AN ALGORITHM TO GENERATE ALL POLYNOMIALS IN THE κ-JET OF A HOLOMORPHIC DISC $\mathbb{D} \to \mathbb{C}^n$ THAT ARE INVARIANT UNDER SOURCE REPARAMETRIZATION

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ABSTRACT. A major unsolved problem (according to Demailly 1997) towards the Kobayashi hyperbolicity conjecture in optimal degree is to understand jet differentials of germs of holomorphic discs that are invariant under any reparametrization of the source. The underlying group action is not reductive, but we provide a complete algorithm to generate all invariants, in arbitrary dimension n and for jets of arbitrary order k.

Two main new situations are studied in great details. For jets of order 4 in dimension 4, we establish that the algebra of Demailly-Semple invariants is generated by 2835 polynomials, while the algebra of bi-invariants is generated by 16 mutually independent polynomials sharing 41 gröbnerized syzygies. Nonconstant entire holomorphic curves valued in an algebraic 3-fold (resp. 4-fold) $X^3 \subset \mathbb{P}^4(\mathbb{C})$ (resp. $X^4 \subset \mathbb{P}^5(\mathbb{C})$) of degree d satisfy global differential equations as soon as $d \geqslant 72$ (resp. $d \geqslant 259$). A useful asymptotic formula for the Euler-Poincaré characteristic of Schur bundles in terms of Giambelli's determinants is derived.

For jets of order 5 in dimension 2, we establish that the algebra of Demailly-Semple invariants is generated by 56 polynomials, while the algebra of bi-invariants is generated by 17 mutually independent polynomials sharing 105 gröbnerized syzygies.

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§1. INTRODUCTION

The Kobayashi hyperbolicity conjecture (1970), in optimal degree and taking account of Brody's theorem (1978), expects that all entire holomorphic curves

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$f : \mathbb{C} \to X$ into a complex projective (algebraic, smooth) hypersurface $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ must be constant if $\deg X \geqslant 2n+1$, provided X is generic. In 1980, Green and Griffiths conjectured that if $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ is of general type, which holds in degree $d \geqslant n+3$, then there is a proper algebraic subvariety $Y \subsetneq X$ which absorbs the image of all nonconstant entire holomorphic maps $f : \mathbb{C} \to X$, namely $f(\mathbb{C}) \subset Y$ necessarily. Correspondingly, an entire holomorphic $f : \mathbb{C} \to X$ will be called *algebraically degenerate* if its image is contained in some proper algebraic subvariety (which might depend on f).

Publications up to 2008 are still quite far from approaching the two optimal degrees $2n+1$ and $n+3$. For $X^2 \subset \mathbb{P}^3(\mathbb{C})$ very generic, such entire f's are known to be algebraically degenerate and even constant, in degree $d \geqslant 21$ (resp. $d \geqslant 18$) according to [6] (resp. [26]). For $X^3 \subset \mathbb{P}^4(\mathbb{C})$ very generic, algebraic degeneracy of such f's holds true in degree $d \geqslant 593$ according to [31]. For $X^4 \subset \mathbb{P}^5(\mathbb{C})$, a forthcoming work [11] applying the results of the present paper will obtain an effective degree lower bound for algebraic degeneracy; other applications to the logarithmic case also are imminent.

Quite unexpectedly, the two conjectures above and other similar problems as well in complex algebraic geometry happened in the last few years to pertain to purely algebraic problems, and not only to rely upon the scope of some soft techniques (pluripotential theory, currents, plurisubharmonic functions, *etc.*). Computational invariant theory should be expressly invoked here, as the present paper will show that what is at stake really is to find a complete description of the algebra of polynomials that are invariant under a certain Lie group action, which is *not* reductive.

Green-Griffiths Jet differentials. How can one figure out that a given nonconstant entire holomorphic map $f = \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is constrained to be somehow degenerate by just being valued in X? Looking at its derivatives $f', f'', \dots, f^{(k)}$ (in some jet-chart), one may expect at first to derive, by means of some suitable elimination process, sufficiently many *differential equations* which might presumably be due to the virtual guidance by some hidden $Y \subsetneq X$ absorbing $f(\mathbb{C})$.

For instance, for $X^2 \subset \mathbb{P}^3(\mathbb{C})$, the entire f's do satisfy (invariant) algebraic differential equations of order $k=2$, resp. $k=3$, resp. $k=4$ when X has degree $d \geqslant 15$, resp. $d \geqslant 11$, resp. $d \geqslant 9$ according to [4, 6], resp. [29], resp. [21]. For $X^3 \subset \mathbb{P}^4(\mathbb{C})$, differential equations of order $k=3$ enjoyed by any entire f exist when X has degree $d \geqslant 97$ ([30]).

Intrinsically speaking, consider the bundle J_k of k-jets of holomorphic curves $f : (\mathbb{D}, 0) \longrightarrow (X, x)$ centered at various points $x = f(0) \in X$. In the seminal article [17] (1980), Green and Griffiths introduced the fiber bundle $\mathsf{E}_{k,m}^{GG} T_X^* \to X$ of jet polynomials of order k and of weighted degree m whose fibers in some jet-chart are complex-valued polynomials $Q\big(f', f'', \dots, f^{(k)}\big)$ satisfying the weighted homogeneity:

$$Q\big(\lambda f',\, \lambda^2 f'',\, \dots,\, \lambda^k f^{(k)}\big) = \lambda^m\, Q\big(f',\, f'',\, \dots,\, f^{(k)}\big),$$

for every $\lambda \in \mathbb{C}^*$. Global sections of $\mathsf{E}_{k,m}^{GG} T_X^*$ over X are differential operators of order k. Elementary reasonings show ([17, 4, 32, 9]) that $\mathsf{E}_{k,m}^{GG} T_X^*$ is in fact a graded *vector* bundle isomorphic to the direct sum:

$$\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_k} T_X^*.$$

Such a grading of $\mathsf{E}_{k,m}^{GG} T_X^*$ enables one ([17]) to derive from Hirzebruch's Riemann-Roch formula ([18]) a sharp asymptotic estimate for its Euler-Poincaré characteristic, namely:

$$\chi\big(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\big) = \frac{m^{(k+1)n-1}}{(k!)^n\,\big((k+1)n-1\big)!} \left(\frac{(-1)^n}{n!}\, \mathsf{c}_1(X)^n\, (\log k)^n + \mathrm{O}\big((\log k)^{n-1}\big)\right) + \\ + \mathrm{O}\big(m^{(k+1)n-2}\big).$$

This formula and the knowledge of the expression of the n-th power of the first Chern class (implicitly integrated over X) in terms of the degree:

$$(-1)^n\, \mathsf{c}_1(X)^n = (d-n-2)^n\, d$$

entails that, as the jet order k tends to ∞, the characteristic $\chi\big(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\big)$ becomes eventually positive for m large enough, as soon as $\deg X \geqslant n+3$. Thus, up to a constant factor, c_1^n *becomes the dominant term of* $\chi\big(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\big)$ *when* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *is of general type*.

Demailly-Semple invariant jet differentials. In 1997, inspired also by an older paper of Semple, Demailly introduced a subbundle of $\mathsf{E}_{k,m}^{GG} T_X^*$ having better positivity properties and exhibiting a nice, stepwise compactification process.

With $\mathbb{D} \subset \mathbb{C}$ denoting any nonempty open disc centered at 0 (possibly $\mathbb{D} = \mathbb{C}$), consider a nonconstant holomorphic curve $f : \mathbb{D} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$. Of course, $f'(\zeta)$ then belongs to the tangent space $T_{X,f(\zeta)}$ for every $\zeta \in \mathbb{D}$. The projectivization $[f'(\zeta)] \in PT_{X,f(\zeta)}$ therefore belongs to the projectivized bundle of tangent lines to X, so that one gratuitously obtains a lifting $f_{[1]} := \big(f,\, [f']\big) : \mathbb{D} \longrightarrow P(T_X)$, at least for all ζ with $f'(\zeta) \neq 0$. Here, $P(T_X)$ is $(2n-1)$-dimensional, but the so lifted holomorphic curve $f_{[1]}$ happens to be guided by a certain n-dimensional subbundle of $P(T_X)$, better seen as follows.

Abstractly and generally speaking, let Y be a complex manifold, let $V \subset T_Y$ be any vector subbundle and call (Y, V) a *directed manifold*. Define $Y' := P(V)$ the projectivized bundle of lines contained in the vector subbundle $V \subset TY$ with of course $\dim Y' = \dim Y + \mathrm{rk}\, V - 1$. It is equipped with a natural projection $\pi : Y' \to Y$ which enables one to introduce the *lifted subbundle* $V' \subset T_{Y'}$, the fiber of which, at an arbitrary point $(x, [v]) \in Y'$, is precisely defined by:

$$V'_{(x,[v])} := \big\{ v' \in T_{X'} :\, d\pi(v') \in \mathbb{C}v \big\},$$

and the rank of which is clearly untouched: $\mathrm{rk}\, V' = \mathrm{rk}\, V$. Most importantly, any nonconstant holomorphic $f : \mathbb{D} \to Y$ constrained to be V-tangent, namely to satisfy $f'(\zeta) \in V_{f(\zeta)}$ for all $\zeta \in \mathbb{D}$, may be shown ([4]) to lift automatically, even at points ζ where $f'(\zeta)$ vanishes, as a map $f_{[1]} : \mathbb{D} \to Y'$ which is also constrained to be V'-tangent, namely which necessarily satisfies $f'_{[1]}(\zeta) \in V'_{f_{[1]}(\zeta)}$ for all $\zeta \in \mathbb{D}$. So lifting to a higher dimensional manifold keeps memory of the original guidance on the base.

Starting therefore with $Y = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ and with $V = T_X$, setting $X_0 := X$, $V_0 := T_X$, one defines first ([4]) $X_1 := P(T_X)$, $V_1 = V'$ and then inductively $(X_l, V_l) := (X'_{l-1}, V'_{l-1})$ with natural projections $\pi_{l,l-1} : X_l \to X_{l-1}$. One then assembles everything for $l = 0$ to $l = k$ as a tower of projectivized bundles with total projection $\pi_{0,k} : X_k \to X$ and with intermediate projections $\pi_{j,l} : X_l \to X_j$, for any $0 \leqslant j \leqslant l \leqslant k$. By applying inductively the above lifting operator $f_{[l]} := \big(f_{[l-1]}\big)_{[1]}$, every nonconstant holomorphic curve $f : \mathbb{D} \to X$ gives rise to lifts $f_{[l]} : \mathbb{D} \to X_l$ for all $l = 0, 1, \dots, k$. Each of these lifts is guided by V_l, namely $f'_{[l]}(\zeta) \in V_{l, f'_{[l]}(\zeta)}$ for all $\zeta \in \mathbb{D}$.

At each level $l \geqslant 1$, we have a tautological line bundle $\mathcal{O}_{X_l}(-1)$ over $X_l = P(V_{l-1})$ whose fiber at a point $\big(x_{l-1}, [v_{l-1}]\big) \in P(V_{l-1})$ just consists of the line $\mathbb{C} \cdot v_{l-1}$ directed by (a representative of) $[v_{l-1}]$, and similarly as in the projective spaces, one may build the basic bundles $\mathcal{O}_{X_l}(q)$ for every $q \in \mathbb{Z}$.

Now, the bundle of invariant jet differentials of order k and of weighted degree m is the subbundle[1] $\mathsf{E}^n_{k,m} T^*_X$ of $\mathsf{E}^{GG}_{k,m} T^*_X$ whose fibers at a point $x \in X$ consist of polynomial differential operators $Q\big(f', f'', \dots, f^{(k)}\big)$ which, under arbitrary local reparametrization $\phi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ of the source with $\phi(0) = 0$, satisfy the general invariancy condition:

$$Q\big((f \circ \phi)',\, (f \circ \phi)'', \dots, (f \circ \phi)^{(k)}\big) = \phi'(0)^m \, Q\big(f', f'', \dots, f^{(k)}\big),$$

not only under rescaling-like changes of coordinates $\zeta \mapsto \lambda \, \zeta$ with $\lambda \in \mathbb{C}^*$. This apparently neat definition hides several algebraic objects which will be inspected and explored in length throughout the present article. Comparing the two bundles:

$$\begin{array}{ccc} X_k & \longrightarrow & \mathcal{O}_{X_k}(m) \\ {\scriptstyle \pi_{0,k}} \downarrow & & \downarrow \\ X & \longrightarrow & \mathsf{E}^n_{k,m} T^*_X \,, \end{array}$$

over X and over X_k, one establishes ([4]) the direct image formula

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}\big(\mathsf{E}^n_{k,m} T^*_X\big).$$

Existence of global algebraic differential equations. What then are the global algebraic differential equations that nonconstant entire maps $f : \mathbb{C} \to X$ could satisfy? As the hypersurface X lives in $\mathbb{P}^{n+1}(\mathbb{C})$, it carries many ample line bundles, *e.g.* any $\mathcal{O}_{X_k}(q)$ with $q \geqslant 1$.

([17, 4]) *Fix an ample line bundle* $A \to X$ *and assume that* $\mathsf{E}^n_{k,m} T^*_X \otimes A^{-1}$ *has nonzero sections, namely:*

$$h^0\big(X,\, \mathsf{E}^n_{k,m} T^*_X \otimes A^{-1}\big) = \dim H^0(\mathsf{same}) \geqslant 1.$$

[1] Because the dimension n will vary often in our study, it must be indicated as an exponent in the notation of the Demailly-Semple bundle.

Then for every global invariant operator $P \in \Gamma\big(X,\, \mathsf{E}_{k,m}^{n} T_X^* \otimes A^{-1}\big)$ valued in A^{-1}, any entire holomorphic curve f must satisfy the algebraic differential equation $P\big(f_{[k]}\big) \equiv 0$. A similar result also holds true for the larger bundle $\mathsf{E}_{k,m}^{GG} T_X^*$.

How then one can guarantee the existence of such sections P? Because X is elementarily seen to be of general type when $d \geqslant n+3$, it is expected ([4, 29]) in a first moment that the Euler-Poincaré characteristic of the Demailly-Semple subbundle $\mathsf{E}_{k,m}^{n} T_X^*$ should behave in a way quite similar to the Green-Griffiths bundle, with c_1^n becoming the dominant term (up to a constant factor) as k and m both tend to ∞, so that $\chi\big(X,\, \mathsf{E}_{k,m}^{n} T_X^*\big)$ should be eventually large, and furthermore in a second moment, it is also expected that the dimension of the space of global sections $H^0\big(X,\, \mathsf{E}_{k,m}^{n} T_X^*\big)$ should be eventually large, due to some vanishing or to some control of the higher order cohomology groups. The truth of such conjectural expectations would presumably open new routes towards a solution in optimal degree to the two above-mentioned conjectures.

Seeking Schur bundle decomposition of $\mathsf{E}_{k,m}^{n} T_X^*$. However, as is written in [4], it is a major unsolved problem to find the decomposition of $\mathsf{E}_{k,m}^{n} T_X^*$ into direct sums of the irreducible Schur bundles $\Gamma^{(\ell_1, \ell_2, \dots, \ell_n)} T_X^*$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n$ that are the basic bricks and whose cohomology is somehow currently available. According to a possible strategy developed for $k = n = 3$ mainly by Rousseau in [29, 30], such a decomposition would yield access to the Euler characteristic $\chi\big(X,\, \mathsf{E}_{k,m}^{n} T_X^*\big)$, and then afterwards, one would attain an effective estimate of $h^0\big(X,\, \mathsf{E}_{k,m}^{n} T_X^*\big)$, provided one controls the other cohomology groups. In fact, the only decompositions known up to now are the following; the second one ([29]) already required a nontrivial argument based on a theorem of Popov about polarization of multilinear invariants.

- For $n = k = 2$ ([4]):

$$\mathsf{E}_{2,m}^{2} T_X^* = \bigoplus_{a+3b=m} \Gamma^{(a+b,\, b)}\, T_X^*.$$

- For $n = k = 3$ and also for $n = 2$, $k = 3$ ([29]):

$$\mathsf{E}_{3,m}^{3} T_X^* = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\, b+c+d,\, d)}\, T_X^*,$$

$$\text{and} \qquad \mathsf{E}_{3,m}^{2} T_X^* = \bigoplus_{a+3b+5c=m} \Gamma^{(a+b+2c,\, b+c)}\, T_X^*.$$

- For $n = 2$, $k = 4$ ([21]):

$$\mathsf{E}_{4,m}^{2} T_X^* = \bigoplus_{a+3b+5c+8e=m} \Gamma^{(a+b+2c+2e,\, b+c+2e)}\, T_X^* \;\bigoplus_{7+a+5c+7d+8e=m} \Gamma^{(3+a+2c+3d+2e,\, 1+c+d+2e)}\, T_X^*.$$

In this paper, we mainly attack the case $n = k = 4$. The complexity increases suddenly and we seem to be still quite far from being able to push the jet order k to ∞.

