ON THE CONVERGENCE OF S-NONDEGENERATE FORMAL CR MAPS BETWEEN REAL ANALYTIC CR MANIFOLDS

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Abstract. A new pointwise nondegeneracy condition about generic real analytic submanifolds in $\mathbb{C}^n$, called *S-nondegeneracy*, is introduced. This condition is intermediate between essential finiteness and holomorphic nondegeneracy. We prove in this paper that a formal biholomorphism between S-nondegenerate and minimal real analytic CR generic manifolds is convergent, generalizing by this earlier results of Chern and Moser (1974) and of Baouendi, Ebenfelt and Rothschild (1997-8-9). More generally, we consider S-nondegenerate formal mappings and establish their convergence. An essential feature of S-nondegeneracy lies in the unsolvability of the components of the map in terms of the antiholomorphic components together with their jets. We conduct convergence results without explicit solvability by means of repeated applications of Artin's approximation theorem.

§0. Introduction

In this paper, we study the convergence of formal CR mappings of smooth complex spaces taking one real analytic submanifold into another one, following [CM] [BER97] [BER99] and extending the results therein. We prove here a

Main Theorem. *Any formal invertible CR mapping between minimal Segre nondegenerate real analytic* ($\mathcal{C}^\omega$) *CR manifolds in* $\mathbb{C}^n$, $n \geq 2$, *must be convergent.*

plus other related generalizations, which cover all analogous results in the literature.

The study of the convergence of formal holomorphic mappings between analytic or formal CR objects began with Chern and Moser, who proved in a celebrated paper ([CM], §2–3, Theorem 3.5) that the unique formal transformation taking a Levi-nondegenerate hypersurface in $\mathbb{C}^n$ ($n \geq 2$) into normal form is convergent. (Incidentally, the problem of convergence of formal maps is much deeper and much more geometric in case one of the object (normal form) is known to be only formal, and the techniques developed by Chern and Moser cover easily the case where both are analytic, see also Ebenfelt [E1,2,3,4].) Later on, in 1997, Baouendi, Ebenfelt and Rothschild proved the main theorem above assuming that the CR manifolds are finitely nondegenerate [BER97], based on the implicit function theorem and on iteration of jet reflection, or essentially finite [BER99], based on the polynomials

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identities which had been introduced by Baouendi, Jacobowitz and Treves in 1985 ([BJT] or [BERbk], Chapter 5). To my knowledge, these are the only works dealing with the regularity problem about formal maps between two $\mathcal{C}^\omega$ CR manifolds, although Trépreau told me in 1995 that Treves posed to him in the eighties the open problem to find a characterization (we provide it in [Mer99c]).

Quite paradoxically, much attention has been devoted to the convergence problem for maps between real analytic objects with *CR singularities*, thanks to the influence of Moser and Webster [MW]. In this article, after the works of Bishop [Bi], of Bedford-Gaveau [BeGa] and of Kenig-Webster's [KW] about the local hull of holomorphy of a (two-dimensional) surface S in $\mathbb{C}^2$, the authors derived a complete system of three quantities giving (local) biholomorphic invariants for S near an (isolated) elliptic complex tangency: such are biholomorphic to $S_0 = \{(z_1, z_2) \in : y_2 = 0, x_2 = z_1\bar{z}_1 + (\gamma + \delta x_2^s)(z_1^2 + \bar{z}_1^2)\}$ (S_0 is algebraic!), where $0 < \gamma < 1/2$ is Bishop's invariant and where $s \in \mathbb{N}$ and $\delta = \pm 1$, or $s = \infty$ and $\delta = 0$. Let us mention further important results. In 1985, Moser treated the case $\gamma = 0$ and showed that a formal power series change of variables can be found so that the surface S can be defined by an equation of the form $\{(z_1, z_2) : z_2 = z_1\bar{z}_1 + z_1^s + \bar{z}_1^s + z_1^{s+1}\varphi(z_1) + \bar{z}_1^{s+1}\bar{\varphi}(\bar{z}_1)\}$, where $\varphi(z_1)$ is a formal power series in z_1, and where s is a biholomorphic invariant of S at 0. In case $s = \infty$, Moser observed that M is equivalent to the intersection $\{y_2 = 0, x_2 = z_1\bar{z}_1\}$ of the unbounded representation of the 3-sphere with a real hyperplane. In 1995, Huang and Krantz [HK] completed the study by showing that such elliptic M with $\gamma = 0$ is biholomorphically equivalent to $\{(z_1, z_2) : z_2 = z_1\bar{z}_1 + z_1^s + \bar{z}_1^s + \sum_{i+j>s} a_{i,j} z_1^i \bar{z}_1^j\}$, $\bar{a}_{i,j} = a_{j,i}$. Again, as in [CM], one has to deal with nonnecessarily convergent normal forms. Because of CR singularity, Moser and Webster could even produce examples of couples of *hyperbolic* $\mathcal{C}^\omega$ surfaces which are formally equivalent but not biholomorphically equivalent, *e.g.* the surface $\{(z_1, z_2) : z_2 = z_1\bar{z}_1 + \gamma\bar{z}_1^2 + \gamma z_1^3\bar{z}_1\}$, $1/2 < \gamma < \infty$, which cannot be biholomorphically transformed into a real hyperplane, although any surface with $1/2 < \gamma < \infty$ such that the solutions μ of $\mu^2 + (2 - \gamma^{-1})\mu + 1 = 0$ are *not* roots of unity can be *formally* transformed into a real hyperplane ([MW] §5–6). This divergence is related to the divergence of the normalization of a pair of involutions τ_1 and τ_2 invariantly attached to the two two-sheeted projections of the complexification S^c of the surface S onto the coordinate axes (*cf.* also [Ben]), especially to the *elliptic* character of the composition $\varphi = \tau_1\tau_2$ which induces a small divisor problem. Later on, Webster showed [W2] that each real analytic Lagrangian surface in $\mathbb{C}^2$ with a nondegenerate complex tangent at 0 is formally equivalent under holomorphic *symplectic* formal series to the quadric $p = 2z\bar{z} + \bar{z}^2$ in $(z, p) \in \mathbb{C}^2$. Again, the non-hyperbolic character (here, the *parabolic* character) of the composition of a similar pair of involution $\tau_1\tau_2$ enabled Gong [Go2] to show that *generically*, Webster's formal normalization is divergent, following a suggestion of Moser about divergence of parabolic systems ([Go2], p.316).

The two special involutions τ_1 and τ_2 attached to the surface S ($n = 2, \dim_{\mathbb{R}} S = 2$) are replaced in the CR hypersurface (or generic) case M ($n \geq 2, \dim_{\mathbb{R}} M = 2n-1$) by the existence of a pair of complexified CR vector fields $\mathcal{L}$ and $\underline{\mathcal{L}}$ (*cf.* [Mer98]) annihilating formal (anti-)holomorphic mappings, the commutator $\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}$ (or higher orders commutators) being the analog of the Levi-form (or of higher order Levi forms, *cf.* Kohn's finite type conditions and Baouendi-Ebenfelt-Rothschild's finite nondegeneracy) of this pair of vector fields (see in [MW], a remark p. 262, which almost implicitely suggests the geometric interpretation of Segre varieties as a double foliation in the complexified space, as was characterized recently by the author in [Mer98]). The hypersurface case however simplifies considerably, due to the constant CR dimension or equivalently, due to the existence of CR vector fields, which, according to ideas of Sussmann [Su], Treves [Trv] and Trépreau [Trp], become the mean of *propagating properties of CR functions*. In fact, in this paper and in works [BER97] [BER99], it is worth noticing that the convergence proofs reduce to the convergence of formal solutions of ordinary analytic differential equations with singularities, after remembering that to any suitably nondegenerate CR hypersurface can be associated a differentiel equation, *cf.* the grounding works of Tresse [Tre] and Cartan [Ca] (although no published article has yet treated the correspondence between hypersurfaces and differential equations in a more degenerate case than the Levi-nondegenerate case; I owe this to Sukhov). Finally, the natural obstruction to convergence is not due to a small divisor problem, but to a geometric condition called *holomorphic nondegeneracy* (*cf.* [BERbk] [Mer99c]).

Our main intention in this article is to study *nonsolvable* mappings between *two* analytic CR manifolds. Thus, the difficulty does not originate from the possible divergence of some normalizations, but from the high degeneracies of the *Segre morphism* of the image CR manifold M' (*cf.* Introduction of [Mer99c], and *cf.* eq. (1.1.7) below). The finite determination of formal mappings by their jets at one point, or equivalently what we call *S-solvability* here, has been studied intensively by Baouendi, Ebenfelt, Rothschild, and Zaitsev and relates strongly to the solvability (through the usual implicit function theorem only) of the mapping in terms of the conjugate mapping together with its conjugate jets. In our analysis, this case appears *a posteriori* to be much simpler (*cf.* §11). We should point out that the main result in the present paper is new even for invertible mappings, but leaves open the holomorphically nondegenerate case (except in codimension 1, *cf.* [Mer99c]) which is more general than the Segre nondegenerate case (*cf.* the closing remark in [BER99]).

§1. Statement of the results

Let us now explain the words and the concepts in our theorem. Let h be a *formal* holomorphic (or CR: this happens to be equivalent) mapping $(\mathbb{C}^n,p)\to_{\mathcal{F}}(\mathbb{C}^{n'},p')$, *i.e.* the components of $h=(h_j(t-p)=p'_j+\sum_{\gamma\in\mathbb{N}^n_*}h_{j,\gamma}(t-p)^\gamma)_{1\le j\le n'}$, $\mathbb{N}^n_*:=\mathbb{N}^n\backslash\{0\}$, $h_{j,\gamma}\in\mathbb{C}$, $1\le j\le n$, are *formal series* centered in p with respect to the variables $(t-p)\in\mathbb{C}^n$, with constant term p'. We say that h is *invertible* or that h is a *formal biholomorphism* or that h *has formal rank* n, if $n=n'$ and the formal Jacobian of h at p is invertible. Let M and M' be real analytic CR manifolds in $\mathbb{C}^n$, $\mathbb{C}^{n'}$, let $p\in M$, $p'\in M'$. We say that the formal map h *maps* (M,p) *formally into* (M',p') and write $h(M,p)\subset_{\mathcal{F}}(M',p')$ or $h\colon(M,p)\to_{\mathcal{F}}(M',p')$, if there exists a $d'\times d$ matrix of formal power series $\mu(t,\tau)$ such that $\rho'(h(t),\bar{h}(\tau))\equiv\mu(t,\tau)\rho(t,\tau)$ as formal power series, where $\rho(t,\bar{t})=0$ and $\rho'(t,\bar{t})=0$ are *real analytic* defining equations for (M,p) and (M',p') respectively. More precisely, our general assumption throughout this paper will be:

($\mathcal{GH}$) The map $h\colon(M,p)\to_{\mathcal{F}}(M',p')$ is a local formal holomorphic map between $\mathcal{C}^\omega$ CR manifolds $M\subset\mathbb{C}^n$, $M'\subset\mathbb{C}^{n'}$, $p\in M$ is some point, $p'\in M'$ is some point, and we assume that M is minimal in the sense of Tumanov at p, or equivalently, of finite type at p in the sense of Kohn and Bloom-Graham.

Let $m = \dim_{CR} M$, $d = \mathrm{codim}_{\mathbb{R}} M$, $m' = \dim_{CR} M'$, $d' = \mathrm{codim}_{\mathbb{R}} M'$, $m + d = n$, $m' + d' = n'$, $m \geq 1$, $m' \geq 1$, $d \geq 1$, $d' \geq 1$. *All our CR manifolds are supposed to be of positive CR dimension and of positive codimension.*

In suitable coordinates t, t', then $p = 0$, $f(p) = 0$, $M = \{t \in U \colon \rho(t, \bar{t}) = 0\}$, $M' = \{t' \in U' \colon \rho'(t', \bar{t}') = 0\}$, U, U' are small polydiscs centered at the origin, $\rho_j(t, \bar{t}) = \sum_{\mu,\nu \in \mathbb{N}^n} \rho_{j,\mu,\nu} t^\mu \bar{t}^\nu$, $1 \leq j \leq d$, $\rho'_j(t', \bar{t}') = \sum_{\mu',\nu' \in \mathbb{N}^{n'}} \rho_{j,\mu',\nu'} t'^{\mu'} \bar{t}'^{\nu'}$, $1 \leq j \leq d'$, are *real analytic*, with $\partial \rho_1 \wedge \cdots \wedge \partial \rho_d(0) \neq 0$, $\partial \rho'_1 \wedge \cdots \wedge \partial \rho'_{d'}(0) \neq 0$.

By convention, we shall still write sometimes p and p' to denote the two reference points which are now the origin in the coordinate systems t and t'.

The assumption that $h \colon (M, p) \to_{\mathcal{F}} (M', p')$ can be also interpreted by saying that $\rho'(h(t), \bar{h}(\tau)) = 0$ when $\rho(t, \tau) = 0$. In particular, h induces a formal holomorphic map $(S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$, where $S_{\bar{p}} := \{t \in U \colon \rho(t, 0) = 0\}$ is the Segre variety of M at $p = 0$ and similarly $S'_{\bar{p}'} := \{t' \colon \rho'(t', 0) = 0\}$ is the Segre variety of M' at $p' = 0$. *It is this map $h \colon (S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$ which governs the tangential CR behavior of f.*

In §1.1 below, we shall introduce three classes of mappings between real analytic CR manifolds, one of which is a new class.

§1.1. Three nondegeneracy conditions on formal CR maps. First, introduce a basis $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_m$, of $T^{0,1}M$ with complexified coefficients analytic in (t, τ).

For instance, after a possible renumbering, we have $\det\left(\left(\frac{\partial \rho_j(0)}{\partial t_k}\right)^{1 \leq j \leq d}_{m+1 \leq k \leq n}\right) \neq 0$ and we can thus simply choose the vector fields

$$(1.1.1) \qquad \underline{\mathcal{L}}_j = \frac{\partial}{\partial \tau_j} - \left(\frac{\partial \rho(t,\tau)}{\partial \tau_j}\right) \left(\frac{\partial \rho_l(t,\tau)}{\partial \tau_k}\right)^{-1}_{1 \leq l \leq d;\ m+1 \leq k \leq n} \left(\frac{\partial}{\partial \tau_k}\right)_{m+1 \leq k \leq n}.$$

Here, $\left(\frac{\partial}{\partial \tau_k}\right)_{m+1 \leq k \leq n}$ is considered as a $d \times 1$ matrix, $\left(\frac{\partial \rho(t,\tau)}{\partial \tau_j}\right)$ as a $d \times 1$ matrix and the $d \times d$ matrix $\left(\frac{\partial \rho_l(t,\tau)}{\partial \tau_k}\right)_{1 \leq l \leq d;\ m+1 \leq k \leq n}$ is invertible, by assumption.

For $\gamma \in \mathbb{N}^m$, denote $|\gamma| := \gamma_1 + \cdots + |\gamma_m|$ and $\underline{\mathcal{L}}^\gamma := \underline{\mathcal{L}}_1^{\gamma_1} \cdots \underline{\mathcal{L}}_m^{\gamma_m}$. Then applying all these derivations to the identity $\rho'(h(t), \bar{h}(\tau)) = 0$, it is well-known that one obtains an infinite family of formal identities

$$(1.1.2) \qquad 0 = \underline{\mathcal{L}}^\gamma [\rho'(h(t), \bar{h}(\tau))] := R'_\gamma(t, \tau, h(t), \nabla^{|\gamma|} \bar{h}(\tau)) = 0,$$

satisfied by $h(t)$ when (t, τ) satisfy $\rho(t, \tau) = 0$. Here, $\nabla^{|\gamma|} \bar{h}(\tau)$ denotes the $n' C^{|\gamma|+n}_{|\gamma|}$-tuple of derivatives $(\partial^\alpha_\tau \bar{h}(\tau))_{|\alpha| \leq |\gamma|}$ of $\bar{h}$ with respect to τ of all orders of lengths $\leq |\gamma|$, or the $|\gamma|$-jet of $\bar{h}$ at τ, and $C^{|\gamma|+n}_{|\gamma|}$ denotes Pascal's binomial coefficient $\frac{(|\gamma|+n)!}{|\gamma|!\ n!}$. Also, one can easily see that the above term $R'_\gamma = R'_\gamma(t, \tau, t', \nabla^{|\gamma|})$ denotes here, by its very definition (1.1.2), a holomorphic mapping from a neighborhood of $0 \times 0 \times 0 \times \nabla^{|\gamma|} \bar{h}(0)$ into $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n'} \times \mathbb{C}^{n' N_{n,|\gamma|}}$ to $\mathbb{C}^{d'}$, where $N_{n,|\gamma|} := C^{n+|\gamma|}_{|\gamma|}$. Indeed, it is clear that there exists a holomorphic term r'_γ such that we can write $\underline{\mathcal{L}}^\gamma \bar{h}(\tau) = r'_\gamma(t, \tau, \nabla^{|\gamma|} \bar{h}(\tau))$, because coefficients of $\underline{\mathcal{L}}^\gamma$ are analytic in (t, τ). Denote $R'_\gamma = (R'^{l'}_\gamma)_{1 \leq l' \leq d'}$ and $R'_\gamma = R'^{l'}_\gamma(t, \tau, t', \nabla^{|\gamma|} \bar{h}(\tau)) = \underline{\mathcal{L}}^\gamma [\rho'_{l'}(h(t), \bar{h}(\tau))]$, $1 \leq l' \leq d'$.

Definition 1.1.3. We will say that the formal mapping h is

• **S-solvable** at p if the holomorphic mapping

$$\mathbb{C}^n \ni t' \mapsto (R'_\gamma(0,0,t',\nabla^{|\gamma|}\bar{h}(0)))_{|\gamma|\leq\kappa_0} \in \mathbb{C}^{d' N_{n,\kappa_0}} \tag{1.1.4}$$

is an immersion at 0, for κ_0 large enough. Then the first integer κ_0 for which the mapping in (1.1.4) is an immersion is in fact a biholomorphic invariant of h under simultaneous changes of coordinates near (M,p) and near (M',p'). Referring to this integer, we shall shortly say that h *is κ_0-solvable*, in order to means that h *is κ_0-solvable in terms of $\bar{h}$ and its jets $\nabla^{\kappa_0}\bar{h}$* (for the complete explanation, see in advance Lemma 3.12 in this article). For instance, it is well-known that h is l'_0-solvable in the following circumstance: when M' is *l'_0-finitely nondegenerate at* p' in the sense of Baouendi, Ebenfelt and Rothschild [BER97] and h is a formal *CR submersive* map, *i.e.* which induces a formal submersion $(S_{\bar{p}},p)\to_{\mathcal{F}}(S_{\bar{p}'},p')$ at p.

• **S-finite** if the complex analytic variety $\mathbb{V}'_p$ defined by

$$\mathbb{V}'_p := \{t'\in\mathbb{C}^{n'} : R'_\gamma(0,0,t',\nabla^{|\gamma|}\bar{h}(0)) = 0, \ \forall\, \gamma\in\mathbb{N}^m\} \tag{1.1.5}$$

is zero-dimensional at the point p': $\dim_{\mathbb{C},p'}\mathbb{V}'_p = 0$ (*i.e.* $\dim_{\mathbb{C},0}\mathbb{V}'_0 = 0$, since $p=0$ and $p'=0$ in our coordinates). (The study of S-finite CR maps is very classical and standard, since the work of Baouendi, Jacobowitz and Treves, see [BJT] [DF88] [BR88] [BR90] [BR95] [BHR96] [CPS98] [BERbk] [CPS99].)

• **S-nondegenerate** if there exist multiindices $\gamma_1,\dots,\gamma_{n'}\in\mathbb{N}^m$ and integers $l'_1,\dots,l'_{n'}$, $1\leq l'_i\leq d'$, such that

$$\det\left(\frac{\partial {R'}^{l'_j}_{\gamma_j}}{\partial t'_k}(t,0,h(t),\nabla^{|\gamma_j|}\bar{h}(0))\right)_{1\leq j,k\leq n'} \not\equiv 0 \tag{1.1.6}$$

when $\rho(t,0)=0$ and where the above formal series should be interpreted as a formal series expressed in terms of a local holomorphic coordinate on the Segre variety $\{\rho(t,0)=0\}$ passing through 0. More precisely, as one can find coordinates $t=(w,z)\in\mathbb{C}^m\times\mathbb{C}^d$, such that M is given by a d-dimensional vectorial equation in the form $z=\bar{z}+i\bar{\Theta}(\bar{w},w,\bar{z})$ (see §2.1, eq. (2.1.1)), the nondegeneracy condition should be understood as meaning the following:

$$\det\left(\frac{\partial {R'}^{l'_j}_{\gamma_j}}{\partial t'_k}(w,i\bar{\Theta}(0,w,0),0,h(w,i\bar{\Theta}(0,w,0)),\nabla^{|\gamma_j|}\bar{h}(0))\right)_{1\leq j,k\leq n'} \not\equiv_w 0. \tag{1.1.7}$$

Remarks. 1. Using the biholomorphic invariance of the Segre varieties attached to M and to M', it can be easily shown that S-solvability, S-finiteness and S-nondegeneracy of a formal CR map do not depend on the choice of some defining functions $(\rho_j)_{1\leq j\leq d}$ for M and $(\rho'_j)_{1\leq j\leq d'}$ for M', and that these conditions are invariant under simultaneous biholomorphic changes of coordinates near M and M' which fix p and p'.

