

# Equivalences of 5-dimensional CR-manifolds V: Six initial frames and coframes; Explicitness obstacles

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**Abstract.** Local CR-generic submanifolds of  $\mathbb{C}^N$  are in one-to-one correspondence with their respective graphing functions, but it is well known that (despite their importance) the Cartan-Hachtroudi-Chern-Moser invariants and coframes for Levi nondegenerate hypersurfaces  $M \subset \mathbb{C}^{n+1}$  have been fully computed in CR dimension  $n \geq 2$  only in special cases which show off a tremendous collapse of computational complexity in comparison to the general case. One of the goals of this Part V is to set up systematic initial data that are essentially explicit in terms of the concerned graphing functions, for the six already studied general classes I, II, III<sub>1</sub>, III<sub>2</sub>, IV<sub>1</sub>, IV<sub>2</sub>. Incredibly, for Class III<sub>1</sub> CR-generic submanifolds  $M^5 \subset \mathbb{C}^4$  that are the geometry-preserving deformations of one of the natural models of Beloshapka, even the initial frame and coframe are not absorbable by an individual personal computer, for some of the concerned coefficient-functions incorporate nearly 100 000 000 of monomials in 165 jet variables, not to mention that the exploration of biholomorphic equivalences yet requires to differentiate such functions at least four times. As will appear later on, deep (archaic) mathematical links are extant between the effective Cartan theory and the famous hyperbolicity conjecture of Kobayashi.

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### 1. $M^3 \subset \mathbb{C}^2$ of general class I: initial frame and coframe in local coordinates

Consider the *most simple case* of:

$$\left( M^3 \subset \mathbb{C}^2 \right) \in \text{General Class I.}$$

Representing as before ([5, 6])  $M$  in coordinates:

$$(z, w) = (x + \sqrt{-1}y, u + \sqrt{-1}v),$$

as a graph:

$$v = \varphi(x, y, u),$$

an associated explicit frame for  $TM$  is:

$$\begin{aligned} X &= \frac{\partial}{\partial x} + \varphi_x(x, y, u) \frac{\partial}{\partial v}, \\ Y &= \frac{\partial}{\partial y} + \varphi_y(x, y, u) \frac{\partial}{\partial v}, \\ U &= \frac{\partial}{\partial u} + \varphi_u(x, y, u) \frac{\partial}{\partial v}, \end{aligned}$$

viewed here *extrinsically*, namely together with the (transversal) coordinate  $v$  which is not internal to  $M$ .

A way of understanding these fields intrinsically is to introduce the auxiliary projection of  $M$  onto its tangent plane at 0:

$$\pi|_M: M \longrightarrow T_0M$$

defined as the restriction to  $M$  of the coordinate projection:

$$\begin{aligned} \pi: \quad \mathbb{R}^4 &\longrightarrow \mathbb{R}^3 \\ (x, y, u, v) &\longmapsto (x, y, u). \end{aligned}$$

Considering then  $\pi|_M: M \longrightarrow \mathbb{R}^3$  as a *chart* on  $M$ , one recovers the intrinsic representation of the tangential fields simply as:

$$\begin{aligned} \pi_* \left( \frac{\partial}{\partial x} + \varphi_x \frac{\partial}{\partial v} \right) &= \frac{\partial}{\partial x}, \\ \pi_* \left( \frac{\partial}{\partial y} + \varphi_y \frac{\partial}{\partial v} \right) &= \frac{\partial}{\partial y}, \\ \pi_* \left( \frac{\partial}{\partial u} + \varphi_u \frac{\partial}{\partial v} \right) &= \frac{\partial}{\partial u}, \end{aligned}$$

and this visibly corresponds just to dropping the  $\frac{\partial}{\partial v}$ -components.

Next, as was shown in [4], a natural local vector field generator of  $T^{1,0}M$  is:

$$\mathcal{L} := \frac{\partial}{\partial z} + \mathbf{A} \frac{\partial}{\partial w},$$

with:

$$\mathbf{A} = \frac{-2\varphi_z}{\sqrt{-1} + \varphi_u},$$

a coefficient-function which is thus *de facto* a function of only  $(x, y, u)$ , independently of  $v$ .

Restricting  $\mathcal{L}$  to  $M^3$ , one must simply and only drop the (extrinsic) vector field  $\frac{\partial}{\partial v}$ :

$$\begin{aligned}\mathcal{L}|_M &= \pi_*(\mathcal{L}) = \frac{\partial}{\partial z} + A \left( \frac{1}{2} \frac{\partial}{\partial u} - \frac{\sqrt{-1}}{2} \frac{\partial}{\partial v} \right) \\ &= \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}.\end{aligned}$$

One uses the notation:

$$\begin{aligned}A &:= \frac{A}{2} \\ &= - \frac{\varphi_z}{\sqrt{-1} + \varphi_u}.\end{aligned}$$

Thus intrinsically on  $M^3$ , the CR-structure induced by the ambient  $\mathbb{C}^2$  on  $M^3$  is encoded by the *intrinsic*  $(1, 0)$  complex-valued vector field:

$$\boxed{\mathcal{L} = \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}},$$

together with its conjugate:

$$\boxed{\overline{\mathcal{L}} = \frac{\partial}{\partial \bar{z}} - \frac{\varphi_{\bar{z}}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}}.$$

By the assumption that  $M$  belongs to the General Class 1:

$$\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]\}$$

constitutes a frame for  $\mathbb{C} \otimes_{\mathbb{R}} TM$ .

Set:

$$\mathcal{T} := \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}],$$

and compute:

$$\begin{aligned}\mathcal{T} &:= \sqrt{-1} \left[ \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{z}} - \frac{\varphi_{\bar{z}}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u} \right] \\ &= \frac{1}{(\sqrt{-1} + \varphi_u)^2 (-\sqrt{-1} + \varphi_u)^2} \left\{ 2\varphi_{z\bar{z}} + 2\varphi_{z\bar{z}}\varphi_u\varphi_u - 2\sqrt{-1}\varphi_{\bar{z}}\varphi_{zu} - \right. \\ &\quad \left. - 2\varphi_{\bar{z}}\varphi_{zu}\varphi_u + 2\sqrt{-1}\varphi_z\varphi_{\bar{z}u} + \right. \\ &\quad \left. + 2\varphi_z\varphi_{\bar{z}}\varphi_{uu} - 2\varphi_z\varphi_{\bar{z}u}\varphi_u \right\} \frac{\partial}{\partial u}.\end{aligned}$$

Abbreviate the appearing coefficient-function as:

$$\ell := \frac{1}{(\sqrt{-1} + \varphi_u)^2 (-\sqrt{-1} + \varphi_u)^2} \left\{ \begin{aligned} &2 \varphi_{z\bar{z}} + 2 \varphi_{z\bar{z}} \varphi_u \varphi_u - 2 \sqrt{-1} \varphi_{\bar{z}} \varphi_{zu} - \\ &- 2 \varphi_{\bar{z}} \varphi_{zu} \varphi_u + 2 \sqrt{-1} \varphi_z \varphi_{\bar{z}u} + \\ &+ 2 \varphi_z \varphi_{\bar{z}} \varphi_{uu} - 2 \varphi_z \varphi_{\bar{z}u} \varphi_u \end{aligned} \right\}.$$

To know the full Lie bracket structure of the frame:

$$\{\mathcal{L}, \overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]\},$$

it yet remains to compute:

$$\begin{aligned} &[\mathcal{L}, \mathcal{T}], \\ &[\overline{\mathcal{L}}, \mathcal{T}], \end{aligned}$$

the second being the conjugate of the first, for, as always,  $\mathcal{T}$  is real:

$$\overline{\mathcal{T}} = \overline{\sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]} = -\sqrt{-1} [\overline{\mathcal{L}}, \mathcal{L}] = \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}] = \mathcal{T}.$$

Since:

$$\mathcal{T} = \ell \cdot \frac{\partial}{\partial u},$$

one has:

$$\begin{aligned} [\mathcal{L}, \mathcal{T}] &= \left[ \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}, \ell \frac{\partial}{\partial u} \right] \\ &= \left\{ \mathcal{L}(\ell) + \mathcal{T} \left( \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \right) \right\} \frac{\partial}{\partial u}, \end{aligned}$$

hence replacing:

$$\frac{\partial}{\partial u} = \frac{1}{\ell} \mathcal{T},$$

one gets:

$$[\mathcal{L}, \mathcal{T}] = \frac{\mathcal{L}(\ell) + \mathcal{T} \left( \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \right)}{\ell} \cdot \mathcal{T}$$

Call this coefficient-function:

$$P := \frac{\mathcal{L}(\ell) + \mathcal{T} \left( \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \right)}{\ell}.$$

**Lemma.** *On a hypersurface:*

$$(M^3 \subset \mathbb{C}^2) \in \text{General Class I},$$

*graphed as:*

$$v = \varphi(x, y, u),$$

with natural generator for  $T^{1,0}M$ :

$$\mathcal{L} = \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}$$

the associated frame for  $\mathbb{C} \otimes_{\mathbb{R}} TM$ :

$$\{\mathcal{L}, \overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]\} = \{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}\}$$

enjoys the Lie structure:

$$[\mathcal{L}, \overline{\mathcal{L}}] = -\sqrt{-1} \mathcal{T},$$

$$[\mathcal{L}, \mathcal{T}] = P \cdot \mathcal{T},$$

$$[\overline{\mathcal{L}}, \mathcal{T}] = \overline{P} \cdot \mathcal{T},$$

where:

$$P = \frac{P \text{ numerator}}{P \text{ denominator}},$$

where:

$$P \text{ denominator} = (1 + \varphi_u \varphi_u)(\sqrt{-1} + \varphi_u) \left( \varphi_{z\bar{z}} + \varphi_{z\bar{z}} \varphi_u \varphi_u - \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zu} - \varphi_{z\bar{z}} \varphi_{zu} \varphi_u + \sqrt{-1} \varphi_z \varphi_{\bar{z}u} - \varphi_z \varphi_{\bar{z}u} \varphi_u + \varphi_z \varphi_{\bar{z}} \varphi_{uu} \right),$$

and where:

$$\begin{aligned} P \text{ numerator} &= \sqrt{-1} \varphi_{zz\bar{z}} - \varphi_{zz} \varphi_{\bar{z}u} + \varphi_{z\bar{z}} \varphi_{zu} + \sqrt{-1} \varphi_z \varphi_{z\bar{z}} \varphi_{uu} + 2 \sqrt{-1} \varphi_z \varphi_{\bar{z}} \varphi_{zuu} + \\ &+ \varphi_{z\bar{z}} \varphi_{zzu} + \sqrt{-1} \varphi_{zz} \varphi_{\bar{z}} \varphi_{uu} - \varphi_z \varphi_z \varphi_{\bar{z}} \varphi_{uu} - 2 \varphi_z \varphi_{z\bar{z}u} - \\ &- \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zu} \varphi_{zu} - \sqrt{-1} \varphi_z \varphi_z \varphi_{\bar{z}uu} - \sqrt{-1} \varphi_z \varphi_{\bar{z}u} \varphi_{zu} + \\ &+ \varphi_u \left( -2 \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zzu} - 4 \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zu} + 4 \varphi_{uu} \varphi_{uu} \varphi_z \varphi_z \varphi_{\bar{z}} + 3 \varphi_z \varphi_{z\bar{z}} \varphi_{uu} + 2 \varphi_z \varphi_{\bar{z}} \varphi_{zuu} - \right. \\ &- 8 \sqrt{-1} \varphi_{zu} \varphi_z \varphi_{\bar{z}} \varphi_{uu} + 4 \sqrt{-1} \varphi_{uu} \varphi_z \varphi_z \varphi_{\bar{z}u} + \varphi_{z\bar{z}} + \varphi_z \varphi_z \varphi_{\bar{z}uu} - 5 \varphi_{z\bar{z}} \varphi_{zu} \varphi_{zu} + \\ &\left. + 5 \varphi_z \varphi_{\bar{z}u} \varphi_{zu} + \varphi_{zz} \varphi_{\bar{z}} \varphi_{uu} \right) + \\ &+ \varphi_u^2 \left( -\sqrt{-1} \varphi_z \varphi_{\bar{z}u} \varphi_{zu} - \sqrt{-1} \varphi_z \varphi_z \varphi_{\bar{z}uu} - 8 \varphi_{zu} \varphi_z \varphi_{\bar{z}} \varphi_{uu} - 4 \varphi_z \varphi_{z\bar{z}u} - \varphi_z \varphi_z \varphi_{\bar{z}} \varphi_{uu} - \right. \\ &- 4 \varphi_{uu} \varphi_z \varphi_z \varphi_{\bar{z}u} - 2 \varphi_{zz} \varphi_{\bar{z}u} + \sqrt{-1} \varphi_{zz} \varphi_{\bar{z}} \varphi_{uu} + 7 \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zu} \varphi_{zu} + \\ &\left. + \sqrt{-1} \varphi_z \varphi_{z\bar{z}} \varphi_{uu} + 2 \sqrt{-1} \varphi_{zz\bar{z}} - 2 \varphi_{z\bar{z}} \varphi_{zu} + 2 \sqrt{-1} \varphi_z \varphi_{\bar{z}} \varphi_{zuu} \right) + \\ &+ \varphi_u^3 \left( -4 \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zu} + \varphi_z \varphi_z \varphi_{\bar{z}uu} - 2 \sqrt{-1} \varphi_{z\bar{z}} \varphi_{zzu} + 3 \varphi_{z\bar{z}} \varphi_{zu} \varphi_{zu} + 2 \varphi_{zz\bar{z}} + \right. \\ &\left. + 5 \varphi_z \varphi_{\bar{z}u} \varphi_{zu} + \varphi_{zz} \varphi_{\bar{z}} \varphi_{uu} + 2 \varphi_z \varphi_{\bar{z}} \varphi_{zuu} + 3 \varphi_z \varphi_{z\bar{z}} \varphi_{uu} \right) + \\ &+ \varphi_u^4 \left( -2 \varphi_z \varphi_{z\bar{z}u} - 3 \varphi_{z\bar{z}} \varphi_{zu} - \varphi_{zz} \varphi_{\bar{z}u} - \varphi_{z\bar{z}} \varphi_{zzu} + \sqrt{-1} \varphi_{zz\bar{z}} \right) + \\ &+ \varphi_u^5 \left( \varphi_{zz\bar{z}} \right). \end{aligned}$$

*Proof.* Just a direct computation.  $\square$

Strikingly (but as is known), the so-called *rigid case* where the graphing function:

$$v = \varphi(x, y)$$

is assumed — as a simplifying assumption — to be *independent* of the CR-transversal coordinate  $u$ , comes up with a *tremendously spectacular collapse of complexity*.

Indeed, then:

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \sqrt{-1} \varphi_z \frac{\partial}{\partial u}, \\ \overline{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} - \sqrt{-1} \varphi_{\bar{z}} \frac{\partial}{\partial u},\end{aligned}$$

hence:

$$\mathcal{T} = 2 \varphi_{z\bar{z}} \frac{\partial}{\partial u},$$

which inverts as:

$$\frac{\partial}{\partial u} = \frac{1}{2 \varphi_{z\bar{z}}} \mathcal{T},$$

so that:

$$\begin{aligned}[\mathcal{L}, \mathcal{T}] &= 2 \varphi_{zz\bar{z}} \frac{\partial}{\partial u} \\ &= \frac{\varphi_{zz\bar{z}}}{\varphi_{z\bar{z}}} \mathcal{T},\end{aligned}$$

*i.e.:*

$$P = \frac{\varphi_{zz\bar{z}}}{\varphi_{z\bar{z}}},$$

which is a *considerable* collapse, indeed!

**Dual coframe and its Darboux structure.** Now, introduce three differential 1-forms, sections of  $\mathbb{C} \otimes_{\mathbb{R}} T^*M$ :

$$\{\rho_0, \bar{\zeta}_0, \zeta_0\}$$

that are dual to the frame:

$$\{\mathcal{T}, \mathcal{L}, \overline{\mathcal{L}}\}$$

namely satisfy:

$$\begin{aligned}\rho_0(\mathcal{T}) &= 1, & \rho_0(\overline{\mathcal{L}}) &= 0, & \rho_0(\mathcal{L}) &= 0, \\ \bar{\zeta}_0(\mathcal{T}) &= 0, & \bar{\zeta}_0(\overline{\mathcal{L}}) &= 1, & \zeta_0(\mathcal{L}) &= 0, \\ \zeta_0(\mathcal{T}) &= 0, & \zeta_0(\overline{\mathcal{L}}) &= 0, & \zeta_0(\mathcal{L}) &= 1,\end{aligned}$$

hence make a *coframe* for:

$$\mathbb{C} \otimes_{\mathbb{R}} T^*M.$$

Explicitly (exercise):

$$\begin{aligned}\rho_0 &= \frac{du - A dz - \bar{A} d\bar{z}}{\ell}, \\ \bar{\zeta}_0 &= d\bar{z}, \\ \zeta_0 &= dz.\end{aligned}$$

**Cartan formula.** Given a differential 1-form:

$$\omega$$

and two vector fields:

$$\mathcal{X}, \quad \mathcal{Y},$$

one has:

$$d\omega(\mathcal{X}, \mathcal{Y}) = \mathcal{X}(\omega(\mathcal{Y})) - \mathcal{Y}(\omega(\mathcal{X})) - \omega([\mathcal{X}, \mathcal{Y}]). \quad \square$$

**Lie structure and Darboux structure.** Given a local frame of vector fields:

$$\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$$

on  $\mathbb{R}^n$  or on  $\mathbb{C}^n$ , and given the dual coframe of differential 1-forms:

$$\{\omega^1, \dots, \omega^n\},$$

namely:

$$\omega^{\nu_1}(\mathcal{X}_{\nu_2}) = \begin{cases} 1 & \text{when } \nu_1 = \nu_2, \\ 0 & \text{when } \nu_1 \neq \nu_2, \end{cases}$$

the Lie structure of the frame is:

$$[\mathcal{X}_{\nu_1}, \mathcal{X}_{\nu_2}] = \sum_{1 \leq \nu_3 \leq n} A_{\nu_1, \nu_2}^{\nu_3} \mathcal{X}_{\nu_3} \quad (1 \leq \nu_1 < \nu_2 \leq n),$$

for certain local functions:

$$A_{\nu_1, \nu_2}^{\nu_3},$$

if and only if the Darboux structure of the coframe is:

$$d\omega^{\nu_1} = - \sum_{1 \leq \nu_2 < \nu_3 \leq n} A_{\nu_2, \nu_3}^{\nu_1} \omega^{\nu_2} \wedge \omega^{\nu_3}.$$

*Proof.* Use the Cartan formula. □

To visually determine the Darboux structure of the dual coframe, introduce an auxiliary array:

$$\begin{array}{rcccl}
 & \mathcal{T} & \overline{\mathcal{L}} & \mathcal{L} & \\
 & \boxed{d\rho_0} & \boxed{d\overline{\zeta_0}} & \boxed{d\zeta_0} & \\
 [\mathcal{T}, \overline{\mathcal{L}}] & = & -\overline{P} \cdot \mathcal{T} & + & 0 & + & 0 & \boxed{\rho_0 \wedge \overline{\zeta_0}} \\
 [\mathcal{T}, \mathcal{L}] & = & -P \cdot \mathcal{T} & + & 0 & + & 0 & \boxed{\rho_0 \wedge \zeta_0} \\
 [\overline{\mathcal{L}}, \mathcal{L}] & = & \sqrt{-1} \cdot \mathcal{T} & + & 0 & + & 0 & \boxed{\overline{\zeta_0} \wedge \zeta_0},
 \end{array}$$

read the three columns *vertically*, and put an overall minus sign:

$$\boxed{
 \begin{array}{l}
 d\rho_0 = P \rho_0 \wedge \zeta_0 + \overline{P} \rho_0 \wedge \overline{\zeta_0} + \sqrt{-1} \zeta_0 \wedge \overline{\zeta_0}, \\
 d\zeta_0 = 0, \\
 d\overline{\zeta_0} = 0.
 \end{array}
 }$$

This is the *initial Darboux structure* for the problem of biholomorphic equivalence in the general class I.

## 2. $M^4 \subset \mathbb{C}^3$ of general class II: initial frame and coframe in local coordinates

Next, consider:

$$\left( M^4 \subset \mathbb{C}^3 \right) \in \text{General Class II.}$$

Representing as before ([5, 6])  $M$  in coordinates:

$$(z, w_1, w_2) = (x + \sqrt{-1}y, u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2),$$

as a graph:

$$v_1 = \varphi_1(x, y, u_1, u_2),$$

$$v_2 = \varphi_2(x, y, u_1, u_2),$$

an associated explicit frame for  $TM$  is:

$$\begin{aligned}
 X &= \frac{\partial}{\partial x} + \varphi_{1,x}(x, y, u_1, u_2) \frac{\partial}{\partial v_1} + \varphi_{2,x}(x, y, u_1, u_2) \frac{\partial}{\partial v_2}, \\
 Y &= \frac{\partial}{\partial y} + \varphi_{1,y}(x, y, u_1, u_2) \frac{\partial}{\partial v_1} + \varphi_{2,y}(x, y, u_1, u_2) \frac{\partial}{\partial v_2}, \\
 U_1 &= \frac{\partial}{\partial u_1} + \varphi_{1,u_1}(x, y, u_1, u_2) \frac{\partial}{\partial v_1} + \varphi_{2,u_1}(x, y, u_1, u_2) \frac{\partial}{\partial v_2}, \\
 U_2 &= \frac{\partial}{\partial u_2} + \varphi_{1,u_2}(x, y, u_1, u_2) \frac{\partial}{\partial v_1} + \varphi_{2,u_2}(x, y, u_1, u_2) \frac{\partial}{\partial v_2}.
 \end{aligned}$$



The projection:

$$\begin{aligned} \mathbb{R}^6 &\longrightarrow \mathbb{R}^4 \\ \pi: (x, y, u_1, v_1, u_2, v_2) &\longmapsto (x, y, u_1, u_2), \end{aligned}$$

makes a *chart* on  $M$  and does:

$$\begin{aligned} \pi_* \left( \frac{\partial}{\partial x} + \varphi_{1,x} \frac{\partial}{\partial v_1} + \varphi_{2,x} \frac{\partial}{\partial v_2} \right) &= \frac{\partial}{\partial x}, \\ \pi_* \left( \frac{\partial}{\partial y} + \varphi_{1,y} \frac{\partial}{\partial v_1} + \varphi_{2,y} \frac{\partial}{\partial v_2} \right) &= \frac{\partial}{\partial y}, \\ \pi_* \left( \frac{\partial}{\partial u_1} + \varphi_{1,u_1} \frac{\partial}{\partial v_1} + \varphi_{2,u_1} \frac{\partial}{\partial v_2} \right) &= \frac{\partial}{\partial u_1}, \\ \pi_* \left( \frac{\partial}{\partial u_2} + \varphi_{1,u_2} \frac{\partial}{\partial v_1} + \varphi_{2,u_2} \frac{\partial}{\partial v_2} \right) &= \frac{\partial}{\partial u_2}. \end{aligned}$$

Next, as was explained in [4], a natural *intrinsic* local vector field generator for  $T^{1,0}M$  is:

$$\mathcal{L} := \frac{\partial}{\partial z} + A_1 \frac{\partial}{\partial u_1} + A_2 \frac{\partial}{\partial u_2},$$

where:

$$\begin{aligned} A_1 &:= \frac{\begin{vmatrix} -\varphi_{1,z} & \varphi_{1,u_2} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}}, \\ A_2 &:= \frac{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,z} \\ \varphi_{2,u_1} & -\varphi_{2,z} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}}. \end{aligned}$$

Set:

$$\Delta := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} \end{vmatrix},$$

whence:

$$\bar{\Delta} := \begin{vmatrix} -\sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} \\ \varphi_{2,u_1} & -\sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}.$$

Also, set:

$$\begin{aligned} \Lambda_1 &:= \begin{vmatrix} -\varphi_{1,z} & \varphi_{1,u_2} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}, \\ \Lambda_2 &:= \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,z} \\ \varphi_{2,u_1} & -\varphi_{2,z} \end{vmatrix}, \end{aligned}$$

whence:

$$\begin{aligned}\bar{\Lambda}_1 &:= \begin{vmatrix} -\varphi_{1,\bar{z}} & \varphi_{1,u_2} \\ -\varphi_{2,\bar{z}} & -\sqrt{-1} + \varphi_{2,u_2} \end{vmatrix}, \\ \bar{\Lambda}_2 &:= \begin{vmatrix} -\sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,\bar{z}} \\ \varphi_{2,u_1} & -\varphi_{2,\bar{z}} \end{vmatrix}.\end{aligned}$$

In these notations:

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2}, \\ \bar{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} + \frac{\bar{\Lambda}_1}{\bar{\Delta}} \frac{\partial}{\partial u_1} + \frac{\bar{\Lambda}_2}{\bar{\Delta}} \frac{\partial}{\partial u_2}.\end{aligned}$$