Theorem. *On a smooth complex algebraic hypersurface $X^4 \subset \mathbb{P}^5(\mathbb{C})$, the graduate m-th part $\mathsf{E}_{4,m}^4 T_X^*$ of the Demailly-Semple bundle $\mathsf{E}_4^4 T_X^* = \oplus_m \mathsf{E}_{4,m}^4 T_X^*$ has the following decomposition in direct sums of Schur bundles:*

$$\mathsf{E}_{4,m}^4 T_X^* = \bigoplus_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\backslash(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n+10p=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T_X^*,$$

where the 41 subsets $\square_i$, $i = 1, 2, \dots, 41$, of $\mathbb{N}^{14} \ni (a, b, \dots, l, m', n)$ are explicitly defined in the complete statement on p. 86.

It is known ([30]) that $\mathsf{E}_{k,m}^3 T_X^*$ has no nonzero sections for jet order $k = 1$ or $k = 2$. More generally ([9]), for jet order $k \leqslant n-1$ strictly smaller than the dimension, sections are never available: $H^0\big(X,\, \mathsf{E}_{k,m}^n T_X^*\big) = 0$. Consequently, even if one may easily deduce from the above theorem a Schur decomposition of $\mathsf{E}_{4,m}^3 T_X^*$, for applications to hyperbolicity in dimension higher than 3, one should always start with jet order k at least equal to the dimension[2]. The case $n = k = 4$ was the first unknown one before.

Asymptotic expansion of Euler-Poincaré characteristic. Because the characteristic is just additive on direct sums of vector bundles, knowing a representation of $\mathsf{E}_{k,m}^n T_X^*$ (for certain values of n, k, *e.g.* for $n = k = 4$) as a direct sum of certain Schur bundle is very convenient, provided of course that one already possesses an asymptotic for the Euler-Poincaré characteristic of the $\Gamma^{(\ell_1,\dots,\ell_n)} T_X^*$ as $\ell_1 + \cdots + \ell_n \to \infty$. Section 13 will derive an explicit asymptotic for which there seems to be no reference with a precise enunciation (compare [1, 28]). Because of the relations $\mathsf{c}_k\big(T_X^*\big) = (-1)^k \mathsf{c}_k\big(T_X\big)$, there is no loss of generality to express everything in terms of the Chern classes of the tangent bundle T_X.

Theorem. *The terms of highest order with respect to $|\ell| = \max_{1\leqslant i\leqslant n} \ell_i$ in the Euler-Poincaré characteristic of the Schur bundle $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)} T_X$ are homogeneous of order $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ and they are given by a sum of ℓ_i'-determinants indexed by all the partitions $(\lambda_1, \dots, \lambda_n)$ of n:*

$$\chi\Big(X,\ \Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\, T_X\Big) = \\ = \sum_{\lambda \text{ partition of } n} \frac{\mathsf{C}_{\lambda^c}}{(\lambda_1+n-1)!\cdots\lambda_n!} \begin{vmatrix} {\ell_1'}^{\lambda_1+n-1} & {\ell_2'}^{\lambda_1+n-1} & \cdots & {\ell_n'}^{\lambda_1+n-1} \\ {\ell_1'}^{\lambda_2+n-2} & {\ell_2'}^{\lambda_2+n-2} & \cdots & {\ell_n'}^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell_1'}^{\lambda_n} & {\ell_2'}^{\lambda_n} & \cdots & {\ell_n'}^{\lambda_n} \end{vmatrix} + \\ + \mathrm{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big),$$

[2] Nonetheless, we ignore whether the case $n = k = 5$ is accessible to us.

where $\ell_i' := \ell_i + n - i$ *for notational brevity, with coefficients* C_{λ^c} *being expressed in terms of the Chern classes* $\mathsf{c}_k\big(T_X\big) = \mathsf{c}_k$ *of* T_X *by means of* Giambelli's determinantal expression *depending upon the* conjugate *partition* λ^c*:*

$$\mathsf{C}_{\lambda^c} = \mathsf{C}_{(\lambda_1^c,\dots,\lambda_n^c)} = \begin{vmatrix} \mathsf{c}_{\lambda_1^c} & \mathsf{c}_{\lambda_1^c+1} & \mathsf{c}_{\lambda_1^c+2} & \cdots & \mathsf{c}_{\lambda_1^c+n-1} \\ \mathsf{c}_{\lambda_2^c-1} & \mathsf{c}_{\lambda_2^c} & \mathsf{c}_{\lambda_2^c+1} & \cdots & \mathsf{c}_{\lambda_2^c+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{c}_{\lambda_n^c-n+1} & \mathsf{c}_{\lambda_n^c-n+2} & \mathsf{c}_{\lambda_n^c-n+3} & \cdots & \mathsf{c}_{\lambda_n^c} \end{vmatrix},$$

on understanding by convention that $\mathsf{c}_k := 0$ *for* $k < 0$ *or* $k > n$*, and that* $\mathsf{c}_0 := 1$.

Effective calculations of characteristics in dimensions 3 and 4. We then perform electronically assisted computations to obtain the desired, quite complicated value of the characteristic of $\mathsf{E}_{4,m}^4 T_X^*$.

Theorem. *If* $X^4 \subset \mathbb{P}^5(\mathbb{C})$ *is a degree* d *smooth algebraic 4-fold, then as* $m \to \infty$*, one has the asymptotic:*

$$\begin{aligned} \chi\big(X,\, \mathsf{E}_{4,m}^4 T_X^*\big) = \ & \frac{m^{16}}{13133178323038943332103356416000000000000000} \cdot d \cdot \\ & \cdot \big(5004851113579703425623 5\, d^4 - \\ & \quad - 6170606622505955255988786\, d^3 - \\ & \quad - 9288869013541411538806247 04\, d + \\ & \quad + 1411704752502476621473639 41\, d^2 + \\ & \quad + 1624908955061039283976041114\big) \\ & + \mathrm{O}\big(m^{15}\big). \end{aligned}$$

Furthermore, the coefficient of m^{16} *here, a factorized polynomial of degree 5 with respect to* d*, is positive in all degrees* $d \geqslant 96$*.*

For $n = k = 3$, based on his above-mentioned Schur decomposition of $\mathsf{E}_{3,m}^3 T_X^*$, Rousseau ([29]) showed that:

$$\chi\big(X,\, \mathsf{E}_{3,m}^3 T_X^*\big) = \frac{m^9}{81648000000} \cdot d \cdot \big(389d^3 - 20739d^2 + 185559d - 358873\big) + \mathrm{O}\big(m^8\big),$$

and that the coefficient of m^9 is positive for all degrees $d \geqslant 43$. Furthermore, in [28], Rousseau showed that $h^2\big(X,\, \mathrm{Sym}^m T_X^*\big) = \big(-\frac{7}{24}d + \frac{1}{8}d^2\big)\, m^5 + \mathrm{O}\big(m^4\big)$ in any degree $d \geqslant 6$, so that one cannot expect second cohomology groups to vanish. Afterwards, as the main objective of the paper [30], he first established the general majoration:

$$h^2\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)} T_X^*\big) \leqslant d(d+13)\, \frac{3(\ell_1+\ell_2+\ell_3)^3}{2}\, (\ell_1-\ell_2)(\ell_1-\ell_3)(\ell_2-\ell_3) + \mathrm{O}\big(|\ell|^5\big).$$

he then deduced by summation from the cited decomposition $\mathsf{E}_{3,m}^3 T_X^* = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\, b+c+d,\, d)} T_X^*$ that:

$$h^2\big(X,\, \mathsf{E}_{3,m}^3 T_X^*\big) \leqslant \frac{49403}{252 \cdot 10^7}\, d(d+13)\, m^9 + \mathrm{O}\big(m^8\big),$$

and finally, by applying the trivial minoration:

$$h^0\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)\geqslant\chi\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)-h^2\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big),$$

stemming from the definition $\chi=h^0-h^1+h^2-h^3$, he immediately deduced the minoration:

$$h^0\big(X,\,\mathsf{E}_{3,m}^3T_X^*\big)\geqslant\frac{m^9}{408240000000}\cdot d\cdot\big(1945\,d^3-103695\,d^2-7075491\,d-105837083\big)+\mathrm{O}\big(m^8\big),$$

in which the coefficient of m^9 is checked (again electronically) to be positive in all degrees $d\geqslant 97$. As a result, nontrivial sections of $\mathsf{E}_{3,m}^3T_X^*$ exist when $\deg X\geqslant 97$.

For jets of order 4 in dimension 3, when applying in dimension 3 our decomposition of $\mathsf{E}_{4,m}^3T_X^*$ into Schur bundles which appears in the theorem on p. 86, a Maple computation using the cited majoration formula for $h^2\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3)}T_X^*\big)$ then provides:

$$h^2\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)\leqslant d(d+13)\,\frac{342988705758851}{29822568148961280000000}\,m^{11}+\mathrm{O}\big(m^{10}\big).$$

Theorem. *The asymptotic, as $m\to\infty$, of the Euler-Poincaré characteristic of the Demailly-Semple bundle $\mathsf{E}_{4,m}^4T_X^*$ on a degree d smooth projective algebraic 3-fold $X^3\subset\mathbb{P}^4(\mathbb{C})$ is given by:*

$$\begin{aligned}\chi\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)=\frac{m^{11}}{206133591045620367360000000}&\cdot d\cdot\big(1029286103034112\,d^3-\\&-38980726828290305\,d^2+299551055917162501\,d-561169562618151944\big)+\\&+\mathrm{O}\big(m^{10}\big),\end{aligned}$$

and the coefficient of m^{11} here is positive in all degrees $d\geqslant 29$. Furthermore, subtracting to this asymptotic the above majorant of $h^2\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)$:

$$\begin{aligned}h^0\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)\geqslant\frac{m^{11}}{206133591045620367360000000}&\cdot d\cdot\big(1029286103034112\,d^3-\\&-38980726828290305\,d^2+207118687828801561\,d-313807627072854674\,00\big)+\\&+\mathrm{O}\big(m^{10}\big),\end{aligned}$$

and the modified coefficient here of m^{11} is now positive in all degrees $d\geqslant 72$.

This last condition $d\geqslant 72$ on the degree insuring the existence of global invariant jet differentials of order $\kappa=4$ on $X^3\subset\mathbb{P}^4(\mathbb{C})$ improves the condition $d\geqslant 97$ obtained in [30] and appears to be slightly better than the condition $d\geqslant 74$ obtained more recently in [9] with another approach. A number of further numerical applications, especially to the logarithmic case, shall appear soon ([11]); as will be seen in a near future, in dimension 4, the lower bound on the degree $d\geqslant 259$ for the existence of sections which will based on the present approach will also improve the bound $d\geqslant 298$ obtained in [9]. Nonetheless, we must stop at this point in order to describe the main contribution of the present article. Last but not least, we cannot go beyond without mentioning that Siu's strategy for establishing algebraic degeneracy ([35, 26, 31]) will also bring further fruits thanks to the recent construction of a global meromorphic framing on the space of vertical n-jets tangent to the universal hypersurface in arbitrary dimension n ([22]).

A problem in invariant theory. Now, how does one obtain Schur decompositions of Demailly-Semple bundles? To begin with, we show how one can understand the condition of being invariant under reparametrization in terms of classical invariant theory.

Let us from now on denote by κ (instead of k) the jet order and let us abbreviate $j^\kappa f = \big(f', f'', \dots, f^{(\kappa)}\big)$.

The group G_κ of κ-jets at the origin of local reparametrizations $\phi(\zeta) = \zeta + \phi''(0)\,\frac{\zeta^2}{2!} + \cdots + \phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!} + \cdots$ that are tangent to the identity, namely which satisfy $\phi'(0) = 1$, may be seen to act linearly on the $n\kappa$-tuples $\big(f'_{j_1}, f''_{j_2}, \dots, f^{(\kappa)}_{j_\kappa}\big)$ by plain matrix multiplication, *i.e.* when we set $g_i^{(\lambda)} := \big(f_i \circ \phi\big)^{(\lambda)}$, a computation applying the chain rule gives for each index i:

$$\begin{pmatrix} g_i' \\ g_i'' \\ g_i''' \\ g_i'''' \\ \vdots \\ g_i^{(\kappa)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi'' & 1 & 0 & 0 & \cdots & 0 \\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0 \\ \phi'''' & 4\phi''' + 3\phi''^2 & 6\phi'' & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} f_i' \circ \phi \\ f_i'' \circ \phi \\ f_i''' \circ \phi \\ f_i'''' \circ \phi \\ \vdots \\ f_i^{(\kappa)} \circ \phi \end{pmatrix} \qquad (i = 1 \cdots n).$$

Polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization satisfy by definition for some integer m:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(j^\kappa (f \circ \phi)\big) = \phi'(0)^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) = \mathsf{P}\big((j^\kappa f) \circ \phi\big),$$

for any ϕ. If we denote by $\mathsf{E}^n_{\kappa,m}$ the vector space consisting of such polynomials, the direct sum $\mathsf{E}^n_\kappa = \oplus_{m \geqslant 1}\, \mathsf{E}^n_{\kappa,m}$ forms an algebra graded by constancy of weights: $\mathsf{E}^n_{\kappa,m_1} \cdot \mathsf{E}^n_{\kappa,m_2} \subset \mathsf{E}^n_{\kappa,m_1+m_2}$.

Then obviously when $\phi'(0) = 1$, the algebra E^n_κ just coincides with the algebra of invariants for the linear group action represented by the group of matrices just written:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{M}_{\phi'', \phi''', \dots, \phi^{(\kappa)}} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big),$$

with $\phi'', \phi''', \dots, \phi^{(\kappa)}$ interpreted as arbitrary complex constants. Such a group clearly has dimension $\kappa - 1$.

But unfortunately, this group of matrices is a subgroup of the full unipotent group, hence it is *non-reductive*, and for this reason, it is impossible to apply almost anything from the so well developed invariant theory of reductive actions ([7]). Moreover, though the invariants of the full unipotent group are well understood, as soon as one looks at a *proper* subgroup of it, formal harmonies happen to be rapidly destroyed.

We ignore whether the algebra of invariants is finitely generated, in general. But in all previously known cases (carefully reminded below) and in all further new cases studied in this paper, E^n_κ is finitely generated. We will establish that the (graded) algebra $\mathsf{E}^4_4 = \oplus_m\, \mathsf{E}^4_{4,m}$ is generated by 2835 invariant polynomials and that $\mathsf{E}^2_5 = \oplus_m\, \mathsf{E}^2_{5,m}$ is generated by 56 invariant polynomials. We will also provide, in the theorem stated in length on p. 57, a *general algorithm* which, *in arbitrary dimension n and for arbitrary jet order κ, generates all invariants by adding a* *new invariant only when it cannot be expressed as a polynomial with respect to the already known invariants, and which stops after a finite number of loops if and only if* $\mathsf{E}_\kappa^n = \oplus_m \mathsf{E}_{\kappa,m}^n$ *is finitely generated as an algebra.*

Insufficiency of bracketing. By definition, a polynomial $\mathsf{P}\big(j^\kappa f\big)$ in the κ-th order jet space which is invariant by reparametrization must satisfy $\mathsf{P}\big(j^\kappa((f\circ\phi)\big) = \phi'^m\,\mathsf{P}\big(j^\kappa f)\circ\phi\big)$ for every biholomorphism $\phi:(\mathbb{D},0)\longrightarrow(\mathbb{D},0)$, where the integer m is called the *weight* of P, and where it is implicitly understood that the base point is the origin. Also, suppose next that Q is another invariant of weight n in the τ-th order jet space, *i.e.* satisfying $\mathsf{Q}\big(j^\tau(f\circ\phi)\big) = \phi'^n\,\mathsf{Q}\big((j^\tau f)\circ\phi\big)$. If $\mathsf{D} := \sum_{k=1}^n \sum_{\lambda\in\mathbb{N}} \frac{\partial(\bullet)}{\partial f_k^{(\lambda)}}\cdot f_k^{(\lambda+1)}$ denotes the *total differentiation operator*, which acts on any polynomial in $f', f'', \dots, f^{(\kappa)}$ as if it differentiated it with respect to the (virtual) source variable $\zeta\in\mathbb{D}$, then the *bracket expression*:

$$\big[\mathsf{P},\,\mathsf{Q}\big] := n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}$$

will easily be checked (in §3) to provide gratuitously another invariant of weight $m+n+1$ in the jet space of order $1+\max(\kappa,\tau)$.

For jet order $\kappa=1$, the algebra of invariants is just $\mathbb{C}\big[f_1', f_2', \dots, f_n'\big]$. For $\kappa=2$, the algebra E_2^n is generated by the f_i' together with the two-dimensional Wronskians $f_i'f_j''-f_i''f_j'$ which identify to the brackets $\big[f_j',\,f_i'\big]$, where $1\leqslant i,j\leqslant n$.

