CARTAN EQUIVALENCES FOR LEVI-NONDEGENERATE HYPERSURFACES M^3 IN $\mathbb{C}^2$ BELONGING TO GENERAL CLASS I

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ABSTRACT. We develope in great computational details the classical Cartan equivalence problem for Levi-nondegenerate $\mathscr{C}^6$-smooth real hypersurfaces M^3 in $\mathbb{C}^2$, performing all calculations effectively in terms of a (local) graphing function φ. In particular, we present explicitly the unique (complex) essential invariant $\mathfrak{J}$ of the problem. Its expansion in terms of the 3-variables function φ incorporates millions of differential monomials, while, when φ is assumed to depend only on 2 variables (rigid case), $\mathfrak{J}$ writes out in two lines (7 monomials).

1. INTRODUCTION

In 1907, Henri Poincaré [19] initiated the question of determining whether two given Cauchy-Riemann (CR for short) local real analytic hypersurfaces in $\mathbb{C}^2$ can be mapped onto each other by a certain (local or global) biholomorphism. This problem was solved later on in 1932 by Élie Cartan [6] in a complete way, by importing techniques from his main original impulse (years 1900–1910) towards general investigations of a large class of problems which nowadays are known as *Cartan equivalence problems*, addressing, in many different contexts, equivalences of submanifolds, of (partial) differential equations, and as well, of several other geometric structures. Unifying the wide variety of these seemingly different equivalence problems into a potentially universal approach, Cartan showed that almost all continuous classification questions can indeed be reformulated in terms of specific adapted coframes.

Seeking an equivalence between coframes usually comprises a certain initial *ambiguity subgroup* $G \subset \mathsf{Gl}(n)$ related to the specific features of the geometry under study. The fundamental general set up is that, for two given coframes $\Omega := \{\omega^1, \dots, \omega^n\}$ and $\Omega' := \{\omega'^1, \dots, \omega'^n\}$ on two certain n-dimensional manifolds M and M', there exists a diffeomorphism $\Phi \colon M \longrightarrow M'$ making a geometric equivalence *if and only if* there is a G-valued function $g \colon M \to G$ such that $\Phi^*(\Omega) = g \cdot \Omega'$.

Cartan's 'algorithm' (the outcomes of which is often unpredictable) comprises three interrelated principal aspects: *absorbtion*; *normalization*; *prolongation*.

In brief outline, starting from:

$$\Omega := g \cdot \Omega', \tag{1}$$

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one has to find the so-called *structure equations* by computing the exterior differential:

$$d\Omega = dg \wedge \Omega' + g \cdot d\Omega'.$$

Inverting (1) as $\Omega' = g^{-1}\,\Omega$, one begins by replacing this in the first term:

$$dg \wedge \Omega' = dg \wedge g^{-1}\,\Omega = \underbrace{dg \cdot g^{-1}}_{\substack{\text{Maurer-Cartan} \\ \text{matrix } \mathsf{MC}_g}} \wedge\, \Omega,$$

with the standard Maurer-Cartan matrix of the matrix group G:

$$\mathsf{MC}_g := \bigg(\sum_{k=1}^n dg_k^i\,\big(g^{-1}\big)_j^k\bigg)_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n} = \sum_{s=1}^r a_{js}^i\,\alpha^s$$

having n^2 entries which express linearly in terms of some basis $\alpha^1,\dots,\alpha^r$ of left-invariant 1-forms on G, with $r := \dim_{\mathbb{R}} G$, by means of certain constants a_{js}^i. Then the structure equations become:

$$d\omega^i = \sum_{j=1}^n \sum_{s=1}^r a_{js}^i\,\alpha^s \wedge \omega^j + g \cdot d\Omega' \qquad\qquad (i = 1 \cdots n).$$

Moreover, one has to express the second term $d\Omega'$ above, which is a 2-form, as a combination of the $\omega^j \wedge \omega^k$. Usually, this step is quite costful, computationally speaking. When one executes this, the appearing (complicated) functions T_{jk}^i, called *torsion coefficients*:

$$d\omega^i = \sum_{j=1}^n \sum_{s=1}^r a_{js}^i\,\alpha^s \wedge \omega^j + \sum_{1\leqslant j<k\leqslant n} T_{jk}^i \cdot \omega^j \wedge \omega^k \qquad\qquad (i = 1 \cdots n),$$

usually reveal *appropriate invariants* of the geometric structure under study.

Then the main thrust of Cartan's approach is that, when one substitutes each Maurer-Cartan form α^s with $\alpha^s + \sum_{j=1}^n z_j^s\,\omega^j$ for arbitrary functions-coefficients z_j^s, while each torsion coefficient T_{jk}^i is simultaneously necessarily replaced by $T_{jk}^i + \sum_{s=1}^r \big(a_{js}^i\,z_k^s - a_{ks}^i\,z_j^s\big)$, and when one does choose the functions-coefficients z_j^s *in order to 'absorb' as many as possible torsion coefficients in the Maurer-Cartan part*, then the remaining, *unabsorbable*, (new, less numerous) torsion coefficients become *true invariants* of the geometric structure under study. Of course, the 'number' of invariant torsion coefficients is 'counted' by means of linear algebra, usually applying the so-called (non-explicit) Cartan's Lemma.

Since the remaining torsion coefficients are essential and invariant, one then *normalizes* them to be equal to a constant, usually 0, 1 or i, simply whether or not the group parameters they contain must be nonzero in the matrix group G to preserve invertibility. Setting these essential torsions equal to 0, 1 or i then determines some entries of the matrix group G, and therefore decreases the dimension of G. In high-level equivalence problems ([14, 17]), these *potentially normalizable* essential torsions are rather numerous and often overdetermined (unfortunately), hence one is forced to enter more deeply in explicit computations if one wants to rigorously settle which group parameters really remain, and which invariants really pop up. Hopefully at the end of a long procedure, one reduces the structure group G to dimension 0, getting a so-called *e-structure*.

But if, as also often occurs, it becomes no longer possible after several absorption-normalization steps to determine a (reduced) set of remaining group parameters, then one has to add the rest of (modified) Maurer-Cartan forms to the initial lifted coframe Ω and to *prolong* the base manifold M as the product $M^{\mathsf{pr}} := M \times G$. Surprisingly, Cartan observed that the solution of the original equivalence problem can be derived from that of M^{pr} equipped with the new coframe. Then, one has to restart the procedure *ab initio* with such a new prolonged problem. This initiates the third essential feature of the equivalence algorithm: the *prolongation*. For a detailed presentation of Cartan's method, the reader is referred to [18, 9, 14].

Cartan's remarkable achievements were encouraging enough to establish his elegant geometries, nowadays known as *Cartan geometries*, a generalization of two seemingly disparate geometries, that of Felix Klein and that of Bernhard Riemann. For the study of hypersurfaces in complex Euclidean spaces, Cartan's method was applied later on by some other mathematicians, *e.g.* Chern-Moser [7] and Tanaka [22], but along two seemingly different ways. In fact, Chern-Moser's work was a fairly direct development of that of Cartan, while Tanaka's was more algebraically-minded, involving Lie algebra cohomology, infinitesimal CR automorphisms, and so-called *Tanaka prolongations*.

Coming to the heart of the matter, let $M^3 \subset \mathbb{C}^2$ be a $\mathscr{C}^6$-smooth Levi-nondegenerate real hypersurface passing through the origin, in some suitable affine holomorphic coordinates $(z, w) = (x + iy, u + i\, v)$ represented as the graph of a certain $\mathscr{C}^6$-smooth defining function:

$$v = \varphi(z, \overline{z}, u) := z\overline{z} + \mathrm{O}(3),$$

satisfying $\varphi(0) = 0$. Our purpose in this paper is to reformulate Cartan's construction of an $\{e\}$-structure associated to such hypersurfaces *effectively in terms of the single datum* φ of the problem.

In [15], inspired by [8], we already performed, within the Tanaka framework, an *effective* construction of a Cartan geometry that is invariantly associated to such $M^3 \subset \mathbb{C}^2$. As the main result there, we explicitly computed the two essential *real* curvature coefficients of the geometry, the vanishing of which characterizes biholomorphic equivalency of M to the *Heisenberg sphere* $v = z\overline{z}$ (*see* Theorem 7.4 in [15]). In the present paper, we have to keep track of how the under consideration Cartan equivalence problem for real hypersurfaces M^3 matches up to their Cartan-Tanaka geometry. In particular, we will explicitly observe a close relationship between the single *complex* essential invariant of the equivalence problem and the two real invariants of the Cartan geometry.

As an outline of this paper, first in section 2, we set up the equivalence problem for Levi-nondegenerate real hypersurfaces $M^3 \subset \mathbb{C}^2$ by constructing the necessary adapted coframe on it. We begin by presenting generators $\mathscr{L}$ and $\overline{\mathscr{L}}$ of $T^{1,0}M$ and of $T^{0,1}M$. Then, the bracket $\mathscr{T} := i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big]$ completes a frame on for $\mathbb{C}\otimes_{\mathbb{R}} TM$. Dually, we deduce an initial complex *co*frame $\{\rho_0, \zeta_0, \overline{\zeta}_0\}$ on $\mathbb{C}\otimes_{\mathbb{R}} T^*M$.

Next, we determine the initial ambiguity group for equivalences under local biholomorphisms:

$$G := \left\{ g := \begin{pmatrix} \mathsf{a} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \overline{\mathsf{b}} & 0 & \overline{\mathsf{c}} \end{pmatrix}, \quad \mathsf{a}\in\mathbb{R}, \quad \mathsf{b},\mathsf{c}\in\mathbb{C} \right\}.$$

In section 3, we proceed to the equivalence algorithm by performing the absorbtion-normalization procedure. After normalizing the group parameter a, we continue in section 4 by performing a first prolongation. Namely, we prolong the equivalence problem of the under consideration CR-manifolds M^3 to that of a certain 7-dimensional prolonged spaces $M^{\mathsf{pr}} := M^3 \times G$ equipped with the initial coframe $\{\rho_0, \zeta_0, \overline{\zeta}_0\}$ to which we add four certain Maurer-Cartan 1-forms $\alpha, \beta, \overline{\alpha}, \overline{\beta}$ — associated to certain four remaining group parameters $\mathsf{b}, \mathsf{c}, \overline{\mathsf{b}}, \overline{\mathsf{c}}$ — and with four new appearing prolonged group parameters $\mathsf{r}, \mathsf{s}, \overline{\mathsf{r}}, \overline{\mathsf{s}}$. Subsequently, we consider this new prolonged equivalence problem *ab initio*.

The well-known Cartan's Lemma (*see* Lemma 4.1) also enables us to temporarily bypass some relatively painful computations (*cf.* Proposition 4.3), that, anyway, we do perform later on. After two absorbtions-normalizations and after one prolongation along the way, the desired equivalence problem transforms to that of some — explicitly computed — eight-dimensional coframe $\{\rho, \zeta, \overline{\zeta}, \alpha, \beta, \overline{\alpha}, \overline{\beta}, \delta\}$ having *e*-structure equations:

$$\begin{aligned}
d\rho &= \alpha\wedge\rho + \overline{\alpha}\wedge\rho + i\,\zeta\wedge\overline{\zeta}, \\
d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta, \\
d\overline{\zeta} &= \overline{\beta}\wedge\rho + \overline{\alpha}\wedge\overline{\zeta}, \\
d\alpha &= \delta\wedge\rho + 2\,i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta, \\
d\beta &= \delta\wedge\zeta + \beta\wedge\overline{\alpha} + \mathfrak{I}\,\overline{\zeta}\wedge\rho, \\
d\overline{\alpha} &= \delta\wedge\rho - 2\,i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\overline{\beta}, \\
d\overline{\beta} &= \delta\wedge\overline{\zeta} + \overline{\beta}\wedge\alpha + \overline{\mathfrak{I}}\,\zeta\wedge\rho, \\
d\delta &= \delta\wedge\alpha + \delta\wedge\overline{\alpha} + i\,\beta\wedge\overline{\beta} + \mathfrak{T}\,\rho\wedge\zeta + \overline{\mathfrak{T}}\,\rho\wedge\overline{\zeta},
\end{aligned}$$

with the single primary complex invariant:

$$\begin{aligned}
\mathfrak{I} := &-\frac{1}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{\mathsf{c}\overline{\mathsf{c}}^3} + \frac{2}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^3} + \frac{1}{2}\frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{\mathsf{c}\overline{\mathsf{c}}^3} - \frac{7}{6}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^3} - \\
&- \frac{1}{6}\frac{\mathscr{L}(\overline{P})\overline{\mathscr{L}}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}^3} + \frac{1}{3}\frac{\mathscr{L}(\overline{P})\overline{P}^2}{\mathsf{c}\overline{\mathsf{c}}^3},
\end{aligned}$$

in which the fundamental function P can expresses explicitly in terms of the single datum φ of the problem as:

$$P := \frac{\ell_z - \ell A_u + A\ell_u}{\ell},$$

where:

$$A := \frac{i\,\varphi_z}{1 - i\,\varphi_u} \qquad \text{and where:} \qquad \ell := i\left(\overline{A}_z + A\,\overline{A}_u - A_{\overline{z}} - \overline{A}\,A_u\right),$$

this last *Levi factor* ℓ being nowhere vanishing, because we assume M to be Levi nondegenerate. Furthermore, the other secondary invariant $\mathfrak{T}$ can be expressed in terms of the first one $\mathfrak{J}$ as:

$$\mathfrak{T} = \frac{1}{\overline{\mathsf{c}}}\left(\overline{\mathscr{L}}(\overline{\mathfrak{J}}) - \overline{P}\,\overline{\mathfrak{J}}\right) - i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\overline{\mathfrak{J}}.$$

Finally in section 5, we turn to a brief discussion of the Cartan-Tanaka geometry of the under consideration hypersurfaces M^3 and — being aware of the results of the papers [8, 15] — we observe that the equivalence problem matches up to their Cartan geometry so that the complex essential primary invariant $\mathfrak{J}$ can be reexpressed effectively in terms of the two (real) essential primary invariants we obtained there (this also matches up with the results of [12]).

Theorem 1.1. *(see Theorem 5.2 at the end) For Levi-nondegenerate $\mathscr{C}^6$-smooth real hypersurfaces $M^3 \subset \mathbb{C}^2$, the following relation holds between the essential complex invariant $\mathfrak{J}$ of their equivalence problem and the essential real invariants $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ of their Cartan geometry:*

$$\mathfrak{J} = \frac{4}{\mathsf{c}\overline{\mathsf{c}}^3}\big(\mathbf{\Delta}_1 + i\,\mathbf{\Delta}_4\big).$$

We close up this introduction by mentioning that, although it is well known that a close relationship exists between equivalences of hypersurfaces $M^3 \subset \mathbb{C}^2$ and second-order ordinary differential equations ([6, 7, 11, 10, 16, 12]), and although the (nonexplicit) geometric features of the results we present here are well known too (but often with hidden computations), a completely effective and systematic presentation of the related (complicated) computational aspects is necessary to understand in a deeper way the core of Cartan's method.

In fact, the present (preliminary) paper was written up in order to serve as a ground-companion to much higher level explorations of equivalence problems for embedded CR structures, that will appear soon ([14, 17]). Intentionally, we endeavour here to develope our systematic computational formalism at first for the simplest known CR structures $M^3 \subset \mathbb{C}^2$, before applying it to more delicate 5-dimensional real analytic CR structures.

The remarkable works of Beloshapka [1, 2, 3, 4, 5] have shown that there exists a wealth of model CR-generic submanifolds whose algebras of infinitesimal CR automorphisms have been computed *explicitly* there, and this paper together with [5, 14, 17] are a very first step in the Cartan-like study of the geometry-preserving deformations of just a few of these models, with a door potentially open towards the exploration of a great number of higher models with a similar emphasis on *effectiveness*.

2. SETTING UP THE EQUIVALENCE PROBLEM

Our aim in this section is to construct — in terms of a certain fundamental graphing function φ — an initial complex coframe on the under consideration three dimensional CR-manifold $M^3 \subset \mathbb{C}^2$, and next to set up the related equivalence problem. First, let us consider this approach dually, namely by constructing a local *frame* on M^3.

2.1. **Local frame adapted to 3-dimensional embedded CR structures.** Consider therefore a local $\mathscr{C}^6$-smooth hypersurface $M^3 \subset \mathbb{C}^2$ passing through the origin. In some suitable affine holomorphic coordinates $(z,w) = (x+iy, u+i\,v)$ adapted so that $T_0 M^3 = \{v = 0\}$, the implicit function theorem enables one to represent M^3 as a graph over the (x,y,u)-space. Since any function of $(x,y,u) = \big(\frac{z+\overline{z}}{2}, \frac{z+\overline{z}}{2i}, u\big)$ can be considered as one of $(z, \overline{z}, u)$, the graph in question may be thought of as being of the form:

$$v = \varphi(z, \overline{z}, u),$$

for some $\mathscr{C}^6$ function φ satisfying $\varphi(0) = \varphi_z(0) = \varphi_{\overline{z}}(0) = \varphi_u(0)$. In the sequel, all appearing invariant objects — vector fields, differential forms, torsion coefficients, essential functions — will depend only on φ and its partial derivatives with respect to the three (complex and real) initial coordinates $(z, \overline{z}, u)$, the latter being understood as *intrinsic* coordinates on M^3.

According to [11, 15], a local $(1,0)$ vector field on $\mathbb{C}^2$ defined near the origin:

$$\mathscr{L} := \frac{\partial}{\partial z} + \mathtt{A}\,\frac{\partial}{\partial w}$$

is *tangent* to M^3 if and only if, on restriction to M^3, its coefficient $\mathtt{A}$ satisfies:

$$\begin{aligned} 0 &= \mathscr{L}\Big(-\tfrac{w-\overline{w}}{2i} + \varphi\big(z, \overline{z}, \tfrac{w+\overline{w}}{2}\big)\Big) \\ &= -\frac{1}{2i}\,\mathtt{A} + \frac{1}{2}\,\mathtt{A}\,\varphi_u + \varphi_z. \end{aligned}$$

For this to hold true, it suffices to set:

$$\mathtt{A} := \frac{-2\ \varphi_z}{i+\ \varphi_u},$$

which is thus *de facto* a function of only $(z, \overline{z}, u)$. Furthermore, restricting $\mathscr{L}$ to M^3, one must simply and only drop the (extrinsic) vector field $\frac{\partial}{\partial v}$:

$$\begin{aligned} \mathscr{L}\big|_M &= \frac{\partial}{\partial z} + \mathtt{A}\left(\frac{1}{2}\,\frac{\partial}{\partial u} - \underline{\frac{i}{2}\,\frac{\partial}{\partial v}}_{\circ}\right) \\ &= \frac{\partial}{\partial z} - \frac{\varphi_z}{i+\varphi_u}\,\frac{\partial}{\partial u}. \end{aligned}$$

Now, it will be convenient to introduce an extra notation for the appearing coefficient of $\frac{\partial}{\partial u}$, say:

$$A := \frac{i\,\varphi_z}{1 - i\,\varphi_u}, \tag{2}$$

not to be confused with $\mathtt{A} = 2\,A$, which, anyway, will be left aside from now on.