2. An S-solvable map is clearly S-finite, but there is no general link between S-finite and S-nondegenerate maps, as shown by simple examples in §5 here.

§**1.2. The main result.** We obtain a general convergence result about formal CR maps between real analytic CR manifolds satisfying each one of the above three nondegeneracy conditions. The third is new and constitutes the core of this article.

Theorem 1.2.1. *Let $h : (M, p) \to_{\mathcal{F}} (M', p')$ be a formal holomorphic map between real analytic CR generic manifolds and assume that M is minimal at p. If*

(i) *h is S-solvable, or if*
(ii) *h is S-finite, or if*
(iii) *h is S-nondegenerate,*

then the power series of the formal mapping h is convergent.

Remarks. 1. An elementary examination of our proof shows that this result extends immediately to M' being any real analytic set through p', which is not necessarily smooth nor CR, provided each one of the nondegeneracy conditions (1.1.4), (1.1.5), or (1.1.6) holds (*cf.* also [CPS99]). However, it seems to be essential in our proof that M is CR generic and minimal and it would be an interesting problem to search for generalizations of the notion of finite type in the category of singular real analytic varieties (*cf.* [BG]).

2. Although not stated in this form, parts (i) and (ii) of Theorem 1.2.1 were essentially proved in [BER97] and [BER99] respectively. Furthermore, the versions of (ii) that are proved in [BER99] followed in fact from [BER97] and earlier techniques developed by Baouendi and Rothschild in the $\mathcal{C}^\infty$-$\mathcal{C}^\omega$ regularity problem [BJT] [BR88] [BR90].

§**1.3. Discussion of the proof.** Our proof of Theorem 1.2.1 (iii) incorporates two essential ingredients. As a first ingredient, we shall derive from the *approximation theorem* of Artin ([A]) a beautiful convergence theorem (Theorem 1.3.2 below). This convergence argument will be applied at the level of Segre varieties and of subsequent Segre chains. The approximation theorem states that formal solutions to analytic equations can be approximated to any order by convergent solutions:

Theorem 1.3.1. (Artin, [A]) *Let $R(w, y) = 0$, $R = (R_1, \dots, R_J)$, where $w \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, $R_j \in \mathcal{O}_{n+m} = \mathbb{C}\{w, y\}$, be a converging system of* holomorphic *equations. Suppose $\hat{g}(w) = (\hat{g}_1(w), \dots, \hat{g}_m(w))$, $\hat{g}_k(w) \in \mathbb{C}[\![w]\!]$, are* formal *power series without constant term which solve $R(w, \hat{g}(w)) \equiv_w 0$ in $\mathbb{C}[\![w]\!]$. Then for every integer $N \in \mathbb{N}$, there exists a* convergent series solution *$g(w) = (g_1(w), \dots, g_m(w))$, i.e. satisfying $R(w, g(w)) \equiv_w 0$, such that $g(w) \equiv_w \hat{g}(w) \pmod{\mathfrak{m}(w)^N}$.*

Here, $\mathfrak{m}(w)$ denotes the maximal ideal of the local ring $\mathbb{C}[\![w]\!]$ of formal power series in w and the congruence relation $g(w) \equiv_w \hat{g}(w) \pmod{\mathfrak{m}(w)^N}$ means that the coefficients of monomials of total degree $< N$ agree in $g(w)$ and $\hat{g}(w)$. We denote by $\mathbb{C}\{w\}$ the local ring of convergent power series in w.

Theorem 1.3.2. *Let $R(w, y) = 0$, $R = (R_1, \dots, R_J)$, where $w \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, $R_j \in \mathcal{O}_{n+m} = \mathbb{C}\{w, y\}$ be a system of* holomorphic *equations. Suppose that $\hat{g}(w) = (\hat{g}_1(w), \dots, \hat{g}_m(w)) \in \mathbb{C}[\![w]\!]^m$ are* formal *power series without constant term solving $R(w, \hat{g}(w)) \equiv_w 0$ in $\mathbb{C}[\![w]\!]$. If $J \geq m$ and if there exist $j_1, \dots, j_m$, $1 \leq j_1 < j_2 < \cdots < j_m \leq J$ such that*

$$\det \left(\frac{\partial R_{j_k}}{\partial y_l}(w, \hat{g}(w)) \right)_{1 \leq k, l \leq m} \not\equiv_w 0 \ \text{ in } \ \mathbb{C}[\![w]\!], \tag{1.3.3}$$

then $\hat{g}(w) \equiv g(w) \in \mathbb{C}\{w\}$ *is convergent.*

This corollary will be of paramount importance in proving Theorem 1.2.1 (iii).

The second main ingredient in our proof of Theorem 1.2.1 (iii) will be an argument about propagation of analyticity which is closely related to the recent works of Baouendi, Ebenfelt and Rothschild and which will be applied here using the formalism that the author have introduced in [Mer98], a formalism which stems from the local theory of foliations by flows of vector fields and appears to be canonical for the following reason (*cf.* [Mer98]).

In the extrinsic complexification $\mathcal{M} = M^c$ of M, the complexifications of Segre varieties give birth to concatenations of Segre varieties, called Segre chains in [Mer98] and which do not coincide *exactly* with the so-called Segre sets introduced by Baouendi, Ebenfelt and Rothschild (*cf.* [BER96] [BER97] [Z97] [BERbk] [BER99]), but coincide up to a change of parametrization. Our formalism interprets these concatenations of Segre varieties as partial orbits of the complexified CR vector fields tangent to M. Applying then an iteration processus giving the analyticity of transversal jets (transversal to subsequent Segre chains), which follows a each step by subsequent applications of Theorem 1.3.2, we shall obtain analyticity of $h^c = (h, \bar{h})$ along all Segre chains. We then conclude by noticing that, if μ_p denote the *Segre type* of $\mathcal{M}$ at p, the Segre sets $\mathcal{S}_p^{2\mu_p}$ and $\underline{\mathcal{S}}_p^{2\mu_p}$ contain an open neighborhood of $\mathcal{M}$ at p, if M is minimal at p (minimality criterion due to Baouendi, Ebenfelt and Rothschild).

In summary, two main steps arise in our proof, as in [BER97] [BER99]. Step I: establishing the analyticity of jets at the level of subsequent Segre chains and Step II: propagating analyticity up to the maximal Segre set. In this paper, the difficulty underlying Step I is hidden behind Artin's theorem, while Step II relies upon known techniques of propagation. After a first reading (*cf.* §11), the reader might notice that, contrary to the very explicit iteration process which may be endeavoured in the case of S-solvable CR maps, as in [BER97], in the S-nondegenerate, the iteration process becomes highly nonexplicit and requires a step by step patient induction.

§1.4. Separate nondegeneracy conditions. Now, we come to some *various separate assumptions on* M, f, M' *which insure that* f *is either S-solvable, S-finite or S-nondegenerate.* As a main point in our definitions, the mapping $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ was considered as a whole object, but some independent hypotheses on M' plus other ones on f are usually made in the literature (see *e.g.* [BER97] [BER99] and the references therein) and we begin by recalling some of them.

Proposition 1.4.1. ([BER97]) *The formal holomorphic mapping* $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ *is S-solvable in each one of the following circumstances:*

(i) *If* $n = n'$, M' *is finitely nondegenerate at* p' *and* h *has formal rank* n *at* p,

(ii) *If* $M \subset \mathbb{C}^n$, $M' \subset \mathbb{C}^{n'}$, $m \geq m'$, M' *is finitely nondegenerate at* p' *and* h *is CR-submersive.*

Remarks. 1. Of course, a formal biholomorphism is CR-submersive, so (ii) $\Rightarrow$ (i). 2. S-solvability of h imposes furthermore a strong nondegeneracy condition on M'. Indeed, one can easily see that it is necessary that M' be finitely nondegenerate at p', but this is far from being sufficient. This is why in (i) and (ii) above, h is assumed to be formally submersive on Segre varieties. Not to mention that there exist many S-solvable mappings which are not CR submersive, see §13, Example 13.1.

Proposition 1.4.2. ([BER99]) *The formal holomorphic mapping* $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ *is S-finite in each one of the following circumstances:*

(i) *If* $n = n'$, M' *is essentially finite at* p' *and* h *has formal rank* n *at* p,
(ii) *If* $n = n'$, M' *is essentially finite at* p' *and* h *induces a finite formal map* $(S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$,
(iii) *If* $M \subset \mathbb{C}^n$, $M' \subset \mathbb{C}^{n'}$, $m \geq m'$, M' *is essentially finite at* p' *and* h *induces a formal map* $(S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$ *of generic rank equal to* $m' = \dim_{\mathbb{C}} S_{\bar{p}'}$.

A formal holomorphic map $h\colon (X,p) \to_{\mathcal{F}} (X',p')$ of complex manifolds is said to have formal generic rank $m' = \dim_{\mathbb{C}} X'$ if in a local chart, a $m' \times m'$ minor of the formal Jacobian matrix of h at p does not vanish identically as a power series. The definition of finite formal maps also can be modeled on the definition of finite holomorphic maps. Then of course (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

The S-finiteness of h imposes a strong nondegeneracy condition on M' at p': *for* h *to be S-finite at* p*, it is necessary that* M' *be essentially finite at* p' *but not at all sufficient* (left to the reader). In fact, the additional conditions that are required in Proposition 1.4.2 are all sufficient to imply that h is S-finite but they do not cover all the cases where h may be S-finite, see §13, Example 13.2.

Let us finally remark that, although S-finite maps are not S-nondegenerate in general, it is a fact that

Proposition 1.4.3. *All the formal maps* $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ *satisfying conditions* (i) *or* (ii) *of Proposition 3 or conditions* (i), (ii) *or* (iii) *of Proposition 4 are, moreover, S-nondegenerate. Furthermore, all the formal maps which appear in* [BER99] *are S-finite* **and** *S-nondegenerate.*

Consequently, with our Theorem 1.2.1 (iii), we recover also all the convergence results in [BER99] (especially Theorem 2.1 there). We can provide here a complete independent proof of Theorem 1.2.1 (i) and (ii) (which is almost contained in [BER97] and [BER99] respectively).

§**1.5. Nondegeneracy conditions for generic manifolds.** Now, we summarize the comparison between various nondegeneracy conditions for real analytic CR manifolds. We shall say that a CR $\mathcal{C}^\omega$ manifold is *S-nondegenerate at* p if the identity map $i\colon (M,p) \to_{\mathcal{F}} (M,p)$ is an S-nondegenerate formal map at p. Recall that M is called *holomorphically nondegenerate at* p if there does not exist a holomorphic vector field tangent to an open piece of M. Also, M is called *finitely nondegenerate at* p if and only if the identity map $i\colon (M,p) \to_{\mathcal{F}} (M,p)$ is S-solvable at p and M is essentially finite at p if and only if $i\colon (M,p) \to_{\mathcal{F}} (M,p)$ is S-finite at p.

Assuming that M is holomorphically nondegenerate, we then have:

(i) The set $\Sigma_{\mathcal{FD}}$ of *finitely degenerate points* of M is a proper real analytic subvariety of M.

(ii) The set $\Sigma_{\mathcal{NESSF}}$ of *non essentially finite points* of M is a proper real analytic subvariety of M.

(iii) The set $\Sigma_{\mathcal{SD}}$ of *S-degenerate points* of M is a proper real analytic subvariety of M.

Furthermore, the following trivial inclusions

$$\Sigma_{\mathcal{FD}} \supset \Sigma_{\mathcal{NESSF}} \supset \Sigma_{\mathcal{SD}} \tag{1.5.1}$$

are all strict in general (see the examples of §13). This shows that the condition of S-nondegeneracy is an intermediate new nondegeneracy condition between essential finiteness and holomorphic nondegeneracy. By the way, our technique in this paper applies well only to S-nondegenerate maps, but we treat elsewhere the holomorphically nondegenerate case in codimension one [MER99c].

To conclude the presentation of our main results, we would like to mention that we can easily obtain an analog of Propositions 1.4.1 and 1.4.2 as follows.

Proposition 1.5.2. *The formal map $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ is S-nondegenerate in each one of the following circumstances:*

(i) *If $n = n'$, M' is S-nondegenerate at p' and h is of formal rank n,*
(ii) *If M' is S-nondegenerate at p' and h induces a formal map $(S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$ of generic rank equal to $m' = \dim_{\mathbb{C}} S_{\bar{p}'}$.*

Finally, it is clear that applying our Theorem 1.2.1 in all of these situations, we thus obtain a collection of seven corollaries that we can summarize in a

Theorem 1.5.3. *Under the assumptions of Propositions 1.4.1, 1.4.2 and 1.5.2, the formal maps $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ are all convergent if M is minimal at p.*

Organization of the paper. We occupy §2-3 with notational ingredients about Segre chains. Despite heaviness of the underlying formalism, the geometric picture can be easily sketched: the Segre chains happen to be just orbits of a system of two m-dimensional vector fields [Su] [Mer98]. This is why a constant use of flows of vector fields is injected in the formalism, especially during the proofs of our main Theorem 1.2.1, (i), (ii) and (iii). Paragraphs §5-6 are devoted to these proofs, using as a main tool the Theorem 1.3.2 to Artin's theorem, a corollary which we will derive in §12 directly from the approximation theorem. We produce in §13 some elementary examples to show that many S-finite maps exist, which are not examplified by separate assumptions on M, h, M' like in Proposition 1.4.2 (*cf.* [BER97] [BER99] [CPS99]) and to show that a S-finite map need not to be S-nondegenerate in general, and vice-versa. Finally, in §15, we propose to the interested reader some *open problems*, which do not appear explicitely in the literature, some of which are left open by our analysis, and others of a wider class.

§2. Real analytic CR manifolds

The next two paragraphs are devoted to a brief presentation of the theory of Segre chains. A reader who is aware of this theory can skip §2 and §3.

§2.1. Equations. Let M be a piece through 0 of a $\mathcal{C}^\omega$ *generic* manifold in $\mathbb{C}^n$, let $m = \dim_{CR} M$, $d = \operatorname{codim}_{\mathbb{R}} M$, with $m + d = n$, $\dim_{\mathbb{R}} M = 2m + n$, and let $\rho = (\rho_1, \ldots, \rho_d)$ be a system of real analytic defining equations for M in a neighborhood U of 0 in $\mathbb{C}^n$, *i.e.* $M = \{t \in U \colon \rho(t, \bar{t}) = 0\}$, $\rho(0) = 0$ and $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d(0) \neq 0$. We shall say that ρ is a *d-vectorial function.* After a sufficiently large dilatation of the coordinates, we can assume that all the series $\rho_j(t, \bar{t}) =$

$\sum_{\mu,\nu\in\mathbb{N}^n}\rho_{j,\mu,\nu}t^\mu\bar{t}^\nu$, $\rho_{j,\mu,\nu}\in\mathbb{C}$, $\rho_{j,\mu,\nu}=\bar{\rho}_{j,\nu,\mu}$, $1\leq j\leq d$, converges uniformly in the polydisc $(4\Delta)^n$, where Δ is the unit disc in $\mathbb{C}$. We let $\tau=(\bar{t})^c$ be the complexified $\bar{t}$ variable, which is an independent variable, and we set $\bar{\rho}(t,\tau)=\sum_{\mu,\nu\in\mathbb{N}^n}\bar{\rho}_{\mu,\nu}t^\mu\tau^\nu$, so that we have $\overline{\rho(t,\tau)}=\bar{\rho}(\bar{t},\bar{\tau})$. Also, it is clear that there exist holomorphic coordinates $(w,z)=(w_1,\dots,w_m,z_1,\dots,z_d)$ near $0\in\mathbb{C}^n$ vanishing at $0\in M$ such that $T_0^cM=\mathbb{C}_w^m\times\{0\}$, $T_0M=\mathbb{C}_w^m\times\mathbb{R}_x^d$, $z=x+iy$. Then M is given by a system of d scalar real analytic equations $y=h(w,\bar{w},x)$ (in vectorial notation), where $h=\sum_{\alpha,\beta,k}h_{k,\beta,\alpha}x^kw^\beta\bar{w}^\alpha$, $h(0)=0$, $dh(0)=0$, $h_{k,\beta,\alpha}\in\mathbb{C}^d$, $\bar{h}_{k,\alpha,\beta}=h_{k,\beta,\alpha}$, $k\in\mathbb{N}^d$, $\beta\in\mathbb{N}^m$, $\alpha\in\mathbb{N}^m$. Again after dilatation of $(w,\bar{w},x)$, we can assume that the d-vectorial power series h converges uniformly in $(4\Delta)^{2m+n}$. Now that this choice of coordinates has been performed, we set $\rho(t,\bar{t}):=y-h(w,\bar{w},x)$ definitely.

In the sequel, the reference point is thought to be the origin and p will denote a possibly varying point of M, close to the origin.

Let σ denote the antiholomorphic involution on $\mathbb{C}_t^n\times\mathbb{C}_\tau^n$ defined by $\sigma(t,\tau)=(\bar{\tau},\bar{t})$, so $\sigma^2=\mathrm{id}$. To the complexification $\rho(t,\tau)$ of $\rho(t,\bar{t})$ is canonically associated an *extrinsic complexification* M^c of M given by $M^c=:\mathcal{M}=\{(t,\tau)\in\Delta^n\times\Delta^n\colon\rho(t,\tau)=0\}\subset\mathbb{C}_t^n\times\mathbb{C}_\tau^n$. Since $\partial\rho_1\wedge\cdots\wedge\partial\rho_d(0)\neq0$, we have $\dim_{\mathbb{C}}\mathcal{M}=2m+n$. We can embed $\mathbb{C}^n$ in $\mathbb{C}_t^n\times\mathbb{C}_\tau^n$ as the totally real plane $\underline{\Lambda}=\{(t,\tau)\in\mathbb{C}^{2n}\colon\tau=\bar{t}\}$, *i.e.* as the graph of $t\mapsto\bar{t}$. Hence M embeds in $\underline{\Lambda}$ as $M=\{(t,\bar{t})\colon t\in M\}$. Notice that M also embeds in $\mathcal{M}$, and thus M can be considered as a maximally real submanifold of $\mathcal{M}$. Notice that $\sigma(t,\bar{t})=(t,\bar{t})$, *i.e.* σ fixes $\underline{\Lambda}$ pointwisely, so $\sigma(M)=M$. If $p\in M$, *i.e.* $t_p\in M\subset\mathbb{C}_t^n$, where p is identified with its coordinates $t_p=(t_{1,p},\dots,t_{n,p})$, or equivalently, if $(t_p,\bar{t}_p)\in M\subset\mathcal{M}\subset\mathbb{C}_t^n\times\mathbb{C}_\tau^n$, let us denote by p^c the point $(t_p,\bar{t}_p)\in\mathcal{M}$. Then $p^c=\pi_t^{-1}(\{p\})\cap\underline{\Lambda}$, where $\pi_t:\mathbb{C}_t^n\times\mathbb{C}_\tau^n\to\mathbb{C}_t^n$, $(t,\tau)\mapsto t$. We also put $\pi_\tau:\mathbb{C}_t^n\times\mathbb{C}_\tau^n\to\mathbb{C}_\tau^n$, $(t,\tau)\mapsto\tau$, so that $p^c=\pi_\tau^{-1}(\{\bar{p}\}\cap\underline{\Lambda})$. Using the reality of the d-vectorial function ρ, namely using that $\rho(t,\tau)\equiv\bar{\rho}(\tau,t)$, one can easily prove that the complex manifold $\mathcal{M}$ is fixed by σ, *i.e.* that $\sigma(\mathcal{M})=\mathcal{M}$, and that there is a one-to-one correspondence between germs of real analytic subsets $\Sigma\subset M$ at 0 and germs at 0 of complex analytic subvarieties $\Sigma_1\subset\mathcal{M}$ satisfying $\sigma(\Sigma_1)=\Sigma_1$ (see [Mer98], §2.2).

