# Rationality in Differential Algebraic Geometry

Joël Merker

**Abstract** Parametric Cartan theory of exterior differential systems, and explicit cohomology of projective manifolds reveal united rationality features of differential algebraic geometry.

## 1 Rationality

The natural integer numbers:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ \dots,\ 2013,\ \dots\dots$$

necessarily hint at some '*invention*' of the Zero. While for the Greeks, the actual '$\infty$' and the actual '0' did not '*exist*', the Babylonians used the symbol '0' in numeration. In India (*cf. e.g.* Brahmagupta), the zero comes from self-subtraction:

$$0 \overset{\text{def}}{:=} \mathbf{a} - \mathbf{a}.$$

In rational numbers:

$$\frac{p}{q} \qquad\qquad {\scriptstyle (q \neq 0)},$$

division by zero is and must be excluded.

The present paper aims at showing that *higher abstract conceptions in advanced mathematics depend upon archetypical rational computational phenomena*.

Several instances of deeper rationality facts will hence be surveyed:

□ In Cartan's theory of exterior differential systems;

□ In Complex Algebraic Geometry.

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## 2 Equivalences of 5-dimensional CR manifolds

> Despite their importance, until now, the invariants of strictly pseudoconvex domains have been fully computed, to our knowledge, only in the case of the unit ball $\mathbb{B}^{n+1}$, where they all vanish!
> Sidney WEBSTER ([73]).

Real analytic ($\mathscr{C}^\omega$) *CR-generic* submanifolds $M \subset \mathbb{C}^{n+c}$ of codimension $c \geqslant 0$ are those satisfying $TM + J(TM) = T\mathbb{C}^{n+c}\big|_M$, where $J\colon T\mathbb{C}^{n+c} \longrightarrow T\mathbb{C}^{n+c}$ denotes the standard complex structure and then $TM \cap J(TM)$ has constant real dimension $2n =: 2\,\mathsf{CRdim}\, M$, while $\mathsf{dim}_{\mathbb{R}} M = 2n + c$; general $\mathscr{C}^\omega$ CR submanifolds $M \subset \mathbb{C}^\nu$, *i.e.* those for which $\mathsf{dim}\,\big(T_pM \cap J(T_pM)\big)$ is constant for $p \in M$, are always locally CR-generic in some complex submanifold [51], hence CR-genericity is not a restriction.

**Problem 1.** *Classify local $\mathscr{C}^\omega$ CR-generic submanifolds $M^{2n+c} \subset \mathbb{C}^{n+c}$ under local biholomorphisms of $\mathbb{C}^{n+c}$ up to dimension $2n + c \leqslant \mathbf{5}$.*

If $c = 0$, then $M \cong \mathbb{C}^n$, where '$\cong$' means '*locally biholomorphic*'; if $n = 0$, then $M \cong \mathbb{R}^c$. Assume therefore $c \geqslant 1$ and $n \geqslant 1$. The possible CR dimensions and real codimensions are:

$$
\begin{aligned}
2n + c = \mathbf{3} &\quad\Longrightarrow\quad \begin{cases} n = \mathbf{1}, \quad c = \mathbf{1}, \end{cases} \\
2n + c = \mathbf{4} &\quad\Longrightarrow\quad \begin{cases} n = \mathbf{1}, \quad c = \mathbf{2}, \end{cases} \\
2n + c = \mathbf{5} &\quad\Longrightarrow\quad \begin{cases} n = \mathbf{1}, \quad c = \mathbf{3}, \\ n = \mathbf{2}, \quad c = \mathbf{1}. \end{cases}
\end{aligned}
$$

In local coordinates $\big(z_1, \dots, z_n, w_1, \dots, w_c\big) = \big(x_1 + \sqrt{-1}\,y_1, \dots, x_n + \sqrt{-1}\,y_n,\, u_1 + \sqrt{-1}\,v_1, \dots, u_c + \sqrt{-1}\,v_c\big)$, represent with graphing $\mathscr{C}^\omega$ functions $\varphi_\bullet$:

$$
\begin{aligned}
M^3 \subset \mathbb{C}^2\colon &\quad \big[\, v = \varphi(x, y, u), \\
M^4 \subset \mathbb{C}^3\colon &\quad \left[\begin{aligned} v_1 &= \varphi_1(x, y, u_1, u_2), \\ v_2 &= \varphi_2(x, y, u_1, u_2), \end{aligned}\right. \\
M^5 \subset \mathbb{C}^4\colon &\quad \left[\begin{aligned} v_1 &= \varphi_1(x, y, u_1, u_2, u_3), \\ v_2 &= \varphi_2(x, y, u_1, u_2, u_3), \\ v_3 &= \varphi_3(x, y, u_1, u_2, u_3), \end{aligned}\right. \\
M^5 \subset \mathbb{C}^3\colon &\quad \big[\, v = \varphi(x_1, y_1, x_2, y_2, u).
\end{aligned}
$$

Before proceeding further, answer (partly) Webster's quote.

## 2.1 Explicit characterization of sphericity

Consider for instance a hypersurface $M^3 \subset \mathbb{C}^2$. As its graphing function $\varphi$ is real analytic, $w$ can be (locally) solves ([51, 43]):

$$w = \Theta\big(z, \overline{z}, \overline{w}\big).$$

Letting the 'round' unit 3-sphere $S^3 \subset \mathbb{C}^2$ be:

$$1 = z\overline{z} + w\overline{w} = x^2 + y^2 + u^2 + v^2,$$

a Cayley transform ([43]) maps $S^3 \backslash \{p_\infty\}$ with $p_\infty := (0, -1)$ biholomorphically onto the *Heisenberg sphere*:

$$w = \overline{w} + 2\sqrt{-1}\, z\overline{z}.$$

An intrinsic local generator for the fundamental subbundle:

$$T^{1,0}M := \big\{X - \sqrt{-1}\, J(X) \colon X \in TM \cap J(TM)\big\}$$

of $TM \otimes_{\mathbb{R}} \mathbb{C}$ is:

$$L = \frac{\partial}{\partial z}.$$

Also, an intrinsic generator for $T^{0,1}M := \overline{T^{1,0}M}$ is:

$$\overline{L} := \frac{\partial}{\partial \overline{z}} - \frac{\Theta_{\overline{z}}(z, \overline{z}, \overline{w})}{\Theta_{\overline{w}}(z, \overline{z}, \overline{w})} \frac{\partial}{\partial \overline{w}}.$$

In the Lie bracket:

$$\big[L, \overline{L}\big] = \left[\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} - \frac{\Theta_{\overline{z}}}{\Theta_{\overline{w}}} \frac{\partial}{\partial \overline{w}}\right] = \left(\frac{-\Theta_{\overline{w}}\Theta_{z\overline{z}} + \Theta_{\overline{z}}\Theta_{z\overline{w}}}{\Theta_{\overline{w}}\Theta_{\overline{w}}}\right) \frac{\partial}{\partial \overline{w}},$$

an explicit *Levi factor in coordinates* appears:

$$\boldsymbol{\frac{-\Theta_{\overline{w}}\Theta_{z\overline{z}} + \Theta_{\overline{z}}\Theta_{z\overline{w}}}{\Theta_{\overline{w}}\Theta_{\overline{w}}}}.$$

The assumption that $M^3 \subset \mathbb{C}^2$ is smooth reads:

$$0 \neq \boldsymbol{\Theta_{\overline{w}}} \text{ vanishes at no point.}$$

The assumption that $M$ is Levi nondegenerate reads:

$$0 \neq \boldsymbol{-\Theta_{\overline{w}}\Theta_{z\overline{z}} + \Theta_{\overline{z}}\Theta_{z\overline{w}}} \text{ also vanishes at no point.}$$

**General principle.** *Various geometric assumptions enter computations in denominator places.*

Here is a first illustration.

**Theorem 1.** ([43]) *An arbitrary real analytic hypersurface $M^3 \subset \mathbb{C}^2$ which is Levi nondegenerate:*

$$w = \Theta\big(z, \bar z, \bar w\big),$$

*is locally biholomorphically equivalent to the Heisenberg sphere if and only if:*

$$0 \equiv \left( \frac{-\Theta_{\bar w}}{\Theta_{\bar z}\Theta_{z\bar w} - \Theta_{\bar w}\Theta_{z\bar z}} \frac{\partial}{\partial \bar z} + \frac{\Theta_{\bar z}}{\Theta_{\bar z}\Theta_{z\bar w} - \Theta_{\bar w}\Theta_{z\bar z}} \frac{\partial}{\partial \bar w} \right)^2 \big[\mathsf{AJ}^4(\Theta)\big]$$

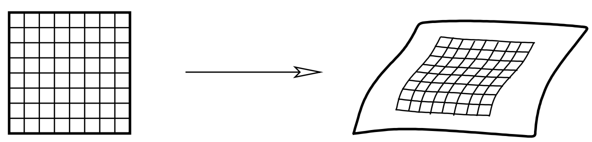
*identically in $\mathbb{C}\{z,\bar z,\bar w\}$, where:*

$$\begin{aligned}
\mathsf{AJ}^4(\Theta) := \frac{1}{[\boldsymbol{\Theta_{\bar z}\Theta_{z\bar w} - \Theta_{\bar w}\Theta_{z\bar z}}]^3} \bigg\{ & \Theta_{zz\bar z\bar z}\left(\Theta_{\bar w}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar w}\\ \Theta_{z\bar z} & \Theta_{z\bar w}\end{vmatrix}\right) - \\
& - 2\Theta_{zz\bar z\bar w}\left(\Theta_{\bar z}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar w}\\ \Theta_{z\bar z} & \Theta_{z\bar w}\end{vmatrix}\right) + \Theta_{zz\bar w\bar w}\left(\Theta_{\bar z}\Theta_{\bar z}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar w}\\ \Theta_{z\bar z} & \Theta_{z\bar w}\end{vmatrix}\right) + \\
& + \Theta_{zz\bar z}\left(\Theta_{\bar z}\Theta_{\bar z}\begin{vmatrix}\Theta_{\bar w} & \Theta_{\bar w\bar w}\\ \Theta_{z\bar w} & \Theta_{z\bar w\bar w}\end{vmatrix} - 2\Theta_{\bar z}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar w} & \Theta_{\bar z\bar w}\\ \Theta_{z\bar w} & \Theta_{z\bar z\bar w}\end{vmatrix} + \Theta_{\bar w}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar w} & \Theta_{\bar z\bar z}\\ \Theta_{z\bar w} & \Theta_{z\bar z\bar z}\end{vmatrix}\right) + \\
& + \Theta_{zz\bar w}\left(-\Theta_{\bar z}\Theta_{\bar z}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar w\bar w}\\ \Theta_{z\bar z} & \Theta_{z\bar w\bar w}\end{vmatrix} + 2\Theta_{\bar z}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar z\bar w}\\ \Theta_{z\bar z} & \Theta_{z\bar z\bar w}\end{vmatrix} - \Theta_{\bar w}\Theta_{\bar w}\begin{vmatrix}\Theta_{\bar z} & \Theta_{\bar z\bar z}\\ \Theta_{z\bar z} & \Theta_{z\bar z\bar z}\end{vmatrix}\right)\bigg\}.
\end{aligned}$$

In fact, $\boldsymbol{\Theta_{\bar w}}$ *also* enters denominator, but erases in the equation '$= 0$'.

## 2.2 Theorema Egregium of Gauss

Briefly, here is a second illustration. On an embedded surface $S^2 \subset \mathbb{R}^3 \ni (x,y,z)$, consider local curvilinear bidimensional coordinates $(u,v)$:



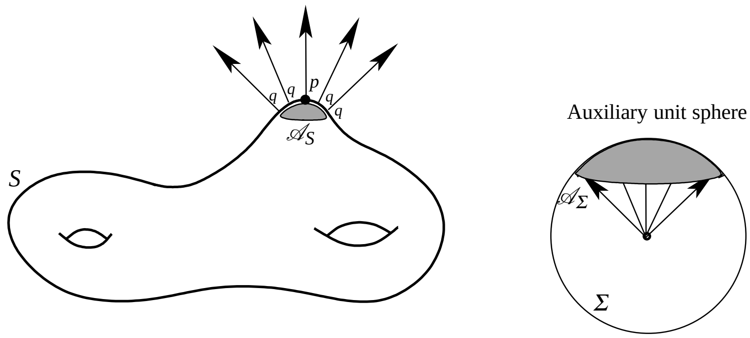
through parametric equations

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v).$$

The flat Pythagorean metric $dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$ induces on $S^2$:

$$ds^2 = \big\|(du,dv)\big\|^2 = E\,du^2 + 2F\,dudv + G\,dv^2,$$

with: $E := x_u^2 + y_u^2 + z_u^2, \qquad F := x_u x_v + y_u y_v + z_u z_v, \qquad G := x_v^2 + y_v^2 + z_v^2.$




The *Gaussian curvature* of $S$ at one of its points $p$ is:

$$\mathsf{Curvature}(p) := \lim_{\mathscr{A}_S \to p} \frac{\text{area of the region } \mathscr{A}_\Sigma \text{ on the auxiliary unit sphere}}{\text{area of the region } \mathscr{A}_S \text{ on the surface}}.$$

When $S$ is graphed as $z = \varphi(x,y)$, a first formula is:

$$\mathsf{Curvature} = \frac{\varphi_{xx}\,\varphi_{yy} - \varphi_{xy}\,\varphi_{xy}}{1 + \varphi_x^2 + \varphi_y^2}.$$

A splendid computation by Gauss provided its intrinsic meaning:

$$\begin{aligned}\mathsf{Curvature} = \frac{1}{\boldsymbol{(EG-F^2)^2}} \bigg\{ & E\left[\frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u}\right)^2\right] + \\ & + F\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial v}\cdot\frac{\partial F}{\partial v} + 4\,\frac{\partial F}{\partial u}\cdot\frac{\partial F}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial u}\right] + \\ & + G\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial u}\cdot\frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v}\right)^2\right] - \\ & - 2\,\big(EG-F^2\big)\left[\frac{\partial^2 E}{\partial v^2} - 2\,\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 G}{\partial u^2}\right]\bigg\},\end{aligned}$$

and, in the denominator appears $E\,G - F^2$ which is $> 0$ in any metric.

## 2.3 Propagation of sphericity across Levi degenerate points

Here is one application of *explicit rational expressions* as above. Developing Pinchuk's techniques of extension along Segre varieties ([61]), Kossovskiy-Shafikov ([33]) showed that local biholomorphic equivalence to a model $(k, n-k)$*-pseudo-sphere:*

$$w = \overline{w} + 2i\big(-z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n\big),$$

propagates on any connected real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ which is Levi nondegenerate outside some $n$-dimensional complex hypersurface $\Sigma \subset M$. A more general statement, not known with Segre varieties techniques, is:

**Theorem 2.** ([45]) *If a connected $\mathscr{C}^\omega$ hypersurface $M \subset \mathbb{C}^{n+1}$ is locally biholomorphic, in a neighborhood of one of its points $p$, to some $(k, n-k)$-pseudo-sphere, then locally at every other Levi nondegenerate point $q \in M \backslash \Sigma_{\mathsf{LD}}$, this hypersurface $M$ is also locally biholomorphic to some Heisenberg $(l, n-l)$-pseudo-sphere, with ([33]), possibly $l \neq k$.*

The proof, suggested by Beloshapka, consists first for $n = 1$ in observing that after expansion, Theorem 1 characterizes sphericity as:

$$0 \equiv \frac{\mathsf{polynomial}\big(\big(\Theta_{z^j \overline{z}^k \overline{w}^l}\big)_{1 \leqslant j+k+l \leqslant 6}\big)}{\big[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}\big]^7},$$

at every Levi nondegenerate point $(z_p, w_p) \in M$ at which the denominator is $\neq 0$. But this means that the numerator is $\equiv 0$ near $(z_p, \overline{z}_p, \overline{w}_p)$, and at every other Levi nondegenerate point $(z_q, w_q) \in M$ close to $(z_p, w_p)$, the numerator is also locally $\equiv 0$ by analytic continuation. Small translations of coordinates are needed; the complete arguments appear in [45].

In dimensions $n \geqslant 2$, the explicit characterization of $(k, n-k)$-pseudo-sphericity is also *rational*. Indeed, in local holomorphic coordinates:

$$t = (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

represent similarly a $\mathscr{C}^\omega$ hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ as:

$$w = \Theta\big(z, \overline{z}, \overline{w}\big) = \Theta\big(z, \overline{t}\big).$$

Introduce the Levi form Jacobian-like determinant:

$$\Delta := \begin{vmatrix} \Theta_{\overline{z}_1} & \cdots & \Theta_{\overline{z}_n} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \cdots & \Theta_{z_1\overline{z}_n} & \Theta_{z_1\overline{w}} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \Theta_{z_n\overline{z}_1} & \cdots & \Theta_{z_n\overline{z}_n} & \Theta_{z_n\overline{w}} \end{vmatrix} = \begin{vmatrix} \Theta_{\overline{t}_1} & \cdots & \Theta_{\overline{t}_n} & \Theta_{\overline{t}_{n+1}} \\ \Theta_{z_1\overline{t}_1} & \cdots & \Theta_{z_1\overline{t}_n} & \Theta_{z_1\overline{t}_{n+1}} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \Theta_{z_n\overline{t}_1} & \cdots & \Theta_{z_n\overline{t}_n} & \Theta_{z_n\overline{t}_{n+1}} \end{vmatrix}.$$

It is nonzero at a point $p = (z_p, \overline{t}_p)$ if and only if $M$ is Levi nondegenerate at $p$. For any index $\mu \in \{1, \dots, n, n+1\}$ and for any index $\ell \in \{1, \dots, n\}$, let also $\Delta^{\mu}_{[0_{1+\ell}]}$ denote the same determinant, but with its $\mu$-th column replaced by the transpose of the line $(0 \cdots 1 \cdots 0)$ with 1 at the $(1+\ell)$-th place, and 0 elsewhere, its other columns being untouched. Similarly, for any indices $\mu, \nu, \tau \in \{1, \dots, n, n+1\}$, denote by $\Delta^{\tau}_{[\overline{t}^\mu \overline{t}^\nu]}$ the same determinant as $\Delta$, but with only its $\tau$-th column replaced by the transpose of the line:

$$\big(\Theta_{\overline{t}^\mu \overline{t}^\nu} \quad \Theta_{z_1 \overline{t}^\mu \overline{t}^\nu} \quad \cdots \quad \Theta_{z_n \overline{t}^\mu \overline{t}^\nu}\big),$$

other columns being again untouched. All these determinants $\Delta$, $\Delta^{\mu}_{[0_{1+\ell}]}$, $\Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]}$ depend upon the third-order jet $J^3_{z,\bar{z},\overline{w}}\Theta$.

**Theorem 3.** ([40, 45]) *A $\mathscr{C}^{\omega}$ hypersurface $M \subset \mathbb{C}^{n+1}$ with $n \geqslant 2$ which is Levi nondegenerate at some point $p = (z_p, \bar{z}_p, \overline{w}_p)$ is $(k, n-k)$-pseudo-spherical at $p$ if and only if, identically for $(z, \bar{z}, \overline{w})$ near $(z_p, \bar{z}_p, \overline{w}_p)$:*

$$
\begin{aligned}
0 \equiv \frac{1}{\Delta^3} \Bigg[ \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \Bigg[ & \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}^{\tau}} \Bigg\} - \\
& - \frac{\delta_{k_1,\ell_1}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}^{\tau}} \Bigg\} - \\
& - \frac{\delta_{k_1,\ell_2}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell'}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}^{\tau}} \Bigg\} - \\
& - \frac{\delta_{k_2,\ell_1}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}^{\tau}} \Bigg\} - \\
& - \frac{\delta_{k_2,\ell_2}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell'}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}^{\tau}} \Bigg\} + \\
& + \frac{1}{(n+1)(n+2)} \cdot \big[ \delta_{k_1,\ell_1} \delta_{k_2,\ell_2} + \delta_{k_2,\ell_1} \delta_{k_1,\ell_2} \big] \cdot \\
& \cdot \sum_{\ell'=1}^{n} \sum_{\ell''=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell''}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^{\mu}\bar{t}^{\nu}]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar{t}^{\tau}} \Bigg\} \Bigg], \\
& \qquad (1 \leqslant k_1, k_2 \leqslant n;\ 1 \leqslant \ell_1, \ell_2 \leqslant n).
\end{aligned}
$$

Then as in the case $n = 1$, propagation of pseudo-sphericity 'jumps' across Levi degenerate points, because above, the denominator $\Delta^3$ locates Levi nondegenerate points. This explicit expression is a translation of Hachtroudi's characterization ([29]) of equivalence to $w'_{z'_{k_1} z'_{k_2}}(z') = 0$ of completely integrable PDE systems:

$$
w_{z_{k_1} z_{k_2}}(z) = \Phi_{k_1,k_2}\big(z, w(z), w_{z_1}(z), \dots, w_{z_n}(z)\big) \qquad (1 \leqslant k_1, k_2 \leqslant n).
$$

**Question still open.** *Compute explicity the Chern-Moser-Webster 1-forms and curvatures ([10, 72]) in terms of a local graphing function for a Levi nondegenerate $M^{2n+1} \subset \mathbb{C}^{n+1}$ (rigid and tube cases are treated in [30]).*

This would, in particular, provide an alternative proof of Theorem 3.