Thus intrinsically on M^3, the CR-structure induced by the ambient $\mathbb{C}^2$ on M^3 is encoded by the complex $(1,0)$ vector field $\mathscr{L}$ and its conjugate $\overline{\mathscr{L}}$:

$$\mathscr{L} = \frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u} \qquad \text{and} \qquad \overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \overline{A}\,\frac{\partial}{\partial u}.$$

In this set up, the non-vanishing property of the Lie bracket:

$$\big[\mathscr{L},\,\overline{\mathscr{L}}\big] = \big(\overline{A}_z + A\,\overline{A}_u - A_{\overline{z}} - \overline{A}\,A_u\big)\,\frac{\partial}{\partial u}$$

at any point of M^3 indicates precisely that M^3 is *Levi nondegenerate* at every point, an assumption that will be held throughout. Since it is slightly better — for convenience reasons — to deal with *real* functions, we introduce the fundamental *Levi factor*:

$$\ell := i\,\big(\overline{A}_z + A\,\overline{A}_u - A_{\overline{z}} - \overline{A}\,A_u\big), \tag{3}$$

so that the reality of $\ell\,\frac{\partial}{\partial u}$ in the first structural Lie bracket relation, viewed again in this abbreviated way $[\mathscr{L},\,\overline{\mathscr{L}}] = -i\,\ell\,\frac{\partial}{\partial u}$, shows now well that the $-i$ mere factor on the right provides the pure imaginarity of the bracket in question:

$$\overline{\big[\mathscr{L},\overline{\mathscr{L}}\big]} = -\,\big[\mathscr{L},\,\overline{\mathscr{L}}\big].$$

For normalization reasons, it is furthermore natural to introduce the auxiliary *real* field:

$$\mathscr{T} := \ell\,\frac{\partial}{\partial u},$$

which is the suitable multiple of $\frac{\partial}{\partial u}$ insuring that the bracket:

$$\big[\mathscr{L},\,\overline{\mathscr{L}}\big] = -i\,\mathscr{T}$$

makes the coefficient-function in front of $\mathscr{T}$ to become a plain constant.

Now, in terms of what will be called the *complex initial frame* on M^3 (written in the following order):

$$\boxed{\begin{aligned} &\mathscr{T} := i\,\big(\overline{A}_z + A\,\overline{A}_u - A_{\overline{z}} - \overline{A}\,A_u\big)\,\frac{\partial}{\partial u}, \\ &\mathscr{L} := \frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u}, \\ &\overline{\mathscr{L}} := \frac{\partial}{\partial \overline{z}} + \overline{A}\,\frac{\partial}{\partial u}, \end{aligned}}$$

it remains to also take up the two remaining — yet uncomputed — brackets.

Simple computations show that we have:

$$\big[\mathscr{T},\,\mathscr{L}\big] = -\,P\,\mathscr{T} \qquad \text{and} \qquad \big[\mathscr{T},\,\overline{\mathscr{L}}\big] = -\,\overline{P}\,\mathscr{T},$$

for a certain (universal) rational function P of the second-order jet $J_{z,\overline{z},u}\big(A,\overline{A}\big)$ given by:

$$P := \frac{\ell_z - \ell A_u + A\ell_u}{\ell}.$$

This function P could be completely expanded in terms of the graphing function φ, for in the notation of [15], one checks that:

$$P = \tfrac{1}{2}\,\Phi_1 - \tfrac{i}{2}\,\Phi_2,$$

with the full, one-page long, expansions of (the numeratorof) Φ_1 and Φ_2 in terms of $J^3_{x,y,u}\varphi$ being provided on page 42 of the extensive arxiv.org version of [15]. Because the computations unavoidably *explode* when one performs them in terms of φ (*cf.* the end of [15]), it is advisable to reset oneself at the level of just P, aiming nevertheless to perform everything which will follow in terms of P, granted that P is explicit with respect to φ.

Notice *passim* that the above two structural bracket relations are conjugate to each other, just because $\overline{\mathscr{T}} = \mathscr{T}$. Furthermore:

Lemma 2.1. *One has the reality condition:*

$$\mathscr{L}(\overline{P}) = \overline{\mathscr{L}}(P).$$

Proof. The already presented expressions simply give:

$$\begin{aligned}
\big[\overline{\mathscr{L}}, \underbrace{[\mathscr{T}, \mathscr{L}]}_{-P\,\mathscr{T}}\big] &= -\overline{\mathscr{L}}(P)\mathscr{T} - P\overline{P}\mathscr{T}, \\
\big[\mathscr{L}, \underbrace{[\overline{\mathscr{L}}, \mathscr{T}]}_{\overline{P}\,\mathscr{T}}\big] &= \mathscr{L}(\overline{P})\mathscr{T} + \overline{P}P,
\end{aligned}$$

and thanks to the Jacobi identity, one obtains:

$$-\overline{\mathscr{L}}(P)\mathscr{T} + \mathscr{L}(\overline{P})\mathscr{T} = \big[\overline{\mathscr{L}}, [\mathscr{T}, \mathscr{L}]\big] + \big[\mathscr{L}, [\overline{\mathscr{L}}, \mathscr{T}]\big] = -\big[\mathscr{T}, \underbrace{[\mathscr{L}, \overline{\mathscr{L}}]}_{\mathscr{T}}\big] = 0,$$

which visibly yields the desired equality $\mathscr{L}(\overline{P}) = \overline{\mathscr{L}}(P)$. □

2.2. **Setting up of an initial Cartan coframe.** All these preliminary normalizations were done in advance to fit dually with a pleasant collection of 1-forms. Indeed, on the natural agreement that the coframe $\{du,\, dz,\, d\overline{z}\}$ is dual to the frame $\{\frac{\partial}{\partial u},\, \frac{\partial}{\partial z},\, \frac{\partial}{\partial \overline{z}}\}$, let us introduce the coframe:

$$\big\{\rho_0,\, \zeta_0,\, \overline{\zeta}_0\big\} \quad \text{which is dual to the frame} \quad \big\{\mathscr{T},\, \mathscr{L},\, \overline{\mathscr{L}}\big\}.$$

that is to say which satisfies by definition:

$$\begin{array}{lll}
\rho_0(\mathscr{T}) = 1 & \rho_0(\mathscr{L}) = 0 & \rho_0(\overline{\mathscr{L}}) = 0, \\
\zeta_0(\mathscr{T}) = 0 & \zeta_0(\mathscr{L}) = 1 & \zeta_0\big(\overline{\mathscr{L}}\big) = 0, \\
\overline{\zeta_0}(\mathscr{T}) = 0 & \overline{\zeta_0}(\mathscr{L}) = 0 & \overline{\zeta_0}\big(\overline{\mathscr{L}}\big) = 1.
\end{array}$$

Using the above expressions of our three vector fields $\mathscr{T}$, $\mathscr{L}$, $\overline{\mathscr{L}}$, we see that the three dual 1-forms have the following simple explicit expressions in terms of the function A — strictly speaking in terms of the defining function φ — :

$$\rho_0 := \frac{du - A\,dz - \overline{A}\,d\overline{z}}{\ell}, \qquad \zeta_0 := dz, \qquad \overline{\zeta}_0 := d\overline{z}. \tag{4}$$

In order to find the exterior differentiations of these initial 1-forms, an application of the so-called Cartan formula $d\omega(\mathscr{X}, \mathscr{Y}) = \mathscr{X}\big(\omega(\mathscr{Y})\big) - \mathscr{Y}\big(\omega(\mathscr{X})\big) - \omega\big([\mathscr{X}, \mathscr{Y}]\big)$ implies that:

Lemma 2.2. *Given a frame $\big\{\mathscr{L}_1, \dots, \mathscr{L}_n\big\}$ on an open subset of $\mathbb{R}^n$ enjoying the Lie structure:*

$$\big[\mathscr{L}_{i_1}, \mathscr{L}_{i_2}\big] = \sum_{k=1}^{n} a_{i_1,i_2}^k \, \mathscr{L}_k \qquad\qquad {\scriptstyle (1 \leqslant i_1 < i_2 \leqslant n)},$$

where the a_{i_1,i_2}^k are functions on $\mathbb{R}^n$, the dual coframe $\{\omega^1, \dots, \omega^n\}$ satisfying by definition $\omega^k\big(\mathscr{L}_i\big) = \delta_i^k$ enjoys a quite similar Darboux-Cartan structure, up to an overall minus sign:

$$d\omega^k = - \sum_{1 \leqslant i_1 < i_2 \leqslant n} a_{i_1,i_2}^k \, \omega^{i_1} \wedge \omega^{i_2} \qquad\qquad {\scriptstyle (k = 1 \cdots n)}.$$

To apply this lemma, it is convenient to consider the auxiliary array:

$$\begin{array}{ccccccccc}
 & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & & \\
 & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & & \\
{[\mathscr{T}, \overline{\mathscr{L}}]} & = & -\overline{P} \cdot \mathscr{T} & + & 0 & + & 0 & & \boxed{\rho_0 \wedge \overline{\zeta_0}} \\
{[\mathscr{T}, \mathscr{L}]} & = & -P \cdot \mathscr{T} & + & 0 & + & 0 & & \boxed{\rho_0 \wedge \zeta_0} \\
{[\overline{\mathscr{L}}, \mathscr{L}]} & = & +i \cdot \mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\zeta_0} \wedge \zeta_0},
\end{array}$$

in which, by reading the three columns, we deduce visually the initial Darboux-Cartan structure in terms of our basic, single function P:

(5)
$$\boxed{\begin{aligned}
& d\rho_0 = P\,\rho_0 \wedge \zeta_0 + \overline{P}\,\rho_0 \wedge \overline{\zeta_0} + i\,\zeta_0 \wedge \overline{\zeta}_0, \\
& d\zeta_0 = 0, \\
& d\overline{\zeta_0} = 0.
\end{aligned}}$$

2.3. **Complex structure on the kernel of the contact 1-form ρ_0.** We end up this preparative part by a thoughtful summary which will offer the natural geometric meaning of ρ_0. The defining equation of M^3 may be understood as:

$$r = 0 \qquad \text{with} \qquad r = r(z, \overline{z}, u, v) := -v + \varphi(z, \overline{z}, u).$$

Given any function $G = G(z, \overline{z}, w, \overline{w})$, one classically defines its $(1,0)$ and $(0,1)$ differentials respectively by:

$$\partial G := G_z \, dz + G_w \, dw \qquad \text{and} \qquad \overline{\partial} G := G_{\overline{z}} \, d\overline{z} + G_{\overline{w}} \, d\overline{w},$$

and one easily checks that its complete real differential:

$$dG = G_x \, dx + G_y \, dy + G_u \, du + G_v \, dv$$

is the plain sum of these two holomorphic and antiholomorphic differentials:

$$dG = \partial G + \overline{\partial} G.$$

Lemma 2.3. *With* $r = 0$ *being* any *real defining equation for a* $\mathscr{C}^1$ *hypersurface* $M^3 \subset \mathbb{C}^2$*, the restriction to* M^3 *of the* $(1,0)$ *form* $i\,\partial r$*, namely:*

$$\varrho := i\,\partial r\big|_M$$

is a real form on M^3*:*

$$\varrho = \overline{\varrho}.$$

Moreover, at every point $p \in M$*, the real kernel of* ϱ *in* T_pM *identifies with the complex tangent bundle at* p*:*

$$\big\{X_p \in T_pM\colon\ \varrho(X_p) = 0\big\} = T_p^cM,$$

while its kernel in the complexified tangent bundle $\mathbb{C} \otimes T_pM$ *identifies with* $\mathbb{C} \otimes T_p^cM$*:*

$$\big\{\mathscr{X}_p \in \mathbb{C} \otimes T_pM\colon\ \varrho(\mathscr{X}_p) = 0\big\} = \mathbb{C} \otimes T_p^cM = T_p^{1,0}M \oplus T_p^{0,1}M.$$

Proof. For the first part of the assertion, since $r|_{M^3} \equiv 0$, then on restriction to M^3 we also have $dr = 0$ which means $\partial r = -\overline{\partial} r$. Hence the i factor in ϱ in front of ∂r makes it real. For the rest, *see* [11], page 25. □

To go into this lemma in detail, with $r(z, \overline{z}, u, v) = -v + \varphi(z, \overline{z}, u)$ and with $w = u + iv$, we have:

$$dw = du + i\,dv = du + i\,d\varphi(z, \overline{z}, u) = du + i\big(\varphi_z\,dz + \varphi_{\overline{z}}\,d\overline{z} + \varphi_u\,du\big),$$

and hence the expression of ϱ can be expressed in terms of the functions φ:

(6)
$$\begin{aligned}
\varrho = i\,\partial r|_{M^3} &= i\big(r_z\,dz + r_w\,dw\big)|_{M^3} \\
&= i\big(\varphi_z\,dz + (\tfrac{1}{2}\,\varphi_u + \tfrac{i}{2})\,dw\big) \\
&= i\big(\varphi_z\,dz + (\tfrac{1}{2}\,\varphi_u + \tfrac{i}{2})(du + i\,\varphi_z\,dz + i\,\varphi_{\overline{z}}\,d\overline{z} + i\,\varphi_u\,du)\big) \\
&= \big(-\tfrac{1}{2} - \tfrac{1}{2}\,(\varphi_u)^2\big)\,du + \big(\tfrac{i}{2}\,\varphi_z - \tfrac{1}{2}\,\varphi_z\,\varphi_u\big)\,dz + \big(-\tfrac{i}{2}\,\varphi_{\overline{z}} - \tfrac{1}{2}\,\varphi_{\overline{z}}\,\varphi_u\big)\,d\overline{z}.
\end{aligned}$$

Furthermore, a plain computations show that (*see* (2), (3) and (4) for the expressions):

$$\rho_0 = -\frac{1}{\ell}\,\frac{2}{1 + \varphi_u^2}\,\varrho. \tag{7}$$

Then, non-vanishing property of the Levi factor ℓ also implies the equality:

$$\mathrm{Ker}(\varrho) = \mathrm{Ker}(\rho_0).$$

2.4. **Differential facts about CR equivalences.** Now, we explain how one may launch Cartan's method in the case under study, namely for deformations of the Heisenberg sphere:

$$w - \overline{w} = 2i\,z\overline{z}, \tag{8}$$

that are *geometry-preserving* in the sense that Levi nondegeneracy is preserved.

Consider therefore two Levi-nondegenerate real hypersurfaces of class $\mathscr{C}^6$, represented in two systems of coordinates (z, w) and (z', w') as graphs:

$$M^3\colon\quad 0 = -v + \varphi(z, \overline{z}, u) \qquad \text{and} \qquad M'^3\colon\quad 0 = -v' + \varphi'(z', \overline{z}', u'),$$

for two certain functions, normalized in advance so that $\varphi := z\overline{z} + \mathrm{O}(3)$ and $\varphi' := z'\overline{z}' + \mathrm{O}(3)$. The general problem is to discover *when*, and if so *how*, the two CR hypersurfaces are equivalent through a local ambient biholomorphic map:

$$(z,w) \longmapsto (z',w') = \big(z'(z,w),\ w'(z,w)\big)$$

of $\mathbb{C}^2$. This is nothing else than saying that such a map should send any point of M^3 to some determinate point of M'^3. In other words, one should have $v' = \varphi'(z',\overline{z}',u')$ as soon as $v = \varphi(z,\overline{z},u)$.

Then a well known simple fact (Lemma 1.2.3 page 47 of [21]) insures that M is sent to M' if and only if there exists a *real-valued* function $a = a(z,w)$ defined in a neighborhood of the origin in $\mathbb{C}^2$ so that:

$$-v' + \varphi'(z',\overline{z}',u')\big|_{(z',w')=(z'(z,w),w'(z,w))} \equiv a(z,,\overline{z},w,\overline{w}) \cdot \big(-v + \varphi(z,\overline{z},u)\big),$$

identically as functions of the *four* real coordinates of $\mathbb{C}^2$. For easier reading, we shall drop the mention of this *pullback* and simply write down:

$$-v' + \varphi'(z',\overline{z}',u') = a\,\big(-v + \varphi(z,\overline{z},u)\big),$$

or even in a shorter way: $r' = a\,r$. We now clearly see that $r = 0$ implies $r' = 0$, namely that points of M^3 are sent to points of M'^3. But now, the two fundamental 1-forms $\varrho = i\,\partial r\big|_M$ and $\varrho' = i\,\partial r'\big|_{M'}$ in the two spaces happen to be *real multiples of each other*:

$$i\,\partial r'\big|_{M'} = a\,i\,\partial r\big|_M + \underline{r\,i\,\partial a\big|_M}_{\circ},$$

through the same function a.

Of course such a function a highly depends on the equivalence $(z,w) \to (z',w')$ between M^3 and M'^3, when it exists, but the idea of Cartan is to consider it as some *unknown*. Taking the relationship (7) into account, the already obtained equality $\varrho' = a\,\varrho$ can be slightly adjusted (with same notation for a new function a) into the form:

$$\rho' := a \cdot \rho$$

for some unknown real-valued function $a := a(z,\overline{z},u)$.