Set:

$$\mathcal{T} := \sqrt{-1} [\mathcal{L}, \bar{\mathcal{L}}],$$

and set:

$$\mathcal{S} := [\mathcal{L}, \mathcal{T}].$$

By hypothesis, the CR-geometric invariant condition:

$$\mathbb{C} \otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]]$$

holds, which means as is known that the 4 fields:

$$\{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{T}, \mathcal{S}\}$$

constitute a frame for:

$$4 = \text{rank}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} TM).$$

**Lemma.** *There are certain uniquely defined coefficient-functions that are polynomials:*

$$\Upsilon_1 = \Upsilon_1\left(\varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}}\right)_{1 \leq j+k+l_1+l_2 \leq 2},$$

$$\Upsilon_2 = \Upsilon_2\left(\varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}}\right)_{1 \leq j+k+l_1+l_2 \leq 2},$$

$$\Pi_1 = \Pi_1\left(\varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}}\right)_{1 \leq j+k+l_1+l_2 \leq 3},$$

$$\Pi_2 = \Pi_2\left(\varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}}\right)_{1 \leq j+k+l_1+l_2 \leq 3},$$

such that:

$$\begin{aligned}\mathcal{T} &= \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_2} \\ &=: Y_1 \frac{\partial}{\partial u_1} + Y_2 \frac{\partial}{\partial u_2}, \\ \mathcal{S} &= \frac{\Pi_1}{\Delta^4 \bar{\Delta}^3} \frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4 \bar{\Delta}^3} \frac{\partial}{\partial u_2} \\ &=: H_1 \frac{\partial}{\partial u_1} + H_2 \frac{\partial}{\partial u_2}.\end{aligned}$$

*Proof.* In fact firstly:

$$\begin{aligned}\sqrt{-1} [\mathcal{L}, \bar{\mathcal{L}}] &= \sqrt{-1} \left[ \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \bar{z}} + \frac{\bar{\Lambda}_1}{\bar{\Delta}} \frac{\partial}{\partial u_1} + \frac{\bar{\Lambda}_2}{\bar{\Delta}} \frac{\partial}{\partial u_2} \right] \\ &= \sqrt{-1} \left\{ \mathcal{L} \left( \frac{\bar{\Lambda}_1}{\bar{\Delta}} \right) - \bar{\mathcal{L}} \left( \frac{\Lambda_1}{\Delta} \right) \right\} \frac{\partial}{\partial u_1} - \sqrt{-1} \left\{ \mathcal{L} \left( \frac{\bar{\Lambda}_2}{\bar{\Delta}} \right) - \bar{\mathcal{L}} \left( \frac{\Lambda_2}{\Delta} \right) \right\} \frac{\partial}{\partial u_2}\end{aligned}$$

Compute:

$$\begin{aligned}\mathcal{L} \left( \frac{\bar{\Lambda}_1}{\bar{\Delta}} \right) &= \frac{\mathcal{L}(\bar{\Lambda}_1)}{\bar{\Delta}} - \frac{\bar{\Lambda}_1 \mathcal{L}(\bar{\Delta})}{\bar{\Delta}^2} \\ &= \frac{\Delta \bar{\Lambda}_{1,z} + \Lambda_1 \bar{\Lambda}_{1,u_1} + \Lambda_2 \bar{\Lambda}_{1,u_2}}{\Delta \bar{\Delta}} - \\ &\quad - \frac{\bar{\Lambda}_1 (\Delta \bar{\Delta}_z + \Lambda_1 \bar{\Delta}_{u_1} + \Lambda_2 \bar{\Delta}_{u_2})}{\Delta \bar{\Delta}^2},\end{aligned}$$

conjugate this:

$$\begin{aligned}\bar{\mathcal{L}} \left( \frac{\Lambda_1}{\Delta} \right) &= \frac{\bar{\Delta} \Lambda_{1,\bar{z}} + \bar{\Lambda}_1 \Lambda_{1,u_1} + \bar{\Lambda}_2 \Lambda_{1,u_2}}{\Delta \bar{\Delta}} - \\ &\quad - \frac{\Lambda_1 (\bar{\Delta} \Delta_{\bar{z}} + \bar{\Lambda}_1 \Delta_{u_1} + \bar{\Lambda}_2 \Delta_{u_2})}{\Delta^2 \bar{\Delta}}\end{aligned}$$

whence after reduction to the common denominator  $\Delta^2 \bar{\Delta}^2$ :

$$\begin{aligned}\sqrt{-1} \left\{ \mathcal{L} \left( \frac{\bar{\Lambda}_1}{\bar{\Delta}} \right) - \bar{\mathcal{L}} \left( \frac{\Lambda_1}{\Delta} \right) \right\} &= \frac{\sqrt{-1}}{\Delta^2 \bar{\Delta}^2} \left\{ \Delta \Delta \bar{\Delta} \bar{\Lambda}_{1,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{1,u_2} - \right. \\ &\quad - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_1 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_1 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_1 - \\ &\quad - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{1,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{1,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_2 \Lambda_{1,u_2} + \\ &\quad \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_1 \right\},\end{aligned}$$

which provides:

$$\begin{aligned} \Upsilon_1 := \sqrt{-1} \left( \Delta \Delta \bar{\Delta} \bar{\Lambda}_{1,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{1,u_2} - \right. \\ \left. - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_1 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_1 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_1 - \right. \\ \left. - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{1,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{1,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_z \Lambda_{1,u_2} + \right. \\ \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_1 \right), \end{aligned}$$

Similarly:

$$\sqrt{-1} \left\{ \mathcal{L} \left( \frac{\bar{\Lambda}_2}{\bar{\Delta}} \right) - \bar{\mathcal{L}} \left( \frac{\Lambda_2}{\Delta} \right) \right\} = \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2},$$

with:

$$\begin{aligned} \Upsilon_2 := \sqrt{-1} \left( \Delta \Delta \bar{\Delta} \bar{\Lambda}_{2,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{2,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{2,u_2} - \right. \\ \left. - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_2 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_2 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_2 - \right. \\ \left. - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{2,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{2,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_z \Lambda_{2,u_2} + \right. \\ \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_2 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_2 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_2 \right). \end{aligned}$$

Secondly:

$$\begin{aligned} [\mathcal{L}, \mathcal{F}] &= \left[ \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2}, \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_2} \right] \\ &= \left\{ \mathcal{L} \left( \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} \right) - \mathcal{F} \left( \frac{\Lambda_1}{\Delta} \right) \right\} \frac{\partial}{\partial u_1} + \left\{ \mathcal{L} \left( \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2} \right) - \mathcal{F} \left( \frac{\Lambda_2}{\Delta} \right) \right\} \frac{\partial}{\partial u_2}. \end{aligned}$$

Compute:

$$\begin{aligned} \mathcal{L} \left( \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} \right) &= \frac{\mathcal{L}(\Upsilon_1)}{\Delta^2 \bar{\Delta}^2} - \frac{\Upsilon_1 2 \mathcal{L}(\Delta)}{\Delta^3 \bar{\Delta}^2} - \frac{\Upsilon_1 2 \mathcal{L}(\bar{\Delta})}{\Delta^2 \bar{\Delta}^3} \\ &= \frac{\Delta \Upsilon_{1,z} + \Lambda_1 \Upsilon_{1,u_1} + \Lambda_2 \Upsilon_{1,u_2}}{\Delta^3 \bar{\Delta}^2} - \\ &\quad - \frac{\Upsilon_1 (2 \Delta_z \Delta + 2 \Lambda_1 \Delta_{u_1} + 2 \Lambda_2 \Delta_{u_2})}{\Delta^4 \bar{\Delta}^2} - \\ &\quad - \frac{\Upsilon_1 (2 \bar{\Delta}_z \Delta + 2 \Lambda_1 \bar{\Delta}_{u_1} + 2 \Lambda_2 \bar{\Delta}_{u_2})}{\Delta^3 \bar{\Delta}^3}, \end{aligned}$$

whence after reduction to the common denominator  $\Delta^4 \bar{\Delta}^3$ :

$$\mathcal{L}\left(\frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2}\right) = \frac{1}{\Delta^4 \bar{\Delta}^3} \left\{ \begin{aligned} &\Delta \Delta \bar{\Delta} \Upsilon_{1,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{1,u_2} - \\ &- 2 \Delta \bar{\Delta} \Delta_z \Upsilon_1 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_1 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_1 - \\ &- 2 \Delta \Delta \bar{\Delta}_z \Upsilon_1 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_1 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_1 \end{aligned} \right\}.$$

Compute also:

$$\begin{aligned} \mathcal{F}\left(\frac{\Lambda_1}{\Delta}\right) &= \frac{\mathcal{F}(\Lambda_1)}{\Delta} - \frac{\Lambda_1 \mathcal{F}(\Delta)}{\Delta^2} \\ &= \frac{\Upsilon_1 \Lambda_{1,u_1} + \Upsilon_2 \Lambda_{1,u_2}}{\Delta^3 \bar{\Delta}^2} - \\ &- \frac{\Lambda_1 (\Upsilon_1 \Delta_{u_1} + \Upsilon_2 \Delta_{u_2})}{\Delta^4 \bar{\Delta}^2} \\ &= \frac{1}{\Delta^4 \bar{\Delta}^3} \left\{ \begin{aligned} &\Delta \bar{\Delta} \Upsilon_1 \Lambda_{1,u_1} + \Delta \bar{\Delta} \Upsilon_2 \Lambda_{1,u_2} - \\ &- \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_1 - \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_1 \end{aligned} \right\}. \end{aligned}$$

In sum:

$$\mathcal{L}\left(\frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2}\right) - \mathcal{F}\left(\frac{\Lambda_1}{\Delta}\right) = \frac{\Pi_1}{\Delta^4 \bar{\Delta}^3},$$

with:

$$\begin{aligned} \Pi_1 &= \Delta \Delta \bar{\Delta} \Upsilon_{1,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{1,u_2} - \\ &- 2 \Delta \bar{\Delta} \Delta_z \Upsilon_1 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_1 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_1 - \\ &- 2 \Delta \Delta \bar{\Delta}_z \Upsilon_1 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_1 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_1 - \\ &- \Delta \bar{\Delta} \Upsilon_1 \Lambda_{1,u_1} - \Delta \bar{\Delta} \Upsilon_2 \Lambda_{1,u_2} + \\ &+ \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_1 + \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_1. \end{aligned}$$

Similarly:

$$\mathcal{L}\left(\frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2}\right) - \mathcal{F}\left(\frac{\Lambda_2}{\Delta}\right) = \frac{\Pi_2}{\Delta^4 \bar{\Delta}^3},$$

with:

$$\begin{aligned} \Pi_2 = & \Delta \Delta \bar{\Delta} \Upsilon_{2,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{2,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{2,u_2} - \\ & - 2 \Delta \bar{\Delta} \Delta_z \Upsilon_2 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_2 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_2 - \\ & - 2 \Delta \Delta \bar{\Delta}_z \Upsilon_2 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_2 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_2 - \\ & - \Delta \bar{\Delta} \Upsilon_1 \Lambda_{2,u_1} - \Delta \bar{\Delta} \Upsilon_2 \Lambda_{2,u_2} + \\ & + \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_2 + \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_2, \end{aligned}$$

which concludes.  $\square$

**Explicitness obstacle.** After reduction to a common minimal denominator, the two numerators in:

$$Y_1 = \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2}, \quad Y_2 = \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2},$$

are both polynomials in the  $2 \cdot 14$  partial derivatives:

$$\left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}} \right)_{1 \leq j+k+l_1+l_2 \leq 2}$$

incorporating:

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monomials, while the next two numerators in:

$$H_1 = \frac{\Pi_1}{\Delta^4 \bar{\Delta}^3}, \quad H_2 = \frac{\Pi_2}{\Delta^4 \bar{\Delta}^3}$$

incorporate both:

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monomials in the  $2 \cdot 34$  partial derivatives:

$$\left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2}} \right)_{1 \leq j+k+l_1+l_2 \leq 3}.$$

*Proof.* Use a computer software.  $\square$

The Class II hypothesis now reads as the nonzeroness:

$$0 \neq \det \begin{pmatrix} \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} & \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2} \\ \frac{\Pi_1}{\Delta^4 \bar{\Delta}^3} & \frac{\Pi_2}{\Delta^4 \bar{\Delta}^3} \end{pmatrix} (x, y, u_1, u_2),$$

at every point.

More precisely, if one preliminarily normalizes coordinates as in [6]:

$$\begin{aligned} v_1 &= z\bar{z} & + z\bar{z} O_2(z, \bar{z}) + z\bar{z} O_1(u_1) + z\bar{z} O_1(u_2), \\ v_2 &= z^2\bar{z} + z\bar{z}^2 & + z\bar{z} O_2(z, \bar{z}) + z\bar{z} O_1(u_1) + z\bar{z} O_1(u_2), \end{aligned}$$

so that at the origin:

$$\begin{aligned}\mathcal{L}|_0 &= \frac{\partial}{\partial z}\Big|_0, \\ \overline{\mathcal{L}}|_0 &= \frac{\partial}{\partial \bar{z}}\Big|_0, \\ [\mathcal{L}, \overline{\mathcal{L}}]|_0 &= -2\sqrt{-1}\frac{\partial}{\partial u_1}\Big|_0, \\ [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]|_0 &= -4\sqrt{-1}\frac{\partial}{\partial u_2}\Big|_0,\end{aligned}$$

the determinant in question:

$$\det \begin{pmatrix} \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} & \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} \\ \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} & \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} \end{pmatrix} (0) = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

becomes quite visibly nonzero, hence also near the origin.

But generally, the disease is, that when (necessarily) re-expressing:

$$\begin{aligned}\frac{\partial}{\partial u_1} &= \frac{\Delta^2 \overline{\Delta}^2 \Pi_2}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \Pi_1 & \Pi_2 \end{vmatrix}} \cdot \mathcal{T} - \frac{\Delta^4 \overline{\Delta}^3 \Upsilon_2}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \Pi_1 & \Pi_2 \end{vmatrix}} \cdot \mathcal{S}, \\ \frac{\partial}{\partial u_2} &= \frac{\Delta^2 \overline{\Delta}^2 \Pi_1}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \Pi_1 & \Pi_2 \end{vmatrix}} \cdot \mathcal{T} - \frac{\Delta^4 \overline{\Delta}^3 \Upsilon_1}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \Pi_1 & \Pi_2 \end{vmatrix}} \cdot \mathcal{S},\end{aligned}$$

some quite huge fractions appear, and it becomes unwieldy, even for a powerful computer, to make explicit the coefficients in the next 4 brackets completing the Lie structure of the frame:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{T}] &= \text{function} \cdot \mathcal{T} + \text{function} \cdot \mathcal{S}, \\ [\mathcal{L}, \mathcal{S}] &= \text{function} \cdot \mathcal{T} + \text{function} \cdot \mathcal{S}, \\ [\overline{\mathcal{L}}, \mathcal{S}] &= \text{function} \cdot \mathcal{T} + \text{function} \cdot \mathcal{S}, \\ [\mathcal{T}, \mathcal{S}] &= \text{function} \cdot \mathcal{T} + \text{function} \cdot \mathcal{S}.\end{aligned}$$

**About a classical simplified context.** In the (special) so-called *rigid* case:

$$\begin{aligned}v_1 &= \varphi_1(x, y), \\ v_2 &= \varphi_2(x, y),\end{aligned}$$

where the graphing functions do not depend upon  $u_1, u_2$ , completely explicit formulas can be typesetted.

Indeed:

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \sqrt{-1} \varphi_{1,z} \frac{\partial}{\partial u_1} + \sqrt{-1} \varphi_{2,z} \frac{\partial}{\partial u_2}, \\ \overline{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} - \sqrt{-1} \varphi_{1,\bar{z}} \frac{\partial}{\partial u_1} - \sqrt{-1} \varphi_{2,\bar{z}} \frac{\partial}{\partial u_2}, \\ \mathcal{T} &= 2 \varphi_{1,z\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,z\bar{z}} \frac{\partial}{\partial u_2}, \\ \mathcal{S} &= 2 \varphi_{1,zz\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,zz\bar{z}} \frac{\partial}{\partial u_2}.\end{aligned}$$

The determinant in question is:

$$4 (\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}),$$

and one has:

$$\begin{aligned}\frac{\partial}{\partial u_1} &= \frac{\varphi_{2,zz\bar{z}}}{2 (\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}})} \cdot \mathcal{T} - \frac{\varphi_{2,z\bar{z}}}{2 (\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}})} \cdot \mathcal{S}, \\ \frac{\partial}{\partial u_2} &= - \frac{\varphi_{1,zz\bar{z}}}{2 (\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}})} \cdot \mathcal{T} + \frac{\varphi_{1,z\bar{z}}}{2 (\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}})} \cdot \mathcal{S}.\end{aligned}$$

Next:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{T}] &= 2 \varphi_{1,zz\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,zz\bar{z}} \frac{\partial}{\partial u_2} \\ &= \left( \frac{\varphi_{1,zz\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,zz\bar{z}} \varphi_{1,zz\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{T} + \left( \frac{-\varphi_{1,zz\bar{z}} \varphi_{2,z\bar{z}} + \varphi_{2,zz\bar{z}} \varphi_{1,z\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{S},\end{aligned}$$

and:

$$\begin{aligned}[\mathcal{L}, \mathcal{S}] &= 2 \varphi_{1,zzz\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,zzz\bar{z}} \frac{\partial}{\partial u_2} \\ &= \left( \frac{\varphi_{1,zzz\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,zzz\bar{z}} \varphi_{1,zz\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{T} + \left( \frac{-\varphi_{1,zzz\bar{z}} \varphi_{2,z\bar{z}} + \varphi_{2,zzz\bar{z}} \varphi_{1,z\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{S},\end{aligned}$$

and also:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{S}] &= 2 \varphi_{1,zzz\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,zzz\bar{z}} \frac{\partial}{\partial u_2} \\ &= \left( \frac{\varphi_{1,zzz\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,zzz\bar{z}} \varphi_{1,zz\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{T} + \left( \frac{-\varphi_{1,zzz\bar{z}} \varphi_{2,z\bar{z}} + \varphi_{2,zzz\bar{z}} \varphi_{1,z\bar{z}}}{\varphi_{1,z\bar{z}} \varphi_{2,zz\bar{z}} - \varphi_{2,z\bar{z}} \varphi_{1,zz\bar{z}}} \right) \cdot \mathcal{S},\end{aligned}$$

while trivially (only in the rigid case!):

$$[\mathcal{S}, \mathcal{T}] = 0.$$

**Symbolic treatment of the general case.** Now, come back to the general case where  $\varphi_1, \varphi_2$  do depend on all variables  $(x, y, u_1, u_2)$ , so that, from the



form of the two defining equations  $v_1 = \varphi^1$  and  $v_2 = \varphi^2$ :

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2}, \\ \overline{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} + \frac{\overline{\Lambda}_1}{\overline{\Delta}} \frac{\partial}{\partial u_1} + \frac{\overline{\Lambda}_2}{\overline{\Delta}} \frac{\partial}{\partial u_2}, \\ \mathcal{T} &= \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_2}, \\ \mathcal{S} &= \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_2}.\end{aligned}$$

**Lemma-Exercise.** *There is a uniquely defined function:*

$$\ell = \ell(x, y, u_1, u_2)$$

*defined on  $M$  near 0 such that the vector field:*

$$\mathcal{N} := \mathcal{L} + \ell \overline{\mathcal{L}}$$

*has the property:*

$$[\mathcal{N}, \mathcal{T}] \equiv 0 \pmod{(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T})}.$$

*Proof.* The solution appears implicitly below. □

Since, as always,  $\mathcal{T}$  is real:

$$\overline{\mathcal{T}} = \mathcal{T},$$

one sees:

$$\overline{\mathcal{T}} = \overline{[\mathcal{L}, \mathcal{T}]} = [\overline{\mathcal{L}}, \mathcal{T}].$$

But of course:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{T}] &= \text{function} \cdot \frac{\partial}{\partial u_1} + \text{function} \cdot \frac{\partial}{\partial u_2} \\ &= \text{function} \cdot \mathcal{T} + \text{function} \cdot \mathcal{S}.\end{aligned}$$

Therefore, by introducing two *fundamental* functions  $A$  and  $B$  of  $(x, y, u_1, u_2)$  so that:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{T}] &= A \mathcal{T} + B \mathcal{S} \\ &= \overline{\mathcal{T}},\end{aligned}$$

these two functions — that will be very fundamental when applying Cartan's method later on — enjoy some relations coming from the comparison

with:

$$\begin{aligned}
[\mathcal{L}, \mathcal{I}] &= \overline{[\mathcal{L}, \mathcal{I}]} \\
&= \overline{A} \mathcal{I} + \overline{B} \underbrace{\overline{\mathcal{I}}}_{\text{replace}} \\
&= \overline{A} \mathcal{I} + \overline{B} (A \mathcal{I} + B \mathcal{I}) \\
&= (\overline{A} + \overline{B} A) \mathcal{I} + B \overline{B} \mathcal{I} \\
&= \mathcal{I} \qquad \qquad \qquad \text{[by definition],}
\end{aligned}$$

whence by identification:

$$\boxed{
\begin{aligned}
B \overline{B} &\equiv 1, \\
\overline{A} + \overline{B} A &\equiv 0.
\end{aligned}
}$$

Later in the Cartan procedure, one should take account of these two relations.

For now:

$$\begin{aligned}
[\mathcal{L} + \ell \overline{\mathcal{L}}, \mathcal{I}] &\equiv [\mathcal{L}, \mathcal{I}] + \ell [\overline{\mathcal{L}}, \mathcal{I}] \pmod{(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{I})}, \\
&\equiv \mathcal{I} + \ell (A \mathcal{I} + B \mathcal{I}) \pmod{(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{I})} \\
&\equiv (1 + \ell B) \mathcal{I},
\end{aligned}$$

hence the solution to the lemma-exercise is:

$$\ell = -\frac{1}{B}.$$

Next, introduce two new *fundamental* functions  $P$  and  $Q$  of  $(x, y, u_1, u_2)$  so that:

$$[\mathcal{L}, \mathcal{I}] = P \mathcal{I} + Q \mathcal{I},$$

whence *passim* after conjugation and replacement:

$$\begin{aligned}
[\overline{\mathcal{L}}, \overline{\mathcal{I}}] &= \overline{P} \mathcal{I} + \overline{Q} \overline{\mathcal{I}} \\
&= \overline{P} \mathcal{I} + \overline{Q} (A \mathcal{I} + B \mathcal{I}) \\
&= (\overline{P} + A \overline{Q}) \mathcal{I} + (B \overline{Q}) \mathcal{I}.
\end{aligned}$$

**Lemma.** *One has the reality:*

$$[\overline{\mathcal{L}}, \mathcal{I}] = [\mathcal{L}, \overline{\mathcal{I}}].$$

*Proof.* Indeed, in full intermediate details:

$$\begin{aligned} [\overline{\mathcal{L}}, \mathcal{S}] &= [\overline{\mathcal{L}}, [\mathcal{L}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]]] \\ &\stackrel{\text{Jacobi}}{=} - [[\mathcal{L}, \overline{\mathcal{L}}], [\overline{\mathcal{L}}, \sqrt{-1}\mathcal{L}]] - [\mathcal{L}, [[\mathcal{L}, \overline{\mathcal{L}}], \sqrt{-1}\mathcal{L}]] \\ &= [\mathcal{L}, [\overline{\mathcal{L}}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]]] \\ &= [\mathcal{L}, \overline{\mathcal{S}}], \end{aligned}$$

considering that the  $\sqrt{-1}$  factor does not perturb Jacobi's identity.  $\square$

As an application, determine:

$$\begin{aligned} [\overline{\mathcal{L}}, \mathcal{S}] &= [\mathcal{L}, \overline{\mathcal{S}}] \\ &= [\mathcal{L}, A\overline{\mathcal{T}} + B\overline{\mathcal{S}}] \\ &= \mathcal{L}(A) \cdot \overline{\mathcal{T}} + \mathcal{L}(B) \cdot \overline{\mathcal{S}} + A \underbrace{[\mathcal{L}, \overline{\mathcal{T}}]}_{\mathcal{S}} + B \underbrace{[\mathcal{L}, \overline{\mathcal{S}}]}_{P\overline{\mathcal{T}} + Q\overline{\mathcal{S}}} \\ &= (\mathcal{L}(A) + BP) \cdot \overline{\mathcal{T}} + (\mathcal{L}(B) + BQ + A) \cdot \overline{\mathcal{S}}. \end{aligned}$$

**Scholium.** *The reality condition:*

$$[\overline{\mathcal{L}}, \mathcal{S}] = [\mathcal{L}, \overline{\mathcal{S}}]$$

entails:

$$\begin{aligned} \mathcal{L}(A) + BP &\equiv \overline{\mathcal{L}}(\overline{A}) + A\overline{\mathcal{L}}(\overline{B}) + \overline{B}\overline{P} + A\overline{B}\overline{Q} + A\overline{A}, \\ \mathcal{L}(B) + BQ + A &\equiv B\overline{\mathcal{L}}(\overline{B}) + B\overline{B}\overline{Q} + \overline{A}B. \end{aligned}$$

*Proof.* Starting from what has been obtained at the moment:

$$[\overline{\mathcal{L}}, \mathcal{S}] = (\mathcal{L}(A) + BP) \cdot \overline{\mathcal{T}} + (\mathcal{L}(B) + BQ + A) \cdot \overline{\mathcal{S}},$$

a plain conjugation gives:

$$\begin{aligned} [\mathcal{L}, \overline{\mathcal{S}}] &= (\overline{\mathcal{L}}(\overline{A}) + \overline{B}\overline{P}) \cdot \overline{\mathcal{T}} + (\overline{\mathcal{L}}(\overline{B}) + \overline{B}\overline{Q} + \overline{A}) \underbrace{\overline{\mathcal{S}}}_{A\overline{\mathcal{T}} + B\overline{\mathcal{S}}} \\ &= (\overline{\mathcal{L}}(\overline{A}) + A\overline{\mathcal{L}}(\overline{B}) + \overline{B}\overline{P} + A\overline{B}\overline{Q} + A\overline{A}) \cdot \overline{\mathcal{T}} + \\ &\quad + (B\overline{\mathcal{L}}(\overline{B}) + B\overline{B}\overline{Q} + \overline{A}B) \cdot \overline{\mathcal{S}}, \end{aligned}$$

and an identification gives the result.  $\square$

Lastly, compute the 6<sup>th</sup> Lie bracket  $[\mathcal{S}, \overline{\mathcal{S}}]$ .