If we replace $y=(z-\bar{z})/2i$, $x=(z+\bar{z})/2$ in the equation $y=h(w,\bar{w},x)$ and solve this equation in terms of z or of $\bar{z}$, using the analytic implicit function theorem, we may obtain two new equivalent systems of d-vectorial equations for M

$$z=\bar{z}+i\bar{\Theta}(\bar{w},w,\bar{z})\qquad\text{or}\qquad\bar{z}=z-i\Theta(w,\bar{w},z),\tag{2.1.1}$$

with $\Theta\in\mathbb{C}\{w,\bar{w},\bar{z}\}$ and thus also, two equivalent systems of equations for $\mathcal{M}$

$$z=\xi+i\bar{\Theta}(\zeta,w,\xi)\qquad\text{or}\qquad\xi=z-i\Theta(w,\zeta,z),\tag{2.1.2}$$

where $\zeta=(\bar{w})^c$ and $\xi=(\bar{z})^c$. Since $h(0)=0$ and $dh(0)=0$, then we have also $\bar{\Theta}(0)=0$ and $d\bar{\Theta}(0)=0$. After a new dilatation of the coordinates, we can assume the convergence in $(4\Delta)^{2m+n}$ of the d-vectorial function $\bar{\Theta}$ above. The fact that these two systems of equations define the same manifold $\mathcal{M}$ (or, equivalently, that $\sigma(\mathcal{M})=\mathcal{M}$) is reflected by the following relations that are obtained by replacing one system of equations of $\mathcal{M}$ into the other

(2.1.3)

$$\Theta(w,\zeta,z)\equiv\bar{\Theta}(\zeta,w,z-i\Theta(w,\zeta,z))\ \text{ and }\ \bar{\Theta}(\zeta,w,\xi)\equiv\Theta(w,\zeta,\xi+i\bar{\Theta}(\zeta,w,\xi)).$$

§**2.2. Vector fields.** We say that L is an m-vector field if L is given by the span of a set of m independent vector fields $L_1, \dots, L_m$ over, say, $\Delta^n \subset \mathbb{C}^n$ or $\Delta^{2n} \subset \mathbb{C}^{2n}$, *such that $L_1, \dots, L_m$ commute with each other.*

For instance, the complexification $\mathcal{M} = M^c$ admits complexified (1,0) and (0,1) tangent m-vector fields which can be written explicitely as

$$\mathcal{L} = \frac{\partial}{\partial w} + i\bar{\Theta}_w(\zeta, w, \xi)\frac{\partial}{\partial z} \quad \text{and} \quad \underline{\mathcal{L}} = \frac{\partial}{\partial \zeta} - i\Theta_\zeta(w, \zeta, z)\frac{\partial}{\partial \xi} \tag{2.2.1}$$

in vectorial notation: here $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ and $\underline{\mathcal{L}} = (\underline{\mathcal{L}}_1, \dots, \underline{\mathcal{L}}_m)$ each form a system of m-vectorial vector fields over Δ^{2n}. These two m-vector fields form together a couple $\mathbb{S} = \{\mathcal{L}, \underline{\mathcal{L}}\}$ of m-vector fields that are clearly the complexifications of the canonical representatives for a basis of $T^{1,0}M$ and $T^{0,1}M$ respectively, namely

$$L = \frac{\partial}{\partial w} + i\bar{\Theta}_w(\bar{w}, w, \bar{z})\frac{\partial}{\partial z} \quad \text{and} \quad \bar{L} = \frac{\partial}{\partial \bar{w}} - i\Theta_{\bar{w}}(w, \bar{w}, z)\frac{\partial}{\partial \bar{z}}, \tag{2.2.2}$$

that is to say $L^c = \mathcal{L}$ and $\bar{L}^c = \underline{\mathcal{L}}$.

Let us recall the following fundamental facts (see [Mer98], §2). The system $\{L, \bar{L}\}$ is orbit-minimal on M if and only if the system $\{\mathcal{L}, \underline{\mathcal{L}}\}$ is orbit-minimal on $\mathcal{M}$ and, more generally, $\mathcal{O}_{L,\bar{L}}(M,p)^c = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$. The main fact is that the family of complexified Segre and conjugate Segre varieties $\mathcal{S}_{\tau_p}$ and $\underline{\mathcal{S}}_{t_p}$ form families of integral manifolds for $\mathcal{L}$ and $\underline{\mathcal{L}}$ respectively which induce their canonical flow foliation. Indeed, this can be observed after writing explicitely the complexifications of the Segre and of the conjugate Segre varieties, namely:

$$S_{\bar{t}_p} : z = \bar{z}_p + i\bar{\Theta}(\bar{w}_p, w, \bar{z}_p) \quad \text{and} \quad \overline{S}_{t_p} : \bar{z} = z_p - i\Theta(w_p, \bar{w}, z_p) \tag{2.2.3}$$

whose complexifications can be seen and written as follows:

$$\begin{aligned} &\mathcal{S}_{\zeta_p,\xi_p} : \zeta = \zeta_p, \xi = \xi_p, z = \xi_p + i\bar{\Theta}(\zeta_p, w, \xi_p) \quad \text{and} \\ &\underline{\mathcal{S}}_{w_p,z_p} : w = w_p, z = z_p, \xi = z_p - i\Theta(w_p, \zeta, z_p), \end{aligned} \tag{2.2.4}$$

and thus it is clear from (2.2.4) that $\mathcal{L}\mathcal{S}_{\zeta_p,\xi_p} \equiv 0$ and $\underline{\mathcal{L}}\underline{\mathcal{S}}_{w_p,z_p} \equiv 0$. For dimensional reasons ($\dim_{\mathbb{C}} \mathcal{S}_{\zeta_p,\xi_p} = \dim_{\mathbb{C}} \underline{\mathcal{S}}_{w_p,z_p} = m$), these complexified Segre varieties coincide with the leaves of the flow foliations induced by $\mathcal{L}$ and by $\underline{\mathcal{L}}$ on $\mathcal{M}$, respectively.

§3. Segre chains

§**3.1. Definitions.** The *orbit k-chains* of the pair $\{\mathcal{L}, \underline{\mathcal{L}}\}$ of m-vector fields on $\mathcal{M}$ are called *Segre k-chains.* The orbit k-chains are almost implicitely defined by Sussmann in [Su] and the reader is referred to [Mer98] for the general construction in the analytic category. Here, we summarize how they can be constructed.

First, let us write $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m)$ and let $w \in \mathbb{C}^m$. We have already observed that the $\mathcal{L}_j$'s commute. Consequently, if we consider the multiple flow mapping

$$\mathbb{C}^m \times \Delta^{2n} \ni (w_1, \dots, w_m, p) \mapsto \exp(w_m\mathcal{L}_m) \circ \cdots \circ \exp(w_1\mathcal{L}_1)(p) \in \Delta^{2n}, \tag{3.1.1}$$

which is defined over a certain domain of $\mathbb{C}^m \times \Delta^{2n}$, we shall have immediately:

(3.1.2)
$$\exp(w_{\varpi(m)}\mathcal{L}_{\varpi(m)}) \circ \cdots \circ \exp(w_{\varpi(1)}\mathcal{L}_{\varpi(1)})(p) = \exp(w_m\mathcal{L}_m) \circ \cdots \circ \exp(w_1\mathcal{L}_1)(p),$$

for every permutation $\varpi : \{1, 2, \dots, m\} \to \{1, 2, \dots, m\}$. The m-vector field $\underline{\mathcal{L}}$ satisfies the same property. We shall simply denote these two multiple flow maps by $(w, p) \mapsto \mathcal{L}_w(p)$ and by $(\zeta, p) \mapsto \underline{\mathcal{L}}_\zeta(p)$. Thus, we can formally work with multiple vector fields as if they were usual vector fields, *i.e.* as if $m = 1$.

Now, let us denote shortly $\mathbb{S} = \{\mathcal{L}, \underline{\mathcal{L}}\}$. Then rk $\mathbb{S} = 2m$ at each point of $\Delta^{2n} \cap \mathcal{M}$. The *codimension* of $\mathbb{S}$ in $\mathcal{M}$ equals $\dim_{\mathbb{C}}\mathcal{M} - 2m = d = \mathrm{codim}_{\mathbb{R}} M$. We fix a *reference* point $p \in M$, $p^c \in \mathcal{M}$, close to the origin 0 (which is the *central* point in our coordinates (t, τ)). It will be interesting to let this point p vary in order to consider the various orbits of different points close to the origin. In this paragraph, the reference point should not be confused with the origin.

If $k \in \mathbb{N}_*$, $\mathbb{L}^k = (L^1, \dots, L^k) \in \mathbb{S}^k$, $w_{(k)} = (w_1, \dots, w_k) \in \mathbb{C}^k$, $p \in (\frac{1}{2}\Delta)^{2n} \cap \mathcal{M}$, let us denote $\mathbb{L}^k_{w_{(k)}}(p) = L^k_{w_k} \circ \cdots \circ L^1_{w_1}(p)$ whenever the composition is defined. By the way, after bounding $k \leq 3(2n)$, it is clear that $\mathbb{L}^k_{w_{(k)}}(p) \in \Delta^{2n} \cap \mathcal{M}$ whenever $w_{(k)} \in (\delta\Delta^m)^k$, $p \in (\frac{1}{2}\Delta)^{2n} \cap \mathcal{M}$, $k \leq 3(2n)$, for $\delta > 0$ small enough.

Because Card $\mathbb{S} = 2$, only two different $\mathbb{S}$-chains exist. They can be naturally called Segre k-chains (we choose this denomination instead of "Segre sets", because they come together with compositions of flow paps, which are concatenations of holomorphic maps). These two families of Segre k-chains of $p = (t_p, \tau_p) \in \mathcal{M}$ are defined by

$$(3.1.3) \quad \begin{aligned}
\mathcal{S}^{2j}_{\tau_p} &= \{\underline{\mathcal{L}}_{\zeta_j} \circ \mathcal{L}_{w_j} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p) \colon w_1, \zeta_1, \dots, w_j, \zeta_j \in \delta\Delta^m\} \\
\mathcal{S}^{2j+1}_{\tau_p} &= \{\mathcal{L}_{w_{j+1}} \circ \underline{\mathcal{L}}_{\zeta_j} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p) \colon w_1, \zeta_1, \dots, \zeta_j, w_{j+1} \in \delta\Delta^m\} \\
\underline{\mathcal{S}}^{2j}_{t_p} &= \{\mathcal{L}_{w_j} \circ \underline{\mathcal{L}}_{\zeta_j} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p) \colon \zeta_1, w_1, \dots, \zeta_j, w_j \in \delta\Delta^m\} \\
\mathcal{S}^{2j+1}_{t_p} &= \{\underline{\mathcal{L}}_{\zeta_{j+1}} \circ \mathcal{L}_{w_j} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p) \colon \zeta_1, w_1, \dots, w_j, \zeta_{j+1} \in \delta\Delta^m\},
\end{aligned}$$

for $k = 2j$ or $k = 2j + 1$, $j \in \mathbb{N}$, $k \leq 3(2n)$. Of course, $\mathcal{S}^k_{\tau_p} \subset \mathcal{M}$ and $\underline{\mathcal{S}}_{t_p} \subset \mathcal{M}$.

The actions of the vectorial m-flows of $\mathcal{L}$ and of $\underline{\mathcal{L}}$ on a point $p \in \mathcal{M}$ with coordinates $(w_p, z_p, \zeta_p, \xi_p) \in \mathcal{M}$ are simply given by

$$(3.1.4) \quad \begin{aligned}
&\mathcal{L}_w(w_p, z_p = \xi_p + i\bar{\Theta}(\zeta_p, w_p, \zeta_p), \zeta_p, \xi_p) = \\
&\qquad = (w_p + w, z = \xi_p + i\bar{\Theta}(\zeta_p, w_p + w, \zeta_p), \zeta_p, \xi_p) \\
&\underline{\mathcal{L}}_\zeta(w_p, z_p, \zeta_p, \xi_p = z_p - i\Theta(w_p, \zeta_p, z_p)) = \\
&\qquad = (w_p, z_p, \zeta + \zeta_p, \xi = z_p - i\Theta(w_p, \zeta + \zeta_p, z_p)).
\end{aligned}$$

We recall the property $\sigma(\mathcal{L}_w(q)) = \underline{\mathcal{L}}_{\bar{w}}(\sigma(q))$, which can be seen easily after applying σ to the members of eq. (3.1.4), so that more generally $\sigma(\mathcal{S}^k_{\tau_p}) = \underline{\mathcal{S}}^k_{\bar{\tau}_p}$ and $\sigma(\underline{\mathcal{S}}^k_{t_p}) = \mathcal{S}^k_{\bar{t}_p}$. This property easily implies that the *a priori* different two minimality types and multitypes of $\mathbb{S}$ must coincide. Accordingly, let us recall how the Segre type and multitype of $\mathcal{M}$ at 0 are defined.

First, let us point out that, if we are given $f : X \to Y$ a holomorphic map of connected complex manifold, then there exists a proper complex subvariety $Z \subset X$

with $\dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} X$ and an integer r such that $\mathrm{rk}_{\mathbb{C},p}(f) = r = \max_{q \in X} \mathrm{rk}_{\mathbb{C},q}(f)$ for all $p \in X \backslash Z$. We shall denote this integer, the *generic rank* of f, by $\mathrm{genrk}_{\mathbb{C}}(f)$.

As Card $\mathbb{S} = 2$, any alternating k-tuple of m-vectors $\mathbb{L}^k \in \mathbb{S}^k$ can be written uniquely $(\cdots, \mathcal{L}, \underline{\mathcal{L}}, \mathcal{L})$ or $(\cdots, \underline{\mathcal{L}}, \mathcal{L}, \underline{\mathcal{L}})$. Accordingly, we shall write $\mathcal{L}\underline{\mathcal{L}}^k$ to denote $(\mathcal{L}\underline{\mathcal{L}})^j$ if $k = 2j$ is even and $\underline{\mathcal{L}}(\mathcal{L}\underline{\mathcal{L}})^j$ if $k = 2j+1$ is odd. Of course, we introduce also the similar notation for $\underline{\mathcal{L}}\mathcal{L}^k$. Also, we denote by $\Gamma_{\underline{\mathcal{L}}\mathcal{L}^k}$ the map $(\delta\Delta^m)^k \to \Delta^{2n}$, $w_{(k)} \mapsto \underline{\mathcal{L}}\mathcal{L}^k_{w_{(k)}}(0)$, where $w_{(k)} = (w_1, \ldots, w_k)$.

In these notations, $\mathcal{S}^k_{\tau_p} = \Gamma_{\underline{\mathcal{L}}\mathcal{L}^k}((\delta\Delta^m)^k)$ and $\underline{\mathcal{S}}^k_{t_p} = \Gamma_{\mathcal{L}\underline{\mathcal{L}}^k}((\delta\Delta^m)^k)$. By slight abuse of notation we shall also denote $\Gamma_{\underline{\mathcal{L}}\mathcal{L}^k}$, $\Gamma_{\mathcal{L}\underline{\mathcal{L}}^k}$ by $\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}$, $\Gamma^k_{\mathcal{L}\underline{\mathcal{L}}}$. Whenever we will be working with coordinates, the central point will be for us the origin and p will denote a varying point of $\mathcal{M}$. When we state an invariant theorem, p is the reference point that we pick in $\mathcal{M}$ or in $\mathcal{M}$ near the origin.

The *Segre multitype of* $\mathcal{M}$ *at* p is defined to be the μ_p-tuple $(m, m, e_1, \ldots, e_{\kappa_p})$, $\mu_p = 2 + \kappa_p$, being the *Segre type of* $\mathcal{M}$ at p satisfying

1. $\mathrm{genrk}_{\mathbb{C}}(\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}) = 2m + e_1 + \cdots + e_k = 2m + e_{\{k\}}$, $2 \leq k \leq \kappa_p$ and
2. $\mathrm{genrk}_{\mathbb{C}}(\Gamma^{\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = \mathrm{genrk}_{\mathbb{C}}(\Gamma^{\mu_p+1}_{\underline{\mathcal{L}}\mathcal{L}}) = 2m + e_{\{\kappa_p\}}$, $\mu_p = 2 + \kappa_p$.

§**3.2. Properties.** All these integers are uniquely and invariantly defined (because the two canonical Segre foliations are biholomorphically invariant). Of course, $\mathrm{genrk}_{\mathbb{C}}(\Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}) = m$ and $\mathrm{genrk}_{\mathbb{C}}(\Gamma^2_{\underline{\mathcal{L}}\mathcal{L}}) = 2m$. Using elementary properties of the flow maps, one can show easily (*cf.* [Mer98]) that $\mathrm{genrk}_{\mathbb{C}}(\Gamma^{\mu_p+k}_{\underline{\mathcal{L}}\mathcal{L}}) = 2m + e_{\{\kappa_p\}}$ for all $k \geq 1$ and that $e_1 > 0, \ldots, e_{\kappa_p} > 0$. The integer μ_p satisfying properties 1 and 2 above is called the *Segre type* of $\mathcal{M}$ at p and the μ_p-tuple $(m, m, e_1, \ldots, e_{\kappa_p})$ is called the *Segre multitype* of $\mathcal{M}$ at p. There is a similar definition of multitype for the maps $\Gamma^k_{\mathcal{L}\underline{\mathcal{L}}}$ instead of $\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}$, but this definition yields the same integers, because of the property $\sigma(\mathcal{L}\underline{\mathcal{L}}^k_{w_{(k)}}(q)) = \underline{\mathcal{L}}\mathcal{L}^k_{\bar{w}_{(k)}}(\sigma(q))$.

Let us summarize the properties of Segre chains in $\mathcal{M}$. For the definition and the presentation of the concept of *orbit* of a system of vector fields, see [Su] [Mer98].

Theorem 3.2.1. *Let* $\mathcal{M} = M^c$, $m = \dim_{CR} M$, $d = \mathrm{codim}_{\mathbb{R}} M \geq 1$, $\dim_{\mathbb{R}} M = 2m + d = \dim_{\mathbb{C}} \mathcal{M}$ *and let* $p^c \in \mathcal{M}$, $p^c = \pi_t^{-1}(\{p\}) \cap \underline{\Lambda}$. *Then there exist an invariant integer* $\mu_p \in \mathbb{N}_*$, $3 \leq \mu_p \leq d + 2$, *the* Segre type *of* $\mathcal{M}$ *at* p^c *and* $(m, m, e_1, \ldots, e_{\kappa_p})$, *the* Segre multitype *of* $\mathcal{M}$ *at* p^c, $\kappa_p = \mu_p - 2$, *and* $w^*_{(\mu_p)} \in (\delta\Delta^m)^{\mu_p}$ *with* $w^*_{\mu_p} = 0$ *and a neighborhood* $\mathcal{W}^*$ *of* $w^*_{(\mu_p)}$ *in* $(\delta\Delta^m)^{\mu_p}$ *such that, putting* $\underline{w}^*_{(\mu_p-1)} = (-w^*_{\mu_p-1}, \ldots, -w^*_1)$, *we have:*

1) $\Gamma^{2\mu_p-1}_{\underline{\mathcal{L}}\mathcal{L}}(\mathcal{W}^* \times \{\underline{w}^*_{(\mu_p-1)}\}) = \mathcal{O}^c_p =$ *constitutes a piece* $\mathcal{O}^c_p$ *of the complexified CR orbit of* M *through* $p^c = \mathcal{N} =$ *a piece* $\mathcal{N}$ *of the orbit* $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$ *through* p^c;

2) $2m + e_1 + \cdots + e_{\kappa_p} = 2m + e_{\{\kappa_p\}} = \dim_{\mathbb{C}} \mathcal{O}^c_p = \dim_{\mathbb{C}} \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c) = \dim_{\mathbb{R}} \mathcal{O}_p$;

3) $\mathrm{genrk}_{\mathbb{C}}(\Gamma^{k+2}_{\underline{\mathcal{L}}\mathcal{L}}) = 2m + e_{\{k\}} = 2m + e_1 + \cdots + e_k = \mathrm{genrk}_{\mathbb{C}}(\Gamma^{k+2}_{\mathcal{L}\underline{\mathcal{L}}})$, $0 \leq k \leq \kappa_p$;

4) $\mathrm{genrk}_{\mathbb{C}}(\Gamma^{k+2}_{\underline{\mathcal{L}}\mathcal{L}}) = 2m + e_{\{\kappa_p\}} = \mathrm{genrk}_{\mathbb{C}}(\Gamma^{k+2}_{\mathcal{L}\underline{\mathcal{L}}})$, $\kappa_p \leq k \leq 3(2n) - 2$.

It is not in this form that we shall use the main properties of Segre chains. A more appropriate theorem which is suitable for our purposes can be stated as follows. This theorem will be mainly used at the very end of the proof of our main Theorem 1.2.1, precisely in Assertion 8.4. More effectively, we will use this theorem in the case of a minimal generic manifold, *i.e.* we will use its Corollary 3.2.6 below.