2.5. **Associated ambiguity matrix.** Next, let us construct the associated *ambiguity matrix* which encodes holomorphic equivalence of two hypersurfaces M^3 and M'^3, recently equipped with two coframes:

$$\big\{\rho_0, dz, d\overline{z}\big\} \qquad \text{and} \qquad \big\{\rho_0', dz', d\overline{z'}\big\}.$$

On restriction to M^3, we have:

$$z' = z'\big(z,\ u + i\,\varphi(z,\overline{z},u)\big),$$

whence differentiation using the general formula $dg = g_z dz + g_{\overline{z}}\,d\overline{z} + g_u\,du$ gives (*see* (6)):

$$\begin{aligned} dz' &= \big(z_z' + i\,z_w'\,\varphi_z\big)\,dz + \big(i\,z_w'\,\varphi_{\overline{z}}\big)\,d\overline{z} + \big(i\,z_w'\,\varphi_u + z_w'\big)\,du \\ &= \big(z_z' + i\,z_w'\,\varphi_z\big)\,dz + z_w'\big\{\underline{i\,\varphi_{\overline{z}}\,d\overline{z} + (i\,\varphi_u + 1)\,du}\big\}. \end{aligned} \tag{9}$$

On the other hand, multiplying by some (innocuous) complex multiple the fundamental 1-form $\varrho = i\,\partial r\big|_M$, we also have:

$$\frac{-2\,(1+i\,\varphi_u)}{1+(\varphi_u)^2}\,\varrho = (1+i\,\varphi_u)\,du + i\,\varphi_{\overline{z}}\,d\overline{z} + \frac{\varphi_z(-i+\varphi_u)}{1-i\,\varphi_u}\,dz,$$

which enables us to substitute the (underlined) 1-form that we left in braces after z'_w just above (we also replace $\varrho = -\frac{1}{2}\,\ell(1+(\varphi_u)^2)\,\rho_0$ in terms of ρ_0, *see* (7)) as:

$$i\,\varphi_{\overline{z}}\,d\overline{z} + (i\,\varphi_u+1)\,du = -\frac{2\,(1+i\,\varphi_u)}{1+(\varphi_u)^2}\,\varrho - \frac{\varphi_z(-i+\varphi_u)}{1-i\,\varphi_u}\,dz = \ell\,(1+i\,\varphi_u)\,\rho_0 - \frac{\varphi_z(-i+\varphi_u)}{1-i\,\varphi_u}\,dz.$$

This implies from (9) that dz' is a linear combination — with some complicated coefficients — of dz and of ρ, *without $d\overline{z}$ component*:

$$dz' = \Big(\underbrace{z'_w\,\ell\,(1+i\,\varphi_u))}_{=:b(z,\overline{z},v)}\Big)\,\rho_0 + \Big(\underbrace{z'_z + i\,z'_w\,\varphi_z - z'_w\,\frac{\varphi_z(-i+\varphi_u)}{1-i\,\varphi_u}}_{=:c(z,\overline{z},v)}\Big)\,dz.$$

We thus have obtained:

Proposition 2.4. *Two local $\mathscr{C}^1$ real hypersurfaces M^3 and M'^3 of $\mathbb{C}^2$ are equivalent through some biholomorphism whenever their two corresponding fundamental coframes:*

$$\big\{\rho_0,\,\zeta_0 = dz,\,\overline{\zeta}_0 = d\overline{z}_0\big\} \qquad \text{and} \qquad \big\{\rho'_0,\,\zeta'_0 = dz',\,\overline{\zeta'_0} = d\overline{z}'_0\big\}$$

are mapped one to another by means of a certain matrix of functions:

$$\begin{pmatrix}\rho'_0\\ \zeta'_0\\ \overline{\zeta'_0}\end{pmatrix} = \begin{pmatrix} a & 0 & 0\\ b & c & 0\\ \overline{b} & 0 & \overline{c}\end{pmatrix}\begin{pmatrix}\rho_0\\ \zeta_0\\ \overline{\zeta}_0\end{pmatrix},$$

in which $a := a(z,\overline{z},v)$ is a real-valued function on M^3, and where $b := b(z,\overline{z},v)$ and $c := c(z,\overline{z},v)$ are both complex-valued. □

2.6. **The related structure group.** As we saw, when a CR equivalence exists, the functions a, b and c depend — in a somewhat complicated way — upon the CR equivalence, whose existence is under question! The gist of Cartan's method is to consider these functions *as new unknowns*, hence to add them as extra *group variables*. So we consider the subgroup of matrices inside $\mathsf{GL}_3(\mathbb{C})$:

$$\begin{pmatrix} \mathsf{a} & 0 & 0\\ \mathsf{b} & \mathsf{c} & 0\\ \overline{\mathsf{b}} & 0 & \overline{\mathsf{c}}\end{pmatrix},$$

where now $\mathsf{a} \in \mathbb{R}$, $\mathsf{b} \in \mathbb{C}$, $\mathsf{c} \in \mathbb{C}$ are arbitrary parameters and we consider the so-called *lifted coframe* on the eight-dimensional space $(z,\overline{z},u,\mathsf{a},\mathsf{b},\overline{\mathsf{b}},\mathsf{c},\overline{\mathsf{c}})$:

$$\begin{pmatrix}\rho\\ \zeta\\ \overline{\zeta}\end{pmatrix} := \begin{pmatrix} \mathsf{a} & 0 & 0\\ \mathsf{b} & \mathsf{c} & 0\\ \overline{\mathsf{b}} & 0 & \overline{\mathsf{c}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \zeta_0\\ \overline{\zeta}_0\end{pmatrix},$$

that is to say:

$$\rho = \mathsf{a}\,\rho_0,$$
$$\zeta = \mathsf{b}\,\rho_0 + \mathsf{c}\,\zeta_0,$$
$$\overline{\zeta} = \overline{\mathsf{b}}\,\rho_0 + \overline{\mathsf{c}}\,\overline{\zeta}_0.$$

Of course, the 1-form ρ is real and the $\overline{\zeta}$ is the conjugate of ζ.

So far, we have provided the necessary data for launching the Cartan algorithm of equivalence. Next, we have to perform *normalization*, *absorption* and *prolongation*.

3. Absorbtion and Normalization

Associated to the equivalence problem for real hypersurface $M^3 \subset \mathbb{C}^2$, we set up the structure matrix group:

$$G := \left\{ g := \begin{pmatrix} \mathsf{a} & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 \\ \overline{\mathsf{b}} & 0 & \overline{\mathsf{c}} \end{pmatrix}, \quad \mathsf{a} \in \mathbb{R}, \quad \mathsf{b}, \mathsf{c} \in \mathbb{C} \right\}.$$

The lifted coframe writes out as:

$$\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix} := g. \begin{pmatrix} \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix} = \begin{pmatrix} \mathsf{a}\rho_0 \\ \mathsf{b}\rho_0 + \mathsf{c}\zeta_0 \\ \overline{\mathsf{b}}\rho_0 + \overline{\mathsf{c}}\overline{\zeta}_0 \end{pmatrix}. \tag{10}$$

Applying the differential operator d to these three equations and next substituting the expressions of $d\rho_0, d\zeta_0, d\overline{\zeta}_0$, presented in (5), give:

$$\begin{cases} d\rho = d\mathsf{a} \wedge \rho_0 + \mathsf{a}i\,\zeta_0 \wedge \overline{\zeta}_0 + \mathsf{a}\overline{P}\,\rho_0 \wedge \overline{\zeta}_0 + \mathsf{a}P\,\rho_0 \wedge \zeta_0 \\ d\zeta = d\mathsf{b} \wedge \rho_0 + d\mathsf{c} \wedge \zeta_0 + \mathsf{b}i\,\zeta_0 \wedge \overline{\zeta}_0 + \mathsf{b}\overline{P}\,\rho_0 \wedge \overline{\zeta}_0 + \mathsf{b}P\,\rho_0 \wedge \zeta_0 \\ d\overline{\zeta} = d\overline{\mathsf{b}} \wedge \rho_0 + d\overline{\mathsf{c}} \wedge \overline{\zeta}_0 + \overline{\mathsf{b}}i\,\zeta_0 \wedge \overline{\zeta}_0 + \overline{\mathsf{b}P}\,\rho_0 \wedge \overline{\zeta}_0 + \overline{\mathsf{b}}P\,\rho_0 \wedge \zeta_0, \end{cases}$$

or equivalently in matrix notation:

$$d\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} d\mathsf{a} & 0 & 0 \\ d\mathsf{b} & d\mathsf{c} & 0 \\ d\overline{\mathsf{b}} & 0 & d\overline{\mathsf{c}} \end{pmatrix}}_{dg} \wedge \begin{pmatrix} \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix} + \begin{pmatrix} \mathsf{a}\,P & \mathsf{a}\,\overline{P} & \mathsf{a}\,i \\ \mathsf{b}\,P & \mathsf{b}\,\overline{P} & \mathsf{b}\,i \\ \overline{\mathsf{b}}\,P & \overline{\mathsf{b}}\,\overline{P} & \overline{\mathsf{b}}i \end{pmatrix} \begin{pmatrix} \rho_0 \wedge \zeta_0 \\ \rho_0 \wedge \overline{\zeta}_0 \\ \zeta_0 \wedge \overline{\zeta}_0 \end{pmatrix}. \tag{11}$$

On the other hand, multiplying both sides of (10) by the inverse matrix:

$$g^{-1} = \begin{pmatrix} \frac{1}{\mathsf{a}} & 0 & 0 \\ -\frac{\mathsf{b}}{\mathsf{ac}} & \frac{1}{\mathsf{c}} & 0 \\ -\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{c}}} & 0 & \frac{1}{\overline{\mathsf{c}}} \end{pmatrix}$$

yields the expressions of $\rho_0, \zeta_0, \overline{\zeta}_0$ in terms of $\rho, \zeta, \overline{\zeta}$:

$$\begin{aligned} \rho_0 &= \frac{1}{\mathsf{a}}\rho \\ \zeta_0 &= -\frac{\mathsf{b}}{\mathsf{ac}}\rho + \frac{1}{\mathsf{c}}\zeta \\ \overline{\zeta}_0 &= -\frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{c}}}\rho + \frac{1}{\overline{\mathsf{c}}}\overline{\zeta}. \end{aligned} \tag{12}$$

We may then compute the three exterior products between these basic 1-forms:

$$
\tag{13}
\begin{cases}
\rho_0 \wedge \zeta_0 = \frac{1}{\mathsf{ac}}\rho \wedge \zeta \\[1em]
\rho_0 \wedge \overline{\zeta}_0 = \frac{1}{\mathsf{a}\overline{\mathsf{c}}}\rho \wedge \overline{\zeta} \\[1em]
\zeta_0 \wedge \overline{\zeta}_0 = \frac{\overline{\mathsf{b}}}{\mathsf{ac}\overline{\mathsf{c}}}\rho \wedge \zeta - \frac{\mathsf{b}}{\mathsf{ac}\overline{\mathsf{c}}}\rho \wedge \overline{\zeta} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\zeta \wedge \overline{\zeta}.
\end{cases}
$$

In addition, one has to replace the first part $dg \wedge (\rho_0, \zeta_0, \overline{\zeta}_0)^t$ in (11) by:

$$
\tag{14}
\underbrace{dg \cdot g^{-1}}_{\omega_{MC}} \wedge \underbrace{g \,.\begin{pmatrix} \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix}}_{(\rho,\zeta,\overline{\zeta})^t},
$$

and finally we obtain from (11), the exterior differentiations of the lifted 1-forms $\rho, \zeta, \overline{\zeta}$:

$$
\tag{15}
d\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 \\ \beta & \alpha & 0 \\ \overline{\beta} & 0 & \overline{\alpha} \end{pmatrix}}_{\omega_{MC}} \wedge \begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix} + \begin{pmatrix} U_1\, \rho \wedge \zeta + \overline{U}_1\, \rho \wedge \overline{\zeta} + U_2 \zeta \wedge \overline{\zeta} \\ V_1\, \rho \wedge \zeta + V_2\, \rho \wedge \overline{\zeta} + V_3\, \zeta \wedge \overline{\zeta} \\ \overline{V}_2\, \rho \wedge \zeta + \overline{V}_1\, \rho \wedge \overline{\zeta} - \overline{V}_3\, \zeta \wedge \overline{\zeta} \end{pmatrix},
$$

which incorporate the following *torsion coefficients*:

$$
\begin{aligned}
U_1 &:= \tfrac{P\,\overline{\mathsf{c}}+\overline{\mathsf{b}}i}{\mathsf{c}\overline{\mathsf{c}}} & U_2 &:= \tfrac{\mathsf{a}i}{\mathsf{c}\overline{\mathsf{c}}} \\
V_1 &:= \tfrac{P\,\mathsf{b}\overline{\mathsf{c}}+\overline{\mathsf{b}}\mathsf{b}i}{\mathsf{a}\mathsf{c}\overline{\mathsf{c}}} & V_2 &:= \tfrac{\overline{P}\,\mathsf{bc}-\mathsf{b}^2 i}{\mathsf{a}\mathsf{c}\overline{\mathsf{c}}} & V_3 &:= \tfrac{\mathsf{b}i}{\mathsf{c}\overline{\mathsf{c}}},
\end{aligned}
$$

and in which the three plain *Maurer-Cartan* 1-forms are:

$$
\alpha := \frac{d\mathsf{c}}{\mathsf{c}}, \quad \beta := \frac{d\mathsf{b}}{\mathsf{a}} - \frac{\mathsf{b}\, d\mathsf{c}}{\mathsf{ac}}, \quad \gamma := \frac{d\mathsf{a}}{\mathsf{a}}.
$$

Here the obtained equations are called the *structure equations* of the problem and moreover the appearing matrix ω_{MC} is the so-called *Maurer-Cartan form* of G.

3.1. **Absorbtion and normalization.** One of the most essential parts of the Cartan (equivalence) algorithm is the absorbtion-normalization step, which, generally speaking, is expressed as follows.

Observation 3.1. *(see* [14]*) Let* $\Theta := \{\theta^1, \ldots, \theta^n\}$ *be a lifted coframe associated to an equivalence problem having structure equations*:

$$
d\theta^i = \sum_{k=1}^{n} \left(\sum_{s=1}^{r} a^i_{ks}\, \alpha^s + \sum_{j=1}^{k-1} T^i_{jk}\, \theta^j \right) \wedge \theta^k \qquad {\scriptstyle (i=1\cdots n)}.
$$

Then, one can replace each Maurer-Cartan form α^s and each torsion coefficient T^i_{jk} with:

$$\tag{16} \boxed{\begin{aligned} &\alpha^s \longmapsto \alpha^s + \sum_{j=1}^n z^s_j\,\theta^j && (s=1\cdots r),\\ &T^i_{jk} \longmapsto T^i_{jk} + \sum_{s=1}^r \big(a^i_{js}\,z^s_k - a^i_{ks}\,z^s_j\big) && (i=1\cdots n;\ 1\leqslant j<k\leqslant n),\end{aligned}}$$

for some arbitrary functions $z^\bullet_\bullet$ on the base manifold M. □

Then one does such a replacement so as to annihilate as many torsion coefficients as possible, by some appropriate determinations of the functions $z^\bullet_\bullet$.

Thus, let us perform the following replacements:

$$\tag{17} \begin{aligned} \alpha &\mapsto \alpha + p_1\,\rho + q_1\,\zeta + r_1\,\overline{\zeta},\\ \beta &\mapsto \beta + p_2\,\rho + q_2\,\zeta + r_2\,\overline{\zeta},\\ \gamma &\mapsto \gamma + p_3\,\rho + q_3\,\zeta + r_3\,\overline{\zeta}.\end{aligned}$$

These substitutions convert the structure equations (15) into the form — from now on and for brevity, we drop presenting the structure equation $d\overline{\zeta}$ since it is just the conjugation of $d\zeta$:

$$\begin{aligned} d\rho &= \gamma\wedge\rho + (U_1 - q_3)\,\rho\wedge\zeta + (\overline{U}_1 - r_3)\,\rho\wedge\overline{\zeta} + U_2\,\zeta\wedge\overline{\zeta},\\ d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta + (V_1 - q_2 + p_1)\,\rho\wedge\zeta + (V_2 - r_2)\,\rho\wedge\overline{\zeta} + (V_3 - r_1)\,\zeta\wedge\overline{\zeta}.\end{aligned}$$

Visually, one sees that by some appropriate determinations of p_i, q_i, r_i, one can annihilate all the (so modified) torsion coefficients, except just one, namely U_2 in front of $\zeta\wedge\overline{\zeta}$ at the end of the first line. Consequently, this torsion coefficient U_2 is *essential*, and the general theory ([18]) shows that U_2 (potentially) provides a *normalization* of some group parameter, and here because U_2 is so simple, normalizing it to be $U_2 := i$ provides the simple group parameter reduction:

$$\boxed{\mathsf{a} := \mathsf{c}\,\overline{\mathsf{c}}.}$$

This then replaces the Maurer-Cartan form $\gamma = \frac{d\mathsf{a}}{\mathsf{a}}$ by $\alpha + \overline{\alpha}$ and transforms the structure equations (15) into the form:

$$\tag{18} \begin{aligned} d\rho &= (\alpha + \overline{\alpha})\wedge\rho + U_1\,\rho\wedge\zeta + \overline{U}_1\,\rho\wedge\overline{\zeta} + i\,\zeta\wedge\overline{\zeta},\\ d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta + V_1\,\rho\wedge\zeta + V_2\,\rho\wedge\overline{\zeta} + V_3\,\zeta\wedge\overline{\zeta},\end{aligned}$$

with new torsion coefficients:

$$\begin{aligned} &U_1 := \tfrac{P\,\overline{\mathsf{c}} + \overline{\mathsf{b}}i}{\mathsf{c}\overline{\mathsf{c}}}\\ &V_1 := \tfrac{P\,\mathsf{b}\overline{\mathsf{c}} + \overline{\mathsf{b}}\mathsf{b}i}{\mathsf{c}^2\overline{\mathsf{c}}^2} \qquad V_2 := \tfrac{\overline{P}\,\mathsf{b}\mathsf{c} - \mathsf{b}^2 i}{\mathsf{c}^2\overline{\mathsf{c}}^2} \qquad V_3 := \tfrac{\mathsf{b}i}{\mathsf{c}\overline{\mathsf{c}}}\end{aligned}$$

and with the new Maurer-Cartan 1-forms:

$$\alpha := \frac{d\mathsf{c}}{\mathsf{c}} \qquad \beta := \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}}.$$

Now, let us try again a second absorbtion-normalization procedure. Doing similar replacements:

$$\alpha \mapsto \alpha + p_1\,\rho + q_1\,\zeta + r_1\,\overline{\zeta},$$
$$\beta \mapsto \beta + p_2\,\rho + q_2\,\zeta + r_2\,\overline{\zeta},$$

one obtains:
(19)

$$d\rho = (\alpha + \overline{\alpha}) \wedge \rho + (U_1 - q_1 - \overline{r}_1)\rho \wedge \zeta + (\overline{U}_1 - r_1 - \overline{q}_1)\rho \wedge \overline{\zeta} + i\,\zeta \wedge \overline{\zeta},$$
$$d\zeta = \beta \wedge \rho + \alpha \wedge \zeta + (V_1 - q_2 + p_1)\rho \wedge \zeta + (V_2 - r_2)\rho \wedge \overline{\zeta} + (V_3 - r_1)\zeta \wedge \overline{\zeta}.$$

Visually, one can annihilate all the (so modified) torsion coefficients by choosing:

$$q_1 := U_1 - \overline{V}_3, \qquad r_1 := V_3,$$
$$q_2 := V_1 + p_1, \qquad r_2 := V_2,$$

while the two remaining functions:

$$p_1 =: \mathsf{s}, \qquad p_2 =: \mathsf{r}$$

can yet be chosen *arbitrarily*.