## 2.4 Zariski-generic $\mathscr{C}^{\omega}$ CR manifolds of dimension $\leqslant 5$

Coming back to $M^{2n+c} \subset \mathbb{C}^{n+c}$ of dimension $2n + c \leqslant 5$, and calling *Zariski-open* any complement $M \backslash \Sigma$ of some *proper* real analytic subset $\Sigma \subsetneq M$, treat at first the:

**Problem 2.** (Accessible subquestion of Problem 1) *Set up all possible initial geometries of connected $\mathscr{C}^{\omega}$ CR-generic submanifolds $M^{2n+c} \subset \mathbb{C}^{n+c}$ at Zariski-generic points.*

For general $M^{2n+c} \subset \mathbb{C}^{n+c}$, recall that the fundamental invariant bundle is:

$$T^{1,0}M := \big\{X - \sqrt{-1}\,J(X) \colon X \in TM \cap J(TM)\big\}.$$

**Lemma 1.** ([46]) *If a CR-generic $M^{2n+c} \subset \mathbb{C}^{n+c}$ is locally graphed as:*

$$v_j = \varphi_j\big(x_1,\dots,x_n,y_1,\dots,y_n,u_1,\dots,u_c\big) \qquad\qquad {\scriptstyle (1 \leqslant j \leqslant c)},$$

*a local frame $\{\mathscr{L}_1,\dots,\mathscr{L}_n\}$ for $T^{1,0}M$ consists of the $n$ vector fields:*

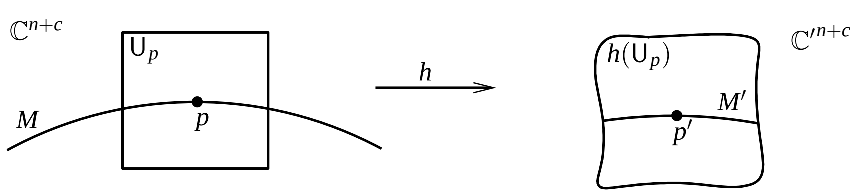
$$\mathscr{L}_i = \frac{\partial}{\partial z_k} + A_i^1\big(x_\bullet,y_\bullet,u_\bullet\big)\,\frac{\partial}{\partial u_1} + \cdots\cdots + A_i^c\big(x_\bullet,y_\bullet,u_\bullet\big)\,\frac{\partial}{\partial u_c} \qquad\qquad {\scriptstyle (1 \leqslant i \leqslant n)},$$

*having* rational *coefficient-functions:*

$$A_i^1 = \frac{\begin{vmatrix} -\varphi_{1,z_i} & \varphi_{1,u_2} & \cdots & \varphi_{1,u_c} \\ -\varphi_{2,z_i} & \sqrt{-1}+\varphi_{2,u_2} & \cdots & \varphi_{2,u_c} \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_{c,z_i} & \varphi_{c,u_2} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}, \;\dots\dots\dots,\; A_i^c = \frac{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & -\varphi_{1,z_i} \\ \varphi_{2,u_1} & \cdots & -\varphi_{2,z_i} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & -\varphi_{c,z_i} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}.$$

Nonvanishing of the denominator is equivalent to CR-genericity of $M$.

Here is how $M$ transfers through local biholomorphisms.




**Lemma 2.** ([46]) *Given a connected $\mathscr{C}^\omega$ CR-generic submanifold $M^{2n+c} \subset \mathbb{C}^{n+c}$ and a local biholomorphism between open subsets:*

$$h\colon \quad \mathsf{U}_p \xrightarrow{\ \sim\ } h(\mathsf{U}_p) = \mathsf{U}'_{p'} \subset \mathbb{C}'^{n+c},$$

*with $p \in M$, $p' = h(p)$, setting:*

$$M' := h(M) \subset \mathbb{C}'^{n+c} \qquad\qquad {\scriptstyle (c = \mathsf{codim}\,M',\ n = \mathsf{CRdim}\,M')},$$

*then for any two local frames:*

$$\big\{\mathscr{L}_1,\dots,\mathscr{L}_n\big\} \quad \textit{for } T^{1,0}M \qquad\qquad \textit{and} \qquad\qquad \big\{\mathscr{L}'_1,\dots,\mathscr{L}'_n\big\} \quad \textit{for } T^{1,0}M',$$

*there exist uniquely defined $\mathscr{C}^\omega$ local coefficient-functions:*

$$a'_{i_1 i_2} \colon \quad M' \longrightarrow \mathbb{C} \qquad\qquad {\scriptstyle (1 \leqslant i_1, i_2 \leqslant n)},$$

*satisfying:*

$$\begin{aligned} h_*(\mathscr{L}_1) &= a'_{11}\,\mathscr{L}'_1 + \cdots + a'_{n1}\,\mathscr{L}'_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ h_*(\mathscr{L}_n) &= a'_{1n}\,\mathscr{L}'_1 + \cdots + a'_{nn}\,\mathscr{L}'_n. \end{aligned}$$

**Definition 1.** Taking any local 1-form $\rho_0 \colon TM \to \mathbb{R}$ whose extension to $\mathbb{C} \otimes_{\mathbb{R}} TM$ satisfies:

$$T^{1,0}M \oplus T^{0,1}M = \big\{\rho_0 = 0\big\},$$

the Hermitian matrix of the *Levi form* of $M$ at various points $p \in M$ is:

$$\begin{pmatrix} \rho_0\big(\sqrt{-1}\,[\mathscr{L}_1, \overline{\mathscr{L}}_1]\big) & \cdots & \rho_0\big(\sqrt{-1}\,[\mathscr{L}_n, \overline{\mathscr{L}}_1]\big) \\ \vdots & \ddots & \vdots \\ \rho_0\big(\sqrt{-1}\,[\mathscr{L}_1, \overline{\mathscr{L}}_n]\big) & \cdots & \rho_0\big(\sqrt{-1}\,[\mathscr{L}_n, \overline{\mathscr{L}}_n]\big) \end{pmatrix}(p),$$

the extra factor $\sqrt{-1}$ being present in order to counterbalance the change of sign:

$$\overline{[\mathscr{L}_j, \overline{\mathscr{L}}_k]} = -\,[\mathscr{L}_k, \overline{\mathscr{L}}_j].$$

As an application, show the invariance of Levi nondegeneracy. For $M^3 \subset \mathbb{C}^2$ equivalent to $M'^3 \subset \mathbb{C}'^2$, whence $n = n' = 1$, introduce local vector field generators:

$$\mathscr{L} \ \text{ for } \ T^{1,0}M \qquad\qquad \text{and} \qquad\qquad \mathscr{L}' \ \text{ for } \ T^{1,0}M'.$$

**Lemma 3.** At every point $q \in M$ near $p$:

$$\begin{gathered} \mathbf{3} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}\big|_q,\ \overline{\mathscr{L}}\big|_q,\ [\mathscr{L}, \overline{\mathscr{L}}]\big|_q\Big) \\ \Updownarrow \\ \mathbf{3} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}'\big|_{h(q)},\ \overline{\mathscr{L}}'\big|_{h(q)},\ [\mathscr{L}', \overline{\mathscr{L}}']\big|_{h(q)}\Big). \end{gathered}$$

*Proof.* By what precedes, there exists a function $a' \colon M' \longrightarrow \mathbb{C}\backslash\{0\}$ with:

$$h_*(\mathscr{L}) = a'\,\mathscr{L}' \qquad\qquad \text{and} \qquad\qquad h_*\big(\overline{\mathscr{L}}\big) = \overline{a}'\,\overline{\mathscr{L}}'.$$

Consequently:

$$\begin{aligned} h_*\big([\mathscr{L}, \overline{\mathscr{L}}]\big) &= \big[h_*(\mathscr{L}), h_*(\overline{\mathscr{L}})\big] \\ &= \big[a'\mathscr{L}', \overline{a}'\overline{\mathscr{L}}'\big] \\ &= a'\overline{a}'\,\big[\mathscr{L}', \overline{\mathscr{L}}'\big] + a'\,\mathscr{L}'(\overline{a}')\cdot\overline{\mathscr{L}}' - \overline{a}'\,\overline{\mathscr{L}}'(a')\cdot\mathscr{L}'. \end{aligned}$$

Dropping the mention of $h_*$, because the *change of frame* matrix:

$$\begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ \big[\mathscr{L},\overline{\mathscr{L}}\big] \end{pmatrix} = \begin{pmatrix} a' & 0 & 0 \\ 0 & \overline{a}' & 0 \\ * & * & a'\overline{a}' \end{pmatrix} \begin{pmatrix} \mathscr{L}' \\ \overline{\mathscr{L}}' \\ \big[\mathscr{L}',\overline{\mathscr{L}}'\big] \end{pmatrix}$$

is visibly of rank 3, the result follows. □

Since $T^{1,0}M$ and $T^{0,1}M$ are Frobenius-involutive ([46]), only iterated Lie brackets between $T^{1,0}M$ and $T^{0,1}M$ are nontrivial. For a $\mathscr{C}^\omega$ connected $M^{2n+c} \subset \mathbb{C}^{n+c}$ with $T^{1,0}M$ having local generators $\mathscr{L}_1, \dots, \mathscr{L}_n$, set:

$$\mathbb{L}^1_{\mathscr{L},\overline{\mathscr{L}}} := \mathscr{C}^\omega\text{-linear combinations of } \mathscr{L}_1, \dots, \mathscr{L}_n,\ \overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbb{L}^{\nu+1}_{\mathscr{L},\overline{\mathscr{L}}} := \mathscr{C}^\omega\text{-linear combinations of vector fields } \mathscr{M}^\nu \in \mathbb{L}^\nu_{\mathscr{L},\overline{\mathscr{L}}}$$
$$\text{and of brackets } \big[\mathscr{L}_k, \mathscr{M}^\nu\big],\ \big[\overline{\mathscr{L}}_k, \mathscr{M}^\nu\big].$$

Set:

$$\mathbb{L}^{\mathsf{Lie}}_{\mathscr{L},\overline{\mathscr{L}}} := \bigcup_{\nu \geqslant 1} \mathbb{L}^\nu_{\mathscr{L},\overline{\mathscr{L}}}.$$

By real analyticity ([51]), there exists an integer $c_M$ with $0 \leqslant c_M \leqslant c$ and a proper real analytic subset $\Sigma \subsetneqq M$ such that at every point $q \in M \backslash \Sigma$:

$$\mathsf{dim}_{\mathbb{C}}\Big(\mathbb{L}^{\mathsf{Lie}}_{\mathscr{L},\overline{\mathscr{L}}}(q)\Big) = 2n + c_M.$$

**Theorem 4.** (Known, [51]) *Every point $q \in M \backslash \Sigma$ has a small open neighborhood $\mathsf{U}_q \subset \mathbb{C}^{n+c}$ in which:*

$$M^{2n+c} \cong \underline{M}^{2n+c}$$

*biholomorphically, with a CR and $\mathscr{C}^\omega$:*

$$\underline{M}^{2n+c} \subset \mathbb{C}^{n+c_M} \times \mathbb{R}^{c-c_M}.$$

The case $c_M \leqslant c-1$ must hence be considered as *degeneration*, excluded in classification of initial geometries at Zariski-generic points.

A well known fact is that, at a Zariski-generic point, a connected $\mathscr{C}^\omega$ hypersurface $M^3 \subset \mathbb{C}^2$ is either $\cong \mathbb{C} \times \mathbb{R}$ or is Levi nondegenerate:

$$3 = \mathsf{rank}_{\mathbb{C}}\big(T^{1,0}M,\ T^{0,1}M,\ \big[T^{1,0}M,\ T^{0,1}M\big]\big).$$

**Theorem 5.** ([46]) *Excluding degenerate CR manifolds, there are precisely* **six** *general classes of nondegenerate connected $M^{2n+c} \subset \mathbb{C}^{n+c}$ having dimension:*

$$2n + c \leqslant \mathbf{5},$$

*hence having CR dimension $n = 1$ or $n = 2$, namely if:*

$$\big\{\mathscr{L}\big\} \qquad \text{or} \qquad \big\{\mathscr{L}_1, \mathscr{L}_2\big\},$$

*denotes any local frame for* $T^{1,0}M$*:*

• **General Class** $\mathsf{I}$**:** *Hypersurfaces* $M^3 \subset \mathbb{C}^2$ *with* $\big\{\mathscr{L}, \overline{\mathscr{L}}, \, [\mathscr{L}, \overline{\mathscr{L}}]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, with:*

$$\text{Model I:} \qquad v = z\bar{z},$$

• **General Class** $\mathsf{II}$**:** *CR-generic* $M^4 \subset \mathbb{C}^3$ *with* $\big\{\mathscr{L}, \overline{\mathscr{L}}, \, [\mathscr{L}, \overline{\mathscr{L}}], \, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, with:*

$$\text{Model II:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2,$$

• **General Class** $\mathsf{III}_1$**:** *CR-generic* $M^5 \subset \mathbb{C}^4$ *with* $\big\{\mathscr{L}, \overline{\mathscr{L}}, \, [\mathscr{L}, \overline{\mathscr{L}}], \, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \, [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, with:*

$$\text{Model III}_1\text{:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2, \qquad v_3 = \sqrt{-1}\,\big(z^2\bar{z} - z\bar{z}^2\big),$$

• **General Class** $\mathsf{III}_2$**:** *CR generic* $M^5 \subset \mathbb{C}^4$ *with* $\big\{\mathscr{L}, \overline{\mathscr{L}}, \, [\mathscr{L}, \overline{\mathscr{L}}], \, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \, [\mathscr{L}, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, while* $\mathbf{4} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}, \overline{\mathscr{L}}, \, [\mathscr{L}, \overline{\mathscr{L}}], \, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], \, [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\Big)$*, with:*

$$\text{Model III}_2\text{:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2, \qquad v_3 = 2\,z^3\bar{z} + 2\,z\bar{z}^3 + 3\,z^2\bar{z}^2,$$

• **General Class** $\mathsf{IV}_1$**:** *Hypersurfaces* $M^5 \subset \mathbb{C}^3$ *with* $\big\{\mathscr{L}_1, \mathscr{L}_2, \overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2, \, [\mathscr{L}_1, \overline{\mathscr{L}}_1]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, and with the Levi-Form of* $M$ *being of rank* 2 *at every point* $p \in M$*, with:*

$$\text{Model(s) IV}_1\text{:} \qquad v = z_1\bar{z}_1 \pm z_2\bar{z}_2,$$

• **General Class** $\mathsf{IV}_2$**:** *Hypersurfaces* $M^5 \subset \mathbb{C}^3$ *with* $\big\{\mathscr{L}_1, \mathscr{L}_2, \overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2, \, [\mathscr{L}_1, \overline{\mathscr{L}}_1]\big\}$ *constituting a frame for* $\mathbb{C} \otimes_{\mathbb{R}} TM$*, with the Levi-Form being of rank* 1 *at every point* $p \in M$ *while the Freeman-Form (defined below) is nondegenerate at every point, with:*

$$\text{Model IV}_2\text{:} \qquad v = \frac{z_1\bar{z}_1 + \frac{1}{2}\,z_1 z_1 \bar{z}_2 + \frac{1}{2}\,z_2\bar{z}_1\bar{z}_1}{1 - z_2\bar{z}_2}.$$

The models $\mathsf{II}$, $\mathsf{III}_1$ appear in the works of Beloshapka ([2, 3]) which exhibit a wealth of higher dimensional models widening the biholomorphic equivalence problem. Before proceeding, explain (only) how the General Classes $\mathsf{IV}_1$ and $\mathsf{IV}_2$ occur.

## *2.5 Concept of Freeman form*

If $M^5 \subset \mathbb{C}^3$ is connected $\mathscr{C}^\omega$ and belongs to Class $\mathsf{IV}_2$, so that:

$$1 = \mathsf{rank}_{\mathbb{C}}\big(\mathsf{Levi\text{-}Form}^M(p)\big) \qquad\qquad {\scriptstyle (\forall\, p \in M)},$$

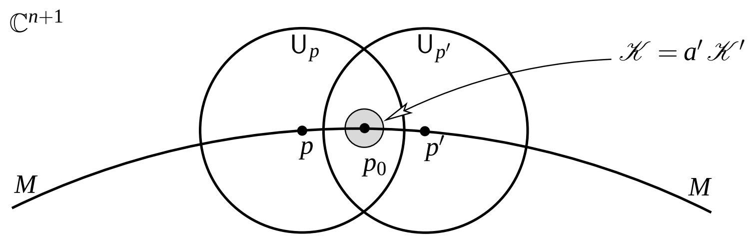
then there exists a unique rank 1 complex vector subbundle:

$$K^{1,0}M \subset T^{1,0}M$$

such that, at every point $p \in M$:

$$K^{1,0}_p M \ni \mathscr{K}_p \iff \mathscr{K}_p \in \mathsf{Kernel}\big(\mathsf{Levi\text{-}Form}^M(p)\big).$$

Local trivializations of $K^{1,0}M$ match up on intersections of balls ([46]).




