-1}\bar{a}_{11}\bar{\mathcal{L}}_1(a_{11}) - \sqrt{-1}\bar{a}_{21}\bar{\mathcal{L}}_2(a_{11}) \right) \cdot \mathcal{L}_1 + \\
&+ \left(a_{21}\bar{a}_{11}D_2 + a_{11}\bar{a}_{21}\bar{B}_2 + a_{21}\bar{a}_{21}E_2 - \sqrt{-1}\bar{a}_{11}\bar{\mathcal{L}}_1(a_{21}) - \sqrt{-1}\bar{a}_{21}\bar{\mathcal{L}}_2(a_{21}) \right) \cdot \mathcal{L}_2.
\end{aligned}$$

The first appearing coefficient-function:

$$c := a_{11}\bar{a}_{11} + a_{21}\bar{a}_{11}A + a_{11}\bar{a}_{21}\bar{A} + a_{21}\bar{a}_{21}C$$

is real-valued and vanishes nowhere (mental exercise):

$$c: M \cap U_p \longrightarrow \mathbb{R} \setminus \{0\}.$$

The two functions (and their conjugates):

$$b_1 := a_{21}\bar{a}_{11}D_1 + a_{11}\bar{a}_{21}\bar{B}_1 + a_{21}\bar{a}_{21}E_1 - \sqrt{-1}\bar{a}_{11}\bar{\mathcal{L}}_1(a_{11}) - \sqrt{-1}\bar{a}_{21}\bar{\mathcal{L}}_2(a_{11}),$$

$$b_2 := a_{21}\bar{a}_{11}D_2 + a_{11}\bar{a}_{21}\bar{B}_2 + a_{21}\bar{a}_{21}E_2 - \sqrt{-1}\bar{a}_{11}\bar{\mathcal{L}}_1(a_{21}) - \sqrt{-1}\bar{a}_{21}\bar{\mathcal{L}}_2(a_{21}),$$

are complex-valued:

$$b_1, b_2 : M \cap U_p \longrightarrow \mathbb{C}.$$

Summary. *Through any local biholomorphic equivalences between CR-generic submanifolds of \mathbb{C}^3 belonging to the General Class IV₁:*

$$M^5 \xrightarrow{\sim} M'^5,$$

for any two choices of pairs of local vector field generators:

$$\{\mathcal{L}_1, \mathcal{L}_2\} \quad \text{for } T^{1,0}M,$$

$$\{\mathcal{L}'_1, \mathcal{L}'_2\} \quad \text{for } T^{1,0}M',$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{L}'_1 \\ \mathcal{L}'_2 \\ \bar{\mathcal{L}}'_1 \\ \bar{\mathcal{L}}'_2 \\ \mathcal{F}' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{a}_{11} & \bar{a}_{21} & 0 \\ 0 & 0 & \bar{a}_{21} & \bar{a}_{22} & 0 \\ b_1 & b_2 & \bar{b}_1 & \bar{b}_2 & c \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \bar{\mathcal{L}}_1 \\ \bar{\mathcal{L}}_2 \\ \mathcal{F} \end{pmatrix},$$

for some seven local functions:

$$c: M \longrightarrow \mathbb{R} \setminus \{0\},$$

$$a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 : M \longrightarrow \mathbb{C},$$

with:

$$0 \neq \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} (q) \quad (\forall q \in M).$$

One easily verifies that the set of matrices:

$$G_{IV_1}^{\text{initial}} := \left\{ \begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{a}_{11} & \bar{a}_{21} & 0 \\ 0 & 0 & \bar{a}_{12} & \bar{a}_{22} & 0 \\ b_1 & b_2 & \bar{b}_1 & \bar{b}_2 & c \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : c \in \mathbb{R} \setminus \{0\}, a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{C} \right\}$$

is a (closed) 13-dimensional real matrix subgroup of the full:

$$GL_5(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proposition. *On a 5-dimensional hypersurface submanifold:*