Theorem 3.2.2. *If μ_p denotes the Segre type of $\mathcal{M}$ at p^c, then there exist a μ_p-tuple $w^*_{(2\mu_p)} \in (\delta\Delta^m)^{2\mu_p}$ and a neighborhood $\mathcal{W}^*$ of $w^*_{(2\mu_p)}$ in $(\delta\Delta^m)^{2\mu_p}$ such that that $\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}(w^*_{(2\mu_p)}) = p$ and, moreover*

$$\mathrm{rk}_{\mathbb{C},w^*_{(2\mu_p)}}(\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = \dim_{\mathbb{C}}\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c), \tag{3.2.3}$$

$$\mathrm{rk}_{\mathbb{C},w^*_{(2\mu_p)}}(\pi_\tau \circ \Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = \mathrm{rk}_{\mathbb{C},w^*_{(2\mu_p)}}(\pi_t \circ \Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = \dim_{\mathbb{C}}\mathcal{O}^{i_c}_p = m + e_{\{\kappa_p\}}, \tag{3.2.4}$$

where $\mathcal{O}^{i_c}_p \subset \mathbb{C}^n_t$ is the intrinsic complexification of a piece $\mathcal{O}_p$ of $\mathcal{O}_{CR}(M,p)$.

Remark. The rank properties (3.2.3) and (3.2.4) above follow in fact more or less directly from the geometric construction in Sussmann's theory of orbits. Furthermore, Theorem 3.2.1 holds for $\delta > 0$ arbitrarily small.

In particular, if M is generic and *minimal* at p, we have

$$\mathrm{rk}_{\mathbb{C},w^*_{(2\mu_p)}}(\pi_\tau \circ \Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = \mathrm{rk}_{\mathbb{C},w^*_{(2\mu_p)}}(\pi_t \circ \Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}) = n. \tag{3.2.5}$$

Hence the appropriate characterization of minimality for our purposes in this article:

Corollary 3.2.6. *If M is minimal at p, for each $\delta > 0$, there exists a μ_p-tuple $(\zeta^*_{\mu_p}, w^*_{\mu_p}, \dots, \zeta^*_1, w^*_1) \in (\delta\Delta^m)^{2\mu_p}$ such that $\underline{\mathcal{L}}_{\zeta^*_{\mu_p}} \circ \mathcal{L}_{w^*_{\mu_p}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta^*_1} \circ \mathcal{L}_{w^*_1}(p^c) = p^c$ and such that the ranks of the two mappings*

$$(\zeta_{\mu_p}, w_{\mu_p}, \dots, \zeta_1, w_1) \mapsto \pi_t \text{ or } \pi_\tau(\underline{\mathcal{L}}_{\zeta_{\mu_p}} \circ \mathcal{L}_{w_{\mu_p}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p^c)) \tag{3.2.7}$$

at the point $(\zeta^*_{\mu_p}, w^*_{\mu_p}, \dots, \zeta^*_1, w^*_1)$ are both equal to n.

Remark. Also, there exists a μ_p-tuple $(w^{**}_{\mu_p}, \zeta^{**}_{\mu_p}, \dots, w^{**}_1, \zeta^{**}_1) \in (\delta\Delta^m)^{2\mu_p}$ such that $\underline{\mathcal{L}}_{\zeta^{**}_{\mu_p}} \circ \mathcal{L}_{w^{**}_{\mu_p}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta^{**}_1} \circ \mathcal{L}_{w^{**}_1}(p^c) = p^c$ and such that the ranks of the two mappings

$$(w_{\mu_p}, \zeta_{\mu_p}, \dots, w_1, \zeta_1) \mapsto \pi_\tau \text{ or } \pi_t(\mathcal{L}_{w_{\mu_p}} \circ \underline{\mathcal{L}}_{\zeta_{\mu_p}} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p^c)) \tag{3.2.8}$$

are both equal to n at the point $(w^{**}_{\mu_p}, \zeta^{**}_{\mu_p}, \dots, w^{**}_1, \zeta^{**}_1)$. In fact, thanks to the property $\sigma(\mathcal{L}_w(p^c)) = \underline{\mathcal{L}}_{\bar{w}}(\sigma(p^c))$, it easy to see that one can choose

$$(w^{**}_{\mu_p}, \zeta^{**}_{\mu_p}, \dots, w^{**}_1, \zeta^{**}_1) := (\bar{\zeta}^*_{\mu_p}, \bar{w}^*_{\mu_p}, \dots, \bar{\zeta}^*_1, \bar{w}^*_1). \tag{3.2.9}$$

This completes the presentation of Segre chains.

§4. The mapping

Let M' be a second real analytic CR manifold, which is generic in $\mathbb{C}^{n'}$, with $\dim_{CR} M' = m' \geq 1$ and $\text{codim}_{\mathbb{R}} M' = d' \geq 1$, $m' + d' = n'$, $\dim_{\mathbb{R}} M' = 2m' + d'$, and let $p' \in M'$. As for M, there exist holomorphic coordinates (w', z') vanishing at p' in which M' is given by two equivalent systems of d' scalar equations:

$$z' = \bar{z}' + i\bar{\Theta}'(\bar{w}', w', \bar{z}') \qquad \text{or} \qquad \bar{z}' = z' - i\Theta'(w', \bar{w}', z'), \tag{4.1}$$

with Θ' converging in $(4\Delta)^{2m'+d'}$ and also, two equivalent systems of d' scalar equations for its extrinsic complexification $\mathcal{M}'$:

$$z' = \xi' + i\bar{\Theta}'(\zeta', w', \xi') \qquad \text{or} \qquad \xi' = z' - i\Theta'(w', \zeta', z'). \tag{4.2}$$

Let now $h : (\mathbb{C}^n, p) \mapsto_{\mathcal{F}} (\mathbb{C}^{n'}, p')$, $h(t) = (\sum_{\alpha \in \mathbb{N}^n_*} h_{1,\alpha} t^\alpha, \dots, \sum_{\alpha \in \mathbb{N}^n_*} h_{n',\alpha} t^\alpha)$ be a formal holomorphic mapping, in our coordinates in which $p = 0$, $p' = 0$, $h(0) = 0$. By definition, h is called *invertible at* 0 (as in (i) of Proposition 1.5.2), if $n' = n$ and if we have $\det(h_{i,\mathbf{1}_j})_{1 \leq i,j \leq n} \neq 0$, if $\mathbf{1}_j$ denotes the n-tuple $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the place of the i-th digit.

We write $h = (h_1, \dots, h_{n'}) = (g_1, \dots, g_{m'}, f_1, \dots, f_{d'}) = (g, f)$ according to the splitting of coordinates (w', z') which is coherent with the disposition of the complex tangent space to M', $T^c_0 M' = \mathbb{C}^{m'}_{w'} \times \{0\}$. In order to provide an aesthetical presentation of all the formalism used throughout the article, we shall maintain a uniform *vectorial convention*, most often omitting certain superfluous indices. For instance, we shall write $\det(\underline{\mathcal{L}} \bar{g})$ instead of $\det(\underline{\mathcal{L}}_j \bar{g}_k)_{1 \leq j,k \leq m}$ (if $n = n'$, $m = m'$).

Without loss of generality, we can assume that the coordinates (w, z) for M and (w', z') for M' are *normal*, *i.e.* that $\Theta(0, \zeta, z) \equiv 0$, $\Theta(w, 0, z) \equiv 0$ and that $\Theta'(0, \zeta', z') \equiv 0$, $\Theta'(w', 0, z') \equiv 0$.

Our main assumption is that $h : (M, 0) \to_{\mathcal{F}} (M', 0)$ maps $(M, 0)$ into $(M', 0)$ *formally*. This means that the following two equivalent formal identities (between formal power series) hold

$$\begin{aligned} f(w, z) &\equiv \left[\bar{f}(\bar{w}, \bar{z}) + i\bar{\Theta}'(\bar{g}(\bar{w}, \bar{z}), g(w, z), \bar{f}(\bar{w}, \bar{z}))\right]_{\bar{z} := z - i\Theta(w, \bar{w}, z)} \\ \bar{f}(\bar{w}, \bar{z}) &\equiv \left[f(w, z) - i\Theta'(g(w, z), \bar{g}(\bar{w}, \bar{z}), f(w, z))\right]_{z := \bar{z} + i\bar{\Theta}(\bar{w}, w, \bar{z})} \end{aligned} \tag{4.3}$$

in $\mathbb{C}[\![w, \bar{w}, z]\!]$ and in $\mathbb{C}[\![\bar{w}, w, \bar{z}]\!]$ respectively. Consequently, h induces a formal map $h^c := (h, \bar{h})$, $h^c(t, \tau) = (h(t), \bar{h}(\tau)) = (g(w, z), f(w, z), \bar{g}(\zeta, \xi), \bar{f}(\zeta, \xi))$ between $(\mathcal{M}, 0)$ and $(\mathcal{M}', 0)$, which can be constructed just by complexifying the two formal identities in (4.3). We obtain thus two equivalent formal identities (between formal power series):

$$\begin{aligned} f(w, z) &= \left[\bar{f}(\zeta, \xi) + i\bar{\Theta}'(\bar{g}(\zeta, \xi), g(w, z), \bar{f}(\zeta, \xi))\right]_{\xi := z - i\Theta(w, \zeta, z)} \\ \bar{f}(\zeta, \xi) &= \left[f(w, z) - i\Theta'(g(w, z), \bar{g}(\zeta, \xi), f(w, z))\right]_{z := \xi + i\bar{\Theta}(\zeta, w, \xi)} \end{aligned} \tag{4.4}$$

in $\mathbb{C}[\![w, \zeta, \xi]\!]$ and in $\mathbb{C}[\![\zeta, w, \xi]\!]$ respectively, after replacing ξ by $z - i\Theta(w, \zeta, z)$ in the first equation of (4.4) and z by $\xi + i\bar{\Theta}(\zeta, w, \xi)$ in the second equation. Then (4.4) means that $h^c(\mathcal{M}, 0) \subset_{\mathcal{F}} (\mathcal{M}', 0)$, *i.e.* that h^c maps $(\mathcal{M}, 0)$ formally into $(\mathcal{M}', 0)$.

In principle, *both the triples* (ζ, w, z) *and* (w, ζ, ξ) can be chosen as coordinates on $\mathcal{M}$. *There is no canonical or preferred choice.* Consequently, we will opt here for a systematic twofold presentation of everything, in coherence with the twofold theory of Segre chains which was built [Mer98]. Furthermore, we shall henceforth work only with the complexified map $h^c : (\mathcal{M}, 0) \to_{\mathcal{F}} (\mathcal{M}', 0)$. Let us finish with a

Lemma 4.5. *Let $h^c = (h, \bar{h})\colon (\mathcal{M}, 0) \to_{\mathcal{F}} (M', 0)$ as above, $h^c(t,\tau) = (h(t), \bar{h}(\tau))$. Then $\sigma' \circ h^c = h^c \circ \sigma$. Also, notice that $h \circ \underline{\mathcal{L}}_\zeta(q(x)) \equiv h(q(x))$ in $\mathbb{C}[\![x,\zeta]\!]$ and that $\bar{h} \circ \mathcal{L}_w(q(x)) \equiv \bar{h}(q(x))$ in $\mathbb{C}[\![x,w]\!]$ for any formal series $q(x) \in \mathbb{C}[\![x]\!]^{2n}$, $q(0)=0$.*

Proof. The first property is trivial:

$$\sigma' \circ h^c(t,\tau) = \sigma'(h(t), \bar{h}(\tau)) = (h(\bar{\tau}), \bar{h}(\bar{t})) = h^c(\bar{\tau}, \bar{t}) = h \circ \sigma(t,\tau). \tag{4.6}$$

The second property follows easily from eq. (3.1.4), from which we see that $\pi_t \circ \underline{\mathcal{L}}_\zeta(q(x)) = \pi_t(q(x))$ and $\pi_\tau \circ \mathcal{L}_w(q(x)) = \pi_\tau(q(x))$ and from $\bar{h}(r(w,x)) = \bar{h} \circ \pi_\tau(r(w,x))$ and $h(r(\zeta,x)) = h \circ \pi_t(s(\zeta,x))$ for any two formal power series $r(w,x) \in \mathbb{C}[\![w,x]\!]$ and $s(\zeta,x) \in \mathbb{C}[\![\zeta,x]\!]$. □

§5. Differentiations

The purpose of this paragraph is to prove Proposition 1.5.2 (i), namely to establish that any formal invertible holomorphic mapping between S-nondegenerate CR manifolds is S-nondegenerate (Proposition 5.13 in this paragraph). We thus work in the equidimensional case, *i.e.* with $m' = m$, $d' = d$ and $n' = n$.

We first derive a convient expression of the *fundamental family of identities* written in eq. (1.1.2) as follows.

Let us consider the derivations $\underline{\mathcal{L}}^\beta = \underline{\mathcal{L}}_1^{\beta_1} \cdots \underline{\mathcal{L}}_{n_1}^{\beta_{n-1}}$. Then applying all these derivations of any order to the identity $\bar{\rho}'(\bar{h}(\tau), h(t))$, *i.e.* to

$$\bar{f}(\zeta,\xi) \equiv f(w,z) - i \sum_{\gamma \in \mathbb{N}_*} \bar{g}(\zeta,\xi)^\gamma \ \Theta'_\gamma(g(w,z), f(w,z)), \tag{5.2}$$

as $(w,z,\zeta,\xi) \in \mathcal{M}$, it is well-known that we obtain an infinite family of formal identities that we collect in an independent statement that we will reprove quickly below, for completeness (*cf.* [BR88] [BR90] [BER97] [BERbk] [CPS2] [Mer99c]). Let $\Theta'_{\zeta'^\beta}(w',\zeta',z')$ denote $\partial_{\zeta'}^\beta \Theta'(w',\zeta',z')$, $\beta \in \mathbb{N}^m$, $\partial_{\zeta'}^\beta = \partial_{\zeta_1'}^{\beta_1} \cdots \partial_{\zeta_m'}^{\beta_m}$.

Proposition 5.1. *Let $h : (M,0) \to_{\mathcal{F}} (M',0)$ be a formal biholomorphism between CR generic $\mathcal{C}^\omega$ manifolds in $\mathbb{C}^n$. For every $\beta \in \mathbb{N}^m$, there exists a collection of $d \times d$ matrices of universal polynomial $\underline{u}_{\beta,\gamma}$, $|\gamma| \le |\beta|$ in $mN_{m,|\beta|}$ variables, where $N_{m,|\beta|} = \frac{(|\beta|+m)!}{|\beta|! \ m!}$, and holomorphic $\mathbb{C}^d$-valued functions $\underline{\Omega}_\beta$ in $(2n - d + n'N_{n,|\beta|})$ variables near $0 \times 0 \times 0 \times (\partial_\xi^{\alpha^1} \partial_\zeta^{\gamma^1} \bar{h}(0))_{|\alpha^1| + |\gamma^1| \le |\beta|}$ in $\mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C}^d \times \mathbb{C}^{n'N_{n,|\beta|}}$ such that*

$$\begin{aligned}
\underline{\theta}'_\beta(w,\zeta,\xi) &:= \Big[\Theta'_{\zeta'^\beta}(g(w,z), \bar{g}(\zeta,\xi), f(w,z))\Big]_{z = \xi + i\bar{\Theta}(\zeta,w,\xi)} \equiv_{w,\zeta,\xi} \\
&\equiv \sum_{|\gamma| \le |\beta|} \frac{\underline{\mathcal{L}}^\gamma \bar{f}(\zeta,\xi) \ \underline{u}_{\beta,\gamma}((\underline{\mathcal{L}}^\delta \bar{g}(\zeta,\xi))_{|\delta| \le |\beta|})}{\underline{\Delta}(w,\zeta,\xi)^{2|\beta|-1}} \\
&\equiv: \underline{\Omega}_\beta(w,\zeta,\xi, (\partial_\xi^{\alpha^1} \partial_\zeta^{\gamma^1} \bar{h}(\zeta,\xi))_{|\alpha^1| + |\gamma^1| \le |\beta|}) \\
&=: \underline{\omega}_\beta(w,\zeta,\xi),
\end{aligned} \tag{5.3}$$

as formal power series in (w,ζ,ξ)*, where*

$$\underline{\Delta}(w,\zeta,\xi)=\underline{\Delta}(w,z,\zeta,\xi)|_{z=\xi+i\bar{\Theta}(\zeta,w,\xi)}=\det(\underline{\mathcal{L}}\bar{g})= \tag{5.4}$$
$$=\det\left(\frac{\partial\bar{g}}{\partial\zeta}(\zeta,\xi)-i\Theta_\zeta(w,\zeta,z)\frac{\partial\bar{g}}{\partial\xi}(\zeta,\xi)\right)|_{z=\xi+i\bar{\Theta}(\zeta,w,\xi)}.$$

Remarks.

1. As $\det(h_{i,\mathbf{1}_j})_{1\le i,j\le n}\neq 0$, then $\det(\underline{\mathcal{L}}\bar{g}(0))\neq 0$ also, so $\underline{\Delta}^{1-2|\beta|}\in\mathbb{C}[\![w,\zeta,\xi]\!]$.

2. Putting $(\zeta,\xi)=(0,0)$, but after perharps shrinking r, we first readily observe that $\underline{\Delta}^{1-2|\beta|}(w,0,0)\in\mathcal{O}((r\Delta)^m,\mathbb{C})$ for some $r>0$, since $\Theta_\zeta(w,0,0)\in\mathcal{O}((r\Delta)^m,\mathbb{C})$ and since $\partial_\zeta^{\gamma^1}\bar{g}(0,0)$ for $|\gamma^1|=1$ and $\partial_\xi^{\alpha^1}\bar{g}(0,0)$ for $|\alpha^1|=1$ are constants, so that we observe finally that

$$\underline{\Omega}_\beta(w,0,0,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(0,0))_{|\alpha^1|+|\gamma^1|\le|\beta|})\in\mathcal{O}((r\Delta)^m,\mathbb{C}), \tag{5.5}$$

for all $\beta\in\mathbb{N}_*^m$, because all the coefficients of the $\underline{\mathcal{L}}^\gamma$ belong to $\mathcal{O}((r\Delta)^m,\mathbb{C})$. Therefore, *the domains of convergence of the* $\underline{\omega}_\beta(w,0,0)$ *are independent of* β*.*

3. $\underline{\Omega}_\beta$ arises after writing $\underline{\mathcal{L}}^\gamma\bar{h}(\zeta,\xi)$ as $\chi_\delta(w,z,\zeta,\xi,(\partial_\xi^{\alpha^1}\partial_\zeta^{\gamma^1}\bar{h}(\zeta,\xi))_{|\alpha^1|+|\gamma^1|\le|\beta|})$ (by noticing that the coefficients of $\underline{\mathcal{L}}$ are analytic in (w,z,ζ,ξ)) and by replacing again z by $\xi+i\bar{\Theta}(\zeta,w,\xi)$).

Proof. Applying the $\underline{\mathcal{L}}_j$, $1\le j\le m$, to eq. (5.1), we obtain

$$\underline{\mathcal{L}}_j\bar{f}=-i\sum_{|\beta|=1}\underline{\mathcal{L}}_j\bar{g}^\beta\ \Theta'_{\zeta'^\beta}(g,\bar{g},f),\qquad 1\le j\le m, \tag{5.6}$$

Then Cramer's rule applied to (5.6) yields immediately (5.3) for $|\beta|=1$.

By induction, applying the $\underline{\mathcal{L}}_j$, $1\le j\le m$, to (5.3), we obtain

$$\frac{1}{\beta!}\sum_{|\beta_1|=1}\underline{\mathcal{L}}_j\bar{g}_{\lrcorner\beta_1}\Theta'_{\zeta'^{\beta+\beta_1}}(g,\bar{g},f)\equiv \tag{5.7}$$
$$\equiv\sum_{|\gamma|\le|\beta|}\frac{\underline{\mathcal{L}}^{\gamma+1_j}\bar{f}\ \underline{u}_{\beta,\gamma}((\underline{\mathcal{L}}^\delta\bar{g})_{|\delta|\le|\beta|})}{\underline{\Delta}^{2|\beta|-1}}+\frac{\underline{\mathcal{L}}^\gamma\bar{f}\ \sum_{X_\delta}\frac{\partial\underline{u}_{\beta,\gamma}}{\partial X_\delta}((\underline{\mathcal{L}}^\delta\bar{g})_{|\delta|\le|\beta|})\ \underline{\mathcal{L}}^{\delta+\mathbf{1}_j}\bar{g}}{\underline{\Delta}^{2|\beta|-1}}+$$
$$+\frac{\underline{\mathcal{L}}^\gamma\bar{f}\ \ \underline{u}_{\beta,\gamma}((\underline{\mathcal{L}}^\delta\bar{g})_{|\delta|\le|\beta|})(1-2|\beta|)\underline{\mathcal{L}}_j(\underline{\Delta})}{\underline{\Delta}^{2|\beta|}},$$

where $\lrcorner\beta$ is the integer k such that $\beta_k=1$ if $|\beta|=1$ and where $((X_\delta)_{|\delta|\le|\beta|})$ denote indeterminates in place of $((\underline{\mathcal{L}}^\delta\bar{g})_{|\delta|\le|\beta|})$ for the functions $u_{\beta,\gamma}=u_{\beta,\gamma}((X_\delta)_{|\delta|\le|\beta|})$.