For $\kappa=3$ in dimension $n=2$, the Demailly-Semple algebra E_3^2 is generated by 5 mutually independent invariants:

$$f_1',\qquad f_2',\qquad \Lambda^3 := \big[f_2',\,f_1'\big],\qquad \Lambda_1^5 := \big[\Lambda^3,\,f_1'\big],\qquad \Lambda_2^5 := \big[\Lambda^3,\,f_2'\big],$$

which all are furnished by just bracketing, according to [29]; (but bracketing did not enter the scene there).

In the next dimension $n=3$ for jets of the same order $\kappa=3$, the Demailly-Semple algebra E_3^3 is generated by 16 mutually independent invariants ([29]), namely the $3+3+9=15$ following ones:

$$f_i',\qquad \Lambda_{i,j}^3 := \big[f_j',\,f_i'\big],\qquad \Lambda_{i,j;\,k}^5 := \big[\Lambda_{i,j}^3,\,f_k'\big],$$

where $1\leqslant i<j\leqslant 3$ and where $1\leqslant k\leqslant 3$, which are clearly all obtained by bracketing some invariants from the preceding jet level, together with the three-dimensional Wronskian:

$$D_{1,2,3}^6 := \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

which also appears, though after some division by f_1', to come from the brackets, for one checks by direct calculation the three relations:

$$\big[\Lambda_{1,2}^3,\,\Lambda_{1,3}^3\big] = -3\,f_1'\,D_{1,2,3}^6,\qquad\qquad \big[\Lambda_{1,2}^3,\,\Lambda_{2,3}^3\big] = -3\,f_2'\,D_{1,2,3}^6,$$
$$\big[\Lambda_{1,3}^3,\,\Lambda_{2,3}^3\big] = -3\,f_3'\,D_{1,2,3}^6.$$

Here, as the reader may have observed already, we always put the weight of every invariant at the upper index place.

Lastly, coming back to the dimension $n = 2$, for jet order $\kappa = 4$, the algebra E_4^2 is generated by the 9 mutually independent invariants ([5, 21]):

$$
\begin{aligned}
&f_1', \qquad f_2', \qquad \Lambda_{1,2}^3, \qquad \Lambda_{1,2;\,1}^5, \qquad \Lambda_{1,2;\,2}^5,\\
&\Lambda_{1,1}^7 := \big[\Lambda_{1,2;\,1}^5,\, f_1'\big], \qquad \Lambda_{1,2}^7 := \big[\Lambda_{1,2;\,1}^5,\, f_2'\big] = \big[\Lambda_{1,2;\,2}^5,\, f_1'\big] = \Lambda_{2,1}^7,\\
&\Lambda_{2,2}^7 := \big[\Lambda_{1,2;\,2}^5,\, f_2'\big], \qquad M^8 := \frac{1}{f_1'}\,\big[\Lambda_{1,2;\,1}^5,\, \Lambda_{1,2}^3\big],
\end{aligned}
$$

coming again all from bracketing, possibly allowing a division by f_1'.

In view of all these positive results, one could believe that bracketing (with possible division) always generate all invariants when passing from one jet level to the subsequent one. In fact, the two so-called *sigma-* and *Omega-processes* are known to generate all the invariants of binary forms in any degree ([25, 7, 27]).

Unfortunately, in [21], we discovered that in dimension $n = 2$ for jet order $\kappa = 5$, many invariants exist which are totally independent from the ones obtained by bracketing the invariants existing at the inferior jet levels $\kappa \leqslant 4$. Section 8 will provide more explanations, emphasizing that *it is by no means possible to derive these further invariants by dividing any incoming bracket-invariant by any other already known (bracket) invariant*.

Nonetheless, there could exist a second (and even a third) algebraically uniform process which would generate gratuitously many other invariants, and which, in cooperation with the bracketing process, would be complete, but regarding such an idea, we must confess our ignorance.

Initial rational expression for invariants. Hopefully, the algorithm we already devised (and hid slightly?) in [21] provides another route. How does it work?

To begin with, we define $\Lambda_{1,i}^3 := \big[f_i', f_1'\big]$ and then by induction for any λ with $3 \leqslant \lambda \leqslant \kappa$:

$$\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1} := \big[\Lambda_{1,i;\,1^{\lambda-3}}^{2\lambda-3},\, f_1'\big].$$

Being built by bracketing, these are invariants of weight $2\lambda - 1$ for any $i = 1, \dots, n$. The power $\lambda - 2$ of 1 counts the number of brackets with f_1', starting from the Wronskian $\Lambda_{1,i}^3$.

The preliminary step is to establish a *rational* representation of any invariant polynomial as a sum of polynomials in terms of f_1' and of the $\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}$, $2 \leqslant i \leqslant n$, $1 \leqslant \lambda \leqslant \kappa$, a representation in which f_1' is allowed to have possibly negative powers $(f_1')^a$ with $-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m$. The following basic statement will appear in §5.

Lemma. *In dimension $n \geqslant 1$ and for jets of order $\kappa \geqslant 1$, every polynomial $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization writes under the form:*

$$
\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a \left(
\begin{array}{ccccc}
f_2', & f_3', & f_4', & \dots, & f_n',\\
\Lambda_{1,2}^3, & \Lambda_{1,3}^3, & \Lambda_{1,4}^3, & \dots, & \Lambda_{1,n}^3,\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,3;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,4;\,1^{\kappa-2}}^{2\kappa-1}, & \dots, & \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}
\end{array}
\right)},
$$

where the integer a *takes all possibly negative values in the interval* $\big[-\frac{\kappa-1}{\kappa}\,m, m\big]$*, for certain weighted homogeneous polynomials:*

$$\mathsf{P}_a = \sum_{\substack{b_2+\cdots+b_n+3c_2+\cdots+3c_n+\\ +\cdots+(2\kappa-1)q_2+\cdots+(2\kappa-1)q_n=m-a}} \mathsf{coeff}\cdot\prod_{i=2}^n \big(F_i\big)^{b_i}\prod_{i=2}^n \big(A_i^3\big)^{c_i}\cdots\prod_{i=2}^n \big(A_i^{2\kappa-1}\big)^{q_i}$$

of weighted degree $m-a$*, namely satisfying:*

$$\mathsf{P}_a\Big(\delta\,F_i,\,\delta^3\,A_i^3,\,\dots,\,\delta^{2\kappa-1}\,A_i^{2\kappa-1}\Big) = \delta^{m-a}\cdot\mathsf{P}_a\Big(F_i,\,A_i^3,\,\dots,\,A_i^{2\kappa-1}\Big).$$

Conversely, for every collection of such weighted homogeneous polynomials P_a *in* $\mathbb{C}\big[F_i, A_i^3,\dots,A_i^{2\kappa-1}\big]$ *of weighted degree* $m-a$ *indexed by an integer* a *running in* $\big[-\frac{\kappa-1}{\kappa}\,m,\,m\big]$ *such that the reduction to the same denominator and the simplification of the finite sum:*

$$\mathsf{R}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\left(\begin{array}{ccccc} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda_{1,2}^3, & \Lambda_{1,3}^3, & \Lambda_{1,4}^3, & \dots, & \Lambda_{1,n}^3, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,3;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,4;\,1^{\kappa-2}}^{2\kappa-1}, & \dots, & \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1} \end{array}\right)$$

yields a true *jet* polynomial *in* $\mathbb{C}\big[j^\kappa f\big]$*, then* $\mathsf{R}\big(j^\kappa f\big)$ *is a polynomial invariant by reparametrization belonging to* $\mathsf{E}^n_{\kappa,m}$.

Next, we summarize briefly the way how our algorithm works; mathematical causalities, motivations and "reasons-why" shall be transparent to any reader who will study the example E^2_4 detailed in Section 6.

Suppose that, setting aside the special invariant f_1', we already know a certain number of invariants $L^{l_1},\dots,L^{l_{k_1}}$, for instance the very initial ones above $f_2',\dots,f_n'$ together with all the $\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}$. The recipe is to compute the ideal of relations between these invariants after setting $f_1'=0$ in them:

$$\mathsf{Ideal\text{-}Rel}\Big(L^{l_1}(j^\kappa f)\big|_{f_1'=0},\,\dots,\,L^{l_{k_1}}(j^\kappa f)\big|_{f_1'=0}\Big).$$

Using any symbolic package for computing Gröbner bases, suppose that, for some monomial ordering, we may dispose of a Gröbner basis for the ideal of relations between these restricted invariants which we shall represent shortly by the following collection of algebraic equations:

$$0\equiv\mathsf{S}_i\Big(L^{l_1}\big|_0,\,\dots,\,L^{l_{k_1}}\big|_0\Big)\qquad\qquad {\scriptstyle (i\,=\,1\,\cdots\,N_1)}.$$

One checks that each S_i may be supposed to be of constant homogeneous weight μ_i, namely:

$$\mathsf{S}_i\big(\delta^{l_1}A_1,\dots,\delta^{l_{k_1}}A_{k_1}\big) = \delta^{\mu_i}\mathsf{S}_i\big(A_1,\dots,A_{k_1}\big)\qquad\qquad {\scriptstyle (i\,=\,1\,\cdots\,N_1)}.$$

Since $\mathsf{S}_i\big(j^\kappa f\big)$ vanishes identically after setting $f_1'=0$, when we do not set $f_1'=0$, there must exist certain (possibly zero) polynomial remainders $\mathsf{R}_i\big(j^\kappa f\big)$ such that we may write in $\mathbb{C}\big[j^\kappa f\big]$:

$$\mathsf{S}_i\big(L^{l_1},\dots,L^{l_{k_1}}\big) = (f_1')^{\nu_i}\,\mathsf{R}_i\big(j^\kappa f\big)\qquad\qquad {\scriptstyle (i\,=\,1\,\cdots\,N_1)},$$

with $\mathsf{R}_i \not\equiv 0$ when $1 \leqslant \nu_i < \infty$ and with $\mathsf{R}_i = 0$ by convention when $\nu_i = \infty$.

Then one easily convinces oneself that every remainder $\mathsf{R}_i\big(j^\kappa f\big)$ also is a polynomial invariant by reparametrization.

Afterwards, one then tests whether the first remainder R_1 belongs to the algebra generated by $L^{l_1}, \dots, L^{l_{k_1}}$. If not, R_1 must be added to the list of invariants. Next, one tests whether R_2 belongs to the algebra generated by $\mathsf{L}^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1$. If not, one adds R_2 to the list, and so on.

At the end, one gets a new list of invariants $L^{l_1}, \dots, L^{k_1}, M^{m_1}, \dots, M^{m_{k_2}}$ and then one restarts a second loop by computing a Gröbner basis for the ideal of relations:

$$\textsf{Ideal-Rel}\Big(L^{l_1}\big|_0,\ \dots,\ L^{l_{k_1}}\big|_0,\ M^{m_1}\big|_0,\ \dots,\ M^{m_{k_2}}\big|_0\Big).$$

Theorem. *For a certain dimension n and for a certain jet order κ, suppose that, after performing a finite number of loops of the algorithm, one possesses a finite number $1+M$ of mutually independent invariants f_1', Λ^{l_1}, $\dots$, $\Lambda^{l_M} \in \mathbb{C}\big[j^\kappa f_1, \dots, j^\kappa f_n\big]$ of weights $1, l_1, \dots, l_M$ belonging to E_κ^n, whose restrictions to $\{f_1' = 0\}$ share an ideal of relations:*

$$\textsf{Ideal-Rel}\Big(\ \Lambda^{l_1}\big|_0,\ \ \dots\dots,\ \Lambda^{l_M}\big|_0\ \Big)$$

generated by a finite number N (often large) of homogeneous syzygies:

$$0 \equiv \mathsf{S}_i\big(\Lambda^{l_1}\big|_0,\ \dots,\ \Lambda^{l_M}\big|_0\big), \qquad (i = 1 \cdots N)$$

of weight μ_i assumed to be represented by a certain reduced Gröbner basis $\big\langle \mathsf{S}_i \big\rangle_{1\leqslant i\leqslant N}$ for a certain monomial order, with the crucial property that no *new invariant appears behind f_1', namely with the property that, without setting $f_1' = 0$, one has N identically satisfied relations:*

$$0 \equiv \mathsf{S}_i\big(\Lambda^{l_1},\ \dots,\ \Lambda^{l_M}\big) - f_1'\,\mathsf{R}_i\big(f_1',\ \Lambda^{l_1},\ \dots,\ \Lambda^{l_M}\big) \qquad (i = 1 \cdots N),$$

for some remainders R_i which all depend polynomially upon the same collection of invariants $f_1', \Lambda^{l_1}, \dots, \Lambda^{l_M}$*, so that no new invariant appears at this stage.*

Then the algorithm terminates and the algebra of invariants coincides with:

$$\boxed{\mathsf{E}_\kappa^n = \mathbb{C}\big[f_1',\ \Lambda^{l_1},\ \dots\dots,\ \Lambda^{l_M}\big]\ \ \textsf{modulo syzygies}}\,.$$

As a standard byproduct of basic Gröbner bases theory, one deduces a unique representation of any polynomial invariant under reparametrization modulo the syzygies.

Indeed, for these values of n and of κ, if one denotes the leading terms (with respect to the monomial order in question) of the above N syzygies by:

$$\mathsf{LT}\big(\mathsf{S}_i(\Lambda)\big) = \big(\Lambda^{l_1}\big)^{\alpha_1^i} \cdots \big(\Lambda^{l_M}\big)^{\alpha_M^i} \qquad (i = 1 \cdots N),$$

for certain specific multiindices $\big(\alpha_1^i, \dots, \alpha_M^i\big) \in \mathbb{N}^M$, and if for $i = 1, \dots, N$ one denotes by:

$$\square_i := \alpha^i + \mathbb{N}^M = \big\{\big(\alpha_1^i + b_1,\ \dots,\ \alpha_M^i + b_M\big) :\ b_1, \dots, b_M \in \mathbb{N}^M\big\}$$

the positive quadrant of $\mathbb{N}^M$ having vertex at α^i, then a general, arbitrary invariant in $\mathsf{E}_{\kappa,m}^n$ of weight m writes *uniquely* under the *normal form*:

$$\sum_{0\leqslant a\leqslant m} (f_1')^a\, \widetilde{\mathsf{P}}_a\big(\Lambda^{l_1},\,\dots,\,\Lambda^{l_M}\big),$$

with summation containing *only positive powers* of f_1', where each $\widetilde{P}_a$ is of weight $m-a$ and is put under *Gröbner-normalized form*:

$$\boxed{\widetilde{\mathsf{P}}_a = \sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\backslash(\square_1\cup\cdots\cup\square_N)\\ l_1b_1+\cdots+l_Mb_M=m-a}} \mathsf{coeff}_{a;\,b_1,\dots,b_M}\cdot\big(\Lambda^{l_1}\big)^{b_1}\cdots\big(\Lambda^{l_M}\big)^{b_M}},$$

with complex coefficients $\mathsf{coeff}_{a;\,b_1,\dots,b_M}$ subjected to no restriction at all.

The kernel algorithm. We would like to mention that, after the paper [21] was completed and submitted, on the occasion of a Workshop about holomorphic extension of CR functions and their removable singularities organized by Berit Stensønes and John-Erik Fornæss at the university of Michigan (Ann Arbor, December 2007), Harm Derksen indicated to us the so-called *Van den Essen's kernel algorithm* for locally nilpotent derivations, the goal of which is to generate all invariants for certain one-dimensional non-reductive actions ([14, 7, 15]). Although applied here to actions of any dimension, our algorithm here is in substance the same, though some features will be dealt with here more explicitly in the quite nontrivial explorations to which the paper is devoted: homogeneity of syzygies; stepwise generation of relations; skirting of Gröbner bases when they fail (due to oversizeness) to compute of the remainders R_i; systematic restriction to $\{f_1'=0\}$ to shorten time computation.

In a near future, we hope to set up a refined algorithm which would almost completely tame the disturbing expression swelling.

Action of $\mathsf{GL}_n(\mathbb{C})$ and unipotent reduction. Lastly, we come back to explaining how one obtains Schur decompositions of Demailly-Semple bundles.

On an arbitrary fiber $\mathsf{E}_{\kappa,m}^n$ of $\mathsf{E}_{\kappa,m}^n T_X^*$ consisting of polynomials $\mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(f', f'', \dots, f^{(\kappa)}\big)$ invariant by reparametrization, one looks at the action of matrices $\mathsf{w}=(w_{ij})\in\mathsf{GL}_n(\mathbb{C})$ which, for each jet level λ with $1\leqslant\lambda\leqslant\kappa$, multiplies by w the n jet-components $f^\lambda := \big(f_1^{(\lambda)},\dots,f_n^{(\lambda)}\big)$, namely which transforms them into $\mathsf{w}\cdot f^\lambda := \big(\sum_{j=1}^n w_{1j}f_j^{(\lambda)},\dots,\sum_{j=1}^n w_{nj}f_j^{(\lambda)}\big)$ with the *same* matrix for each jet level $\lambda=1,2,\dots,\kappa$.