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class IV}_1,$$

having biholomorphically invariant $(1, 0)$ CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of a pair of local vector field generators:

$$\{\mathcal{L}_1, \mathcal{L}_2\} \quad \text{for } T^{1,0}M,$$

the associated frame:

$$\{\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \sqrt{-1}[\mathcal{L}_1, \overline{\mathcal{L}}_1]\} =: \{\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \mathcal{T}\}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $GL_5(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 13-dimensional subgroup:

$$G_{IV_1}^{\text{initial}} := \left\{ \begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{a}_{11} & \bar{a}_{21} & 0 \\ 0 & 0 & \bar{a}_{12} & \bar{a}_{22} & 0 \\ b_1 & b_2 & \bar{b}_1 & \bar{b}_2 & c \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : c \in \mathbb{R} \setminus \{0\}, a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{C} \right\}. \quad \square$$

7. General class IV_2

Equip \mathbb{C}^3 with coordinates:

$$(z_1, z_2, w) \in \mathbb{C}^3.$$

Let a connected hypersurface:

$$M^5 \subset \mathbb{C}^3$$

be of smoothness:

$$\mathcal{C}^\kappa \ (\kappa \geq 4), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Pick a point:

$$p \in M,$$

and take a (small) open neighborhood:

$$p \in U_p \subset \mathbb{C}^3.$$

Take a local frame:

$$\{\mathcal{L}_1, \mathcal{L}_2\} \quad \text{for } T^{1,0}(M \cap U_p).$$

Recall ([2, 3]) that when:

$$(M^5 \subset \mathbb{C}^3) \in \text{General Class } IV_2,$$

the kernel, in $T^{1,0}M$, of the Levi form is an everywhere of rank 1 complex subbundle:

$$K^{1,0}M \subset T^{1,0}M$$

hence has a vector field generator denoted:

$$\mathcal{H},$$

so that, renumbering if necessary, it is more natural to take:

$$\{\mathcal{H}, \mathcal{L}_1\}$$

as a local frame for $T^{1,0}M$.

Then without loss of generality:

$$\{\mathcal{H}, \mathcal{L}_1, \overline{\mathcal{H}}, \overline{\mathcal{L}}_1, \mathcal{F}\}$$

constitutes a (local) frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, setting as before:

$$\mathcal{F} := \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1].$$

Because one knows ([2]) that:

$$\begin{aligned} [\mathcal{H}, \overline{\mathcal{L}}_1] &\equiv 0 \quad \text{mod } (T^{1,0}M \oplus T^{0,1}M), \\ [\mathcal{H}, \overline{\mathcal{H}}] &\equiv 0 \quad \text{mod } (K^{1,0}M \oplus K^{0,1}M), \end{aligned}$$

alltogether the brackets are:

$$\begin{aligned}\sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1] &= \mathcal{I}, \\ \sqrt{-1} [\mathcal{K}, \overline{\mathcal{L}}_1] &= A \cdot \mathcal{K} + B \cdot \mathcal{L}_1 + C \cdot \overline{\mathcal{K}} + d \cdot \overline{\mathcal{L}}_1, \\ \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{K}}] &= \overline{C} \cdot \mathcal{K} + \overline{D} \cdot \mathcal{L}_1 + \overline{A} \cdot \overline{\mathcal{K}} + \overline{B} \cdot \overline{\mathcal{L}}_1, \\ \sqrt{-1} [\mathcal{K}, \overline{\mathcal{K}}] &= E \cdot \mathcal{K} + \overline{E} \cdot \overline{\mathcal{K}}\end{aligned}$$

for certain local functions:

$$A, B, C, D, E : M \longrightarrow \mathbb{C}.$$

Next, take any (local) biholomorphism:

$$h : U_p \xrightarrow{\sim} U'_{p'} = h(U_p) \quad (p' = h(p)),$$

which, when U_p is small enough, certainly transfers $M \cap U_p$ to a certain hypersurface:

$$M'^5 := h(M \cap U_p) \subset \mathbb{C}^3,$$

having the same smoothness (mental exercise):

$$\mathcal{C}^\kappa \ (\kappa \geq 4), \quad \text{or} \quad \mathcal{C}^\infty, \quad \text{or} \quad \mathcal{C}^\omega.$$