Chosing first these last two functions to be 0, and coming back to the explicit expressions of U_1, V_1, V_2, V_3, we see by introducing the following two *modified Maurer Cartan forms*:

$$\alpha_0 = \frac{d\mathsf{c}}{\mathsf{c}} - \frac{P\overline{\mathsf{c}} + 2\,i\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\zeta - \frac{i\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\overline{\zeta},$$
$$\beta_0 = \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{P\mathsf{b}\overline{\mathsf{c}} + i\,\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\zeta - \frac{\overline{P}\mathsf{b}\mathsf{c} - i\,\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2}\overline{\zeta}, \tag{20}$$

that the whole torsion is absorbed so that the structure equations receive the very simple form:

$$d\rho = (\alpha_0 + \overline{\alpha}_0) \wedge \rho + i\,\zeta \wedge \overline{\zeta},$$
$$d\zeta = \beta_0 \wedge \rho + \alpha \wedge \zeta. \tag{21}$$

At this stage, no torsion coefficient can be used anymore to reduce the structure group.

In fact, one verifies that the two complex parameters r and s and their conjugations are precisely the *free variables* in the absorption equations, and consequently, according to the general procedure, one has to *prolong* the equivalence problem.

4. Prolongation of the equivalence problem

4.1. **Prolongation procedure.** If one therefore encodes the general remaining ambiguity in the choice of α_0 and β_0 by setting:

$$\alpha := \alpha_0 + \mathsf{s}\rho,$$
$$\beta := \beta_0 + \mathsf{r}\,\rho + \mathsf{s}\,\zeta, \tag{22}$$

one will still have that the absorbed equations look the same (without lower index '$_0$'):

$$\boxed{\begin{aligned} d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \beta \wedge \rho + \alpha \wedge \zeta. \end{aligned}} \tag{23}$$

At this moment, one has to launch the prolongation procedure. This part of Cartan's algorithm relies on the following general result (*see* [18], page 395 Proposition 12.13):

Proposition 4.1. *Let Θ and Θ' be lifted coframes of an equivalence problem which admits a non-involutive system of structure equations and which has a positive degree of indeterminancy. Let Λ and Λ' be the modified Maurer-Cartan forms after the last absorbtion-normalization step. Then, there exists a diffeomorphism $\Phi : M \longrightarrow M'$ mapping Θ to Θ' for some choice of the group parameters if and only if there is a diffeomorphism $\Psi : M \times G \longrightarrow M' \times G'$ mapping the coframe (Θ, Λ) to (Θ', Λ') for some choice of the prolonged group parameters.*

This permits us to change our concentration on the original equivalence problem of the three dimensional hypersurfaces M^3 equipped with the lifted coframes $\{\rho, \zeta, \overline{\zeta}\}$ to that, along the same lines, of the *prolonged manifolds* $M^{\mathsf{pr}} := M^3 \times G$ with the lifted coframe — living on the product $M^{\mathsf{pr}} \times G^{\mathsf{pr}} = (M^3 \times G) \times G^{\mathsf{pr}}$ — of the seven 1-forms $\rho, \zeta, \overline{\zeta}, \psi, \varphi, \overline{\psi}, \overline{\varphi}$, defined as follows:

$$\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \\ \alpha \\ \beta \\ \overline{\alpha} \\ \overline{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathsf{s} & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathsf{r} & \mathsf{s} & 0 & 0 & 1 & 0 & 0 \\ \overline{\mathsf{s}} & 0 & 0 & 0 & 0 & 1 & 0 \\ \overline{\mathsf{r}} & 0 & \overline{\mathsf{s}} & 0 & 0 & 0 & 1 \end{pmatrix}}_{g_{\mathsf{pr}} \in G^{\mathsf{pr}}} \cdot \begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \\ \alpha_0 \\ \beta_0 \\ \overline{\alpha}_0 \\ \overline{\beta}_0 \end{pmatrix}, \tag{24}$$

with the new structure group G^{pr}, a subgroup of $\mathsf{GL}_{3+4}(\mathbb{C}) = \mathsf{GL}_7(\mathbb{C})$ constituted by the *prolonged group parameters* r, s and their conjugates.

Remark 4.2. This prolonged group (24) resembles much the equations (3.3) on page 7 of the paper [10], devoted to the equivalence problem for second order ordinary differential equations. In fact, there exists for known reasons (*cf. e.g.* [16, 12]), a certain *transfert principle* showing that these two seemingly different equivalence problems will follow fairly the same lines of resolution. Our main goal here is to go beyond the so-called — usually less costful — *non-parametric* approach and to perform all computations effectively in terms of the single function P, hence in terms of the graphing function $\varphi(z, \overline{z}, u)$ of our hypersurface. In fact, with our choice $\{\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}\}$ of an initial frame for TM^3, which is explicit in terms of φ, we deviate from the common approaches.

With the obtained four supplementary 1-forms α, β, $\overline{\alpha}$, $\overline{\beta}$, we can now start the first loop of absorbtion and normalization on the 7-dimensional prolonged space.

Letting a group element $g_{\sf pr} \in G^{\sf pr}$ be in (24) and abbreviating:

$$\Omega_0 := \big(\rho, \zeta, \overline{\zeta}, \alpha_0, \beta_0, \overline{\alpha}_0, \overline{\beta}_0\big), \qquad\qquad \Omega := \big(\rho, \zeta, \overline{\zeta}, \alpha, \beta, \overline{\alpha}, \overline{\beta}\big)$$

the first simple computation shows that the associated structure equations:

$$d\Omega = \big(dg_{\sf pr} \cdot g_{\sf pr}^{-1}\big) \wedge \Omega + g_{\sf pr} \cdot d\Omega_0,$$

read as:

$$d\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \\ \alpha \\ \beta \\ \overline{\alpha} \\ \overline{\beta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 & 0 \\ \overline{\delta} & 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{\gamma} & 0 & \overline{\delta} & 0 & 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \\ \alpha \\ \beta \\ \overline{\alpha} \\ \overline{\beta} \end{pmatrix} + \begin{pmatrix} d\rho \\ d\zeta \\ d\overline{\zeta} \\ {\sf s}\, d\rho + d\alpha_0 \\ {\sf r}\, d\rho + {\sf s}\, d\zeta + d\beta_0 \\ \overline{\sf s}\, d\rho + d\overline{\alpha}_0 \\ \overline{\sf r}\, d\rho + \overline{\sf s}\, d\overline{\zeta} + d\overline{\beta}_0 \end{pmatrix} \tag{25}$$

for two new basic Maurer-Cartan 1-forms:

$$\gamma := d{\sf r}, \qquad\qquad \delta := d{\sf s}.$$

To explicitly find the torsion coefficients which should come from the last four rows of the rightmost 7×1 matrix, one needs to express the exterior derivations of α_0 and β_0 in terms of the lifted 1-forms, and this task is costful, computationally speaking. Instead of performing this directly, let us at first employ a well-know indirect tool (*cf.* [9, 18]) which temporarily bypasses this computational obstacle and has the virtue of enabling one to better predict the way the final structure equations will look like after absorption.

Cartan's (elementary) Lemma. *Let* $\{\omega^1, \dots, \omega^k\}$ *be a set of linearly independent local 1-forms on some manifold. Then,* k *arbitrary 1-forms* $\theta^1, \dots, \theta^k$ *satisfy* $\sum_{i=1}^k \theta^i \wedge \omega^i = 0$ *if and only if they express* $\theta^i = \sum_{j=1}^k A_j^i\, \omega^j$ *for some symmetric matrix of local functions with* $A_j^i = A_i^j$. □

The truth here is that one intentionally leaves aside the question of how these A_j^i could be expressed in terms of $\theta^1, \dots, \theta^k, \omega^1, \dots, \omega^k$.

Now, using the standard differentiation formula for the exterior product of two 1-forms λ and μ (mind the minus sign!):

$$d\big(\lambda \wedge \mu\big) = d\lambda \wedge \mu - \lambda \wedge d\mu,$$

the differentiation of the two equations (23) gives:

$$\begin{cases} d^2\rho = 0 \equiv \big(\underbrace{(d\alpha + 2\,i\,\overline{\beta} \wedge \zeta + i\,\beta \wedge \overline{\zeta}) + (d\overline{\alpha} - 2\,i\,\beta \wedge \overline{\zeta} - i\,\overline{\beta} \wedge \zeta)}_{=:\Xi_1}\big) \wedge \rho, \\ d^2\zeta = 0 \equiv (\underbrace{d\alpha + 2\,i\,\overline{\beta} \wedge \zeta + i\,\beta \wedge \overline{\zeta}}_{=:\Xi_2}) \wedge \zeta + (\underbrace{d\beta - \beta \wedge \overline{\alpha}}_{=:\Xi_3}) \wedge \rho, \end{cases} \tag{26}$$

noticing as a 'trick' that the redundant term $2i\,\overline{\beta} \wedge \zeta$ in Ξ_1 helps us to insure the reality relation:

$$\Xi_1 = \Xi_2 + \overline{\Xi}_2,$$

which will be useful for our next:

Proposition 4.3. *The exterior differentials of the new prolonged lifted* 1*-forms* α *and* β *can be read as:*

$$
\tag{27}
\begin{cases}
d\alpha = \delta^{\mathsf{modified}} \wedge \rho + \\
\qquad +2\,i\,\zeta \wedge \overline{\beta} + i\,\overline{\zeta} \wedge \beta + W\,\zeta \wedge \overline{\zeta} \\
\\
d\beta = \gamma^{\mathsf{modified}} \wedge \rho + \delta^{\mathsf{modified}} \wedge \zeta + \\
\qquad +\beta \wedge \overline{\alpha}
\end{cases}
$$

for a certain torsion coefficient W *which is* real, *and for some two modified Maurer-Cartan* 1*-forms* $\delta^{\mathsf{modified}}$ *and* $\gamma^{\mathsf{modified}}$.

Proof. Applying Cartan's Lemma 4.1 to (26) brings the following expressions of Ξ_1, Ξ_2, Ξ_3 for some three 1-forms $\mathscr{A}_{ij}, \mathscr{B}_{ij}, \mathscr{C}$:

$$
\begin{aligned}
\Xi_1 &= -\mathscr{C} \wedge \rho, \\
\Xi_2 &= \mathscr{A}_{11} \wedge \zeta + \mathscr{A}_{12} \wedge \rho, \qquad\qquad \Xi_3 = \mathscr{A}_{12} \wedge \zeta + \mathscr{A}_{22} \wedge \rho.
\end{aligned}
$$

The relation $\Xi_2 + \overline{\Xi}_2 - \Xi_1 = 0$ we 'trickily' insured then reads as:

$$
\mathscr{A}_{11} \wedge \zeta + \mathscr{B}_{11} \wedge \overline{\zeta} + \big(\mathscr{A}_{12} + \overline{\mathscr{A}}_{12} + \mathscr{C}\big) \wedge \rho \equiv 0.
$$

Again, a further application of Cartan's Lemma yields the (non-explicit) expressions:

$$
\begin{bmatrix}
\mathscr{A}_{11} = R_{11}\zeta + R_{12}\overline{\zeta} + R_{13}\rho, \\
\\
\mathscr{B}_{11} = R_{12}\zeta + R_{22}\overline{\zeta} + R_{23}\rho, \\
\\
\mathscr{A}_{12} + \overline{\mathscr{A}}_{12} + \mathscr{C} = R_{13}\zeta + R_{23}\overline{\zeta} + R_{33}\rho,
\end{bmatrix}
$$

by means of some complex functions R_{ij}, $i, j = 1, 2, 3$. If we now denote the two 1-forms $\mathscr{A}_{12}$ and $\mathscr{A}_{22}$ by $\delta^{\mathsf{modified}}$ and $\gamma^{\mathsf{modified}}$ (respectively), then the expressions of Ξ_1, Ξ_2, Ξ_3 change into:

$$
\begin{aligned}
&\Xi_1 = \overline{\delta^{\mathsf{modified}}} \wedge \rho + \delta^{\mathsf{modified}} \wedge \rho - R_{13}\zeta \wedge \rho - R_{23}\overline{\zeta} \wedge \rho, \\
&\Xi_2 = R_{12}\overline{\zeta} \wedge \zeta + R_{13}\rho \wedge \zeta + \delta^{\mathsf{modified}} \wedge \rho, \\
&\Xi_3 = \delta^{\mathsf{modified}} \wedge \zeta + \gamma^{\mathsf{modified}} \wedge \rho.
\end{aligned}
$$

Comparing with the initial expressions of Ξ_2, Ξ_3, Ξ_3 in (26) implies that:

$$
\tag{28}
\begin{cases}
d\alpha = -2\,i\,\overline{\beta} \wedge \zeta - i\,\beta \wedge \overline{\zeta} + \big(\delta^{\mathsf{modified}} - R_{13}\zeta\big) \wedge \rho - R_{12}\zeta \wedge \overline{\zeta}, \\
\\
d\beta = \beta \wedge \overline{\alpha} + \delta^{\mathsf{modified}} \wedge \zeta + \gamma^{\mathsf{modified}} \wedge \rho \\
\\
d\overline{\alpha} = 2i\,\beta \wedge \overline{\zeta} + i\,\overline{\beta} \wedge \zeta + \big(\overline{\delta^{\mathsf{modified}}} - R_{23}\overline{\zeta}\big) \wedge \rho + R_{12}\zeta \wedge \overline{\zeta}.
\end{cases}
$$

Now granted the equality $\overline{d\alpha} = d\overline{\alpha}$, one obtains the following equation, after plain simplifications:

$$
-\overline{R_{13}}\,\overline{\zeta} \wedge \rho + \overline{R}_{12}\zeta \wedge \overline{\zeta} = -R_{23}\overline{\zeta} \wedge \rho + R_{12}\zeta \wedge \overline{\zeta}.
$$

Taking account of the linearly independency between $\overline{\zeta} \wedge \rho$ and $\zeta \wedge \overline{\zeta}$, one immediately concludes that:

$$
R_{23} = \overline{R}_{13} \qquad\qquad \text{and} \qquad\qquad R_{12} = \overline{R}_{12}.
$$

In other words, R_{12} is a *real* function and also one can replace R_{23} with $\overline{R}_{13}$ in the expression of $d\overline{\alpha}$. Lastly, the equations (28) can be transformed as follows after the substitution $\delta^{\sf modified} - R_{13}\zeta \mapsto \delta^{\sf modified}$ and putting $W := -R_{12}$:

$$
\tag{29}
\left\{
\begin{aligned}
d\alpha &= -2\,i\,\overline{\beta}\wedge\zeta - i\,\beta\wedge\overline{\zeta} + (\underbrace{\delta^{\sf modified} - R_{13}\zeta}_{\mapsto\delta^{\sf modified}})\wedge\rho \underbrace{-R_{12}}_{+W}\zeta\wedge\overline{\zeta},\\
d\beta &= \beta\wedge\overline{\alpha} + (\underbrace{\delta^{\sf modified} - R_{13}\zeta}_{\mapsto\delta^{\sf modified}})\wedge\zeta + \gamma^{\sf modified}\wedge\rho,\\
d\overline{\alpha} &= 2\,i\,\beta\wedge\overline{\zeta} + i\,\overline{\beta}\wedge\zeta + (\underbrace{\overline{\delta}^{\sf modified} - \overline{R}_{13}\overline{\zeta}}_{\mapsto\overline{\delta}^{\sf modified}})\wedge\rho \underbrace{+R_{12}}_{-W}\zeta\wedge\overline{\zeta}.
\end{aligned}
\right.
$$

This completes the proof. □

The two equations (27) (together with their unwritten conjugates) and the three equations of (23) constitute the new structure equations of the problem with $\delta^{\sf modified}$ and $\gamma^{\sf modified}$ as the *modified* Maurer-Cartan forms after maximal absorbtion of torsion. Thus, thanks to the above (non-explicit) proposition, one has bypassed some painful computations, keeping track of some relevant, somewhat sufficient information, as Cartan usually did in his papers. Nevertheless, we will present just at the moment the explicit expressions of $\delta^{\sf modified}$ and $\gamma^{\sf modified}$.

Before doing this, let us present the following assertion which permits one to consider some two fixed expressions of $\delta^{\sf modified}$ and $\gamma^{\sf modified}$, enjoying (27).

Lemma 4.4. *Let* $\delta^{\sf modified}, \gamma^{\sf modified}$ *and* $\delta_0^{\sf modified}, \gamma_0^{\sf modified}$ *be two couples of 1-forms satisfying both the same equations* (27)*:*

$$
\left[
\begin{aligned}
d\alpha &= \delta^{\sf modified}\wedge\rho + 2i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta + W\,\zeta\wedge\overline{\zeta},\\
d\beta &= \gamma^{\sf modified}\wedge\rho + \delta^{\sf modified}\wedge\zeta + \beta\wedge\overline{\alpha}.
\end{aligned}
\right.
\qquad
\left[
\begin{aligned}
d\alpha &= \delta_0^{\sf modified}\wedge\rho + 2i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta + W\,\zeta\wedge\overline{\zeta},\\
d\beta &= \gamma_0^{\sf modified}\wedge\rho + \delta_0^{\sf modified}\wedge\zeta + \beta\wedge\overline{\alpha}.
\end{aligned}
\right.
$$

Then necessarily:

$$
\tag{30}
\begin{aligned}
\delta^{\sf modified} &= \delta_0^{\sf modified} + {\sf p}\,\rho,\\
\gamma^{\sf modified} &= \gamma_0^{\sf modified} + {\sf p}\,\zeta + {\sf q}\,\rho,
\end{aligned}
$$

for some arbitrary complex functions ${\sf p}$ *and* ${\sf q}$.

Proof. A plain subtraction yields:

$$
\begin{aligned}
0 &\equiv (\delta^{\sf modified} - \delta_0^{\sf modified})\wedge\rho,\\
0 &\equiv (\gamma^{\sf modified} - \gamma_0^{\sf modified})\wedge\rho + (\delta^{\sf modified} - \delta_0^{\sf modified})\wedge\zeta.
\end{aligned}
$$

Now, Cartan's lemma applied to the first equation immediately gives the first equation of (30). Putting then this into the second equation obtained by subtraction yields, again by means of Cartan's lemma, the conclusion. □

Next, a straightforward computation provides a general lemma, unavoidably required when one wants to perform all computations explicitly.