**Lemma.** *One has the Jacobi-type Lie bracket relation:*

$$[\mathcal{S}, \overline{\mathcal{S}}] = \sqrt{-1} [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{S}}]] - \sqrt{-1} [\mathcal{L}, [\overline{\mathcal{L}}, \overline{\mathcal{S}}]].$$

*Proof.* Indeed, in full intermediate details:

$$\begin{aligned}
[\mathcal{S}, \mathcal{T}] &= [\mathcal{S}, \sqrt{-1}[\overline{\mathcal{L}}, \overline{\mathcal{L}}]] \\
&= \sqrt{-1}[\mathcal{S}, [\overline{\mathcal{L}}, \overline{\mathcal{L}}]] \\
&\stackrel{\text{Jacobi}}{=} -\sqrt{-1}[\overline{\mathcal{L}}, [\mathcal{S}, \overline{\mathcal{L}}]] - \sqrt{-1}[\overline{\mathcal{L}}, [\overline{\mathcal{L}}, \mathcal{S}]] \\
&= \sqrt{-1}[\overline{\mathcal{L}}, [\mathcal{L}, \mathcal{S}]] - \sqrt{-1}[\overline{\mathcal{L}}, [\overline{\mathcal{L}}, \mathcal{S}]],
\end{aligned}$$

which is so.  $\square$

Consequently:

$$\begin{aligned}
[\mathcal{S}, \mathcal{T}] &= \sqrt{-1}[\overline{\mathcal{L}}, P\mathcal{T} + Q\mathcal{S}] - \sqrt{-1}[\overline{\mathcal{L}}, (\mathcal{L}(A) + BP)\mathcal{T} + (\mathcal{L}(B) + BQ + A)\mathcal{S}] \\
&= \sqrt{-1}\overline{\mathcal{L}}(P) \cdot \mathcal{T} + \sqrt{-1}\overline{\mathcal{L}}(Q) \cdot \mathcal{S} + \underbrace{\sqrt{-1}P[\overline{\mathcal{L}}, \mathcal{T}]}_{A\mathcal{T}+B\mathcal{S}} + \underbrace{\sqrt{-1}Q[\overline{\mathcal{L}}, \mathcal{S}]}_{\substack{(\mathcal{L}(A)+BP)\mathcal{T}+ \\ +(\mathcal{L}(B)+BQ+A)\mathcal{S}}} - \\
&\quad - \sqrt{-1}(\mathcal{L}(\mathcal{L}(A)) + P\mathcal{L}(B) + B\mathcal{L}(P)) \cdot \mathcal{T} - \\
&\quad - \sqrt{-1}(\mathcal{L}(\mathcal{L}(B)) + Q\mathcal{L}(B) + B\mathcal{L}(Q) + \mathcal{L}(A)) \cdot \mathcal{S} - \\
&= \sqrt{-1}(\mathcal{L}(A) + BP) \underbrace{[\overline{\mathcal{L}}, \mathcal{T}]}_{\mathcal{S}} - \sqrt{-1}(\mathcal{L}(B) + BQ + A) \underbrace{[\overline{\mathcal{L}}, \mathcal{S}]}_{P\mathcal{T}+Q\mathcal{S}} \\
&= \left( -\sqrt{-1}\mathcal{L}(\mathcal{L}(A)) - \sqrt{-1}P\mathcal{L}(B) + \sqrt{-1}\overline{\mathcal{L}}(P) + \sqrt{-1}Q\mathcal{L}(A) - \right. \\
&\quad \left. - \sqrt{-1}P\mathcal{L}(B) - \sqrt{-1}B\mathcal{L}(P) \right) \cdot \mathcal{T} + \\
&\quad + \left( -\sqrt{-1}\mathcal{L}(\mathcal{L}(B)) - \sqrt{-1}Q\mathcal{L}(B) - \sqrt{-1}B\mathcal{L}(Q) - \right. \\
&\quad \left. - 2\sqrt{-1}\mathcal{L}(A) + \sqrt{-1}\overline{\mathcal{L}}(Q) \right) \cdot \mathcal{S}.
\end{aligned}$$

In order to abbreviate, it will be convenient to set:

$$\begin{aligned}
E_{\text{rpl}} &:= \mathcal{L}(A) + BP, \\
F_{\text{rpl}} &:= \mathcal{L}(B) + BQ + A, \\
G_{\text{rpl}} &:= \sqrt{-1}\mathcal{L}(\mathcal{L}(A)) + \sqrt{-1}P\mathcal{L}(B) - \sqrt{-1}\overline{\mathcal{L}}(P) - \sqrt{-1}Q\mathcal{L}(A) + \sqrt{-1}P\mathcal{L}(B) + \sqrt{-1}B\mathcal{L}(P), \\
H_{\text{rpl}} &:= \sqrt{-1}\mathcal{L}(\mathcal{L}(B)) + \sqrt{-1}Q\mathcal{L}(B) + \sqrt{-1}B\mathcal{L}(Q) + 2\sqrt{-1}\mathcal{L}(A) - \sqrt{-1}\overline{\mathcal{L}}(Q),
\end{aligned}$$

the lower index "rpl" meaning that one should replace these letters by their expression in terms of the **4** fundamental functions:

$$A, \quad B, \quad P, \quad Q.$$

In other words, *the four functions*  $E_{\text{rpl}}, F_{\text{rpl}}, G_{\text{rpl}}, H_{\text{rpl}}$  *are not fundamental.*

**Summary.** One has the 6 Lie bracket relations:

$$\begin{aligned} [\mathcal{S}, \mathcal{T}] &= -G_{\text{rpl}} \cdot \mathcal{T} - H_{\text{rpl}} \cdot \mathcal{S}, \\ [\mathcal{S}, \overline{\mathcal{L}}] &= -E_{\text{rpl}} \cdot \mathcal{T} - F_{\text{rpl}} \cdot \mathcal{S}, \\ [\mathcal{S}, \mathcal{L}] &= -P \cdot \mathcal{T} - Q \mathcal{S}, \\ [\mathcal{T}, \overline{\mathcal{L}}] &= -A \cdot \mathcal{T} - B \mathcal{S}, \\ [\mathcal{T}, \mathcal{L}] &= -\mathcal{S}, \\ [\overline{\mathcal{L}}, \mathcal{L}] &= \sqrt{-1} \mathcal{T}. \end{aligned}$$

**Initial Darboux structure of the dual coframe.** The coframe:

$$\{du_2, du_1, dz, d\bar{z}\}$$

is clearly dual to the frame:

$$\left\{ \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right\}.$$

Introduce then the coframe:

$$\{\sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\}$$

which is dual to the frame:

$$\{\mathcal{S}, \mathcal{T}, \overline{\mathcal{L}}, \mathcal{L}\},$$

namely:

$$\begin{array}{cccc} \sigma_0(\mathcal{S}) = 1 & \sigma_0(\mathcal{T}) = 0 & \sigma_0(\overline{\mathcal{L}}) = 0 & \sigma_0(\mathcal{L}) = 0, \\ \rho_0(\mathcal{S}) = 0 & \rho_0(\mathcal{T}) = 1 & \rho_0(\overline{\mathcal{L}}) = 0 & \rho_0(\mathcal{L}) = 0, \\ \overline{\zeta}_0(\mathcal{S}) = 0 & \overline{\zeta}_0(\mathcal{T}) = 0 & \overline{\zeta}_0(\overline{\mathcal{L}}) = 1 & \overline{\zeta}_0(\mathcal{L}) = 0, \\ \zeta_0(\mathcal{S}) = 0 & \zeta_0(\mathcal{T}) = 0 & \zeta_0(\overline{\mathcal{L}}) = 0 & \zeta_0(\mathcal{L}) = 1. \end{array}$$

One has (exercice):

$$\zeta_0 = dz \quad \text{and} \quad \overline{\zeta}_0 = d\bar{z}.$$

Proceeding as in the launching file [3], and as in [9], one determines the Darboux structure of this initial coframe by reading vertically the following

convenient auxiliary array, putting an overall minus sign:

	$\mathcal{S}$	$\mathcal{T}$	$\overline{\mathcal{L}}$	$\mathcal{L}$	
	$d\sigma_0$	$d\rho_0$	$d\overline{\zeta}_0$	$d\zeta_0$	
$[\mathcal{S}, \mathcal{T}]$	$-H_{\text{rpl}}$	$+ G_{\text{rpl}}$	$+ 0$	$+ 0$	$\sigma_0 \wedge \rho_0$
$[\mathcal{S}, \overline{\mathcal{L}}]$	$-F_{\text{rpl}}$	$+ E_{\text{rpl}}$	$+ 0$	$+ 0$	$\sigma_0 \wedge \overline{\zeta}_0$
$[\mathcal{S}, \mathcal{L}]$	$-Q$	$+ P$	$+ 0$	$+ 0$	$\sigma_0 \wedge \zeta_0$
$[\mathcal{T}, \overline{\mathcal{L}}]$	$-B$	$+ A$	$+ 0$	$+ 0$	$\rho_0 \wedge \overline{\zeta}_0$
$[\mathcal{T}, \mathcal{L}]$	$-1$	$+ 0$	$+ 0$	$+ 0$	$\rho_0 \wedge \zeta_0$
$[\overline{\mathcal{L}}, \mathcal{L}]$	$0$	$+ \sqrt{-1}$	$+ 0$	$+ 0$	$\overline{\zeta}_0 \wedge \zeta_0$

This gives:

$$\begin{aligned}
 d\sigma_0 &= H_{\text{rpl}} \cdot \sigma_0 \wedge \rho_0 + F_{\text{rpl}} \cdot \sigma_0 \wedge \overline{\zeta}_0 + Q \cdot \sigma_0 \wedge \zeta_0 + B \cdot \rho_0 \wedge \overline{\zeta}_0 + \rho_0 \wedge \zeta_0, \\
 d\rho_0 &= G_{\text{rpl}} \cdot \sigma_0 \wedge \rho_0 + E_{\text{rpl}} \cdot \sigma_0 \wedge \overline{\zeta}_0 + P \cdot \sigma_0 \wedge \zeta_0 + A \cdot \rho_0 \wedge \overline{\zeta}_0 + \sqrt{-1} \zeta_0 \wedge \overline{\zeta}_0, \\
 d\overline{\zeta}_0 &= 0, \\
 d\zeta_0 &= 0.
 \end{aligned}$$

### 3. $M^5 \subset \mathbb{C}^4$ of general class III<sub>1</sub>: initial frame and coframe in local coordinates

Next, consider:

$$(M^5 \subset \mathbb{C}^4) \in \text{General Class III}_1.$$

Representing as before ([5, 6])  $M$  in coordinates:

$$(z, w_1, w_2, w_3) = (x + \sqrt{-1}y, u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2, u_3 + \sqrt{-1}v_3),$$

as a graph:

$$\begin{aligned}
 v_1 &= \varphi_1(x, y, u_1, u_2, u_3), \\
 v_2 &= \varphi_2(x, y, u_1, u_2, u_3), \\
 v_3 &= \varphi_3(x, y, u_1, u_2, u_3),
 \end{aligned}$$

an associated explicit frame for  $TM$  is:

$$\begin{aligned} X &= \frac{\partial}{\partial x} + \varphi_{1,x}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_1} + \varphi_{2,x}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_2} + \varphi_{3,x}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_3}, \\ Y &= \frac{\partial}{\partial y} + \varphi_{1,y}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_1} + \varphi_{2,y}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_2} + \varphi_{3,y}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_3}, \\ U_1 &= \frac{\partial}{\partial u_1} + \varphi_{1,u_1}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_1} + \varphi_{2,u_1}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_2} + \varphi_{3,u_1}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_3}, \\ U_2 &= \frac{\partial}{\partial u_2} + \varphi_{1,u_2}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_1} + \varphi_{2,u_2}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_2} + \varphi_{3,u_2}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_3}, \\ U_3 &= \frac{\partial}{\partial u_3} + \varphi_{1,u_3}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_1} + \varphi_{2,u_3}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_2} + \varphi_{3,u_3}(x, y, u_1, u_2, u_3) \frac{\partial}{\partial v_3}. \end{aligned}$$

The projection:

$$\begin{aligned} \pi: \quad \mathbb{R}^8 &\longrightarrow \mathbb{R}^5 \\ (x, y, u_1, v_1, u_2, v_2, u_3, v_3) &\longmapsto (x, y, u_1, u_2, u_3), \end{aligned}$$

makes a *chart* on  $M$  and does:

$$\begin{aligned} \pi_* \left( \frac{\partial}{\partial x} + \varphi_{1,x} \frac{\partial}{\partial v_1} + \varphi_{2,x} \frac{\partial}{\partial v_2} + \varphi_{3,x} \frac{\partial}{\partial v_3} \right) &= \frac{\partial}{\partial x}, \\ \pi_* \left( \frac{\partial}{\partial y} + \varphi_{1,y} \frac{\partial}{\partial v_1} + \varphi_{2,y} \frac{\partial}{\partial v_2} + \varphi_{3,y} \frac{\partial}{\partial v_3} \right) &= \frac{\partial}{\partial y}, \\ \pi_* \left( \frac{\partial}{\partial u_1} + \varphi_{1,u_1} \frac{\partial}{\partial v_1} + \varphi_{2,u_1} \frac{\partial}{\partial v_2} + \varphi_{3,u_1} \frac{\partial}{\partial v_3} \right) &= \frac{\partial}{\partial u_1}, \\ \pi_* \left( \frac{\partial}{\partial u_2} + \varphi_{1,u_2} \frac{\partial}{\partial v_1} + \varphi_{2,u_2} \frac{\partial}{\partial v_2} + \varphi_{3,u_2} \frac{\partial}{\partial v_3} \right) &= \frac{\partial}{\partial u_2}, \\ \pi_* \left( \frac{\partial}{\partial u_3} + \varphi_{1,u_3} \frac{\partial}{\partial v_1} + \varphi_{2,u_3} \frac{\partial}{\partial v_2} + \varphi_{3,u_3} \frac{\partial}{\partial v_3} \right) &= \frac{\partial}{\partial u_3}. \end{aligned}$$

Next, a natural *intrinsic* local vector field generator for  $T^{1,0}M$  is:

$$\mathcal{L} := \frac{\partial}{\partial z} + A_1 \frac{\partial}{\partial u_1} + A_2 \frac{\partial}{\partial u_2} + A_3 \frac{\partial}{\partial u_3},$$

where:

$$A_1 := \begin{array}{c} \left| \begin{array}{ccc} -\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ -\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{array} \right| \\ \hline \left| \begin{array}{ccc} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{array} \right| \end{array},$$

$$A_2 := \frac{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,z} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & -\varphi_{2,z} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & -\varphi_{3,z} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}},$$

$$A_3 := \frac{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & -\varphi_{1,z} \\ \varphi_{2,u_1} & \varphi_{2,u_2} & -\varphi_{2,z} \\ \varphi_{3,u_1} & \sqrt{-1} + \varphi_{3,u_2} & -\varphi_{3,z} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}}.$$

Set:

$$\Delta := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

whence:

$$\bar{\Delta} := \begin{vmatrix} -\sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & -\sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & -\sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}.$$

Also, set:

$$\Lambda_1 := \begin{vmatrix} -\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ -\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

$$\Lambda_2 := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,z} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & -\varphi_{2,z} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & -\varphi_{3,z} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

$$\Lambda_3 := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & -\varphi_{1,z} \\ \varphi_{2,u_1} & \varphi_{2,u_2} & -\varphi_{2,z} \\ \varphi_{3,u_1} & \sqrt{-1} + \varphi_{3,u_2} & -\varphi_{3,z} \end{vmatrix},$$



whence:

$$\begin{aligned}\bar{\Lambda}_1 &:= \begin{vmatrix} -\varphi_{1,\bar{z}} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ -\varphi_{2,\bar{z}} & -\sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ -\varphi_{3,\bar{z}} & \varphi_{3,u_2} & -\sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}, \\ \bar{\Lambda}_2 &:= \begin{vmatrix} -\sqrt{-1} + \varphi_{1,u_1} & -\varphi_{1,\bar{z}} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & -\varphi_{2,\bar{z}} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & -\varphi_{3,\bar{z}} & -\sqrt{-1} + \varphi_{3,u_3} \end{vmatrix}, \\ \bar{\Lambda}_3 &:= \begin{vmatrix} -\sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & -\varphi_{1,\bar{z}} \\ \varphi_{2,u_1} & \varphi_{2,u_2} & -\varphi_{2,\bar{z}} \\ \varphi_{3,u_1} & -\sqrt{-1} + \varphi_{3,u_2} & -\varphi_{3,\bar{z}} \end{vmatrix}.\end{aligned}$$

In these notations:

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2} + \frac{\Lambda_3}{\Delta} \frac{\partial}{\partial u_3}, \\ \bar{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} + \frac{\bar{\Lambda}_1}{\bar{\Delta}} \frac{\partial}{\partial u_1} + \frac{\bar{\Lambda}_2}{\bar{\Delta}} \frac{\partial}{\partial u_2} + \frac{\bar{\Lambda}_3}{\bar{\Delta}} \frac{\partial}{\partial u_3}.\end{aligned}$$

Set:

$$\mathcal{T} := \sqrt{-1} [\mathcal{L}, \bar{\mathcal{L}}],$$

and set:

$$\mathcal{S} := [\mathcal{L}, \mathcal{T}],$$

whence:

$$\bar{\mathcal{S}} = [\bar{\mathcal{L}}, \mathcal{T}].$$

By hypothesis, the CR-geometric invariant condition:

$$\begin{aligned}\mathbb{C} \otimes_{\mathbb{R}} TM &= T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] \\ &\quad + [T^{0,1}M, [T^{1,0}M, T^{0,1}M]]\end{aligned}$$

holds, which means as is known that the 5 fields:

$$\{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \bar{\mathcal{S}}\}$$

constitute a frame for:

$$\mathfrak{5} = \text{rank}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} TM).$$

**Lemma.** *There are certain uniquely defined coefficient-functions that are polynomials:*

$$\begin{aligned}\Upsilon_1 &= \Upsilon_1 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 2}, \\ \Upsilon_2 &= \Upsilon_2 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 2}, \\ \Upsilon_3 &= \Upsilon_3 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 2},\end{aligned}$$

$$\begin{aligned}\Pi_1 &= \Pi_1 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 3}, \\ \Pi_2 &= \Pi_2 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 3}, \\ \Pi_3 &= \Pi_3 \left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 3},\end{aligned}$$

such that:

$$\begin{aligned}\mathcal{F} &= \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^2 \bar{\Delta}^2} \frac{\partial}{\partial u_3} \\ &=: Y_1 \frac{\partial}{\partial u_1} + Y_2 \frac{\partial}{\partial u_2} + Y_3 \frac{\partial}{\partial u_3}, \\ \mathcal{S} &= \frac{\Pi_1}{\Delta^4 \bar{\Delta}^3} \frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4 \bar{\Delta}^3} \frac{\partial}{\partial u_2} + \frac{\Pi_3}{\Delta^4 \bar{\Delta}^3} \frac{\partial}{\partial u_3} \\ &=: H_1 \frac{\partial}{\partial u_1} + H_2 \frac{\partial}{\partial u_2} + H_3 \frac{\partial}{\partial u_3}, \\ \bar{\mathcal{F}} &= \frac{\bar{\Pi}_1}{\Delta^3 \bar{\Delta}^4} \frac{\partial}{\partial u_1} + \frac{\bar{\Pi}_2}{\Delta^3 \bar{\Delta}^4} \frac{\partial}{\partial u_2} + \frac{\bar{\Pi}_3}{\Delta^3 \bar{\Delta}^4} \frac{\partial}{\partial u_3} \\ &=: \bar{H}_1 \frac{\partial}{\partial u_1} + \bar{H}_2 \frac{\partial}{\partial u_2} + \bar{H}_3 \frac{\partial}{\partial u_3}.\end{aligned}$$

*Proof.* The proof goes as for the preceding General Class II, and it provides firstly:

$$\begin{aligned}\Upsilon_1 &:= \sqrt{-1} \left( \Delta \Delta \bar{\Delta} \bar{\Lambda}_{1,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{1,u_2} + \Delta \bar{\Delta} \Lambda_3 \bar{\Lambda}_{1,u_3} - \right. \\ &\quad - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_1 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_1 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_1 - \Delta \Lambda_3 \bar{\Delta}_{u_3} \bar{\Lambda}_1 - \\ &\quad - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{1,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{1,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_2 \Lambda_{1,u_2} - \Delta \bar{\Delta} \bar{\Lambda}_3 \Lambda_{1,u_3} + \\ &\quad \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_1 + \bar{\Delta} \bar{\Lambda}_3 \Delta_{u_3} \Lambda_1 \right),\end{aligned}$$

$$\begin{aligned}\Upsilon_2 &:= \sqrt{-1} \left( \Delta \Delta \bar{\Delta} \bar{\Lambda}_{2,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{2,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{2,u_2} + \Delta \bar{\Delta} \Lambda_3 \bar{\Lambda}_{2,u_3} - \right. \\ &\quad - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_2 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_2 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_2 - \Delta \Lambda_3 \bar{\Delta}_{u_3} \bar{\Lambda}_2 - \\ &\quad - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{2,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{2,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_2 \Lambda_{2,u_2} - \Delta \bar{\Delta} \bar{\Lambda}_3 \Lambda_{2,u_3} + \\ &\quad \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_2 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_2 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_2 + \bar{\Delta} \bar{\Lambda}_3 \Delta_{u_3} \Lambda_2 \right),\end{aligned}$$

$$\Upsilon_3 := \sqrt{-1} \left( \Delta \Delta \bar{\Delta} \bar{\Lambda}_{3,z} + \Delta \bar{\Delta} \Lambda_1 \bar{\Lambda}_{3,u_1} + \Delta \bar{\Delta} \Lambda_2 \bar{\Lambda}_{3,u_2} + \Delta \bar{\Delta} \Lambda_3 \bar{\Lambda}_{3,u_3} - \right. \\ \left. - \Delta \Delta \bar{\Delta}_z \bar{\Lambda}_3 - \Delta \Lambda_1 \bar{\Delta}_{u_1} \bar{\Lambda}_3 - \Delta \Lambda_2 \bar{\Delta}_{u_2} \bar{\Lambda}_3 - \Delta \Lambda_3 \bar{\Delta}_{u_3} \bar{\Lambda}_3 - \right. \\ \left. - \Delta \bar{\Delta} \bar{\Delta} \Lambda_{3,\bar{z}} - \Delta \bar{\Delta} \bar{\Lambda}_1 \Lambda_{3,u_1} - \Delta \bar{\Delta} \bar{\Lambda}_2 \Lambda_{3,u_2} - \Delta \bar{\Delta} \bar{\Lambda}_3 \Lambda_{3,u_3} + \right. \\ \left. + \bar{\Delta} \bar{\Delta} \Delta_{\bar{z}} \Lambda_3 + \bar{\Delta} \bar{\Lambda}_1 \Delta_{u_1} \Lambda_3 + \bar{\Delta} \bar{\Lambda}_2 \Delta_{u_2} \Lambda_3 + \bar{\Delta} \bar{\Lambda}_3 \Delta_{u_3} \Lambda_3 \right),$$

and secondly:

$$\Pi_1 = \Delta \Delta \bar{\Delta} \Upsilon_{1,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{1,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{1,u_2} + \Delta \bar{\Delta} \Lambda_3 \Upsilon_{1,u_3} - \\ - 2 \Delta \bar{\Delta} \Delta_z \Upsilon_1 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_1 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_1 - 2 \bar{\Delta} \Lambda_3 \Delta_{u_3} \Upsilon_1 - \\ - 2 \Delta \Delta \bar{\Delta}_z \Upsilon_1 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_1 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_1 - 2 \Delta \Lambda_3 \bar{\Delta}_{u_3} \Upsilon_1 - \\ - \Delta \bar{\Delta} \Upsilon_1 \Lambda_{1,u_1} - \Delta \bar{\Delta} \Upsilon_2 \Lambda_{1,u_2} - \Delta \bar{\Delta} \Upsilon_3 \Lambda_{1,u_3} + \\ + \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_1 + \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_1 + \bar{\Delta} \Upsilon_3 \Delta_{u_3} \Lambda_1,$$

$$\Pi_2 = \Delta \Delta \bar{\Delta} \Upsilon_{2,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{2,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{2,u_2} + \Delta \bar{\Delta} \Lambda_3 \Upsilon_{2,u_3} - \\ - 2 \Delta \bar{\Delta} \Delta_z \Upsilon_2 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_2 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_2 - 2 \bar{\Delta} \Lambda_3 \Delta_{u_3} \Upsilon_2 - \\ - 2 \Delta \Delta \bar{\Delta}_z \Upsilon_2 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_2 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_2 - 2 \Delta \Lambda_3 \bar{\Delta}_{u_3} \Upsilon_2 - \\ - \Delta \bar{\Delta} \Upsilon_1 \Lambda_{2,u_1} - \Delta \bar{\Delta} \Upsilon_2 \Lambda_{2,u_2} - \Delta \bar{\Delta} \Upsilon_3 \Lambda_{2,u_3} + \\ + \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_2 + \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_2 + \bar{\Delta} \Upsilon_3 \Delta_{u_3} \Lambda_2,$$

$$\Pi_3 = \Delta \Delta \bar{\Delta} \Upsilon_{3,z} + \Delta \bar{\Delta} \Lambda_1 \Upsilon_{3,u_1} + \Delta \bar{\Delta} \Lambda_2 \Upsilon_{3,u_2} + \Delta \bar{\Delta} \Lambda_3 \Upsilon_{3,u_3} - \\ - 2 \Delta \bar{\Delta} \Delta_z \Upsilon_3 - 2 \bar{\Delta} \Lambda_1 \Delta_{u_1} \Upsilon_3 - 2 \bar{\Delta} \Lambda_2 \Delta_{u_2} \Upsilon_3 - 2 \bar{\Delta} \Lambda_3 \Delta_{u_3} \Upsilon_3 - \\ - 2 \Delta \Delta \bar{\Delta}_z \Upsilon_3 - 2 \Delta \Lambda_1 \bar{\Delta}_{u_1} \Upsilon_3 - 2 \Delta \Lambda_2 \bar{\Delta}_{u_2} \Upsilon_3 - 2 \Delta \Lambda_3 \bar{\Delta}_{u_3} \Upsilon_3 - \\ - \Delta \bar{\Delta} \Upsilon_1 \Lambda_{3,u_1} - \Delta \bar{\Delta} \Upsilon_2 \Lambda_{3,u_2} - \Delta \bar{\Delta} \Upsilon_3 \Lambda_{3,u_3} + \\ + \bar{\Delta} \Upsilon_1 \Delta_{u_1} \Lambda_3 + \bar{\Delta} \Upsilon_2 \Delta_{u_2} \Lambda_3 + \bar{\Delta} \Upsilon_3 \Delta_{u_3} \Lambda_3,$$

which concludes.  $\square$

**Explicitness obstacle.** After reduction to a common minimal denominator, the three numerators in:

$$Y_1 = \frac{\Upsilon_1}{\Delta^2 \bar{\Delta}^2}, \quad Y_2 = \frac{\Upsilon_2}{\Delta^2 \bar{\Delta}^2}, \quad Y_3 = \frac{\Upsilon_3}{\Delta^2 \bar{\Delta}^2},$$

are both polynomials in the  $3 \cdot 20$  partial derivatives:

$$\left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 2}$$

incorporating:

monomials, while the next three numerators in:

$$H_1 = \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3}, \quad H_2 = \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3}, \quad H_3 = \frac{\Pi_3}{\Delta^4 \overline{\Delta}^3},$$

would incorporate approximately more than:

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monomials in the  $3 \cdot 55$  partial derivatives:

$$\left( \varphi_{1,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{2,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}}, \varphi_{3,x^j y^k u_1^{l_1} u_2^{l_2} u_3^{l_3}} \right)_{1 \leq j+k+l_1+l_2+l_3 \leq 3},$$

since e.g.  $\Pi_1$  includes:

$$\Delta \Delta \overline{\Delta} \Upsilon_{1,z},$$

inside which  $\Delta \Delta \overline{\Delta}$  incorporates:

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monomials while  $\Upsilon_{1,z}$  incorporates:

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monomials.

*Proof.* The product is really unwieldy, even on a powerful computer software, because the monomials themselves are almost all of an already large size, looking like e.g. the five following ones:

$$\begin{aligned} & - (\varphi_{3,u_1})^2 (\varphi_{1,u_3})^2 (\varphi_{2,u_2})^2 \varphi_{2,\bar{z}u} \varphi_{3,u_2} \varphi_{2,z} \varphi_{1,u_2}, \\ & - \sqrt{-1} (\varphi_{2,u_2})^2 \varphi_{3,u_1} \varphi_{1,u_2} \varphi_{2,\bar{z}u_3} \varphi_{1,z} \varphi_{3,u_3}, \\ & \sqrt{-1} (\varphi_{1,u_1})^2 (\varphi_{2,u_2})^3 (\varphi_{3,u_3})^2 \varphi_{2,u_1} \varphi_{1,u_3} \varphi_{3,u_2} \varphi_{1,zz}, \\ & 6 \sqrt{-1} (\varphi_{1,u_1})^2 \varphi_{2,u_3} \varphi_{3,u_2} \varphi_{1,z} \varphi_{3,u_3} \varphi_{3,zu_1} \varphi_{1,u_3} \varphi_{2,u_2}, \\ & - 4 \sqrt{-1} \varphi_{1,u_1} \varphi_{2,u_1} \varphi_{2,u_1} (\varphi_{1,u_2})^2 \varphi_{2,u_3} \varphi_{3,z} \varphi_{2,zu_2} \varphi_{3,u_3}, \end{aligned}$$

so that complete explicitnes — even at the basic level of the coefficients of the initial frame! — seems to be out of reach.  $\square$

The Class III<sub>1</sub> hypothesis now reads as the nonzeroness:

$$0 \neq \det \begin{pmatrix} \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} & \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} & \frac{\Upsilon_3}{\Delta^2 \overline{\Delta}^2} \\ \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} & \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} & \frac{\Pi_3}{\Delta^4 \overline{\Delta}^3} \\ \frac{\overline{\Pi}_1}{\Delta^3 \overline{\Delta}^4} & \frac{\overline{\Pi}_2}{\Delta^3 \overline{\Delta}^4} & \frac{\overline{\Pi}_3}{\Delta^3 \overline{\Delta}^4} \end{pmatrix} (x, y, u_1, u_2, u_3),$$

at every point.

More precisely, if one preliminarily normalizes coordinates as in [6]:

$$\begin{aligned} v_1 &= z\bar{z} && + z\bar{z} O_2(z, \bar{z}) + z\bar{z} O_1(u_1) + z\bar{z} O_1(u_2) + z\bar{z} O_1(u_3), \\ v_2 &= z^2\bar{z} + z\bar{z}^2 && + z\bar{z} O_2(z, \bar{z}) + z\bar{z} O_1(u_1) + z\bar{z} O_1(u_2) + z\bar{z} O_1(u_3), \\ v_3 &= \sqrt{-1}(z^2\bar{z} - z\bar{z}^2) && + z\bar{z} O_2(z, \bar{z}) + z\bar{z} O_1(u_1) + z\bar{z} O_1(u_2) + z\bar{z} O_1(u_3), \end{aligned}$$

so that at the origin:

$$\begin{aligned} \mathcal{L}|_0 &= \frac{\partial}{\partial z}|_0, \\ \overline{\mathcal{L}}|_0 &= \frac{\partial}{\partial \bar{z}}|_0, \\ [\mathcal{L}, \overline{\mathcal{L}}]|_0 &= -2\sqrt{-1} \frac{\partial}{\partial u_1}|_0, \\ [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]|_0 &= -4\sqrt{-1} \frac{\partial}{\partial u_2}|_0 + 4 \frac{\partial}{\partial u_3}|_0, \\ [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]|_0 &= -4\sqrt{-1} \frac{\partial}{\partial u_2}|_0 - 4 \frac{\partial}{\partial u_3}|_0, \end{aligned}$$

the determinant in question:

$$\det \begin{pmatrix} \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} & \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} & \frac{\Upsilon_3}{\Delta^2 \overline{\Delta}^2} \\ \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} & \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} & \frac{\Pi_3}{\Delta^4 \overline{\Delta}^3} \\ \frac{\overline{\Pi}_1}{\Delta^3 \overline{\Delta}^4} & \frac{\overline{\Pi}_2}{\Delta^3 \overline{\Delta}^4} & \frac{\overline{\Pi}_3}{\Delta^3 \overline{\Delta}^4} \end{pmatrix} (0) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 4\sqrt{-1} \\ 0 & 4 & -4\sqrt{-1} \end{vmatrix}$$

becomes quite visibly nonzero, hence also near the origin.

But generally, the disease is, that when (necessarily) re-expressing:

$$\begin{aligned} \frac{\partial}{\partial u_1} &= \frac{\Delta^2 \overline{\Delta}^2 \begin{vmatrix} \Pi_2 & \Pi_3 \\ \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \mathcal{I} - \frac{\Delta^4 \overline{\Delta}^3 \begin{vmatrix} \Upsilon_2 & \Upsilon_3 \\ \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \mathcal{J} + \frac{\Delta^4 \overline{\Delta}^3 \begin{vmatrix} \Upsilon_2 & \Upsilon_3 \\ \Pi_2 & \Pi_3 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \overline{\mathcal{I}}, \\ \frac{\partial}{\partial u_2} &= \frac{\Delta^2 \overline{\Delta}^2 \begin{vmatrix} \Pi_3 & \Pi_1 \\ \overline{\Pi}_3 & \overline{\Pi}_1 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \mathcal{I} - \frac{\Delta^4 \overline{\Delta}^3 \begin{vmatrix} \Upsilon_3 & \Upsilon_1 \\ \overline{\Pi}_3 & \overline{\Pi}_1 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \mathcal{J} + \frac{\Delta^4 \overline{\Delta}^3 \begin{vmatrix} \Upsilon_3 & \Upsilon_1 \\ \Pi_3 & \Pi_1 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \overline{\Pi}_1 & \overline{\Pi}_2 & \overline{\Pi}_3 \end{vmatrix}} \cdot \overline{\mathcal{I}}, \end{aligned}$$

$$\frac{\partial}{\partial u_3} = \frac{\Delta^2 \bar{\Delta}^2 \begin{vmatrix} \Pi_1 & \Pi_2 \\ \bar{\Pi}_1 & \bar{\Pi}_2 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \bar{\Pi}_1 & \bar{\Pi}_2 & \bar{\Pi}_3 \end{vmatrix}} \cdot \mathcal{I} - \frac{\Delta^4 \bar{\Delta}^3 \begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \bar{\Pi}_1 & \bar{\Pi}_2 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \bar{\Pi}_1 & \bar{\Pi}_2 & \bar{\Pi}_3 \end{vmatrix}} \cdot \mathcal{I} + \frac{\Delta^4 \bar{\Delta}^3 \begin{vmatrix} \Upsilon_1 & \Upsilon_2 \\ \Pi_1 & \Pi_2 \end{vmatrix}}{\begin{vmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \bar{\Pi}_1 & \bar{\Pi}_2 & \bar{\Pi}_3 \end{vmatrix}} \cdot \bar{\mathcal{I}},$$

some quite huge fractions appear, and it becomes (very, very) impossible for a powerful computer, to make explicit the coefficients in the next 7 brackets completing the Lie structure of the frame:

$$\begin{aligned} [\mathcal{L}, \mathcal{I}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\mathcal{L}, \bar{\mathcal{I}}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\bar{\mathcal{L}}, \mathcal{I}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\bar{\mathcal{L}}, \bar{\mathcal{I}}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\mathcal{I}, \mathcal{I}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\mathcal{I}, \bar{\mathcal{I}}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}, \\ [\bar{\mathcal{I}}, \bar{\mathcal{I}}] &= \text{function} \cdot \mathcal{I} + \text{function} \cdot \mathcal{I} + \text{function} \cdot \bar{\mathcal{I}}. \end{aligned}$$

**Rigid collapse.** In the (special) so-called *rigid* case:

$$\begin{aligned} v_1 &= \varphi_1(x, y), \\ v_2 &= \varphi_2(x, y), \\ v_2 &= \varphi_3(x, y), \end{aligned}$$

where the graphing functions do not depend upon  $u_1, u_2, u_3$ , completely explicit formulas can be typesetted.

Indeed:

$$\begin{aligned} \mathcal{L} &= \frac{\partial}{\partial z} + \sqrt{-1} \varphi_{1,z} \frac{\partial}{\partial u_1} + \sqrt{-1} \varphi_{2,z} \frac{\partial}{\partial u_2} + \sqrt{-1} \varphi_{3,z} \frac{\partial}{\partial u_3}, \\ \bar{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} - \sqrt{-1} \varphi_{1,\bar{z}} \frac{\partial}{\partial u_1} - \sqrt{-1} \varphi_{2,\bar{z}} \frac{\partial}{\partial u_2} - \sqrt{-1} \varphi_{3,\bar{z}} \frac{\partial}{\partial u_3}, \\ \mathcal{I} &= 2 \varphi_{1,z\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,z\bar{z}} \frac{\partial}{\partial u_2} + 2 \varphi_{3,z\bar{z}} \frac{\partial}{\partial u_3}, \\ \mathcal{I} &= 2 \varphi_{1,zz\bar{z}} \frac{\partial}{\partial u_1} + 2 \varphi_{2,zz\bar{z}} \frac{\partial}{\partial u_2} + 2 \varphi_{3,zz\bar{z}} \frac{\partial}{\partial u_3}, \\ \bar{\mathcal{I}} &= 2 \varphi_{1,\bar{z}\bar{z}z} \frac{\partial}{\partial u_1} + 2 \varphi_{2,\bar{z}\bar{z}z} \frac{\partial}{\partial u_2} + 2 \varphi_{3,\bar{z}\bar{z}z} \frac{\partial}{\partial u_3}. \end{aligned}$$







**Symbolic treatment of the general case.** Now, come back to the general case where  $\varphi_1, \varphi_2, \varphi_3$  do depend on all variables  $(x, y, u_1, u_2, u_3)$ , so that:

$$\begin{aligned}\mathcal{L} &= \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2} + \frac{\Lambda_3}{\Delta} \frac{\partial}{\partial u_3}, \\ \overline{\mathcal{L}} &= \frac{\partial}{\partial \bar{z}} + \frac{\overline{\Lambda}_1}{\overline{\Delta}} \frac{\partial}{\partial u_1} + \frac{\overline{\Lambda}_2}{\overline{\Delta}} \frac{\partial}{\partial u_2} + \frac{\overline{\Lambda}_3}{\overline{\Delta}} \frac{\partial}{\partial u_3}, \\ \mathcal{T} &= \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_3}, \\ \mathcal{S} &= \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_2} + \frac{\Pi_3}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_3}, \\ \overline{\mathcal{S}} &= \frac{\overline{\Pi}_1}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_1} + \frac{\overline{\Pi}_2}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_2} + \frac{\overline{\Pi}_3}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_3}.\end{aligned}$$

Since, as always,  $\mathcal{T}$  is real, one sees:

$$\overline{\mathcal{S}} = \overline{[\mathcal{L}, \mathcal{T}]} = [\overline{\mathcal{L}}, \mathcal{T}].$$

Now, introduce named functions so that:

$$\begin{aligned}[\mathcal{L}, \mathcal{S}] &= P \cdot \mathcal{T} + Q \cdot \mathcal{S} + R \cdot \overline{\mathcal{S}}, \\ [\overline{\mathcal{S}}, \mathcal{S}] &= A \cdot \mathcal{T} + B \cdot \mathcal{S} + C \cdot \overline{\mathcal{S}}.\end{aligned}$$

**Lemma.** *One has the reality condition:*

$$[\overline{\mathcal{L}}, \mathcal{S}] = [\mathcal{L}, \overline{\mathcal{S}}].$$

*Proof.* Indeed, a consequence of the Jacobi identity shows that for any two words  $h_1$  and  $h_2$  in a free Lie algebra:

$$[h_2, [h_1, [h_1, h_2]]] = [h_1, [h_2, [h_1, h_2]]].$$

Apply:

$$\begin{aligned}[\overline{\mathcal{L}}, \mathcal{S}] &= [\overline{\mathcal{L}}, [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]] \\ &= \sqrt{-1} [\overline{\mathcal{L}}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]] \\ &= \sqrt{-1} [\mathcal{L}, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]] \\ &= [\mathcal{L}, [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]] \\ &= [\mathcal{L}, \overline{\mathcal{S}}]\end{aligned}$$

which is so. □

Hence necessarily:

$$\begin{aligned}A &= \overline{A}, \\ C &= \overline{B},\end{aligned}$$

and gathering:

$$\boxed{\begin{aligned} [\mathcal{L}, \mathcal{S}] &= P \cdot \mathcal{T} + Q \cdot \mathcal{S} + R \cdot \overline{\mathcal{S}}, \\ [\overline{\mathcal{L}}, \mathcal{S}] &= A \cdot \mathcal{T} + B \cdot \mathcal{S} + \overline{B} \cdot \overline{\mathcal{S}}, \\ [\mathcal{L}, \overline{\mathcal{S}}] &= A \cdot \mathcal{T} + B \cdot \mathcal{S} + \overline{B} \cdot \overline{\mathcal{S}}, \\ [\overline{\mathcal{L}}, \overline{\mathcal{S}}] &= \overline{P} \cdot \mathcal{T} + \overline{R} \cdot \mathcal{S} + \overline{Q} \cdot \overline{\mathcal{S}}. \end{aligned}}$$

These five functions:

$$A, \quad B, \quad P, \quad Q, \quad R,$$

will appear to be the only fundamental ones.

Indeed, it yet remains to compute the three brackets:

$$\begin{aligned} &[\mathcal{T}, \mathcal{S}], \\ &[\mathcal{T}, \overline{\mathcal{S}}], \\ &[\mathcal{S}, \overline{\mathcal{S}}], \end{aligned}$$

using the Jacobi identity.

**Lemma.** *The coefficients of the two Lie brackets:*

$$\begin{aligned} [\mathcal{T}, \mathcal{S}] &= E_{\text{rpl}} \cdot \mathcal{T} + F_{\text{rpl}} \cdot \mathcal{S} + G_{\text{rpl}} \cdot \overline{\mathcal{S}}, \\ [\mathcal{T}, \overline{\mathcal{S}}] &= \overline{E}_{\text{rpl}} \cdot \mathcal{T} + \overline{G}_{\text{rpl}} \cdot \mathcal{S} + \overline{F}_{\text{rpl}} \cdot \overline{\mathcal{S}} \end{aligned}$$

express in terms of  $A, B, P, Q, R$  as:

$$\begin{aligned} E_{\text{rpl}} &= \sqrt{-1} \left( \mathcal{L}(A) - \overline{\mathcal{L}}(P) + A\overline{B} + BP - AQ - \overline{P}R \right), \\ F_{\text{rpl}} &= \sqrt{-1} \left( \mathcal{L}(B) - \overline{\mathcal{L}}(Q) + A + B\overline{B} - R\overline{R} \right), \\ G_{\text{rpl}} &= \sqrt{-1} \left( \mathcal{L}(\overline{B}) - \overline{\mathcal{L}}(R) + \overline{B}\overline{B} + BR - P - \overline{B}Q - R\overline{Q} \right). \end{aligned}$$

*Proof.* Indeed, for  $h_1$  and  $h_2$  any two words in a free Lie algebra, the Jacobi identity gives:

$$\begin{aligned} \left[ [h_1, h_2], [h_1, [h_1, h_2]] \right] &= - \left[ \left[ [h_1, [h_1, h_2]], h_1 \right], h_2 \right] - \left[ \left[ h_2, [h_1, [h_1, h_2]] \right], h_1 \right] \\ &= - \left[ h_2, \left[ h_1, [h_1, [h_1, h_2]] \right] \right] + \left[ h_1, \left[ h_2, [h_1, [h_1, h_2]] \right] \right]. \end{aligned}$$

Apply this to  $h_1 := \mathcal{L}$  and  $h_2 := \overline{\mathcal{L}}$  and get:

$$\begin{aligned} -\sqrt{-1} [\mathcal{T}, \mathcal{S}] &= \left[ [\mathcal{L}, \overline{\mathcal{L}}], [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]] \right] \\ &= - \left[ \overline{\mathcal{L}}, \underbrace{\left[ \mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]] \right]}_{[\mathcal{L}, \mathcal{S}]} \right] + \left[ \mathcal{L}, \underbrace{\left[ \overline{\mathcal{L}}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]] \right]}_{[\overline{\mathcal{L}}, \mathcal{S}]} \right], \end{aligned}$$

that is to say after replacement:

$$\begin{aligned}
-\sqrt{-1} [\mathcal{I}, \mathcal{I}] &= - \left[ \overline{\mathcal{L}}, P \cdot \mathcal{I} + Q \cdot \mathcal{I} + R \cdot \overline{\mathcal{I}} \right] + \left[ \mathcal{L}, A \cdot \mathcal{I} + B \cdot \mathcal{I} + \overline{B} \cdot \overline{\mathcal{I}} \right] \\
&= - \overline{\mathcal{L}}(P) \cdot \mathcal{I} - \overline{\mathcal{L}}(Q) \cdot \mathcal{I} - \overline{\mathcal{L}}(R) \cdot \overline{\mathcal{I}} + \\
&\quad + \mathcal{L}(A) \cdot \mathcal{I} + \mathcal{L}(B) \cdot \mathcal{I} + \mathcal{L}(C) \cdot \overline{\mathcal{I}} - \\
&\quad - P \underbrace{[\overline{\mathcal{L}}, \mathcal{I}]_{\overline{\mathcal{I}}}}_{\overline{\mathcal{I}}} - Q \underbrace{[\overline{\mathcal{L}}, \mathcal{I}]_{A\mathcal{I}+B\mathcal{I}+\overline{B}\overline{\mathcal{I}}}}_{A\mathcal{I}+B\mathcal{I}+\overline{B}\overline{\mathcal{I}}} - R \underbrace{[\overline{\mathcal{L}}, \mathcal{I}]_{\overline{P}\overline{\mathcal{I}}+\overline{R}\mathcal{I}+\overline{Q}\overline{\mathcal{I}}}}_{\overline{P}\overline{\mathcal{I}}+\overline{R}\mathcal{I}+\overline{Q}\overline{\mathcal{I}}} + \\
&\quad + A \underbrace{[\mathcal{L}, \mathcal{I}]_{\mathcal{I}}}_{\mathcal{I}} + B \underbrace{[\mathcal{L}, \mathcal{I}]_{P\mathcal{I}+Q\mathcal{I}+R\overline{\mathcal{I}}}}_{P\mathcal{I}+Q\mathcal{I}+R\overline{\mathcal{I}}} + \overline{B} \underbrace{[\mathcal{L}, \mathcal{I}]_{\overline{A}\overline{\mathcal{I}}+\overline{B}\mathcal{I}+\overline{B}\overline{\mathcal{I}}}}_{\overline{A}\overline{\mathcal{I}}+\overline{B}\mathcal{I}+\overline{B}\overline{\mathcal{I}}},
\end{aligned}$$

which collects as:

$$\begin{aligned}
-\sqrt{-1} [\mathcal{I}, \mathcal{I}] &= \left( \mathcal{L}(A) - \overline{\mathcal{L}}(P) - AQ - \overline{P}R + BP + A\overline{B} \right) \cdot \mathcal{I} + \\
&\quad + \left( \mathcal{L}(B) - \overline{\mathcal{L}}(Q) + A + B\overline{B} - R\overline{R} \right) \cdot \mathcal{I} + \\
&\quad + \left( \mathcal{L}(C) - \overline{\mathcal{L}}(R) - P + BR - \overline{B}Q - R\overline{Q} + \overline{B}\overline{B} \right) \cdot \overline{\mathcal{I}},
\end{aligned}$$

as stated.  $\square$

Notice that the last remaining Lie bracket is purely imaginary:

$$[\overline{\mathcal{I}}, \overline{\mathcal{I}}] = [\mathcal{I}, \mathcal{I}] = -[\mathcal{I}, \overline{\mathcal{I}}].$$