Furthermore, three known involutiveness conditions hold ([46]):

$$\begin{aligned} \big[K^{1,0}M, K^{1,0}M\big] &\subset K^{1,0}M, \\ \big[K^{0,1}M, K^{0,1}M\big] &\subset K^{0,1}M, \\ \big[K^{1,0}M, K^{0,1}M\big] &\subset K^{1,0}M \oplus K^{0,1}M. \end{aligned} \tag{1}$$

In local coordinates, $M^5 \subset \mathbb{C}^3$ is graphed as:

$$v = \varphi\big(x_1, x_2, y_1, y_2, u\big),$$

and two local generators of $T^{1,0}M$ with their conjugates are:

$$\begin{aligned} \mathscr{L}_1 &= \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial u}, & \overline{\mathscr{L}}_1 &= \frac{\partial}{\partial \overline{z}_1} + \overline{A_1} \frac{\partial}{\partial u}, \\ \mathscr{L}_2 &= \frac{\partial}{\partial z_2} + A_2 \frac{\partial}{\partial u}, & \overline{\mathscr{L}}_2 &= \frac{\partial}{\partial \overline{z}_2} + \overline{A_2} \frac{\partial}{\partial u}, \end{aligned}$$

with:

$$\begin{aligned} A_1 &:= -\frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u}, \\ A_2 &:= -\frac{\varphi_{z_2}}{\sqrt{-1} + \varphi_u}. \end{aligned}$$

Taking as a 1-form $\rho_0$ with $\{\rho_0 = 0\} = T^{1,0}M \oplus T^{0,1}M$:

$$\rho_0 := -A_1\, dz_1 - A_2\, dz_2 - \overline{A_1}\, d\overline{z}_1 - \overline{A_2}\, d\overline{z}_2 + du,$$

the top-left entry of the:

$$\text{Levi-Matrix} = \begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_2\big]\big) \end{pmatrix}$$

$$(2) \qquad = \begin{pmatrix} \sqrt{-1}\big(\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)\big) & \sqrt{-1}\big(\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)\big) \\ \sqrt{-1}\big(\mathscr{L}_1\big(\overline{A_2}\big) - \overline{\mathscr{L}}_2\big(A_1\big)\big) & \sqrt{-1}\big(\mathscr{L}_2\big(\overline{A_2}\big) - \overline{\mathscr{L}}_2\big(A_2\big)\big) \end{pmatrix},$$

expresses as:

$$\sqrt{-1}\,\big(\mathscr{L}_1(\overline{A_1}) - \overline{\mathscr{L}}_1(A_1)\big) = \frac{1}{(\sqrt{-1}+\varphi_u)^2(-\sqrt{-1}+\varphi_u)^2}\Big\{2\,\varphi_{z_1\bar{z}_1} + 2\,\varphi_{z_1\bar{z}_1}\varphi_u\varphi_u - 2\sqrt{-1}\,\varphi_{\bar{z}_1}\varphi_{z_1u} - 2\,\varphi_{\bar{z}_1}\varphi_{z_1u}\varphi_u + 2\sqrt{-1}\,\varphi_{z_1}\varphi_{\bar{z}_1u} + 2\,\varphi_{z_1}\varphi_{\bar{z}_1}\varphi_{uu} - 2\,\varphi_{z_1}\varphi_{\bar{z}_1u}\varphi_u\Big\},$$

with quite similar expressions for the remaining three entries. As $M$ belongs to Class $\mathsf{IV}_2$:

$$0 \equiv \det\begin{pmatrix} \sqrt{-1}\big(\overline{A_1}_{z_1} + A_1\overline{A_1}_u - A_{1\overline{z_1}} - \overline{A_1}A_{1u}\big) & \sqrt{-1}\big(\overline{A_1}_{z_2} + A_2\overline{A_1}_u - A_{2\overline{z_1}} - \overline{A_1}A_{2u}\big) \\ \sqrt{-1}\big(\overline{A_2}_{z_1} + A_1\overline{A_2}_u - A_{1\overline{z_2}} - \overline{A_2}A_{1u}\big) & \sqrt{-1}\big(\overline{A_2}_{z_2} + A_2\overline{A_2}_u - A_{2\overline{z_2}} - \overline{A_2}A_{2u}\big) \end{pmatrix},$$

that is to say in terms of $\varphi$:

$$(3) \qquad \begin{aligned} 0 \equiv \frac{4}{(\sqrt{-1}+\varphi_u)^3(-\sqrt{-1}+\varphi_u)^3}\Big\{ & \varphi_{z_2\bar{z}_2}\varphi_{z_1\bar{z}_1} - \varphi_{z_2\bar{z}_1}\varphi_{z_1\bar{z}_2} + \\ & + \varphi_{z_2\bar{z}_1}\varphi_{\bar{z}_2}\varphi_{z_1u}\varphi_u - \varphi_{z_2\bar{z}_1}\varphi_{\bar{z}_2}\varphi_{z_1}\varphi_{uu} - \varphi_{\bar{z}_1}\varphi_{z_2u}\varphi_{z_1}\varphi_{\bar{z}_2u} + \varphi_{\bar{z}_1}\varphi_{z_2u}\varphi_u\varphi_{z_1\bar{z}_2} - \\ & - \varphi_{z_2}\varphi_{\bar{z}_1u}\varphi_{\bar{z}_2}\varphi_{z_1u} - \varphi_{z_2}\varphi_{\bar{z}_1}\varphi_{uu}\varphi_{z_1\bar{z}_2} + \varphi_{z_2}\varphi_{\bar{z}_1u}\varphi_u\varphi_{z_1\bar{z}_2} - \varphi_{z_2\bar{z}_2}\varphi_{\bar{z}_1}\varphi_{z_1u}\varphi_u + \\ & + \varphi_{z_2\bar{z}_2}\varphi_{z_1}\varphi_{\bar{z}_1}\varphi_{uu} - \varphi_{z_2\bar{z}_2}\varphi_{z_1}\varphi_{\bar{z}_1u}\varphi_u + \varphi_{z_2\bar{z}_1}\varphi_{z_1}\varphi_{\bar{z}_2u}\varphi_u + \varphi_{z_2}\varphi_{\bar{z}_2u}\varphi_{\bar{z}_1}\varphi_{z_1u} - \\ & - \varphi_{z_2}\varphi_{\bar{z}_2u}\varphi_{z_1\bar{z}_1}\varphi_u + \varphi_{\bar{z}_2}\varphi_{z_2u}\varphi_{z_1}\varphi_{\bar{z}_1u} - \varphi_{\bar{z}_2}\varphi_{z_2u}\varphi_u\varphi_{z_1\bar{z}_1} + \varphi_{\bar{z}_2}\varphi_{z_2}\varphi_{uu}\varphi_{z_1\bar{z}_1} + \\ & + \sqrt{-1}\big(\varphi_{z_2\bar{z}_2}\varphi_{z_1}\varphi_{\bar{z}_1u} + \varphi_{\bar{z}_1}\varphi_{z_2u}\varphi_{z_1\bar{z}_2} + \varphi_{z_2\bar{z}_1}\varphi_{\bar{z}_2}\varphi_{z_1u} + \varphi_{z_2}\varphi_{\bar{z}_2u}\varphi_{z_1\bar{z}_1}\big) - \\ & - \sqrt{-1}\big(\varphi_{\bar{z}_2}\varphi_{z_2u}\varphi_{z_1\bar{z}_1} + \varphi_{z_2\bar{z}_1}\varphi_{z_1}\varphi_{\bar{z}_2u} + \varphi_{z_2}\varphi_{\bar{z}_1u}\varphi_{z_1\bar{z}_2} + \varphi_{z_2\bar{z}_2}\varphi_{\bar{z}_1}\varphi_{z_1u}\big) - \\ & - \varphi_{z_2\bar{z}_1}\varphi_{z_1\bar{z}_2}\varphi_u\varphi_u + \varphi_{z_2\bar{z}_2}\varphi_{z_1\bar{z}_1}\varphi_u\varphi_u \Big\}. \end{aligned}$$

Since the rank of the Levi Matrix (2) equals 1 everywhere, after permutation, its top-left entry vanishes nowhere. Hence a local generator for $K^{1,0}M$ is:

$$\mathscr{K} = k\,\mathscr{L}_1 + \mathscr{L}_2,$$

with:

$$k = -\frac{\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)}{\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)},$$

namely:

$$k = \frac{\varphi_{z_2\bar{z}_1} + \varphi_{z_2\bar{z}_1}\varphi_u\varphi_u - \sqrt{-1}\varphi_{\bar{z}_1}\varphi_{z_2u} - \varphi_{\bar{z}_1}\varphi_{z_2u}\varphi_u + \sqrt{-1}\varphi_{z_2}\varphi_{\bar{z}_1u} + \varphi_{z_2}\varphi_{\bar{z}_1}\varphi_{uu} - \varphi_{z_2}\varphi_{\bar{z}_1u}\varphi_u}{-\varphi_{z_1\bar{z}_1} - \varphi_{z_1\bar{z}_1}\varphi_u\varphi_u + \sqrt{-1}\varphi_{\bar{z}_1}\varphi_{z_1u} + \varphi_{\bar{z}_1}\varphi_{z_1u}\varphi_u - \sqrt{-1}\varphi_{z_1}\varphi_{\bar{z}_1u} - \varphi_{z_1}\varphi_{\bar{z}_1}\varphi_{uu} + \varphi_{z_1}\varphi_{\bar{z}_1u}\varphi_u},$$

and there is a surprising computational fact that this function $k$ happens to be also equal to the other two quotients ([46], II, p. 82):

$$k = -\frac{\mathscr{L}_2\big(\overline{A_1}\big)}{\mathscr{L}_1\big(\overline{A_1}\big)} = -\frac{-\overline{\mathscr{L}}_1\big(A_2\big)}{-\overline{\mathscr{L}}_1\big(A_1\big)}. \tag{4}$$

Heuristically, this fact becomes transparent when $\varphi = \varphi(x_1, x_2, y_1, y_2)$ is independent of $u$, whence the Levi matrix becomes:

$$\begin{pmatrix} 2\,\varphi_{z_1\bar{z}_1} & 2\,\varphi_{z_2\bar{z}_1} \\ 2\,\varphi_{z_1\bar{z}_2} & 2\,\varphi_{z_2\bar{z}_2} \end{pmatrix},$$

and clearly:

$$k = -\frac{2\,\varphi_{z_2\bar{z}_1}}{2\,\varphi_{z_1\bar{z}_1}} = -\frac{\varphi_{z_2\bar{z}_1}}{\varphi_{z_1\bar{z}_1}} = -\frac{\mathscr{L}_2(\varphi_{\bar{z}_1})}{\mathscr{L}_1(\varphi_{\bar{z}_1})} = -\frac{-\overline{\mathscr{L}}_1(\varphi_{z_2})}{-\overline{\mathscr{L}}_1(\varphi_{z_1})}.$$

**Proposition 1.** ([46]) *In any system of holomorphic coordinates, for any choice of Levi-kernel adapted local $T^{1,0}M$-frame $\big\{\mathscr{L}_1, \mathscr{K}\big\}$ satisfying:*

$$K^{1,0}M = \mathbb{C}\mathscr{K},$$

*and for any choice of differential* 1*-forms $\big\{\rho_0, \kappa_0, \zeta_0\big\}$ satisfying:*

$$\begin{aligned} \big\{0 = \rho_0\big\} &= T^{1,0}M \oplus T^{0,1}M, \\ \big\{0 = \rho_0 = \kappa_0 = \overline{\kappa}_0 = \overline{\zeta}_0\big\} &= K^{1,0}M, \end{aligned}$$

*the quantity:*

$$\kappa_0\big(\big[\mathscr{K}, \overline{\mathscr{L}}_1\big]\big),$$

*is, at one fixed point $p \in M$, either zero or nonzero, independently of any choice.*

**Definition 2.** The *Freeman form* at a point $p \in M$ is the value of $\kappa_0\big(\big[\mathscr{K}, \overline{\mathscr{L}}_1\big]\big)(p)$, and it depends only on $\kappa_0(p)$, $\mathscr{K}\big|_p$, $\overline{\mathscr{L}}_1\big|_p$.

With a $T^{1,0}M$-frame $\{\mathscr{K}, \mathscr{L}_1\}$ satisfying $K^{1,0}M = \mathbb{C}\mathscr{K}$, define quite equivalently:

$$\mathsf{Freeman\text{-}Form}^M(p)\colon \begin{bmatrix} K_p^{1,0}M \times \big(T_p^{1,0}M \mod K_p^{1,0}M\big) \longrightarrow \mathbb{C} \\ \big(\mathscr{K}_p, \mathscr{L}_{1p}\big) \longmapsto \big[\mathscr{K}, \overline{\mathscr{L}}_1\big](p) \\ \mod \big(K^{1,0}M \oplus T^{0,1}M\big), \end{bmatrix}$$

the result being independent of vector field extensions $\mathscr{K}\big|_p = \mathscr{K}_p$ and $\mathscr{L}_1\big|_p = \mathscr{L}_{1p}$.

**General Classes $\mathsf{IV}_1$, $\mathsf{IV}_2$.** For a connected $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$, if the Levi form is of rank 2 at one point, it is of rank 2 at every Zariski-generic point. Excluding Levi degenerate points, this brings $\mathsf{IV}_1$.

If the Levi form is identically zero, then as is known $M \cong \mathbb{C}^2 \times \mathbb{R}$.

If the Levi form is of rank 1, the Freeman form creates bifurcation:

**Proposition 2.** ([46]) *A $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ having at every point $p$:*

$$\mathsf{rank}_{\mathbb{C}}\big(\mathsf{Levi\text{-}Form}^M(p)\big) = \mathbf{1}$$

*has an identically vanishing:*

$$\mathsf{Freeman\text{-}Form}^M(p) \equiv \mathbf{0},$$

*if and only if it is locally biholomorphic to a product:*

$$M^5 \cong M^3 \times \mathbb{C}$$

*with a $\mathscr{C}^\omega$ hypersurface $M^3 \subset \mathbb{C}^2$.*

*Proof.* With:

$$\mathscr{K} = k\,\mathscr{L}_1 + \mathscr{L}_2 = k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u},$$

the involutiveness (1):

$$\big[\mathscr{K}, \overline{\mathscr{K}}\big] = \text{function}\cdot\mathscr{K} + \text{function}\cdot\overline{\mathscr{K}},$$

and the fact that this bracket does not contain either $\partial/\partial z_2$ or $\partial/\partial \overline{z}_2$:

$$\begin{aligned}\big[\mathscr{K}, \overline{\mathscr{K}}\big] &= \bigg[k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u},\ \overline{k}\,\frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_2} + \big(\overline{k}\overline{A}_1 + \overline{A}_2\big)\,\frac{\partial}{\partial u}\bigg] \\ &= \mathscr{K}\big(\overline{k}\big)\,\frac{\partial}{\partial \overline{z}_1} - \overline{\mathscr{K}}\big(k\big)\,\frac{\partial}{\partial z_1} + \Big(\mathscr{K}\big(\overline{k}\overline{A}_1 + \overline{A}_2\big) - \overline{\mathscr{K}}\big(kA_1 + A_2\big)\Big)\,\frac{\partial}{\partial u},\end{aligned}$$

entail:

$$0 \equiv \overline{\mathscr{K}}(k) \equiv \mathscr{K}\big(\overline{k}\big).$$

Next, identical vanishing of the Freeman form:

$$\big[\mathscr{K}, \overline{\mathscr{L}}_1\big] \equiv 0 \qquad \text{mod}\ \big(\mathscr{K}, \overline{\mathscr{K}}, \overline{\mathscr{L}}_1\big),$$

with:

$$\begin{aligned}\big[\mathscr{K}, \overline{\mathscr{L}}_1\big] &= \bigg[k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u},\ \frac{\partial}{\partial \overline{z}_1} + \overline{A}_1\,\frac{\partial}{\partial u}\bigg] \\ &= \overline{\mathscr{L}}_1(k)\,\frac{\partial}{\partial z_1} + \text{something}\,\frac{\partial}{\partial u},\end{aligned}$$

reads as:

$$0 \equiv \overline{\mathscr{L}}_1(k),$$

and since $\{\overline{\mathscr{K}}, \overline{\mathscr{L}}_1\}$ is a $T^{0,1}M$-frame,

*The $\mathscr{C}^\omega$ slanting function $k$ is a CR function!*

Moreover, the last coefficient-function of:

$$\mathscr{K} = k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u}$$

is *also* a CR function, namely it is annihilated by $\overline{\mathscr{L}}_1$, $\overline{\mathscr{L}}_2$, because firstly:

$$0 \overset{?}{=} \overline{\mathscr{L}}_1\big(kA_1 + A_2\big) = k\overline{\mathscr{L}}_1(A_1) + \overline{\mathscr{L}}_1(A_2) \overset{\text{ok}}{=} 0,$$

thanks to (4), and secondly because a direct computation gives:

$$\begin{aligned} 0 &\overset{?}{=} \overline{\mathscr{L}}_2\big(kA_1 + A_2\big) \\ &= k\overline{\mathscr{L}}_2(A_1) + \overline{\mathscr{L}}_2(A_2) \\ &= \frac{-\,\text{numerator of the Levi determinant}}{(\sqrt{-1} + \varphi_u)\,[\varphi_{z_1\bar{z}_1} + \varphi_{z_1\bar{z}_1}\varphi_u\varphi_u - \sqrt{-1}\,\varphi_{\bar{z}_1}\varphi_{z_1u} - \varphi_{\bar{z}_1}\varphi_{z_1u}\varphi_u + \sqrt{-1}\,\varphi_{z_1}\varphi_{\bar{z}_1u}\varphi_u + \varphi_{z_1}\varphi_{\bar{z}_1}\varphi_{uu}]} \\ &\equiv 0. \end{aligned}$$

In conclusion, the $(1,0)$ field $\mathscr{K}$ has $\mathscr{C}^\omega$ CR coefficient-functions, hence $\mathscr{K}$ is locally extendable to a neighborhood of $M$ in $\mathbb{C}^3$ as a $(1,0)$ field having holomorphic coefficients, and a straightening $\mathscr{K} = \partial/\partial z_2$ yields $M^5 \cong M^3 \times \mathbb{C}$. □

Excluding therefore such a degeneration, and focusing attention on a Zariski-generic initial classification, it therefore remains only the General Class $\mathsf{IV}_2$.

## 2.6 Existence of the six General Classes I, II, III$_1$, III$_2$, IV$_1$, IV$_2$

Graphing functions are essentially free and arbitrary.

**Proposition 3.** ([46]) *Every CR-generic submanifold belonging to the four classes* I, II, III$_1$, IV$_1$ *may be represented in suitable local holomorphic coordinates as:*

$$\underline{(\mathsf{I})}: \quad \big[v = z\bar{z} + z\bar{z}\,\mathrm{O}_1\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u),$$

$$\underline{(\mathsf{II})}: \quad \begin{cases} v_1 = z\bar{z} + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2), \\ v_2 = z^2\bar{z} + z\bar{z}^2 + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2), \end{cases}$$

$$\underline{(\mathsf{III})_1}: \quad \begin{cases} v_1 = z\bar{z} + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \\ v_2 = z^2\bar{z} + z\bar{z}^2 + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \\ v_3 = \sqrt{-1}\,\big(z^2\bar{z} - z\bar{z}^2\big) + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \end{cases}$$

$$\underline{\mathsf{(IV)_1}}\colon\qquad \big[v = z_1\overline{z}_1 \pm z_2\overline{z}_2 + \mathrm{O}_3\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big),$$

*with arbitrary remainder functions. For class:*

$$\underline{\mathsf{(III)_2}}\colon\quad \begin{aligned} v_1 &= z\overline{z} + c_1\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_3 &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \end{aligned}$$

*the* 3 *graphing functions* $\varphi_1$*,* $\varphi_2$*,* $\varphi_3$ *are subjected to the identical vanishing:*

$$0 \equiv \left| \begin{array}{ccc} \mathscr{L}(\overline{A}_1)-\overline{\mathscr{L}}(A_1) & \mathscr{L}(\overline{A}_2)-\overline{\mathscr{L}}(A_2) & \mathscr{L}(\overline{A}_3)-\overline{\mathscr{L}}(A_3) \\ \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_1))-2\mathscr{L}(\overline{\mathscr{L}}(A_1))+\\ +\overline{\mathscr{L}}(\mathscr{L}(A_1))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_2))-2\mathscr{L}(\overline{\mathscr{L}}(A_2))+\\ +\overline{\mathscr{L}}(\mathscr{L}(A_2))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_3))-2\mathscr{L}(\overline{\mathscr{L}}(A_3))+\\ +\overline{\mathscr{L}}(\mathscr{L}(A_3))\end{array} \\ \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_1))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_1))-\\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_1))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_2))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_2))-\\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_2))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_3))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_3))-\\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_3))\end{array} \end{array} \right|.$$

*Lastly, for class:*

$$\underline{\mathsf{(IV)_2}}\colon\qquad \big[v = z_1\overline{z}_1 + \tfrac{1}{2}\,z_1z_1\overline{z}_2 + \tfrac{1}{2}\,z_2\overline{z}_1\overline{z}_1 + \mathrm{O}_4\big(z_1,z_2,\overline{z}_1,\overline{z}_2\big) + u\,\mathrm{O}_2\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big),$$

*the graphing function* $\varphi$ *is subjected to the identical vanishing of the Levi determinant* (3)*.*

## *2.7 Cartan equivalences and curvatures*

**Problem 3.** (Subproblem of Problem 1) *Perform Cartan's local equivalence procedure for these six general classes* $\mathsf{I}$*,* $\mathsf{II}$*,* $\mathsf{III_1}$*,* $\mathsf{III_2}$*,* $\mathsf{IV_1}$*,* $\mathsf{IV_2}$ *of nondegenerate CR manifolds up to dimension* **5***.*

**Class** $\mathsf{I}$ **equivalences.** Consider firstly Class $\mathsf{I}$ hypersurfaces $M^3 \subset \mathbb{C}^2$ graphed as $v = \varphi(x,y,u)$ with:

$$\left\{ \mathscr{L} = \frac{\partial}{\partial z} - \frac{\varphi_z}{\sqrt{-1}+\varphi_u}\,\frac{\partial}{\partial u}, \quad \overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} - \frac{\varphi_{\overline{z}}}{-\sqrt{-1}+\varphi_u}\,\frac{\partial}{\partial u}, \quad \mathscr{T} := \sqrt{-1}\,\big[\mathscr{L},\overline{\mathscr{L}}\big] = \ell\,\frac{\partial}{\partial u} \right\}$$

making a frame for $\mathbb{C}\otimes_{\mathbb{R}} TM$, where the Levi-factor (nonvanishing) function:

(5)

$$\begin{aligned}\ell &:= \sqrt{-1}\Big(\mathscr{L}\big(\overline{A}\big) - \overline{\mathscr{L}}(A)\Big)\\ &= \frac{1}{(\sqrt{-1}+\varphi_u)^2\,(-\sqrt{-1}+\varphi_u)^2}\Big\{2\,\varphi_{z\overline{z}} + 2\,\varphi_{z\overline{z}}\,\varphi_u\,\varphi_u - 2\,\sqrt{-1}\,\varphi_{\overline{z}}\,\varphi_{zu} - 2\,\varphi_{\overline{z}}\,\varphi_{zu}\,\varphi_u + \\ &\qquad + 2\,\sqrt{-1}\,\varphi_z\,\varphi_{\overline{z}u} + 2\,\varphi_z\,\varphi_{\overline{z}}\,\varphi_{uu} - 2\,\varphi_z\,\varphi_{\overline{z}u}\,\varphi_u\Big\}\end{aligned}$$

*will, notably, enter computations in denominator place*.