Lemma 4.5. *The exterior differential:*

$$
dG = \mathscr{L}(G)\cdot\zeta_0 + \overline{\mathscr{L}}(G)\cdot\overline{\zeta}_0 + \mathscr{T}(G)\cdot\rho_0
$$

of some function $G(z, \overline{z}, u)$ of class at least $\mathscr{C}^1$ on the base manifold $M \subset \mathbb{C}^2$ reexpresses, in terms of the lifted coframe, as:

$$dG = \bigg(\frac{1}{\mathsf{c}}\mathscr{L}(G)\bigg)\cdot\zeta + \bigg(\frac{1}{\overline{\mathsf{c}}}\overline{\mathscr{L}}(G)\bigg)\cdot\overline{\zeta} + \bigg(-\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{L}(G) - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(G) + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\mathscr{T}(G)\bigg)\cdot\rho. \quad \square \tag{31}$$

Thus, we may now compare and inspect the two separate expressions of $d\alpha$ in (27) and (25), namely:

$$\begin{aligned} d\alpha &= \delta^{\mathsf{modified}} \wedge \rho + 2\, i\, \zeta \wedge \overline{\beta} + i\, \overline{\zeta} \wedge \beta + W\, \zeta \wedge \overline{\zeta}, \\ d\alpha &= d\alpha_0 + \gamma \wedge \rho + \mathsf{s}\, d\rho. \end{aligned} \tag{32}$$

Here, we must compute the differential $d\alpha_0$ of α_0 given in (20):

$$\begin{aligned} d\alpha_0 = \underline{d\big(\tfrac{d\mathsf{c}}{\mathsf{c}}\big)}_{\circ} &- \bigg(\frac{1}{\mathsf{c}}\, dP - P\, \frac{1}{\mathsf{c}\mathsf{c}}\, d\mathsf{c} + 2i\, \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\, d\overline{\mathsf{b}} - 2i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\, d\mathsf{c} - 2i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\, d\overline{\mathsf{c}}\bigg) \wedge \zeta - \\ &- \bigg(\frac{1}{\mathsf{c}}\, P + 2i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\bigg) d\zeta - \bigg(i\, \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\, d\mathsf{b} - i\, \frac{\mathsf{b}}{\mathsf{c}\mathsf{c}\overline{\mathsf{c}}}\, d\mathsf{c} - i\, \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{c}}}\bigg) \wedge \overline{\zeta} - i\, \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\, d\overline{\zeta}. \end{aligned}$$

Now, thanks to the expressions (22) and (20), one obtains:

$$\begin{aligned} d\mathsf{c} &= \mathsf{c}\, \alpha_0 + \frac{P\overline{\mathsf{c}} + 2\, i\, \overline{\mathsf{b}}}{\overline{\mathsf{c}}}\zeta + \frac{i\, \mathsf{b}}{\overline{\mathsf{c}}}\overline{\zeta} \\ &= \mathsf{c}\, \alpha - \mathsf{c}\mathsf{s}\, \rho + \frac{P\overline{\mathsf{c}} + 2\, i\, \overline{\mathsf{b}}}{\overline{\mathsf{c}}}\zeta + \frac{i\, \mathsf{b}}{\overline{\mathsf{c}}}\overline{\zeta}, \\ d\mathsf{b} &= \mathsf{c}\overline{\mathsf{c}}\beta_0 + \mathsf{b}\alpha_0 + \frac{2\, P\mathsf{b}\overline{\mathsf{c}} + 3\, i\, \mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\zeta + \frac{\overline{P}\mathsf{b}}{\overline{\mathsf{c}}}\overline{\zeta} \\ &= \mathsf{b}\, \alpha + \mathsf{c}\overline{\mathsf{c}}\, \beta - \big(\mathsf{c}\overline{\mathsf{c}}\mathsf{r} + \mathsf{b}\mathsf{s}\big)\, \rho + \Big(\frac{2\, P\mathsf{b}\overline{\mathsf{c}} + 3\, i\, \mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} - \mathsf{s}\mathsf{c}\overline{\mathsf{c}}\Big)\zeta + \frac{\overline{P}\mathsf{b}}{\overline{\mathsf{c}}}\overline{\zeta}. \end{aligned} \tag{33}$$

These equations together with (31) and (23), enable one to transform the second expression of $d\alpha$ in (32) into:

$$\begin{aligned} d\alpha = \bigg\{ &\bigg(\frac{\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}}\mathscr{L}(P) + \frac{\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(P) - \frac{1}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{T}(P) - \frac{P\mathsf{s}}{\mathsf{c}} + 2i\, \mathsf{r} - 2i\, \frac{\mathsf{s}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\bigg) \cdot \rho + \\ &+ \bigg(- \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\overline{\mathscr{L}}(P) + i\, \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - 2i\, \frac{\overline{P\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2} - 4\, \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + i\, \mathsf{s}\bigg) \cdot \overline{\zeta} - 2i\, \overline{\beta}\bigg\} \wedge \zeta + \\ + \bigg\{ &\bigg(i\, \mathsf{r} + i\, \frac{\mathsf{b}\overline{\mathsf{s}}}{\mathsf{c}\overline{\mathsf{c}}}\bigg) \cdot \rho + \bigg(- i\, \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} + 2\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + i\, \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + i\, \mathsf{s}\bigg) \cdot \zeta + i\, \beta\bigg\} \wedge \overline{\zeta} + \\ + \bigg\{ &\bigg(- \frac{P}{\mathsf{c}} - 2i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} + \mathsf{s}\bigg) \cdot \beta + \bigg(- i\, \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + \mathsf{s}\bigg) \cdot \overline{\beta} + \gamma\bigg\} \wedge \rho. \end{aligned}$$

Chasing then just the coefficient of $\zeta \wedge \overline{\zeta}$ in this last (long) expression, which is the function we called W, we therefore obtain the explicit expression of this single essential torsion coefficient:

$$W = \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\, \overline{\mathscr{L}}(P) - 2i\, \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}P + 2i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{P} + 6\, \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + 2\, i\, \mathsf{s} - 2\, i\, \overline{\mathsf{s}}. \tag{34}$$

Thanks to Lemma 2.1, one easily realizes that W is a real function as was already mentioned in Proposition 4.3.

Furthermore, collecting together the coefficients of $\bullet\wedge\rho$ from these two expressions of $d\alpha$, one also finds the explicit expression of $\delta^{\mathsf{modified}}$:

(35)

$$\begin{aligned}\delta^{\mathsf{modified}} = {} & \bigg(\frac{1}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{T}(P)-\frac{\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}}\mathscr{L}(P)-\frac{\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(P)+\frac{\mathsf{s}}{\mathsf{c}}P+2i\,\frac{\overline{\mathsf{b}}\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}}-2i\,\overline{\mathsf{r}}\bigg)\cdot\zeta+\bigg(i\frac{\mathsf{b}\overline{\mathsf{s}}}{\mathsf{c}\overline{\mathsf{c}}}-i\,\mathsf{r}\bigg)\cdot\overline{\zeta}+ \\ & +\mathsf{s}\,\alpha-\bigg(\frac{1}{\mathsf{c}}P+2i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\bigg)\cdot\beta+\mathsf{s}\,\overline{\alpha}-i\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\,\overline{\beta}+ \\ & +d\mathsf{s}.\end{aligned}$$

Likewise, let us consider the two separate expressions:

$$\begin{aligned}d\beta &= \gamma^{\mathsf{modified}}\wedge\rho+\delta^{\mathsf{modified}}\wedge\zeta+\beta\wedge\overline{\alpha}, \\ d\beta &= d\beta_0+\delta\wedge\rho+\mathsf{r}d\rho+\gamma\wedge\zeta+\mathsf{s}\,d\zeta,\end{aligned} \tag{36}$$

of $d\beta$ in (27) and (25), with $d\beta_0$ being the differentiation of β_0 in (20) as follows:

$$\begin{aligned}d\beta_0 = {} & \bigg(-\frac{1}{\mathsf{c}\overline{\mathsf{c}}^2}d\overline{\mathsf{c}}\wedge d\mathsf{b}+\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\overline{\mathsf{c}}\wedge d\mathsf{c}\bigg)-\bigg(\frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}+i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\bigg)d\zeta+ \\ & +\bigg(-\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}dP-\frac{P}{\mathsf{c}^2\overline{\mathsf{c}}}d\mathsf{b}+\frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\overline{\mathsf{c}}+2\,\frac{P\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}}d\mathsf{c}-i\frac{\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\mathsf{b}-i\,\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\overline{\mathsf{b}}+2i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^2}d\mathsf{c}+2i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3}d\overline{\mathsf{c}}\bigg)\wedge\zeta- \\ & +\bigg(-\frac{\overline{P}\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}+i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2}\bigg)d\overline{\zeta}+ \\ & +\bigg(-\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}d\overline{P}-\frac{\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^2}d\mathsf{b}+\frac{\overline{P}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\mathsf{c}+2\,\frac{\overline{P}\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^3}d\overline{\mathsf{c}}+2i\,\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\mathsf{b}-2i\,\frac{\mathsf{b}^2}{\mathsf{c}^3\overline{\mathsf{c}}^2}d\mathsf{c}-2i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^3}d\overline{\mathsf{c}}\bigg)\wedge\overline{\zeta}.\end{aligned}$$

Performing lines of (rather lengthy) computations similar to those we already did, we can extract the coefficients of $\bullet\wedge\rho$ from the two equal expressions of $d\beta$ in (36) and we find:

(37)

$$\begin{aligned}\gamma^{\mathsf{modified}} = {} & \bigg(\frac{\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2}\mathscr{T}(P)-\frac{\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2}\mathscr{L}(P)-\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3}\overline{\mathscr{L}}(P)+\frac{\mathsf{b}\mathsf{s}}{\mathsf{c}^2\overline{\mathsf{c}}}P-\frac{\mathsf{r}}{\mathsf{c}}P+i\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{s}}{\mathsf{c}^2\overline{\mathsf{c}}^2}-2i\,\frac{\overline{\mathsf{b}}\mathsf{r}}{\mathsf{c}\overline{\mathsf{c}}}-i\,\frac{\mathsf{b}\overline{\mathsf{r}}}{\mathsf{c}\overline{\mathsf{c}}}+\mathsf{s}\overline{\mathsf{s}}\bigg)\cdot\zeta+ \\ & +\bigg(\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^3}\mathscr{T}(\overline{P})-\frac{\mathsf{b}^2}{\mathsf{c}^3\overline{\mathsf{c}}^3}\mathscr{L}(\overline{P})-\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^4}\overline{\mathscr{L}}(\overline{P})+\frac{\mathsf{b}\overline{\mathsf{s}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{P}-i\,\frac{\mathsf{b}^2\overline{\mathsf{s}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\bigg)\cdot\overline{\zeta}+ \\ & +\mathsf{r}\,\alpha-\bigg(\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}P+i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}-\mathsf{s}+\overline{\mathsf{s}}\bigg)\cdot\beta+2\mathsf{r}\,\overline{\alpha}+\bigg(-\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{P}+i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2}\bigg)\cdot\overline{\beta}- \\ & +d\mathsf{r}.\end{aligned}$$

From now on and for the sake of simplicity and compatibility among the notations, let us drop the word "modified" from $\delta^{\mathsf{modified}}$ and $\gamma^{\mathsf{modified}}$ and denote them simply by δ and γ. Summarizing the results, now the structure equations (25) is transformed into:

$$\begin{aligned}d\rho &= \alpha\wedge\rho+\overline{\alpha}\wedge\rho+i\,\zeta\wedge\overline{\zeta}, \\ d\zeta &= \beta\wedge\rho+\alpha\wedge\zeta, \\ d\overline{\zeta} &= \overline{\beta}\wedge\rho+\overline{\alpha}\wedge\overline{\zeta}, \\ d\alpha &= \delta\wedge\rho+2\,i\,\zeta\wedge\overline{\beta}+i\,\overline{\zeta}\wedge\beta+W\,\zeta\wedge\overline{\zeta}, \\ d\beta &= \gamma\wedge\rho+\delta\wedge\zeta+\beta\wedge\overline{\alpha}, \\ d\overline{\alpha} &= \overline{\delta}\wedge\rho-2\,i\,\overline{\zeta}\wedge\beta-i\,\zeta\wedge\overline{\beta}-\overline{W}\,\zeta\wedge\overline{\zeta}, \\ d\overline{\beta} &= \overline{\gamma}\wedge\rho+\overline{\delta}\wedge\overline{\zeta}+\overline{\beta}\wedge\alpha,\end{aligned} \tag{38}$$

with the already modified Maurer-Cartan forms δ and γ given by (35) and (37), and with some relevant real torsion coefficient W given by (34).

4.2. **Absorbtion-normalization.** After having re-shaped so the structure equations, one has to apply again the absorbtion-normalization procedure by considering the substitutions:

$$\begin{aligned}\delta &\mapsto \delta + p_1\,\rho + q_1\,\zeta + r_1\,\overline{\zeta} + s_1\,\alpha + t_1\,\overline{\alpha} + u_1\,\beta + v_1\,\overline{\beta},\\ \gamma &\mapsto \gamma + p_2\,\rho + q_2\,\zeta + r_2\,\overline{\zeta} + s_2\,\alpha + t_2\,\overline{\alpha} + u_2\,\beta + v_2\,\overline{\beta}.\end{aligned}$$

One easily verifies by elementary linear algebra computations that here the single torsion coefficient W is, as guessed, indeed normalizable.

Normalizing then this coefficient to zero determines $\overline{\mathsf{s}}$ as:

$$\boxed{\overline{\mathsf{s}} = \mathsf{s} - \frac{i}{2}\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\overline{\mathscr{L}}(P) - \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}P + \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{P} - 3i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}.} \tag{39}$$

Consequently, one has to differentiate this equation:

$$\begin{aligned}d\overline{\mathsf{s}} = d\mathsf{s} - \Big\{ & 3i\,\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\overline{\mathsf{b}} + 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\mathsf{b} - 6i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^2}d\mathsf{c} - 6i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3}d\overline{\mathsf{c}} + \frac{P}{\mathsf{c}^2\overline{\mathsf{c}}}d\mathsf{b} + \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}dP - 2\,\frac{P\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}}d\mathsf{c} - \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\overline{\mathsf{c}} -\\ & - \frac{\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^2}d\overline{\mathsf{b}} - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}d\overline{P} + \frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}d\mathsf{c} + 2\frac{\overline{P}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^3}d\overline{\mathsf{c}} - \frac{i}{2\mathsf{c}^2\overline{\mathsf{c}}}\overline{\mathscr{L}}(P)d\mathsf{c} - \frac{i}{2\mathsf{c}\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(P)d\overline{\mathsf{c}} + + \frac{i}{2\mathsf{c}\overline{\mathsf{c}}}d\overline{\mathscr{L}}(P)\Big\},\end{aligned}$$

in which similarly to (31), one has:

(40)

$$d(\overline{\mathscr{L}}(P)) = \left(\frac{1}{\mathsf{c}}\mathscr{L}(\overline{\mathscr{L}}(P))\right)\cdot\zeta + \left(\frac{1}{\overline{\mathsf{c}}}\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))\right)\cdot\overline{\zeta} + \left(-\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{L}(\overline{\mathscr{L}}(P)) - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\mathscr{T}(\overline{\mathscr{L}}(P))\right)\cdot\rho.$$

Then, putting the expressions (33) of $d\mathsf{b}, d\mathsf{c}$ into the above equation expression of $d\overline{\mathsf{s}}$ changes it into the following form after simplification:

$$\begin{aligned}d\overline{\mathsf{s}} = d\mathsf{s} + \Bigg(& -\frac{\overline{P}\,\overline{\mathsf{r}}}{\overline{\mathsf{c}}} + \frac{P\,\mathsf{r}}{\mathsf{c}} - 9\frac{\mathsf{b}^2\overline{\mathsf{b}}^2}{\mathsf{c}^4\overline{\mathsf{c}}^4} + \frac{\mathscr{L}(P)\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} + \frac{PP\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} + \frac{\overline{P}P\overline{\mathsf{b}}^2}{\mathsf{c}^2\overline{\mathsf{c}}^4} - \frac{\overline{\mathsf{b}}^2\overline{\mathscr{L}}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}^4} - \frac{1}{4}\frac{\mathscr{L}(\overline{P})\mathscr{L}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}^2} + i\frac{\overline{\mathscr{L}}(\mathscr{L}(P))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} +\\ & + i\,\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - 3\frac{\mathscr{L}(\overline{P})\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} - 2\frac{P\mathsf{b}\mathsf{s}}{\mathsf{c}^2\overline{\mathsf{c}}} - 2\,\frac{P\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} + 2\,\frac{\overline{P}\overline{\mathsf{b}}\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}^2} + i\,\frac{P\mathscr{L}(\overline{P})\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} - i\,\frac{\mathscr{L}(\overline{P})\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{\overline{P}\mathscr{L}(\overline{P})\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} -\\ & - \frac{i}{2}\frac{\mathscr{L}(\mathscr{L}(\overline{P}))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + 3i\,\frac{\overline{\mathsf{b}}\mathsf{r}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{i}{2}\frac{\mathscr{T}(\overline{\mathscr{L}}(P))}{\mathsf{c}^2\overline{\mathsf{c}}^2} - 6i\,\frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}^2}{\mathsf{c}^3\overline{\mathsf{c}}^4} + 3i\,\frac{\mathsf{b}\overline{\mathsf{r}}}{\mathsf{c}\overline{\mathsf{c}}} - 6\,\frac{\mathsf{b}\overline{\mathsf{b}}\mathsf{s}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + 6i\,\frac{P\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^4\overline{\mathsf{c}}^3} - \frac{i}{2}\,\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}^3}\Bigg)\cdot\rho +\\ & + \left(\frac{P\mathsf{s}}{\mathsf{c}} - \frac{i}{2}\frac{\mathscr{L}(\mathscr{L}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{\mathscr{L}(P)\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}} + 3i\,\frac{\overline{\mathsf{b}}\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{i}{2}\frac{P\mathscr{L}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}} + 3\,\frac{\mathsf{b}\overline{\mathsf{b}}^2}{\mathsf{c}^3\overline{\mathsf{c}}^3} + \frac{1}{2}\frac{\mathscr{L}(\overline{P})\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - 3i\,\frac{P\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^2}\right)\cdot\zeta +\\ & + \left(-\frac{\overline{P}^2\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^3} + 6\,\frac{\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} + i\,\frac{\mathscr{L}(\overline{P})\,\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{\overline{\mathscr{L}}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}^3} + 3i\,\frac{\mathsf{b}\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}} - 3i\,\frac{P\mathsf{b}^2}{\mathsf{c}^3\overline{\mathsf{c}}^2} - \frac{\overline{P}\mathsf{s}}{\overline{\mathsf{c}}} + 3i\,\frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} + \frac{\mathscr{L}(\overline{P})\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{P\overline{P}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{i}{2}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))}{\mathsf{c}\overline{\mathsf{c}}^2}\right)\cdot\overline{\zeta} +\\ & + \left(3i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{i}{2}\frac{\mathscr{L}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\right)\cdot\alpha + \left(-\frac{P}{\mathsf{c}} - 3i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}}\right)\cdot\beta + \left(\frac{3i\,\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{i}{2}\frac{\mathscr{L}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\right)\cdot\overline{\alpha} + \left(\frac{\overline{P}}{\overline{\mathsf{c}}} - 3i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\right)\cdot\overline{\beta}.\end{aligned}$$