**Lemma.** *The coefficients of:*

$$[\mathcal{I}, \overline{\mathcal{I}}] = \sqrt{-1} J_{\text{rpl}} \cdot \mathcal{I} + K_{\text{rpl}} \cdot \mathcal{I} - \overline{K}_{\text{rpl}} \cdot \overline{\mathcal{I}}$$

express in terms of  $A, B, P, Q, R$  as:

$$\begin{aligned}
J_{\text{rpl}} &= \frac{1}{2} \left( \overline{\mathcal{L}}(\overline{\mathcal{L}}(P)) + \mathcal{L}(\mathcal{L}(\overline{P})) - \overline{\mathcal{L}}(\mathcal{L}(A)) - \mathcal{L}(\overline{\mathcal{L}}(A)) + \right. \\
&\quad + Q \overline{\mathcal{L}}(A) + \overline{Q} \mathcal{L}(A) + 2A \overline{\mathcal{L}}(Q) + 2A \mathcal{L}(\overline{Q}) + R \overline{\mathcal{L}}(\overline{P}) + \overline{R} \mathcal{L}(P) + \\
&\quad + 2\overline{P} \overline{\mathcal{L}}(R) + 2P \mathcal{L}(\overline{R}) - 2P \overline{\mathcal{L}}(B) - 2\overline{P} \mathcal{L}(\overline{B}) - B \overline{\mathcal{L}}(P) - \overline{B} \mathcal{L}(\overline{P}) - \\
&\quad - \overline{B} \overline{\mathcal{L}}(A) - B \mathcal{L}(A) - 2A \overline{\mathcal{L}}(\overline{B}) - 2A \mathcal{L}(B) - \\
&\quad - 2AA - 2A\overline{B}\overline{B} - BBP - \overline{B}\overline{B}\overline{P} - B\overline{P}R - \overline{B}P\overline{R} + PQ\overline{R} + \overline{P}\overline{Q}R + \\
&\quad \left. + 2AR\overline{R} + 2P\overline{P} + \overline{B}\overline{P}Q + BP\overline{Q} \right),
\end{aligned}$$

which is real, and:

$$\begin{aligned}
K_{\text{rpl}} = \frac{1}{2} & \left( \overline{\mathcal{L}}(\overline{\mathcal{L}}(Q)) - \overline{\mathcal{L}}(\mathcal{L}(B)) - \mathcal{L}(\overline{\mathcal{L}}(B)) + \mathcal{L}(\mathcal{L}(\overline{R})) + \right. \\
& + 2\overline{R}\overline{\mathcal{L}}(R) + R\overline{\mathcal{L}}(\overline{R}) + B\overline{\mathcal{L}}(Q) + 2\mathcal{L}(\overline{P}) + \overline{R}\mathcal{L}(Q) + 2Q\mathcal{L}(\overline{R}) + \overline{Q}\mathcal{L}(B) - Q\overline{\mathcal{L}}(B) + \\
& + 2B\mathcal{L}(\overline{Q}) - 2\overline{\mathcal{L}}(A) - \overline{B}\overline{\mathcal{L}}(B) - 2B\overline{\mathcal{L}}(\overline{B}) - 3B\mathcal{L}(B) - 2\overline{R}\mathcal{L}(\overline{B}) - \overline{B}\mathcal{L}(\overline{R}) - \\
& - 3AB - BBQ - \overline{B}\overline{R} - 2BB\overline{B} - \overline{B}\overline{P} + A\overline{Q} + \overline{P}Q + \\
& \left. + QQ\overline{R} + BQ\overline{Q} + BR\overline{R} + 2P\overline{R} + \overline{Q}R\overline{R} \right).
\end{aligned}$$

*Proof.* In subsection 7.3 of [1]), in a free Lie containing  $h_1$  and  $h_2$ , one has:

$$\begin{aligned}
0 &= [[h_1, [h_1, h_2]], [h_2, [h_1, h_2]]] - [[h_1, h_2], [h_1, [h_2, [h_1, h_2]]]] - \\
&\quad - [h_2, [h_2, [h_1, [h_1, [h_1, h_2]]]]] + [h_2, [h_1, [h_1, [h_2, [h_1, h_2]]]]], \\
0 &= [[h_1, [h_1, h_2]], [h_2, [h_1, h_2]]] + [[h_1, h_2], [h_1, [h_2, [h_1, h_2]]]] + \\
&\quad + [h_1, [h_2, [h_1, [h_2, [h_1, h_2]]]]] - [h_1, [h_1, [h_2, [h_2, [h_1, h_2]]]]].
\end{aligned}$$

Adding and replacing  $h_1 := \mathcal{L}$  and  $h_2 := \overline{\mathcal{L}}$ :

$$\begin{aligned}
-2 \left[ [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]] \right] &= - \left[ \overline{\mathcal{L}}, \left[ \overline{\mathcal{L}}, [\mathcal{L}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]]] \right] \right] + \\
&\quad + \left[ \overline{\mathcal{L}}, \left[ \mathcal{L}, [\mathcal{L}, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]] \right] \right] + \\
&\quad + \left[ \mathcal{L}, \left[ \overline{\mathcal{L}}, [\mathcal{L}, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]] \right] \right] - \\
&\quad - \left[ \mathcal{L}, \left[ \mathcal{L}, [\overline{\mathcal{L}}, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]]] \right] \right].
\end{aligned}$$

Taking account of some  $\sqrt{-1}$ :

$$\begin{aligned}
[\mathcal{S}, \overline{\mathcal{S}}] &= \left[ [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]] \right] \\
&= - \left[ [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{L}}]], [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{L}}]] \right] \\
&= \frac{\sqrt{-1}}{2} \left( + \left[ \overline{\mathcal{L}}, \left[ \overline{\mathcal{L}}, \underbrace{[\mathcal{L}, [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]}_{[\mathcal{L}, \mathcal{S}]} \right] \right] \right) - \\
&\quad - \left[ \overline{\mathcal{L}}, \left[ \mathcal{L}, \underbrace{[\mathcal{L}, [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]}_{[\mathcal{L}, \overline{\mathcal{S}}]} \right] \right] - \\
&\quad - \left[ \mathcal{L}, \left[ \overline{\mathcal{L}}, \underbrace{[\mathcal{L}, [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]}_{[\mathcal{L}, \overline{\mathcal{S}}]} \right] \right] + \\
&\quad + \left[ \mathcal{L}, \left[ \mathcal{L}, \underbrace{[\overline{\mathcal{L}}, [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]}_{[\overline{\mathcal{L}}, \overline{\mathcal{S}}]} \right] \right] \Big).
\end{aligned}$$

that is to say:

$$\begin{aligned}
[\mathcal{S}, \overline{\mathcal{S}}] &= \sqrt{-1} J_{\text{rpl}} \cdot \mathcal{S} + K_{\text{rpl}} \cdot \mathcal{S} - \overline{K}_{\text{rpl}} \cdot \overline{\mathcal{S}} \\
&= \frac{\sqrt{-1}}{2} \left( \left[ \overline{\mathcal{L}}, [\overline{\mathcal{L}}, P\mathcal{S} + Q\mathcal{S} + R\overline{\mathcal{S}}] \right] - \left[ \overline{\mathcal{L}}, [\mathcal{L}, A\mathcal{S} + B\mathcal{S} + \overline{B}\overline{\mathcal{S}}] \right] - \right. \\
&\quad \left. - \left[ \mathcal{L}, [\overline{\mathcal{L}}, A\mathcal{S} + B\mathcal{S} + \overline{B}\overline{\mathcal{S}}] \right] + \left[ \mathcal{L}, [\mathcal{L}, \overline{P}\mathcal{S} + \overline{R}\mathcal{S} + \overline{Q}\overline{\mathcal{S}}] \right] \right).
\end{aligned}$$

On preliminarily computes:

$$\begin{aligned}
\left[ \overline{\mathcal{L}}, P\mathcal{S} + Q\mathcal{S} + R\overline{\mathcal{S}} \right] &= \overline{\mathcal{L}}(P) \mathcal{S} + \overline{\mathcal{L}}(Q) \mathcal{S} + \overline{\mathcal{L}}(R) \overline{\mathcal{S}} + \\
&\quad + P [\overline{\mathcal{L}}, \mathcal{S}] + Q [\overline{\mathcal{L}}, \mathcal{S}] + R [\overline{\mathcal{L}}, \overline{\mathcal{S}}] \\
&= (\overline{\mathcal{L}}(P) + AQ + \overline{P}R) \cdot \mathcal{S} + \\
&\quad + (\overline{\mathcal{L}}(Q) + BQ + R\overline{R}) \cdot \mathcal{S} + \\
&\quad + (\overline{\mathcal{L}}(R) + P + \overline{B}Q + \overline{Q}R) \cdot \overline{\mathcal{S}}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\left[ \mathcal{L}, A\mathcal{S} + B\mathcal{S} + \overline{B}\overline{\mathcal{S}} \right] &= (\mathcal{L}(A) + BP + A\overline{B}) \cdot \mathcal{S} + \\
&\quad (\mathcal{L}(B) + A + BQ + B\overline{B}) \cdot \mathcal{S} + \\
&\quad (\mathcal{L}(\overline{B}) + BR + \overline{B}\overline{B}) \cdot \overline{\mathcal{S}},
\end{aligned}$$

with conjugate:

$$\begin{aligned} \left[ \overline{\mathcal{L}}, A\mathcal{I} + B\mathcal{J} + \overline{B}\overline{\mathcal{I}} \right] &= (\overline{\mathcal{L}}(A) + \overline{B}P + AB) \cdot \mathcal{I} + \\ &\quad (\overline{\mathcal{L}}(B) + \overline{B}R + BB) \cdot \mathcal{J} + \\ &\quad (\overline{\mathcal{L}}(\overline{B}) + B\overline{B} + A + \overline{B}Q) \cdot \overline{\mathcal{I}}, \end{aligned}$$

and lastly:

$$\begin{aligned} \left[ \mathcal{L}, \overline{P}\mathcal{I} + \overline{R}\mathcal{J} + \overline{Q}\overline{\mathcal{I}} \right] &= (\mathcal{L}(\overline{P}) + P\overline{R} + A\overline{Q}) \cdot \mathcal{I} + \\ &\quad (\mathcal{L}(\overline{R}) + \overline{P} + Q\overline{R} + B\overline{Q}) \cdot \mathcal{J} + \\ &\quad (\mathcal{L}(\overline{Q}) + R\overline{R} + \overline{B}Q) \cdot \overline{\mathcal{I}}, \end{aligned}$$

One can therefore computes:

$$\begin{aligned} \left[ \overline{\mathcal{L}}, \left[ \overline{\mathcal{L}}, [\mathcal{L}, \mathcal{I}] \right] \right] &= \left[ \overline{\mathcal{L}}, \left[ \overline{\mathcal{L}}, P\mathcal{I} + Q\mathcal{J} + R\overline{\mathcal{I}} \right] \right] \\ &= \left[ \overline{\mathcal{L}}, (\overline{\mathcal{L}}(P) + AQ + \overline{P}R) \cdot \mathcal{I} + \right. \\ &\quad \left. + (\overline{\mathcal{L}}(Q) + BQ + R\overline{R}) \cdot \mathcal{J} + \right. \\ &\quad \left. + (\overline{\mathcal{L}}(R) + P + \overline{B}Q + \overline{Q}R) \cdot \overline{\mathcal{I}} \right] \\ &= \left( \overline{\mathcal{L}}(\overline{\mathcal{L}}(P)) + Q\overline{\mathcal{L}}(A) + A\overline{\mathcal{L}}(Q) + R\overline{\mathcal{L}}(\overline{P}) + \overline{P}\overline{\mathcal{L}}(R) \right) \cdot \mathcal{I} + \\ &\quad + \left( \overline{\mathcal{L}}(\overline{\mathcal{L}}(Q)) + Q\overline{\mathcal{L}}(B) + B\overline{\mathcal{L}}(Q) + \overline{R}\overline{\mathcal{L}}(R) + R\overline{\mathcal{L}}(\overline{R}) \right) \cdot \mathcal{J} + \\ &\quad + \left( \overline{\mathcal{L}}(\overline{\mathcal{L}}(R)) + \overline{L}(P) + Q\overline{\mathcal{L}}(B) + \overline{B}\overline{\mathcal{L}}(Q) + R\overline{\mathcal{L}}(\overline{Q}) + \overline{Q}\overline{\mathcal{L}}(R) \right) \cdot \overline{\mathcal{I}} + \\ &\quad + (\overline{\mathcal{L}}(P) + AQ + \overline{P}R) \underbrace{[\overline{\mathcal{L}}, \mathcal{I}]}_{\overline{\mathcal{I}}} + \\ &\quad + (\overline{\mathcal{L}}(Q) + BQ + R\overline{R}) \underbrace{[\overline{\mathcal{L}}, \mathcal{J}]}_{A\mathcal{I} + B\mathcal{J} + \overline{B}\overline{\mathcal{I}}} + \\ &\quad + (\overline{\mathcal{L}}(R) + P + \overline{B}Q + \overline{Q}R) \underbrace{[\overline{\mathcal{L}}, \overline{\mathcal{I}}]}_{\overline{P}\mathcal{I} + \overline{R}\mathcal{J} + \overline{Q}\overline{\mathcal{I}}}, \end{aligned}$$

that is to say after collecting:

$$\begin{aligned} [\overline{\mathcal{L}}, [\overline{\mathcal{L}}, [\mathcal{L}, \mathcal{S}]]] = & \mathcal{F} \cdot \left( \begin{array}{l} \overline{\mathcal{L}}(\overline{\mathcal{L}}(P)) + Q\overline{\mathcal{L}}(A) + A\overline{\mathcal{L}}(Q) + R\overline{\mathcal{L}}(\overline{P}) + \overline{P}\overline{\mathcal{L}}(R) + \\ + A\overline{\mathcal{L}}(Q) + \overline{P}\overline{\mathcal{L}}(R) + \\ + ABQ + AR\overline{R} + P\overline{P} + \overline{B}\overline{P}Q + \overline{P}\overline{Q}R \end{array} \right) + \\ & + \mathcal{S} \cdot \left( \begin{array}{l} \overline{\mathcal{L}}(\overline{\mathcal{L}}(Q)) + Q\overline{\mathcal{L}}(B) + B\overline{\mathcal{L}}(Q) + \overline{R}\overline{\mathcal{L}}(R) + R\overline{\mathcal{L}}(\overline{R}) + \\ + B\overline{\mathcal{L}}(Q) + \overline{R}\overline{\mathcal{L}}(R) + \\ + BBQ + BR\overline{R} + P\overline{R} + \overline{B}Q\overline{R} + \overline{Q}R\overline{R} \end{array} \right) + \\ & + \overline{\mathcal{S}} \cdot \left( \begin{array}{l} \overline{\mathcal{L}}(\overline{\mathcal{L}}(R)) + \overline{\mathcal{L}}(P) + Q\overline{\mathcal{L}}(B) + \overline{B}\overline{\mathcal{L}}(Q) + R\overline{\mathcal{L}}(\overline{Q}) + \overline{Q}\overline{\mathcal{L}}(R) + \\ + \overline{\mathcal{L}}(P) + \overline{B}\overline{\mathcal{L}}(Q) + \overline{Q}\overline{\mathcal{L}}(R) + \\ + AQ + \overline{P}R + B\overline{B}Q + \overline{B}R\overline{R} + P\overline{Q} + \overline{B}Q\overline{Q} + \overline{Q}\overline{Q}R \end{array} \right), \end{aligned}$$

Quite similar computations the remaining three provide (mind minus signs):

$$\begin{aligned} -[\overline{\mathcal{L}}, [\mathcal{L}, [\mathcal{L}, \overline{\mathcal{S}}]]] = & \mathcal{F} \cdot \left( \begin{array}{l} -\overline{\mathcal{L}}(\mathcal{L}(A)) - P\overline{\mathcal{L}}(B) - B\overline{\mathcal{L}}(P) - \overline{B}\overline{\mathcal{L}}(A) - A\overline{\mathcal{L}}(\overline{B}) - \\ - A\mathcal{L}(B) - \overline{P}\mathcal{L}(\overline{B}) - \\ - AA - ABQ - AB\overline{B} - B\overline{P}R - \overline{B}BP \end{array} \right) + \\ & + \mathcal{S} \cdot \left( \begin{array}{l} -\overline{\mathcal{L}}(\mathcal{L}(B)) - \overline{\mathcal{L}}(A) - Q\overline{\mathcal{L}}(B) - B\overline{\mathcal{L}}(Q) - \overline{B}\overline{\mathcal{L}}(B) - B\overline{\mathcal{L}}(\overline{B}) - \\ - B\mathcal{L}(B) - \overline{R}\mathcal{L}(\overline{B}) - \\ - AB - BBQ - BB\overline{B} - BR\overline{R} - \overline{B}B\overline{R} \end{array} \right) + \\ & + \overline{\mathcal{S}} \cdot \left( \begin{array}{l} -\overline{\mathcal{L}}(\mathcal{L}(\overline{B})) - R\overline{\mathcal{L}}(B) - B\overline{\mathcal{L}}(R) - 2\overline{B}\overline{\mathcal{L}}(\overline{B}) - \\ - \mathcal{L}(A) - \overline{B}\mathcal{L}(B) - \overline{Q}\mathcal{L}(\overline{B}) - \\ - BP - 2A\overline{B} - B\overline{B}Q - BB\overline{B} - B\overline{Q}R - \overline{B}B\overline{Q} \end{array} \right) \\ -[\mathcal{L}, [\overline{\mathcal{L}}, [\mathcal{L}, \overline{\mathcal{S}}]]] = & \mathcal{F} \cdot \left( \begin{array}{l} -\mathcal{L}(\overline{\mathcal{L}}(A)) - \overline{P}\mathcal{L}(\overline{B}) - \overline{B}\mathcal{L}(\overline{P}) - B\mathcal{L}(A) - A\mathcal{L}(B) - \\ - A\overline{\mathcal{L}}(\overline{B}) - P\overline{\mathcal{L}}(B) - \\ - AA - A\overline{B}Q - AB\overline{B} - \overline{B}P\overline{R} - BBP \end{array} \right) + \\ & + \mathcal{S} \cdot \left( \begin{array}{l} -\mathcal{L}(\overline{\mathcal{L}}(B)) - \overline{R}\mathcal{L}(\overline{B}) - \overline{B}\mathcal{L}(\overline{R}) - 2B\mathcal{L}(B) - \\ - \overline{\mathcal{L}}(A) - B\overline{\mathcal{L}}(\overline{B}) - Q\overline{\mathcal{L}}(B) - \\ - \overline{B}P - 2AB - B\overline{B}Q - BB\overline{B} - \overline{B}Q\overline{R} - BBQ \end{array} \right) + \\ & + \overline{\mathcal{S}} \cdot \left( \begin{array}{l} -\mathcal{L}(\overline{\mathcal{L}}(\overline{B})) - \mathcal{L}(A) - \overline{Q}\mathcal{L}(\overline{B}) - \overline{B}\mathcal{L}(\overline{Q}) - B\mathcal{L}(\overline{B}) - \overline{B}\mathcal{L}(B) - \\ - \overline{B}\overline{\mathcal{L}}(\overline{B}) - R\overline{\mathcal{L}}(B) - \\ - A\overline{B} - B\overline{B}Q - BB\overline{B} - \overline{B}R\overline{R} - BBR \end{array} \right), \end{aligned}$$

$$\begin{aligned}
[\mathcal{L}, [\mathcal{L}, [\overline{\mathcal{L}}, \overline{\mathcal{F}}]]] &= \mathcal{F} \cdot \left( \begin{aligned} &\mathcal{L}(\mathcal{L}(\overline{P})) + \overline{R}\mathcal{L}(P) + P\mathcal{L}(\overline{R}) + \overline{Q}\mathcal{L}(A) + A\mathcal{L}(\overline{Q}) + \\ &+ P\mathcal{L}(\overline{R}) + A\mathcal{L}(\overline{Q}) + \\ &+ P\overline{P} + PQ\overline{R} + BP\overline{Q} + AR\overline{R} + A\overline{B}\overline{Q} \end{aligned} \right) + \\
&+ \mathcal{L} \cdot \left( \begin{aligned} &\mathcal{L}(\mathcal{L}(\overline{R})) + \mathcal{L}(\overline{P}) + \overline{R}\mathcal{L}(Q) + Q\mathcal{L}(\overline{R}) + \overline{Q}\mathcal{L}(B) + B\mathcal{L}(\overline{Q}) + \\ &+ \mathcal{L}(\overline{P}) + Q\mathcal{L}(\overline{R}) + B\mathcal{L}(\overline{Q}) + \\ &+ P\overline{R} + A\overline{Q} + \overline{P}Q + QQ\overline{R} + BQ\overline{Q} + BR\overline{R} + B\overline{B}\overline{Q} \end{aligned} \right) + \\
&+ \overline{\mathcal{F}} \cdot \left( \begin{aligned} &\mathcal{L}(\mathcal{L}(\overline{Q})) + \overline{R}\mathcal{L}(R) + R\mathcal{L}(\overline{R}) + \overline{Q}\mathcal{L}(\overline{B}) + \overline{B}\mathcal{L}(\overline{Q}) + \\ &+ R\mathcal{L}(\overline{R}) + \overline{B}\mathcal{L}(\overline{Q}) + \\ &+ \overline{P}R + QR\overline{R} + B\overline{Q}R + \overline{B}R\overline{R} + \overline{B}\overline{B}\overline{Q} \end{aligned} \right).
\end{aligned}$$

Adding these four expressions, one obtains  $J_{\text{rpl}}$  and  $K_{\text{rpl}}$ .  $\square$

**Summary.** One has the 10 Lie bracket relations:

$$\begin{aligned}
[\overline{\mathcal{F}}, \mathcal{F}] &= \overline{K}_{\text{rpl}} \cdot \overline{\mathcal{F}} - K_{\text{rpl}} \cdot \mathcal{F} - \sqrt{-1} J_{\text{rpl}} \cdot \mathcal{F}, \\
[\overline{\mathcal{F}}, \overline{\mathcal{F}}] &= -\overline{F}_{\text{rpl}} \cdot \overline{\mathcal{F}} - \overline{G}_{\text{rpl}} \cdot \mathcal{F} - \overline{E}_{\text{rpl}} \cdot \mathcal{F}, \\
[\overline{\mathcal{F}}, \overline{\mathcal{L}}] &= -\overline{Q} \cdot \overline{\mathcal{F}} - \overline{R} \cdot \mathcal{F} - \overline{P} \cdot \mathcal{F}, \\
[\overline{\mathcal{F}}, \mathcal{L}] &= -\overline{B} \cdot \overline{\mathcal{F}} - B \cdot \mathcal{F} - A \cdot \mathcal{F}, \\
[\mathcal{F}, \overline{\mathcal{F}}] &= -G_{\text{rpl}} \cdot \overline{\mathcal{F}} - F_{\text{rpl}} \cdot \mathcal{F} - E_{\text{rpl}} \cdot \mathcal{F}, \\
[\mathcal{F}, \overline{\mathcal{L}}] &= -\overline{B} \cdot \overline{\mathcal{F}} - B \cdot \mathcal{F} - A \cdot \mathcal{F}, \\
[\mathcal{F}, \mathcal{L}] &= -R \cdot \overline{\mathcal{F}} - Q \cdot \mathcal{F} - P \cdot \mathcal{F}, \\
[\mathcal{F}, \overline{\mathcal{L}}] &= -\overline{\mathcal{F}}, \\
[\mathcal{F}, \mathcal{L}] &= -\mathcal{F}, \\
[\overline{\mathcal{L}}, \mathcal{L}] &= \sqrt{-1} \mathcal{F}.
\end{aligned}$$

**Initial Darboux structure of the dual coframe.** The coframe:

$$\{du_3, du_2, du_1, dz, d\overline{z}\}$$

is clearly dual to the frame:

$$\left\{ \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right\}.$$

Introduce then the coframe:

$$\{\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\}$$



which is dual to the frame:

$$\{\overline{\mathcal{F}}, \mathcal{S}, \mathcal{T}, \overline{\mathcal{L}}, \mathcal{L}\},$$

namely:

$$\begin{array}{ccccc} \overline{\sigma}_0(\overline{\mathcal{F}}) = 1 & \overline{\sigma}_0(\mathcal{S}) = 0 & \overline{\sigma}_0(\mathcal{T}) = 0 & \overline{\sigma}_0(\overline{\mathcal{L}}) = 0 & \overline{\sigma}_0(\mathcal{L}) = 0, \\ \sigma_0(\overline{\mathcal{F}}) = 0 & \sigma_0(\mathcal{S}) = 1 & \sigma_0(\mathcal{T}) = 0 & \sigma_0(\overline{\mathcal{L}}) = 0 & \sigma_0(\mathcal{L}) = 0, \\ \underline{\rho}_0(\overline{\mathcal{F}}) = 0 & \underline{\rho}_0(\mathcal{S}) = 0 & \underline{\rho}_0(\mathcal{T}) = 1 & \underline{\rho}_0(\overline{\mathcal{L}}) = 0 & \underline{\rho}_0(\mathcal{L}) = 0, \\ \underline{\zeta}_0(\overline{\mathcal{F}}) = 0 & \underline{\zeta}_0(\mathcal{S}) = 0 & \underline{\zeta}_0(\mathcal{T}) = 0 & \underline{\zeta}_0(\overline{\mathcal{L}}) = 1 & \underline{\zeta}_0(\mathcal{L}) = 0, \\ \zeta_0(\overline{\mathcal{F}}) = 0 & \zeta_0(\mathcal{S}) = 0 & \zeta_0(\mathcal{T}) = 0 & \zeta_0(\overline{\mathcal{L}}) = 0 & \zeta_0(\mathcal{L}) = 1. \end{array}$$

One has:

$$\zeta_0 = dz \quad \text{and} \quad \overline{\zeta}_0 = d\bar{z}.$$

Organize the ten Lie brackets as a convenient auxiliary array:

	$\overline{\mathcal{F}}$	$\mathcal{S}$	$\mathcal{T}$	$\overline{\mathcal{L}}$	$\mathcal{L}$	
	$d\overline{\sigma}_0$	$d\sigma_0$	$d\rho_0$	$d\overline{\zeta}_0$	$d\zeta_0$	
$[\overline{\mathcal{F}}, \mathcal{S}]$	$\overline{K}_{\text{rpl}} \cdot \overline{\mathcal{F}}$	$-K_{\text{rpl}} \cdot \mathcal{S}$	$-\sqrt{-1} J_{\text{rpl}} \cdot \mathcal{T}$	0	0	$\overline{\sigma}_0 \wedge \sigma_0$
$[\overline{\mathcal{F}}, \mathcal{T}]$	$-\overline{F}_{\text{rpl}} \cdot \overline{\mathcal{F}}$	$-\overline{G}_{\text{rpl}} \cdot \mathcal{S}$	$-\overline{E}_{\text{rpl}} \cdot \mathcal{T}$	0	0	$\overline{\sigma}_0 \wedge \rho_0$
$[\overline{\mathcal{F}}, \overline{\mathcal{L}}]$	$-\overline{Q} \cdot \overline{\mathcal{F}}$	$-\overline{R} \cdot \mathcal{S}$	$-\overline{P} \cdot \mathcal{T}$	0	0	$\overline{\sigma}_0 \wedge \overline{\zeta}_0$
$[\overline{\mathcal{F}}, \mathcal{L}]$	$-\overline{B} \cdot \overline{\mathcal{F}}$	$-B \cdot \mathcal{S}$	$-A \cdot \mathcal{T}$	0	0	$\overline{\sigma}_0 \wedge \zeta_0$
$[\mathcal{S}, \mathcal{T}]$	$-G_{\text{rpl}} \cdot \overline{\mathcal{F}}$	$-F_{\text{rpl}} \cdot \mathcal{S}$	$-E_{\text{rpl}} \cdot \mathcal{T}$	0	0	$\sigma_0 \wedge \rho_0$
$[\mathcal{S}, \overline{\mathcal{L}}]$	$-\overline{B} \cdot \overline{\mathcal{F}}$	$-B \cdot \mathcal{S}$	$-A \cdot \mathcal{T}$	0	0	$\sigma_0 \wedge \overline{\zeta}_0$
$[\mathcal{S}, \mathcal{L}]$	$-R \cdot \overline{\mathcal{F}}$	$-Q \cdot \mathcal{S}$	$-P \cdot \mathcal{T}$	0	0	$\sigma_0 \wedge \zeta_0$
$[\mathcal{T}, \overline{\mathcal{L}}]$	$-\overline{\mathcal{F}}$	0	0	0	0	$\rho_0 \wedge \overline{\zeta}_0$
$[\mathcal{T}, \mathcal{L}]$	0	$-\mathcal{S}$	0	0	0	$\rho_0 \wedge \zeta_0$
$[\overline{\mathcal{L}}, \mathcal{L}]$	0	0	$\sqrt{-1} \mathcal{T}$	0	0	$\overline{\zeta}_0 \wedge \zeta_0$

Read *vertically* and put an overall minus sign:

$$\begin{aligned} d\overline{\sigma}_0 &= -\overline{K}_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \sigma_0 + \overline{F}_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \rho_0 + \overline{Q} \cdot \overline{\sigma}_0 \wedge \overline{\zeta}_0 + \overline{B} \cdot \overline{\sigma}_0 \wedge \zeta_0 + \\ &\quad + G_{\text{rpl}} \cdot \sigma_0 \wedge \rho_0 + \overline{B} \cdot \sigma_0 \wedge \overline{\zeta}_0 + R \cdot \sigma_0 \wedge \zeta_0 + \rho_0 \wedge \overline{\zeta}_0, \\ d\sigma_0 &= K_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \sigma_0 + \overline{G}_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \rho_0 + \overline{R} \cdot \overline{\sigma}_0 \wedge \overline{\zeta}_0 + B \cdot \overline{\sigma}_0 \wedge \zeta_0 + \\ &\quad + F_{\text{rpl}} \cdot \sigma_0 \wedge \rho_0 + B \cdot \sigma_0 \wedge \overline{\zeta}_0 + Q \cdot \sigma_0 \wedge \zeta_0 + \rho_0 \wedge \zeta_0, \\ d\rho_0 &= \sqrt{-1} J_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \sigma_0 + \overline{E}_{\text{rpl}} \cdot \overline{\sigma}_0 \wedge \rho_0 + \overline{P} \cdot \overline{\sigma}_0 \wedge \overline{\zeta}_0 + A \cdot \overline{\sigma}_0 \wedge \zeta_0 + \\ &\quad + E_{\text{rpl}} \cdot \sigma_0 \wedge \rho_0 + A \cdot \sigma_0 \wedge \overline{\zeta}_0 + P \cdot \sigma_0 \wedge \zeta_0 - \sqrt{-1} \overline{\zeta}_0 \wedge \zeta_0, \\ d\overline{\zeta}_0 &= 0, \\ d\zeta_0 &= 0. \end{aligned}$$

4.  $M^5 \subset \mathbb{C}^4$  of general class III<sub>2</sub>:  
initial frame and coframe in local coordinates

Next, consider:

$$\left(M^5 \subset \mathbb{C}^4\right) \in \text{General Class III}_2.$$

As for the class III<sub>1</sub>, represent  $M$  in coordinates:

$$(z, w_1, w_2, w_3) = (x + \sqrt{-1}y, u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2, u_3 + \sqrt{-1}v_3),$$

as a graph:

$$v_1 = \varphi_1(x, y, u_1, u_2, u_3),$$

$$v_2 = \varphi_2(x, y, u_1, u_2, u_3),$$

$$v_3 = \varphi_3(x, y, u_1, u_2, u_3),$$

One assumes at every point that the following biholomorphically invariant geometric condition holds:

$$\begin{aligned} \mathbf{3} &= \text{rank}_{\mathbb{C}} \left( T^{1,0}M, T^{0,1}M, [T^{1,0}M, T^{0,1}M] \right), \\ \mathbf{4} &= \text{rank}_{\mathbb{C}} \left( T^{1,0}M, T^{0,1}M, [T^{1,0}M, T^{0,1}M], \right. \\ &\quad \left. [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] \right), \\ \mathbf{4} &= \text{rank}_{\mathbb{C}} \left( T^{1,0}M, T^{0,1}M, [T^{1,0}M, T^{0,1}M], \right. \\ &\quad \left. [T^{1,0}M, [T^{1,0}M, T^{0,1}M]], [T^{0,1}M, [T^{1,0}M, T^{0,1}M]] \right), \\ \mathbf{5} &= \text{rank}_{\mathbb{C}} \left( T^{1,0}M, T^{0,1}M, [T^{1,0}M, T^{0,1}M], [T^{1,0}M, [T^{1,0}M, T^{0,1}M]], \right. \\ &\quad \left. [T^{1,0}M, [T^{1,0}M, [T^{1,0}M, T^{0,1}M]]] \right), \end{aligned}$$

the third rank condition being an exceptional degeneracy feature, because here, 5 fields have only rank 4.

There is a unique (local) generator of  $T^{1,0}M$  of the form:

$$\mathcal{L} = \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta} \frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta} \frac{\partial}{\partial u_2} + \frac{\Lambda_3}{\Delta} \frac{\partial}{\partial u_3},$$

having conjugate:

$$\overline{\mathcal{L}} = \frac{\partial}{\partial \bar{z}} + \frac{\overline{\Lambda}_1}{\overline{\Delta}} \frac{\partial}{\partial u_1} + \frac{\overline{\Lambda}_2}{\overline{\Delta}} \frac{\partial}{\partial u_2} + \frac{\overline{\Lambda}_3}{\overline{\Delta}} \frac{\partial}{\partial u_3},$$

with coefficient-functions that have exactly the same expressions as in the class III<sub>1</sub> treated in the preceding section.

Furthermore, the three fields — the first  $\overline{\mathcal{F}} = \mathcal{F}$  being real —:

$$\begin{aligned}\mathcal{F} &:= \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}], \\ \mathcal{S} &:= [\mathcal{L}, \mathcal{F}] = [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]], \\ \mathcal{R} &:= [\mathcal{L}, \mathcal{S}] = \left[ \mathcal{L}, [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]] \right]\end{aligned}$$

are of the form:

$$\begin{aligned}\mathcal{F} &= \frac{\Upsilon_1}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^2 \overline{\Delta}^2} \frac{\partial}{\partial u_3}, \\ \mathcal{S} &= \frac{\Pi_1}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_2} + \frac{\Pi_3}{\Delta^4 \overline{\Delta}^3} \frac{\partial}{\partial u_3}, \\ \mathcal{R} &= \frac{\Sigma_1}{\Delta^6 \overline{\Delta}^4} \frac{\partial}{\partial u_1} + \frac{\Sigma_2}{\Delta^6 \overline{\Delta}^4} \frac{\partial}{\partial u_2} + \frac{\Sigma_3}{\Delta^6 \overline{\Delta}^4} \frac{\partial}{\partial u_3},\end{aligned}$$

with coefficient-functions that depend explicitly upon  $\varphi_1, \varphi_2, \varphi_3$ , although computers seem too weak to do that.

One therefore *decides* not to keep an explicit track of these dependencies.

One also has the conjugate field:

$$\begin{aligned}\overline{\mathcal{F}} &= \overline{[\mathcal{L}, \mathcal{F}]} \\ &= [\overline{\mathcal{L}}, \mathcal{F}] \\ &= \frac{\overline{\Pi}_1}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_1} + \frac{\overline{\Pi}_2}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_2} + \frac{\overline{\Pi}_3}{\Delta^3 \overline{\Delta}^4} \frac{\partial}{\partial u_3}.\end{aligned}$$

**Reformulation of the geometric assumptions:**

$$\begin{aligned}3 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{F}), \\ 4 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{F}, \mathcal{S}), \\ 4 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{F}, \mathcal{S}, \overline{\mathcal{F}}), \\ 5 &= \text{rank}_{\mathbb{C}}(\mathcal{L}, \overline{\mathcal{L}}, \mathcal{F}, \mathcal{S}, \mathcal{R}).\end{aligned}$$

Consequently, there are 2 functions  $A, B$  so that:

$$\boxed{\begin{aligned}\overline{\mathcal{F}} &= A \cdot \mathcal{F} + B \cdot \mathcal{S} \\ &= [\overline{\mathcal{L}}, \mathcal{F}].\end{aligned}}$$

Hence:

$$\begin{aligned}\mathcal{S} &= [\mathcal{L}, \mathcal{F}] = \overline{[\overline{\mathcal{L}}, \mathcal{F}]} = \overline{A \cdot \mathcal{F} + B \cdot \underbrace{\overline{\mathcal{F}}}_{\text{replace}}} \\ &= \overline{A \cdot \mathcal{F} + \overline{B}A \cdot \mathcal{F} + \overline{B}B \cdot \mathcal{S}} \\ &= (\overline{A} + \overline{B}A) \cdot \mathcal{F} + B\overline{B} \cdot \mathcal{S},\end{aligned}$$

whence by identification:

$$\boxed{\begin{array}{l} 0 = \overline{A} + \overline{B}A, \\ 1 = B\overline{B}. \end{array}}$$

Next, remind that the Jacobi identity gives:

$$\left[ \mathcal{L}, [\overline{\mathcal{L}}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]] \right] = \left[ \overline{\mathcal{L}}, [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]] \right],$$

that is to say:

$$[\overline{\mathcal{L}}, \mathcal{I}] = [\mathcal{L}, \overline{\mathcal{I}}].$$

Consequently:

$$\begin{aligned} [\overline{\mathcal{L}}, \mathcal{I}] &= [\mathcal{L}, \overline{\mathcal{I}}] = [\mathcal{L}, A\mathcal{I} + B\mathcal{I}] \\ &= \mathcal{L}(A) \cdot \mathcal{I} + \mathcal{L}(B) \cdot \mathcal{I} + A \underbrace{[\mathcal{L}, \mathcal{I}]_{\mathcal{I}}} + B \underbrace{[\mathcal{L}, \mathcal{I}]_{\mathcal{R}}} \\ &= \mathcal{L}(A) \cdot \mathcal{I} + (\mathcal{L}(B) + A) \cdot \mathcal{I} + B \cdot \mathcal{R}, \end{aligned}$$

*i.e.:*

$$\boxed{[\mathcal{I}, \overline{\mathcal{L}}] = -\mathcal{L}(A) \cdot \mathcal{I} - (\mathcal{L}(B) + A) \cdot \mathcal{I} - B \cdot \mathcal{R}.}$$

The 2 functions  $A, B$  will be fundamental, plus 3 other functions  $E, F, G$  only in:

$$\boxed{[\mathcal{L}, \mathcal{R}] = E \cdot \mathcal{I} + F \cdot \mathcal{I} + G \cdot \mathcal{R},}$$

because the other Lie brackets will be expressed in terms of:

$$A, \quad B, \quad E, \quad F, \quad G.$$

Indeed, between the **5** elements:

$$\{\mathcal{R}, \mathcal{I}, \mathcal{I}, \overline{\mathcal{L}}, \mathcal{L}\},$$

we yet have to determine **4** amongst all the **10** possible Lie brackets

$$\begin{aligned}
[\mathcal{R}, \mathcal{S}] &= \text{something}, \\
[\mathcal{R}, \mathcal{T}] &= \text{something}, \\
[\mathcal{R}, \overline{\mathcal{L}}] &= \text{something}, \\
[\mathcal{R}, \mathcal{L}] &= -E \cdot \mathcal{T} - F \cdot \mathcal{S} - G \cdot \mathcal{R}, \\
[\mathcal{S}, \mathcal{T}] &= \text{something}, \\
[\mathcal{S}, \overline{\mathcal{L}}] &= -\mathcal{L}(A) \cdot \mathcal{T} - (\mathcal{L}(B) + A) \cdot \mathcal{S} - B \cdot \mathcal{R}, \\
[\mathcal{S}, \mathcal{L}] &= -\mathcal{R}, \\
[\mathcal{T}, \overline{\mathcal{L}}] &= -A \cdot \mathcal{T} - B \cdot \mathcal{S}, \\
[\mathcal{T}, \mathcal{L}] &= -\mathcal{S}, \\
[\overline{\mathcal{L}}, \mathcal{L}] &= i \mathcal{T},
\end{aligned}$$

and these **4** remaining will express in terms of:

$$A, \quad B, \quad E, \quad F, \quad G.$$

**Preparatory lemma.** *The conjugate  $\overline{\mathcal{R}}$  expresses as:*

$$\begin{aligned}
\overline{\mathcal{R}} &= (\overline{\mathcal{L}}(A) + B \mathcal{L}(A) + AA) \cdot \mathcal{T} + \\
&\quad + (\overline{\mathcal{L}}(B) + B \mathcal{L}(B) + 2AB) \cdot \mathcal{S} + \\
&\quad + (BB) \cdot \mathcal{R}.
\end{aligned}$$

*Proof.* Indeed, from:

$$\mathcal{R} = [\mathcal{L}, \mathcal{S}],$$

by conjugating:

$$\begin{aligned}
\overline{\mathcal{R}} &= [\overline{\mathcal{L}}, \overline{\mathcal{S}}] \\
&= [\overline{\mathcal{L}}, A \mathcal{T} + B \mathcal{S}] \\
&= \overline{\mathcal{L}}(A) \cdot \mathcal{T} + \overline{\mathcal{L}}(B) \cdot \mathcal{S} + \\
&\quad + A \underbrace{[\overline{\mathcal{L}}, \mathcal{T}]}_{A \mathcal{T} + B \mathcal{S}} + B \underbrace{[\overline{\mathcal{L}}, \mathcal{S}]}_{\substack{\mathcal{L}(A) \mathcal{T} + \\ (\mathcal{L}(B) + A) \mathcal{S} \\ + BR}},
\end{aligned}$$

which, collecting, is so. □

**Lemma.** *The 3 functions in the Lie bracket:*

$$[\overline{\mathcal{L}}, \mathcal{R}] =: H_{\text{rpl}} \cdot \mathcal{T} + J_{\text{rpl}} \cdot \mathcal{S} + K_{\text{rpl}} \cdot \mathcal{R}$$

express in terms of  $A, B, E, F, G$  as:

$$K_{\text{rpl}} = 2B\overline{\mathcal{L}}(\overline{B}) + BB\mathcal{L}(\overline{B}) + B\overline{\mathcal{L}}(\overline{B}) + 2AB + \overline{G},$$

$$\begin{aligned} J_{\text{rpl}} = & B\overline{\mathcal{L}}(\mathcal{L}(\overline{B})) + \overline{\mathcal{L}}(\overline{\mathcal{L}}(\overline{B})) - \\ & - 2BB\mathcal{L}(\overline{B})\overline{\mathcal{L}}(\overline{B}) - BBB\mathcal{L}(\overline{B})\mathcal{L}(\overline{B}) - 2BB\mathcal{L}(\overline{B})\overline{\mathcal{L}}(\overline{B}) - \\ & - 2ABB\mathcal{L}(\overline{B}) - B\mathcal{L}(\overline{B})\overline{G} - 2B\overline{\mathcal{L}}(\overline{B})\overline{\mathcal{L}}(\overline{B}) - \overline{G}\overline{\mathcal{L}}(\overline{B}) - \\ & - 4AB\overline{\mathcal{L}}(\overline{B}) - 2ABB\mathcal{L}(\overline{B}) - 2AB\overline{\mathcal{L}}(\overline{B}) + 3\overline{\mathcal{L}}(A) + B\mathcal{L}(A) + \\ & + \overline{BF} - 3AAB - 2A\overline{G}, \end{aligned}$$

and, without full replacements:

$$\begin{aligned} H_{\text{rpl}} = & -AJ_{\text{rpl}} - K_{\text{rpl}}\mathcal{L}(A) - K_{\text{rpl}}\overline{B}\overline{\mathcal{L}}(A) + K_{\text{rpl}}AA + \\ & + \mathcal{L}(\overline{\mathcal{L}}(A)) + B\mathcal{L}(\mathcal{L}(A)) + \\ & + \mathcal{L}(B)\mathcal{L}(A) + 2A\mathcal{L}(A) + BBE. \end{aligned}$$

*Proof.* Start with computing:

$$\begin{aligned} [\mathcal{L}, \overline{\mathcal{R}}] = & \left[ \mathcal{L}, (\overline{\mathcal{L}}(A) + B\mathcal{L}(A) + AA) \cdot \mathcal{T} + \right. \\ & + (\overline{\mathcal{L}}(B) + B\mathcal{L}(B) + 2AB) \cdot \mathcal{S} + \\ & \left. + (BB) \cdot \mathcal{R} \right] \\ = & \left( \mathcal{L}(\overline{\mathcal{L}}(A)) + B\mathcal{L}(\mathcal{L}(A)) + \mathcal{L}(B)\mathcal{L}(A) + 2A\mathcal{L}(A) \right) \cdot \mathcal{T} + \\ & + \left( \mathcal{L}(\overline{\mathcal{L}}(B)) + B\mathcal{L}(\mathcal{L}(B)) + \mathcal{L}(B)\mathcal{L}(B) + 2B\mathcal{L}(A) + 2A\mathcal{L}(B) \right) \cdot \mathcal{S} + \\ & + (2B\mathcal{L}(B)) \cdot \mathcal{R} + \\ & + (\overline{\mathcal{L}}(A) + B\mathcal{L}(A) + AA) \cdot \underbrace{[\mathcal{L}, \mathcal{T}]_{\mathcal{S}}} + \\ & + (\overline{\mathcal{L}}(B) + B\mathcal{L}(B) + 2AB) \cdot \underbrace{[\mathcal{L}, \mathcal{S}]_{\mathcal{R}}} + \\ & + BB \cdot \underbrace{[\mathcal{L}, \mathcal{R}]_{E\mathcal{T}+F\mathcal{S}+G\mathcal{R}}}, \end{aligned}$$

which, collecting, becomes:

$$\begin{aligned}
[\mathcal{L}, \overline{\mathcal{R}}] &= \mathcal{I} \cdot \left( \begin{aligned} &\mathcal{L}(\overline{\mathcal{L}}(A)) + B \mathcal{L}(\mathcal{L}(A)) + \\ &+ \mathcal{L}(B) \mathcal{L}(A) + 2 A \mathcal{L}(A) + BBE \end{aligned} \right) + \\
&+ \mathcal{I} \cdot \left( \begin{aligned} &\mathcal{L}(\overline{\mathcal{L}}(B)) + B \mathcal{L}(\mathcal{L}(B)) + \\ &+ \mathcal{L}(B) \mathcal{L}(B) + 2 B \mathcal{L}(A) + 2 A \mathcal{L}(B) + \overline{\mathcal{L}}(A) + B \mathcal{L}(A) + \\ &+ AA + BBF \end{aligned} \right) + \\
&+ \mathcal{R} \cdot \left( \begin{aligned} &2 B \mathcal{L}(B) + \overline{\mathcal{L}}(B) + B \mathcal{L}(B) + \\ &+ 2 AB + BBG \end{aligned} \right).
\end{aligned}$$

On the other hand, conjugating:

$$[\overline{\mathcal{L}}, \mathcal{R}] =: H_{\text{rpl}} \cdot \mathcal{I} + J_{\text{rpl}} \cdot \mathcal{I} + K_{\text{rpl}} \cdot \mathcal{R},$$

one receives:

$$\begin{aligned}
[\mathcal{L}, \overline{\mathcal{R}}] &= \overline{[\overline{\mathcal{L}}, \mathcal{R}]} \\
&= \overline{H_{\text{rpl}}} \cdot \mathcal{I} + \overline{J_{\text{rpl}}} \cdot \overline{\mathcal{I}} + \overline{K_{\text{rpl}}} \cdot \overline{\mathcal{R}} \\
&= \overline{H_{\text{rpl}}} \cdot \mathcal{I} + \overline{J_{\text{rpl}}} (A \cdot \mathcal{I} + B \cdot \mathcal{I}) + \\
&\quad + \overline{K_{\text{rpl}}} \left\{ (\overline{\mathcal{L}}(A) + B \mathcal{L}(A) + AA) \cdot \mathcal{I} + \right. \\
&\quad \quad \quad \left. + (\overline{\mathcal{L}}(B) + B \mathcal{L}(B) + 2 AB) \cdot \mathcal{I} + \right. \\
&\quad \quad \quad \left. + (BB) \cdot \mathcal{R} \right\},
\end{aligned}$$

that is to say:

$$\begin{aligned}
[\overline{\mathcal{L}}, \mathcal{R}] &= \mathcal{I} \cdot \left( \overline{H_{\text{rpl}}} + A \overline{J_{\text{rpl}}} + \overline{K_{\text{rpl}}} \overline{\mathcal{L}}(A) + \overline{K_{\text{rpl}}} B \mathcal{L}(A) + \overline{K_{\text{rpl}}} AA \right) + \\
&+ \mathcal{I} \cdot \left( B \overline{J_{\text{rpl}}} + \overline{K_{\text{rpl}}} \overline{\mathcal{L}}(B) + \overline{K_{\text{rpl}}} B \mathcal{L}(B) + 2 \overline{K_{\text{rpl}}} AB \right) + \\
&+ \mathcal{R} \cdot \left( BB \overline{K_{\text{rpl}}} \right).
\end{aligned}$$

Equate the coefficients of  $\mathcal{R}$  in these two expressions of  $[\mathcal{L}, \overline{\mathcal{R}}]$ :

$$\begin{aligned}
BB \overline{K_{\text{rpl}}} &= 2 B \mathcal{L}(B) + \overline{\mathcal{L}}(B) + B \mathcal{L}(B) + \\
&+ 2 AB + BBG.
\end{aligned}$$

Recall:

$$B\overline{B} \equiv 1,$$

hence multiply both sides by  $\overline{BB}$  to get:

$$\begin{aligned}
\overline{K_{\text{rpl}}} &= 2 \overline{B} \mathcal{L}(B) + \overline{BB} \overline{\mathcal{L}}(B) + \overline{B} \mathcal{L}(B) + \\
&+ 2 A\overline{B} + G.
\end{aligned}$$

Conjugate this, and get  $K_{\text{rpl}}$  as stated.

Next, identify the coefficients of  $\mathcal{S}$  in the two expressions of  $[\mathcal{L}, \overline{\mathcal{R}}]$ , and get:

$$\begin{aligned} B\overline{J}_{\text{rpl}} &= -\overline{K}_{\text{rpl}}\overline{\mathcal{L}}(B) - \overline{K}_{\text{rpl}}B\mathcal{L}(B) - 2\overline{K}_{\text{rpl}}AB + \\ &\quad + \mathcal{L}(\overline{\mathcal{L}}(B)) + B\mathcal{L}(\mathcal{L}(B)) + \\ &\quad + \mathcal{L}(B)\mathcal{L}(B) + 2B\mathcal{L}(A) + 2A\mathcal{L}(B) + \overline{\mathcal{L}}(A) + B\mathcal{L}(A) + \\ &\quad + AA + BBF. \end{aligned}$$

Multiply both sides by  $\overline{B}$ , conjugate, and get  $J_{\text{rpl}}$  are stated.