Introduce also the dual coframe for $\mathbb{C}\otimes_{\mathbb{R}} T^*M$:

$$\big\{\rho_0,\,\overline{\zeta}_0,\,\zeta_0\big\},$$

satisfying:

$$\begin{array}{lll}\rho_0(\mathscr{T})=1 & \rho_0(\overline{\mathscr{L}})=0 & \rho_0(\mathscr{L})=0,\\ \overline{\zeta}_0(\mathscr{T})=0 & \overline{\zeta}_0(\overline{\mathscr{L}})=1 & \overline{\zeta}_0\big(\mathscr{L}\big)=0,\\ \zeta_0(\mathscr{T})=0 & \zeta_0(\overline{\mathscr{L}})=0 & \zeta_0\big(\mathscr{L}\big)=1.\end{array}$$

Since:

$$\big[\mathscr{L},\,\mathscr{T}\big] = \bigg[\frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u},\,\ell\,\frac{\partial}{\partial u}\bigg] = \Big(\ell_z + A\,\ell_u - \ell A_u\Big)\,\frac{\partial}{\partial u} = \overbrace{\frac{\ell_z + A\,\ell_u - \ell A_u}{\ell}}^{=:P}\,\mathscr{T},$$

the *initial Darboux structure* reads dually as:

$$\begin{aligned}d\rho_0 &= P\,\rho_0\wedge\zeta_0 + \overline{P}\,\rho_0\wedge\overline{\zeta}_0 + \sqrt{-1}\,\zeta_0\wedge\overline{\zeta}_0,\\ d\overline{\zeta}_0 &= 0,\\ d\zeta_0 &= 0,\end{aligned}$$

with a *single* fundamental function $P$. Élie Cartan in 1932 performed his equivalence procedure (well presented in [58]) for such $M^3\subset\mathbb{C}^2$, but the completely explicit aspects must be endeavoured once again for systematic treatment of Problem 3. Within Tanaka's theory, Ezhov-McLaughlin-Schmalz ([22]) already constructed a Cartan connection on a certain principal bundle $N^8\to M^3$, whose effective aspects have been explored further in [52, 53].

Indeed, as already observed in Lemma 3 (*see* also [46, 53]), the initial *G-structure* for (local) biholomorphic equivalences of such hypersurfaces is:

$$\mathsf{G}^{\mathsf{initial}}_{\mathsf{IV}_2} := \left\{\begin{pmatrix}\mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \mathsf{b} & 0 & \mathsf{a}\end{pmatrix}\in\mathsf{GL}_3(\mathbb{C})\colon\quad \mathsf{a}\in\mathbb{C}\backslash\{0\},\ \mathsf{b}\in\mathbb{C}\right\}.$$

Cartan's method is to deal with the so-called *lifted coframe:*

$$\begin{pmatrix}\rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \mathsf{b} & 0 & \mathsf{a}\end{pmatrix} \begin{pmatrix}\rho_0\\ \overline{\zeta}_0\\ \zeta_0\end{pmatrix},$$

in the space of $(x, y, u, \mathsf{a}, \overline{\mathsf{a}}, \mathsf{b}, \overline{\mathsf{b}})$. After two absorbtions-normalizations and after one prolongation:

**Theorem 6.** (M.-SABZEVARI, [53]) *The biholomorphic equivalence problem for $M^3 \subset \mathbb{C}^2$ transforms some explicit eight-dimensional coframe:*

$$\{\rho, \overline{\zeta}, \zeta, \alpha, \beta, \overline{\alpha}, \overline{\beta}, \delta\},$$

*on a certain manifold $N^8 \longrightarrow M^3$ having $\{e\}$-structure equations:*

$$\begin{aligned}
d\rho &= \alpha\wedge\rho + \overline{\alpha}\wedge\rho + \sqrt{-1}\,\zeta\wedge\overline{\zeta},\\
d\overline{\zeta} &= \overline{\beta}\wedge\rho + \overline{\alpha}\wedge\overline{\zeta},\\
d\zeta &= \beta\wedge\rho + \alpha\wedge\zeta,\\
d\alpha &= \delta\wedge\rho + 2\sqrt{-1}\,\zeta\wedge\overline{\beta} + \sqrt{-1}\,\overline{\zeta}\wedge\beta,\\
d\beta &= \delta\wedge\zeta + \beta\wedge\overline{\alpha} + \mathfrak{I}\,\overline{\zeta}\wedge\rho,\\
d\overline{\alpha} &= \delta\wedge\rho - 2\sqrt{-1}\,\overline{\zeta}\wedge\beta - \sqrt{-1}\,\zeta\wedge\overline{\beta},\\
d\overline{\beta} &= \delta\wedge\overline{\zeta} + \overline{\beta}\wedge\alpha + \overline{\mathfrak{I}}\,\zeta\wedge\rho,\\
d\delta &= \delta\wedge\alpha + \delta\wedge\overline{\alpha} + \sqrt{-1}\,\beta\wedge\overline{\beta} + \mathfrak{T}\,\rho\wedge\zeta + \overline{\mathfrak{T}}\,\rho\wedge\overline{\zeta},
\end{aligned}$$

*with the single primary complex invariant:*

$$\begin{aligned}
\mathfrak{I} := \frac{1}{6}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\Big( &- 2\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big)\big) + 3\,\overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big)\big) - 7\,\overline{P}\,\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big) + \\
&+ 4\,\overline{P}\,\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big) - \mathscr{L}\big(\overline{P}\big)\,\overline{\mathscr{L}}\big(\overline{P}\big) + 2\,\overline{P}\,\overline{P}\,\mathscr{L}\big(\overline{P}\big)\Big),
\end{aligned}$$

*and with one secondary invariant:*

$$\mathfrak{T} = \frac{1}{\overline{\mathsf{a}}}\Big(\overline{\mathscr{L}}(\overline{\mathfrak{I}}) - \overline{P}\,\overline{\mathfrak{I}}\Big) - \sqrt{-1}\,\frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}}\,\overline{\mathfrak{I}}.$$

**Explicitness obstacle.** In terms of the function $P$, the formulas for $\mathfrak{I}$, for $\mathfrak{T}$ and for the 1-forms constituting the $\{e\}$-structure are writable, but when expressing everything in terms of the graphing function $\varphi$, because $P$ involves the Levi factor $\ell$ in denominator place, formulas 'explode'.

Indeed, the real and imaginary parts $\Delta_1$ and $\Delta_2$ in:

$$\mathfrak{I} = \frac{4}{\mathsf{a}\overline{\mathsf{a}}^3}\big(\boldsymbol{\Delta}_1 + \sqrt{-1}\,\boldsymbol{\Delta}_4\big)$$

have numerators containing respectively ([53]):

**1 553 198** and **1 634 457**

monomials in the differential ring in $\binom{6+3}{3} - 1 = 83$ variables:

$$\mathbb{Z}\big[\varphi_x, \varphi_y, \varphi_{x^2}, \varphi_{y^2}, \varphi_{u^2}, \varphi_{xy}, \varphi_{xu}, \varphi_{yu}, \dots\dots, \varphi_{x^6}, \varphi_{y^6}, \varphi_{u^6}, \dots\big].$$

Strikingly, though, in the so-called *rigid case* (often useful as a case of study-exploration) where $\varphi = \varphi(x,y)$ is independent of $u$ so that:

$$P = \frac{\varphi_{zz\overline{z}}}{\varphi_{z\overline{z}}}, \qquad \text{Levi factor at denominator} = \ell = \varphi_{z\overline{z}},$$

$\mathfrak{I}$ is easily writable:

$$\begin{aligned}\mathfrak{I}\Big|_{\substack{\text{rigid}\\ \text{case}}} = \frac{1}{6}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\bigg(&\frac{\varphi_{z^2\overline{z}^4}}{\varphi_{z\overline{z}}} - 6\,\frac{\varphi_{z^2\overline{z}^3}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^2} - \frac{\varphi_{z\overline{z}^4}\,\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^2} - 4\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^2} + \\ &+ 10\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^3} + 15\,\frac{(\varphi_{z\overline{z}^2})^2\,\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^3} - 15\,\frac{(\varphi_{z\overline{z}^2})^3\,\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^4}\bigg),\end{aligned}$$

and this therefore shows that *there is a spectacular contrast of computational complexity when passing from the rigid case to the general case*. The reason of this contrast mainly comes from the size of the Levi factor $\ell$ in (5) *appearing at denominator place in subsequent differentiations*.

**Class $\mathsf{IV}_2$ equivalences.** Consider secondly a Class $\mathsf{IV}_2$ hypersurface $M^5 \subset \mathbb{C}^3$, and let a Levi-kernel adapted frame for $\mathbb{C}\otimes_{\mathbb{R}} TM$ be:

$$\begin{aligned}&\big\{\mathscr{T}, \overline{\mathscr{L}}_1, \overline{\mathscr{K}}, \mathscr{L}_1, \mathscr{K}\big\}, && \overline{\mathscr{T}} = \mathscr{T}, \\ &\mathscr{T} := \sqrt{-1}\,\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big] = \ell\,\frac{\partial}{\partial u}, && \ell := \sqrt{-1}\,\big(\mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1(A_1)\big).\end{aligned}$$

**Lemma 4.** ([46, 62]) *The initial Lie structure of this frame consists of* $10 = \binom{5}{2}$ *brackets:*

$$\begin{aligned}\big[\mathscr{T}, \overline{\mathscr{L}}_1\big] &= -\overline{P}\cdot\mathscr{T}, \\ \big[\mathscr{T}, \overline{\mathscr{K}}\big] &= \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\mathscr{T} + \mathscr{T}\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1, \\ \big[\mathscr{T}, \mathscr{L}_1\big] &= -P\cdot\mathscr{T}, \\ \big[\mathscr{T}, \mathscr{K}\big] &= \mathscr{L}_1(k)\cdot\mathscr{T} + \mathscr{T}(k)\cdot\mathscr{L}_1, \\ \big[\overline{\mathscr{L}}_1, \overline{\mathscr{K}}\big] &= \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1, \\ \big[\overline{\mathscr{L}}_1, \mathscr{L}_1\big] &= \sqrt{-1}\,\mathscr{T}, \\ \big[\overline{\mathscr{L}}_1, \mathscr{K}\big] &= \overline{\mathscr{L}}_1(k)\cdot\mathscr{L}_1, \\ \big[\overline{\mathscr{K}}, \mathscr{L}_1\big] &= -\mathscr{L}_1\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1, \\ \big[\overline{\mathscr{K}}, \mathscr{K}\big] &= 0, \\ \big[\mathscr{L}_1, \mathscr{K}\big] &= \mathscr{L}_1(k)\cdot\mathscr{L}_1,\end{aligned}$$

*in terms of the* 2 *fundamental functions:*

$$k := -\frac{\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)}{\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)}, \qquad\qquad P := \frac{\ell_{z_1} + A_1\,\ell_u - \ell\,A_{1,u}}{\sqrt{-1}\,\big(\mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_1\big)\big)}$$

*(in [62], $M^5$ is graphed as $u = F(x_1, y_1, x_2, y_2, v)$ instead, hence P changes).*

Introduce then the coframe:

$$\big\{\rho_0,\, \overline{\kappa}_0,\, \overline{\zeta}_0,\, \kappa_0,\, \zeta_0\big\}$$

which is dual to the frame:

$$\big\{\mathscr{T}, \overline{\mathscr{L}}_1, \overline{\mathscr{K}}, \mathscr{L}_1, \mathscr{K}\big\},$$

the notations being the same as in Proposition 1, that is to say:

$$\begin{aligned}
\rho_0 &= \frac{du - A_1dz_1 - A_2dz_2 - \overline{A}_1d\overline{z}_1 - \overline{A}_2d\overline{z}_2}{\ell},\\
\kappa_0 &= dz_1 - k\,dz_2,\\
\zeta_0 &= dz_2.
\end{aligned}$$

The initial Darboux structure is:

$$\begin{aligned}
d\rho_0 &= \overline{P}\cdot\rho_0\wedge\overline{\kappa}_0 - \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\rho_0\wedge\overline{\zeta}_0 + P\cdot\rho_0\wedge\kappa_0 - \mathscr{L}_1(k)\cdot\rho_0\wedge\zeta_0 + \sqrt{-1}\,\kappa_0\wedge\overline{\kappa}_0,\\
d\overline{\kappa}_0 &= -\,\mathscr{T}\big(\overline{k}\big)\cdot\rho_0\wedge\overline{\zeta}_0 - \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\overline{\kappa}_0\wedge\overline{\zeta}_0 + \mathscr{L}_1\big(\overline{k}\big)\cdot\overline{\zeta}_0\wedge\kappa_0,\\
d\overline{\zeta}_0 &= 0,\\
d\kappa_0 &= -\,\mathscr{T}(k)\cdot\rho_0\wedge\zeta_0 - \overline{\mathscr{L}}_1(k)\cdot\overline{\kappa}_0\wedge\zeta_0 - \mathscr{L}_1(k)\cdot\kappa_0\wedge\zeta_0,\\
d\zeta_0 &= 0.
\end{aligned}$$

The initial associated $G$-structure is:

$$\mathsf{G}^{\mathsf{initial}}_{\mathsf{IV}_2} := \left\{ \begin{pmatrix} \mathsf{c} & 0 & 0 & 0 & 0\\ \mathsf{b} & \mathsf{a} & 0 & 0 & 0\\ 0 & 0 & \overline{\mathsf{c}} & 0 & 0\\ 0 & 0 & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \mathsf{e} & \mathsf{d} & \overline{\mathsf{e}} & \overline{\mathsf{d}} & \mathsf{a}\overline{\mathsf{a}} \end{pmatrix} \in \mathscr{M}_{5\times 5}(\mathbb{C}) :\ \mathsf{a}, \mathsf{c} \in \mathbb{C}\backslash\{0\},\, \mathsf{b}, \mathsf{d}, \mathsf{e} \in \mathbb{C} \right\}.$$

**Theorem 7.** (POCCHIOLA, [62]) *Two fundamental explicit invariants both having denominators related to the nondegeneracy of the Freeman form:*

$$\begin{aligned}
W :=\ & \frac{2}{3}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\overline{k})\big)}{\mathscr{L}_1(\overline{k})} +\\
& + \frac{1}{3}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\,\mathscr{K}\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^3} - \frac{1}{3}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\big)}{\overline{\mathscr{L}}_1(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}}_1(k)},
\end{aligned}$$

*and:*

$$
\begin{aligned}
J := &\frac{5}{18}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)^2}{\mathscr{L}_1(\bar{k})^2} + \frac{1}{3}\,P\,\mathscr{L}_1(P) - \frac{1}{9}\,P^2\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)}{\mathscr{L}_1(\bar{k})} + \frac{20}{27}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)^3}{\mathscr{L}_1(\bar{k})^3} - \\
&- \frac{5}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\,\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)}{\mathscr{L}_1(\bar{k})^2} + \frac{1}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\,\mathscr{L}_1(P)}{\mathscr{L}_1(\bar{k})} - \\
&- \frac{1}{6}\,P\,\frac{\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)}{\mathscr{L}_1(\bar{k})} - \frac{2}{27}\,P^3 - \frac{1}{6}\,\mathscr{L}_1\big(\mathscr{L}_1(P)\big) + \frac{1}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)\big)}{\mathscr{L}_1(\bar{k})},
\end{aligned}
$$

*occur in the local biholomorphic equivalence problem for Class* $\mathsf{IV}_2$ *real analytic hypersurfaces* $M^5 \subset \mathbb{C}^3$*. Such a* $M^5$ *is locally biholomorphically equivalent to the light cone model:*

$$
(\mathsf{LC}) \qquad u = \frac{z_1\bar{z}_1 + \frac{1}{2}\,z_1 z_1 \bar{z}_2 + \frac{1}{2}\,z_2\bar{z}_1\bar{z}_1}{1 - z_2\bar{z}_2} \underset{\substack{\text{locally}\\ \text{by [24]}}}{\cong} (\mathsf{Re}\,z_1')^2 - (\mathsf{Re}\,z_2')^2 - (\mathsf{Re}\,z_3')^2,
$$

*having* 10*-dimensional symmetry group* $\mathsf{Aut}_{CR}(\mathsf{LC}) \cong \mathsf{Sp}(4,\mathbb{R})$*, if and only if:*

$$
0 \equiv W \equiv J.
$$

*Moreover, if either* $W \not\equiv 0$ *or* $J \not\equiv 0$*, an absolute parallelism is constructed on* $M$ *(after relocalization), and in this case, the local Lie group of* $\mathscr{C}^\omega$ *CR automorphisms of* $M$ *always has:*

$$
\dim \mathsf{Aut}_{CR}(M) \leqslant 5.
$$

The latter dimension drop was obtained by Fels-Kaup ([25]) under the assumption that $\mathsf{Aut}_{CR}(M)$ is locally transitive, while Cartan's method embraces *all* Class $\mathsf{IV}_2$ hypersurfaces $M^5 \subset \mathbb{C}^3$.

Reduction to an absolute parallelism on a 10-dimensional bundle $N^{10} \to M$ has been obtained previously by Isaev-Zaitsev ([31]) and Medori-Spiro ([37]). The explicitness of $W$ and $J$, the equivalence bifurcation $W \not\equiv 0$ or $J \not\equiv 0$ and the dimension drop $10 \to 5$ provide a complementary aspect. Furthermore, here is an application of the explicit rational expressions of $J$ and $W$ in the spirit of Theorem 2.

**Corollary 1.** *Let* $M^5 \subset \mathbb{C}^3$ *be a connected* $\mathscr{C}^\omega$ *hypersurface whose Levi form is of rank* 1 *at Zariski-generic points, possibly of rank* 0 *somewhere, and whose Freeman form is also nondegenerate at Zariski-generic points. If* $M$ *is locally biholomorphic to the light cone model* $(\mathsf{LC})$ *at some Freeman nondegenerate point, then* $M$ *is also locally biholomorphic to* $(\mathsf{LC})$ *at every other Freeman nondegenerate point.*

**Equivalences of remaining Classes $\mathsf{II}$, $\mathsf{III}_1$, $\mathsf{III}_2$, $\mathsf{IV}_1$.** Class $\mathsf{II}$ has been treated optimally by Beloshapka-Ezhov-Schmalz ([4]) who directly constructed a Cartan connection on a principal bundle $N^5 \to M^4$ with fiber $\cong \mathbb{R}$. Recently, Pocchiola ([63]) provided an alternative construction the elements of which are computed deeper.

Class $\mathsf{IV}_1$ is reduced to an absolute parallelism with a Cartan connection by Chern-Moser ([10]) inspired by Hachtroudi ([29]), though not explicity in terms of a local graphing function (question still open).