According to elementary representation theory, $\mathsf{E}_{k,m}^n$ then decomposes into a certain direct sum of irreducible GL_n-representations, which are nothing but the Schur representations $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}$ indexed by integers $\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n$. General reasons ([4]) insure that such a decomposition on fibers globalizes coherently as a decomposition between bundles over $X\subset\mathbb{P}^{n+1}(\mathbb{C})$. How then does one determine the appearing Schur components? It suffices to look at the so-called *vectors of highest weight*, which in our situation are just the polynomials invariant by reparametrization $\mathsf{P} \in \mathsf{E}_{\kappa,m}^n$ which are *unipotent-invariant*, namely which are left untouched after multiplication by any unipotent matrix:

$$\mathsf{u} \cdot \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big) \qquad \text{for every } \ \mathsf{u} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & 1 \end{pmatrix}.$$

Then the full space $\mathsf{E}_{\kappa,m}^n$ is obtained as just the $\mathsf{GL}_n(\mathbb{C})$-orbit of $\mathsf{UE}_{\kappa,m}^n$, and this will correspond to somehow *polarizing* the lower indices of bi-invariants, *see* below. We then call *bi-invariants* the polynomials which are both invariant under reparametrization and under the unipotent action:

$$\boxed{\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) \qquad \text{and} \qquad \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)}.$$

Thus, the bi-invariants are nothing but vectors of highest weight for this representation of $\mathsf{GL}_n(\mathbb{C})$. According to the general theory, to each vector of highest weight corresponds one and only one irreducible Schur representation $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}$. How does one finds the integers ℓ_i?

Suppose that, after executing the algorithm, one already knows that UE_κ^n is generated by a finite number $f_1', \Lambda^{l_1}, \dots, \Lambda^{l_M}$ of bi-invariants of weights $1, l_1, \dots, l_M$, and suppose that we have a *unique* writing:

$$\sum_{(a,b_1,\dots,b_M)\in\mathcal{N}} \mathsf{coeff}_{a,b_1,\dots,b_M} \, (f_1')^a \left(\Lambda^{l_1}\right)^{b_1} \cdots \left(\Lambda^{l_M}\right)^{b_M}$$

of an arbitrary, general bi-invariant modulo the syzygies, for a certain monomial order, where $\mathcal{N} \subset \mathbb{N}^{1+M}$ denotes the complement of the union of quadrants having vertex at leading exponents. Then for every $(a, b_1, \dots, b_M)$, the single monomial $(f_1')^a \left(\Lambda^{l_1}\right)^{b_1} \cdots \left(\Lambda^{l_M}\right)^{b_M}$ is a vector of highest weight, and if one lets a general diagonal matrix:

$$\mathsf{x} := \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix}$$

act on it, the theory says it necessarily is an eigenvector, and the eigenvalue:

$$\mathsf{x} \cdot (f_1')^a \left(\Lambda^{l_1}\right)^{b_1} \cdots \left(\Lambda^{l_M}\right)^{b_M} = x_1^{\ell_1} \cdots x_n^{\ell_n} \, (f_1')^a \left(\Lambda^{l_1}\right)^{b_1} \cdots \left(\Lambda^{l_M}\right)^{b_M},$$

exhibits the wanted ℓ_i's which necessarily satisfy $\ell_1 \geqslant \cdots \geqslant \ell_n$.

In conclusion, *both in order to understand invariants and in order to make Euler-characteristic computations, the very main goal is to explore algebras of bi-invariants*.

By requiring unipotent-invariance, the initial rational expression for bi-invariants will depend upon certain determinants defined as follows in terms of the initial invariants $\Lambda_{1,i:\,1^{\lambda-2}}^{2\lambda-1}$.

Theorem. *In dimension $n \geqslant 1$ and for jets of arbitrary order $\kappa \geqslant 1$, every bi-invariant polynomial $\mathsf{BP} = \mathsf{BP}\big(j^\kappa f\big)$ invariant by reparametrization and invariant under the unipotent action writes under the form:*

$$\mathsf{BP}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{BP}_a \left(\left| \begin{array}{cccc} \Lambda_{1,2}^{2\lambda_2-1} & \Lambda_{1,3}^{2\lambda_2-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_2-1} \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \end{array} \right|_{n_1=1,2\ldots,n}^{2\leqslant\lambda_2,\ldots,\lambda_{n_1}\leqslant\kappa} \right),$$

for certain specific polynomials BP_a which depend upon $\mathsf{BP}(j^\kappa f)$.

Algebras of bi-invariants. As announced in the abstract, we finalized two main applications of our algorithm. Only one bi-invariant, namely Y^{27}, was missed in [21], an article which pointed out that bracketing was insufficient.

Theorem. *In dimension $n = 2$ for jet order $\kappa = 5$, the algebra UE_5^2 of jet polynomials $\mathsf{P}\big(j^5 f_1, j^5 f_2\big)$ invariant by reparametrization and invariant under the unipotent action is generated by 17 mutually independent bi-invariants explicitly defined in Section 10:*

$$\boxed{\begin{array}{ccccccccc} f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & \Lambda^9, & M^8, & M^{10}, & K^{12}, & \\ N^{12}, & H^{14}, & F^{16}, & X^{18}, & X^{19}, & X^{21}, & X^{23}, & X^{25}, & Y^{27} \end{array}}.$$

As a consequence, the full algebra E_5^2 of jet polynomials $\mathsf{P}\big(j^5 f\big)$ invariant by reparametrization is generated by the polarizations:

$$\boxed{\begin{array}{ccccccccc} f_i', & \Lambda^3, & \Lambda_i^5, & \Lambda_{i,j}^7, & \Lambda_{i,j,k}^9, & M^8, & M_i^{10}, & K_{i,j}^{12}, & \\ N^{12}, & H_i^{14}, & F_{i,j}^{16}, & X_{i,j,k}^{18}, & X_i^{19}, & X^{21}, & X_i^{23}, & X_{i,j}^{25}, & Y_{i,j,k}^{27} \end{array}}$$

of these 17 bi-invariants, where the indices i, j, k vary in $\{1, 2\}$, whence the total number of these invariants equals:

$$2+1+2+4+8+1+2+4+1+2+4+8+2+1+2+4+8 = \boxed{\mathbf{56}}.$$

Secondly, we obtain the following new result in dimension 4. We must confess that we were unable to discover some harmonious algebraic structures which could probably (in)exist?

Theorem. *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UE_4^4 of jet polynomials $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by 16 mutually independent bi-invariants explicitly defined*[3] *in Section 11:*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

[3] The bi-invariant X^{21} here is different from the X^{21} of the preceding theorem.

whose restriction to $\{f_1' = 0\}$ *has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the 41 syzygies written on p. 73.*

Furthermore, any bi-invariant of weight m *writes uniquely in the finite polynomial form:*

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,\dots,n)\in\mathbb{N}^{14}\backslash(\square_1\cup\cdots\cup\square_{41})\\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot$$
$$\cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h$$
$$\big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ *subjected to no restriction, where* $\square_1, \dots, \square_{41}$ *denote the quadrants in* $\mathbb{N}^{14}$ *having vertex at the leading terms of the 41 syzygies in question.*

As a consequence, the full algebra E_4^4 *of jet polynomials* $\mathsf{P}\big(j^4 f\big)$ *invariant by reparametrization is generated by the polarizations of the 16 bi-invariants:*

$$\boxed{\begin{array}{c} W^{10},\quad f_i',\quad \Lambda^3_{[i,j]},\quad \Lambda^5_{[i,j];\,\alpha},\quad \Lambda^7_{[i,j];\,\alpha,\beta},\quad D^6_{[i,j,k]},\\ D^8_{[i,j,k];\,\alpha},\quad N^{10}_{[i,j,k];\,\alpha,\beta},\quad M^8_{[i,j],[k,l]},\quad E^{10}_{[i,j,k],[p,q]},\quad L^{12}_{[i,j,k],[p,q];\,\alpha},\\ Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta},\quad R^{15}_{[i,j,k],[p,q,r];\,\alpha},\quad U^{17}_{[i,j,k],[p,q,r],[s,t]},\\ V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha},\quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta},\end{array}}$$

These polarized invariants are skew-symmetric with respect to each collection of bracketed indices $[i,j,k]$, $[p,q,r]$, $[s,t]$, *where the roman indices satisfy* $1 \leqslant i < j < k \leqslant 4$, *where* $1 \leqslant p < q < r \leqslant 4$, *where* $1 \leqslant s < r \leqslant 4$ *and where the two greek indices* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 4$ *without restriction and finally the total number of these invariants generating the Demailly-Semple algebra* E_4^4 *equals:*

$$\begin{aligned} &1+4+6+24+96+4+16+64+\\ &+36+24+96+384+64+96+384+1536 = \boxed{\mathbf{2835}}. \end{aligned}$$

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Finally, the theorem on p. 75 which describes the structure of the algebra of bi-invariants for $n = k = 4$ was firmly gained during the author's stay at the Mittag-Leffler Institute in April 2008.

§2. INVARIANT POLYNOMIALS AND COMPOSITE DIFFERENTATION

Fixing basic notations. Let X be a smooth n-dimensional complex algebraic hypersurface of $\mathbb{P}^{n+1}(\mathbb{C})$, let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and consider an arbitrary holomorphic disc $f : \mathbb{D} \to X$ valued in X, for instance the restriction to $\mathbb{D}$ of some entire holomorphic curve $\mathbb{C} \to X$. In some local chart on $X \simeq \mathbb{D}^n$ centered at $f(0)$, the κ-jet $j_0^\kappa f$ of f at $0 \in \mathbb{D}$ is represented by the collection of all the derivatives, with respect to the variable $\zeta \in \mathbb{D}$, of the n components $f_1, \dots, f_n$ of f, up to order κ, that is to say:

$$j^\kappa f = \big(f_1', \dots, f_n', \, f_1'', \dots, f_n'', \, \dots\dots, \, f_1^{(\kappa)}, \dots, f_n^{(\kappa)}\big);$$

from the beginning and throughout this study, we shall in fact constantly omit to denote the base point $0 \in \mathbb{D}$.

Polynomials invariant by reparametrization. For $\kappa \geqslant 1$, we consider polynomials in all the jet variables:

$$\mathsf{P} = \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(f_{j_1}', \, f_{j_2}'', \dots, \, f_{j_\kappa}^{(\kappa)}\big),$$

where the indices $j_1, j_2, \dots, j_\kappa$ run in $\{1, \dots, n\}$. An open problem in Demailly's strategy towards the Kobayashi hyperbolicity conjecture ([4, 6]) was to describe those polynomials $\mathsf{P}\big(j^\kappa f\big)$ enjoying the property that a change of variable $\mathbb{D} \ni \zeta \longmapsto \phi(\zeta) \in \mathbb{C}$ in the source affects the polynomial only through multiplication by some power of the first derivative of ϕ:

$$\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot P\big((j^\kappa f) \circ \phi\big),$$

where $m \geqslant 1$ is an integer which shall be called here the *weight* of P.

Choosing in particular ϕ to be simply a dilation $\zeta \mapsto \delta \cdot z$ by a constant nonzero complex factor δ, one sees that such polynomials must at least (*cf.* [17]) be *weighted homogeneous of order* m with respect to the weighted anisotropic dilations:

$$\mathsf{P}\big(\delta \cdot f_{j_1'}, \, \delta^2 \cdot f_{j_2}'', \, \dots, \, \delta^\kappa \cdot f_{j_\kappa}^{(\kappa)}\big) \equiv \delta^m \cdot \mathsf{P}\big(f_{j_1}', \, f_{j_2}'', \, \dots, \, f_{j_\kappa}^{(\kappa)}\big).$$

As a useful mnemonic, weight therefore always counts the total number of primes.

By $\mathsf{E}_{\kappa,m}^n$, we will thus denote the vector space consisting of all such polynomials. The direct sum $\mathsf{E}_\kappa^n := \bigoplus_{m \geqslant 1} \mathsf{E}_{\kappa,m}^n$ forms an algebra which is graded by constancy of weights, for the definition yields:

$$\mathsf{E}_{\kappa,m_1}^n \cdot \mathsf{E}_{\kappa,m_2}^n \subset \mathsf{E}_{\kappa,m_1+m_2}^n.$$

Following a nowadays established terminology, a polynomial $\mathsf{P}\big(j^\kappa f\big)$ in this algebra will be said to be *invariant by reparametrization*. The present article aims to describe a complete algorithm generating all such polynomials, sometimes briefly called *invariants*.

Example. For $\kappa = 1$, the components f_i' for $i = 1, \dots, n$ of the jet satisfy:

$$\big(f_i \circ \phi\big)' = \phi' \cdot f_i',$$

hence every polynomial $\mathsf{P} = \mathsf{P}\big(f_1', \dots, f_n'\big)$ which depends only upon the first order jet $j^1 f$ is invariant by reparametrization. So E_1^n coincides with the plain polynomial algebra $\mathbb{C}\big[f_1', \dots, f_n'\big]$.

Example. For $\kappa = 2$, aside from the monomials $f_1', \dots, f_n'$ coming from the preceding jet level $\kappa = 1$, there are yet the 2×2 determinants (clearly of weight 3):

$$\Delta_{i,j}^{',''} := \begin{vmatrix} f_i' & f_j' \\ f_i'' & f_j'' \end{vmatrix},$$

for one easily checks, thanks to row linear dependence, that:

$$\begin{vmatrix} (f_i \circ \phi)' & (f_j \circ \phi)' \\ (f_i \circ \phi)'' & (f_j \circ \phi)'' \end{vmatrix} = \begin{vmatrix} \phi' f_i' & \phi' f_j' \\ \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \end{vmatrix} = \phi'^3 \cdot \begin{vmatrix} f_i' & f_j' \\ f_i'' & f_j'' \end{vmatrix}.$$

It is a theorem, to be stated below, that the f_i' and the $\Delta_{j,k}^{',''}$ generate the algebra E_n^2.

Composite differentiation up to order $\kappa = 5$. Setting $g_i := f_i \circ \phi$ for $i = 1, \dots, n$, the elementary chain rule provides derivatives of g_i with respect to the source variable $\zeta \in \mathbb{D}$:

$$\begin{aligned}
g_i' &= \phi' f_i', \\
g_i'' &= \phi'' f_i' + \phi'^2 f_i'', \\
g_i''' &= \phi''' f_i' + 3\,\phi''\phi' f_i'' + \phi'^3 f_i''', \\
g_i'''' &= \phi'''' f_i' + 4\,\phi'''\phi' f_i'' + 3\,\phi''^2 f_i'' + 6\,\phi''\phi'^2 f_i''' + \phi'^4 f_i'''', \\
g_i''''' &:= \phi''''' f_i' + 5\,\phi''''\phi' f_i'' + 10\,\phi'''\phi'' f_i'' + 15\,\phi''^2\phi' f_i''' + \\
&\qquad + 10\,\phi'''\phi'^2 f_i''' + 10\,\phi''\phi'^3 f_i'''' + \phi'^5 f_i'''''.
\end{aligned}$$

Thus with $\kappa = 5$ for instance, the goal is to find all polynomials $\mathsf{P} = \mathsf{P}\big(j^5 g\big)$ which, after replacing g_i', g_i'', g_i''', g_i'''' and g_i''''' by these expressions, have the property of *cancelling* the derivatives ϕ'', ϕ''', ϕ'''' and ϕ''''' of ϕ whose order is $\geqslant 2$, so that $\mathsf{P}\big(j^5 g\big) = \phi'^m \mathsf{P}\big(j^5 f\big)$ for a certain $m \in \mathbb{N}$.

For the sake of completeness, let us present the classical *Faà di Bruno*, well known in the case of one variable $\zeta \in \mathbb{C}$.