Now, applying Cramer's rule to (5.7), we get (5.3) with $\beta+\beta_1$ instead of β, for all $|\beta_1|=1$, which completes the proof of Proposition 5.2. □

Remark. Analogously, chosing instead the derivation $\mathcal{L}^\beta$, we have

$$\theta'_\beta(\zeta,w,z):=\left[\bar{\Theta}'_{w^\beta}(\bar{g}(\zeta,\xi),g(w,z),\bar{f}(\zeta,\xi))\right]_{\xi=i\Theta(w,\zeta,z)}\equiv_{\zeta,w,z}\equiv \tag{5.8}$$
$$\sum_{|\gamma|\le|\beta|}\frac{\mathcal{L}^\gamma f(w,z)u_{\beta,\gamma}((\mathcal{L}^\delta g(w,z))_{|\delta|\le|\beta|})}{\Delta(\zeta,w,z)^{2|\beta|-1}}$$
$$=\Omega_\beta(\zeta,w,z,(\partial_z^\alpha\partial_w^\gamma h(w,z))_{|\alpha|+|\gamma|\le|\beta|})$$

as formal power series in (ζ, w, z), where

$$\Delta(\zeta, w, z) = \Delta(\zeta, \xi, w, z)|_{\xi = z - i\Theta(w,\zeta,z)} = \det(\mathcal{L}g) = \det\left(\frac{\partial g}{\partial w}(w,z) + i\bar{\Theta}_w(\zeta, w, \xi)\frac{\partial g}{\partial z}(w,z)\right)_{\xi = z - i\Theta(w,\zeta,z)}. \tag{5.9}$$

But these new identities offer no real new information, because

Lemma 5.10. *We have*
$\underline{\overline{\theta}}'_\beta(\zeta, w, z) \equiv \theta'_\beta(\zeta, w, z)$,
$\underline{\overline{\Delta}}(\zeta, w, z) \equiv \Delta(\zeta, w, z)$,
$\underline{\overline{\Omega}}_\beta(\zeta, w, z, (\partial_z^\alpha \partial_w^\gamma h(w,z))_{|\alpha|+|\gamma|\le|\beta|}) \equiv \Omega_\beta(\zeta, w, z, (\partial_z^\alpha \partial_w^\gamma h(w,z))_{|\alpha|+|\gamma|\le|\beta|})$,
$\underline{\overline{u}}_{\beta,\gamma}((\mathcal{L}^\delta g(w,z))_{|\delta|\le|\beta|}) \equiv u_{\beta,\gamma}((\mathcal{L}^\delta g(w,z))_{|\delta|\le|\beta|})$.

Proof. When $\zeta = \bar{w}$ and $\xi = \bar{z}$, the equations (5.8) and (5.9) are just conjugates of (5.3) and (5.4) respectively. □

We now give a characterization of S-nondegeneracy of a $\mathcal{C}^\omega$ generic M'.

Proposition 5.11. *Assume that M' is given in normal coordinates. Then M' is S-nondegenerate if and only if there exist $\beta_1, \dots, \beta_{m'} \in \mathbb{N}_*^{m'}$ and integers $l_1, \dots, l_{m'}$, $1 \le l_i \le d'$, such that*

$$\det\left(\frac{\partial {\Theta'}^{l_i}_{\zeta^{\beta_i}}}{\partial {w'}_j}(w', 0, 0)\right)_{1\le i,j\le m'} \not\equiv_{w'} 0 \qquad \text{in } \mathbb{C}[\![w']\!]. \tag{5.12}$$

Proof. This condition can be seen after replacing t' by $(w', i\bar{\Theta}'(0, w', 0)) = (w', 0)$ in eq. (1.1.6) of the introduction and $\rho'(t', \tau')$ by $\xi' - z' + i\Theta'(w', \zeta', z')$. Indeed, if $h = (g, f) = \text{Id} = (w', z')$, we have $\rho'(h(w', 0), \bar{h}(0)) = f(w', 0)$, so in the determinant (1.1.6), we can already include the terms $\rho'(h(w', 0), \bar{h}(0)) = f(w', 0)$ amongst all the equations R'_γ, and these terms satisfy the nice minor property $\frac{\partial \rho'}{\partial z'} = \text{Id}_{d'\times d'}$. Thus, it only remains to find equations for $g(w', 0)$ similar to (1.1.6). But the set $\{R'_{\beta'} = 0\}_{\beta' \in \mathbb{N}_*^{m'}}$ coincides with the set $\{\Theta'_{\zeta'^{\beta'}} - \underline{\Omega}_{\beta'} = 0\}_{\beta' \in \mathbb{N}_*^{m'}}$ in eq. (5.3). Since $g(w', 0) = w'$, then condition (1.1.7) reduces exactly to (5.12). □

Proposition 5.13. *If $n = n'$, if M' is S-nondegenerate at p' and if h is of formal rank n, then the formal map $h\colon (M, p) \to_{\mathcal{F}} (M', p')$ is S-nondegenerate.*

Proof. This is immediate, after using the characterization given in Proposition 5.11, the equations given in eq. (5.3) and the fact that $\det\left(\frac{\partial g_j}{\partial w_k}{}_{1\le j,k\le m}(0,0)\right) \neq 0$. □

§6. Proof of Proposition 1.5.2 (ii)

The purpose of this paragraph is to prove Proposition 1.5.2 (ii), namely to establish

Proposition 6.1. *If $m \geq m'$, if M' is S-nondegenerate at p' and if h induces a formal map $(S_{\bar{p}}, p) \to_{\mathcal{F}} (S_{\bar{p}'}, p')$ of generic rank equal to $m' = \dim_{\mathbb{C}} S_{\bar{p}'}$, then the formal map $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ is S-nondegenerate.*

Proof. Thus, assume that the formal map $h\colon (S_{\bar{p}}, p) \to_{\mathcal{F}} (S'_{\bar{p}'}, p')$ is of generic rank equal to m' (hence in particular, m must satisfy $m \geq m'$). After renumbering the *normal* coordinates (w', z') for M' near 0, we claim that can assume that the minor $\det(\underline{\mathcal{L}}\bar{g}(\zeta,\xi))^{\square} := \det\left(\underline{\mathcal{L}}_j \bar{g}_k(\zeta,\xi)\right)_{1\leq j,k\leq m'}$ satisfies $\det(\underline{\mathcal{L}}\bar{g}(\zeta,0))^{\square} \not\equiv 0$ in $\mathbb{C}[\![\zeta]\!]$, *i.e.* more precisely, that $[\det(\underline{\mathcal{L}}\bar{g}(\zeta,\xi))^{\square}]_{w=0,z=0,\xi=0} \not\equiv_{\zeta} 0$ in $\mathbb{C}[\![\zeta]\!]$. Indeed, for any power series $a(\zeta)$, then $\underline{\mathcal{L}}^{\gamma} a(\zeta)|_{w=z=\xi=0} = (\partial^{|\gamma|}/\partial\zeta^{\gamma})a(\zeta)$, because $\underline{\mathcal{L}} = \partial/\partial\zeta$ at $(w,z,\zeta,\xi) = (0,0,\zeta,0)$. This property can be again readily seen by noticing that the induced mapping $h|_{S_0}\colon (S_0,0) \to_{\mathcal{F}} (S'_0,0)$ is simply given by $(\zeta,0) \mapsto_{\mathcal{F}} (g(\zeta),0)$.

Now, it is easy to observe after using the adjoint matrix of $(\underline{\mathcal{L}}\bar{g}(\zeta,\xi))^{\square}$ (instead of its inverse in case it is invertible) that the same calculation as the one that we did in Proposition 5.2 yields immediately

(6.2)
$$\begin{aligned}&(\det(\underline{\mathcal{L}}\bar{g}(\zeta,\xi))^{\square})^{2|\beta|-1}\left[\Theta'_{\zeta'^{\beta}}(g(w,z),\bar{g}(\zeta,\xi),f(w,z))\right]_{z=\xi+i\bar{\Theta}(\zeta,w,\xi)} \equiv_{w,\zeta,\xi}\\ &\equiv \sum_{|\gamma|\leq|\beta|} \underline{\mathcal{L}}^{\gamma}\bar{f}(\zeta,\xi)\, \underline{u}_{\beta,\gamma}((\underline{\mathcal{L}}^{\delta}\bar{g}(\zeta,\xi))_{|\delta|\leq|\beta|}) =: \underline{\mathcal{L}}^{\gamma^{\sharp}(2|\beta|-1)}\underline{\Omega}^{1}_{\beta}/(\underline{\mathcal{L}}^{\gamma^{\sharp}(2|\beta|-1)}\Delta^{2|\beta|+1})\end{aligned}$$

Now the determinant $\underline{\Delta}(w,\zeta,\xi) := \det(\underline{\mathcal{L}}\bar{g}(\zeta,\xi))^{\square}$ satisfies $\underline{\Delta}(0,\zeta,0) \not\equiv 0$ by assumption. This holds if and only if there exists a multiindex $\gamma_{\sharp} \in \mathbb{N}^m_*$ such that $[\underline{\mathcal{L}}^{\gamma_{\sharp}}\underline{\Delta}(w,\zeta,\xi)]|_{w=z=\xi=0} \neq 0$. Indeed, $\underline{\mathcal{L}}^{\gamma}a(\zeta)|_{\zeta=0} = (\partial^{|\gamma|}/\partial\zeta^{\gamma})a(\zeta)|_{\zeta=0}$ because $\underline{\mathcal{L}} = \partial/\partial\zeta$ at $(w,\zeta,z) = (0,\zeta,0)$.

Moreover, we can choose $\gamma^{\sharp}$ to be minimal with respect to the lexicographic order.

By applying now the derivation $(\underline{\mathcal{L}}^{\gamma_{\sharp}})^{2|\beta|-1}$ to eq. (6.2), a derivation which satisfies $(\underline{\mathcal{L}}^{\gamma_{\sharp}})^{2|\beta|-1}(\underline{\Delta}(w,\zeta,\xi)^{2|\beta|-1})|_{w=z=\zeta=\xi=0} \neq 0$, but

$$[\underline{\mathcal{L}}^{\gamma}(\underline{\Delta}(w,\zeta,\xi)^{2|\beta|-1})]|_{w=\zeta=\xi=0} = 0, \quad \forall\ \gamma < (\gamma_{\sharp})(2|\beta|-1) \tag{6.3}$$

(since $\gamma^{\sharp}$ is minimal with respect to the lexicographic order), we get that there exists a holomorphic remainder C_{β} such that we can write

(6.4)
$$\begin{aligned}&\left[\Theta'_{\zeta'^{\beta}}(g(w,z),\bar{g}(\zeta,\xi),f(w,z))\right]_{z=\xi+i\bar{\Theta}(\zeta,w,\xi)} + C_{\beta}(g(w,z),f(w,z),\nabla^{|\gamma_{\sharp}|(2|\beta|-1)}\bar{g}\\ &\bar{g}(\zeta,\xi)) = \underline{\Xi}_{\beta}(w,\zeta,\xi,\nabla^{|\beta|+|\gamma_{\sharp}|(2|\beta|-1)}\bar{h}(\zeta,\xi)) = \underline{\mathcal{L}}^{\gamma^{\sharp}(2|\beta|-1)}\underline{\Omega}^{1}_{\beta}/(\underline{\mathcal{L}}^{\gamma^{\sharp}(2|\beta|-1)}\Delta)\end{aligned}$$

a remainder which has the property

$$C_{\beta}(g(w,0),f(w,0),\nabla^{|\gamma_{\sharp}|(2|\beta|-1)}\bar{g}(0,0)) \equiv 0. \tag{6.5}$$

In summary, amongst the equations ${R'}^{l}_{\beta}(w,0,0,h(w,0),\nabla^{|\beta|}\bar{h}(0)) \equiv 0$ for $\beta \neq 0$, we have obtained equations of the form

$$\Theta'_{\zeta'^{\beta}}(g(w,0),\bar{g}(0),0) = \underline{\Xi}_{\beta}(w,0,0,\nabla^{|\beta|+|\gamma_{\sharp}|(2|\beta|-1)}\bar{h}(0,0)) \tag{6.6}$$

(recall $f(w,0) \equiv 0$). Then condition (1.1.6) defining S-nondegeneracy is clearly satisfied, because (5.12) holds and $\det(\mathcal{L}g(w,0))^{\square} \not\equiv 0$. $\square$

§7. Propagation of analyticity along Segre chains

Let us recall that we started in the introdution with the family of equations

$$0 = \underline{\mathcal{L}}^\gamma \rho'(h(t), \bar{h}(\tau)) := R'_\gamma(t, \tau, h(t), \nabla^{|\gamma|}\bar{h}(\tau)) = 0 \tag{7.1}$$

satisfied by $h(t)$ as $\rho(t,\tau) = 0$.

To treat S-nondegenerate maps, we delineate the following statement.

Theorem 7.2. *Let $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ be a formal holomorphic map with M given by equations* (2.1.1) *and minimal at p. Assume that there exist $\kappa_0 \in \mathbb{N}_*$ and a finite family of relations of the form*

$$\mathcal{X}'_\lambda(t, \tau, h(t), \nabla^{\kappa_0}\bar{h}(\tau)) \equiv 0 \quad \text{on} \quad \mathcal{M} = \{\rho(t,\tau) = 0\}, \tag{7.3}$$

for $\lambda = 1, \ldots, \Lambda$, $\Lambda \in \mathbb{N}_*$, the $\mathcal{X}'_\lambda = \mathcal{X}'_\lambda(t, \tau, t', \nabla^{\kappa_0})$ being holomorphic from a neighborhood of $0 \times 0 \times 0 \times \nabla^{\kappa_0}\bar{h}(0)$ in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n'} \times \mathbb{C}^{n' N_{n,\kappa_0}}$, and $\mathbb{C}$-valued, such that the following nondegeneracy condition holds:

(∗) *There exist $\lambda_1, \ldots, \lambda_{n'} \in [\![1, \Lambda]\!]$ such that*

$$\det\left(\frac{\partial \mathcal{X}'_{\lambda_j}}{\partial t'_k}(w, i\bar{\Theta}(0,w,0), 0, h(w, i\bar{\Theta}(0,w,0)), \nabla^{\kappa_0}\bar{h}(0))\right)_{1\leq j,k\leq n'} \not\equiv_w 0 \quad \text{in} \quad \mathbb{C}[\![w]\!]. \tag{7.4}$$

Then h is convergent.

To treat S-finite maps and in the same time S-solvable maps, we delineate the following statement.

Theorem 7.5. [BER99] *Let $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ be a formal holomorphic map, with M given by equation (2.1.1) and minimal at p. Assume that*

(∗∗) *There exist $\kappa_0 \in \mathbb{N}_*$, $N'_j \in \mathbb{N}_*$, $1 \leq j \leq n'$ and monic Weierstrass polynomials of the form $P'_j(t, \tau, t'_j, \nabla^{\kappa_0})$,*

$$P'_j(t, \tau, t'_j, \nabla^{\kappa_0}) = {t'}_j^{N'_j} + \sum_{1\leq k\leq N'_j} A'_{j,k}(t, \tau, \nabla^{\kappa_0}) {t'}_j^{N'_j - k}, \tag{7.6}$$

the $A'_{j,k}$ being holomorphic from a neighborhood of $0 \times 0 \times \nabla^{\kappa_0}\bar{h}(0)$ in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n' N_{n,\kappa_0}}$, $1 \leq j \leq n'$, and $\mathbb{C}$-valued, such that the following formal identities

$$P'_j(t, \tau, h_j(t), \nabla^{\kappa_0}\bar{h}(\tau)) \equiv 0, \qquad 1 \leq j \leq n', \tag{7.7}$$

hold if $\rho(t,\tau) = 0$. Then h is convergent.

To apply Theorem 7.5 to S-finite formal maps, we shall consider a transformation of the complex analytic subset of a neighborhood of $0 \times 0 \times 0 \times \nabla^{\kappa_0}\bar{h}(0)$ in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{n'} \times \mathbb{C}^{n' N_{n,\kappa_0}}$ defined by the equations

$$\mathbb{S}^1 := \{(t, \tau, t', \nabla^{\kappa_0})\colon \rho(t,\tau) = 0, R'_\gamma(t, \tau, t', \nabla^{\kappa_0}\bar{h}(\tau)) = 0 \ \forall |\gamma| \leq \kappa_0\}, \tag{7.8}$$

where κ_0 is large enough.

Thanks to the assumption that $\dim_{p'}(\mathbb{V}'_p) = 0$ ($\mathbb{V}'_p = \mathbb{S}^1 \cap (0 \times 0 \times \mathbb{C}^{n'} \times \nabla^{\kappa_0}\bar{h}(0))$), we can utilize a classical transformation of the equations of $\mathbb{S}^1$ which consists in replacing the R'_γ's by the canonical defining functions of the ramified analytic cover $\pi\colon S^1 \to \mathbb{C}^n \times \mathbb{C}^n \times 0 \times \mathbb{C}^{n' N_{n,\kappa_0}}$ (Whitney's contruction, [Ch1], Chapter 1, paragraph 4, pp. 42-51) to obtain:

Lemma 7.9. *If h is S-finite at p, then there exist $N \in \mathbb{N}_*$, $\kappa_0 \in \mathbb{N}_*$ and Weierstrass polynomials $P'_j(t,\tau,t'_j,\nabla^{\kappa_0}) = {t'_j}^N + \sum_{1\leq k\leq N} A_{j,k}(t,\tau,\nabla^{\kappa_0}){t'_j}^{N-k}$, with $A_{j,k}$ being holomorphic in a neighborhood of $0\times 0\times \nabla^{\kappa_0}\bar{h}(0)$ in $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^{n'N_{n,\kappa_0}}$, $1\leq j\leq n'$, such that $\mathbb{S}^1$ is contained in the complex analytic set*

$$\mathbb{E}^1 := \{(t,\tau,t',\nabla^{\kappa_0})\colon P'_j(t,\tau,t',\nabla^{\kappa_0}) = 0\} \tag{7.10}$$

and the following formal equalities

$$P'_j(t,\tau,h_j(t),\nabla^{\kappa_0}\bar{h}(\tau)) \equiv 0, \qquad 1\leq j\leq n', \tag{7.11}$$

hold if (t,τ) satisfy $\rho(t,\tau)=0$.

Proof. Existence of $\mathbb{E}^1\supset\mathbb{S}^1$ follows by taking a suitable subset of the set of canonical defining functions of the analytic cover $\pi\colon \mathbb{S}^1\to\mathbb{C}^n\times\mathbb{C}^n\times 0\times\mathbb{C}^{n'N_{n,\kappa_0}}$, namely the set of functions which are polynomial in a single variable t'_j (see [Ch1], p.42).

Let $\mathcal{I}_{\mathbb{S}^1}$ denote the ideal $(R'_\gamma)_{|\gamma|\leq\kappa_0}$ in $\mathcal{O}_t\times\mathcal{O}_\tau\times\mathcal{O}_{t'}\times\mathcal{O}_{\nabla^{\kappa_0}}$. Each P'_j vanishes over $\mathbb{S}^1$. Thus, by the Nullstellensatz, there exist integers $M_j\in\mathbb{N}_*$ such that ${P'}_j^{M_j}\in\mathcal{I}_{\mathbb{S}^1}$, $1\leq j\leq n'$. We deduce $P'_j(t,\tau,h(t),\nabla^{\kappa_0}\bar{h}(\tau))^{M_j}\equiv 0$ when $\rho(t,\tau)=0$, whence $P'_j(t,\tau,h(t),\nabla^{\kappa_0}\bar{h}(\tau))\equiv 0$ as desired. □

The main feature of S-finite maps is that after applying Lemma 7.9, we see that each component $h_j(t)$ is entire over $\bar{h}(\tau)$ and its jets, that is to say, it satisfies a monic polynomial equation (7.11).

In particular, as S-solvable CR maps are S-finite, they satisfy Lemma 7.9 above. In truth, they satisfy a much more aesthetic relation.

Lemma 7.12. *If h is S-solvable at $p\in M$, then there exist $N\in\mathbb{N}_*$, $\kappa_0\in\mathbb{N}_*$, and analytic functions $A'_j(t,\tau,\nabla^{\kappa_0})$ holomorphic in a neighborhood of $0\times 0\times\nabla^{\kappa_0}\bar{h}(0)$ in $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^{n'N_{n,\kappa_0}}$, such that the following formal equalities*

$$h_j(t)\equiv A'_j(t,\tau,\nabla^{\kappa_0}\bar{h}(\tau)), \qquad 1\leq j\leq n', \tag{7.13}$$

hold if (t,τ) satisfy $\rho(t,\tau)=0$.