Recently, jointly with Sabzevari, Class $\mathsf{III}_1$ has been recently settled. Beloshapka (*see* also [1]), proved that the Lie algebra $\mathfrak{aut}_{CR} = 2\,\mathrm{Re}\,\mathfrak{hol}$ of $\mathrm{Aut}_{CR}$ of the cubic:

$$\text{Model }\mathsf{III}_1: \qquad v_1 = z\overline{z}, \qquad v_2 = z^2\overline{z} + z\overline{z}^2, \qquad v_3 = \sqrt{-1}\left(z^2\overline{z} - z\overline{z}^2\right),$$

is 7-dimensional generated by:

$$\begin{aligned}
T &:= \partial_{w_1}, \\
S_1 &:= \partial_{w_2}, \\
S_2 &:= \partial_{w_3}, \\
L_1 &:= \partial_z + (2\sqrt{-1}\,z)\,\partial_{w_1} + (2\sqrt{-1}\,z^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}, \\
L_2 &:= \sqrt{-1}\,\partial_z + (2\,z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2\sqrt{-1}\,z^2 - 4w_1)\,\partial_{w_3}, \\
D &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}, \\
R &:= \sqrt{-1}\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned}$$

Hence for a CR-generic $M^5 \subset \mathbb{C}^4$ belonging to Class $\mathsf{III}_1$ graphed as:

$$v_1 = \varphi_1(x,y,u_1,u_2,u_3), \qquad v_2 = \varphi_2(x,y,u_1,u_2,u_3), \qquad v_3 = \varphi_3(x,y,u_1,u_2,u_3),$$

reduction to an absolute parallelism on a certain bundle $N^7 \to M^5$ can be expected, and in fact, similarly as in Theorem 7, finer equivalence bifurcations will occur.

Lemma 1 showed that a generator for $T^{1,0}M$ is:

$$\mathscr{L} = \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta}\,\frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta}\,\frac{\partial}{\partial u_2} + \frac{\Lambda_3}{\Delta}\,\frac{\partial}{\partial u_3},$$

where:

$$\Delta := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

and where:

$$\Lambda_1 := \begin{vmatrix} -\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ -\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

with similar $\Lambda_2$, $\Lambda_3$. By definition, on a Class $\mathsf{III}_1$ CR-generic $M^5 \subset \mathbb{C}^4$ the fields:

$$\Big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\Big\},$$

where:

$$\mathscr{T} := \sqrt{-1}\,\big[\mathscr{L}, \overline{\mathscr{L}}\big], \qquad \mathscr{S} := \big[\mathscr{L},\, \mathscr{T}\big], \qquad \overline{\mathscr{S}} := \big[\overline{\mathscr{L}},\, \mathscr{T}\big],$$

make up a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM^5$. Computing these (iterated) Lie brackets, there are certain coefficient-polynomials:

$$\Upsilon_i = \Upsilon_i\Big(\varphi_{1,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{2,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{3,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}}\Big)_{1\leqslant j+k+l_1+l_2+l_3\leqslant 2} \qquad (i=1,2,3),$$

$$\Pi_i = \Pi_i\Big(\varphi_{1,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{2,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{3,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}}\Big)_{1\leqslant j+k+l_1+l_2+l_3\leqslant 3} \qquad (i=1,2,3),$$

so that (mind exponents in denominators):

$$\begin{aligned}
\mathscr{T} &= \frac{\Upsilon_1}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_3},\\
\mathscr{S} &= \frac{\Pi_1}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_2} + \frac{\Pi_3}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_3},\\
\overline{\mathscr{S}} &= \frac{\overline{\Pi}_1}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_1} + \frac{\overline{\Pi}_2}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_2} + \frac{\overline{\Pi}_3}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_3}.
\end{aligned}$$

**Explicitness obstacle.** The expansions of $\Upsilon_1$, $\Upsilon_2$, $\Upsilon_3$ as polynomials in their $3\cdot 20$ variables incorporate 41 964 monomials while those of $\Pi_1$, $\Pi_2$, $\Pi_3$ as polynomials in their $3\cdot 55$ variables would incorporate more than (no computer succeeded) 100 000 000 terms.

*Hence renouncement to full expliciteness is necessary.*

Between the 5 fields $\big\{\overline{\mathscr{S}}, \mathscr{S}, \mathscr{T}, \overline{\mathscr{L}}, \mathscr{L}\big\}$, there are $10 = \binom{5}{2}$ Lie brackets. Assign therefore formal names to the uncomputable appearing coefficient-functions.

**Lemma 5.** *([54]) In terms of* 5 *fundamental coefficient-functions:*

$$P, \quad Q, \quad R, \quad A, \quad B,$$

*the 10 Lie bracket relations write as:*

$$\begin{aligned}
\big[\overline{\mathscr{S}}, \mathscr{S}\big] &= \overline{K}_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} - K_{\mathsf{rpl}}\cdot\mathscr{S} - \sqrt{-1}\,J_{\mathsf{rpl}}\cdot\mathscr{T},\\
\big[\overline{\mathscr{S}}, \mathscr{T}\big] &= -\overline{F}_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} - \overline{G}_{\mathsf{rpl}}\cdot\mathscr{S} - \overline{E}_{\mathsf{rpl}}\cdot\mathscr{T},\\
\big[\overline{\mathscr{S}}, \overline{\mathscr{L}}\big] &= -\overline{Q}\cdot\overline{\mathscr{S}} - \overline{R}\cdot\mathscr{S} - \overline{P}\cdot\mathscr{T},\\
\big[\overline{\mathscr{S}}, \mathscr{L}\big] &= -\overline{B}\cdot\overline{\mathscr{S}} - B\cdot\mathscr{S} - A\cdot\mathscr{T},\\
\big[\mathscr{S}, \mathscr{T}\big] &= -G_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} - F_{\mathsf{rpl}}\cdot\mathscr{S} - E_{\mathsf{rpl}}\cdot\mathscr{T},\\
\big[\mathscr{S}, \overline{\mathscr{L}}\big] &= -\overline{B}\cdot\overline{\mathscr{S}} - B\cdot\mathscr{S} - A\cdot\mathscr{T},\\
\big[\mathscr{S}, \mathscr{L}\big] &= -R\cdot\overline{\mathscr{S}} - Q\cdot\mathscr{S} - P\cdot\mathscr{T},\\
\big[\mathscr{T}, \overline{\mathscr{L}}\big] &= -\overline{\mathscr{S}},\\
\big[\mathscr{T}, \mathscr{L}\big] &= -\mathscr{S},\\
\big[\overline{\mathscr{L}}, \mathscr{L}\big] &= \sqrt{-1}\,\mathscr{T},
\end{aligned}$$

*the coefficient-functions* $E_{\mathsf{rpl}}$, $G_{\mathsf{rpl}}$, $H_{\mathsf{rpl}}$, $J_{\mathsf{rpl}}$, $K_{\mathsf{rpl}}$ *being secondary:*

$$
\begin{aligned}
E_{\mathsf{rpl}} &= \sqrt{-1}\Big(\mathscr{L}(A) - \overline{\mathscr{L}}(P) + A\overline{B} + BP - AQ - \overline{P}R\Big),\\
F_{\mathsf{rpl}} &= \sqrt{-1}\Big(\mathscr{L}(B) - \overline{\mathscr{L}}(Q) + A + B\overline{B} - R\overline{R}\Big),\\
G_{\mathsf{rpl}} &= \sqrt{-1}\Big(\mathscr{L}(\overline{B}) - \overline{\mathscr{L}}(R) + \overline{B}\overline{B} + BR - P - \overline{B}Q - R\overline{Q}\Big),
\end{aligned}
$$

*with similar, longer expressions for* $J_{\mathsf{rpl}}$, $K_{\mathsf{rpl}}$.

Introduce then the coframe:

$$\big\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\big\},$$

which is dual to the frame:

$$\big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\}.$$

Organize the ten Lie brackets as a convenient auxiliary array:

$$
\begin{array}{rcccccccccccc}
 & & \overline{\mathscr{S}} & & \mathscr{S} & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & \\
 & & \boxed{d\overline{\sigma_0}} & & \boxed{d\sigma_0} & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & \\
[\overline{\mathscr{S}},\mathscr{S}] & = & \overline{K}_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} & + & -K_{\mathsf{rpl}}\cdot\mathscr{S} & + & -\sqrt{-1}J_{\mathsf{rpl}}\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0}\wedge\sigma_0}\\
[\overline{\mathscr{S}},\mathscr{T}] & = & -\overline{F}_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} & + & -\overline{G}_{\mathsf{rpl}}\cdot\mathscr{S} & + & -\overline{E}_{\mathsf{rpl}}\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0}\wedge\rho_0}\\
[\overline{\mathscr{S}},\overline{\mathscr{L}}] & = & -\overline{Q}\cdot\overline{\mathscr{S}} & + & -\overline{R}\cdot\mathscr{S} & + & -\overline{P}\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0}\wedge\overline{\zeta_0}}\\
[\overline{\mathscr{S}},\mathscr{L}] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0}\wedge\zeta_0}\\
[\mathscr{S},\mathscr{T}] & = & -G_{\mathsf{rpl}}\cdot\overline{\mathscr{S}} & + & -F_{\mathsf{rpl}}\cdot\mathscr{S} & + & -E_{\mathsf{rpl}}\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0\wedge\rho_0}\\
[\mathscr{S},\overline{\mathscr{L}}] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0\wedge\overline{\zeta_0}}\\
[\mathscr{S},\mathscr{L}] & = & -R\cdot\overline{\mathscr{S}} & + & -Q\cdot\mathscr{S} & + & -P\cdot\mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0\wedge\zeta_0}\\
[\mathscr{T},\overline{\mathscr{L}}] & = & -\overline{\mathscr{S}} & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\rho_0\wedge\overline{\zeta_0}}\\
[\mathscr{T},\mathscr{L}] & = & 0 & + & -\mathscr{S} & + & 0 & + & 0 & + & 0 & \boxed{\rho_0\wedge\zeta_0}\\
[\overline{\mathscr{L}},\mathscr{L}] & = & 0 & + & 0 & + & \sqrt{-1}\,\mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\zeta_0}\wedge\zeta_0}
\end{array}
$$

Read *vertically* and put an overall minus sign to get the *initial Darboux structure*:

$$
\begin{aligned}
d\overline{\sigma_0} &= -\overline{K}_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\sigma_0 + \overline{F}_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\rho_0 + \overline{Q}\cdot\overline{\sigma_0}\wedge\overline{\zeta}_0 + \overline{B}\cdot\overline{\sigma_0}\wedge\zeta_0 +\\
&\qquad + G_{\mathsf{rpl}}\cdot\sigma_0\wedge\rho_0 + \overline{B}\cdot\sigma_0\wedge\overline{\zeta}_0 + R\cdot\sigma_0\wedge\zeta_0 + \rho_0\wedge\overline{\zeta}_0,\\
d\sigma_0 &= K_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\sigma_0 + \overline{G}_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\rho_0 + \overline{R}\cdot\overline{\sigma_0}\wedge\overline{\zeta}_0 + B\cdot\overline{\sigma_0}\wedge\zeta_0 +\\
&\qquad + F_{\mathsf{rpl}}\cdot\sigma_0\wedge\rho_0 + B\cdot\sigma_0\wedge\overline{\zeta}_0 + Q\cdot\sigma_0\wedge\zeta_0 + \rho_0\wedge\zeta_0,\\
d\rho_0 &= \sqrt{-1}J_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\sigma_0 + \overline{E}_{\mathsf{rpl}}\cdot\overline{\sigma_0}\wedge\rho_0 + \overline{P}\cdot\overline{\sigma_0}\wedge\overline{\zeta}_0 + A\cdot\overline{\sigma_0}\wedge\zeta_0 +\\
&\qquad + E_{\mathsf{rpl}}\cdot\sigma_0\wedge\rho_0 + A\cdot\sigma_0\wedge\overline{\zeta}_0 + P\cdot\sigma_0\wedge\zeta_0 - \sqrt{-1}\,\overline{\zeta}_0\wedge\zeta_0,\\
d\overline{\zeta}_0 &= 0,\\
d\zeta_0 &= 0.
\end{aligned}
$$

The initial $G$-structure is:

$$\mathsf{G}_{\mathsf{III}_1}^{\mathsf{initial}} := \left\{ \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 & 0 \\ 0 & \overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{b} & \overline{\mathsf{b}} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{e} & \mathsf{d} & \mathsf{c} & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \overline{\mathsf{e}} & \overline{\mathsf{c}} & 0 & \mathsf{a}\overline{\mathsf{a}\mathsf{a}} \end{pmatrix} \in \mathscr{M}_{5\times 5}(\mathbb{C}) : \ \mathsf{a} \in \mathbb{C}\backslash\{0\},\ \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} \in \mathbb{C} \right\}.$$

The lifted coframe is:

$$\begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a} \end{pmatrix} \begin{pmatrix} \overline{\sigma_0} \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta_0} \\ \zeta_0 \end{pmatrix}.$$

Performing absorption of torsion and normalization of group variables thanks to remaining essential torsion ([58, 54]), the coefficient-function $R$ is an invariant which creates bifurcation. Even in terms of $P$, $Q$, $R$, $A$, $B$, the expressions of some of the curvatures happen to be large and the study of their mutual independencies requires to take account of iterated Jacobi identities, an aspect of the subject which remains invisible in non-parametric Cartan method.

**Theorem 8.** (M.-SABZEVARI, [54]) *Within the branch $R = 0$, the biholomorphic equivalence problem for $M^5 \subset \mathbb{C}^4$ in Class $\mathsf{III}_1$ reduces to various absolute parallelisms namely to $\{e\}$-structures on certain manifolds of dimension* 6*, or directly on the* 5*-dimensional basis $M$, unless all existing essential curvatures vanish identically, in which case $M$ is (locally) biholomorphic to the cubic model with a characterization of such a condition being explicit in terms of the five fundamental functions $P$, $Q$, $R$, $A$, $B$.*

*Within the branch $R \not\equiv 0$, reduction to an absolute parallelism on the* 5*-dimensional basis $M$ always takes place, whence:*

$$\dim \mathsf{Aut}_{CR}(M) \leqslant 5.$$

Class $\mathsf{III}_2$ was recently settled also.

**Theorem 9.** (POCCHIOLA, [64]) *If $M^5 \subset \mathbb{C}^4$ is a local $\mathscr{C}^\omega$ CR-generic submanifold belonging to Class $\mathsf{III}_1$, then there exists a* 6*-dimensional principal bundle $P^6 = M^5 \times \mathbb{R}^*$ and there exists a coframe on $P^6$:*

$$\varpi := \big(\lambda,\, \tau,\, \sigma,\, \rho,\, \zeta,\, \overline{\zeta}\big)$$

*such that any local $\mathscr{C}^\omega$ CR-diffeomorphism $\mathsf{H}_M \colon M \to M$ lifts as a bundle isomorphism $\widehat{\mathsf{H}}_M \colon P \to P$ which satisfies $\mathsf{H}^*(\varpi) = \varpi$. Moreover, the structure equations of $\varpi$ on $P$ are of the form:*

$$
\begin{aligned}
d\tau &= 4\lambda\wedge\tau + I_1\,\tau\wedge\zeta - I_1\,\tau\wedge\overline{\zeta} + 3I_1\,\sigma\wedge\rho + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta},\\
d\sigma &= 3\lambda\wedge\sigma + I_2\,\tau\wedge\rho + I_3\,\tau\wedge\zeta + \overline{I}_3\,\tau\wedge\overline{\zeta} + I_4\,\sigma\wedge\rho -\\
&\qquad - \tfrac{1}{2}I_1\,\sigma\wedge\zeta + \tfrac{1}{2}I_1\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta},\\
d\rho &= 2\lambda\wedge\rho + I_5\,\tau\wedge\sigma + I_6\,\tau\wedge\rho + I_7\,\tau\wedge\zeta + \overline{I}_7\,\tau\wedge\overline{\zeta} + I_8\,\sigma\wedge\rho +\\
&\qquad + I_9\,\sigma\wedge\zeta + \overline{I}_9\,\sigma\wedge\overline{\zeta} - \tfrac{1}{2}I_1\,\rho\wedge\zeta + \tfrac{1}{2}I_1\,\rho\wedge\overline{\zeta} + \sqrt{-1}\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \lambda\wedge\zeta + I_{10}\,\tau\wedge\sigma + I_{11}\,\tau\wedge\rho + I_{12}\,\tau\wedge\zeta + I_{13}\,\tau\wedge\overline{\zeta} + I_{14}\,\sigma\wedge\rho + I_{15}\,\sigma\wedge\zeta,
\end{aligned}
$$

*for function $I_\bullet$, $J_{\bullet\bullet}$ on $P$ together with:*

$$
d\lambda = \sum_{\nu,\mu} J_{\nu\mu}\,\nu\wedge\mu \qquad\qquad (\mu,\nu=\tau,\sigma,\rho,\zeta,\overline{\zeta}).
$$

For both Classes $\mathsf{III}_1$ and $\mathsf{III}_2$, there also exist canonical Cartan connections naturally related to the final $\{e\}$-structures ([64, 50]).

*These works complete the program of performing* parametric *Cartan equivalences for all CR manifolds up to dimension* 5*.*

In dimension 6, Ezhov-Isaev-Schmalz ([23]) treated elliptic and hyperbolic $M^6 \subset \mathbb{C}^4$. A wealth of higher dimensional biholomorphic equivalence problems exists, *e.g.* ([35]) for CR-generic $M^{2+c} \subset \mathbb{C}^{1+c}$ in relation to classification of nilpotent Lie algebras ([26]).

## 3 Kobayashi hyperbolicity

The dominant theme is the *interplay* between the extrinsic projective geometry of algebraic subvarieties of $\mathbb{P}^n(\mathbb{C})$ and their intrinsic geometric features. Phillip GRIFFITHS.

In local Cartan theory, as seen in what precedes, denominators therefore play a central role in the differential ring generated by derivatives of the fundamental graphing functions $\varphi_j$. Similarly, in arithmetics of rational numbers $p/q$, like *e.g.* in multizeta calculus ([44]) involving a wealth of nested Cramer-type determinants, a growing complexity, potentially infinite, exists, and in fact, the complexity of rational numbers *also enters* high order covariant derivatives of Cartan curvatures, as an expression of the unity of mathematics. It is now time to show how *explicit rationality* also concerns the core of global algebraic geometry.

Let $X$ be a compact complex $n$-dimensional projective manifold that is of *general type*, namely whose canonical bundle $K_X = \Lambda^n T_X^*$ is *big* in the sense that $\dim H^0\big(X,(K_X)^{\otimes m}\big) \geqslant c\,m^n$ when $m\to\infty$ for some constant $c>0$. It is known that smooth hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ are so if and only if their degree $d$ is $\geqslant n+3$, since the *adjunction formula* shows that $K_X \cong \mathscr{O}_X(d-n-2)$ (the related rationality aspects will be discussed later).

A conjecture of Green-Griffiths-Lang expects that all nonconstant entire holomorphic curves $f\colon \mathbb{C}\to X$ should in fact land (be ‘canalized’) inside a certain proper subvariety $Y \subsetneqq X$. The current state of the art is still quite (very) far from reaching such a statement in this optimality. Furthermore, a companion Picard-type conjecture dating back to Kobayashi 1970 expects that all entire curves valued in Zariski-generic hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geqslant 2n+1$ should necessarily be *constant*, and the state of the art is also still quite far from understanding properly when this occurs, even in dimension 2, because of the lack of an appropriate explicit rational theory.

The current general method towards these conjectures consists as a first step in setting up a great number of nonzero differential equations $P\big(f, f', f'', \dots, f^{(\kappa)}\big) = 0$ of some order $\kappa$ (necessarily $\geqslant n$) satisfied by all nonconstant $f\colon \mathbb{C} \to X$, and then as a second step, in trying to *eliminate* from such numerous differential equations the true derivatives $f', f'', \dots, f^{(\kappa)}$ in order to receive certain purely algebraic nonzero equations $Q(f) = 0$ involving no derivatives anymore.