Theorem. *For every integer $\kappa \geqslant 1$, the derivative of order κ of each composite function $g_i(z) := f_i \circ \phi(z)$ $(1 \leqslant i \leqslant n)$ with respect to the variable $\zeta \in \mathbb{C}$ is a polynomial with integer coefficients in the derivatives of f_i (same index i) and in* *the derivatives of* ϕ:

$$\boxed{g_i^{(\kappa)} = \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant 1,\dots,\mu_e\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\,\mu_1!\cdots(\lambda_e!)^{\mu_e}\,\mu_e!}\,\big(\phi^{(\lambda_1)}\big)^{\mu_1}\cdots\cdots\big(\phi^{(\lambda_e)}\big)^{\mu_e}\,f_i^{(\mu_1+\cdots+\mu_e)}}.$$

To read this general formula with the help of the formulas specialized above, let us observe that the general monomial $\big(\phi^{(\lambda_1)}\big)^{\mu_1}\cdots\cdots\big(\phi^{(\lambda_e)}\big)^{\mu_e}$ in the reparametrization jet gathers derivatives of increasing orders $\lambda_1<\lambda_2<\cdots<\lambda_e$, with $\mu_1,\mu_2,\dots,\mu_e$ counting their respective numbers. Then the function f_i is subjected to a partial differentiation of order $\mu_1+\mu_2+\cdots+\mu_e$, the total number of derivatives $\phi^{(\lambda_k)}$ in the monomial in question. Finally, in the permutation group $\mathfrak{S}_\kappa$ of $\{1,2,\dots,\kappa\}$ whose cardinality clearly equals $\kappa!$, the quantity $(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!$ counts the number of permutations which possess μ_1 cycles of length λ_1, μ_2 cycles of length λ_2, *etc.*, μ_e cycles of length λ_e, so that the fractional coefficient $\frac{\kappa!}{(\lambda_1!)^{\mu_1}\,\mu_1!\cdots(\lambda_e!)^{\mu_e}\,\mu_e!}$ with $\kappa=\mu_1\lambda_1+\mu_2\lambda_2+\cdots+\mu_e\lambda_e$ is an integer which provides the cardinality of the (left or right) coset of $\mathfrak{S}_\kappa$ modulo such a subgroup permutations. Notice that all these observations are confirmed by the formulas developed above up to $\kappa=5$.

With such a formula, the problem of finding all polynomials invariant by reparametrization can be interpreted in terms of invariant theory ([4, 29]).

Indeed, the group G_κ of κ-jets at the origin of local reparametrizations:

$$\phi(\zeta)=\zeta+\phi''(0)\,\frac{\zeta^2}{2!}+\cdots+\phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!}+\cdots$$

that are tangent to the identity, namely $\phi'(0)=1$, may be seen, thanks to the above formulas, to act linearly on the $n\kappa$-tuples $\big(f'_{j_1},f''_{j_2},\dots,f^{(\kappa)}_{j_\kappa}\big)$ just by matrix multiplication:

$$\begin{pmatrix} g_i' \\ g_i'' \\ g_i''' \\ g_i'''' \\ \vdots \\ g_i^{(\kappa)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi'' & 1 & 0 & 0 & \cdots & 0 \\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0 \\ \phi'''' & 4\phi'''+3\phi''^2 & 6\phi'' & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} f_i' \\ f_i'' \\ f_i''' \\ f_i'''' \\ \vdots \\ f_i^{(\kappa)} \end{pmatrix} \qquad {\scriptstyle (i\,=\,1\,\cdots\,n)}.$$

Polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization coincide with the invariants for this linear group action, an action which is clearly unipotent, hence non-reductive. In such a context, no general theory or algorithm exists to decide whether the algebra of invariants is finitely generated (*cf.* Problem 2 p. 2 in [8]). In fact, we will attack the problem from another point of view.

§3. Bracketing process and syzygies:

JACOBI, PLÜCKER 1 AND PLÜCKER 2

Cross product between two invariants polynomials. A natural process known to Demailly and to El Goul (*cf.* [5] and [21]) is as follows. Suppose that we know two reparametrization-invariant polynomials $\mathsf{P} = \mathsf{P}\big(j^\kappa g\big)$ of weight m and $\mathsf{Q} = \mathsf{Q}\big(j^\tau f\big)$ of weight n, namely we have:

$$\begin{aligned}\mathsf{P}\big(j^\kappa g\big) &= \phi'^m\, \mathsf{P}\big((j^\kappa f)\circ\phi\big),\\ \mathsf{Q}\big(j^\tau g\big) &= \phi'^n\, \mathsf{Q}\big((j^\tau f)\circ\phi\big),\end{aligned}$$

where we have again set $g := f\circ\phi$. To differentiate a polynomial with respect to the source variable $\zeta\in\mathbb{C}$ amounts to apply to it the *total differentiation operator*:

$$\mathsf{D} := \sum_{k=1}^{n}\sum_{\lambda\in\mathbb{N}} \frac{\partial(\bullet)}{\partial f_k^{(\lambda)}}\cdot f_k^{(\lambda+1)},$$

which gives here:

$$\begin{aligned}\big[\mathsf{DP}\big]\big(j^{\kappa+1}g\big) &= m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^\kappa f)\circ\phi\big) + \phi'^m\,\phi'\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big),\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}g\big) &= n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big) + \phi'^n\,\phi'\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big),\end{aligned}$$

and in order to remove the second order derivative ϕ'', it suffices to perform a *cross-product*, namely to form the 2×2 determinant:

$$\begin{aligned}&\left|\begin{array}{cc}\big[\mathsf{DP}\big]\big(j^{\kappa+1}g\big) & m\,\mathsf{P}\big(j^\kappa g\big)\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}g\big) & n\,\mathsf{Q}\big(j^\tau g\big)\end{array}\right| =\\ &= \left|\begin{array}{cc} m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^\kappa f)\circ\phi\big)+\phi'^{m+1}\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^m\,\mathsf{P}\big((j^\kappa f)\circ\phi\big)\\ n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big)+\phi'^{n+1}\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^n\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big)\end{array}\right|\\ &= \left|\begin{array}{cc}\phi'^{m+1}\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^m\,\mathsf{P}\big((j^\kappa f)\circ\phi\big)\\ \phi'^{n+1}\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^n\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big)\end{array}\right|\\ &= \phi'^{m+n+1}\left|\begin{array}{cc}\big[\mathsf{DP}\big]\big(j^{\kappa+1}f\big) & m\,\mathsf{P}\big(j^\kappa f\big)\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}f\big) & n\,\mathsf{Q}\big(j^\tau f\big)\end{array}\right|,\end{aligned}$$

which therefore happens to constitute a new invariant of weight $m+n+1$ in the jet space of order $1+\max(\kappa,\tau)$ *increased by one unit*.

Bracket operator $\big[\cdot,\cdot\big]$ and its accompanying syzygies. Thus, *every pair of invariants automatically produces a new invariant*:

$$\boxed{\big[\mathsf{P},\,\mathsf{Q}\big] := n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}},$$

which is obviously skew-symmetric with respect to the pair $\big(\mathsf{P},\mathsf{Q}\big)$. For instance, we recover in this way *all* the invariants of (jet) order 2 mentioned above:

$$\big[f_i',\,f_j'\big] = \mathsf{D}f_i'\cdot f_j' - f_i'\mathsf{D}f_j' = f_i''f_j' - f_i'f_j'' = -\Delta_{i,j}^{',''},$$

and again, we notice that bracketing increases jet order by one unit.

Certainly, as soon as at least 3 pairwise distinct invariants P, Q and R are known, a *Jacobi-type* identity (checked on pp. 867–868 of [21]) must hold:

$$\boxed{(\mathcal{J}ac): \qquad 0 \equiv \big[[\mathsf{P},\,\mathsf{Q}],\,\mathsf{R}\big] + \big[[\mathsf{R},\,\mathsf{P}],\,\mathsf{Q}\big] + \big[[\mathsf{Q},\,\mathsf{R}],\,\mathsf{P}\big]}\,.$$

Although such relations give nothing for jet order $\kappa = 2$, because the jet order of an iterated bracket $\big[[\cdot,\,\cdot],\,\cdot\big]$ is in any case $\geqslant 3$, if we introduce the following new bracket-type invariants:

$$\begin{aligned}\big[\Delta_{i,j}^{\prime,\prime\prime},\,f_k'\big] &= \mathsf{D}\big(f_i'f_j'' - f_i''f_j'\big)\cdot f_k' - 3\,\big(f_i'f_j'' - f_i''f_j'\big)\cdot f_k'' \\ &= \big(f_i'f_j''' - f_i'''f_j'\big)\cdot f_k' - 3\,\big(f_i'f_j'' - f_i''f_j'\big)\cdot f_k'',\end{aligned}$$

then we gratuitously have Jacobi-type relations which will hold true at the next jet level $\kappa = 3$:

$$0 \equiv \big[\Delta_{i,j}^{\prime,\prime\prime},\,f_k'\big] + \big[\Delta_{k,i}^{\prime,\prime\prime},\,f_j'\big] + \big[\Delta_{j,k}^{\prime,\prime\prime},\,f_i'\big].$$

On the other hand, we remind that the 2×2 minors $\mathsf{a}_{1,2}^{j_1,j_2} := \det\big(\mathsf{a}_i^j\big)_{i=1,2}^{j=j_1,j_2}$ of an arbitrary $2\times N$ complex-valued matrix $\big(\mathsf{a}_i^j\big)_{i=1,2}^{1\leqslant j\leqslant N}$ are known to enjoy ([21], p. 883) the so-called *quadratic Plücker relations* which are usually organized in two families[4]:

$$\begin{aligned}(\mathcal{P}lck_1): &\qquad 0 \equiv \mathsf{a}_1^{j_1}\cdot\mathsf{a}_{1,2}^{j_2,j_3} + \mathsf{a}_1^{j_3}\cdot\mathsf{a}_{1,2}^{j_1,j_2} + \mathsf{a}_1^{j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_1}, \\ (\mathcal{P}lck_2): &\qquad 0 \equiv \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4} + \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4} + \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4},\end{aligned}$$

and which may be checked by expanding the minors, just observing cancellations[5]. We then deduce that our bracketing process, when interpreted as computing the minors of an auxiliary matrix:

$$\begin{pmatrix} m\,\mathsf{P} & n\,\mathsf{Q} & o\,\mathsf{R} & p\,\mathsf{S} & \cdots \\ \mathsf{DP} & \mathsf{DQ} & \mathsf{DR} & \mathsf{DS} & \cdots \end{pmatrix},$$

whose first line lists known invariants multiplied by their own weight, and whose second line lists their total derivatives, we immediately deduce that our bracketing process introduces the following two supplementary families of identically satisfied *Plückerian-like* relations:

$$\boxed{\begin{aligned}(\mathcal{P}lck_1): &\qquad 0 \equiv m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big] + o\,\mathsf{R}\,\big[\mathsf{P},\,\mathsf{Q}\big] + n\,\mathsf{Q}\,\big[\mathsf{R},\,\mathsf{P}\big], \\ (\mathcal{P}lck_2): &\qquad 0 \equiv \big[\mathsf{P},\,\mathsf{Q}\big]\cdot\big[\mathsf{R},\,\mathsf{S}\big] + \big[\mathsf{S},\,\mathsf{P}\big]\cdot\big[\mathsf{R},\,\mathsf{Q}\big] + \big[\mathsf{Q},\,\mathsf{S}\big]\cdot\big[\mathsf{R},\,\mathsf{P}\big]\end{aligned}}\,.$$

Throughout the text, identically satisfied relations between polynomials will often be called *syzygies*, following the terminology of classical invariant theory ([25]).

---

[4] In the first line, the sum bears upon circular permutations of (j_1, j_2, j_3); in the second line, j_3 is fixed and the sum bears upon circular permutations of (j_1, j_2, j_4). Equivalently, one could have fixed j_4 and considered circular permutations of (j_1, j_2, j_3).

[5] In fact, only these relations appear in the ideal of syzygies between the a_1^j and the $\mathsf{a}_{1,2}^{j_1,j_2}$, for an appropriate monomial order ([24], p. 277).

For instance, at the jet level $\kappa = 2$, we plainly have:

$$
\begin{aligned}
0 &\equiv \Delta_{i,j}^{',''} \cdot f_k' + \Delta_{k,i}^{',''} \cdot f_j' + \Delta_{j,k}^{',''} \cdot f_i', \\
0 &\equiv \Delta_{i,j}^{',''} \cdot \Delta_{k,l}^{',''} + \Delta_{l,i}^{',''} \cdot \Delta_{k,j}^{',''} + \Delta_{j,l}^{',''} \cdot \Delta_{k,i}^{',''},
\end{aligned}
$$

for all indices $i, j, k, l = 1, \dots, n$.

A general notation for Wronskian-like determinants. It will be quite useful to abbreviate the explicit denotation of the further, rather complicated invariants that we shall have to deal with in the sequel by introducing the minors:

$$
\Delta_{i,\,j}^{(\alpha),(\beta)} := \begin{vmatrix} f_i^{(\alpha)} & f_j^{(\alpha)} \\ f_i^{(\beta)} & f_j^{(\beta)} \end{vmatrix}, \qquad \Delta_{i,\,j,\,k}^{(\alpha),(\beta),(\gamma)} := \begin{vmatrix} f_i^{(\alpha)} & f_j^{(\alpha)} & f_k^{(\alpha)} \\ f_i^{(\beta)} & f_j^{(\beta)} & f_k^{(\beta)} \\ f_i^{(\gamma)} & f_j^{(\gamma)} & f_k^{(\gamma)} \end{vmatrix}, \qquad \textit{etc.}
$$

extracted from the jet matrix $\big(f_i^{(\lambda)}\big)$. Top indices list derivative orders, appearing in rows.

Thanks to skew-symmetry, after some row or column permutations, one can always write these determinants in such a way that the lower, dimensional indices satisfy $1 \leqslant i < j < k \leqslant n$ and similarly, the upper, derivative indices also satisfy $1 \leqslant \alpha < \beta < \gamma \leqslant \kappa$ at the same time.

In fact, the already mentioned observation that $\Delta_{i,j}^{',''}$ always provides an invariant easily generalizes, for if we set:

$$
g_i^{(\lambda)} := \big(f_i \circ \phi\big)^{(\lambda)},
$$

then by either manipulating the Faà di Bruno formula written above, or by using a less explicit intermediate inductive assertion in order to pass from one jet level to the next jet level, one may subject the determinants to row linear combinations in order to establish the following:

Lemma. *For every λ with $1 \leqslant \lambda \leqslant \kappa$ and for all indices $i_1, i_2, \dots, i_\lambda = 1, \dots, n$, one has:*

$$
\begin{vmatrix} g_{i_1}' & g_{i_2}' & \cdots & g_{i_\lambda}' \\ g_{i_1}'' & g_{i_2}'' & \cdots & g_{i_\lambda}'' \\ \vdots & \vdots & \ddots & \vdots \\ g_{i_1}^{(\lambda)} & g_{i_2}^{(\lambda)} & \cdots & g_{i_\lambda}^{(\lambda)} \end{vmatrix} = (\phi')^{2\lambda-1} \cdot \begin{vmatrix} f_{i_1}' & f_{i_2}' & \cdots & f_{i_\lambda}' \\ f_{i_1}'' & f_{i_2}'' & \cdots & f_{i_\lambda}'' \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_1}^{(\lambda)} & f_{i_2}^{(\lambda)} & \cdots & f_{i_\lambda}^{(\lambda)} \end{vmatrix} = (\phi')^{2\lambda-1} \cdot \Delta_{i_1,i_2,\dots,i_\lambda}^{',\,'',\,\dots,(\lambda)},
$$

hence all the Wronskian-like determinants $\Delta_{i_1,i_2,\dots,i_\lambda}^{',\,'',\dots,(\lambda)}$ always are invariant by reparametrization.

Here, it is crucial that the derivative order starts from 1 at the first row and increases by one unit exactly while descending stepwise along the rows; otherwise, we would *not in any case* get a true invariant; for instance in the expression:

$$
\begin{aligned}
\begin{vmatrix} g_i'' & g_j'' \\ g_i''' & g_j''' \end{vmatrix} &= \begin{vmatrix} \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \\ \phi''' f_i' + 3\,\phi''\phi' f_i'' + \phi'^3 f_i''' & \phi''' f_j' + 3\,\phi''\phi' f_j'' + \phi'^3 f_j''' \end{vmatrix} \\
&= \begin{vmatrix} \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \\ \phi''' f_i' - 2\phi'^3 f_i''' & \phi''' f_j' - 2\phi'^3 f_j''' \end{vmatrix},
\end{aligned}
$$

no further simplification enables to get rid of ϕ'', ϕ''' and such an obstruction happens to hold generally.

Combinatorics of the subalgebra generated by the Wronskians. Thus at least, we know a large family of invariants. The following statement goes back to the nineteenth century.