Proof. It suffices to apply the analytic implicit function theorem to the collection of equations

$$0 = \underline{\mathcal{L}}^\gamma\rho'(t',\bar{h}(\tau)) = R'_\gamma(t,\tau,t',\nabla^{|\gamma|}\bar{h}(\tau)), \qquad |\gamma|\leq\kappa_0, \tag{7.14}$$

to solve them in t' as $t'_j = A'_j(t,\tau,\nabla^{\kappa_0})$. In fact, h is S-solvable if and only if eq. (7.14) is solvable (implicit function theorem) in t' for κ_0 large enough. □

Heuristically we would like to say that this relation (7.13), compared with the relation (7.7), shows *a posteriori* that in a certain sense, a S-finite maps is *almost solvable* in terms $\bar{h}$ and its jets $\nabla^\kappa\bar{h}$. *However, in the S-nondegenerate case, the map h is neither solvable nor entire over $\bar{h}$ and its jets $\nabla^\kappa\bar{h}$, nor almost solvable in any sense.*

§8. Proof of Theorem 7.2: Step I

This paragraph is devoted to the first step in the proof of Theorem 7.2.

Proof of Theorem 7.2. First of all, we extract from the system of $(\mathcal{X}'_\lambda)_{1\leq\lambda\leq\Lambda}$ the subsystem $(\mathcal{X}'_{\lambda_j})_{1\leq j\leq n'}$, we denote it by $(\mathcal{X}'_j)_{1\leq j\leq n'}$ and we show that if h satisfies

$$\mathcal{X}'_j(t,\tau,h(t),\nabla^{\kappa_0}\bar{h}(\tau))\equiv 0 \qquad \text{on } \ \mathcal{M}=\{\rho(t,\tau)=0\}, \tag{8.1}$$

for $j=1,\dots,n'$, with $\mathcal{X}'_1,\dots,\mathcal{X}'_{n'}$ satisfying eq. (7.4), then h is convergent.

We conduct the proof by induction on the length $k\in\mathbb{N}$ of a Segre chain $\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}$ up to $k=2\mu_p$, where μ_p denotes the Segre type of $\mathcal{M}$ at p (see Theorem 3.2.2) and the induction processus $(\mathcal{I}_k)\Rightarrow(\mathcal{I}_{k+1})$ will be divided in two essential steps.

Here is the k-th induction assumption:

$(\mathcal{I}_k)$ There exists a $\mathbb{C}^{2n'}$-valued holomorphic map $\Psi^k\colon(\delta\Delta^m)^k\to\Delta^{2n'}$, $\delta>0$, such that the composition

$$h^c\circ\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}(w_1,\dots,w_k)\equiv\Psi^k(w_1,\dots,w_k)\in\mathbb{C}\{w_1,\dots,w_k\}^{2n'} \tag{8.2}$$

is convergent for $w_1,\dots,w_k\in\delta\Delta^m$.

It will be clear soon that $(\mathcal{I}_{2\mu_p})$ implies that h^c is convergent in a neighborhood of p in $\mathcal{M}$. Recall that $h^c=(h,\bar{h})$, so eq. (8.2) above is a statement about the formal complexified map $h^c\colon(\mathcal{M},p^c)\to_{\mathcal{F}}(\mathcal{M}',p'^c)$.

As announced above, the proof that $(\mathcal{I}_k)\Rightarrow(\mathcal{I}_{k+1})$ involves two essential steps:

First step: $[(\mathcal{I}_k)$ and $(*_{k-1})]\Rightarrow(*_k)$, and

second step: $(*_k)\Rightarrow(\mathcal{I}_{k+1})$, where

$(*_k)$ For all $\kappa\in\mathbb{N}$ and all $\beta\in\mathbb{N}^n$ with $|\beta|\leq\kappa$, there exist $\mathbb{C}^{2n'}$-valued holomorphic maps $\Psi^k_\beta\colon(\delta\Delta^m)^k\to\Delta^{2n'}$, $\delta>0$, such that the composition

$$(\nabla^\kappa h^c)\circ\Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}(w_1,\dots,w_k)\equiv(\Psi^k_\beta(w_1,\dots,w_k))_{|\beta|\leq\kappa}\in\mathbb{C}\{w_1,\dots,w_k\}^{2n'N_{n,\kappa}} \tag{8.3}$$

is convergent for $w_1,\dots,w_k\in\delta\Delta^m$.

Remark. The positive number δ may shrink to 0 as κ goes to ∞.

The proof of the first step occupies this §8 and the proof of step two is postponed to §9. Now, let us explain how we end-up the proof.

Assertion 8.4. *If $(\mathcal{I}_{2\mu_p})$ is satisfied, then $h\colon(M,p)\to_{\mathcal{F}}(M',p')$ is convergent.*

Proof. We know by Corollary 3.2.6 that for each $\delta>0$, there exist $w^*_{(2\mu_p)}\in(\delta\Delta^m)^{2\mu_p}$ such that $\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}(w^*_{(2\mu_p)})=p^c$ and such that the rank of $\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}$ at $w^*_{(2\mu_p)}$ equals $\dim_{\mathbb{C}}\mathcal{M}$. (Recall $\mathcal{M}$ is $\{\mathcal{L},\underline{\mathcal{L}}\}$-minimal at p^c.) Thus $\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}$ can provide holomorphic coordinates over a small neighborhood of p^c in $\mathcal{M}$. By $(\mathcal{I}_{2\mu_p})$, we have that $h^c\circ\Gamma^{2\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}$ is convergent. This proves that $h^c\colon(\mathcal{M},p)\to_{\mathcal{F}}(\mathcal{M}',p')$ is convergent. Thus h is convergent. □

Remark. According to Theorem 2.2.6, we know that $\text{gen-rk}_{\mathbb{C}}(\Gamma^{\mu_p}_{\underline{\mathcal{L}}\mathcal{L}})=2m+d=\dim_{\mathbb{C}}\mathcal{M}$ already and we shall prove in particular that $h\circ\Gamma^{\mu_p}_{\underline{\mathcal{L}}\mathcal{L}}$ is convergent (this is

$(\mathcal{I}_{\mu_p})$). Does it imply already that h converges? Yes, and the result is due to Eakin and Harris [EH]: *Let $a(y)$ be a formal power series and let $\varphi\colon (\mathbb{C}_x^\nu, 0) \to (\mathbb{C}_y^\mu, 0)$ be a local holomorphic map of generic rank ν. If $a \circ \varphi(x) \in \mathbb{C}\{x\}$ is convergent, then $a(y)$ is convergent.* But the iteration process up to the $2\mu_p$-th step is free here and we can conclude the end of the proof in a much more elementary way after using Assertion 8.4 instead of applying the theorem of Eakin and Harris...!

First of all, we prove $(\mathcal{I}_1)$ by applying Theorem 1.3.2 to the family of equations

$$\mathcal{X}_j'(\Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1), h \circ \Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1), (\nabla^{\kappa_0}\bar{h}) \circ \Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1)) \equiv 0, \qquad 1 \leq j \leq n', \tag{8.5}$$

or equivalently, since $\Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1) = \mathcal{L}_{w_1}(0) = (w_1, i\bar{\Theta}(0, w_1, 0), 0, 0)$,

$$\mathcal{X}_j'(w_1, i\bar{\Theta}(0, w_1, 0), h(w_1, i\bar{\Theta}(0, w_1, 0)), \nabla^{\kappa_0}\bar{h}(0)) \equiv 0, \qquad 1 \leq j \leq n'. \tag{8.6}$$

The main assumption (7.4) of Theorem 7.2 fits the hypothesis of Theorem 1.3.2 exactly. We thus get that the series $h(w_1, i\bar{\Theta}(0, w_1, 0)) = h \circ \Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1) \equiv \Psi^1(w_1) \in \mathbb{C}\{w_1\}$ is convergent. On the other hand, $\bar{h} \circ \Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1) = \bar{h}(0)$ is clearly convergent, which completes our checking of $(\mathcal{I}_1)$.

To prove that $[(\mathcal{I}_k)$ and $(*)_{k-1}] \Rightarrow (*_k)$, we first make preliminary considerations.

Let us introduce two systems of d-vector fields Υ and $\underline{\Upsilon}$ which are both complementary to the subsystem $\{\mathcal{L}, \underline{\mathcal{L}}\}$ (in order that each system $\{\mathcal{L}, \underline{\mathcal{L}}, \Upsilon\}$ and $\{\mathcal{L}, \underline{\mathcal{L}}, \underline{\Upsilon}\}$ spans $T\mathcal{M}$) and which can be written

$$\Upsilon = \frac{\partial}{\partial z} + (1 - i\Theta_z(w, \zeta, z))\frac{\partial}{\partial \xi} \qquad \text{and} \qquad \underline{\Upsilon} = \frac{\partial}{\partial \xi} + (1 + i\bar{\Theta}_\xi(\zeta, w, \xi))\frac{\partial}{\partial z}. \tag{8.7}$$

We observe immediately that $\mathcal{L}$ and $\underline{\Upsilon}$ commute and that $\underline{\mathcal{L}}$ and Υ commute also. Of course, this commutation property can be written in terms of their flows:

$$\mathcal{L}_{w_1} \circ \underline{\Upsilon}_{\xi_1}(p) = \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_1}(p) \quad \text{and} \quad \underline{\mathcal{L}}_{\zeta_1} \circ \Upsilon_{z_1}(p) = \Upsilon_{z_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p), \tag{8.8}$$

for all $w_1, \zeta_1 \in \delta\Delta^m$, $z_1, \xi_1 \in \delta\Delta^d$, $p \in \mathcal{M}$ and $\delta > 0$ small. Let $q(x)$ denote a formal $2n$-vectorial power series in $\mathbb{C}[\![x]\!]^{2n}$ vanishing at 0, where $x \in \mathbb{C}^\nu$ and $\nu \in \mathbb{N}_*$.

From this observation and from Lemma 4.5, we deduce,

$$\begin{aligned} h \circ \Upsilon_{z_1} \circ \underline{\mathcal{L}}_{\zeta_1}(q(x)) &\equiv h \circ \underline{\mathcal{L}}_{\zeta_1} \circ \Upsilon_{z_1}(q(x)) \equiv h \circ \Upsilon_{z_1}(q(x)), \\ \bar{h} \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_1}(q(x)) &\equiv \bar{h} \circ \mathcal{L}_{w_1} \circ \underline{\Upsilon}_{\xi_1}(q(x)) \equiv \bar{h} \circ \underline{\Upsilon}_{\xi_1}(q(x)). \end{aligned} \tag{8.9}$$

More generally and similarly

$$\begin{aligned} (\nabla^\kappa h) \circ \Upsilon_{z_1} \circ \underline{\mathcal{L}}_{\zeta_1}(q(x)) &\equiv (\nabla^\kappa h) \circ \Upsilon_{z_1}(q(x)), \\ (\nabla^\kappa \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_1}(q(x)) &\equiv (\nabla^\kappa \bar{h}) \circ \underline{\Upsilon}_{\xi_1}(q(x)). \end{aligned} \tag{8.10}$$

Now, we remark that the fundamental identity (7.4) also comes together with a conjugate identity

$$\bar{\mathcal{X}}_j'(\tau, t, \bar{h}(\tau), \nabla^{\kappa_0} h(t)) \equiv 0 \qquad \text{on} \;\; \mathcal{M} = \{\rho(t, \tau) = 0\}, \tag{8.11}$$

for $j = 1, \ldots, n'$.

By the way, this second identity can be simply derived from (7.4) by specifying $\tau = \bar{t}$ in (7.4), by conjugating (7.4), which yields $\bar{\mathcal{X}}_j'(\bar{t}, t, \bar{h}(\bar{t}), \nabla^{\kappa_0} h(t)) \equiv 0$ on $M = \{\rho(t, \bar{t}) = 0\}$ and then complexifying $(\bar{t})^c = \tau$, using that $M \subset \mathcal{M}$ is a maximally real manifold. By this, we achieve a duplication, in accordance with the heuristic principle that there is no privilegiate choice between h and $\bar{h}$.

This second identity will be used to achieve the second inductive step $(*_k) \Rightarrow (\mathcal{I}_{k+1})$ in case k is odd and to achieve the first inductive step $[(\mathcal{I}_k)$ and $(*_{k-1})] \Rightarrow (*_k)$ in case k is even.

We shall only prove $[(\mathcal{I}_k)$ and $(*_{k-1})] \Rightarrow (*_k)$ and $(*_k) \Rightarrow (\mathcal{I}_{k+1})$ in case k is odd. The even case is completely similar. We would like to mention that the scheme of our proof differs in an essential way from the proofs given in [BER97] [BER99]. Whereas a solution $h_j(t) = A_j'(t, \tau, \nabla^{\kappa_0} \bar{h}(\tau))$ or $P_j'(t, \tau, h_j(t), \nabla^{\kappa_0} \bar{h}(\tau)) = 0$ (see §11) provides an immediate explicit iteration processus along Segre chains, we cannot here iterate directly h in terms of $\nabla^{\kappa_0} \bar{h}$ along Segre chains. But we can use step by step the powerfulness of Theorem 1.3.2 and we shall have to check step by step the convergence of jets of h without such an explicit relation of solvability.

As k is odd, we write $k = 2l + 1$, $l \in \mathbb{N}$, and develope the expression of $\Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+1}$ in the long explicit form (just by definition)

$$\Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+1}(w_{l+1}, \zeta_l, w_l, \ldots, \zeta_1, w_1) = \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0). \tag{8.12}$$

This expression in terms of vector fields is geometric, invariant and appropriate for our understanding of the sequel, especially for the calculation of high order derivatives. We shall therefore keep such long explicit forms in the formalism to perform the calculations below. Thus, the remainder of §8 is devoted to the proof of $[\mathcal{I}_{2l+1}$ and $(*_{2l})] \Rightarrow (*_{2l+1})$.

By substituting the series

$$\underline{\Upsilon}_{\xi_1} \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+1}(w_{l+1}, \zeta_l, w_l, \ldots, \zeta_1, w_1) \in \mathbb{C}[\![w_1, \zeta_1, \ldots, w_l, \zeta_l, w_{l+1}]\!]^{2n} \tag{8.13}$$

for the variables $(t, \tau) \in \mathcal{M}$ in the fundamental identities (7.3), we can read (7.3) as

$$\begin{aligned} &\mathcal{X}_j'(\underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ h \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), \\ &(\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv 0, \quad 1 \leq j \leq n' \end{aligned} \tag{8.14}$$

or after a crucial simplification of the last term:

$$\begin{aligned} &\mathcal{X}_j'(\underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ h \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), \\ &(\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv 0, \quad 1 \leq j \leq n' \end{aligned} \tag{8.15}$$

in view of identity (8.10). The goal is to prove that the jets of h on the $2l+1$-th Segre chain all converge, *i.e.* that for all $\kappa \in \mathbb{N}$, then $(\nabla^{\kappa} h) \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+1}(w_{l+1}, \zeta_l, w_l, \ldots, \zeta_1, w_1)$ is convergent. To this aim, we claim that it suffices to prove that

$$(\mathcal{L}^{\gamma} \underline{\Upsilon}^{\delta} h) \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+1}(w_{l+1}, \zeta_l, w_l, \ldots, \zeta_1, w_1) \in \mathbb{C}\{w_1, \zeta_1, \ldots, w_l, \zeta_l, w_{l+1}\} \tag{8.16}$$

for all $\gamma \in \mathbb{N}^m$, $\delta \in \mathbb{N}^d$, with $|\gamma| + |\delta| \leq \kappa$. In fact, it is easy to see that property (8.16) is equivalent to all $(\nabla^\kappa h) \circ \Gamma^{2l+1}_{\underline{\mathcal{L}}\mathcal{L}}(w_1, \zeta_1, \dots, w_l, \zeta_l, w_{l+1})$ being convergent, since $\{\mathcal{L}, \underline{\Upsilon}\}$ spans the (w,z)-space and has analytic coefficients.

Recall that by definition of the flow, for a power series, $a\colon \mathcal{M} \to \mathbb{C}$, $a(p) = 0$, one has $\frac{\partial}{\partial w_1}(a \circ \mathcal{L}_{w_1}(q(x))) = (\mathcal{L}a) \circ \mathcal{L}_{w_1}(q(x))$ and $\frac{\partial}{\partial \xi_1}|_{\xi_1 = 0}(a \circ \underline{\Upsilon}_{\xi_1}(q(x))) = (\underline{\Upsilon}a)(q(x))$ (in symbolic notations, omitting indices for $w_{1,1}, \dots, w_{1,m}$ and $\xi_{1,1}, \dots, \xi_{1,d}$).

Of course, (8.16) is satisfied for $\gamma = 0$ and $\delta = 0$, since we have assumed that $(\mathcal{I}_{2l+1})$ holds true.

Now, assume by induction that (8.16) is satisfied for all γ, δ with $|\gamma| + |\delta| \leq \lambda$, some $\lambda \in \mathbb{N}_*$. Applying to equation (8.14) the derivatives

$$\left[\frac{\partial^{|\gamma|+1}}{\partial w_{l+1}^{\gamma + \mathbf{1}_r^m}} \frac{\partial^{|\delta|}}{\partial \xi_1^\delta} \ \bullet \ \right]_{\xi_1 = 0} \qquad \text{and} \qquad \left[\frac{\partial^{|\gamma|}}{\partial w_{l+1}^{\gamma}} \frac{\partial^{|\delta|+1}}{\partial \xi_1^{\delta + \mathbf{1}_s^d}} \ \bullet \ \right]_{\xi_1 = 0} \tag{8.17}$$

where $\mathbf{1}_r^m = (0, \dots, 1, \dots, 0) \in \mathbb{N}^m$ with 1 at the r-th place, and where $\mathbf{1}_s^d = (0, \dots, 1, \dots, 0)$, with 1 at the s-th place, we will obtain after applying the induction assumption that there exist two families of analytic series $A_{j,\gamma,\delta,\mathbf{1}_r^m}$ and $B_{j,\gamma,\delta,\mathbf{1}_s^d}$ such that

$$\begin{gathered} \sum_{k=1}^{n'} \frac{\partial \mathcal{X}'_j}{\partial t'_k}(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \\ (\nabla^{\kappa_0} \bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \mathcal{L}_{w_1}(0))(\mathcal{L}^{\gamma + \mathbf{1}_r^m} \underline{\Upsilon}^\delta h_k) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) + \\ + A_{j,\gamma,\delta,\mathbf{1}_r^m}(w_1, \zeta_1, \dots, w_l, \zeta_l, w_{l+1}) \equiv 0, \quad 1 \leq j \leq n' \end{gathered} \tag{8.18}$$

and

$$\begin{gathered} \sum_{k=1}^{n'} \frac{\partial \mathcal{X}'_j}{\partial t'_k}(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \\ (\nabla^{\kappa_0} \bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \mathcal{L}_{w_1}(0))(\mathcal{L}^{\gamma} \underline{\Upsilon}^{\delta + \mathbf{1}_s^d} h_k) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) + \\ + B_{j,\gamma,\delta,\mathbf{1}_s^d}(w_1, \zeta_1, \dots, w_l, \zeta_l, w_{l+1}) \equiv 0, \quad 1 \leq j \leq n'. \end{gathered} \tag{8.19}$$

Indeed, all the terms appearing in $A_{j,\gamma,\delta,\mathbf{1}_r^m}$ (or $B_{j,\gamma,\delta,\mathbf{1}_s^d}$) involve three sorts of terms.

Firstly, they involve derivatives of h of the form $(\mathcal{L}^\alpha \underline{\Upsilon}^\beta h) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)$ with $|\alpha| + |\beta| \leq |\gamma| + |\delta|$, which are already analytic, by our induction assumption on $\lambda = |\gamma| + |\delta|$, together with derivatives of the form

$$\begin{gathered} \sum_{|\alpha_1|+|\beta_1|+|\gamma_1|+|\delta_1| \leq |\gamma|+|\delta|+1} \frac{\partial^{|\alpha_1|+|\beta_1|+|\gamma_1|+|\delta_1|} \mathcal{X}'_j}{\partial t^{\alpha_1} \partial \tau^{\beta_1} \partial t'^{\gamma_1} (\partial \nabla^{\kappa_0})^{\delta_1}}(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \\ h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \ (\nabla^{\kappa_0} \bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)), \end{gathered} \tag{8.20}$$

which are all clearly already analytic, because $h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)$ is analytic by the induction assumption $(\mathcal{I}_{2l+1})$ and because $(\nabla^{\kappa_0}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$ is analytic by the induction assumption $(*_{2l})$.

Secondly, they also involve terms of the form

$$\left[\frac{\partial^{|\alpha|}}{\partial w_{l+1}^{\alpha}} \frac{\partial^{|\beta|}}{\partial \xi_1^{\beta}} \quad \left(\underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)\right)\right]_{\xi_1=0}, \qquad |\alpha| \leq |\gamma|+1, \ |\beta| \leq |\delta|, \tag{8.21}$$

which are obviously analytic.