In 1979, by computing the Euler-Poincaré characteristic of a natural vector bundle nowadays called the *Green-Griffiths jet bundle* $\mathscr{E}_{\kappa,m}^{\rm GG} T_X^* \to X$, and by relying upon a $H^2$-cohomology vanishing theorem due to Bogomolov, Green and Griffiths ([27]) showed the existence of differential equations satisfied by entire curves valued in smooth *surfaces* $X^2 \subset \mathbb{P}^3$ of degree $d \geqslant 5$. In 1996, a breakthrough article by Siu-Yeung ([71]) showed Kobayashi-hyperbolicity of complements $\mathbb{P}^2 \backslash X^1$ of generic curves of degree $\geqslant 10^{13}$ (rounding off). Around 2000, McQuillan [34], by importing ideas from (multi)foliation theory considered entire maps valued in compact surfaces of general type having Chern numbers $\mathsf{c}_1^2 - \mathsf{c}_2 > 0$, which, for the case of $X^2 \subset \mathbb{P}^3$, improved very substantially the degree bound to $d \geqslant 36$, and this was followed in a work of Demailly and El Goul (*see* [16]) by the improvement $d \geqslant 21$. Later, using the Demailly-Semple bundle of jets that are invariant under reparametrization of the source $\mathbb{C}$, Rousseau was the first to treat in great details threefolds $X^3 \subset \mathbb{P}^4$ and he established algebraic degeneracy of entire curves in degree $d \geqslant 593$ ([65]). Previously, in two conference proceedings of the first 2000 years (ICM [68] and Abel Symposium [69]), Siu showed the existence of differential equations on hypersurfaces $X^n \subset \mathbb{P}^{n+1}$; the recent publication [70] of his extended preprint of that time confirmed the validity and the strength of his approach, which will be pursued *infra*.

Invariant jets used in [16, 65, 39, 19] are in fact deeply connected to rationality.

Indeed, another instance of the key role of denominators appears in computational invariant theory. Starting with an ideal $\mathscr{I} \subset \mathbb{C}[X_1, \dots, X_n]$ and with a nonzero $f \in \mathbb{C}[X_1, \dots, X_n]$, the *f-saturation* of $\mathscr{I}$ is:

$$\mathscr{I}^{\sf sat} \;\equiv\; \frac{\mathscr{I}}{f^\infty} \stackrel{\rm def}{=} \big\{ g \in \mathbb{C}[X] \colon \;\; f^m g \in \mathscr{I} \text{ for some } m \in \mathbb{N} \big\},$$

with increasing union stabilizing by noetherianity:

$$\frac{\mathscr{I}}{f} \subset \frac{\mathscr{I}}{f^2} \subset \frac{\mathscr{I}}{f^3} \subset \cdots \subset \frac{\mathscr{I}}{f^m} = \frac{\mathscr{I}}{f^{m+1}} = \cdots.$$

The *Kernel algorithm*, discovered in the $19^{\rm th}$ Century, consists in:

$$\begin{aligned}\left\langle \mathsf{g}_0,\ldots,\mathsf{g}_{n_0}\right\rangle &= \mathsf{Initial\ ideal},\\ \left\langle \mathsf{g}_0,\ldots,\mathsf{g}_{n_0},\ldots,\mathsf{g}_{n_1}\right\rangle &= \mathsf{Saturation}_f \left\langle \mathsf{g}_0,\ldots,\mathsf{g}_{n_0}\right\rangle,\\ \left\langle \mathsf{g}_0,\ldots,\mathsf{g}_{n_0},\ldots,\mathsf{g}_{n_1},\ldots,\mathsf{g}_{n_2}\right\rangle &= \mathsf{Saturation}_f \left\langle \mathsf{g}_0,\ldots,\mathsf{g}_{n_0},\ldots,\mathsf{g}_{n_1}\right\rangle, \qquad \textit{etc.},\end{aligned}$$

and it has been applied in [39] to set up an algorithm wich generates all polynomials in the $\kappa$-jet of a local holomorphic map $\mathbb{D} \to \mathbb{C}^n$, $\zeta \longmapsto (f_1(\zeta),\ldots,f_n(\zeta))$ from the unit disc $\mathbb{D} = \{|\zeta| < 1\}$ that are invariant under all biholomorphic reparametrizations of $\mathbb{D}$ with saturation with respect to the first derivative $f_1'$. The explicit generators for $n = 4 = \kappa$ and for $n = 2$, $\kappa = 5$ given in [39] show well that saturation (division) by $f_1'$ generates some unpredictable complexity, a well known phenomenon in invariant theory.

Later, Berczi and Kirwan ([7]), by developing concepts and tools from reductive geometric invariant theory, showed that the concerned algebra is always finitely generated. A challenging still open question is to get information about the number of generators and about the structure of relations they share. In any case, the prohibitive complexity of these algebras still prevents to hope for reaching arbitrary dimension $n \geqslant 2$ and jet order $\kappa \geqslant n$ for Green-Griffiths and Kobayashi conjectures with invariant jets.

Around the same time, under the direction of Demailly and using an algebraic version of holomorphic Morse inequalities delineated by Trapani, Diverio studied a certain *subbundle* of the bundle of invariant jets, already introduced before by Demailly in [16]. This, for the first time after Siu, opened the door to arbitrary dimension $n \geqslant 2$, though this was clearly not sufficient to reach the first step towards the Green-Griffiths-Lang conjecture. In fact, an inspection of [19] shows that on hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d$ approximately $\geqslant 2^{n^5}$ (rounding off), many differential equations exist with jet order $\kappa = n$ equal to the dimension, but when increasing the jet order $\kappa = n+1, n+2, \ldots$, an unpleasant stabilization of the degree gain occurs, so that there is absolutely no hope to reach the optimal $d \geqslant n+3$ for the first step towards the Green-Griffiths-Lang conjecture (as did Green-Griffiths in 1979 in dimension $n = 2$) with Diverio's technique (even) for hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$.

*For all of these reasons, it became undoubtedly clear that the whole theory had to come back to the bundle of (plain) Green-Griffiths jets.*

On an $n$-dimensional complex manifod $X^n$, for a jet order $\kappa \geqslant 1$ and for an homogeneous order $m \geqslant 1$, the *Green-Griffiths bundle* $\mathscr{E}_{\kappa,m}^{\mathsf{GG}} T_X^* \longrightarrow X$ in a local chart $(z_1,\ldots,z_n)\colon \mathsf{U} \longrightarrow \mathbb{C}^n$ with $\mathsf{U} \subset X$ open, has general local holomorphic sections which are polynomials in the derivatives $z'$, $z''$, $\ldots$, $z^{(\kappa)}$ of the $z_i$ (considered as functions of a single variable $\zeta \in \mathbb{D}$) of the form:

$$\sum_{|\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m} P_{\alpha_1,\ldots,\alpha_\kappa}(\mathrm{z}) \left(\mathrm{z}'\right)^{\alpha_1} \left(\mathrm{z}''\right)^{\alpha_2} \cdots \left(\mathrm{z}^{(\kappa)}\right)^{\alpha_\kappa},$$

the $P_{\alpha_1,\ldots,\alpha_\kappa}$ being holomorphic in $\mathsf{U}$ (this local definition ignores rationality features of the $P_{\alpha,\ldots,\alpha_\kappa}$ which will be explored *infra*).

The memoir [42] established that on a hypersurface $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree:

$$d \geqslant n+3,$$

if $\mathscr{A} \to X$ is any ample line bundle — take *e.g.* simply $\mathscr{A} := \mathscr{O}_X(1)$ —, then:

$$h^0\big(X, \mathscr{E}^{GG}_{\kappa,m} T^*_X \otimes \mathscr{A}^{-1}\big) \geqslant \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big\{\frac{(\log\kappa)^n}{n!}d(d-n-2)^n - \mathsf{Constant}_{n,d}\cdot(\log\kappa)^{n-1}\Big\} - \mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2},$$

a formula in which the right-hand side minorant visibly tends to $\infty$, as soon as both $\kappa \geqslant \kappa^0_{n,d}$ and $m \geqslant m^0_{n,d,\kappa}$ do (no explicit expressions of the constants was provided there). This, then, generalized to dimension $n \geqslant 2$ the Green and Griffiths surface theorem, by estimating the asymptotic quantititative behavior of weighted Young diagrams and by applying partial (good enough) results of Brückmann ([9]) concerning the cohomology of Schur bundles $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X$.

Also coming back to plain Green-Griffiths jets, but developing completely different elaborate negative jet curvature estimates which go back to an article of Cowen and Griffiths ([11]) and which had been 'in the air' for some time, though 'blocked for deep reasons' by the untractable algebraic complexity of invariant jets, Demailly ([17]) realized the next significant advance towards the conjecture by establishing, under the sole assumption that $X$ be of general type (not necessarily a hypersurface), that nonconstant entire holomorphic curves $f\colon \mathbb{C} \longrightarrow X$ always satisfy (many) nonzero differential equations. The Bourbaki Seminar 1061 by Paŭn ([60]) is a useful guide to enter the main concepts and techniques of the topic.

However, according to [21], it is impossible to reach the Green-Griffiths conjecture for *all* general type compact complex manifolds by applying the jet differential technique. This justifies to restrict attention to hypersurfaces or to complete intersections in the projective space, and in this case, as recently highlighted once again by Siu ([70]), the only convincing strategy towards a first solution to Kobayashi's conjecture in arbitrary dimension $n \geqslant 1$ is to develope a new systematic theory of explicit *rational* holomorphic sections of jet bundles.

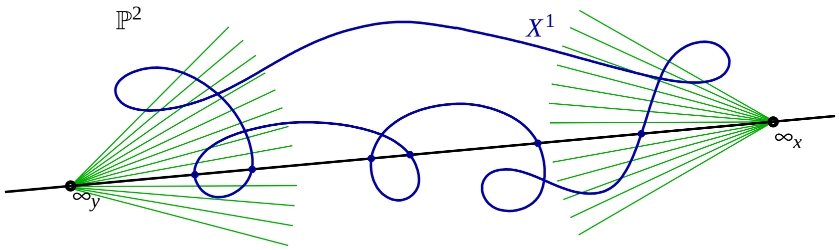
## *3.1 Holomorphic jet differentials*

Let $[X_0\colon X_1\colon \cdots\colon X_n] \in \mathbb{P}^n(\mathbb{C})$ be homogenous coordinates. Recall that for $t \in \mathbb{N}$, holomorphic sections of $\mathscr{O}_{\mathbb{P}^n}(t)$ are represented on $\mathsf{U}_i = \{X_i \neq 0\}$ as quotients:

$$\ell_i([X]) := \frac{P(X_0\colon X_1\colon \cdots\colon X_n)}{(X_i)^t},$$

for some polynomial $P \in \mathbb{C}[X]$ homogeneous of degree $t$, while, for $t \in \mathbb{Z}\backslash\mathbb{N}$, meromorphic sections are:

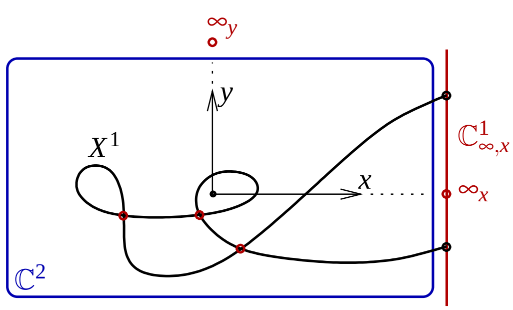
$$\ell_i([X]) := \frac{(X_i)^t}{P(X_0 \colon X_1 \colon \cdots \colon X_n)}.$$




To review another global rationality phenomenon, consider a complex algebraic curve $X^1$ smooth or with simple normal crossings in $\mathbb{P}^2(\mathbb{C})$ of degree $d \geqslant 1$, choose two points $\infty_x, \infty_y \notin X$ so that the line $\overline{\infty_x \infty_y}$ intersects $X^1$ transversally in $d$ distinct points, and adapt homogeneous coordinates $[T \colon X \colon Y] \in \mathbb{P}^2$ with affine $x := \frac{X}{T}$ and $y := \frac{Y}{T}$ so that $\overline{\infty_x \infty_y} = \mathbb{P}^\infty_1 = \{[0 \colon X \colon Y]\}$, $\infty_x = [0 \colon 1 \colon 0]$, $\infty_y = [0 \colon 0 \colon 1]$, whence:

$$X^1 \cap \mathbb{C}^2_{(x,y)} = \big\{(x,y) \in \mathbb{C}^2 \colon R(x,y) = 0\big\},$$

for some polynomial $R \in \mathbb{C}[x,y]$ of degree $d$.




Within the intrinsic theory, in terms of the ambient line bundles $\mathscr{O}_{\mathbb{P}^2}(t) \longrightarrow \mathbb{P}^2$ ($t \in \mathbb{Z}$), the adjunction formula tells:

$$T_X^* \cong \mathscr{O}_X(d-3) \overset{\text{def}}{=} \mathscr{O}_{\mathbb{P}^2}(d-3)\big|_X.$$

The genus formula is of great importance, because it exposes the relationship between the '*intrinsic*' topological invariant $g$ of the curve $X^1$ and the '*extrinsic*' quantity $d$. Phillip GRIFFITHS.

**Theorem 10. (Inspirational)** *On a smooth degree $d$ algebraic curve $X^1 \subset \mathbb{P}^2$:*

$$\tfrac{(d-1)(d-2)}{2} = \dim H^0\big(X, T_X^*\big) = \mathsf{genus}(X) = g.$$

But the extrinsic theory ([28]) tells more. Differentiating once $0 = R(x,y)$:

$$0 = R_x\, dx + R_y\, dy \qquad \Longleftrightarrow \qquad \frac{dy}{R_x} = -\frac{dx}{R_y},$$

*denominators must appear*. If $X^1$ is smooth, $X^1 \cap \mathbb{C}^2 = \{R_x \neq 0\} \cup \{R_y \neq 0\}$, and global holomorphic sections of $T_X^*$ are represented by multiplications:

$$G(x,y)\left(\frac{dy}{R_x} = -\frac{dx}{R_y}\right),$$

with $G \in \mathbb{C}[x,y]$ having degree $\leqslant d-3$, the space of such $G$ being of dimension $\frac{(d-3+2)(d-3+1)}{2}$, since changing affine chart in order to capture $\mathbb{P}^1_\infty \backslash \{\infty_x\}$:

$$(x,y) \longmapsto \left(\frac{x}{y}, \frac{1}{y}\right) =: \big(x_2, y_2\big), \qquad\qquad R_2(x_2,y_2) := (y_2)^d\, R\left(\frac{x_2}{y_2}, \frac{1}{y_2}\right),$$

knowing $dy = -\,dy_2/(y_2)^2$, the left side $G\,dy/R_x$ transfers to:

$$G(x,y)\,\frac{dy}{R_x} = G\left(\frac{x_2}{y_2}, \frac{1}{y_2}\right)(y_2)^{d-3}\,\frac{-\,dy_2}{R_{2,x_2}(x_2,y_2)},$$

the denominator $R_{2,x_2}(x_2,y_2)$ being nonzero on $X$ at every point of $\{y_2 = 0\} = \mathbb{C}^1_{\infty,y}$, while $(y_2)^{d-3}$ compensates the poles of $G\big(\frac{x_2}{y_2}, \frac{1}{y_2}\big)$ as soon as $\deg G \leqslant d-3$.

Next, the *intrinsic* Riemann-Roch theorem states that, given any divisor $D$ on a compact, abstract, Riemann surface $S$, if $\mathscr{O}_D$ denotes the sheaf of meromorphic functions $f \in \Gamma(\mathscr{M}_S)$ with $\mathrm{div}\, f \geqslant -D$, then:

$$\dim H^0\big(S, \mathscr{O}_D\big) - \dim H^1\big(S, \mathscr{O}_D\big) = \deg D - \mathrm{genus}(S) + 1.$$

For compact Riemann surfaces $S$, there exists a satisfactory correspondence between intrinsic features and extrinsic embeddings: all $S$ admit a representation as a curve $X^1 \subset \mathbb{P}^2$, smooth or having normal crossings singularities.

Using Brill-Noether duality, the Riemann-Roch theorem can be proved ([28]) for such $X^1 \subset \mathbb{P}^2$ by means of two inequalities:

$$\begin{aligned}
\deg D - \mathrm{g}(S) + 1 &\leqslant \dim \big\{ f \in \mathscr{M}(S) \colon\ \mathrm{div}(f) \geqslant -D \big\} = \dim H^0\big(S, \mathscr{O}_D\big), \\
-\deg D + \mathrm{g}(S) - 1 &\leqslant \dim \big\{ \omega \in \mathscr{M}T_X^*(S) \colon\ \mathrm{div}(\omega) \geqslant +D \big\} = \dim H^1\big(S, \mathscr{O}_D\big).
\end{aligned}$$

For instance, the second inequality is proved by means of *extrinsic* meromorphic differential forms:

$$\frac{G}{H}\left(\frac{dy}{R_x} = -\frac{dx}{R_y}\right),$$

with $G, H \in \mathbb{C}[x,y]$ subjected to appropriate conditions with respect to $D$.

**Jets of order** 2. Next, consider second order jets of holomorphic maps $\mathbb{D} \to X^1 \subset \mathbb{P}^2$, use $x'$, $y'$ instead of $dx$, $dy$, and $x''$, $y''$. Differentiate $0 \equiv R\big(x(\zeta), y(\zeta)\big)$ twice:

$$0 = x' R_x + y' R_y, \qquad\qquad 0 = x'' R_x + y'' R_y + (x')^2 R_{xx} + 2\, x' y' R_{xy} + (y')^2 R_{yy},$$

divide by $R_x R_y$, solve for $y''$, replace $y'$ on the right:

$$\frac{y'}{R_x} = -\frac{x'}{R_y}, \qquad\qquad \frac{y''}{R_x} = -\frac{x''}{R_y} - \frac{(x')^2}{R_y}\left[\frac{R_{xx}}{\boxed{R_x}} - 2\,\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\,\frac{R_{yy}}{R_y}\right].$$

To erase the division by $R_x$, multiply the first equation by $x' \frac{R_{xx}}{R_x}$ and subtract:

$$\frac{y''}{R_x} - \frac{y'x'}{R_x}\,\frac{R_{xx}}{R_x} = -\frac{x''}{R_y} - \frac{(x')^2}{R_y}\left[-2\,\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\,\frac{R_{yy}}{R_y}\right].$$

Little further, this expression can be symmetrized ([48]):

$$\frac{y''}{R_x} + \frac{(y')^2}{R_x}\left[-\frac{R_{xy}}{R_x} + \frac{R_y}{R_x}\,\frac{R_{xx}}{R_x}\right] = -\frac{x''}{R_y} - \frac{(x')^2}{R_y}\left[-\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\,\frac{R_{yy}}{R_y}\right],$$

and this provides *second order holomorphic jet differentials* on $X^1 \subset \mathbb{P}^2$ when $d \geqslant 4$, after checking holomorphicity on the $\mathbb{P}^1_\infty$. For jets of order 3 ([48]):

$$\begin{aligned}
&\frac{y'''}{R_x} + \frac{y''y'}{R_x}\left[-3\,\frac{R_{xy}}{R_x} + 3\left(\frac{R_y}{R_x}\right)\frac{R_{xx}}{R_x}\right] + \\
&\quad + \frac{(y')^3}{R_x}\left[-6\left(\frac{R_y}{R_x}\right)\frac{R_{xy}}{R_x}\,\frac{R_{xx}}{R_x} + 3\left(\frac{R_y}{R_x}\right)^2\frac{R_{xx}}{R_x}\,\frac{R_{xx}}{R_x} + 3\left(\frac{R_y}{R_x}\right)\frac{R_{xxy}}{R_x} - \left(\frac{R_y}{R_x}\right)^2\frac{R_{xxx}}{R_x}\right] = \\
&= -\frac{x'''}{R_y} - \frac{x''x'}{R_y}\left[-3\,\frac{R_{xy}}{R_y} + 3\left(\frac{R_x}{R_y}\right)\frac{R_{yy}}{R_y}\right] - \\
&\quad - \frac{(x')^3}{R_y}\left[-6\left(\frac{R_x}{R_y}\right)\frac{R_{xy}}{R_y}\,\frac{R_{yy}}{R_y} + 3\left(\frac{R_x}{R_y}\right)^2\frac{R_{yy}}{R_y}\,\frac{R_{yy}}{R_y} + 3\left(\frac{R_x}{R_y}\right)\frac{R_{xyy}}{R_y} - \left(\frac{R_x}{R_y}\right)^2\frac{R_{yyy}}{R_y}\right].
\end{aligned}$$

Strikingly, and quite interestingly, there appear explicit rational expressions belonging to $\mathbb{Z}\big[\frac{R_{\cdot\cdot}}{R_x}\big]$ on the left, and to $\mathbb{Z}\big[\frac{R_{\cdot\cdot}}{R_y}\big]$ on the right.