Proposition. ([19, 4, 24]) *For jets of order $\kappa = 2$ in arbitrary dimension $n \geqslant 2$, the algebra E_2^n consists of the algebra generated by the $n + \frac{n(n-1)}{2}$ fundamental invariants:*

$$f_k' \qquad \text{and} \qquad \Delta_{i,j}^{',''} = \left| \begin{array}{cc} f_i' & f_j' \\ f_i'' & f_j'' \end{array} \right|$$

and their syzygy ideal is generated by the two families of Plückerian relations written above:

$$\begin{cases} 0 \equiv \Delta_{i,j}^{',''} \cdot f_k' + \Delta_{k,i}^{',''} \cdot f_j' + \Delta_{j,k}^{',''} \cdot f_i', \\ 0 \equiv \Delta_{i,j}^{',''} \cdot \Delta_{k,l}^{',''} + \Delta_{l,i}^{',''} \cdot \Delta_{k,j}^{',''} + \Delta_{j,l}^{',''} \cdot \Delta_{k,i}^{',''}. \end{cases}$$

§4. SURVEY OF KNOWN DESCRIPTIONS OF E_κ^n IN LOW DIMENSIONS FOR SMALL JET LEVELS

The above-defined algebra E_κ^n of jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization is understood only in certain specific situations.

Demailly 1997. At first, in dimension $n \geqslant 2$ for jet level $\kappa = 2$, the $n + \frac{n(n-1)}{2}$ generators of the proposition just above appear on p. 341 of [4], namely every polynomial P in E_2^n writes:

$$\mathsf{P}\big(j^2 f\big) \equiv \mathcal{P}_{\mathsf{P}}\big(f_1', \dots, f_n', \Delta_{1,2}^{',''}, \dots, \Delta_{n-1,n}^{',''}\big)$$

having as arguments the basic invariants in question.

In the particular case of surfaces, namely for $n = 2$, no syzygy exists between f_1', f_2' and $\Delta_{1,2}^{',''}$, hence E_2^2 coincides with a plain polynomial algebra:

$$\mathsf{E}_2^2 = \mathbb{C}\big[f_1', f_2', \Delta_{1,2}^{',''}\big].$$

Basic notions of invariant theory. For higher n's and κ's, unpredictable syzygies will obscure the picture, but before pursuing, we must fix a suitable terminology. We formulate these concepts for E_κ^n, but they hold quite more generally.

Definition. If, for certain values of n and κ, there are finitely many invariants $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ in E_κ^n with the property that every polynomial $\mathsf{P}\big(j^\kappa f\big) \in \mathsf{E}_\kappa^n$ invariant by reparametrization can be written as a polynomial:

$$\mathsf{P}\big(j^\kappa f\big) \equiv \mathcal{P}_{\mathsf{P}}\big(\Lambda_1, \dots, \Lambda_{\mathsf{last}}\big)$$

having $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ as arguments, we shall say that E_κ^n is *generated* (as an algebra) *by* $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$.

Definition. Further, we shall say that $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ are *mutually independent* if, for every middle index with $1 \leqslant \mathsf{middle} \leqslant \mathsf{last}$, there does not exist any polynomial $\mathcal{P}$ such that $\Lambda_{\mathsf{middle}}$ identifies to a polynomial:

$$\Lambda_{\mathsf{middle}} = \mathcal{P}\big(\Lambda_1, \dots, \widehat{\Lambda_{\mathsf{middle}}}, \dots, \Lambda_{\mathsf{last}}\big)$$

in the other remaining invariants. Then $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ will be called *fundamental invariants generating* E_κ^n (for such values of n, κ) and an indivivual $\Lambda_{\mathsf{middle}}$ will be called a *basic invariant*.

For a fixed E_κ^n, all sets of fundamental invariants, either finite or infinite, have the same cardinality.

Weights always appear as upper indices. Also, we want for later use to introduce the new notation:

$$\Lambda_{1,2}^3 := \Delta_{1,2}^{','' },$$

where we specify the row indices $1, 2$ and where we specially emphasize the weight 3, counting the total number of primes. In fact, throughout the whole paper, *we shall systematically write the weight of every basic invariant as its upper index*. We thus can continue the survey.

Demailly 2004; Rousseau 2006. Next, in dimension $n = 2$ for jet level $\kappa = 3$, it is shown[6] in [29] that the algebra E_3^2 is generated by the three invariants f_1', f_2' and $\Delta_{1,2}^{','' }$ (already known from the preceding jet level) to which one adds the two further invariants of weight 5:

$$\Lambda_{1,2;\,1}^5 := \big[\Lambda_{1,2}^3,\, f_1'\big] = \Delta_{1,3}^{','' } f_1' - 3\,\Delta_{1,2}^{','' } f_1'' \qquad \text{and} \qquad \Lambda_{1,2;\,2}^5 := \big[\Lambda_{1,2}^3,\, f_2'\big] = \Delta_{1,3}^{','' } f_2' - 3\,\Delta_{1,2}^{','' } f_2'',$$

the only possible brackets, as one checks. Moreover, these five invariants f_1', f_2', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ and $\Lambda_{1,2;\,2}^5$ are mutually independent and their syzygy ideal is *principal*, generated by the single quadratic relation:

$$0 \equiv 3\,\Lambda_{1,2}^3\Lambda_{1,2}^3 - f_2'\Lambda_{1,2;\,1}^5 + f_1'\Lambda_{1,2;\,2}^5.$$

One sees that this syzygy just comes $(\mathcal{P}lck_1)$. In fact, $(\mathcal{J}ac)$ and $(\mathcal{P}lck_1)$ give nothing.

Rousseau 2006. Now, in dimension $n = 3$ and for jet level $\kappa = 3$, applying a theorem of Popov, Rousseau ([29], p. 403) deduced that the algebra E_3^3 is generated by all the invariants known in dimension $n - 1 = 2$ whose lower indices are *polarized* in all possible ways, namely the 15 invariants:

$$f_1', \quad f_2', \quad f_3', \quad \Lambda_{1,2}^3, \quad \Lambda_{1,3}^3, \quad \Lambda_{2,3}^3,$$
$$\Lambda_{1,2;\,1}^5, \quad \Lambda_{1,3;\,1}^5, \quad \Lambda_{2,3;\,1}^5, \quad \Lambda_{1,2;\,2}^5, \quad \Lambda_{1,3;\,2}^5, \quad \Lambda_{2,3;\,2}^5, \quad \Lambda_{1,2;\,3}^5, \quad \Lambda_{1,3;\,3}^5, \quad \Lambda_{2,3;\,3}^5,$$

[6] The result was known to Demailly (unpublished).

together with a single further invariant, the Wronskian:

$$D^6_{1,2,3} := \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}.$$

This makes 16 invariants in sum. An alternative, direct proof of this result may be found in [21], and will also pop up again in the present paper.

We must mention that the Wronskian $D^6_{1,2,3}$ also appears in fact in terms of brackets, for one checks the following three relations:

$$\big[\Lambda^3_{1,2},\,\Lambda^3_{1,3}\big] = -3\,f_1'\,D^6_{1,2,3}, \qquad \big[\Lambda^3_{1,2},\,\Lambda^3_{2,3}\big] = -3\,f_2'\,D^6_{1,2,3},$$
$$\big[\Lambda^3_{1,3},\,\Lambda^3_{2,3}\big] = -3\,f_3'\,D^6_{1,2,3}.$$

A Maple computation ([28]) also provided the ideal of relations between these 16 invariants. Among the 62 generators of the reduced Gröbner basis supplied by Maple after ~ 15 hours of symbolic computations, 30 appear to be minimal generators of the ideal of relations, the 32 remaining ones being further automatically generated S-polynomials which are required to complete the basis. Remarkably, it may be checked ([21]) that *each one of the 30 minimal syzygies* in question is included among the collection of syzygies deduced from our three fundamental families by inserting f_i' and $\Lambda^3_{j,k}$ in all possible ways in place of P, Q, R and T:

$$(\mathcal{J}ac): \quad \Big\{ 0 \equiv \big[[f_i',\,f_j'],\,f_k'\big] + \big[[f_k',\,f_i'],\,f_j'\big] + \big[[f_j',\,f_k'],\,f_i'\big],$$

$$(\mathcal{P}lck_1): \quad \begin{cases} 0 \equiv f_i'\big[f_j',\,f_k'\big] + f_k'\big[f_i',\,f_j'\big] + f_j'\big[f_k',\,f_i'\big], \\ 0 \equiv f_i'\big[f_j',\,\Lambda^3_{k,l}\big] + 3\,\Lambda^3_{k,l}\big[f_i',\,f_j'\big] + f_j'\big[\Lambda^3_{k,l},\,f_i'\big], \\ 0 \equiv f_i'\big[\Lambda^3_{j,k},\,\Lambda^3_{l,m}\big] + \Lambda^3_{l,m}\big[f_i',\,\Lambda^3_{j,k}\big] + \Lambda^3_{j,k}\big[\Lambda^3_{l,m},\,f_i'\big], \\ 0 \equiv \Lambda^3_{i,j}\big[\Lambda^3_{k,l},\,\Lambda^3_{m,n}\big] + \Lambda^3_{m,n}\big[\Lambda^3_{i,j},\,\Lambda^3_{k,l}\big] + \Lambda^3_{k,l}\big[\Lambda^3_{m,n},\,\Lambda^3_{i,j}\big], \end{cases}$$

$$(\mathcal{P}lck_2): \quad \begin{cases} 0 \equiv \big[f_i',\,f_j'\big]\cdot\big[f_k',\Lambda^3_{l,m}\big] + \big[\Lambda^3_{l,m},\,f_i'\big]\cdot\big[f_k',f_j'\big] + \big[f_j',\,\Lambda^3_{l,m}\big]\cdot\big[f_k',f_i'\big], \\ 0 \equiv \big[f_i',\,f_j'\big]\cdot\big[\Lambda^3_{k,l},\,\Lambda^3_{m,n}\big] + \big[\Lambda^3_{m,n},\,f_i'\big]\cdot\big[\Lambda^3_{k,l},\,f_j'\big] + \big[f_j',\,\Lambda^3_{m,n}\big]\cdot\big[\Lambda^3_{k,l},\,f_i'\big], \\ 0 \equiv \big[f_i',\,\Lambda^3_{j,k}\big]\cdot\big[\Lambda^3_{l,m},\,\Lambda^3_{n,p}\big] + \big[\Lambda^3_{n,p},\,f_i'\big]\cdot\big[\Lambda^3_{l,m},\,\Lambda^3_{j,k}\big] + \big[\Lambda^3_{j,k},\,\Lambda^3_{n,p}\big]\cdot\big[\Lambda^3_{l,m},\,f_i'\big], \end{cases}$$

where the indices i, j, k, l, m, n, and p take all the values $1, 2, 3$.

Demailly-El Goul 2004; Rousseau 2007; M. 2007. Finally[7], for jets of order $\kappa = 4$ in dimension $n = 2$, the algebra E^2_4 is generated by the five invariants:

$$f_1', \qquad f_2', \qquad \Lambda^3_{1,2}, \qquad \Lambda^5_{1,2;\,1}, \qquad \Lambda^5_{1,2;\,2}$$

already known from the preceding jet level, to which one adds the four further invariants gently provided by bracketing:

$$\Lambda^7_{1,1} := \big[\Lambda^5_{1,2;\,1},\,f_1'\big], \qquad \Lambda^7_{1,2} := \big[\Lambda^5_{1,2;\,1},\,f_2'\big] = \big[\Lambda^5_{1,2;\,2},\,f_1'\big] = \Lambda^7_{2,1}, \qquad \Lambda^7_{2,2} := \big[\Lambda^5_{1,2;\,2},\,f_2'\big],$$
$$M^8 := \frac{1}{f_1'}\,\big[\Lambda^5_{1,2;\,1},\,\Lambda^3_{1,2}\big].$$

[7] The result was known (unpublished) to experts; a proof appears in [21].

This in sum makes 9 fundamental invariants. Notice the (necessary) division by f_1' to get M^8. The two missing brackets[8]:

$$\big[\Lambda^5_{1,2;\,2},\,\Lambda^3_{1,2}\big]=f_2'\,M^8\qquad\text{and}\qquad\big[\Lambda^5_{1,2;\,1},\,\Lambda^5_{1,2;\,2}\big]=\Lambda^3_{1,2}\,M^8$$

appear to in fact belong already to the algebra generated by these 9 invariants.

Now, we lighten a little the notation by dropping some of the lower indices, especially in the $\Delta^{(\alpha),(\beta)}_{1,2}\equiv:\ \Delta^{(\alpha),(\beta)}$, because in dimension $n=2$, by skew-symmetry of determinants, only $(1,2)$ can appear at the bottom.

Theorem. ([21]) *For jets of order $\kappa=4$ in dimension $n=2$, the algebra E^2_4 is generated by 9 mutually independent fundamental invariants explicitly defined by:*

$$f_1',\qquad f_2',\qquad \Lambda^3:=\Delta^{','\,''},$$

$$\begin{aligned}\Lambda^5_1&:=\Delta^{','''}\,f_1'-3\,\Delta^{','\,''}\,f_1'',\\ \Lambda^5_2&:=\Delta^{','''}\,f_2'-3\,\Delta^{','\,''}\,f_2'',\end{aligned}$$

$$\begin{aligned}\Lambda^7_{1,1}&:=\big(\Delta^{','''''}+4\,\Delta^{'','''}\big)\,f_1'f_1'-10\,\Delta^{','''}\,f_1'f_1''+15\,\Delta^{','\,''}\,f_1''f_1'',\\ \Lambda^7_{1,2}&:=\big(\Delta^{','''''}+4\,\Delta^{'','''}\big)\,f_1'f_2'-5\,\Delta^{','''}\big(f_1''f_2'+f_2''f_1'\big)+15\,\Delta^{','\,''}\,f_1''f_2'',\\ \Lambda^7_{2,2}&:=\big(\Delta^{','''''}+4\,\Delta^{'','''}\big)\,f_2'f_2'-10\,\Delta^{','''}\,f_2'f_2''+15\,\Delta^{','\,''}\,f_2''f_2'',\end{aligned}$$

$$M^8:=3\,\Delta^{','''''}\,\Delta^{','\,''}+12\,\Delta^{'','''}\,\Delta^{','\,''}-5\,\Delta^{','''}\,\Delta^{','''}$$

whose ideal of relations is generated by 9 fundamental syzygies:

$$\Big[0\overset{1}{\equiv}f_2'\,\Lambda^5_1-f_1'\,\Lambda^5_2-3\,\Lambda^3\,\Lambda^3,$$

$$\left[\begin{aligned}0&\overset{2}{\equiv}f_2'\,\Lambda^7_{1,1}-f_1'\,\Lambda^7_{1,2}-5\,\Lambda^3\,\Lambda^5_1,\\ 0&\overset{3}{\equiv}f_2'\,\Lambda^7_{1,2}-f_1'\,\Lambda^7_{2,2}-5\,\Lambda^3\,\Lambda^5_2,\end{aligned}\right.$$

$$\left[\begin{aligned}0&\overset{4}{\equiv}f_1'\,f_1'\,M^8-3\,\Lambda^3\,\Lambda^7_{1,1}+5\,\Lambda^5_1\,\Lambda^5_1,\\ 0&\overset{5}{\equiv}f_1'\,f_2'\,M^8-3\,\Lambda^3\,\Lambda^7_{1,2}+5\,\Lambda^5_1\,\Lambda^5_2,\\ 0&\overset{6}{\equiv}f_2'\,f_2'\,M^8-3\,\Lambda^3\,\Lambda^7_{2,2}+5\,\Lambda^5_2\,\Lambda^5_2,\end{aligned}\right.$$

$$\left[\begin{aligned}0&\overset{7}{\equiv}f_1'\,\Lambda^3\,M^8-\Lambda^5_1\,\Lambda^7_{1,2}+\Lambda^5_2\,\Lambda^7_{1,1},\\ 0&\overset{8}{\equiv}f_2'\,\Lambda^3\,M^8-\Lambda^5_1\,\Lambda^7_{2,2}+\Lambda^5_2\,\Lambda^7_{1,2},\end{aligned}\right.$$

$$\Big[0\overset{9}{\equiv}5\,\Lambda^3\,\Lambda^3\,M^8-\Lambda^7_{2,2}\,\Lambda^7_{1,1}+\Lambda^7_{1,2}\,\Lambda^7_{1,2},$$

which are all obtained by means of the three families of automatic relations $(\mathcal{J}ac)$, $(\mathcal{P}lck_1)$ and $(\mathcal{P}lck_2)$.

---

[8] Details of computations may be found in [21], pp. 870–871 and also pp. 882–886.

Summary and induction. Thus, all known descriptions of algebras of jet polynomials invariant by reparametrization were obtained by starting with the trivial list:

$$f_1',\, f_2',\, \dots,\, f_n'$$

of invariants of order 1, and bracketing them again and again in order to lift oneself to higher jet levels. The principle of induction then dictates to continue such a process.