Lastly, to get  $H_{\text{rpl}}$ , identify the conjugated coefficients of  $\mathcal{R}$ .  $\square$

Thanks to all this:

$$\begin{aligned} [\mathcal{S}, \mathcal{T}] &= [\mathcal{S}, \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}]] \\ &= \sqrt{-1}[\overline{\mathcal{L}}, [\mathcal{L}, \mathcal{S}]] - \sqrt{-1}[\mathcal{L}, [\overline{\mathcal{L}}, \mathcal{S}]] \\ &= \sqrt{-1}[\overline{\mathcal{L}}, \mathcal{R}] - \sqrt{-1}[\mathcal{L}, \mathcal{L}(A)\mathcal{T} + (\mathcal{L}(B) + A)\mathcal{S} + B\mathcal{R}] \\ &= \sqrt{-1}H_{\text{rpl}} \cdot \mathcal{T} + \sqrt{-1}J_{\text{rpl}} \cdot \mathcal{S} + \sqrt{-1}K_{\text{rpl}} \cdot \mathcal{R} - \\ &\quad - \sqrt{-1}\mathcal{L}(\mathcal{L}(A)) \cdot \mathcal{T} - \sqrt{-1}\mathcal{L}(\mathcal{L}(B)) \cdot \mathcal{S} - \sqrt{-1}\mathcal{L}(A) \cdot \mathcal{S} - \sqrt{-1}\mathcal{L}(B) \cdot \mathcal{R} - \\ &\quad - \underbrace{\sqrt{-1}\mathcal{L}(A)[\mathcal{L}, \mathcal{T}]}_{\mathcal{S}} - \underbrace{\sqrt{-1}(\mathcal{L}(B) + A)[\mathcal{L}, \mathcal{R}]}_{\mathcal{R}} - \underbrace{\sqrt{-1}B[\mathcal{L}, \mathcal{R}]}_{E\mathcal{T} + F\mathcal{S} + G\mathcal{R}}, \end{aligned}$$

which gives:

$$\begin{aligned} [\mathcal{S}, \mathcal{T}] &= \mathcal{T} \cdot (-\sqrt{-1}\mathcal{L}(\mathcal{L}(A)) + \sqrt{-1}H_{\text{rpl}} - \sqrt{-1}BE) + \\ &\quad + \mathcal{S} \cdot (-\sqrt{-1}\mathcal{L}(\mathcal{L}(B)) - 2\sqrt{-1}\mathcal{L}(A) + \sqrt{-1}J_{\text{rpl}} - \sqrt{-1}BF) + \\ &\quad + \mathcal{R} \cdot (-2\sqrt{-1}\mathcal{L}(B) + \sqrt{-1}K_{\text{rpl}} - \sqrt{-1}A - \sqrt{-1}BG). \end{aligned}$$

By similar reasonings based on the Jacobi identity (exercise):

$$\begin{aligned} [\mathcal{R}, \mathcal{T}] &= \mathcal{T} \cdot (\sqrt{-1}\overline{\mathcal{L}}(E) - \sqrt{-1}\mathcal{L}(K_{\text{rpl}}) + \sqrt{-1}F\mathcal{L}(A) + \sqrt{-1}AE + \\ &\quad + \sqrt{-1}GH_{\text{rpl}} - \sqrt{-1}EK_{\text{rpl}}) + \\ &\quad + \mathcal{S} \cdot (\sqrt{-1}\overline{\mathcal{L}}(F) - \sqrt{-1}\mathcal{L}(J_{\text{rpl}}) + \sqrt{-1}F\mathcal{L}(B) + \sqrt{-1}BE + \sqrt{-1}AF + \\ &\quad + \sqrt{-1}GJ_{\text{rpl}} - \sqrt{-1}H_{\text{rpl}} - \sqrt{-1}FK_{\text{rpl}}) + \\ &\quad + \mathcal{R} \cdot (\sqrt{-1}\overline{\mathcal{L}}(G) - \sqrt{-1}\mathcal{L}(K_{\text{rpl}}) + \sqrt{-1}BF + \sqrt{-1}GH_{\text{rpl}} - \sqrt{-1}J_{\text{rpl}} - \sqrt{-1}GH_{\text{rpl}}). \end{aligned}$$



It is advisable to abbreviate the just computed complicated Lie brackets by naming functions:

$$\begin{array}{lll} L_{\text{rpl}}, & M_{\text{rpl}}, & N_{\text{rpl}}, \\ O_{\text{rpl}}, & P_{\text{rpl}}, & Q_{\text{rpl}}, \\ R_{\text{rpl}}, & S_{\text{rpl}}, & T_{\text{rpl}}, \end{array}$$

with:

$$\begin{aligned} [\mathcal{S}, \mathcal{T}] &= -L \cdot \mathcal{T} - M \cdot \mathcal{S} - N \cdot \mathcal{R}, \\ [\mathcal{R}, \mathcal{T}] &= -O_{\text{rpl}} \cdot \mathcal{T} - P_{\text{rpl}} \cdot \mathcal{S} - Q_{\text{rpl}} \cdot \mathcal{R}, \\ [\mathcal{R}, \mathcal{S}] &= -R_{\text{rpl}} \cdot \mathcal{T} - S_{\text{rpl}} \cdot \mathcal{S} - T_{\text{rpl}} \cdot \mathcal{R}; \end{aligned}$$

in fact lastly:

$$\begin{aligned} [\mathcal{R}, \mathcal{S}] &= \left[ [\mathcal{L}, [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]]], [\mathcal{L}, \sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]] \right] \\ &= [\mathcal{R}, [\mathcal{L}, \mathcal{T}]] \\ &= -[\mathcal{T}, [\mathcal{R}, \mathcal{L}]] - [\mathcal{L}, [\mathcal{T}, \mathcal{R}]] \\ &= [\mathcal{L}, E \mathcal{T} + F \mathcal{S} + G \mathcal{R}] - \\ &\quad - [\mathcal{L}, O_{\text{rpl}} \mathcal{T} + P_{\text{rpl}} \mathcal{S} + Q_{\text{rpl}} \mathcal{R}] \\ &= \mathcal{T}(E) \cdot \mathcal{T} + \mathcal{T}(F) \cdot \mathcal{S} + \mathcal{T}(G) \cdot \mathcal{R} + \\ &\quad + E \underbrace{[\mathcal{T}, \mathcal{T}]}_{\text{replace}} + F \underbrace{[\mathcal{T}, \mathcal{S}]}_{\text{replace}} + G \underbrace{[\mathcal{T}, \mathcal{R}]}_{\text{replace}} - \\ &\quad - \mathcal{L}(O_{\text{rpl}}) \cdot \mathcal{T} - \mathcal{L}(P_{\text{rpl}}) \cdot \mathcal{S} - \mathcal{L}(Q_{\text{rpl}}) \cdot \mathcal{R} - \\ &\quad - O_{\text{rpl}} \underbrace{[\mathcal{L}, \mathcal{S}]}_{\mathcal{S}} - P_{\text{rpl}} \underbrace{[\mathcal{L}, \mathcal{S}]}_{\mathcal{R}} - Q_{\text{rpl}} \underbrace{[\mathcal{L}, \mathcal{R}]}_{E\mathcal{T}+F\mathcal{S}+G\mathcal{R}}, \end{aligned}$$

so that in:

$$[\mathcal{R}, \mathcal{S}] = -R_{\text{rpl}} \cdot \mathcal{T} - S_{\text{rpl}} \cdot \mathcal{S} - T_{\text{rpl}} \cdot \mathcal{R},$$

the three appearing coefficient-functions indeed express — lengthily if expanded which is not done here — in terms of the five fundamental functions  $A, B, E, F, G$  and their  $\{\mathcal{L}, \overline{\mathcal{L}}\}$ -derivatives.

**Summary.** In the 10 Lie brackets, "rpl" functions are thought of as *replaced* by their values in terms of the five fundamental functions  $A, B, E, F, G$ :

$$\begin{aligned}
[\mathcal{R}, \mathcal{S}] &= -R_{\text{rpl}} \cdot \mathcal{I} - S_{\text{rpl}} \cdot \mathcal{S} - T_{\text{rpl}} \cdot \mathcal{R}, \\
[\mathcal{R}, \mathcal{I}] &= -O_{\text{rpl}} \cdot \mathcal{I} - P_{\text{rpl}} \cdot \mathcal{S} - Q_{\text{rpl}} \cdot \mathcal{R}, \\
[\mathcal{R}, \overline{\mathcal{L}}] &= -H_{\text{rpl}} \cdot \mathcal{I} - J_{\text{rpl}} \cdot \mathcal{S} - K_{\text{rpl}} \cdot \mathcal{R}, \\
[\mathcal{R}, \mathcal{L}] &= -E \cdot \mathcal{I} - F \cdot \mathcal{S} - G \cdot \mathcal{R}, \\
[\mathcal{S}, \mathcal{I}] &= -L_{\text{rpl}} \cdot \mathcal{I} - M_{\text{rpl}} \cdot \mathcal{S} - N_{\text{rpl}} \cdot \mathcal{R}, \\
[\mathcal{S}, \overline{\mathcal{L}}] &= -\mathcal{L}(A) \cdot \mathcal{I} - (\mathcal{L}(B) + A) \cdot \mathcal{S} - B \cdot \mathcal{R}, \\
[\mathcal{S}, \mathcal{L}] &= -\mathcal{R}, \\
[\mathcal{I}, \overline{\mathcal{L}}] &= -A \cdot \mathcal{I} - B \cdot \mathcal{S}, \\
[\mathcal{I}, \mathcal{L}] &= -\mathcal{S}, \\
[\overline{\mathcal{L}}, \mathcal{L}] &= \sqrt{-1} \mathcal{I}.
\end{aligned}$$

**Initial Darboux structure of the dual coframe.** Dualize and introduce a dual coframe:

$$\{\mathcal{R}, \mathcal{S}, \mathcal{I}, \overline{\mathcal{L}}, \mathcal{L}\} \xleftrightarrow{\text{dual}} \{\tau_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\}.$$

The convenient auxiliary array that one should read vertically then is:

	$\mathcal{R}$	$\mathcal{S}$	$\mathcal{I}$	$\overline{\mathcal{L}}$	$\mathcal{L}$	
	$d\tau_0$	$d\sigma_0$	$d\rho_0$	$d\overline{\zeta}_0$	$d\zeta_0$	
$[\mathcal{R}, \mathcal{S}]$	$= -T_{\text{rpl}} +$	$-S_{\text{rpl}} +$	$-R_{\text{rpl}} +$	$0 +$	$0$	$\tau_0 \wedge \sigma_0$
$[\mathcal{R}, \mathcal{I}]$	$= -Q_{\text{rpl}} +$	$-P_{\text{rpl}} +$	$-O_{\text{rpl}} +$	$0 +$	$0$	$\tau_0 \wedge \rho_0$
$[\mathcal{R}, \overline{\mathcal{L}}]$	$= -K_{\text{rpl}} +$	$-J_{\text{rpl}} +$	$-H_{\text{rpl}} +$	$0 +$	$0$	$\tau_0 \wedge \overline{\zeta}_0$
$[\mathcal{R}, \mathcal{L}]$	$= -G +$	$-F +$	$-E +$	$0 +$	$0$	$\tau_0 \wedge \zeta_0$
$[\mathcal{S}, \mathcal{I}]$	$= -N_{\text{rpl}} +$	$-M_{\text{rpl}} +$	$-L_{\text{rpl}} +$	$0 +$	$0$	$\sigma_0 \wedge \rho_0$
$[\mathcal{S}, \overline{\mathcal{L}}]$	$= -B +$	$-\mathcal{L}(B) - A +$	$-\mathcal{L}(A) +$	$0 +$	$0$	$\sigma_0 \wedge \overline{\zeta}_0$
$[\mathcal{S}, \mathcal{L}]$	$= -1 +$	$0 +$	$0 +$	$0 +$	$0$	$\sigma_0 \wedge \zeta_0$
$[\mathcal{I}, \overline{\mathcal{L}}]$	$= 0 +$	$-B +$	$-A +$	$0 +$	$0$	$\rho_0 \wedge \overline{\zeta}_0$
$[\mathcal{I}, \mathcal{L}]$	$= 0 +$	$-1 +$	$0 +$	$0 +$	$0$	$\rho_0 \wedge \zeta_0$
$[\overline{\mathcal{L}}, \mathcal{L}]$	$= 0 +$	$0 +$	$i +$	$0 +$	$0$	$\overline{\zeta}_0 \wedge \zeta_0$

and one obtains the initial Darboux structure for class  $III_2$  CR manifolds:

$$\begin{aligned}
d\tau_0 &= T_{\text{rpl}} \tau_0 \wedge \sigma_0 + Q_{\text{rpl}} \tau_0 \wedge \rho_0 + K_{\text{rpl}} \tau_0 \wedge \bar{\zeta}_0 + G \tau_0 \wedge \zeta_0 + \\
&\quad + N_{\text{rpl}} \sigma_0 \wedge \rho_0 + B \sigma_0 \wedge \bar{\zeta}_0 + \sigma_0 \wedge \zeta_0, \\
d\sigma_0 &= S_{\text{rpl}} \tau_0 \wedge \sigma_0 + P_{\text{rpl}} \tau_0 \wedge \rho_0 + J_{\text{rpl}} \tau_0 \wedge \bar{\zeta}_0 + F \tau_0 \wedge \zeta_0 + M_{\text{rpl}} \sigma_0 \wedge \rho_0 + \\
&\quad + (\mathcal{L}(B) + A) \sigma_0 \wedge \bar{\zeta}_0 + B \rho_0 \wedge \bar{\zeta}_0 + \rho_0 \wedge \zeta_0, \\
d\rho_0 &= R_{\text{rpl}} \tau_0 \wedge \sigma_0 + O_{\text{rpl}} \tau_0 \wedge \rho_0 + H_{\text{rpl}} \tau_0 \wedge \bar{\zeta}_0 + E \tau_0 \wedge \zeta_0 + L_{\text{rpl}} \sigma_0 \wedge \rho_0 + \\
&\quad + \mathcal{L}(A) \sigma_0 \wedge \bar{\zeta}_0 + A \rho_0 \wedge \bar{\zeta}_0 + \sqrt{-1} \zeta_0 \wedge \bar{\zeta}_0, \\
d\bar{\zeta}_0 &= 0, \\
d\zeta_0 &= 0.
\end{aligned}$$

5.  $M^5 \subset \mathbb{C}^3$  of general class  $IV_1$ :  
initial frame and coframe in local coordinates

Consider:

$$(M^5 \subset \mathbb{C}^3) \in \text{General Class } IV_2.$$

Represent  $M$  as a (local) graph:

$$v = \varphi(x_1, x_2, y_1, y_2, u),$$

with:

$$\varphi(0) = 0,$$

without (necessarily) requiring that  $T_0M = \{v = 0\}$ .

Provided only that:

$$0 \neq \sqrt{-1} + \varphi_u(0),$$

two local generators for  $T^{1,0}M$  are:

$$\begin{aligned}
\mathcal{L}_1 &= \frac{\partial}{\partial z_1} - \frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}, \\
\mathcal{L}_2 &= \frac{\partial}{\partial z_2} - \frac{\varphi_{z_2}}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u},
\end{aligned}$$

having conjugates:

$$\begin{aligned}
\bar{\mathcal{L}}_1 &= \frac{\partial}{\partial \bar{z}_1} - \frac{\varphi_{\bar{z}_1}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}, \\
\bar{\mathcal{L}}_2 &= \frac{\partial}{\partial \bar{z}_2} - \frac{\varphi_{\bar{z}_2}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}.
\end{aligned}$$

Set as before in [5]:

$$A_1 := \frac{-\varphi_{z_1}}{\sqrt{-1} + \varphi_u},$$

$$A_2 := \frac{-\varphi_{z_2}}{\sqrt{-1} + \varphi_u},$$

two functions which are hence smooth near 0.

Choose:

$$\rho_0 := du - A_1 dz_1 - A_2 dz_2 - \bar{A}_1 d\bar{z}_1 - \bar{A}_2 d\bar{z}_2,$$

so that, in  $\mathbb{C} \otimes_{\mathbb{R}} TM$ :

$$T^{1,0}M \oplus T^{0,1}M = \{\rho_0 = 0\}.$$

Compute:

$$\begin{aligned} \mathcal{F} &:= \sqrt{-1} [\mathcal{L}_1, \bar{\mathcal{L}}_1] \\ &= \sqrt{-1} \left[ \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{z}_1} + \bar{A}_1 \frac{\partial}{\partial u} \right] \\ &= \sqrt{-1} \left( \mathcal{L}_1(\bar{A}_1) - \bar{\mathcal{L}}_1(A_1) \right) \frac{\partial}{\partial u}. \end{aligned}$$

After a possible  $\mathrm{GL}_2(\mathbb{C})$ -change of coordinates:

$$(z_1, z_2) \longmapsto (\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2),$$

one may assume that at the origin:

$$0 \neq \sqrt{-1} \left( \mathcal{L}_1(\bar{A}_1) - \bar{\mathcal{L}}_1(A_1) \right) (0),$$

hence in a neighborhood too.

Set:

$$\ell_1 := \sqrt{-1} \left( \mathcal{L}_1(\bar{A}_1) - \bar{\mathcal{L}}_1(A_1) \right),$$

so that:

$$\bar{\ell}_{11} = \ell_{11}.$$

Dually to the frame:

$$\{\mathcal{F}, \bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2, \mathcal{L}_1, \mathcal{L}_2\},$$

one has the coframe:

$$\{\rho_0, \bar{\zeta}_{01}, \bar{\zeta}_{02}, \zeta_{01}, \zeta_{02}\},$$

where:

$$\begin{aligned}\rho_0 &:= \frac{du - \bar{A}_1 d\bar{z}_1 - \bar{A}_2 d\bar{z}_2 - A_1 dz_1 - A_2 dz_2}{\ell_{11}}, \\ d\bar{\zeta}_{01} &:= d\bar{z}_1, \\ d\bar{\zeta}_{02} &:= d\bar{z}_2, \\ d\zeta_{01} &:= dz_1, \\ d\zeta_{02} &:= dz_2.\end{aligned}$$

Next, determine the Lie structure, namely:

$$\begin{aligned}[\mathcal{T}, \bar{\mathcal{L}}_1], \quad [\mathcal{T}, \bar{\mathcal{L}}_2], \quad [\mathcal{T}, \mathcal{L}_1], \quad [\mathcal{T}, \mathcal{L}_2], \\ [\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2], \quad [\bar{\mathcal{L}}_1, \mathcal{L}_1], \quad [\bar{\mathcal{L}}_1, \mathcal{L}_2], \\ [\bar{\mathcal{L}}_2, \mathcal{L}_1], \quad [\bar{\mathcal{L}}_2, \mathcal{L}_2], \\ [\mathcal{L}_1, \mathcal{L}_2].\end{aligned}$$

Compute:

$$\begin{aligned}[\mathcal{T}, \bar{\mathcal{L}}_1] &= \left[ \ell_{11} \frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{z}_1} + \bar{A}_1 \frac{\partial}{\partial u} \right] \\ &= - \left( \ell_{11, \bar{z}_1} + \bar{A}_1 \ell_{11, u} - \ell_{11} \bar{A}_{1, u} \right) \frac{\partial}{\partial u} \\ &= - \left( \frac{\ell_{11, \bar{z}_1} + \bar{A}_1 \ell_{11, u} - \ell_{11} \bar{A}_{1, u}}{\ell_{11}} \right) \mathcal{T} \\ &=: -\bar{P}_1 \cdot \mathcal{T},\end{aligned}$$

with (remind reality of  $\ell_{11}$ ):

$$P_1 := \frac{\ell_{11, z_1} + A_1 \ell_{11, u} - \ell_{11} A_{1, u}}{\ell_{11}}.$$

Similarly:

$$\begin{aligned}[\mathcal{T}, \bar{\mathcal{L}}_2] &= \left[ \ell_{11} \frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{z}_2} + \bar{A}_2 \frac{\partial}{\partial u} \right] \\ &= - \left( \ell_{11, \bar{z}_2} + \bar{A}_2 \ell_{11, u} - \ell_{11} \bar{A}_{2, u} \right) \frac{\partial}{\partial u} \\ &= - \left( \frac{\ell_{11, \bar{z}_2} + \bar{A}_2 \ell_{11, u} - \ell_{11} \bar{A}_{2, u}}{\ell_{11}} \right) \mathcal{T} \\ &=: -\bar{P}_2 \cdot \mathcal{T},\end{aligned}$$

with:

$$P_2 := \frac{\ell_{11, z_2} + A_2 \ell_{11, u} - \ell_{11} A_{2, u}}{\ell_{11}}.$$

Conjugating:

$$\begin{aligned} [\mathcal{I}, \mathcal{L}_1] &= -P_1 \cdot \mathcal{I}, \\ [\mathcal{I}, \mathcal{L}_2] &= -P_2 \cdot \mathcal{I}. \end{aligned}$$

Next, since:

$$\begin{aligned} [T^{1,0}M, T^{1,0}M] &\subset T^{1,0}M, \\ [T^{0,1}M, T^{0,1}M] &\subset T^{0,1}M, \end{aligned}$$

one has here (cf. the Scholium on pp. 26–28 in [5]):

$$\begin{aligned} [\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2] &= 0, \\ [\mathcal{L}_1, \mathcal{L}_2] &= 0. \end{aligned}$$

Also:

$$\begin{aligned} [\overline{\mathcal{L}}_1, \mathcal{L}_2] &= \left( \overline{\mathcal{L}}_1(A_2) - \mathcal{L}_2(\overline{A}_1) \right) \frac{\partial}{\partial u}, \\ [\overline{\mathcal{L}}_2, \mathcal{L}_1] &= \left( \overline{\mathcal{L}}_2(A_1) - \mathcal{L}_1(\overline{A}_2) \right) \frac{\partial}{\partial u}, \\ [\overline{\mathcal{L}}_2, \mathcal{L}_2] &= \left( \overline{\mathcal{L}}_2(A_2) - \mathcal{L}_2(\overline{A}_2) \right) \frac{\partial}{\partial u}. \end{aligned}$$

Setting:

$$\begin{aligned} \ell_{11} &:= \sqrt{-1} \left( \mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1) \right), & \ell_{12} &:= \sqrt{-1} \left( \mathcal{L}_2(\overline{A}_1) - \overline{\mathcal{L}}_1(A_2) \right), \\ \ell_{21} &:= \sqrt{-1} \left( \mathcal{L}_1(\overline{A}_2) - \overline{\mathcal{L}}_2(A_1) \right), & \ell_{22} &:= \sqrt{-1} \left( \mathcal{L}_2(\overline{A}_2) - \overline{\mathcal{L}}_2(A_2) \right), \end{aligned}$$

observing:

$$\begin{aligned} \overline{\ell}_{11} &= \ell_{11}, \\ \overline{\ell}_{12} &= \ell_{21}, \\ \overline{\ell}_{22} &= \ell_{22}, \end{aligned}$$

one therefore has:

$$\begin{aligned} [\overline{\mathcal{L}}_1, \mathcal{L}_1] &= \sqrt{-1} \ell_{11} \frac{\partial}{\partial u} \\ &= \sqrt{-1} \mathcal{I}, \\ [\overline{\mathcal{L}}_1, \mathcal{L}_2] &= \sqrt{-1} \ell_{12} \frac{\partial}{\partial u} \\ &= \sqrt{-1} \frac{\ell_{12}}{\ell_{11}} \mathcal{I}, \\ [\overline{\mathcal{L}}_2, \mathcal{L}_1] &= \sqrt{-1} \ell_{21} \frac{\partial}{\partial u} \\ &= \sqrt{-1} \frac{\ell_{21}}{\ell_{11}} \mathcal{I}, \end{aligned}$$

$$\begin{aligned} [\overline{\mathcal{L}}_2, \mathcal{L}_2] &= \sqrt{-1} \ell_{22} \frac{\partial}{\partial u} \\ &= \sqrt{-1} \frac{\ell_{22}}{\ell_{11}} \mathcal{I}. \end{aligned}$$