Thirdly, they involve terms of the form

$$\left[\frac{\partial^{|\alpha|}}{\partial w_{l+1}^{\alpha}} \frac{\partial^{|\beta|}}{\partial \xi_1^{\beta}} \quad \left((\nabla^{\kappa_0}\bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \cdots \circ \mathcal{L}_{w_1}(0)\right)\right]_{\xi_1=0} = \tag{8.22}$$

$$= (\underline{\Upsilon}^{\beta}\nabla^{\kappa_0}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \cdots \circ \mathcal{L}_{w_1}(0) = Q_{\alpha,\beta}(\underline{\mathcal{L}}_{\zeta_l} \circ \cdots \circ \mathcal{L}_{w_1}(0), (\nabla^{\kappa_0+|\beta|}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \cdots \circ \mathcal{L}_{w_1}(0)),$$

with $Q_{\alpha,\beta}$ holomorphic in its variables and $Q_{\alpha,\beta} \equiv 0$ if $|\alpha| > 0$, and these terms are firmly analytic thanks to the induction assumption $(*_{2l})$.

Now, we consider equations (8.18) (or (8.19)) as a $n' \times n'$ matrix equation with the unknowns $(\mathcal{L}^{\gamma+\mathbf{1}_r^m}\underline{\Upsilon}^{\delta}h_l) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) := H_k$, $1 \leq k \leq n'$, written in the form $XH + A = 0$. Here, X and A have analytic coefficients, H is formal. But det X as a convergent pover series does not vanish identically, because

$$\det X = \det\left(\frac{\partial \mathcal{X}'_j}{\partial t'_k}\left(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \ h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0),\right.\right. \tag{8.23}$$

$$\left.\left.(\nabla^{\kappa_0}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \mathcal{L}_{w_1}(0)\right)_{1\leq j,k\leq n'}\right),$$

and when we specify the variables $(w_1, \zeta_1, \ldots, w_l, \zeta_l, w_{l+1})$ in (8.23) above, as $(0, 0, \ldots, 0, 0, w_{l+1})$, we obtain

$$\det\left(\frac{\partial \mathcal{X}'_j}{\partial t'_k}\left(\mathcal{L}_{w_{l+1}}(0), h \circ \mathcal{L}_{w_l+1}(0), \nabla^{\kappa_0}\bar{h}(0)\right)\right)_{1\leq j,k\leq n'}\right) \not\equiv_{w_{l+1}} 0 \quad \text{in } \mathbb{C}[\![w_{l+1}]\!], \tag{8.24}$$

by our main assumption (7.4) in Theorem (7.2). Let us denote by X^T the classical adjoint matrix of X that satisfies by definition $X^T X = X X^T = (\det X)\mathrm{Id}_{n'\times n'}$. Then X^T has convergent power series entries. Also, $(\det X)H = -X^T A$. We deduce that for each $k = 1, \ldots, n'$, there exists converging power series b and a_k, $1 \leq k \leq n'$, $a_k \in \mathbb{C}\{w_1, \ldots, w_{l+1}\}$ with $b = \det X$ such that

$$b(w_1, \ldots, w_{l+1})\, \mathcal{L}^{\gamma+\mathbf{1}_r^m}\underline{\Upsilon}^{\delta}h_k(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)) = a_k(w_1, \ldots, w_{l+1}), \quad 1 \leq k \leq n', \tag{8.25}$$

with $b \not\equiv 0$. It is then a consequence of the Weierstrass division theorem that (8.25) implies that there exist $c_j \in \mathbb{C}\{w_1, \ldots, w_{l+1}\}$ such that $a_k = b\, c_k$. In conclusion, $\mathcal{L}^{\gamma+\mathbf{1}_r^m}\underline{\Upsilon}^{\delta}h_k(\mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)) \in \mathbb{C}\{w_1, \ldots, w_{l+1}\}$, $1 \leq k \leq n'$.

We thus have proved by induction on $\lambda = |\gamma| + |\delta|$ that, for all multiindices γ, δ, then $(\mathcal{L}^\gamma \underline{\Upsilon}^\delta h) \circ \Gamma^{2l+1}_{\underline{\mathcal{L}}\mathcal{L}}(w_{l+1}, \zeta_l, w_l, \dots, \zeta_1, w_1) \in \mathbb{C}\{w_1, \zeta_1, \dots, w_l, \zeta_l, w_{l+1}\}$, which proves that $(\nabla^\kappa h) \circ \Gamma^{2l+1}_{\underline{\mathcal{L}}\mathcal{L}}(w_{l+1}, \zeta_l, w_l, \dots, \zeta_1, w_1)$ is convergent for all $\kappa \in \mathbb{N}$.

Finally, it is clear that $(\nabla^\kappa \bar{h}) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) = (\nabla^\kappa \bar{h}) \circ \underline{\mathcal{L}}_{\zeta_l} \circ \cdots \circ \mathcal{L}_{w_1}(0) \in \mathbb{C}\{w_1, \dots, \zeta_l\}$ converges for all κ, by the induction assumption $(*_{2l})$.

In conclusion, $\nabla^\kappa h^c = (\nabla^\kappa h, \nabla^\kappa \bar{h})$ converges on the $(2l+1)$-th Segre chain, which completes the proof of the implication $[\mathcal{I}_{2l+1}$ and $(*_{2l})] \Rightarrow (*_{2l+1})$. □

§9. Proof of Theorem 7.2: step II

End of proof of Theorem 7.2. Now, it remains to prove that $(*_{2l+1}) \Rightarrow (\mathcal{I}_{2l+2})$, using $(*_{2l+1})$ that we just have proved.

For that purpose, we start with the conjugate fundamental identities (8.11) in which we substitute for the variables $(t, \tau) \in \mathcal{M}$ the series $\underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$, obtaining

$$\text{(9.1)}\quad \bar{\mathcal{X}}'_j(\bar{\sigma} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ \bar{h} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),$$
$$(\nabla^{\kappa_0} h) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv 0, \qquad 1 \le j \le n',$$

since again $(\nabla^{\kappa_0} h) \circ \underline{\mathcal{L}}_{\zeta_{l+1}}(q(x)) \equiv \nabla^{\kappa_0} h(q(x))$ by Lemma 4.5. In eq. (9.1), by $(*_{2l+1})$, all the arguments of $\bar{\mathcal{X}}'_j$ are convergent, except the formal unknowns $\bar{h}_k \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0)$. In order to apply Theorem 1.3.2 to deduce that these unknowns are convergent, it suffices to observe that the determinant

$$\text{(9.2)}\quad \det \left(\frac{\partial \bar{\mathcal{X}}_j}{\partial \tau''_k} \Big(\bar{\sigma} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \bar{h} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0), \right.$$
$$\left. (\nabla^{\kappa_0} h) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \Big)_{1 \le j,k \le n'} \right)$$

does not vanish identically in $\mathbb{C}[\![w_1, \zeta_1, \dots, w_{l+1}, \zeta_{l+1}]\!]$, because, when we specify $(w_1, \zeta_1, \dots, w_{l+1}, \zeta_{l+1})$ as $(0, 0, \dots, 0, \zeta_{l+1})$ in (9.2), we see that

$$\text{(9.3)}\quad \det \left(\frac{\partial \bar{\mathcal{X}}_j}{\partial \tau'_k} \Big(\bar{\sigma} \circ \underline{\mathcal{L}}_{\zeta_{l+1}}(0), \bar{h} \circ \underline{\mathcal{L}}_{\zeta_{l+1}}(0), \nabla^{\kappa_0} h(0) \Big)_{1 \le j,l \le n'} \right) \not\equiv 0 \quad \text{in } \mathbb{C}[\![\zeta_{l+1}]\!],$$

according to our main assumption (3.4).

Finally, it is clear that $h_j \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) \equiv h_j \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \mathcal{L}_{w_1}(0) \in \mathbb{C}\{w_1, \dots, w_{l+1}\}$ by the induction assumption $(\mathcal{I}_{2l+1})$. In conclusion, $h^c = (h, \bar{h})$ converges on the $(2l+2)$-th Segre chain, which completes the proof of the implication $(*_{2l+1}) \Rightarrow (\mathcal{I}_{2l+2})$. □

The proof of Theorem 7.2 is complete. □

§10. Proof of Theorem 7.5: steps I and II

Proof of Theorem 7.5. For the propagation processus, we shall need essentially two lemmas.

Lemma 10.1. *Let $w \in \mathbb{C}^\mu$, $z \in \mathbb{C}^d$ and suppose that a power series $h(w,z) \in \mathbb{C}[\![w,z]\!]$ formally satisfies a polynomial equation of the form*

$$h(w,z)^N + \sum_{1\leq k\leq N} a_k(w,z)\, h(w,z)^{N-k}, \tag{10.2}$$

where $N \in N_*$ and where each power series $a_k(w,z) = \sum_{\alpha\in\mathbb{N}^d} z^\alpha\, a_{k,\alpha}(w) \in \mathbb{C}[\![w,z]\!]$ has all its derivatives with respect to z at 0 being convergent power series, i.e.

$$(1/\alpha!)\ \partial_z^\alpha a_k(w,0) = a_{k,\alpha}(w) \in \mathbb{C}\{w\}, \qquad 1 \leq k \leq N,\ \forall\ \alpha \in \mathbb{N}^d. \tag{10.3}$$

Then $h(w,z) = \sum_{\alpha\in\mathbb{N}^d} z^\alpha\, h_\alpha(w)$ also satisfies $h_\alpha(w) \in \mathbb{C}\{w\}$, $\forall\ \alpha$, i.e.

$$(1/\alpha!)\ \partial_z^\alpha h(w,0) = h_\alpha(w) \in \mathbb{C}\{w\} \qquad \forall\ \alpha \in \mathbb{N}^d. \tag{10.4}$$

We shall observe in a while that the case $d = 1$ implies the general case easily. Let $d = 1$. Putting $z = 0$ in (10.2), we deduce that $h(w,0) \in \mathbb{C}\{w\}$ by applying the following consequence of Artin's theorem.

Lemma 10.5. *Let $w \in \mathbb{C}^\mu$, $h \in \mathbb{C}[\![w]\!]$, assume that $P(h(w),w) \equiv 0$, where*

$$P(X,w) = \sum_{j=0}^{N} a_j(w) X^j, \qquad a_j \in \mathbb{C}\{w\}, \quad a_N(w) \not\equiv 0. \tag{10.6}$$

Then $h \in \mathbb{C}\{w\}$.

Proof. If $\partial P/\partial X(h(w),w) \equiv 0$, we can of course replace P by $\partial P/\partial X$. By induction and since $\partial^N P/\partial X^N = N!\, a_N(w) \not\equiv 0$, we can assume that $P(h(w),w) \equiv 0$ and $\partial P/PX(h(w),w) \not\equiv 0$. Finally, applying Theorem 1.3.2, we get $h(w) \in \mathbb{C}\{w\}$. □

Proof of Lemma 10.1. Thus $h(w,0) \in \mathbb{C}\{w\}$. Recall that $d = 1$. By induction, let us assume that $h_0(w), \ldots, h_l(w) \in \mathbb{C}\{w\}$ and prove that $h_{l+1}(w) \in \mathbb{C}\{w\}$. It is clear that if we replace h by $\tilde{h}(w,z) := h(w,z) - h_0(w) - \cdots - z^l h_l(w)$ in (10.2), we get immediately that $\tilde{h}$ satisfies a similar polynomial equation and we have $\tilde{h} = z^{l+1}\tilde{h}_1$. Thus, we can assume after coming back to the previous notation h that $h = z^{l+1} h_1$ and thus, we must prove that $h_1(w,0) \in \mathbb{C}\{w\}$.

Let us write uniquely $a_k(w,z) = z^{\lambda_k} c_k(w,z)$, where $\lambda_k \in \mathbb{N}$, $c_k(w,0) \not\equiv 0$ if $a_k(w,z) \not\equiv 0$ or $\lambda_k = \infty$ if $a_k(w,z) \equiv 0$. Now, in the identity

$$z^{(l+1)N}\, h_1(w,z)^N + \sum_{1\leq k\leq N} z^{\lambda_k + (N-k)(l+1)}\, c_k(w,z)\ h_1(w,z)^{N-k} \equiv 0, \tag{10.7}$$

if we select the term behind z^ϖ, where

$$\varpi := \inf\ \left((l+1)N, \inf_{1\leq k\leq N} \lambda_k + (N-k)(l+1) \right) < \infty \tag{10.8}$$

we shall immediately get that $h_1(w,0)$ satisfies a nontrivial polynomial equation as in Lemma 10.5. In conclusion, $h_1(w,0) \in \mathbb{C}\{w\}$ and we are done.

To deduce the general case from the case $d = 1$, it suffices to apply the case $d = 1$ to functions $\tilde{h}_c(w, \zeta) := h(w, c\zeta)$, $w \in \mathbb{C}^\mu$, $\zeta \in \mathbb{C}$ for all possible complex lines $\mathbb{C} \ni \zeta \mapsto (c_1\zeta, \dots, c_d\zeta) \in \mathbb{C}^d$, $(c_1, \dots, c_d) \in \mathbb{C}^d$. □

End of proof of Theorem 7.5. First, taking $\tau = 0$ and $\rho(t, 0) = 0$ in (7.7), we deduce that $h \circ \Gamma^1_{\underline{\mathcal{L}}\mathcal{L}}(w_1) \in \mathbb{C}\{w_1\}$ by a straightforward application of Lemma 10.1 above.

As in the proof of Theorem 7.2, it suffices to establish that $[(\mathcal{I}_k)$ and $(*_{k-1})] \Rightarrow (*_k)$ and that $(*_k) \Rightarrow (\mathcal{I}_{k+1})$ in case $k = 2l + 1$ is odd.

Firstly, for the first step, we substitute the series

$$\underline{\Upsilon}_{\xi_1} \circ \Gamma^{2l+1}_{\underline{\mathcal{L}}\mathcal{L}}(w_{l+1}, \zeta_l, w_l, \dots, \zeta_1, w_1) \in \mathbb{C}\{w_1, \zeta_1, \dots, w_l, \zeta_l, w_{l+1}, \xi_1\}^{2n} \tag{10.9}$$

for the variables $(t, \tau) \in \mathcal{M}$ in the fundamental identity (7.7) and get

$$\begin{aligned} P'_j(\underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ h_j \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), \\ (\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \equiv 0, \quad 1 \le j \le n', \end{aligned} \tag{10.10}$$

after the simplification $(\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}}(q(x)) \equiv (\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1}(q(x))$. Since all derivatives

$$\left[\frac{\partial^{|\beta|}}{\partial w^\beta_{l+1}} \frac{\partial^{|\alpha|}}{\partial \xi_1^\alpha} \left((\nabla^{\kappa_0} \bar{h}) \circ \underline{\Upsilon}_{\xi_1} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \right) \right]_{\xi_1 = 0} \tag{10.11}$$

are converging, by assumption $(*_{2l})$, we can apply Lemma 10.1 to deduce that the derivatives

$$\left[\frac{\partial^{|\beta|}}{\partial w^\beta_{l+1}} \frac{\partial^{|\alpha|}}{\partial \xi_1^\alpha} \left(h_j \circ \underline{\Upsilon}_{\xi_1} \circ \mathcal{L}_{w_{l+1}} \circ \underline{\mathcal{L}}_{\zeta_l} \circ \mathcal{L}_{w_l} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \right) \right]_{\xi_1 = 0} \tag{10.12}$$

are in $\mathbb{C}\{w_1, \dots, \zeta_l, w_{l+1}\}$, for all $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^m$: this is $(*_{2l+1})$.

Secondly, for the second step, considering the conjugate identities (8.11), we get

$$\begin{aligned} \bar{P}_j(\bar{\sigma} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ \bar{h}_j \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), \\ (\nabla^{\kappa_0} h) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv 0, \quad 1 \le j \le n'. \end{aligned} \tag{10.13}$$

In (10.13) above, the formal series in $(\nabla^{\kappa_0} h) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$ are converging, by assumtion $(*_{2l+1})$. Thus, applying Lemma 10.5, we deduce that $\bar{h} \circ \Gamma^{2l+1}_{\underline{\mathcal{L}}\mathcal{L}}(w_1, \zeta_1, \dots, w_{l+1}, \zeta_{l+1}) \in \mathbb{C}\{w_1, \zeta_1, \dots, w_{l+1}, \zeta_{l+1}\}^{n'}$.

Finally, it is clear that $h \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) = h \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \in \mathbb{C}\{w_1, \zeta_1, \dots, w_{l+1}\}^{n'}$ by $(\mathcal{I}_{2l+1})$. In conclusion, $h^c = (h, \bar{h})$ converges on the $(2l + 2)$-th Segre chain.

This completes the proof of Theorem 7.5. □

§11 Simplifications in the S-solvable case

In the case where h is S-solvable at 0, since we can solve h in terms of $\bar{h}$ and its jets $\nabla^\kappa \bar{h}$ by the usual analytic implicit function theorem, which takes place instead of Artin's approximation theorem, then the propagation process can be highly simplified.

Proof of Theorem 1.2.1 (i). Indeed, according to Lemma 7.12, we can write

$$h_j'(\zeta,\xi) \equiv \Psi_j(w,z,\zeta,\xi,\nabla^{\kappa_0}\bar{h}(\tau)), \qquad 1 \leq j \leq n', \tag{11.1}$$

for holomorphic Ψ_j in terms of their arguments, when (t,τ) satisfy $\rho(t,\tau)=0$. Furthermore, it is possible (*cf.* [BER97]) to obtain after applying the d-vector fields Υ and $\underline{\Upsilon}$ to the above identity (11.1) to obtains for all $\kappa \in \mathbb{N}$ the existence of holomorphic Ψ^κ and $\underline{\Psi}^\kappa$ such that

$$\nabla^\kappa \bar{h}(\zeta,\xi) \equiv_{\zeta,w,\xi} \left[\Psi^\kappa(w,z,\zeta,\xi,\nabla^{\kappa+\kappa_0}h(w,z))\right]_{z:=\xi+i\bar{\Theta}(\zeta,w,\xi)} \tag{11.2}$$

$$\nabla^\kappa h(w,z) \equiv_{w,\zeta,z} \left[\underline{\Psi}^\kappa(\zeta,\xi,w,z,\nabla^{\kappa+\kappa_0}\bar{h}(\zeta,\xi))\right]_{\xi:=z-i\Theta(w,\zeta,z)}.$$

But starting with the more intrinsic (equivalent) writing of (11.2) as follows

$$(\nabla^\kappa \bar{h}) \circ \underline{\mathcal{L}}_\zeta(p) = \underline{\Psi}^\kappa(\underline{\mathcal{L}}_\zeta(p), \nabla^{\kappa+\kappa_0}h(p)), \tag{11.3}$$

$$(\nabla^\kappa h) \circ \mathcal{L}_w(p) = \Psi^\kappa(\mathcal{L}_w(p), \nabla^{\kappa+\kappa_0}\bar{h}(p)), \qquad p = p^c \in \mathcal{M} \cap \underline{\Lambda}, \quad \kappa \in \mathbb{N},$$

we can make immediate iterations of identities (11.3), for instance twice:

$$(\nabla^\kappa h) \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p) = \Psi^\kappa(\mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p), \nabla^{\kappa+\kappa_0}\bar{h} \circ \underline{\mathcal{L}}_{\zeta_1}(p) = \tag{11.4}$$

$$\Psi^\kappa(\mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p), \Psi^{\kappa+\kappa_0}(\underline{\mathcal{L}}_{\zeta_1}(p), \nabla^{\kappa+2\kappa_0}h(p))),$$

$$= \psi_2^\kappa(w_1,\zeta_1,\nabla^{\kappa+2\kappa_0}h(p)) \in \mathbb{C}\{w_1,\zeta_1\}$$

so that to perform the propagation process, we only have to iterate (*i.e.* replace them into themselves) the identities (11.2), and thus, we immediately obtain

$$(\nabla^\kappa \bar{h}) \circ \underline{\mathcal{L}}_{\zeta_{\mu_p}} \circ \mathcal{L}_{w_{\mu_p}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p) \equiv \underline{\psi}^\kappa_{2\mu_p}(\zeta_1,w_1,...,\zeta_{\mu_p},w_{\mu_p},\nabla^{\kappa+2\mu_p\kappa_0}\bar{h}(p)), \tag{11.5}$$

$$(\nabla^\kappa h) \circ \mathcal{L}_{w_{\mu_p}} \circ \underline{\mathcal{L}}_{\zeta_{\mu_p}} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p) \equiv \psi^\kappa_{2\mu_p}(w_1,\zeta_1,...,w_{\mu_p},\zeta_{\mu_p},\nabla^{\kappa+2\mu_p\kappa_0}h(p)),$$

which, together with Theorem 3.2.2, completes the proof of Theorem 1.2.1 (i). □

§12. Analyticity of formal solutions

In this very short paragraph, we prove Theorem 1.3.2, which has been applied several times during the proof of Theorem 1.2.1. By $\mathcal{V}_X(p)$, we mean a small open neighborhood, equivalent to a polydisc, of the point p in the complex manifold X equivalent to a polydisc.