Jets of order $\kappa = 5$ in dimension $n = 2$. Bracketing all invariants from the preceding jet level $\kappa = 4$ amounts to compute all the 2×2 minors of the following 2×9 matrix:

$$\left\| \begin{array}{ccccccccc} f_1' & f_2' & 3\,\Lambda^3 & 5\,\Lambda_1^5 & 5\,\Lambda_2^5 & 7\,\Lambda_{1,1}^7 & 7\,\Lambda_{1,2}^7 & 7\,\Lambda_{2,2}^7 & 8\,M^8 \\ \mathsf{D}f_1' & \mathsf{D}f_2' & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda_1^5 & \mathsf{D}\Lambda_2^5 & \mathsf{D}\Lambda_{1,1}^7 & \mathsf{D}\Lambda_{1,2}^7 & \mathsf{D}\Lambda_{2,2}^7 & \mathsf{D}M^8 \end{array} \right\|,$$

which in sum makes a total of $\frac{9!}{2!\,7!} = 36$ brackets. But taking account of the fact that the $\frac{5!}{2!\,3!} = 10$ minors of the first 5 columns correspond to the already known passage from $\kappa = 3$ to $\kappa = 4$, just a few less brackets, namely $36 - 10 = 26$ have to be computed, namely the eight families:

$$\begin{array}{ll} \big[\Lambda_{i,j}^7,\, f_k'\big], & \big[M^8,\, f_i'\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda^3\big], & \big[M^8,\, \Lambda^3\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda_k^5\big], & \big[M^8,\, \Lambda_i^5\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda_{k,l}^7\big], & \big[M^8,\, \Lambda_{i,j}^7\big]. \end{array}$$

In [21], this task was achieved, thoroughly and in great details, the obtained brackets being all written in terms of the $\Delta^{(\alpha),(\beta)}$. Furthermore, by inspecting systematically the first fundamental family[9] of syzygies $(\mathcal{J}ac)$, some superfluous brackets that are certain polynomials in terms of previously known invariants were left out.

Theorem. ([21]) *For jets of order $\kappa = 5$ in dimension $n = 2$, the algebra of bracket invariants in E_5^2 is generated by exactly* **24** *mutually independent fundamental invariants:*

$$\boxed{\begin{array}{r} f_1',\;\; f_2',\;\; \Lambda^3,\;\; \Lambda_1^5,\;\; \Lambda_2^5,\;\; \Lambda_{1,1}^7,\;\; \Lambda_{1,2}^7,\;\; \Lambda_{2,2}^7,\;\; M^8, \\ \Lambda_{1,1,1}^9,\;\; \Lambda_{1,2,1}^9,\;\; \Lambda_{2,1,2}^9,\;\; \Lambda_{2,2,2}^9,\;\; M_1^{10},\;\; M_2^{10}, \\ N^{12},\;\; K_{1,1}^{12},\;\; K_{1,2}^{12} = K_{2,1}^{12},\;\; K_{2,2}^{12}, \\ H_1^{14},\;\; H_2^{14},\;\; F_{1,1}^{16},\;\; F_{1,2}^{16},\;\; F_{2,2}^{16}, \end{array}}$$

[9] The other two families of syzygies $(\mathcal{P}lck_1)$ and $(\mathcal{P}lck_2)$ having all their terms quadratic, no resolved relation for any bracket invariant Π of the form $\Pi = \mathsf{polynomial}\,(\Lambda^1, \dots, \Lambda_{\mathsf{last}})$ can arise from them.

among which the pure order 5 brackets are defined by:

$$
\begin{aligned}
\Lambda^9_{i,j,k} &:= \big[\Lambda^7_{i,j},\ f'_k\big]\\
M^{10}_i &:= \big[M^8,\ f'_i\big]\\
N^{12} &:= \big[M^8,\ \Lambda^3\big]\\
K^{12}_{i,j} &:= \big[\Lambda^7_{i,j},\ \Lambda^5_1\big]\big/f'_i\\
H^{14}_i &:= \big[M^8,\ \Lambda^5_i\big]\\
F^{16}_{i,j} &:= \big[M^8,\ \Lambda^7_{i,j}\big]
\end{aligned}
$$

and are explicitly given by the following normalized formulas:

$$
\begin{aligned}
\Lambda^9_{i,j,k} := {} & \Delta^{','''''}\, f'_i f'_j f'_k + 5\,\Delta^{'',''''}\, f'_i f'_j f'_k -\\
& - 4\,\Delta^{',''''}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k - 7\,\Delta^{',''''}\, f'_i f'_j f''_k -\\
& - 16\,\Delta^{'','''}\big(f''_i f'_j + f'_i f''_j\big) f'_k - 28\,\Delta^{'','''}\, f'_i f'_j f''_k -\\
& - 5\,\Delta^{','''}\big(f'''_i f'_j + f'_i f'''_j\big)\, f'_k + 35\,\Delta^{','''}\big(f''_i f''_j f'_k + f''_i f'_j f''_k + f'_i f''_j f''_k\big) -\\
& - 105\,\Delta^{',''}\, f''_i f''_j f''_k,
\end{aligned}
$$

$$
\begin{aligned}
M^{10}_i := {} & \big[3\,\Delta^{','''''}\,\Delta^{',''} + 15\,\Delta^{'',''''}\,\Delta^{',''} - 7\,\Delta^{',''''}\,\Delta^{','''} + 2\,\Delta^{'','''}\,\Delta^{','''}\big]\, f'_i -\\
& - \big[24\,\Delta^{',''''}\,\Delta^{',''} + 96\,\Delta^{'','''}\,\Delta^{',''} - 40\,\Delta^{','''}\,\Delta^{','''}\big] f''_i,
\end{aligned}
$$

$$
\begin{aligned}
N^{12} := {} & 9\,\Delta^{','''''}\,\Delta^{',''}\,\Delta^{',''} + 45\,\Delta^{'',''''}\,\Delta^{',''}\,\Delta^{',''} - 45\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} -\\
& - 90\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} + 40\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''},
\end{aligned}
$$

$$
\begin{aligned}
K^{12}_{i,j} := {} & f'_i f'_j\Big(5\,\Delta^{','''''}\,\Delta^{','''} + 25\,\Delta^{'',''''}\,\Delta^{','''} - 7\,\Delta^{',''''}\,\Delta^{',''''} - 56\,\Delta^{'','''}\,\Delta^{',''''} - 112\,\Delta^{'','''}\,\Delta^{'','''}\Big) +\\
& + \frac{(f'_i f''_j + f''_i f'_j)}{2}\Big(- 15\,\Delta^{','''''}\,\Delta^{',''} - 75\,\Delta^{'',''''}\,\Delta^{',''} + 65\,\Delta^{',''''}\,\Delta^{','''} + 110\,\Delta^{'','''}\,\Delta^{','''}\Big) +\\
& + \frac{(f'_i f'''_j + f'''_i f'_j)}{2}\Big(- 50\,\Delta^{','''}\,\Delta^{','''}\Big) +\\
& + f''_i f''_j\Big(- 25\,\Delta^{','''}\,\Delta^{','''} + 15\,\Delta^{',''''}\,\Delta^{',''} + 60\,\Delta^{'','''}\,\Delta^{',''}\Big),
\end{aligned}
$$

$$
\begin{aligned}
H^{14}_i := {} & \Big(15\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{',''} + 75\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{',''} + 5\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{','''} +\\
& + 170\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{','''} - 24\,\Delta^{',''''}\,\Delta^{',''''}\,\Delta^{',''} - 192\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{',''} -\\
& - 384\,\Delta^{'','''}\,\Delta^{'','''}\,\Delta^{',''}\Big)\, f'_i + \Big(- 45\,\Delta^{','''''}\,\Delta^{',''}\,\Delta^{',''} - 225\,\Delta^{'',''''}\,\Delta^{',''}\,\Delta^{',''} +\\
& + 225\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 450\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 200\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''}\Big)\, f''_i,
\end{aligned}
$$

$$
\begin{aligned}
F^{16}_{i,j} := {} & \Big(- 3\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{',''} - 15\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{',''} - 12\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{',''} +\\
& + 40\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{','''} - 60\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{',''} + 200\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{','''} -\\
& - 49\,\Delta^{',''''}\,\Delta^{',''''}\,\Delta^{','''} - 422\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''} - 904\,\Delta^{'','''}\,\Delta^{'','''}\,\Delta^{','''}\Big) f'_i f'_j +
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-105\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}-525\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}+205\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}-\\
&-230\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}+96\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}+768\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}+\\
&+1536\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\Big)\big(f_i''f_j'+f_i'f_j''\big)+\\
&\quad+\Big(-200\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\Big)\big(f_i'''f_j'+f_i'f_j'''\big)+\\
&\quad+\Big(315\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\,\Delta^{\prime,\prime\prime}+1575\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\,\Delta^{\prime,\prime\prime}-1575\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}-\\
&\quad-3150\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}+1400\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\Big)f_i''f_j'',
\end{aligned}
$$

where the indices i, j and k run in $\{1,2\}$. Furthermore, the ideal of relations between these 24 fundamental bracket invariants consists of all the syzygies that one obtains[10] *by substituting in $(\mathcal{P}lck_1)$ or in $(\mathcal{P}lck_2)$ for* P, Q, R, T *three or four among the nine invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$, M^8, in all possible ways, which makes in sum:*

$$
\tfrac{9!}{3!\,6!}+\tfrac{9!}{4!\,5!}=84+126=210
$$

generating syzygies.

It is now great time to offer ideas, arguments, principles of computations, and also proofs.

§5. Initial invariants in dimension n for arbitrary jet level $\kappa \geqslant 1$

Reparametrizing by f_1^{-1}. To fix ideas and to better offer the intuition of our computations, we shall firstly work in dimension $n=2$ until everything about the first basic step becomes clear, so that afterwards, the description of the birth of the initial invariants in the higher dimensions $n=3$ and $n=4$ shall present no real difficulty.

Thus, let $\mathsf{P}\big(j^\kappa f_1,\,j^\kappa f_2\big)$ be a polynomial of weight m that is invariant by reparametrization. By definition,

$$
(*)\qquad \mathsf{P}\big(j^\kappa(f\circ\phi)\big)=\phi'^{m}\,\mathsf{P}\big((j^\kappa f)\circ\phi\big),
$$

for every local biholomorphism of $\mathbb{C}$ fixing 0. Following a trick of Rousseau ([29]), we will apply this formula to the inverse mapping $\phi := f_1^{-1}$ of the first coordinate map $f_1 : \mathbb{C}\to\mathbb{C}$, assuming that $f_1'(0)\neq 0$, whence $\phi' = \frac{1}{f_1'}\circ f_1^{-1}$. We will explain in a moment that the assumption $f_1'(0)\neq 0$ is harmless for the result.

At first, we trivially have $f_1\circ f_1^{-1}=\mathrm{Id}$, whence $\big(f_1\circ f_1^{-1}\big)'=1$ and $\big(f_1\circ f_1^{-1}\big)^{(\lambda)}=0$ for all $\lambda\geqslant 2$. Next, by some direct computations, the derivatives of

---

[10] The data of our manuscript are not reproduced here.

the reparametrization of f_2 happen to be:

$$\begin{aligned}
\big(f_2\circ f_1^{-1}\big)' &= \left(\frac{f_2'}{f_1'}\right)\circ f_1^{-1},\\
\big(f_2\circ f_1^{-1}\big)'' &= \left[\frac{f_2''}{(f_1')^2}-\frac{f_1''f_2'}{(f_1')^3}\right]\circ f_1^{-1} = \left[\frac{f_1'f_2''-f_1''f_2'}{(f_1')^3}\right]\circ f_1^{-1}\\
&= \frac{\Lambda^3}{(f_1')^3}\circ f_1^{-1},
\end{aligned}$$

where we recognize here our favorite Wronskian $\Lambda^3=\Delta_{1,2}^{','' }$. Furthermore, by pursuing as we should the computations with the help of our beloved total differentiation operator, we next get:

$$\begin{aligned}
\big(f_2\circ f_1^{-1}\big)''' &= \left(\frac{\mathsf{D}\Lambda^3}{(f_1')^4}-3\,\frac{\Lambda^3\,f_1''}{(f_1')^5}\right)\circ f_1^{-1} = \frac{\big[\Lambda^3,\,f_1'\big]}{(f_1')^5}\circ f_1^{-1}\\
&= \frac{\Lambda_1^5}{(f_1')^5}\circ f_1^{-1},\\
\big(f_2\circ f_1^{-1}\big)''''\circ f_1^{-1} &= \frac{\big[\Lambda_1^5,\,f_1'\big]}{(f_1')^7}\circ f_1^{-1}\\
&= \frac{\Lambda_{1,1}^7}{(f_1')^7}\circ f_1^{-1},
\end{aligned}$$

and so on, with the now clear formal facts that numerators should be constructed by successively bracketing with f_1', their weight being visible as just the power of f_1' in the denominator.

With indices, we may therefore define inductively the collection of *initial invariants* (including f_1' and Λ^3):

$$\begin{aligned}
\Lambda_{1^{\lambda-2}}^{2\lambda-1} &:= \big[\Lambda_{1^{\lambda-3}}^{2\lambda-3},\,f_1'\big]\\
&= \mathsf{D}\Lambda_{1^{\lambda-3}}^{2\lambda-3}\cdot f_1'-(2\lambda-3)\Lambda_{1^{1-\lambda_3}}^{2\lambda-3}\cdot f_1'',
\end{aligned}$$

for all λ with $3\leqslant\lambda\leqslant\kappa$, where at the bottom of $\Lambda_{1^\ell}^{\bullet}$, the notation 1^ℓ stands for ℓ copies of 1. We then get by induction:

$$\boxed{\big(f_2\circ f_1^{-1}\big)^{(\lambda)} = \frac{\Lambda_{1^{\lambda-2}}^{2\lambda-1}}{(f_1')^{2\lambda-1}}\circ f_1^{-1}}\,.$$

It would not be a so straightforward task to find a general explicit expression of these invariants $\Lambda_{1^{\kappa-2}}^{2\kappa-1}$ in terms of $j^\kappa f$ for arbitrary jet order. For instance, the invariant $\Lambda_{1,1,1}^9$, obtained by specializing $i=j=k=1$ in the expression given in the theorem stated above (and by simplifying) reads:

$$\begin{aligned}
\Lambda_{1,1,1}^9 = \big(\Delta^{',''''' }+5\,\Delta^{','''' }\big)\,f_1'f_1'f_1' - \big(15\,\Delta^{','''' }+60\,\Delta^{'','''}\big)\,f_1'f_1'f_1''-\\
-10\,\Delta^{',''' }\,f_1'f_1'f_1'''+105\,\Delta^{',''' }\,f_1'f_1''f_1''-105\,\Delta^{','' }\,f_1''f_1''f_1''.
\end{aligned}$$

Nonetheless, we will in fact not really need to expand the expressions of these initial invariants.

Fact. *The invariants* f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, $\dots$, $\Lambda_{1^{\kappa-2}}^{2\kappa-1}$ *are mutually algebraically independent*.

This is just because $\Lambda^{2\lambda-1}_{1^{\lambda-2}}$ is a polynomial in $\big(j^\lambda f_1, j^\lambda f_2\big)$ while $\Lambda^{2\lambda+1}_{1^{\lambda-1}}$ contains the higher jet monomial $f_2^{(\lambda+1)}[f_1']^\lambda$.