**Jets of arbitrary order.** The *Green-Griffiths* bundle $\mathscr{E}^{\rm GG}_{\kappa,m}T^*_{X^1} \longrightarrow X^1 \subset \mathbb{P}^2$ consists, for a jet order $\kappa \geqslant 1$, in the $m$-homogeneous polynomialization of the bundle $J^\kappa\big(\mathbb{D}, X^1\big)$, and is a *vector* bundle of:

$$\mathsf{rank}\left(\mathscr{E}^{\rm GG}_{\kappa,m}T^*_{X^1}\right) = \mathsf{Card}\Big\{\big(m_1, m_2, \dots, m_\kappa\big) \in \mathbb{N}^\kappa \colon\, m_1 + 2\,m_2 + \cdots + \kappa\, m_\kappa = m\Big\}.$$

It admits a natural filtration whose associated graded vector bundle is ([42]):

$$\mathsf{Gr}^\bullet\mathscr{E}^{\rm GG}_{\kappa,m}T^*_{X^1} \cong \bigoplus_{\substack{m_1+\cdots+\kappa m_\kappa = m \\ m_1 \geqslant 0, \dots, m_\kappa \geqslant 0}} \Big(\mathsf{Sym}^{m_1}T^*_X \otimes \cdots \otimes \mathsf{Sym}^{m_\kappa}T^*_X\Big),$$

whence:

$$\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{\mathrm{GG}} T_{X^1}^* \cong \bigoplus_{\substack{m_1+\cdots+\kappa m_\kappa = m \\ m_1 \geqslant 0, \ldots, m_\kappa \geqslant 0}} \mathscr{O}_X\Big(\big(m_1+\cdots+m_\kappa\big)\,(d-3)\Big).$$

Knowing that for $t \geqslant d$:

$$\mathsf{dim}\, H^0\big(X, \mathscr{O}_X(t)\big) = \binom{t+2}{2} - \binom{t-d+2}{2},$$

and knowing $H^1\big(X, \mathscr{O}_X(t)\big) = 0$, it follows:

$$\begin{aligned}\mathsf{dim}\, H^0\big(X, \mathscr{E}_{\kappa,m}^{\mathrm{GG}} T_{X^1}^*\big) = \sum_{m_1+\cdots+\kappa m_\kappa = m} \Bigg\{ & \binom{(m_1+\cdots+m_\kappa)(d-3)+2}{2} - \\ & - \binom{(m_1+\cdots+m_\kappa)(d-3)-d+2}{2} \Bigg\},\end{aligned}$$

which, asymptotically, becomes:

$$\mathsf{dim}\, H^0\big(X, \mathscr{E}_{\kappa,m}^{\mathrm{GG}} T_X^*\big) \geqslant \frac{m^\kappa}{\kappa!\,\kappa!}\Big[d^2 \log \kappa + d^2\,\mathrm{O}(1) + \mathrm{O}(d)\Big] + \mathrm{O}\big(m^{\kappa-1}\big).$$

**Theorem 11.** ([48]) *Given an arbitrary jet order $\kappa \geqslant 1$, for every $1 \leqslant \lambda \leqslant \kappa$, if* $\mathsf{deg}\, R \geqslant \kappa + 3$, *there exist perfectly symmetric expressions:*

$$\mathsf{J}_R^\lambda := \begin{cases} \dfrac{y^{(\lambda)}}{R_x} + \displaystyle\sum_{\mu_1+\cdots+(\lambda-1)\mu_{\lambda-1}=\lambda} \frac{(y')^{\mu_1}\cdots\big(y^{(\lambda-1)}\big)^{\mu_{\lambda-1}}}{R_x}\, \mathscr{J}_{\mu_1,\ldots,\mu_{\lambda-1}}^\lambda \left(\frac{R_y}{R_x}, \left(\frac{R_{x^i y^j}}{R_x}\right)_{\substack{2\leqslant i+j \leqslant \\ \leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}}}\right), \\ -\dfrac{x^{(\lambda)}}{R_y} - \displaystyle\sum_{\mu_1+\cdots+(\lambda-1)\mu_{\lambda-1}=\lambda} \frac{(x')^{\mu_1}\cdots\big(x^{(\lambda-1)}\big)^{\mu_{\lambda-1}}}{R_y}\, \mathscr{J}_{\mu_1,\ldots,\mu_{\lambda-1}}^\lambda \left(\frac{R_x}{R_y}, \left(\frac{R_{y^i x^j}}{R_y}\right)_{\substack{2\leqslant i+j \leqslant \\ \leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}}}\right), \\ 0 \qquad \text{on } X^1 \cap \mathbb{P}_\infty^1, \end{cases}$$

*which define generating holomorphic jet differentials on the smooth curve $X^1 \subset \mathbb{P}^2$, notably on the two open subsets:*

$$\begin{aligned} &\big\{R_x \neq 0\big\} \quad \text{where the bundle } J^\kappa(\mathbb{D}, X^1) \text{ has intrinsic coordinates:} \\ &\qquad\qquad \big(y; y', y'', \ldots, y^{(\kappa)}\big), \\ &\big\{R_y \neq 0\big\} \quad \text{where the bundle } J^\kappa(\mathbb{D}, X^1) \text{ has intrinsic coordinates:} \\ &\qquad\qquad \big(x; x', x'', \ldots, x^{(\kappa)}\big). \end{aligned}$$

*All $J_R^\lambda$ vanish on the ample divisor $X^1 \cap \mathbb{P}_\infty^1$, and involve universal polynomials:*

$$\mathscr{J}_{\mu_1,\ldots,\mu_{\lambda-1}}^\lambda = \mathscr{J}_{\mu_1,\ldots,\mu_{\lambda-1}}^\lambda\Big(\mathsf{R}_{0,1},\ \big(\mathsf{R}_{i,j}\big)_{2\leqslant i+j\leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}}\Big)$$

*with coefficients in $\mathbb{Z}$, and in terms of these generating jet differentials, holomorphic sections of $\mathscr{E}^{\rm GG}_{\kappa,m}T^*_{X^1}$ are generally represented as:*

$$\boxed{\sum_{m_1+2m_2+\cdots+\kappa m_\kappa=m} \big(\mathsf{J}^1_R\big)^{m_1}\big(\mathsf{J}^2_R\big)^{m_2}\cdots\big(\mathsf{J}^\kappa_R\big)^{m_\kappa}\cdot\mathsf{G}_{m_1,m_2,\dots,m_\kappa}(x,y),}$$

*with polynomials:*

$$\mathsf{G}_{m_1,m_2,\dots,m_\kappa}=\mathsf{G}_{m_1,m_2,\dots,m_\kappa}(x,y)$$

*of degrees:*

$$\deg\mathsf{G}_{m_1,m_2,\dots,m_\kappa}\leqslant\underbrace{m_1(d-3)+m_2(d-4)+\cdots+m_\kappa\big(d-\kappa-2\big)}_{=:\delta},$$

*which belong to the quotient spaces:*

$$\mathbb{C}_\delta[x,y]\Big/R\cdot\mathbb{C}_{\delta-d}[x,y],$$

*the total number of such sections being equal to:*

$$\sum_{m_1+\cdots+\kappa m_\kappa=m}\bigg\{\binom{m_1(d-3)+\cdots+m_\kappa(d-\kappa-2)+2}{2}-\binom{m_1(d-3)+\cdots+m_\kappa(d-\kappa-2)-d+2}{2}\bigg\},$$

*that is to say, also asymptotically equal to:*

$$\frac{m^\kappa}{\kappa!\,\kappa!}\Big[d^2\log\kappa+d^2\,\mathrm{O}(1)+\mathrm{O}(d)\Big]+\mathrm{O}\big(m^{\kappa-1}\big).$$

This generalizes directly to the case of complete intersection *curves* $X^1\subset\mathbb{P}^{1+c}(\mathbb{C})$:

$$0=R^1\big(z_1,\dots,z_c,z_{c+1}\big),\ \ \cdots\cdots\cdots,\ \ 0=R^c\big(z_1,\dots,z_c,z_{c+1}\big),$$

minors of the Jacobian matrix naturally occupying denominator places:

$$\frac{z_1'}{\begin{vmatrix}R^1_{z_2}&\cdots&R^1_{z_{c+1}}\\ \cdot\cdot&\cdots&\cdot\cdot\\ R^c_{z_2}&\cdots&R^c_{z_{c+1}}\end{vmatrix}}=\cdots\cdots=(-1)^c\,\frac{z_{c+1}'}{\begin{vmatrix}R^1_{z_1}&\cdots&R^1_{z_c}\\ \cdot\cdot&\cdots&\cdot\cdot\\ R^c_{z_1}&\cdots&R^c_{z_c}\end{vmatrix}}.$$

**Holomorphic sections of the canonical bundle.** For $X^n\subset\mathbb{P}^{n+1}(\mathbb{C})$ a smooth hypersurface:

$$0=R\big(z_1,\dots,z_n,z_{n+1}\big),$$

affinely represented as the zero-set of a degree $d\geqslant 1$ polynomial, whence:

$$0=R_{z_1}\,dz_1+\cdots+R_{z_n}\,dz_n+R_{z_{n+1}}\,dz_{n+1},$$

holomorphic sections of the *canonical bundle* $K_X := \Lambda^n T_X^*$ (which generalizes the cotangent $T_X^*$ of $X^1 \subset \mathbb{P}^2$), are represented by the equalities:

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{R_{z_{n+1}}} = -\frac{dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dz_{n+1}}{R_{z_n}} = \cdots\cdots\cdots = (-1)^n \frac{dz_2 \wedge \cdots \wedge dz_{n+1}}{R_{z_1}},$$

that are always holomorphic on the $\mathbb{P}^n_\infty$ as soon as $d \geqslant n+3$.

**Question.** *For $X^n \subset \mathbb{P}^{n+c}$ a complete intersection $\{0 = R^1 = \cdots = R^c\}$ of codimension $c$, are there explicit rational holomorphic sections of $\mathscr{E}^{\mathrm{GG}}_{\kappa,m} T_X^*$ having as denominators the appropriate minors of the Jacobian matrix $\big(R^j_{z_k}\big)$ and numerators in $\mathbb{Z}\big[R^j_{z_1^{\alpha_1}\cdots z_{n+c}^{\alpha_{n+c}}}\big]$?*

**Surfaces $X^2 \subset \mathbb{P}^3$.** Let $X^2 \subset \mathbb{P}^3$ be a smooth surface represented in affine coordinates $(x,y,z) \in \mathbb{C}^3 \subset \mathbb{P}^3$ as:

$$0 = R(x,y,z),$$

for some polynomial $R \in \mathbb{C}[x,y,z]$ of degree $d \geqslant 1$. Differentiate this once:

$$0 = x' R_x + y' R_y + z' R_z.$$

Three natural open sets $\{R_x \neq 0\}$, $\{R_y \neq 0\}$, $\{R_z \neq 0\}$ cover $X^2$, by smoothness. On $X^2 \cap \{R_x \neq 0\}$, coordinates are $(y,z)$, cotangent (fiber) coordinates are $(y',z')$. The change of trivialization for $T_X^* \cong J^1(\mathbb{D},X)$ from above $X^2 \cap \{R_x \neq 0\}$ to above $X^2 \cap \{R_y \neq 0\}$:

$$\big(y,z,y',z'\big) \longmapsto \big(x,z,x',z'\big)$$

amounts to just solving:

$$y' = -x' \frac{R_x}{R_y} - z' \frac{R_z}{R_y}.$$

Inspired by what precedes for curves $X^1 \subset \mathbb{P}^2$, seek global holomorphic sections of:

$$\mathscr{E}^{\mathrm{GG}}_{1,m} T_X^* \cong \mathsf{Sym}^m T_X^*$$

(symmetric differentials) under the form:

$$\sum_{j+k=m} \mathsf{coeff}_{j,k} \cdot (y')^j \, (z')^k \quad \underset{\text{trivialization}}{\overset{\text{change of}}{\longmapsto}} \quad \sum_{j+k=m} \mathsf{coeff}_{\widetilde{j,k}} \cdot (x')^j \, (z')^k,$$

with all coefficient-functions $\mathsf{coeff}_{j,k}(y,z)$ being holomorphic on $X^2 \cap \{R_x \neq 0\}$ and all $\mathsf{coeff}_{\widetilde{j,k}}(x,z)$ being holomorphic on $X^2 \cap \{R_y \neq 0\}$. What sort of coefficients? A proposal of answer, inspired by $X^1 \subset \mathbb{P}^2$, is that they belong to:

$$\frac{1}{R_x} \mathbb{Z}\bigg[\frac{R_y}{R_x}, \frac{R_z}{R_x}\bigg] \qquad \text{and to:} \qquad \frac{1}{R_y} \mathbb{Z}\bigg[\frac{R_x}{R_y}, \frac{R_z}{R_y}\bigg],$$

because then, such jet differentials would vanish on the $\mathbb{P}^2_\infty$, as soon as $\deg R \geqslant 2m+2$. Of course, in this special case, since the intrinsic theory ([20, 8]) kowns that whenever $\kappa < \frac{n}{c}$, on a complete intersection $X^n \subset \mathbb{P}^{n+c}$:

$$0 = H^0\big(X, \mathscr{E}^{\mathrm{GG}}_{\kappa,m} T^*_X\big),$$

whence with $n = 2$, $c = 1$ here, inexistence is expectable. As a confirmation:

**Proposition 4.** *For every $m \geqslant 1$, if polynomials $\Pi_{j,k} \in \mathbb{Z}[\mathsf{U},\mathsf{V}]$ are such that:*

$$\begin{aligned}\sum_{j+k=m} \frac{(y')^j\,(z')^k}{R_x}\,\Pi_{j,k}\bigg(\frac{R_y}{R_x}, \frac{R_z}{R_x}\bigg) &= \sum_{j+k=m} \frac{\big(-x'\,\frac{R_x}{R_y} - z'\,\frac{R_z}{R_y}\big)^j\,(z')^k}{R_x}\,\Pi_{j,k}\bigg(\frac{R_y}{R_x}, \frac{R_z}{R_x}\bigg) \\ &= \sum_{j+k=m} \frac{(x')^j\,(z')^k}{R_y}\,\Pi^{\sim}_{j,k}\bigg(\frac{R_x}{R_y}, \frac{R_z}{R_y}\bigg)\end{aligned}$$

*rewrites, after transitioning, only with $\frac{1}{R_y}$-denominators, then all $\Pi_{j,k} \equiv 0$.*

However, the intrinsic theory knows already ([71, 16, 68, 69, 65, 19, 20, 8]) that for $X^n \subset \mathbb{P}^{n+c}$, global holomorphic sections of $\mathscr{E}^{\mathrm{GG}}_{\kappa,m} T^*_X$ exist when $\kappa \geqslant \frac{n}{c}$.

Hence for $n = 2$, $c = 1$, $\kappa = 2$, differentiate *twice:*

$$\begin{aligned}0 &= x'\,R_x + y'\,R_y + z'\,R_z, \\ 0 &= x''\,R_x + y''\,R_y + z''\,R_z + (x')^2\,R_{xx} + (y')^2\,R_{yy} + (z')^2\,R_{zz} + 2x'y'\,R_{xy} + 2x'z'\,R_{xz} + 2y'z'\,R_{yz}.\end{aligned}$$

The interesting question (to which no answer is known not up to date) is whether there exist holomorphic jet differentials of the rational form:

$$\sum_{j_1+k_1+j_2+k_2=m} \frac{(y')^{j_1}\,(z')^{k_1}\,(y'')^{j_2}\,(z'')^{k_2}}{R_x} \cdot \Pi_{j_1k_1j_2k_2}\bigg(\frac{R_y}{R_x}, \frac{R_z}{R_x}, \frac{R_{xx}}{R_x}, \frac{R_{yy}}{R_x}, \frac{R_{zz}}{R_x}, \frac{R_{xy}}{R_x}, \frac{R_{xz}}{R_x}, \frac{R_{yz}}{R_x}\bigg),$$

having the property that, after replacement of:

$$\begin{aligned}y' &= -x'\,\frac{R_x}{R_y} - z'\,\frac{R_z}{R_y}, \\ y'' &= -x''\,\frac{R_x}{R_y} - z''\,\frac{R_z}{R_y} - (x')^2\,\frac{R_{xx}}{R_y} - (y')^2\,\frac{R_{yy}}{R_y} - (z')^2\,\frac{R_{zz}}{R_y} - 2x'y'\,\frac{R_{xy}}{R_y} - 2x'z'\,\frac{R_{xz}}{R_y} - 2y'z'\,\frac{R_{yz}}{R_y},\end{aligned}$$

after expansion and after reorganization, a similar jet-rational expression is got:

$$\sum_{j_1+k_1+j_2+k_2=m} \frac{(x')^{j_1}\,(z')^{k_1}\,(y'')^{j_2}\,(z'')^{k_2}}{R_y} \cdot \Pi^{\sim}_{j_1k_1j_2k_2}\bigg(\frac{R_x}{R_y}, \frac{R_z}{R_y}, \frac{R_{xx}}{R_y}, \frac{R_{yy}}{R_y}, \frac{R_{zz}}{R_y}, \frac{R_{xy}}{R_y}, \frac{R_{xz}}{R_y}, \frac{R_{yz}}{R_y}\bigg)$$

which involves division by only $R_y$. The number of variables becomes 8 (large):

$$\Pi_{j_1k_1j_2k_2} \in \frac{1}{R_x}\cdot\mathbb{Z}\bigg[\frac{R_y}{R_x},\frac{R_z}{R_x},\frac{R_{xx}}{R_x},\frac{R_{yy}}{R_x},\frac{R_{zz}}{R_x},\frac{R_{xy}}{R_x},\frac{R_{xz}}{R_x},\frac{R_{yz}}{R_x}\bigg],$$
$$\Pi^{\sim}_{j_1k_1j_2k_2} \in \frac{1}{R_y}\cdot\mathbb{Z}\bigg[\frac{R_x}{R_y},\frac{R_z}{R_y},\frac{R_{xx}}{R_y},\frac{R_{yy}}{R_y},\frac{R_{zz}}{R_y},\frac{R_{xy}}{R_y},\frac{R_{xz}}{R_y},\frac{R_{yz}}{R_y}\bigg].$$

By anticipation, for $X^n \subset \mathbb{P}^{n+1}$ and for jets of order $\kappa = n$:

$$\#\Big(\text{partial derivatives}\,R_{z_1^{\alpha_1}\cdots z_n^{\alpha_n}z_{n+1}^{\alpha_{n+1}}}\Big) = \binom{n+1+n}{n} \sim 2^{2n+1}\,\frac{1}{\sqrt{\pi n}},$$

hence something is *intimately exponential* in the subject. For $X^2 \subset \mathbb{P}^3$ of degree $d \gg 1$, a patient cohomology sequences chasing shows that there exist nonzero second-order holomorphic jet differentials in $H^0\big(X,\mathscr{E}^{\mathrm{GG}}_{2,m}T^*_X\big)$ only when:

$$m \geqslant 14$$

(similarly, by [8], for $X^2 \subset \mathbb{P}^4$ of bidegrees $d_1, d_2 \gg 1$, it is necessary that $m \geqslant 10$). Hence combinatorially, there is a complexity obstacle, and moreover, an inspection of what holds true for curves $X^1 \subset \mathbb{P}^2$ shows that it is quite probable that the degrees of the $\Pi_{j_1k_1j_2k_2}$ are about to be approximately equal to $m \geqslant 14$, whence the total number of monomials they involve:

$$\binom{14+8}{8} = 319\,770,$$

would be already rather large to determine in a really effective way whether they exist.