Next, by a careful glance on the expression of δ and its conjugation (*see* (35)), we realize that having $d\overline{\mathsf{s}}$ in terms of $d\mathsf{s}$ and the lifted 1-forms $\rho, \zeta, \overline{\zeta}, \alpha, \beta, \overline{\alpha}, \overline{\beta}$ enables us to express $\overline{\delta}$ in terms of δ and the lifted coframe (*cf.* (35)). More precisely, our computations show that we have — the coefficients of $\alpha, \beta, \overline{\alpha}, \overline{\beta}$ vanish identically after simplification:

$$\overline{\delta} := \delta + i\,W_1\,\rho + W_2\,\zeta - \overline{W}_2\,\overline{\zeta}, \tag{41}$$

with the coefficients:

$$W_1 := -\frac{1}{2}\frac{\mathscr{T}(\overline{\mathscr{L}}(P))}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - \frac{1}{2}\frac{\mathscr{L}((\overline{P}))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} - \frac{1}{2}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} + \frac{\overline{\mathscr{L}}(\mathscr{L}(P))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} - i\,\frac{\mathscr{L}(P)\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} + i\,\frac{\overline{\mathscr{L}}(\overline{P})\overline{\mathsf{b}}^2}{\mathsf{c}^2\overline{\mathsf{c}}^4} +$$
$$+\left(-\frac{1}{2}\frac{\mathscr{L}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - 3\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - i\,\frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\right)\mathsf{s} + \left(3\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{P}{\mathsf{c}}\right)\mathsf{r} + \left(-\frac{1}{2}\frac{\mathscr{L}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - 3\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - i\,\frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\right)\overline{\mathsf{s}} + \left(3\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{\overline{P}}{\overline{\mathsf{c}}}\right)\overline{\mathsf{r}}$$
$$W_2 := i\,\frac{\overline{\mathscr{L}}(\mathscr{L}(P))}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{3}{2}i\,\frac{\mathscr{L}(\mathscr{L}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{3}{2}\,\frac{\mathscr{L}(\overline{P})\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{i}{2}\,\frac{P\mathscr{L}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}} - 3i\,\frac{P\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + 3\,\frac{\mathsf{b}\overline{\mathsf{b}}^2}{\mathsf{c}^3\overline{\mathsf{c}}^3} + 3i\,\overline{\mathsf{r}}.$$

(We notice *passim* that the first torsion coefficient W_1 is real.)

Further, after determining $\overline{\mathsf{s}}$ in (39), the expressions of $\overline{\alpha}$ and $\overline{\beta}$ change and are not anymore the conjugates of α and β. Hence, we replace the notations $\overline{\alpha}$ and $\overline{\beta}$ by $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. Putting this new expression of $\overline{\delta}$ into the last structure equation (38) changes it into the form:

$$\begin{aligned}
d\rho &= \alpha\wedge\rho + \widetilde{\alpha}\wedge\rho + i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta,\\
d\overline{\zeta} &= \widetilde{\beta}\wedge\rho + \widetilde{\alpha}\wedge\overline{\zeta},\\
d\alpha &= \delta\wedge\rho + 2\,i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta,\\
d\beta &= \gamma\wedge\rho + \delta\wedge\zeta + \beta\wedge\overline{\alpha},\\
d\widetilde{\alpha} &= \delta\wedge\rho - 2\,i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\widetilde{\beta} + W_2\,\zeta\wedge\rho - \overline{W}_2\overline{\zeta}\wedge\rho,\\
d\widetilde{\beta} &= \overline{\gamma}\wedge\rho + \delta\wedge\overline{\zeta} + \widetilde{\beta}\wedge\alpha + i\,W_1\,\rho\wedge\overline{\zeta} + W_2\,\zeta\wedge\overline{\zeta}.
\end{aligned} \tag{42}$$

4.3. **Absorbtion-normalization of the latest structure equation.** To determine essential torsion coefficients, similarly as before, we make substitutions of the kind:

$$\delta \mapsto \delta + p_1\,\rho + q_1\,\zeta + r_1\,\overline{\zeta} + s_1\,\alpha + t_1\,\overline{\alpha} + u_1\,\beta + v_1\,\overline{\beta},$$
$$\gamma \mapsto \gamma + p_2\,\rho + q_2\,\zeta + r_2\,\overline{\zeta} + s_2\,\alpha + t_2\,\overline{\alpha} + u_2\,\beta + v_2\,\overline{\beta}.$$

This converts the structure equations into the form:

$$\begin{aligned}
d\alpha &= \delta\wedge\rho + q_1\,\zeta\wedge\rho + r_1\,\overline{\zeta}\wedge\rho + s_1\,\alpha\wedge\rho + t_1\,\widetilde{\alpha}\wedge\rho + u_1\,\beta\wedge\rho + v_1\,\widetilde{\beta}\wedge\rho + 2i\,\zeta\wedge\widetilde{\beta} + i\,\overline{\zeta}\wedge\beta,\\
d\beta &= \gamma\wedge\rho + \delta\wedge\zeta + (q_2 - p_1)\zeta\wedge\rho + r_2\,\overline{\zeta}\wedge\rho + s_2\,\alpha\wedge\rho + t_2\,\widetilde{\alpha}\wedge\rho + u_2\,\beta\wedge\rho + v_2\,\widetilde{\beta}\wedge\rho + r_1\,\overline{\zeta}\wedge\zeta +\\
&\quad + s_1\,\alpha\wedge\zeta + t_1\,\widetilde{\alpha}\wedge\zeta + u_1\,\beta\wedge\zeta + v_1\,\widetilde{\beta}\wedge\zeta + \beta\wedge\widetilde{\alpha},\\
d\widetilde{\alpha} &= \delta\wedge\rho + (q_1 + W_2)\zeta\wedge\rho + (r_1 - \overline{W}_2)\overline{\zeta}\wedge\rho + s_1\,\alpha\wedge\rho + t_1\,\widetilde{\alpha}\wedge\rho + u_1\,\beta\wedge\rho + v_1\,\widetilde{\beta}\wedge\rho - 2i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\widetilde{\beta},\\
d\widetilde{\beta} &= \overline{\gamma}\wedge\rho + \delta\wedge\overline{\zeta} + (\overline{q}_2 - p_1 - i\,W_1)\overline{\zeta}\wedge\rho + \overline{r}_2\,\zeta\wedge\rho + \overline{s}_2\,\widetilde{\alpha}\wedge\rho + \overline{t}_2\,\alpha\wedge\rho + \overline{u}_2\,\widetilde{\beta}\wedge\rho + \overline{v}_2\,\beta\wedge\rho +\\
&\quad + (q_1 + W_2)\zeta\wedge\overline{\zeta} + s_1\,\alpha\wedge\overline{\zeta} + t_1\,\widetilde{\alpha}\wedge\overline{\zeta} + u_1\,\beta\wedge\overline{\zeta} + v_1\,\widetilde{\beta}\wedge\overline{\zeta}.
\end{aligned}$$

In order to annihilate as much as possible the appearing (modified) torsion coefficients, we have to solve the following system of homogeneous equations:

$$0 = q_1 = r_1 = s_1 = t_1 = u_1 = v_1, \quad 0 = r_2 = s_2 = t_2 = u_2 = v_2,$$
$$0 = q_2 - p_1, \quad 0 = q_1 + W_2, \quad 0 = r_1 - \overline{W}_2, \quad 0 = \overline{q}_2 - p_1 - i\,W_1.$$

One readily realizes that besides the following determinations:

$$q_1 = 0, \quad r_i = s_i = t_i = u_i = v_i = 0, \; i = 1, 2,$$
$$q_2 = p_1, \qquad \mathrm{Im}(p_1) = -\frac{1}{2}W_1,$$

the homogeneous system will be satisfied if and only if we also have:

$$0 \equiv W_2.$$

In other words, W_2 is the only normalizable expression of this step. A careful glance at the expression of this function shows that it will be normalized to zero as soon as we put:

$$\boxed{\mathsf{r} := -\frac{1}{3}\,\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{1}{2}\,\frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))}{\mathsf{c}\overline{\mathsf{c}}^2} - \frac{i}{2}\,\frac{\overline{\mathscr{L}}(P)\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{1}{6}\,\frac{\overline{P}\,\overline{\mathscr{L}}(P)}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - i\,\frac{\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3}.} \tag{43}$$

With this expression of r which reduces the group dimension, the only remaining (inessential) torsion coefficient W_1 takes the form:

$$\begin{aligned}
W_1 = & -\frac{1}{2}\frac{\mathscr{T}(\overline{\mathscr{L}}(P))}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\overline{\mathscr{L}}(\mathscr{L}(P))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - \frac{1}{2}\,\frac{\mathscr{L}(\mathscr{L}(\overline{P}))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + \frac{i}{3}\,\frac{P\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{i}{3}\,\frac{\overline{P}\,\overline{\mathscr{L}}(\mathscr{L}(P))}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{i}{2}\,\frac{\overline{P}\mathscr{L}(\overline{\mathscr{L}}(P))}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \\
& - \frac{i}{2}\,\frac{P\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{3}{2}\,\frac{\mathscr{L}(\overline{\mathscr{L}}(P))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + 3i\,\frac{\overline{\mathscr{L}}(P)\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} + \frac{i}{6}\,\frac{P\overline{P}\mathscr{L}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{i}{6}\,\frac{P\overline{P}\,\overline{\mathscr{L}}(P)}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{i}{4}\,\frac{\mathscr{L}(\overline{P})\overline{\mathscr{L}}(P)}{\mathsf{c}^2\overline{\mathsf{c}}^2} + i\,\frac{\overline{\mathscr{L}}(\overline{P})\overline{\mathsf{b}}^2}{\mathsf{c}^2\overline{\mathsf{c}}^4} - \\
& - i\,\frac{\mathscr{L}(P)\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} - \frac{\overline{P}\,\overline{\mathscr{L}}(P)\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - \frac{\overline{P}\mathscr{L}(\overline{P})\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} + 2i\,\frac{P\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} - i\,\frac{\overline{P}^2\overline{\mathsf{b}}^2}{\mathsf{c}^2\overline{\mathsf{c}}^4} - 4\,\frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}^2}{\mathsf{c}^3\overline{\mathsf{c}}^4} - i\,\frac{P^2\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} + 8\,\frac{P\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^4\overline{\mathsf{c}}^3} + 9i\,\frac{\mathsf{b}^2\overline{\mathsf{b}}^2}{\mathsf{c}^4\overline{\mathsf{c}}^4} + \\
& + \left(-\frac{\mathscr{L}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}} + 2i\,\frac{P\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - 2i\,\frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2} - 6\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} \right)\mathsf{s}.
\end{aligned} \tag{44}$$

After determining so the group parameter r, we have to re-compute γ which can now be expressed as a combination of the lifted coframe $\rho, \zeta, \overline{\zeta}, \alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}$ independently of $d\mathsf{r}$, *cf.* (37). For this, first we need the expression of $d\mathsf{r}$, not only of r.

Differentiating r in (43) gives:

$$\begin{aligned}
d\mathsf{r} = & -\frac{1}{3\,\mathsf{c}\overline{\mathsf{c}}^2}\, d\mathscr{L}(\overline{\mathscr{L}}(\overline{P})) + \frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{3\,\mathsf{c}^2\overline{\mathsf{c}}^2}\, d\mathsf{c} + \frac{2\,\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{3\,\mathsf{c}\overline{\mathsf{c}}^3} d\overline{\mathsf{c}} + \frac{1}{2\,\mathsf{c}\overline{\mathsf{c}}^2} d\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) - \frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))}{2\,\mathsf{c}^2\overline{\mathsf{c}}^2} d\mathsf{c} - \frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))}{\mathsf{c}\overline{\mathsf{c}}^3} d\overline{\mathsf{c}} - \\
& - \frac{i}{2}\,\frac{\overline{\mathscr{L}}(P)}{\mathsf{c}^2\overline{\mathsf{c}}^2} d\mathsf{b} - \frac{i}{2}\,\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2} d\overline{\mathscr{L}}(P) + i\,\frac{\overline{\mathscr{L}}(P)\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} d\mathsf{c} + i\,\frac{\overline{\mathscr{L}}(P)\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^3} d\overline{\mathsf{c}} - \frac{\overline{P}}{6\,\mathsf{c}\overline{\mathsf{c}}^2} d\overline{\mathscr{L}}(P) - \frac{\overline{\mathscr{L}}(P)}{6\,\mathsf{c}\overline{\mathsf{c}}^2} d\overline{P} + \frac{\overline{P}\,\overline{\mathscr{L}}(P)}{6\,\mathsf{c}^2\overline{\mathsf{c}}^2} d\mathsf{c} + \frac{\overline{P}\,\overline{\mathscr{L}}(P)}{3\,\mathsf{c}\overline{\mathsf{c}}^3} d\overline{\mathsf{c}} + \\
& + \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} d\overline{P} + \frac{\overline{P}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^3} d\overline{\mathsf{b}} + \frac{\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} d\mathsf{b} - 2\,\frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} d\mathsf{c} - 3\,\frac{\overline{P}\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^4} d\overline{\mathsf{c}} - 2i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} d\mathsf{b} - i\,\frac{\mathsf{b}^2}{\mathsf{c}^3\overline{\mathsf{c}}^3} d\overline{\mathsf{b}} + 3i\,\frac{\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^4\overline{\mathsf{c}}^3} d\mathsf{c} + 3i\,\frac{\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^4},
\end{aligned}$$

in which, similarly to the expressions (31) and (40), one has to replace the differentials:

$$\begin{aligned}
d\mathscr{L}(\overline{\mathscr{L}}(\overline{P})) = & \left(\frac{1}{\mathsf{c}}\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))\right) \cdot \zeta + \left(\frac{1}{\overline{\mathsf{c}}}\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))\right) \cdot \overline{\zeta} + \\
& + \left(-\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))) - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))) + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\mathscr{T}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))) \right) \cdot \rho, \\
d\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)) = & \left(\frac{1}{\mathsf{c}}\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)))\right) \cdot \zeta + \left(\frac{1}{\overline{\mathsf{c}}}\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)))\right) \cdot \overline{\zeta} + \\
& + \left(-\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\mathscr{L}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))) - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\overline{\mathscr{L}}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))) + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\mathscr{T}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P))) \right) \cdot \rho.
\end{aligned}$$

Then thanks to the expressions (33), one can re-express $d\mathsf{r}$ in terms of the lifted coframe $\rho, \zeta, \overline{\zeta}, \alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}$. Because of the length of the result, we do not present this intermediate computation here. After all, replacing r and $d\mathsf{r}$ in the Maurer-Cartan form γ in (37) re-shapes its expression under the form:

$$\gamma := V_1\,\rho + V_2\,\zeta + V_3\,\overline{\zeta}, \tag{45}$$

with three certain functions given by:

$$\begin{aligned}
V_1 := & -\frac{1}{3}\frac{\mathscr{T}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{c^2\overline{c}^3} + \frac{\mathscr{T}(\overline{\mathscr{L}}(\overline{\mathscr{L}}(P)))}{c^2\overline{c}^3} + \frac{1}{3}\frac{\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))b}{c^3\overline{c}^3} + \frac{1}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))\overline{b}}{c^2\overline{c}^4} - \frac{1}{2}\frac{\overline{b}\,\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{c^2\overline{c}^4} - \frac{1}{2}\frac{\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{P})))b}{c^3\overline{c}^3} + \\
& + \frac{i}{6}\frac{\overline{P}\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{P})))}{c^2\overline{c}^3} - \frac{i}{6}\frac{\overline{P}\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{c^2\overline{c}^3} - 3i\,\frac{b^2\overline{b}s}{c^3\overline{c}^3} - \frac{i}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(P))b^2}{c^4\overline{c}^3} - \frac{5i}{2}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))b\overline{b}}{c^3\overline{c}^4} - \frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))s}{c\overline{c}^2} + \frac{2}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))Pb}{c^3\overline{c}^3} - \\
& - \frac{1}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{P}\overline{b}}{c^2\overline{c}^4} + \frac{3}{2}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))s}{c\overline{c}^2} - \frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))Pb}{c^3\overline{c}^3} + \frac{2}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{P}\overline{b}}{c^2\overline{c}^4} - \frac{1}{3}\frac{\mathscr{L}(\mathscr{L}(\overline{P}))\overline{P}b}{c^3\overline{c}^3} + \frac{1}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(P))\overline{P}b}{c^3\overline{c}^3} + i\,\frac{\mathscr{L}(\mathscr{L}(\overline{P}))b^2}{c^4\overline{c}^3} - \\
& + \frac{7i}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))b\overline{b}}{c^3\overline{c}^4} - \frac{i}{12}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\mathscr{L}(\overline{P})}{c^2\overline{c}^3} - \frac{3i}{2}\frac{\mathscr{L}(\overline{P})bs}{c^2\overline{c}^2} - 5\,\frac{b^3\overline{b}^2}{c^5\overline{c}^5} - \frac{\mathscr{L}(\overline{P})\overline{P}s}{c\overline{c}^2} - \frac{1}{6}\frac{\mathscr{L}(\overline{P})\overline{\mathscr{L}}(\overline{P})\overline{b}}{c^2\overline{c}^4} + \frac{1}{2}\frac{\mathscr{L}(\overline{P})P\overline{P}b}{c^3\overline{c}^3} - \\
& - \frac{1}{6}\frac{\mathscr{L}(\overline{P})\overline{P}^2\overline{b}}{c^2\overline{c}^4} - \frac{\overline{\mathscr{L}}(\overline{P})bb^2}{c^3\overline{c}^5} - 4\,\frac{\mathscr{L}(\overline{P})b^2\overline{b}}{c^4\overline{c}^4} + 3\,\frac{\overline{P}b\overline{b}s}{c^2\overline{c}^3} + \frac{\overline{P}^2b\overline{b}^2}{c^3\overline{c}^5} - \frac{1}{12}\frac{\mathscr{L}(\overline{P})\mathscr{L}(\overline{P})b}{c^3\overline{c}^3} - 3\,\frac{P\overline{P}b^2\overline{b}}{c^4\overline{c}^4} + 3i\,\frac{Pb^3\overline{b}}{c^5\overline{c}^4} + \frac{5i}{6}\frac{P\mathscr{L}(\overline{P})b^2}{c^4\overline{c}^3} - \\
& - 6i\,\frac{\overline{P}b^2\overline{b}^2}{c^4\overline{c}^5} + \frac{i}{12}\frac{\mathscr{L}(\overline{P})\mathscr{L}(\overline{P})\overline{P}}{c^2\overline{c}^3} - \frac{5i}{6}\frac{\mathscr{L}(\overline{P})\overline{P}b\overline{b}}{c^3\overline{c}^4}, \\
V_2 := & \frac{1}{2}\frac{\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{c^2\overline{c}^2} - \frac{1}{3}\frac{\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{c^2\overline{c}^2} - \frac{P\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))}{c^2\overline{c}^2} + \frac{2}{3}\frac{P\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{c^2\overline{c}^2} - \frac{2i}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(P))b}{c^3\overline{c}^2} - \frac{1}{6}\frac{\mathscr{L}(\mathscr{L}(\overline{P}))\overline{P}}{c^2\overline{c}^2} - i\,\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{b}}{c^2\overline{c}^3} - \\
& + \frac{2i}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{b}}{c^2\overline{c}^3}s^2 - \frac{\mathscr{L}(P)b^2}{c^4\overline{c}^2} + \frac{1}{3}\frac{P\mathscr{L}(\overline{P})\overline{P}}{c^2\overline{c}^2} + \frac{i}{3}\frac{\mathscr{L}(\overline{P})\overline{P}\overline{b}}{c^2\overline{c}^3} + \frac{2i}{3}\frac{\mathscr{L}(\overline{P})Pb}{c^3\overline{c}^2} - \frac{1}{6}\frac{\mathscr{L}(\overline{P})\mathscr{L}(\overline{P})}{c^2\overline{c}^2} - 2i\,\frac{Pb^2\overline{b}}{c^4\overline{c}^3} + 2\,\frac{b^2\overline{b}^2}{c^4\overline{c}^4}, \\
V_3 := & \frac{1}{2}\frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{c\overline{c}^3} - \frac{1}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{c\overline{c}^3} + \frac{2}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{P}}{c\overline{c}^3} - \frac{7}{6}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{P}}{c\overline{c}^3} - \frac{1}{6}\frac{\mathscr{L}(\overline{P})\overline{\mathscr{L}}(\overline{P})}{c\overline{c}^3} + \frac{1}{3}\frac{\mathscr{L}(\overline{P})\overline{P}^2}{c\overline{c}^3}.
\end{aligned}$$

One should notice that V_2 depends on the group parameter s, while V_1 and V_3 do not.

Now, substituting this new expression of γ into the lastly achieved structure equation (42), changes it into the form (remind that W_2 vanishes after determining r):

$$\begin{aligned}
d\rho &= \alpha\wedge\rho + \widetilde{\alpha}\wedge\rho + i\,\zeta\wedge\overline{\zeta}, \\
d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta, \\
d\overline{\zeta} &= \widetilde{\beta}\wedge\rho + \widetilde{\alpha}\wedge\overline{\zeta}, \\
d\alpha &= \delta\wedge\rho + 2\,i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta, \\
d\beta &= \delta\wedge\zeta + \beta\wedge\overline{\alpha} + V_2\,\zeta\wedge\rho + V_3\,\overline{\zeta}\wedge\rho \\
&= \big(\delta - V_2\,\rho\big)\wedge\zeta + \beta\wedge\overline{\alpha} + V_3\,\overline{\zeta}\wedge\rho, \\
d\widetilde{\alpha} &= \delta\wedge\rho - 2\,i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\widetilde{\beta}, \\
d\widetilde{\beta} &= \delta\wedge\overline{\zeta} + \widetilde{\beta}\wedge\alpha + i\,W_1\,\rho\wedge\overline{\zeta} + \overline{V}_3\,\zeta\wedge\rho + \overline{V}_2\,\overline{\zeta}\wedge\rho \\
&= \big(\delta + i\,W_1\,\rho - \overline{V}_2\,\rho\big)\wedge\overline{\zeta} + \widetilde{\beta}\wedge\alpha + \overline{V}_3\,\zeta\wedge\rho.
\end{aligned} \tag{46}$$

At present, we have just one group parameter s. The complete absorption will be rigorously possible only if the seemingly *implausible* identity:

$$V_2 = -iW_1 + \overline{V}_2,$$

would be satisfied, because it would enable us to modify-rename:

$$\begin{aligned}
\delta &:= \delta - V_2\rho \\
&= \delta + \big(i\,W_1 - \overline{V}_2\big)\,\rho
\end{aligned}$$

such a substitution for δ having no effect on the preceding wedge product $\delta \wedge \rho$ in $d\alpha$ and $d\widetilde{\alpha}$.
We claim that the desired identity holds. In fact after simplification, we obtain:
(47)
$$\overline{V}_2 - iW_1 - V_2 = \frac{1}{3\,\mathsf{c}^2\overline{\mathsf{c}}^2}\Big(-3\,\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))) + 3\,\overline{\mathscr{L}}(\mathscr{L}(\mathscr{L}(\overline{P}))) + \mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))) - \overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(P))) + \\ + P\overline{\mathscr{L}}(\mathscr{L}(\overline{P})) - P\mathscr{L}(\overline{\mathscr{L}}(\overline{P})) - \overline{P}\mathscr{L}(\mathscr{L}(\overline{P})) + \overline{P}\overline{\mathscr{L}}(\mathscr{L}(P))\Big).$$

Serendipitously, this imaginary expression is much simplified and it does not include the group parameter s. To show that it vanishes identically, we need the following result:

Lemma 4.6. *(*[15]*, Proposition 6.1) Let H_1 and H_2 be two vector fields on a manifolds M satisfying:*
$$[H_1,[H_1,H_2]] = \Phi_1[H_1,H_2], \quad [H_2,[H_1,H_2]] = \Phi_2[H_1,H_2],$$
for some two certain functions Φ_1 and Φ_2. Then the following four identities involving third-order derivatives are satisfied:
$$0 \overset{\text{I}}{\equiv} -H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)),$$
$$0 \overset{\text{II}}{\equiv} -H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)),$$
$$0 \overset{\text{III}}{\equiv} -H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)),$$
$$0 \overset{\text{IV}}{\equiv} H_2(H_2(H_1(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_1(H_2(H_2(\Phi_2))) - \Phi_2\,H_2(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)). \quad \square$$

Corollary 4.7. *The above expression* (47) *of $\overline{V}_2 - iW_1 - V_2$ in fact vanishes identically.*

Proof. Subtracting the equation II from I gives:
$$0 \equiv 3\,H_2(H_1(H_1(\Phi_2))) - 3\,H_1(H_2(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) + H_1(H_1(H_2(\Phi_2))) - \\ - \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)) + \Phi_1\,H_2(H_1(\Phi_2)) - \Phi_1\,H_1(H_2(\Phi_2)).$$
Now, it suffices to put $\Phi_1 := P, \Phi_2 := \overline{P}$ and $H_1 := \mathscr{L}, H_2 := \overline{\mathscr{L}}$ into the above equation, taking account of the reality condition $\mathscr{L}(\overline{P}) = \overline{\mathscr{L}}(P)$. □

Consequently, the equality $\delta - V_2\rho = \delta + iW_1\rho - \overline{V}_2\rho$ permits us to apply the substitution $\delta \mapsto \delta - V_2\rho$. After renaming the single torsion coefficient V_3 as $\mathfrak{I}$, the structure equations (46) received the much simplified form:

(48)
$$\begin{aligned} d\rho &= \alpha\wedge\rho + \widetilde{\alpha}\wedge\rho + i\,\zeta\wedge\overline{\zeta}, \\ d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta, \\ d\overline{\zeta} &= \widetilde{\beta}\wedge\rho + \widetilde{\alpha}\wedge\overline{\zeta}, \\ d\alpha &= \delta\wedge\rho + 2\,i\,\zeta\wedge\overline{\beta} + i\,\overline{\zeta}\wedge\beta, \\ d\beta &= \delta\wedge\zeta + \beta\wedge\overline{\alpha} + \mathfrak{I}\,\overline{\zeta}\wedge\rho, \\ d\widetilde{\alpha} &= \delta\wedge\rho - 2\,i\,\overline{\zeta}\wedge\beta - i\,\zeta\wedge\widetilde{\beta}, \\ d\widetilde{\beta} &= \delta\wedge\overline{\zeta} + \widetilde{\beta}\wedge\alpha + \overline{\mathfrak{I}}\,\zeta\wedge\rho, \end{aligned}$$

with the single (modified) Maurer-Cartan form δ (after simplification):

(49)

$$
\begin{aligned}
\delta = {} & d\mathsf{s}+ \\
& + \bigg(- \mathsf{s}^2 + \frac{1}{3}\,\frac{\mathscr{L}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{1}{2}\,\frac{\mathscr{L}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{2}{3}\,\frac{P\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{2i}{3}\,\frac{\overline{\mathscr{L}}(\mathscr{L}(P))\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + \frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))P}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \\
& + i\,\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} + \frac{1}{6}\,\frac{\mathscr{L}(\mathscr{L}(\overline{P}))\overline{P}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{2i}{3}\,\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} - \frac{2i}{3}\,\frac{\mathscr{L}(\overline{P})P\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}^2} + \frac{\mathscr{L}(P)\mathsf{b}^2}{\mathsf{c}^4\overline{\mathsf{c}}^2} - \\
& - \frac{1}{3}\,\frac{\mathscr{L}(\overline{P})P\overline{P}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{i}{3}\,\frac{\mathscr{L}(\overline{P})\overline{P}\,\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^3} + \frac{1}{6}\,\frac{\mathscr{L}(\overline{P})\mathscr{L}(\overline{P})}{\mathsf{c}^2\overline{\mathsf{c}}^2} - 2\,\frac{\mathsf{b}^2\overline{\mathsf{b}}^2}{\mathsf{c}^4\overline{\mathsf{c}}^4} + 2i\,\frac{P\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^4\overline{\mathsf{c}}^3} \bigg) \cdot \rho + \\
& + \bigg(\frac{P\mathsf{s}}{\mathsf{c}} + 2i\,\frac{\mathsf{s}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{i}{3}\,\frac{\overline{\mathscr{L}}(\mathscr{L}(P))}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{i}{3}\,\frac{\mathscr{L}(\overline{P})P}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{\mathscr{L}(P)\mathsf{b}}{\mathsf{c}^3\overline{\mathsf{c}}} + 2\,\frac{\mathsf{b}\overline{\mathsf{b}}^2}{\mathsf{c}^3\overline{\mathsf{c}}^3} - 2i\,\frac{P\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^2} \bigg) \cdot \zeta + \\
& + \bigg(i\,\frac{\mathsf{b}\mathsf{s}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{i}{2}\,\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{i}{3}\,\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))}{\mathsf{c}\overline{\mathsf{c}}^2} + \frac{i}{6}\,\frac{\mathscr{L}(\overline{P})\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^2} + 2\,\frac{\mathsf{b}^2\overline{\mathsf{b}}}{\mathsf{c}^3\overline{\mathsf{c}}^3} - i\,\frac{P\mathsf{b}^2}{\mathsf{c}^3\overline{\mathsf{c}}^2} \bigg) \cdot \overline{\zeta} + \mathsf{s}\,\alpha - \bigg(\frac{P}{\mathsf{c}} + 2i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} \bigg) \cdot \beta + \mathsf{s}\,\widetilde{\alpha} - i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\,\widetilde{\beta}.
\end{aligned}
$$

As mentioned before, $\mathfrak{I}$ is independent of the only remaining group parameter s, hence it is impossible to normalize it. Consequently, this torsion coefficient is actually an *essential invariant* of the problem.

4.4. **Second prolongation.** In the situation that we have still one undetermined group parameter s without the possibility of normalizing the single essential torsion coefficient $\mathfrak{I}$, we have to prolong the latest structure equations (48) by adding the group parameter s to the set of base variables $z, \overline{z}, u, \mathsf{b}, \overline{\mathsf{b}}, \mathsf{c}, \overline{\mathsf{c}}$ and adding the 1-form δ to the coframe $\{\rho, \zeta, \overline{\zeta}, \alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}\}$. Before starting this step, let us present the following result:

Lemma 4.8. *The above modified 1-form δ is the unique one which enjoys the structure equations* (48).

Proof. Assume that δ and δ' are two forms satisfying the structure equations, simultaneously. A subtraction immediately gives:

$$
0 \equiv (\delta - \delta') \wedge \rho, \qquad\qquad 0 \equiv (\delta - \delta') \wedge \zeta,
$$

which according to Cartan's lemma implies that $\delta - \delta'$ must be a combination of only ρ and of only ζ, which clearly implies $\delta - \delta' = 0$. □

This shows that we do not encounter any new (prolonged) group parameter while starting the next prolongation. In other words, the prolonged structure group will be automatically reduced to an e-*structure*. Hence it only remains to compute $d\delta$.

Proposition 4.9. *The exterior differentiation $d\delta$ has the form:*

$$
d\delta = \delta \wedge \alpha + \delta \wedge \widetilde{\alpha} + i\,\beta \wedge \widetilde{\beta} + \mathfrak{T}\,\rho \wedge \zeta + \overline{\mathfrak{T}}\,\rho \wedge \overline{\zeta}, \tag{50}
$$

for a certain complex function $\mathfrak{T}$.

Proof. Differentiating $d\alpha$ in the last structure equation (48) gives:

$$
\begin{aligned}
0 \equiv {} & d\delta \wedge \rho - \delta \wedge \alpha \wedge \rho - \delta \wedge \overline{\alpha} \wedge \rho \underline{- i\,\delta \wedge \zeta \wedge \overline{\zeta}}_{\mathsf{a}} \underline{- 2i\,\delta \wedge \overline{\zeta} \wedge \zeta}_{\mathsf{a}} \underline{- 2i\,\overline{\beta} \wedge \alpha \wedge \zeta}_{\mathsf{b}} + \\
& \underline{+ 2\,i\,\overline{\beta} \wedge \alpha \wedge \zeta}_{\mathsf{b}} + \underline{2i\,\widetilde{\beta} \wedge \beta \wedge \rho}_{\mathsf{c}} \underline{- i\,\delta \wedge \zeta \wedge \overline{\zeta}}_{\mathsf{a}} \underline{- i\,\beta \wedge \overline{\alpha} \wedge \overline{\zeta}}_{\mathsf{d}} \underline{+ i\,\beta \wedge \overline{\alpha} \wedge \overline{\zeta}}_{\mathsf{d}} + \underline{i\,\beta \wedge \overline{\beta} \wedge \rho}_{\mathsf{c}},
\end{aligned}
$$

in which the underlined terms can be simplified together and bring the following simple equality:

$$(d\delta - \delta \wedge \alpha - \delta \wedge \overline{\alpha} - i\,\beta \wedge \overline{\beta}) \wedge \rho \equiv 0. \tag{51}$$

On the other hand, from differentiating $d\beta$ and $d\overline{\beta}$ we also find:

(52)

$$(d\delta - \delta \wedge \alpha - \delta \wedge \overline{\alpha} - i\,\beta \wedge \overline{\beta}) \wedge \zeta + (\underbrace{d\mathfrak{I} \wedge \overline{\zeta} - 3\,\mathfrak{I}\,\overline{\alpha} \wedge \overline{\zeta} + \mathfrak{I}\,\alpha \wedge \overline{\zeta}}_{\Gamma}) \wedge \rho \equiv 0,$$

$$(d\delta - \delta \wedge \alpha - \delta \wedge \overline{\alpha} - i\,\beta \wedge \overline{\beta}) \wedge \overline{\zeta} + (\underbrace{d\overline{\mathfrak{I}} \wedge \zeta - 3\,\overline{\mathfrak{I}}\,\alpha \wedge \zeta + \overline{\mathfrak{I}}\,\overline{\alpha} \wedge \zeta}_{\overline{\Gamma}}) \wedge \rho \equiv 0,$$

after a slight simplification. Now, applying the Cartan's Lemma 4.1 to the equality (51) gives:

$$d\delta = \delta \wedge \alpha + \delta \wedge \overline{\alpha} + i\,\beta \wedge \overline{\beta} + \xi \wedge \rho, \tag{53}$$

for some 1-form ξ. Putting then this expression of $d\delta$ into (52) brings:

$$\begin{aligned} (\xi \wedge \zeta - \Gamma) \wedge \rho &= 0, \\ (\xi \wedge \overline{\zeta} - \overline{\Gamma}) \wedge \rho &= 0. \end{aligned} \tag{54}$$

Applying again the Cartan's Lemma to the first equation, we get:

$$\xi \wedge \zeta - \Gamma = \mathscr{A} \wedge \rho,$$

for some 1-form $\mathscr{A}$, or equivalently:

$$\xi \wedge \zeta - (d\mathfrak{I} - 3\,\mathfrak{I}\overline{\alpha} + \mathfrak{I}\alpha) \wedge \overline{\zeta} - \mathscr{A} \wedge \rho = 0.$$

Applying the Cartan's Lemma, this time to the last equality, we obtain:

$$\xi = A_1\zeta + A_2\overline{\zeta} + A_3\rho, \tag{55}$$

for some certain functions A_1, A_2, A_3. Subtracting the conjugation of the second equation in (54) from the first one also gives:

$$(\xi \wedge \zeta - \overline{\xi} \wedge \zeta) \wedge \rho \equiv 0,$$

and hence there is a 1-form $\mathscr{C}$ with:

$$(\xi - \overline{\xi}) \wedge \zeta + \mathscr{C} \wedge \rho \equiv 0.$$

We apply again the Cartan's lemma and this time we obtain the following equation for two certain complex functions B_1 and B_2:

$$\xi - \overline{\xi} = B_1\zeta + B_2\rho. \tag{56}$$

The left-hand side of this equality is imaginary and hence the coefficient of ζ must vanish: $B_1 = 0$. On the other hand, according to (55) we have:

$$\xi - \overline{\xi} = (A_1 - \overline{A}_2)\zeta + (A_2 - \overline{A}_1)\overline{\zeta} + (A_3 - \overline{A}_3)\rho.$$

Comparing this equation with (56) then immediately implies that $A_2 = \overline{A}_1$. Hence, denoting $-A_1$ by $\mathfrak{T}$ gives the following expression for the 2-form $\xi \wedge \rho$ according to (55):

$$\xi \wedge \rho = \mathfrak{T}\rho \wedge \zeta + \overline{\mathfrak{T}}\,\rho \wedge \overline{\zeta}.$$

To complete the proof, it is now enough to put the above expression into (53). □

Consequently we will have the following (prolonged) structure equations after adding the differentiation of the new lifted 1-form δ to the previous ones:

$$\begin{aligned}
d\rho &= \alpha\wedge\rho+\widetilde{\alpha}\wedge\rho+i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \beta\wedge\rho+\alpha\wedge\zeta,\\
d\overline{\zeta} &= \widetilde{\beta}\wedge\rho+\widetilde{\alpha}\wedge\overline{\zeta},\\
d\alpha &= \delta\wedge\rho+2\,i\,\zeta\wedge\overline{\beta}+i\,\overline{\zeta}\wedge\beta,\\
d\beta &= \delta\wedge\zeta+\beta\wedge\overline{\alpha}+\mathfrak{I}\,\overline{\zeta}\wedge\rho,\\
d\widetilde{\alpha} &= \delta\wedge\rho-2\,i\,\overline{\zeta}\wedge\beta-i\,\zeta\wedge\widetilde{\beta},\\
d\widetilde{\beta} &= \delta\wedge\overline{\zeta}+\widetilde{\beta}\wedge\alpha+\overline{\mathfrak{I}}\,\zeta\wedge\rho,\\
d\delta &= \delta\wedge\alpha+\delta\wedge\widetilde{\alpha}+i\,\beta\wedge\widetilde{\beta}+\mathfrak{T}\,\rho\wedge\zeta+\overline{\mathfrak{T}}\,\rho\wedge\overline{\zeta}.
\end{aligned}\tag{57}$$

These equations provide the final e-structure.