Abbreviate:

$$\begin{aligned} B &:= \frac{\ell_{21}}{\ell_{11}}, \\ A &:= \frac{\ell_{22}}{\ell_{11}}, \end{aligned}$$

so that:

$$[\overline{\mathcal{L}}_1, \mathcal{L}_1] = \sqrt{-1} \cdot \mathcal{I},$$

$$[\overline{\mathcal{L}}_1, \mathcal{L}_2] = \sqrt{-1} B \cdot \mathcal{I}.$$

$$[\overline{\mathcal{L}}_2, \mathcal{L}_1] = \sqrt{-1} \overline{B} \cdot \mathcal{I}.$$

$$[\overline{\mathcal{L}}_2, \mathcal{L}_2] = \sqrt{-1} A \cdot \mathcal{I}.$$

**Summary.** The 10 Lie brackets gather as:

$$[\mathcal{I}, \overline{\mathcal{L}}_1] = -\overline{P}_1 \cdot \mathcal{I},$$

$$[\mathcal{I}, \overline{\mathcal{L}}_2] = -\overline{P}_2 \cdot \mathcal{I},$$

$$[\mathcal{I}, \mathcal{L}_1] = -P_1 \cdot \mathcal{I},$$

$$[\mathcal{I}, \mathcal{L}_2] = -P_2 \cdot \mathcal{I},$$

$$[\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2] = 0,$$

$$[\overline{\mathcal{L}}_1, \mathcal{L}_1] = \sqrt{-1} \cdot \mathcal{I},$$

$$[\overline{\mathcal{L}}_1, \mathcal{L}_2] = \sqrt{-1} B \cdot \mathcal{I},$$

$$[\overline{\mathcal{L}}_2, \mathcal{L}_1] = \sqrt{-1} \overline{B} \cdot \mathcal{I},$$

$$[\overline{\mathcal{L}}_2, \mathcal{L}_2] = \sqrt{-1} A \cdot \mathcal{I},$$

$$[\mathcal{L}_1, \mathcal{L}_2] = 0,$$

and incorporate 4 fundamental functions:

$$A, \quad B, \quad P_1, \quad P_2.$$

Organize then the ten Lie brackets as a convenient auxiliary array:

	$\mathcal{I}$	$\overline{\mathcal{L}}_1$	$\overline{\mathcal{L}}_2$	$\mathcal{L}_1$	$\mathcal{L}_2$	
	$d\rho_0$	$d\overline{\zeta}_{01}$	$d\overline{\zeta}_{02}$	$d\zeta_{01}$	$d\zeta_{02}$	
$[\mathcal{I}, \overline{\mathcal{L}}_1]$	$= -\overline{P}_1 \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\rho_0 \wedge \overline{\zeta}_{01}$
$[\mathcal{I}, \overline{\mathcal{L}}_2]$	$= -\overline{P}_1 \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\rho_0 \wedge \overline{\zeta}_{02}$
$[\mathcal{I}, \mathcal{L}_1]$	$= -P_1 \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\rho_0 \wedge \zeta_{01}$
$[\mathcal{I}, \mathcal{L}_2]$	$= -P_2 \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\rho_0 \wedge \zeta_{02}$
$[\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2]$	$= 0$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\overline{\zeta}_{01} \wedge \overline{\zeta}_{02}$
$[\overline{\mathcal{L}}_1, \mathcal{L}_1]$	$= \sqrt{-1} \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\overline{\zeta}_{01} \wedge \zeta_{01}$
$[\overline{\mathcal{L}}_1, \mathcal{L}_2]$	$= \sqrt{-1} B \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\overline{\zeta}_{01} \wedge \zeta_{02}$
$[\overline{\mathcal{L}}_2, \mathcal{L}_1]$	$= \sqrt{-1} \overline{B} \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\overline{\zeta}_{02} \wedge \zeta_{01}$
$[\overline{\mathcal{L}}_2, \mathcal{L}_2]$	$= \sqrt{-1} A \cdot \mathcal{I}$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\overline{\zeta}_{02} \wedge \zeta_{02}$
$[\mathcal{L}_1, \mathcal{L}_2]$	$= 0$	$+ 0$	$+ 0$	$+ 0$	$+ 0$	$\zeta_{01} \wedge \zeta_{02}$

Hence the Darboux structure is:

$$\begin{aligned}
d\rho_0 &= \overline{P}_1 \cdot \rho_0 \wedge \overline{\zeta}_{01} + \overline{P}_2 \cdot \rho_0 \wedge \overline{\zeta}_{02} + P_1 \cdot \rho_0 \wedge \zeta_{01} + P_2 \cdot \rho_0 \wedge \zeta_{02} + \\
&\quad + \sqrt{-1} \cdot \zeta_{01} \wedge \overline{\zeta}_{01} + \sqrt{-1} B \cdot \zeta_{02} \wedge \overline{\zeta}_{01} + \sqrt{-1} \overline{B} \cdot \zeta_{01} \wedge \overline{\zeta}_{02} + \sqrt{-1} A \cdot \zeta_{02} \wedge \overline{\zeta}_{02}, \\
d\overline{\zeta}_{01} &= 0, \\
d\overline{\zeta}_{02} &= 0, \\
d\zeta_{01} &= 0, \\
d\zeta_{02} &= 0.
\end{aligned}$$

## 6. $M^5 \subset \mathbb{C}^3$ of general class $IV_2$ : initial frame and coframe in local coordinates

Consider:

$$(M^5 \subset \mathbb{C}^3) \in \text{General Class } IV_2.$$

Represent  $M$  as a (local) graph:

$$v = \varphi(x_1, x_2, y_1, y_2, u),$$

with:

$$\varphi(0) = 0,$$

without (necessarily) requiring that  $T_0M = \{v = 0\}$ .

Provided only that:

$$0 \neq \sqrt{-1} + \varphi_u(0),$$



two local generators for  $T^{1,0}M$  are:

$$\mathcal{L}_1 = \frac{\partial}{\partial z_1} - \frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u},$$

$$\mathcal{L}_2 = \frac{\partial}{\partial z_2} - \frac{\varphi_{z_2}}{\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u},$$

having conjugates:

$$\overline{\mathcal{L}}_1 = \frac{\partial}{\partial \bar{z}_1} - \frac{\varphi_{\bar{z}_1}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u},$$

$$\overline{\mathcal{L}}_2 = \frac{\partial}{\partial \bar{z}_2} - \frac{\varphi_{\bar{z}_2}}{-\sqrt{-1} + \varphi_u} \frac{\partial}{\partial u}.$$

Set as before in [5]:

$$A_1 := \frac{-\varphi_{z_1}}{\sqrt{-1} + \varphi_u},$$

$$A_2 := \frac{-\varphi_{z_2}}{\sqrt{-1} + \varphi_u},$$

two functions which are hence smooth near 0.

Choose:

$$\rho_0 := du - A_1 dz_1 - A_2 dz_2 - \overline{A}_1 d\bar{z}_1 - \overline{A}_2 d\bar{z}_2,$$

so that, in  $\mathbb{C} \otimes_{\mathbb{R}} TM$ :

$$T^{1,0}M \oplus T^{0,1}M = \{\rho_0 = 0\}.$$

Hence ([5]):

$$\text{Levi-Matrix}_{\mathcal{L}, \overline{\mathcal{L}}}^M = \begin{pmatrix} \sqrt{-1}(\mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1)) & \sqrt{-1}(\mathcal{L}_2(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1)) \\ \sqrt{-1}(\mathcal{L}_1(\overline{A}_2) - \overline{\mathcal{L}}_2(A_1)) & \sqrt{-1}(\mathcal{L}_2(\overline{A}_2) - \overline{\mathcal{L}}_2(A_1)) \end{pmatrix}.$$

By hypothesis, this matrix is everywhere of rank 1.

After a possible  $GL_2(\mathbb{C})$ -change of coordinates:

$$(z_1, z_2) \mapsto (\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2),$$

one may assume that at the origin:

$$0 \neq \sqrt{-1}(\mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1))(0),$$

hence in a neighborhood too.

So:

$$1 = \text{rank}_{\mathbb{C}} \left( \text{Levi-Matrix}_{\mathcal{L}, \overline{\mathcal{L}}}^M \right),$$

which now means:

$$0 \equiv \begin{vmatrix} \mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1) & \mathcal{L}_2(\overline{A}_1) - \overline{\mathcal{L}}_1(A_2) \\ \mathcal{L}_1(\overline{A}_2) - \overline{\mathcal{L}}_2(A_1) & \mathcal{L}_2(\overline{A}_2) - \overline{\mathcal{L}}_2(A_2) \end{vmatrix}.$$

Introduce the very fundamental function which is the negative of the quotient of the two entries of the first line:

$$k := - \frac{\mathcal{L}_2(\overline{A}_1) - \overline{\mathcal{L}}_1(A_2)}{\mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1)}.$$

A (mental) exercise convinces that the kernel of the Levi matrix is generated by the vector-valued function:

$$\begin{pmatrix} k \\ 1 \end{pmatrix}$$

namely:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1) & \mathcal{L}_2(\overline{A}_1) - \overline{\mathcal{L}}_1(A_2) \\ \mathcal{L}_1(\overline{A}_2) - \overline{\mathcal{L}}_2(A_1) & \mathcal{L}_2(\overline{A}_2) - \overline{\mathcal{L}}_2(A_2) \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Introduce the  $(1, 0)$  vector field:

$$\mathcal{K} := k \mathcal{L}_1 + \mathcal{L}_2.$$

By [5],  $\mathcal{K}$  is a generator for the Levi-kernel subbundle:

$$K^{1,0}M \subset T^{1,0}M,$$

and it is invariant through biholomorphisms, with conjugate:

$$K^{0,1}M \subset T^{0,1}M.$$

Furthermore ([5], pp. 72–73):

$$\begin{aligned} [K^{1,0}M, K^{1,0}M] &\subset K^{1,0}M, \\ [K^{0,1}M, K^{0,1}M] &\subset K^{0,1}M, \\ [K^{1,0}M, K^{0,1}M] &\subset K^{1,0}M \oplus K^{0,1}M. \end{aligned}$$

Now, since:

$$\begin{aligned} \mathcal{K} &= k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + (k A_1 + A_2) \frac{\partial}{\partial u}, \\ \overline{\mathcal{K}} &= \overline{k} \frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_2} + (\overline{k} \overline{A}_1 + \overline{A}_2) \frac{\partial}{\partial u}, \end{aligned}$$

it is visible that the Lie bracket:

$$[\mathcal{K}, \overline{\mathcal{K}}] = \text{function} \cdot \mathcal{K} + \text{function} \cdot \overline{\mathcal{K}}$$

contains no  $\frac{\partial}{\partial z_2}$  and no  $\frac{\partial}{\partial \overline{z}_2}$ , whence necessarily:

$$[\mathcal{K}, \overline{\mathcal{K}}] = 0,$$

which gives:

$$\begin{aligned} 0 &= \left[ k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + (k A_1 + A_2) \frac{\partial}{\partial u}, \bar{k} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2} + (\bar{k} \bar{A}_1 + \bar{A}_2) \frac{\partial}{\partial u} \right] \\ &= \mathcal{H}(\bar{k}) \frac{\partial}{\partial \bar{z}_1} - \overline{\mathcal{H}}(k) \frac{\partial}{\partial z_1} + \text{something} \cdot \frac{\partial}{\partial u}, \end{aligned}$$

and yields:

$$\begin{aligned} 0 &\equiv \mathcal{H}(\bar{k}), \\ 0 &\equiv \overline{\mathcal{H}}(k). \end{aligned}$$

Now, the natural choice of a frame for  $\mathbb{C} \otimes_{\mathbb{R}} TM$  includes  $\mathcal{H}$ :

$$\{\mathcal{T}, \overline{\mathcal{L}}_1, \overline{\mathcal{H}}, \mathcal{L}_1, \mathcal{H}\},$$

where:

$$\mathcal{T} := \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1]$$

is, as always, real.

To determine the Lie structure of this frame, 10 bracket are extant:

$$\begin{aligned} [\mathcal{H}, \mathcal{L}_1], \quad [\mathcal{H}, \overline{\mathcal{H}}], \quad [\mathcal{H}, \overline{\mathcal{L}}_1], \quad [\mathcal{H}, \mathcal{T}], \\ [\mathcal{L}_1, \overline{\mathcal{H}}], \quad [\mathcal{L}_1, \overline{\mathcal{L}}_1], \quad [\mathcal{L}_1, \mathcal{T}], \\ [\overline{\mathcal{H}}, \overline{\mathcal{L}}_1], \quad [\overline{\mathcal{H}}, \mathcal{T}], \\ [\overline{\mathcal{L}}_1, \mathcal{T}]. \end{aligned}$$

Firstly:

$$\begin{aligned} [\mathcal{H}, \mathcal{L}_1] &= [k \mathcal{L}_1 + \mathcal{L}_2, \mathcal{L}_1] \\ &\quad - \mathcal{L}_1(k) \cdot \mathcal{L}_1. \end{aligned}$$

Secondly, as seen:

$$[\mathcal{H}, \overline{\mathcal{H}}] = 0.$$

Thirdly:

$$\begin{aligned} [\mathcal{H}, \overline{\mathcal{L}}_1] &= [k \mathcal{L}_1 + \mathcal{L}_2, \overline{\mathcal{L}}_1] \\ &\quad - \overline{\mathcal{L}}_1(k) \cdot \mathcal{L}_1 + \\ &\quad + k [\mathcal{L}_1, \overline{\mathcal{L}}_1] + [\mathcal{L}_2, \overline{\mathcal{L}}_1], \end{aligned}$$

the last two terms disappearing by the definition of  $k$ , for:

$$\begin{aligned} [\mathcal{L}_1, \overline{\mathcal{L}}_1] &= \left( \mathcal{L}_1(\bar{A}_1) - \overline{\mathcal{L}}_1(A_1) \right) \frac{\partial}{\partial u}, \\ [\mathcal{L}_2, \overline{\mathcal{L}}_1] &= \left( \mathcal{L}_2(\bar{A}_1) - \overline{\mathcal{L}}_1(A_2) \right) \frac{\partial}{\partial u}. \end{aligned}$$

Fourthly, using Jacobi:

$$\begin{aligned}
[\mathcal{H}, \mathcal{F}] &= \sqrt{-1} [\mathcal{H}, [\mathcal{L}_1, \overline{\mathcal{L}}_1]] \\
&= -\sqrt{-1} \left[ \overline{\mathcal{L}}_1, \underbrace{[\mathcal{H}, \mathcal{L}_1]}_{-\mathcal{L}_1(k) \cdot \mathcal{L}_1} \right] - \sqrt{-1} \left[ \mathcal{L}_1, \underbrace{[\overline{\mathcal{L}}_1, \mathcal{H}]}_{\overline{\mathcal{L}}_1(k) \cdot \mathcal{L}_1} \right] \\
&= \sqrt{-1} [\overline{\mathcal{L}}_1, \mathcal{L}_1(k) \mathcal{L}_1] - \sqrt{-1} [\mathcal{L}_1, \overline{\mathcal{L}}_1(k) \mathcal{L}_1] \\
&= \sqrt{-1} \left( \overline{\mathcal{L}}_1(\mathcal{L}_1(k)) - \mathcal{L}_1(\overline{\mathcal{L}}_1(k)) \right) \cdot \mathcal{L}_1 + \\
&\quad + \mathcal{L}_1(k) \sqrt{-1} [\overline{\mathcal{L}}_1, \mathcal{L}_1] - \sqrt{-1} \overline{\mathcal{L}}_1(k) [\mathcal{L}_1, \overline{\mathcal{L}}_1] \\
&= -\mathcal{F}(k) \cdot \mathcal{L}_1 - \mathcal{L}_1(k) \cdot \mathcal{F},
\end{aligned}$$

that is to say:

$$[\mathcal{H}, \mathcal{F}] = -\mathcal{F}(k) \cdot \mathcal{L}_1 - \mathcal{L}_1(k) \cdot \mathcal{F}.$$

Fifthly, conjugating an equation that precedes:

$$[\mathcal{L}_1, \overline{\mathcal{H}}] = \mathcal{L}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1.$$

Sixthly:

$$[\mathcal{L}_1, \overline{\mathcal{L}}_1] = -\sqrt{-1} \mathcal{F}.$$

Seventhly, abbreviating:

$$\mathcal{F} = \ell \frac{\partial}{\partial u},$$

with:

$$\ell := \sqrt{-1} (\mathcal{L}_1(\overline{A}_1) - \overline{\mathcal{L}}_1(A_1)),$$

compute:

$$\begin{aligned}
[\mathcal{L}_1, \mathcal{F}] &= \left[ \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial u}, \ell \frac{\partial}{\partial u} \right] \\
&= \left( \ell_{z_1} + A_1 \ell_u - \ell A_{1,u} \right) \frac{\partial}{\partial u} \\
&= \frac{\ell_{z_1} + A_1 \ell_u - \ell A_{1,u}}{\ell} \mathcal{F}.
\end{aligned}$$

One therefore comes to the second and last fundamental function:

$$P := \frac{\ell_{z_1} + A_1 \ell_u - \ell A_{1,u}}{\ell},$$

and gets:

$$[\mathcal{L}_1, \mathcal{F}] = P \cdot \mathcal{F}.$$

Eighthly:

$$[\overline{\mathcal{H}}, \overline{\mathcal{L}}_1] = -\overline{\mathcal{L}}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1.$$

Ninthly:

$$[\overline{\mathcal{H}}, \mathcal{T}] = -\overline{\mathcal{T}}(\overline{k}) \cdot \overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_1(\overline{k}) \cdot \overline{\mathcal{T}}.$$

Tenthly:

$$[\overline{\mathcal{L}}_1, \mathcal{T}] = \overline{P} \cdot \mathcal{T}.$$

**Summary.** One has the 10 Lie bracket relations:

$$\begin{aligned} [\mathcal{T}, \overline{\mathcal{L}}_1] &= -\overline{P} \cdot \mathcal{T}, \\ [\mathcal{T}, \overline{\mathcal{H}}] &= \overline{\mathcal{L}}_1(\overline{k}) \cdot \mathcal{T} + \mathcal{T}(\overline{k}) \cdot \overline{\mathcal{L}}_1, \\ [\mathcal{T}, \mathcal{L}_1] &= -P \cdot \mathcal{T}, \\ [\mathcal{T}, \mathcal{H}] &= \mathcal{L}_1(k) \cdot \mathcal{T} + \mathcal{T}(k) \cdot \mathcal{L}_1, \\ [\overline{\mathcal{L}}_1, \overline{\mathcal{H}}] &= \overline{\mathcal{L}}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1, \\ [\overline{\mathcal{L}}_1, \mathcal{L}_1] &= \sqrt{-1} \mathcal{T}, \\ [\overline{\mathcal{L}}_1, \mathcal{H}] &= \overline{\mathcal{L}}_1(k) \cdot \mathcal{L}_1, \\ [\overline{\mathcal{H}}, \mathcal{L}_1] &= -\mathcal{L}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1, \\ [\overline{\mathcal{H}}, \mathcal{H}] &= 0, \\ [\mathcal{L}_1, \mathcal{H}] &= \mathcal{L}_1(k) \cdot \mathcal{L}_1. \end{aligned}$$

**Initial Darboux structure of the dual coframe.** Introduce then the coframe:

$$\{\rho_0, \overline{\kappa}_0, \overline{\zeta}_0, \kappa_0, \zeta_0\}$$

which is dual to the frame:

$$\{\mathcal{T}, \overline{\mathcal{L}}_1, \overline{\mathcal{H}}, \mathcal{L}_1, \mathcal{H}\},$$

namely:

$$\begin{array}{ccccc} \rho_0(\mathcal{T}) = 1 & \rho_0(\overline{\mathcal{L}}_1) = 0 & \rho_0(\overline{\mathcal{H}}) = 0 & \rho_0(\mathcal{L}_1) = 0 & \rho_0(\mathcal{H}) = 0, \\ \overline{\kappa}_0(\mathcal{T}) = 1 & \overline{\kappa}_0(\overline{\mathcal{L}}_1) = 0 & \overline{\kappa}_0(\overline{\mathcal{H}}) = 0 & \overline{\kappa}_0(\mathcal{L}_1) = 0 & \overline{\kappa}_0(\mathcal{H}) = 0, \\ \overline{\zeta}_0(\mathcal{T}) = 1 & \overline{\zeta}_0(\overline{\mathcal{L}}_1) = 0 & \overline{\zeta}_0(\overline{\mathcal{H}}) = 0 & \overline{\zeta}_0(\mathcal{L}_1) = 0 & \overline{\zeta}_0(\mathcal{H}) = 0, \\ \kappa_0(\mathcal{T}) = 1 & \kappa_0(\overline{\mathcal{L}}_1) = 0 & \kappa_0(\overline{\mathcal{H}}) = 0 & \kappa_0(\mathcal{L}_1) = 0 & \kappa_0(\mathcal{H}) = 0, \\ \zeta_0(\mathcal{T}) = 1 & \zeta_0(\overline{\mathcal{L}}_1) = 0 & \zeta_0(\overline{\mathcal{H}}) = 0 & \zeta_0(\mathcal{L}_1) = 0 & \zeta_0(\mathcal{H}) = 0. \end{array}$$

One has:

$$\begin{aligned} \rho_0 &= \frac{du - A_1 dz_1 - A_2 dz_2 - \overline{A}_1 d\overline{z}_1 - \overline{A}_2 d\overline{z}_2}{\ell}, \\ \kappa_0 &= dz_1 - k dz_2, \\ \zeta_0 &= dz_2. \end{aligned}$$

Organize the ten Lie brackets as a convenient auxiliary array:

	$\mathcal{I}$	$\overline{\mathcal{L}}_1$	$\overline{\mathcal{K}}$	$\mathcal{L}_1$	$\mathcal{H}$	
	$d\rho_0$	$d\overline{\kappa}_0$	$d\overline{\zeta}_0$	$d\kappa_0$	$d\zeta_0$	
$[\mathcal{I}, \overline{\mathcal{L}}_1]$	$-\overline{P} \cdot \mathcal{I}$	0	0	0	0	$\rho_0 \wedge \overline{\kappa}_0$
$[\mathcal{I}, \overline{\mathcal{K}}]$	$\overline{\mathcal{L}}_1(\overline{k}) \cdot \mathcal{I}$	$\mathcal{I}(\overline{k}) \cdot \overline{\mathcal{L}}_1$	0	0	0	$\rho_0 \wedge \overline{\zeta}_0$
$[\mathcal{I}, \mathcal{L}_1]$	$-P \cdot \mathcal{I}$	0	0	0	0	$\rho_0 \wedge \kappa_0$
$[\mathcal{I}, \mathcal{H}]$	$\mathcal{L}_1(k) \cdot \mathcal{I}$	0	0	$\mathcal{I}(k) \cdot \mathcal{L}_1$	0	$\rho_0 \wedge \zeta_0$
$[\overline{\mathcal{L}}_1, \overline{\mathcal{K}}]$	0	$\overline{\mathcal{L}}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1$	0	0	0	$\overline{\kappa}_0 \wedge \overline{\zeta}_0$
$[\overline{\mathcal{L}}_1, \mathcal{L}_1]$	$\sqrt{-1} \cdot \mathcal{I}$	0	0	0	0	$\overline{\kappa}_0 \wedge \kappa_0$
$[\overline{\mathcal{L}}_1, \mathcal{H}]$	0	0	0	$\overline{\mathcal{L}}_1(k) \cdot \mathcal{L}_1$	0	$\overline{\kappa}_0 \wedge \zeta_0$
$[\overline{\mathcal{K}}, \mathcal{L}_1]$	0	$-\mathcal{L}_1(\overline{k}) \cdot \overline{\mathcal{L}}_1$	0	0	0	$\overline{\zeta}_0 \wedge \kappa_0$
$[\overline{\mathcal{K}}, \mathcal{H}]$	0	0	0	0	0	$\overline{\zeta}_0 \wedge \zeta_0$
$[\mathcal{L}_1, \mathcal{H}]$	0	0	0	$\mathcal{L}_1(k) \cdot \mathcal{L}_1$	0	$\kappa_0 \wedge \zeta_0$

Read *vertically* and put an overall minus sign:

$$\begin{aligned}
 d\rho_0 &= \overline{P} \cdot \rho_0 \wedge \overline{\kappa}_0 - \overline{\mathcal{L}}_1(\overline{k}) \cdot \rho_0 \wedge \overline{\zeta}_0 + P \cdot \rho_0 \wedge \kappa_0 - \mathcal{L}_1(k) \cdot \rho_0 \wedge \zeta_0 + \sqrt{-1} \kappa_0 \wedge \overline{\kappa}_0, \\
 d\overline{\kappa}_0 &= -\mathcal{I}(\overline{k}) \cdot \rho_0 \wedge \overline{\zeta}_0 - \overline{\mathcal{L}}_1(\overline{k}) \cdot \overline{\kappa}_0 \wedge \overline{\zeta}_0 + \mathcal{L}_1(\overline{k}) \cdot \overline{\zeta}_0 \wedge \kappa_0, \\
 d\overline{\zeta}_0 &= 0, \\
 d\kappa_0 &= -\mathcal{I}(k) \cdot \rho_0 \wedge \zeta_0 - \overline{\mathcal{L}}_1(k) \cdot \overline{\kappa}_0 \wedge \zeta_0 - \mathcal{L}_1(k) \cdot \kappa_0 \wedge \zeta_0, \\
 d\zeta_0 &= 0.
 \end{aligned}$$

This together with the related matrix ambiguity group ([7]):

$$\mathbb{G}_{\mathbb{V}_2}^{\text{initial}} := \left\{ \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & \overline{c} & 0 & 0 \\ 0 & 0 & \overline{b} & \overline{a} & 0 \\ e & d & \overline{e} & \overline{d} & a\overline{a} \end{pmatrix} \in \mathcal{M}_{5 \times 5}(\mathbb{C}) : a, c \in \mathbb{C} \setminus \{0\}, b, d, e \in \mathbb{C} \right\}$$

initializes (launches) an explicit application of Cartan's equivalence method achieved by Pocchiola in [10].

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