It happens to be a bit easier to work with the Wronskians:

$$\square := \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} = y'z'' - z'y'', \qquad\qquad \Delta := \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} = z'x'' - x'z''.$$

Two fundamental transition formulas are:

$$\begin{aligned}
\frac{y'}{R_x} &= -\frac{x'}{R_y} - \frac{z'}{R_y}\,\frac{R_z}{R_x},\\
\frac{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}}{R_x} &= \frac{\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}}{R_y} - \frac{(x')^2z'}{R_y}\left[\left(\frac{R_x}{R_y}\right)\frac{R_{yy}}{R_y} - 2\,\frac{R_{xy}}{R_y} + \frac{R_{xx}}{R_x}\right] -\\
&\quad - 2\,\frac{x'(z')^2}{R_y}\left[\left(\frac{R_z}{R_y}\right)\frac{R_{yy}}{R_y} - \frac{R_{yz}}{R_y} + \frac{R_{xz}}{R_x} - \left(\frac{R_z}{R_y}\right)\frac{R_{xy}}{R_x}\right] -\\
&\quad - \frac{(z')^3}{R_y}\left[\left(\frac{R_z}{R_y}\right)^2\frac{R_{yy}}{R_x} - 2\left(\frac{R_z}{R_y}\right)\frac{R_{yz}}{R_x} + \frac{R_{zz}}{R_x}\right].
\end{aligned}$$

Set as abbreviated new notations:

$$
\begin{aligned}
\frac{R_x}{R_y} &=: r_x, & \frac{R_z}{R_y} &=: r_z,\\
\frac{R_{xx}}{R_y} &=: r_{yy}, & \frac{R_{xy}}{R_y} &=: r_{xy},\\
\frac{R_{yy}}{R_y} &=: r_{zz}, & \frac{R_{xz}}{R_y} &=: r_{xz},\\
\frac{R_{zz}}{R_y} &=: r_{xx}, & \frac{R_{yz}}{R_y} &=: r_{yz}.
\end{aligned}
$$

Rewrite:

$$
\begin{aligned}
y' &= -x'\,r_x - z'\,r_z,\\
\square &= \Delta\, r_x - (x')^2 z' \Big[r_x\, r_x\, r_{yy} - 2\, r_x\, r_{xy} + r_{xx}\Big] -\\
&\qquad - 2\,x'(z')^2 \Big[r_x\, r_z\, r_{yy} - r_x\, r_{yz} + r_{xz} - r_z\, r_{xy}\Big] -\\
&\qquad - (z')^3 \Big[r_z\, r_z\, r_{yy} - 2\, r_z\, r_{yz} + r_{zz}\Big].
\end{aligned}
$$

Divide both sides by:

$$
R_x = r_x\, R_y,
$$

and obtain:

$$
\begin{aligned}
\frac{y'}{R_x} &= -\frac{x'}{R_y} - \frac{z'}{R_y}\,\frac{r_z}{r_x},\\
\frac{\square}{R_x} &= \frac{\Delta}{R_y} - \frac{(x')^2 z'}{R_y}\Big[r_x\, r_{yy} - 2\, r_{xy} + \frac{r_{xx}}{r_x}\Big] -\\
&\qquad - 2\,\frac{x'(z')^2}{R_y}\Big[r_z\, r_{yy} - r_{yz} + \frac{r_{xz}}{r_x} - \frac{r_z\, r_{xy}}{r_x}\Big] -\\
&\qquad \frac{(z')^3}{R_y}\Big[\frac{r_z\, r_z\, r_{yy}}{r_x} - 2\,\frac{r_z\, r_{yz}}{r_x} + \frac{r_{zz}}{r_x}\Big].
\end{aligned}
$$

In the case $n = 2 = \kappa$, $c = 1$, the question formulated above becomes:

**Question.** *Do there exist nontrivial linear combinations of:*

$$
\frac{\big(-x'r_x - z'r_z\big)^j (z')^k \begin{pmatrix} \Delta r_x - (x')^2 z' \big[r_x r_x r_{yy} - 2 r_x r_{xy} + r_{xx}\big] \\ \quad - 2x'(z')^2 \big[r_x r_z r_{yy} - r_x r_{yz} + r_{xz} - r_z r_{xy}\big] - \\ \quad - (z')^3 \big[r r_z r_z r_{yy} - 2 r_z r_{yz} + r_{zz}\big] \end{pmatrix}^l}{R_x\, r_x} \times
$$
$$
\times \left(\frac{1}{r_x}\right)^a \left(\frac{r_z}{r_x}\right)^b \left(\frac{r_{xx}}{r_x}\right)^c \left(\frac{r_{yy}}{r_x}\right)^d \left(\frac{r_{zz}}{r_x}\right)^e \left(\frac{r_{xy}}{r_x}\right)^f \left(\frac{r_{xz}}{r_x}\right)^g \left(\frac{r_{yz}}{r_x}\right)^h,
$$

*for some nonnegative integers:*

$$
j, k, l, \qquad a, b, c, d, e, f, g, h,
$$

*in which any $\frac{1}{r_x}$ would have disappeared?*

## 3.2 Slanted vector fields

To construct holomorphic jet differentials on a hypersurface $X^n \subset \mathbb{P}^{n+1}$ defined as:

$$0 = R(z_1, \dots, z_n, z_{n+1}) = \sum_{\alpha_1+\cdots+\alpha_n+\alpha_{n+1}\leqslant d} a_{\alpha_1\dots\alpha_n\alpha_{n+1}}\, z_1^{\alpha_1}\cdots z_n^{\alpha_n} z_{n+1}^{\alpha_{n+1}},$$

two strategies exist, the first one (still open) being to work (only) in the ring (of fractions) of all partial derivatives of $R$:

$$\mathbb{Z}\Big[\Big(R_{z_1^{\beta_1}\dots z_{n+1}^{\beta_{n+1}}}\Big)_{\beta_1+\cdots+\beta_{n+1}\leqslant\kappa}\Big],$$

and the second one (currently active) being to work in the ring of coefficients:

$$\mathbb{Z}\Big[\Big(a_{\alpha_1\dots\alpha_{n+1}}\Big)_{\alpha_1+\cdots+\alpha_{n+1}\leqslant d}\Big].$$

For instance, differentiate $0 = \sum_\alpha a_\alpha z^\alpha$ up to order, say, 4:

$$
\begin{aligned}
0 &= \sum_\alpha a_\alpha z^\alpha \\
0 &= \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}} z'_{j_1}\bigg) \\
0 &= \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}} z''_{j_1} + \sum_{j_1,\,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} z'_{j_1} z'_{j_2}\bigg) \\
0 &= \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}} z'''_{j_1} + \sum_{j_1,\,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} 3 z'_{j_1} z''_{j_2} + \sum_{j_1,\,j_2,\,j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}} z'_{j_1} z'_{j_2} z'_{j_3}\bigg) \\
0 &= \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}} z''''_{j_1} + \sum_{j_1,\,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} \big(4 z'_{j_1} z'''_{j_2} + 3 z''_{j_1} z''_{j_2}\big) + \\
&\qquad + \sum_{j_1,\,j_2,\,j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}} 6 z'_{j_1} z'_{j_2} z''_{j_3} + \sum_{j_1,\,j_2,\,j_3,\,j_4} \frac{\partial^4(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}\partial z_{j_4}} z'_{j_1} z'_{j_2} z'_{j_3} z'_{j_4}\bigg).
\end{aligned}
$$

**Lemma 6.** ([41]) *The equation obtained by differentiating the condition $R\big(f(\zeta)\big) \equiv 0$ up to an arbitrary order $\kappa \geqslant 1$ reads in closed form as follows:*

$$0 = \sum_{\alpha\in\mathbb{N}^{n+1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant1,\dots,\mu_e\geqslant1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!}$$
$$\sum_{j_1^1,\dots,j_{\mu_1}^1=1}^{n+1}\cdots\sum_{j_1^e,\dots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}} z_{j_1^1}^{(\lambda_1)}\cdots z_{j_{\mu_1}^1}^{(\lambda_1)}\cdots z_{j_1^e}^{(\lambda_e)}\cdots z_{j_{\mu_e}^e}^{(\lambda_e)}.$$

These equations for $\kappa = 0, 1, \dots, \kappa$ define a certain (projectivizable) subvariety:

$$J_{\text{vert}}^{\kappa} \subset \mathbb{C}^{n+1}_{(z_k)} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}}_{(a_\alpha)},$$

complete intersection of codimension $\kappa+1$ outside $\{z_1' = \cdots = z_{n+1}' = 0\}$. Vector fields tangent to $J_{\text{vert}}^{\kappa}$ write under the general form:

$$\mathsf{T} = \sum_{i=1}^{n+1} \mathsf{Z}_i \frac{\partial}{\partial z_i} + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha} + \sum_{k=1}^{n+1} \mathsf{Z}_k' \frac{\partial}{\partial z_k'} + \sum_{k=1}^{n+1} \mathsf{Z}_k'' \frac{\partial}{\partial z_k''} + \cdots + \sum_{k=1}^{n+1} \mathsf{Z}_k^{(\kappa)} \frac{\partial}{\partial z_k^{(\kappa)}}.$$

Notably, the next theorem works with the quotient ring of $\mathbb{Z}[a_\alpha]$, not of $\mathbb{Z}[R_{z\beta}]$.

**Theorem 12.** ([69, 41, 55, 70]) *With $\kappa \leqslant d$, at every point of $J_{\text{vert}}^{\kappa}\big\backslash\{z_i' = 0\}$, there exist $j_{n,\kappa}^d := \dim J_{\text{vert}}^{\kappa}$ global holomorphic sections $\mathsf{T}_1, \dots, \mathsf{T}_{j_{n,\kappa}^d}$ of the twisted tangent bundle:*

$$T_{J_{\text{vert}}^{\kappa}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(\kappa^2 + 2\,\kappa\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(1),$$

*which generate the tangent space:*

$$\mathbb{C}\mathsf{T}_1\big|_p \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j_{n,\kappa}^d}\big|_p = T_{J_{\text{vert}}^{n},\,p}.$$

According to Siu ([69, 70]), these fields can be used to show that, for $X$ generic, entire curves $f \colon \mathbb{C} \to X$ land in the base locus of *all* global algebraic jet differentials belonging to the space:

$$H^0\big(X, E_{n,m}^{\text{GG}} T_X^* \otimes K_X^{-\delta m}\big) \neq 0, \tag{7}$$

which is shown in [19] to be nonzero for small enough $\delta \in \mathbb{Q}_{>0}$, for $\kappa = n$, for $m \gg 1$, provided:

$$d = \deg X \geqslant 2^{n^5}.$$

More precisely, by an abstract argument, extend locally any such jet differential:

$$\mathsf{P}(z,a) = \sum_{|\gamma_1|+\cdots+n|\gamma_n|=m} \mathsf{p}_\gamma(z,a)\,\big(z'\big)^{\gamma_1}\cdots\big(z^{(n)}\big)^{\gamma_n},$$

for $a$ generic. Use the vector fields of Theorem 12 to eliminate $(z')^{\gamma_1}\dots(z^{(n)})^{\gamma_n}$, and get the:

**Proposition 5.** *Nonconstant entire curves algebraically degenerate inside:*

$$Y := \big\{z \in X\colon\ \underbrace{\mathsf{p}_\gamma\big(z,a\big) = 0,\ \ \forall\,|\gamma_1| + \cdots + n\,|\gamma_n| = m}_{\text{all coefficients, very numerous}}\big\}.$$

Here, the total number of algebraic equations $p_\gamma(z,a) = 0$ is exponentially large $\approx m^n \gg (2^{n^5})^n$. Naturally, the common zero-set should conjecturally be empty, whence Kobayashi's conjecture — not in optimal degree — seems to be almost established. However, all *intrinsic* techniques which provide global holomorphic sections like (7) above, namely either a decomposition of jet bundles in Schur bundles, or asymptotic Morse inequalities, or else probabilistic curvature estimates, are up to now unable to provide a partial explicit expression of even a single algebraic coefficient $\mathsf{p}_\gamma\big(z,a\big)$.

*This is why a refoundation towards rational effectiveness is necessary.*

At least before refounding the construction of holomorphic jet differentials, such intrinsic approaches may be pushed further to improve the degree bound $d \geqslant 2^{n^5}$, and to treat new geometric situations.

Brotbek ([8]) produced holomorphic jet differentials on general complete intersections $X^n \subset \mathbb{P}^{n+c}$ of multidegrees $d_1, \dots, d_c \gg 1$. Mourougane ([55]) showed that for general moving enough families of high enough degree hypersurfaces in $\mathbb{P}^{n+1}$, there is a proper algebraic subset of the total space that contains the image of all sections.

Yet getting information about 'high enough' degrees represents a substantial computational work.

The most substantial recent progress concerning degree bounds is mainly due to Berczi ([5]), in the case of $f\colon \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$, with $d \geqslant n^{8n}$, instead of $d \geqslant 2^{n^5}$. Using the above vector fields and probabilistic curvature estimates for Green-Griffiths jets, Demailly obtained in [18], still in the case $f\colon \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$:

$$d \geqslant \frac{n^4}{3}\,\big(n \log\big(n\,\big(\log(24\,n)\big)\big)^n.$$

**Theorem 13.** (DARONDEAU, [13]) *Suppose that the jet order $\kappa$ is Let $X^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ be a smooth complex projective algebraic hypersurface of degree:*

$$d\ \geqslant\ (5\,n)^2\,n^n.$$

*If $X^{n-1}$ is Zariski-generic, then there exists a proper algebraic subvariety $Y \subsetneqq \mathbb{P}^n$ of codimension $\geqslant 2$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n \backslash X$ actually lands in $Y$, namely $f(\mathbb{C}) \subset Y$.*

Consider again the *universal family* of degree $d$ hypersurfaces of $\mathbb{P}^n$:

$$\mathscr{H} := \Big\{\big([Z],[A]\big) \in \mathbb{P}^n \times \mathbb{P}^{\binom{n+d}{d}-1}\colon\ \sum A_\alpha\, Z^\alpha = 0\Big\}.$$

With an additional variable $W \in \mathbb{C}$, introduce the family of hypersurfaces of $\mathbb{P}^{n+1}$:

$$W^d = \sum_{\alpha} A_\alpha Z^\alpha.$$

The space $J^\kappa_{\mathrm{vert}}(-\log)$ of *vertical logarithmic $\kappa$-jets* is associated to jets of local holomorphic maps $f: \mathbb{D} \to \mathbb{P}^n \backslash \mathscr{H}_A$ valued in the complement of a hypersurface $\mathscr{H}_A$ corresponding to a fixed $A$ and having a certain determined behavior near $\{W = 0\}$. The counterpart of Theorem 12 useful *infra* is:

**Theorem 14.** (DARONDEAU, [14]) *With $\kappa \leqslant d$, the twisted tangent bundle to the space of logarithmic $\kappa$-jets:*

$$T_{J^\kappa_{\mathrm{vert}}(-\log)} \otimes \mathscr{O}_{\mathbb{P}^n}\big(\kappa^2 + 2\kappa\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+d)!}{n!\,d!}}}(1)$$

*is generated by its global holomorphic sections at every point not in $\{W = 0\} \cup \{Z_i' = 0\}$.*

## 3.3 Prescribing the Base locus of Siu-Yeung jet differentials

In [71, 49], it is shown that a surface $X^2 \subset \mathbb{P}^3$ having affine equation: $z^d = R(x,y)$, where $R \in \mathbb{C}[x,y]$ is a generic polynomial of high enough degree $d \gg 1$, the following holds. For every collection of polynomials $A_{j,k,p,q} \in \mathbb{C}[x,y]$ having degrees $\deg A_{j,k,p,q} \leqslant d - 3m - 1$, the meromorphic jet differential:

$$\frac{\mathsf{J}(x,y,x',y',x'',y'')}{R_y \cdot z^{m(d-1)}} = \frac{1}{R_y \cdot z^{m(d-1)}} \sum_{j+k+p+3q=m} A_{j,k,p,q}\,(x')^j\,(y')^k\,(R')^p \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^q (R)^{m-p-q},$$

where:

$$R' := R_x x' + R_y y', \qquad R'' := R_x x'' + R_y y'' + R_{xx}(x')^2 + 2\,R_{xy} x' y' + R_{yy}(y')^2,$$

possesses a restriction to $X^2$ which is a *holomorphic* section of the bundle of the Green-Griffiths jet bundle $\mathscr{E}^{\mathrm{GG}}_{2,m} T^*_X$, provided only that the polynomial numerator:

$$\mathsf{J}(x,y,x',y',x'',y'') \equiv R_y(x,y)\,\widetilde{\mathsf{J}}(x,y,x',y',x'',y'')$$

is divisible by $R_y$, which happens to be satisfiable for $m = 81$ and $d = 729$, and more generally whenever $m \geqslant 81$ and $d \geqslant 3m$.

An expansion yields:

$$\mathsf{J} = \sum_{\alpha+\beta+3\gamma=m} \Lambda_{\alpha,\beta,\gamma}\big(A_\bullet, J^2_{x,y}R\big)\,(x')^\alpha\,(y')^\beta \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^\gamma,$$

in terms of some $\Lambda_\bullet$ that are linear in the $A_\bullet$ and polynomial in the 2-jet $J^2_{x,y}R$.

Since the vector fields of Theorem 14 have a maximal pole order 8 here, lowering $\deg A_{j,k,p,q} \leqslant d - 11\,m - 1$ enables to conclude, as in Proposition 5, that for a generic curve $\{R = 0\} \subset \mathbb{P}^2$, nonconstant entire holomorphic maps $f \colon \mathbb{C} \to \mathbb{P}^2 \backslash \{R = 0\}$ land inside the common zero set of all the $\Lambda_\bullet$, for $d \geqslant 2916$.

**Open Problem.** *Control or prescribe the base locus of coordinate jet differentials.*

A conjecturally accessible strategy is as follows, of course extendable to arbitrary dimensions. For convenience, replace $m \mapsto 3\,m$. Decompose $\mathsf{J} = \mathsf{J}^{\mathsf{top}} + \mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$, where:

$$\mathsf{J}^{\mathsf{top}} := 1 \cdot \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^m (R)^{2m} = \left( \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^m R_y + (x')^3 R_{xx} + 2\,(x')^2 y' R_{xy} + x' (y')^2 R_{yy} \right)^m (R)^{2m},$$

$$\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}} := \sum_{\substack{j+k+p+3q=3m \\ q \leqslant m-1}} A_{j,k,p,q}(x,y)\,(x')^j\,(y')^k\,(R')^p \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^q (R)^{m-p-q}.$$

Since $\big((x'y'' - y'x'')\,R_y\big)^m$ in $\mathsf{J}^{\mathsf{top}}$ is divisible by $R_y$, Proposition 5 would show that entire curves land in $\{(R_y)^{m-1} = 0\}$, and exchanging $x \leftrightarrow y$, in $\{(R_x)^{m-1} = 0\}$, hence are constant because $\emptyset = \{0 = R = R_x = R_y\}$ by smothness of $\{R = 0\}$.

However, *all* $\Lambda_{\alpha,\beta,\gamma}$, not just $\Lambda_{0,0,m}$, should be divisible by $R_y$ in order that the restriction to the projectivization of $\{z^d = R(x,y)\}$ of $\mathsf{J} \big/ \big(R_y\, z^{3m(d-1)}\big)$ be a holomorphic jet differential, because modulo $R_y$:

$$\mathsf{J}^{\mathsf{top}} \equiv \Big( (x')^3 R_{xx} + 2\,(x')^2 y' R_{xy} + x' (y')^2 R_{yy} \Big)^m (R)^{2m}$$

is nonzero. The strategy is to use $\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$ in order to *correct* this remainder. Conjecturally, the linear map which, to the $A_\bullet$ of $\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$, associates the coefficients of a basis $x^h y^i$ of $\mathbb{C}[x,y]/\langle R_y \rangle$ in all the monomials $(x')^\alpha (y')^\beta$ with $\alpha + \beta = 3m$ is submersive, also in arbitrary dimension, which would hence terminate.

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