Consider the inverse of h :

$$h' : U'_{p'} \xrightarrow{\sim} U_p \quad (p = h'(p')).$$

Take similarly a local frame for $TM' \otimes_{\mathbb{R}} \mathbb{C}$:

$$\{\mathcal{K}', \mathcal{L}'_1, \overline{\mathcal{K}'}, \overline{\mathcal{L}'}_1, \mathcal{I}'\}$$

with:

$$\mathbb{C}\mathcal{K}' = K^{1,0}M',$$

and:

$$\mathcal{I}' := \sqrt{-1} [\mathcal{L}'_1, \overline{\mathcal{L}'}_1].$$

The invariance of the Levi-kernel:

$$h'_*(K^{1,0}M') = K^{1,0}M$$

yields, as on page 83 of [2]:

$$h'_*(\mathcal{K}') = c \cdot \mathcal{K},$$

for a certain local function:

$$c : M \longrightarrow \mathbb{C} \setminus \{0\},$$

while of course:

$$h'_*(\mathcal{L}'_1) = a \cdot \mathcal{L}_1 + b \cdot \mathcal{K},$$

for two certain functions:

$$\begin{aligned} a: M &\longrightarrow \mathbb{C} \setminus \{0\}, \\ b: M &\longrightarrow \mathbb{C}. \end{aligned}$$

Now, compute:

$$\begin{aligned} h'_*(\mathcal{T}') &= h'_*(\sqrt{-1} [\mathcal{L}'_1, \overline{\mathcal{L}'_1}]) \\ &= \sqrt{-1} [h'_*(\mathcal{L}'_1), h'_*(\overline{\mathcal{L}'_1})] \\ &= \sqrt{-1} [a\mathcal{L}_1 + b\mathcal{K}, \overline{a\mathcal{L}_1 + b\mathcal{K}}] \\ &= a\overline{a}\sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}_1}] + b\overline{a}\sqrt{-1} [\mathcal{K}, \overline{\mathcal{L}_1}] + a\overline{b}\sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{K}}] + b\overline{b}\sqrt{-1} [\mathcal{K}, \overline{\mathcal{K}}] + \\ &\quad + \sqrt{-1} (a\mathcal{L}_1(\overline{a}) + b\mathcal{K}(\overline{a})) \cdot \overline{\mathcal{L}_1} + \sqrt{-1} (a\mathcal{L}_1(\overline{b}) + b\mathcal{K}(\overline{b})) \cdot \overline{\mathcal{K}} - \\ &\quad - \sqrt{-1} (\overline{a\mathcal{L}_1}(a) - \overline{b\mathcal{K}}(a)) - \sqrt{-1} (\overline{a\mathcal{L}_1}(b) - \overline{b\mathcal{K}}(b)), \end{aligned}$$

which is:

$$\begin{aligned} h'_*(\mathcal{T}') &= (a\overline{a}) \cdot \mathcal{T} + \\ &\quad + \left(b\overline{a}D + a\overline{b}B + \sqrt{-1}a\mathcal{L}_1(\overline{a}) + \sqrt{-1}b\mathcal{K}(\overline{a}) \right) \cdot \overline{\mathcal{L}_1} + \\ &\quad + \left(b\overline{a}C + a\overline{b}A + b\overline{b}E + \sqrt{-1}a\mathcal{L}_1(\overline{b}) + \sqrt{-1}b\mathcal{K}(\overline{b}) \right) \cdot \overline{\mathcal{K}} + \\ &\quad + \left(b\overline{a}B + a\overline{b}D - \sqrt{-1}\overline{a\mathcal{L}_1}(a) - \sqrt{-1}\overline{b\mathcal{K}}(a) \right) \cdot \mathcal{L}_1 + \\ &\quad + \left(b\overline{a}A + a\overline{b}C + b\overline{b}E - \sqrt{-1}\overline{a\mathcal{L}_1}(b) - \sqrt{-1}\overline{b\mathcal{K}}(b) \right) \cdot \mathcal{K}. \end{aligned}$$