Proof of Theorem 1.3.2. Fix $j_1, \ldots, j_m$ with

$$\det\left(\frac{\partial R_{j_k}}{\partial y_l}(w, \hat{g}(w))\right)_{1\leq k,l\leq m} \not\equiv_w 0 \ \text{ in } \ \mathbb{C}[\![w]\!] \tag{12.1}$$

and simply write $R_1, \ldots, R_m$. We show that the solution $(\hat{g}_1(w), \ldots, \hat{g}_m(w))$ of the m equations $R_1(w, \hat{g}(w)) \equiv_w 0, \ldots, R_m(w, \hat{g}(w)) \equiv_w 0$ is itself converging.

Let Γ be the germ at 0 of the complex analytic set

$$\Gamma = \{(w, y) \in \mathcal{V}_{\mathbb{C}^n}(0) \times \mathcal{V}_{\mathbb{C}^m}(0) \colon R_1(w, y) = 0, \ldots, R_m(w, y) = 0\}. \tag{12.2}$$

For each $N \in \mathbb{N}$, there exists an analytic solution $g_N(w)$, *i.e.* $R(w, g_N(w)) \equiv_w 0$, with $g - g_N = 0 \,(\text{mod}\ \mathfrak{m}(w)^N)$. Consider the graph of g_N:

$$\text{gr}(g_N) = \{(w, y) \in \mathcal{V}_{\mathbb{C}^n}(0) \times \mathcal{V}_{\mathbb{C}^n}(0) \colon y = g_N(w)\}. \tag{12.3}$$

For N large enough, say $N \geq N_0$, $N_0 \in \mathbb{N}_*$, by (12.1),

$$\det\left(\frac{\partial R_k}{\partial y_l}(w, g_N(w))\right)_{1\leq k,l\leq m} \not\equiv_w 0, \tag{12.4}$$

so that there exists w_0 close to 0 in $\mathbb{C}^n$ such that

$$\det\left(\frac{\partial R_k}{\partial y_l}(w_0, g_N(w_0))\right)_{1\leq k,l\leq m} \neq 0. \tag{12.5}$$

By (12.5), a neighborhood of $(w_0, g_N(w_0))$ in Γ is of dimension n. Therefore,

$$\text{gr}(g_N) \cap (\mathcal{V}_{\mathbb{C}^n}(w_0) \times \mathcal{V}_{\mathbb{C}^m}(g_N(w_0))) \equiv \Gamma \cap (\mathcal{V}_{\mathbb{C}^n}(w_0) \times \mathcal{V}_{\mathbb{C}^m}(g_N(w_0))). \tag{12.6}$$

As a consequence, *for each $N \geq N_0$,* $\text{gr}(g_N)$ *is an irreducible component of* Γ. Since Γ has finitely many irreducible components, a subsequence of g_N, *i.e.* of $\text{gr}(g_N)$, is constant. Since $g_N \to \hat{g}$ in the Krull topology, $\hat{g}$ is convergent. □

Remark. In fact, more is *a posteriori* true above: it is clear then that there exists an integer N_0, depending only on the R_k's, $1 \leq k \leq m$, such that for all $N, N' \geq N_0$, $g_N = g_{N'}$ near 0 (the component stabilizes). More generally, we observe:

Lemma 12.7. *Let $R_1(w, y), \ldots, R_m(w, y) \in \mathbb{C}\{w, y\}$ and assume that there exists $g^1(w) \in \mathbb{C}[\![w]\!]^m$ satisfying*

$$R_j(w, g^1(w)) \equiv_w 0 \quad \textit{and} \quad \det\left(\frac{\partial R_k}{\partial y_l}(w, g^1(w))\right)_{1\leq k,l\leq m} \not\equiv_w 0 \ \text{ in } \ \mathbb{C}[\![w]\!] \tag{12.8}$$

Then there exists a positive integer $\underline{\nu} = \underline{\nu}(R)$ such that if $g^2 \in \mathbb{C}[\![w]\!]$ satisfies

$$\begin{aligned} &R_k(g^2(w), w) \equiv 0 \quad \textit{in} \ \mathbb{C}[\![w]\!], \quad \forall\ 1 \leq k \leq m \\ &\partial_w^\alpha g^1(0) = \partial_w^\alpha g^2(0) \quad \forall\ |\alpha| \leq \underline{\nu}(R), \end{aligned} \tag{12.9}$$

then $g^2(w) \equiv g^1(w)$.

Proof. By the above proof of Theorem 1.3.2, $\text{gr}(g^1)$ is then a fixed irreducible component Γ^1 of the complex analytic set Γ defined by (12.2). It is also clear that there exists a sufficiently large integer $\underline{\nu}$ depending only on R such that $\partial_w^\alpha g^1(0) = \partial_w^\alpha g^2(0)$, $\forall\ |\alpha| \leq \underline{\nu}$ implies that $\det\left(\frac{\partial R_k}{\partial y_l}(w, g^2(w))\right)_{1\leq k,l\leq m} \not\equiv_w 0$. In this case, $\text{gr}(g^2)$ occurs to be also an irreducible component Γ^2 of Γ. Furthermore, $\Gamma^1 = \Gamma^2$ if $\underline{\nu} = \underline{\nu}(R)$ is large enough. Finally, $g^1(w) \equiv g^2(w)$. □

In particular, a direct corollary is as follows (*cf.* [BER99], Lemma 4.3):

Corollary 12.10. *Let $P(X,w)$ be of the form*

$$P(X,w)=\sum_{j=0}^{N} a_j(w)X^j, \qquad a_j\in\mathbb{C}\{w\}, \quad a_N(w)\not\equiv 0. \tag{12.11}$$

Then there exists a positive integer $\underline{\nu}(P)$ such that if $h^1,h^2\in\mathbb{C}[\![w]\!]$ satisfy

$$\begin{aligned} &P(h^1(w),w)\equiv P(h^2(w),w)\equiv 0 \qquad \text{in } \mathbb{C}[\![w]\!] \\ &\partial_w^\alpha h^1(0)=\partial_w^\alpha h^2(0) \qquad \forall\ |\alpha|\le\underline{\nu}(P), \end{aligned} \tag{12.13}$$

then $h^1(w)\equiv h^2(w)$.

Proof. The only thing we have to check is that we can assume in addition that $\partial_X P(h^1(w),w)\not\equiv 0$. We leave this to the reader. □

§13. Examples

Example 13.1. *An S-solvable or S-finite formal map $h\colon (M,p)\to_{\mathcal{F}}(M',p')$ such that the formal generic rank of $h\colon (S_{\bar p},p)\to_{\mathcal{F}}(S'_{\bar p'},p')$ is strictly less than m'.* Take $(z_1,z_3)\mapsto(z_1,0,z_3)$, $\mathbb{C}^2_{z_1,z_3}\to\mathbb{C}^3_{z_1',z_2',z_3'}$, $M=\{z_3=\bar z_3+iz_1\bar z_1\}$ and $M'=\{z_3'=\bar z_3'+iz_1'\bar z_1'+i{z'}_1^a\bar{z'}_2^a+i\bar{z'}_1^a{z'}_2^a$, $a\in N_*$. If $a=1$, then h is S-solvable at the origin. If $a\ge 2$, then h is S-finite but not S-solvable at the origin (of course, M' is essentially finite at the origin.)

Example 13.2. *An S-finite but not S-nondegenerate formal map.* Take $(z_1,z_3)\mapsto(z_1,0,z_3)$, $\mathbb{C}^2_{z_1,z_3}\to\mathbb{C}^3_{z_1',z_2',z_3'}$, $M=\{z_3=\bar z_3+iz_1^2\bar z_1^2\}$ and $M'=\{z_3'=\bar z_3'+i{z'}_1^2\bar{z'}_1^2+iz_1'\bar{z'}_2^2+i\bar{z'}_1{z'}_2^2\}$.

Example 13.3. *An S-nondegenerate but not S-finite formal map.* Perhaps the simplest example is the identity map of $M=\{z_3=\bar z_3+iz_1\bar z_1(1+z_2\bar z_2)\}$ (*cf.* [MM2]).

Example 13.4. *A holomorphically nondegenerate but not S-nondegenerate real hypersurface.* $M\colon\ y_3=|z_1|^2|1+z_1\bar z_2|^2(1+\mathrm{Re}(z_1\bar z_2))^{-1}-x_3\,\mathrm{Im}(z_1\bar z_2)(1+\mathrm{Re}(z_1\bar z_2))^{-1}$. This seems to be the simplest example of such (*cf.* [BER99] [Mer99c]).

Example 13.5. *A class of S-finite or S-nondegenerate real hypersurfaces in $\mathbb{C}^n$.* Here are the two most naive examples: The hypersurface $y=|w_1|^{2r_1}+\cdots+|w_{n-1}|^{2r_{n-1}}+xh(w,\bar w,x)$ where h is any analytic remainder in normal form and where $r_1>0,\dots,r_{n-1}>0$, is essentially finite (S-finite). The hypersurface $y=\sum_{k=1}^{\mu}\prod_{j=1}^{n-1}|w_j|^{2r_{j,k}}(1+xh(w,\bar w,x))$, $\mu\ge n-1$, where h is any analytic remainder and where the exponents $r_{j,k}$ are subject to the condition that the generic rank of the Jacobian matrix $\left(\frac{\partial\prod_{j=1}^{n-1}w_j^{r_{j,k}}}{\partial w_l}\right)_{k,l}$ equals n and all the multiindinces $\beta_k:=(r_{j,k})_{1\le j\le n-1}\in\mathbb{N}_*^{n-1}$ are pairwise distinct.

§14. A uniqueness principle

The purpose of this paragraph is to establish the following uniqueness principle, first derived in the S-solvable case in [BER97] (*cf.* also [BER99], Theorem 2.5):

Theorem 14.1. *Let $h^1\colon (M,p) \to_{\mathcal{F}} (M',p')$ be a real analytic holomorphic mapping between real analytic CR generic manifolds, assume that M is minimal at p, and assume that h^1 is either S-solvable, or S-finite, or S-nondegenerate. Then there exists an integer $\underline{\kappa} \in \mathbb{N}_*$ with the following property. If $h^2\colon (M,p) \to_{\mathcal{F}} (M',p')$ is a formal holomorphic mapping sending (M,p) into (M',p'), and if*

$$\partial_t^\alpha h^2(p) = \partial_t^\alpha h^1(p) \qquad \forall\ |\alpha| \leq \underline{\kappa}, \tag{14.2}$$

then $h^1(t-p) \equiv h^2(t-p)$ in $\mathbb{C}[\![t-p]\!]$.

First, here an immediate consequence in the invertible case:

Corollary 14.3. *Let M be a real analytic generic submanifold through the origin in $\mathbb{C}^n$. Assume that $(M,0)$ is minimal and S-solvable, or essentially finite, or S-nondegenerate at 0. Then there exists an integer $\underline{\kappa}$ with the following property. If M' is a real analytic generic submanifold through the origin in $\mathbb{C}^n$ of the same dimension as M and if $h^1, h^2\colon (\mathbb{C}^n,0) \to_{\mathcal{F}} (\mathbb{C}^n,0)$ are formal invertible mappings sending $(M,0)$ into $(M',0)$ which satisfy*

$$\partial_t^\alpha h^1(t) = \partial_t^\alpha h^2(0) \qquad \forall\ |\alpha| \leq \underline{\kappa}, \tag{14.4}$$

then $h^1(t) \equiv h^2(t)$ in $\mathbb{C}[\![t]\!]$.

Proof. We claim that it suffices to take $\underline{\kappa}$ to be the integer given by Theorem 14.1 with $M = M'$ and $h^1 = \mathrm{Id}$, *i.e.* $h^1(t) \equiv t$. To see this, let M', h^1, h^2 be as in Corollary 14.3 and observe that if (14.4) holds, then $\partial_t^\alpha((h^1)^{-1} \circ h^2)(0) = 0$ for all $|\alpha| \leq \underline{\kappa}$. By Theorem 14.1, and the choice of $\underline{\kappa}$, we deduce that $((h^1)^{-1} \circ h^2)(t) \equiv t$ and hence, the conclusion of Corollary 14.3. □

Proof of Theorem 14.1. To prove Theorem 14.1, we follow the steps of the proof of Theorem 1.3.2 thoroughly. It suffices to treat S-finite and S-nondegenerate maps parallely (S-solvable maps being S-finite). We shall concentrate on the S-nondegenerate case only (to treat the S-finite case, use Corollary 12.10 instead of Lemma 12.7). In what follows, $\mathcal{F}(M,M')$ we denote the set of all formal mappings $(\mathbb{C}^n,0) \to_{\mathcal{F}} (\mathbb{C}^{n'},0)$ that send $(M,0)$ into $(M',0)$. We consider the following property for $k \in \mathbb{N}$ and $\kappa \in \mathbb{N}$.

$(**)_{k,\kappa}$ *There exists $K(k,\kappa) \in \mathbb{N}$ such that for any $h^2 \in \mathcal{F}(M,M')$ with*

$$\partial_t^\alpha h^1(0) = \partial_t^\alpha h^2(0), \qquad \forall\ |\alpha| \leq K(k,\kappa), \tag{14.5}$$

the following holds

$$(\nabla^\kappa h^{1c}) \circ \Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}(w_1,\dots,w_k) \equiv (\nabla^\kappa h^{2c}) \circ \Gamma^k_{\underline{\mathcal{L}}\mathcal{L}}(w_1,\dots,w_k). \tag{14.6}$$

Observe that $(**)_{0,\kappa}$ holds with $K(0,\kappa) = \kappa$, since $\Gamma^0_{\underline{\mathcal{L}}\mathcal{L}} \equiv 0$ is the constant null mapping. We shall prove that $(**)_{k,\kappa}$ holds for all k and κ by double induction on

k and κ. First, let us assume that $(**)_{k',\kappa'}$ holds for all $0 \le k' \le 2l+1$ and all κ' and prove that $(**)_{2l+2,0}$ holds (as in §8-9, the even-to-odd induction is similar). Precisely, we must show the existence of the integer $K(2l+2,0)$ in $(**)_{2l+2,0}$. Let $\bar{\mathcal{X}}_j'(\tau, t, \tau', \nabla^{\kappa_0})$, $1 \le j \le n'$, be the (conjugate, *cf.* eq. (8.11)) analytic functions given in the assumptions of Theorem 7.2. Pick an integer $\tilde{K}$ and consider the formal mappings $h^2 \in \mathcal{F}(M, M')$ satisfying

$$\partial_t^\alpha h^1(0) = \partial_t^\alpha h^2(0), \qquad \forall\, |\alpha| \le \tilde{K}. \tag{14.7}$$

By *a priori* requiring $\tilde{K} \ge K(2l+1, \kappa_0)$, we may assume that any $h^2 \in \mathcal{F}(M, M')$ that satisfies (14.7) above also satisfies the identities (8.11) *on the* $2l+2$*-th Segre chain*, with $\bar{h}^1$ replaced by $\bar{h}^2$, and the same jets $\nabla^{\kappa_0} h^1$, *i.e.* that

$$\bar{\mathcal{X}}_j'(\bar{\sigma} \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0),\ \bar{h}^2 \circ \underline{\mathcal{L}}_{\zeta_{l+1}} \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), \tag{14.8}$$

$$(\nabla^{\kappa_0} h^1) \circ \mathcal{L}_{w_{l+1}} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv 0, \qquad 1 \le j \le n',$$

Hence, if $\underline{\nu}^{2l+2}$ is the integer given by Lemma 12.7 for equations (14.8), and if we choose $K(2l+2,0) = \max(\tilde{K}, \underline{\nu}^{2l+2})$, then the identity $h^1 \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+2} = h^2 \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2l+2}$ follows from Lemma 12.7. The property $(**)_{2l+2,0}$ is proved.

We now fix $k = 2l+1$ and an integer κ. We complete the induction by assuming that $(**)_{k',\kappa'}$ for all pairs (k', κ') satisfying either $0 \le k' < 2l+1$ or $k' = 2l+1$ and $\kappa' \le \kappa$ (the case $k = 2l$ is similar). We shall prove $(**)_{2l+1,\kappa+1}$. Now, consider those $h^2 \in \mathcal{F}(M, M')$ satisfying (14.7) with $\tilde{K} \ge \max\{\tilde{K}(2l, \kappa_0 + \kappa + 1), \tilde{K}(2l+1, \kappa)\}$. Using the induction property, we see that, as above, the derivatives $\mathcal{L}^{\gamma + \mathbf{1}_r^m} \underline{\Upsilon}^\delta h^2$ and $\mathcal{L}^\gamma \underline{\Upsilon}^{\delta + \mathbf{1}_s^d} h^2$ of such an h^2 satisfy the same identities (8.18) and (8.19) as $\mathcal{L}^{\gamma + \mathbf{1}_r^m} \underline{\Upsilon}^\delta h^1$ and $\mathcal{L}^\gamma \underline{\Upsilon}^{\delta + \mathbf{1}_s^d} h^1$ on the $2l+1$-th Segre chain (with the same $A_{j,\gamma,\delta,\mathbf{1}_r^m}$ and $B_{j,\gamma,\delta,\mathbf{1}_s^d}$ for both), for all γ, δ with $|\gamma| + |\delta| \le \kappa$. It is then clear that after taking the inverse of the matrix (8.23) as in the end of §8, we immediately get the agreement of $\mathcal{L}^{\gamma + \mathbf{1}_r^m} \underline{\Upsilon}^\delta h^2$ and $\mathcal{L}^\gamma \underline{\Upsilon}^{\delta + \mathbf{1}_s^d} h^2$ with $\mathcal{L}^{\gamma + \mathbf{1}_r^m} \underline{\Upsilon}^\delta h^1$ and $\mathcal{L}^\gamma \underline{\Upsilon}^{\delta + \mathbf{1}_s^d} h^1$ on the $2l+1$-th Segre chain. This completes the induction and proves $(**)_{k,\kappa}$ for all k and κ.

To complete the proof of Theorem 14.1, it suffices to remember the full rank property of $\Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2\mu_0}$ and to apply it to $(**)_{2\mu_0,0}$: $h^1 \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2\mu_0} = h^2 \circ \Gamma_{\underline{\mathcal{L}}\mathcal{L}}^{2\mu_0}$, whence $h^1 = h^2$. This completes the proof of Theorem 14.1. □

§15. Open problems

First, we wonder whether an analog of Artin's theorem for CR maps holds true:

Problem 15.1. *Let* $h\colon (X,p) \to_{\mathcal{F}} (X', p')$ *be a formal holomorphic map between two germs of real analytic sets in* $(\mathbb{C}^n, p)$ *and* $(\mathbb{C}^{n'}, p')$ *respectively. Prove (or disprove) that for every* $N \in \mathbb{N}$ *there exists a convergent mapping* $h_N\colon (X,p) \to (X', p')$ *such that* $h(t) \equiv h_N(t) \pmod{\mathfrak{m}(t)^N}$.

Our analysis leaves open at least four more specialized questions:

Problem 15.2. *Prove that a formal holomorphic mapping* $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ *between real analytic CR manifolds, such that* (M,p) *is minimal and* (M',p') *does not contain complex analytic sets of positive dimension through* p'*, is convergent.*

Remark. The hypersurface case is treated in [BER99]. It seems that a plausible adaptation of double reflection of jets as in [Z98] [D] [Mer99b] would yield the result modulo some unavoidable technicalities.

A CR manifold M is called *transversally nondegenerate* if it is not biholomorphically equivalent, locally around a generic point, to a product $\underline{M} \times I$ of a CR manifold $\underline{M} \subset \mathbb{C}^{n-1}$ with a real segment $I \subset \mathbb{R}$ ([Mer99a]). A hypersurface M is transversally nondegenerate if and only if it is Levi-nonflat.

Problem 15.3. *Prove* (*or disprove*) *that any formal invertible CR map between holomorphically nondegenerate transversally nondegenerate real analytic CR manifolds is convergent. Study formal invertible self maps of nonminimal hypersurfaces.*

Remark. Even in codimension one, this problem is new and widely open. In codimension two, perharps the simplest example to study would be self maps of $(M,p) = (M',p') = (N,0)$ where $N : z_3 = \bar{z}_3,\ z_2 = \bar{z}_2 + iz_1\bar{z}_1 + i(z_1\bar{z}_1)^2\bar{z}_3$.

Problem 15.4. *Prove that the reflection mapping associated with a formal CR map from* (M,p) $\mathcal{C}^\omega$ *minimal into any* $\mathcal{C}^\omega$ (M',p') *is always convergent* (*without any nondegeneracy condition on* (M',p')).

Problem 15.5. *Find purely algebraic proofs of the convergence results in the circumstance where both* (M,p) *and* (M',p') *are algebraic.*

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