Our ultimate task is to find the expression of the new coefficient $\mathfrak{T}$. For this aim, we employ the same procedure as that of finding the expression of W in (34). At first, we have to compute the exterior differentiatial of δ in (49). Unfortunately, this expression is extensive (almost 2 pages long), hence we do not present it here.

Another much shorter path is to carefully compare this expression of $d\delta$ to that from (57). Considering the coefficient of $\rho\wedge\zeta$ reveals a compact expression for $\mathfrak{T}$, granted the four equations I–IV of Lemma 4.6 and their first order derivations with respect to the operators $\mathscr{L}$ and $\overline{\mathscr{L}}$. Then one finds out that the desired function $\mathfrak{T}$ can be expressed in terms of the essential invariant $\mathfrak{I}$ as:

$$\mathfrak{T}=\frac{1}{\mathsf{c}}\left(\overline{\mathscr{L}}(\mathfrak{I})-\overline{P}\,\mathfrak{I}\right)-i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}}\mathfrak{I}.$$

Now, from standard features of the theory, we conclude:

Theorem 4.1. *The equivalence problem for strongly pseudoconvex Levi-nondegenerate hypersurfaces $M^3\subset\mathbb{C}^2$ has a single essential primary invariant:*

$$\begin{aligned}
\mathfrak{I}=&-\frac{1}{3}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{\mathscr{L}}(\overline{P})))}{\mathsf{c}\overline{\mathsf{c}}^3}+\frac{2}{3}\frac{\mathscr{L}(\overline{\mathscr{L}}(\overline{P}))\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^3}+\frac{1}{2}\frac{\overline{\mathscr{L}}(\overline{\mathscr{L}}(\mathscr{L}(\overline{P})))}{\mathsf{c}\overline{\mathsf{c}}^3}-\frac{7}{6}\frac{\overline{\mathscr{L}}(\mathscr{L}(\overline{P}))\overline{P}}{\mathsf{c}\overline{\mathsf{c}}^3}-\\
&-\frac{1}{6}\frac{\mathscr{L}(\overline{P})\overline{\mathscr{L}}(\overline{P})}{\mathsf{c}\overline{\mathsf{c}}^3}+\frac{1}{3}\frac{\mathscr{L}(\overline{P})\overline{P}^2}{\mathsf{c}\overline{\mathsf{c}}^3},
\end{aligned}$$

in which the fundamental function $P:=P(z,\overline{z},u)$ expresses explicitly in terms of the graphing function φ as:

$$P:=\frac{\ell_z-\ell A_u+A\ell_u}{\ell},$$

where:

$$A:=\frac{i\,\varphi_z}{1-i\,\varphi_u}\quad\text{and}\quad \ell:=i\left(\overline{A}_z+A\,\overline{A}_u-A_{\overline{z}}-\overline{A}\,A_u\right).$$

In particular, this invariant vanishes when and only when M^3 is biholomorphic to the model Heisenberg sphere defined as the graph of the function:

$$v=z\overline{z}.$$

Proof. It is only necessary to observe that with the assumption $\varphi(z, \overline{z}, u) := z\overline{z}$, one immediately gets $P \equiv 0$, and hence $\mathfrak{I} \equiv 0$.

Conversely, if $\mathfrak{I} = 0$, whence also $\mathfrak{T} = 0$, the constructed e-structure identifies with the Maurer-Cartan equations of the real projective group, and one recovers the Heisengerg sphere as the orbit of the origin under the action of this group. □

5. A brief comparison to the Cartan-Tanaka geometry of real hypersurfaces $M^3 \subset \mathbb{C}^2$

We now turn to a brief discussion of Cartan geometry of the under consideration real hypersurfaces $M^3 \subset \mathbb{C}^2$ which is much pertinent to their problem of equivalence. It helps us to understand better the generally close relationship between the equivalence problems and Cartan geometries. Here, we borrow the results, notations and terminology from the recent paper [15] (*see* also [20]).

Definition 5.1. Let G be a Lie group with a closed subgroup H, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. A *Cartan geometry of type* (G, H) on a manifold M is a principal H-bundle:

$$\pi : \mathscr{G} \longrightarrow M$$

together with a $\mathfrak{g}$-valued 1-form ω, called the corresponding *Cartan connection*, on $\mathscr{G}$ subjected to the following three conditions:

- **(i)** $\omega_p : T_p\mathscr{G} \longrightarrow \mathfrak{g}$ is a linear isomorphism at every point $p \in \mathscr{G}$;
- **(ii)** if $R_h(p) := ph$ is the right translation on $\mathscr{G}$ by any $h \in H$, then:
$$R_h^* \omega = \mathrm{Ad}(h^{-1}) \circ \omega;$$
- **(iii)** $\omega(H^\dagger) = \mathsf{h}$ for every $\mathsf{h} \in \mathfrak{h}$, where:
$$H^\dagger|_p := \tfrac{d}{dt}\big|_0 \big((R_{\exp(t\mathsf{h})}(p)\big)$$
is the left-invariant vector field on $\mathscr{G}$ corresponding to h.

Among Cartan geometries of type (G, H), the most symmetric one, called *Klein geometry of type* (G, H), arises when $M = G/H$, when $\pi \colon G \to G/H$ is the projection onto left-cosets, and when $\omega = \omega_{MC} \colon TG \to \mathfrak{g}$ is the *Maurer-Cartan form* on G.

In general, with a Cartan connection ω as above, if we associate the vector field $\widehat{X} := \omega^{-1}(\mathsf{x})$ on $\mathscr{G}$ to an arbitrary element x of $\mathfrak{g}$, then the infinitesimal version of condition **(ii)** reads as:

$$[\widehat{X}, \widehat{Y}] = \widehat{[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}},$$

whenever y belongs to $\mathfrak{h}$. But in the special case of Klein geometries, this equality holds moreover for any arbitrary element y of $\mathfrak{g}$. This difference motivates one to define the *curvature function*:

$$\kappa \colon \quad \mathscr{G} \longrightarrow \mathrm{Hom}\big(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}\big)$$

associated to the Cartan connection ω by:

$$\kappa_p(\mathsf{x}, \mathsf{y}) := \omega_p\big([\widehat{X}, \widehat{Y}]\big) - [\mathsf{x}, \mathsf{y}]_{\mathfrak{g}} \qquad (p \in \mathscr{G}, \ \mathsf{x}, \mathsf{y} \in \mathfrak{g}/\mathfrak{h}).$$

In a way, the curvature function measures how far a Cartan geometry is from its corresponding Klein geometry. In particular, a Cartan geometry is locally equivalent to its corresponding Klein geometry if and only if its curvature function vanishes identically (*see* [20]).

Now, let us return to the Levi-nondegenerate real hypersurfaces M^3 regarded as deformations of the Heisenberg sphere $\mathbb{H}^3$. In [15], we built a regular normal Cartan connection of type (G, H) in which G is the projective group associated to the 8-dimensional projective Lie algebra:

$$\mathfrak{g} := \mathfrak{aut}(\mathbb{H}^3) = \mathsf{Span}_{\mathbb{R}}(\mathsf{t}, \mathsf{h}_1, \mathsf{h}_2, \mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j})$$

of infinitesimal CR-automorphisms of $\mathbb{H}^3$ equipped with the full commutator table:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

Moreover, H is the subgroup of G associated to the 5-dimensional subalgebra:

$$\mathfrak{h} := \mathsf{Span}_{\mathbb{R}}(\mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}).$$

The Lie algebra $\mathfrak{g}$ is in fact graded, in the sense of Tanaka [22]:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{h}},$$

with $\mathfrak{g}_{-2} := \mathsf{Span}_{\mathbb{R}}(\mathsf{t})$, with $\mathfrak{g}_{-1} := \mathsf{Span}_{\mathbb{R}}(\mathsf{h}_1, \mathsf{h}_2)$, with $\mathfrak{g}_0 := \mathsf{Span}_{\mathbb{R}}(\mathsf{d}, \mathsf{r})$, with $\mathfrak{g}_1 := \mathsf{Span}_{\mathbb{R}}(\mathsf{i}_1, \mathsf{i}_2)$ and with $\mathfrak{g}_2 := \mathsf{Span}_{\mathbb{R}}(\mathsf{j})$. Here $\mathfrak{g}_- = \mathfrak{g}\big/\mathfrak{h}$ is in fact the Levi-Tanaka symbol algebra of any Levi nondegenerate $M^3 \subset \mathbb{C}^2$.

According to this grading, the curvature function κ decomposes into homogeneous components:

$$\kappa := \kappa^{(0)} + \cdots + \kappa^{(5)}$$

where $\kappa^{(s)}$ assigns to each pair $(\mathsf{p}_{j_1}, \mathsf{p}_{j_2}) \in \Lambda^2 \mathfrak{g}_-$, for $\mathsf{p}_{j_1} \in \mathfrak{g}_{j_i}, j_i = -2, -1$, an element of $\mathfrak{g}_{j_1+j_2+s}$. It turns out that each curvature component $\kappa^{(s)}$ can be formulated in the form:

$$\kappa^{(s)} = \sum_{s=j-(j_1+j_2)} \kappa_{q_j}^{p_{j_1} p_{j_2}} \, \mathsf{p}_{j_1}^* \wedge \mathsf{p}_{j_2}^* \otimes \mathsf{q}_j, \tag{58}$$

where $\kappa_{q_j}^{p_{j_1} p_{j_2}}(p)$ is the real-valued function defined on an arbitrary point p of $\mathscr{G}$ as the coefficient of q_j in $\kappa(p)(\mathsf{p}_{j_1}, \mathsf{p}_{j_2})$, where $\mathsf{p}_{j_1} \in \mathfrak{g}_{j_1}, \mathsf{p}_{j_2} \in \mathfrak{g}_{j_2}$, $\mathsf{q}_j \in \mathfrak{g}_j$ are some mentioned basis elements of $\mathfrak{g}$, for $j_1, j_2 = -2, -1$ and $j = -2, -1, 0, 1, 2$.

In fact, the process of construction the sought Cartan geometry in [15] has mainly consisted in annihilating as many curvature components as possible, and finally we were able to annihilate $\kappa^{(0)}$ (easiest thing), $\kappa^{(1)}$, $\kappa^{(2)}$ and $\kappa^{(3)}$ by an appropriate progressive building of ω which requires somewhat hard elimination computations. Such computations have been done in the framework of the powerful algorithm of Tanaka [22] which involves some modern concepts such as Lie algebras of infinitesimal CR-automorphisms, Lie algebra cohomology, Tanaka prolongation and so on. Finally we found out that (Proposition 7.3 and Theorem 7.4 of [15]):

Theorem 5.1. *The Cartan geometry associated to any $\mathscr{C}^6$-smooth Levi nondegenerate deformation $M^3 \subset \mathbb{C}^2$ of the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$ has the curvature function:*

$$\begin{aligned}\kappa &= \kappa^{(4)} + \kappa^{(5)} = \\ &= \kappa_{i_1}^{h_1 t} \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1 t} \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2 t} \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \\ &+ \kappa_{i_2}^{h_2 t} \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1 t} \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t} \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},\end{aligned} \tag{59}$$

with:

$$\begin{aligned}\kappa_{i_1}^{h_1 t} &= -\,\mathbf{\Delta_1}\, c^4 - 2\,\mathbf{\Delta_4}\, c^3 d - 2\,\mathbf{\Delta_4}\, c d^3 + \mathbf{\Delta_1}\, d^4, \\ \kappa_{i_2}^{h_1 t} &= -\,\mathbf{\Delta_4}\, c^4 + 2\,\mathbf{\Delta_1}\, c^3 d + 2\,\mathbf{\Delta_1}\, c d^3 + \mathbf{\Delta_4}\, d^4, \\ \kappa_{i_1}^{h_2 t} &= \kappa_{i_2}^{h_1 t}, \qquad \kappa_{i_2}^{h_2 t} = -\,\kappa_{i_1}^{h_1 t}, \\ \kappa_{j}^{h_1 t} &= \widehat{H}_1\big(\kappa_{i_2}^{h_2 t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1 t}\big), \qquad \kappa_{j}^{h_2 t} = -\widehat{H}_1\big(\kappa_{i_1}^{h_2 t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1 t}\big)\end{aligned}$$

and with the essential invariants, explicitly expressed in terms of the defining function φ, as:

(60)

$$\begin{aligned}\mathbf{\Delta_1} = \tfrac{1}{384}\Big[&H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\, H_1(H_2(H_1(\Phi_2))) - 11\, H_2(H_1(H_2(\Phi_1))) + 6\,\Phi_2\, H_2(H_1(\Phi_1)) - \\ &- 6\,\Phi_1\, H_1(H_2(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_2)) + 3\,\Phi_1\, H_2(H_2(\Phi_1)) - 3\,\Phi_1\, H_1(H_1(\Phi_1)) + 3\,\Phi_2\, H_2(H_2(\Phi_2)) - \\ &- H_1(\Phi_1)\, H_1(\Phi_1) + H_2(\Phi_2)\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_1(\Phi_1) + 2\,(\Phi_1)^2\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_2(\Phi_2) + 2\,(\Phi_1)^2\, H_1(\Phi_1)\Big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_4} = \tfrac{1}{384}\Big[&-3\, H_2(H_1(H_2(\Phi_2))) - 3\, H_1(H_2(H_1(\Phi_1))) + 5\, H_1(H_2(H_2(\Phi_2))) + 5\, H_2(H_1(H_1(\Phi_1))) + 4\,\Phi_1\, H_1(H_1(\Phi_2)) + \\ &+ 4\,\Phi_2\, H_2(H_1(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_1)) - 3\,\Phi_1\, H_2(H_2(\Phi_2)) - 7\,\Phi_2\, H_1(H_2(\Phi_2)) - 7\,\Phi_1\, H_2(H_1(\Phi_1)) - \\ &- 2\, H_1(\Phi_1)\, H_1(\Phi_2) - 2\, H_2(\Phi_2)\, H_2(\Phi_1) + 4\,\Phi_1\Phi_2\, H_1(\Phi_1) + 4\,\Phi_1\Phi_2\, H_2(\Phi_2)\Big].\end{aligned}$$

This geometry is equivalent to that of its model $\mathbb{H}^3$ if and only if its two essential curvatures $\kappa_{i_1}^{h_1 t}$ and $\kappa_{i_2}^{h_1 t}$ vanish identically; equivalently, the two explicit real functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$ of only the three horizontal real variables (x, y, u), with $z = x + iy, w = u + iv$, vanish identically.

Inspecting the method of construction of the fundamental vector fields H_1 and H_2 in section 5 of [15] shows that they are in fact the real and imaginary parts of the tangent vector field $2\overline{\mathscr{L}}$, introduced in this paper. Moreover, checking the expressions of T, Φ_1, Φ_2 in [15], enjoying the equalities:

$$[H_1, H_2] = 4T, \quad [H_1, T] = \Phi_1\, T, \quad [H_2, T] = \Phi_2\, T,$$

specifies that we have:

$$\mathscr{L} = \frac{1}{2}\, H_1 - \frac{i}{2}\, H_2, \quad \overline{\mathscr{L}} = \frac{1}{2}\, H_1 + \frac{i}{2}\, H_2, \quad \mathscr{T} = -4\, T,$$
$$P = \frac{1}{2}\,\Phi_1 - \frac{i}{2}\,\Phi_2.$$

Now, putting the above complex expressions of $\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}, P$, into the single complex essential invariant $\mathfrak{J}$ of the equivalence problem of real hypersurfaces $M^3 \subset \mathbb{C}^2$ and comparing them carefully to the above real expressions of the essential invariants $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_4$ of their Cartan geometries surprisingly reveals that:

Theorem 5.2. *The following relation holds between essential invariants of the equivalence problem and Cartan geometry of the Levi-nondegenerate $\mathscr{C}^6$-smooth real hypersurfaces $M^3 \subset \mathbb{C}^2$:*

$$\mathfrak{J} = \frac{4}{\mathsf{c}\overline{\mathsf{c}}^3}\big(\mathbf{\Delta}_1 + i\,\mathbf{\Delta}_4\big).$$

This result shows that how much *explicitly* the two concepts of equivalence problem and of Cartan geometry match up.

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