Two coefficient-function appear (together with their conjugates):

$$\begin{aligned} d &:= b\overline{a}B + a\overline{b}D - \sqrt{-1}\overline{a\mathcal{L}_1}(a) - \sqrt{-1}\overline{b\mathcal{K}}(a), \\ e &:= b\overline{a}A + a\overline{b}C + b\overline{b}E - \sqrt{-1}\overline{a\mathcal{L}_1}(b) - \sqrt{-1}\overline{b\mathcal{K}}(b), \end{aligned}$$

of course complex-valued:

$$d, e : M \longrightarrow \mathbb{C}.$$

Summary. *Through any local biholomorphic equivalences between CR-generic submanifolds of \mathbb{C}^3 belonging to the General Class IV_2 :*

$$M^5 \xrightarrow{\sim} M'^5,$$

for any choices of local vector field generators of the Levi-kernel bundles:

$$\begin{aligned} \{\mathcal{K}\} &\quad \text{for } T^{1,0}M, \\ \{\mathcal{K}'\} &\quad \text{for } T^{1,0}M', \end{aligned}$$

and for any choice of completion frame:

$$\begin{aligned} \{\mathcal{H}, \mathcal{L}_1\} & \text{ for } T^{1,0}M, \\ \{\mathcal{H}', \mathcal{L}'_1\} & \text{ for } T^{1,0}M', \end{aligned}$$

the transfer of frame obeys the rule:

$$\begin{pmatrix} \mathcal{H}' \\ \mathcal{L}'_1 \\ \overline{\mathcal{H}'} \\ \overline{\mathcal{L}'_1} \\ \mathcal{I}' \end{pmatrix} = \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & \bar{b} & \bar{a} & 0 \\ e & d & \bar{e} & \bar{d} & a\bar{a} \end{pmatrix} \begin{pmatrix} \mathcal{H} \\ \mathcal{L}_1 \\ \overline{\mathcal{H}} \\ \overline{\mathcal{L}_1} \\ \mathcal{I} \end{pmatrix},$$

for some five local functions:

$$\begin{aligned} a, c: M & \longrightarrow \mathbb{C} \setminus \{0\}, \\ b, d, e: M & \longrightarrow \mathbb{C}. \quad \square \end{aligned}$$

One easily verifies that the set of matrices:

$$G_{IV_2}^{\text{initial}} := \left\{ \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & \bar{b} & \bar{a} & 0 \\ e & d & \bar{e} & \bar{d} & a\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a, c \in \mathbb{C} \setminus \{0\}, b, d, e \in \mathbb{C} \right\}$$

is a (closed) 10-dimensional real matrix subgroup of the full:

$$GL_5(\mathbb{C}) := \left\{ \pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : 0 \neq \det \pi \right\}.$$

Proposition. On a 5-dimensional hypersurface submanifold:

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class IV}_2,$$

having biholomorphically invariant (1, 0) CR bundle:

$$T^{1,0}M \subset \mathbb{C} \otimes_{\mathbb{R}} TM,$$

for any choice of a pair of vector field generators:

$$\begin{aligned} \{\mathcal{H}\} & \text{ for } K^{1,0}M, \\ \{\mathcal{H}, \mathcal{L}_1\} & \text{ for } T^{1,0}M, \end{aligned}$$

the associated frame:

$$\{\mathcal{H}, \mathcal{L}_1, \overline{\mathcal{H}}, \overline{\mathcal{L}_1}, \sqrt{-1}[\mathcal{L}_1, \overline{\mathcal{L}_1}]\} =: \{\mathcal{H}, \mathcal{L}_1, \overline{\mathcal{H}}, \overline{\mathcal{L}_1}, \mathcal{I}\}$$

for the full complexified tangent bundle:

$$\mathbb{C} \otimes_{\mathbb{R}} TM$$

performs a reduction of the full $\mathrm{GL}_5(\mathbb{C})$ -structure of $\mathbb{C} \otimes_{\mathbb{R}} TM$ to the 10-dimensional subgroup:

$$G_{IV_2}^{\text{initial}} := \left\{ \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & \bar{b} & \bar{a} & 0 \\ e & d & \bar{e} & \bar{d} & a\bar{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a, c \in \mathbb{C} \setminus \{0\}, b, d, e \in \mathbb{C} \right\}. \quad \square$$

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