Initial rational expression for invariant polynomials. A general polynomial $\mathsf{P}\big(j^\kappa f\big)$ of weight m in $\mathsf{E}^2_{\kappa,m}$ writes in expanded form:

$$\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big) = \sum_{a_1^1+a_2^1+2a_1^2+2a_2^2+\cdots+\kappa a_1^\kappa+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot\big(f_1'\big)^{a_1^1}\,\big(f_2'\big)^{a_2^1}\,\big(f_1''\big)^{a_1^2}\,\big(f_2''\big)^{a_2^2} \cdots\,\big(f_1^{(\kappa)}\big)^{a_1^\kappa}\,\big(f_2^{(\kappa)}\big)^{a_2^\kappa},$$

where by “coeff” we mean varying, but notationally unspecified complex numbers. Reparametrizing by $\phi := f_1^{-1}$ by an application of the definition $(*)$, we should have the relation:

$$\frac{1}{(f_1'\circ f_1^{-1})^m}\cdot\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big)\circ f_1^{-1} = \mathsf{P}\big(j^\kappa(f_1\circ f_1^{-1}),\, j^\kappa(f_2\circ f_1^{-1})\big),$$

in the open subset $\{f_1'\neq 0\}$ of the jet space $J^\kappa(\mathbb{C},\,\mathbb{C}^n)$. Thanks to the preparatory computations above, we may replace each monomial in the right hand side, and this gives us a quite interesting representation:

$$\frac{1}{(f_1'\circ f_1^{-1})^m}\cdot\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big)\circ f_1^{-1} =$$

$$= \Bigg[\sum_{a_1^1+a_2^1+2a_1^2+2a_2^2+\cdots+\kappa a_1^\kappa+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot(1)^{a_1^1}\left(\frac{f_2'}{f_1'}\right)^{a_2^1}(0)^{a_1^2}\left(\frac{\Lambda^3}{(f_1')^3}\right)^{a_2^2} \cdots\,(0)^{a_1^\kappa}\left(\frac{\Lambda^{2\kappa-1}_{1^{\kappa-2}}}{(f_1')^{2\kappa-1}}\right)^{a_2^\kappa}\Bigg]\circ f_1^{-1}.$$

Immediately, we reparametrize this identity by f_1, which then simply erases all the appearing f^{-1}, we see that monomials with positive exponent $a_1^\lambda \geqslant 1$ for some λ with $2\leqslant\lambda\leqslant\kappa$ automatically vanish, and we reduce monomials to the same denominator:

$$\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big) = \sum_{a_1^1+a_2^1+2a_2^2+\cdots+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot\frac{\big(f_2'\big)^{a_2^1}\,\big(\Lambda^3\big)^{a_2^2}\cdots\big(\Lambda^{2\kappa-1}_{1^{\kappa-2}}\big)^{a_2^\kappa}}{\big(f_1'\big)^{-m+a_2^1+3a_2^2+\cdots+(2\kappa-1)a_2^\kappa}}.$$

What is the largest power of f_1' as a denominator in the monomials of the right hand side? Supposing for a while that the quantities a_i^j are nonnegative real numbers, instead of integers, we may simplify step by step the definition of this maximum:

$$
\begin{aligned}
&\max_{a_1^1+a_2^1+2a_2^2+\cdots+\kappa a_2^\kappa=m} \Big(-m + a_2^1 + 3a_2^2 + \cdots + (2\kappa-1)a_2^\kappa\Big) = \\
&= \max\Big|_{a_1^1=0} \Big(\text{substitute } a_2^1 = m - 2a_2^2 - \cdots - \kappa a_2^\kappa \text{ in the same quantity}\Big) \\
&= \max_{a_2^1+2a_2^2+3a_2^3+\cdots+\kappa a_2^\kappa=m} \Big(a_2^2 + 2a_2^3 + \cdots + (\kappa-1)a_2^\kappa\Big) \qquad \text{[divide and multiply by 2]} \\
&= \frac{1}{2}\cdot \max_{2a_2^2+3a_2^3+\cdots+\kappa a_2^\kappa=m} \Big(2a_2^2 + 4a_2^3 + \cdots + (2\kappa-2)a_2^\kappa\Big) \qquad \text{[substitute } 2a_2^2\text{]} \\
&= \frac{m}{2} + \frac{1}{2}\cdot \max_{3a_2^3+4a_2^4+\cdots+\kappa a_2^\kappa=m} \Big(a_2^3 + 2a_2^4 + \cdots + (\kappa-2)a_2^\kappa\Big) \qquad \text{[divide and multiply by 3]} \\
&= \frac{m}{2} + \frac{1}{2\cdot 3}\cdot \max_{3a_2^3+4a_2^4+\cdots+\kappa a_2^\kappa=m} \Big(3a_2^3 + 6a_2^4 + \cdots + 3(\kappa-2)a_2^\kappa\Big) \qquad \text{[substitute } 3a_2^3\text{]} \\
&= \frac{m}{2} + \frac{m}{6} + \frac{1}{3\cdot 4}\cdot \max_{4a_2^4+5a_2^5+\cdots+\kappa a_2^\kappa=m} \Big(4a_2^4 + 8a_2^5 + \cdots + 4(\kappa-3)a_2^\kappa\Big) \\
&= \frac{2}{3}\,m + \frac{1}{12}\,m + \frac{1}{4\cdot 5}\cdot \max_{5a_2^5+6a_2^6+\cdots+\kappa a_2^\kappa=m} \Big(5a_2^6 + 10a_2^6 + \cdots + 5(\kappa-4)a_2^\kappa\Big) \\
&= \frac{3}{4}\,m + \frac{1}{20}\,m + \frac{1}{5\cdot 6}\cdot \max_{6a_2^6+7a_2^7+\cdots+\kappa a_2^\kappa=m} \Big(6a_2^6 + 12a_2^7 + \cdots + 6(\kappa-5)a_2^\kappa\Big) \\
&= \frac{4}{5}\,m + \cdots \qquad \text{[observe the induction]} \\
&= \frac{\kappa-1}{\kappa}\,m.
\end{aligned}
$$

Thus, when the a_2^i are restricted to be integers, we in any case deduce that the maximally negative power of f_1' is $\geqslant -\frac{(\kappa-1)}{\kappa}\,m$. Reorganizing the result, we then obtain a representation of $\mathsf{P}\big(j^\kappa f\big)$, valid by construction in the subset $\{f_1' \neq 0\}$ of the jet space $J^\kappa\big(C,\,\mathbb{C}^n\big)$, as a sum of powers of f_1' :

$$
\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}\,m\leqslant a\leqslant m} (f_1')^a \cdot \mathsf{P}_a\Big(f_2',\,\Lambda^3,\,\Lambda_1^5,\,\Lambda_{1,1}^7,\,\dots,\,\Lambda_{1^{\kappa-2}}^{2\kappa-1}\Big),
$$

multiplied by certain polynomials P_a which depend upon P and are *not* arbitrary. In fact, by reduction to the same denominator, we may write:

$$
\mathsf{P}\big(j^\kappa f\big) = \frac{\mathsf{Q}\big(f_1',\,f_2',\,\Lambda^3,\,\dots,\,\Lambda_{1^{\kappa-2}}^{2\kappa-1}\big)}{(f_1')^{-a_0}},
$$

where a_0 is the smallest exponent a of f_1' above. Chasing the denominator in the case where a_0 is negative (this would be unnecessary if $a_0 \geqslant 0$), we get an identity $(f_1')^{a_0}\cdot\mathsf{P} \equiv \mathsf{Q}$ between two polynomials valid in $\{f_1' \neq 0\}$, hence everywhere by the principle of analytic continuation. Thus, the restriction $f_1' \neq 0$ is removed.

Weighted homogeneities. Let $\mu \in \mathbb{Z}$ be an integer, possibly negative. A rational expression $\mathsf{R}\big(j^\kappa f\big) \in \mathrm{Frac}\big(\mathbb{C}[j^\kappa f]\big)$ will be said to be of *weighted homogeneous* *degree* μ when for every complex weighted δ-dilation which acts in accordance with the number of primes:

$$\delta\cdot j^\kappa f := \big(\delta\, f_1',\, \delta f_2',\, \delta^2\, f_1'',\, \delta^2\, f_2'', \cdots, \delta^\kappa\, f_1^{(\kappa)},\, \delta^\kappa\, f_2^{(\kappa)}\big),$$

the dilation factor escapes the parentheses to exactly the μ-th power:

$$\mathsf{R}\big(\delta\cdot j^\kappa f\big) = \delta^\mu\cdot\mathsf{R}\big(j^\kappa f\big).$$

When R is a polynomial, μ is then the total, constant number of primes of each monomial.

By choosing the reparametrization ϕ to just be a δ-dilation in the source, with nonzero $\delta\in\mathbb{C}$, we immediataly see that our original jet polynomial $\mathsf{P}\in\mathsf{E}^2_{\kappa,m}$ — hence also its rational expression obtained above — must in particular be weighted homogeneous of degree m:

$$\mathsf{P}\big(\delta\cdot j^\kappa f\big) = \delta^m\cdot\mathsf{P}\big(j^\kappa f\big).$$

In addition and in particular, using the definition $\Lambda^{2\lambda-1}_{1^{\lambda-2}} = \big[\Lambda^{2\lambda-3}_{1^{\lambda-3}},\, f_1'\big]$, one easily verifies by induction that the invariant $\Lambda^{2\lambda-1}_{1^{\lambda-2}}$ is homogeneous of degree equal to its weight $2\lambda-1$, an integer which we had already specified as the upper index:

$$\Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(\delta\cdot j^\kappa f\big) = \delta^{2\lambda-1}\cdot\Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(j^\lambda f\big).$$

In an analogous fashion, introducing some new extra independent variables $F_1,\, F_2,\, A^3,\, \dots,\, A^{2\kappa-1}$ corresponding to $f_1', f_2', \Lambda^3,\, \dots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}$, a rational expression $\mathsf{T}\in\mathrm{Frac}\big(\mathbb{C}[F_1, F_2, A^3, \dots, A^{2\kappa-1}]\big)$ will be said to be of *weighted homogeneous degree* μ when it enjoys:

$$\mathsf{T}\big(\delta F_1,\, \delta F_2,\, \delta^3 A^3,\, \delta^5 A^5,\, \dots,\, \delta^{2\kappa-1}A^{2\kappa-1}\big) = \delta^\mu\cdot\mathsf{T}\big(F_1,\, F_2,\, A^3,\, A^5,\, \dots,\, A^{2\kappa-1}\big),$$

for every $\delta\in\mathbb{C}$.

Lemma. *In dimension $n=2$ for jets of order $\kappa\geqslant 2$, every jet polynomial $\mathsf{P}=\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization writes under the form:*

$$\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^7_{1,1},\, \cdots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big)},$$

where the integer a takes possibly negative values in the interval $\big[-\frac{\kappa-1}{\kappa}\,m,\, m\big]$, for certain weighted homogeneous polynomials:

$$\mathsf{P}_a = \sum_{b_2+3c_3+\cdots+(2\kappa-1)c_{2\kappa-1}=m-a} \mathsf{coeff}\cdot\,\big(F_2\big)^{b_2}\,\big(A^3\big)^{c_3}\,\cdots\,\big(A^{2\kappa-1}\big)^{c_{2\kappa-1}}$$

of weighted degree $m-a$.

Conversely, for every collection of such weighted homogeneous polynomials P_a in $\mathbb{C}\big[F_2,\, A^3,\, \dots,\, A^{2\kappa-1}\big]$ of weighted degree $m-a$ indexed by an integer a running in $\big[-\frac{\kappa-1}{\kappa}\,m,\, m\big]$ such that the reduction to the same denominator and the simplification of the finite sum:

$$\mathsf{R}\big(j^\kappa f\big) := \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^7_{1,1},\, \cdots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big)$$

yields a true *jet* polynomial *in* $\mathbb{C}\big[j^\kappa f\big]$*, then* $\mathsf{R}(j^\kappa f)$ *is a polynomial invariant by reparametrization belonging to* $\mathsf{E}^2_{\kappa,m}$.

Proof. We saw that P is homogeneous of degree m, namely:

$$\sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} \delta^a\big(f_1'\big)^a\,\mathsf{P}_a\Big(\delta f_2',\,\delta^3\Lambda^3,\,\dots,\,\delta^{2\kappa-1}\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big) = \\ = \delta^m\cdot\sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m}\big(f_1'\big)^a\,\mathsf{P}_a\Big(f_2',\,\Lambda^3,\,\dots,\,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big).$$

By algebraic independency of f_1' with respect to $\mathbb{C}\big[f_2',\Lambda^3,\dots,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\big]$, we then may identify powers of f_1', getting for each a:

$$\mathsf{P}_a\Big(\delta f_2',\,\delta^3\Lambda^3,\dots,\delta^{2\kappa-1}\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big) = \delta^{m-a}\,\mathsf{P}_a\Big(f_2',\Lambda^3,\dots,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big).$$

Further, the algebraic independency of $f_2',\Lambda^3,\dots,\Lambda^{2\kappa-1}_{1^{\kappa-2}}$ then entails that the homogeneities:

$$\mathsf{P}_a\Big(\delta F_2,\delta^3A^3,\dots,\delta^{2\kappa-1}A^{2\kappa-1}\Big) = \delta^{m-a}\cdot\mathsf{P}_a\Big(F_2,A^3,\dots,A^{2\kappa-1}\Big)$$

hold in the polynomial algebra $\mathbb{C}\big[F_2,A^3,\dots,A^{2\kappa-1}\big]$. This gives the claimed representation of any $\mathsf{P}\in\mathsf{E}^2_{\kappa,m}$.

Conversely, assuming that the P_a are homogeneous in this way, then for any reparametrization ϕ, setting $g_i = f_i\circ\phi$ for $i=1,2$ and recalling:

$$\Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(j^\lambda g\big) = (\phi')^{2\lambda-1}\,\Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(j^\lambda f\big),$$

we immediately deduce that:

$$\mathsf{P}_a\Big(g_2',\Lambda^3\big(j^3g\big),\,\dots,\,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\big(j^\kappa g\big)\Big) = \\ = \mathsf{P}_a\Big(\phi'\cdot f_2'\circ\phi,\,(\phi')^3\cdot\Lambda^3\circ\phi,\,\dots,\,(\phi')^{2\kappa-1}\cdot\Lambda^{2\kappa-1}_{1^{\kappa-2}}\circ\phi\Big) \\ = (\phi')^{m-a}\cdot\mathsf{P}_a\Big(f_2',\,\Lambda^3,\,\dots,\,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big),$$

whence multiplication by $(g_1')^a = (\phi')^a\,(f_1')^a$ and summation gives:

$$\sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} g_1'^{\,a}\,\mathsf{P}_a\Big(g_2',\,\Lambda^3\big(j^3g\big),\,\dots,\,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\big(j^\kappa g\big)\Big) = \\ (\phi')^m\cdot\sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m}(f_1')^a\,\mathsf{P}_a\Big(f_2',\,\Lambda^3,\,\dots,\,\Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big)\circ\phi,$$

which exactly means, as soon as such a rational sum represents a true polynomial, that it belongs to $\mathsf{E}^2_{\kappa,m}$, *quod erat demonstrandum.* □

Arbitrary dimension. To generalize the preceding proposition, suppose now that $n \geqslant 2$ is arbitrary. The same trick of reparametrizing each f_i by $\phi = f_1^{-1}$ gives birth to a collection of *initial invariants* appearing as numerators of:

$$\big(f_i \circ f_1^{-1}\big)^{(\lambda)} = \frac{\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}\big(j^\lambda f\big)}{(f_1')^{2\lambda-1}},$$

for $i = 2, 3, \dots, n$, where the Λ-invariants depending on i and on λ are defined inductively by successively bracketing with f_1':

$$\Lambda_{1,i}^3 := \big[f_i',\, f_1'\big], \qquad \Lambda_{1,i;\,1}^5 := \big[\Lambda_{1,i}^3,\, f_1'\big],$$

$$\text{and generally:} \qquad \Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1} := \big[\Lambda_{1,i;\,1^{\lambda-3}}^{2\lambda-3},\, f_1'\big], \qquad \text{for } 3 \leqslant \lambda \leqslant \kappa.$$

Our considerations about brackets show that these polynomials are effectively invariant by reparametrization. Furthermore:

Fact. *The* $n + (n-1)(\kappa-1)$ *invariants:*

$$\begin{array}{cccccc} f_1', & f_2', & f_3', & f_4', & \dots, & f_n', \\ & \Lambda_{1,2}^3, & \Lambda_{1,3}^3, & \Lambda_{1,4}^3, & \dots, & \Lambda_{1,n}^3, \\ & \Lambda_{1,2;\,1}^5, & \Lambda_{1,3;\,1}^5, & \Lambda_{1,4;\,1}^5, & \dots, & \Lambda_{1,n;\,1}^5, \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,3;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,4;\,1^{\kappa-2}}^{2\kappa-1}, & \dots, & \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1} \end{array}$$

are mutually algebraically independent.

Indeed, $\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}$ contains the monomial $f_i^{(\lambda)} {f_1'}^{\lambda-1}$, while the invariants $\Lambda_{1,j;\,1^{\lambda-3}}^{2\lambda-3}$ only depend upon $j^{\lambda-1}f$.

Reasonings similar to the ones developed above yield the following lemma, valuable for any $n \geqslant 1$ and any $\kappa \geqslant 1$.

Lemma. *In dimension* $n \geqslant 1$ *and for jets of order* $\kappa \geqslant 1$, *every polynomial* $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ *invariant by reparametrization writes under the form:*

$$\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a \left(\begin{array}{ccccc} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda_{1,2}^3, & \Lambda_{1,3}^3, & \Lambda_{1,4}^3, & \dots, & \Lambda_{1,n}^3, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,3;\,1^{\kappa-2}}^{2\kappa-1}, & \Lambda_{1,4;\,1^{\kappa-2}}^{2\kappa-1}, & \dots, & \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1} \end{array} \right)},$$

where the integer a *takes all possibly negative values in the interval* $\big[-\frac{\kappa-1}{\kappa}\, m, m\big]$, *for certain weighted homogeneous polynomials:*

$$\mathsf{P}_a = \sum_{\substack{b_2+\cdots+b_n+3c_2+\cdots+3c_n+ \\ +\cdots+(2\kappa-1)q_2+\cdots+(2\kappa-1)q_n = m-a}} \mathsf{coeff} \cdot \prod_{i=2}^n \big(F_i\big)^{b_i} \prod_{i=2}^n \big(A_i^3\big)^{c_i} \cdots \prod_{i=2}^n \big(A_i^{2\kappa-1}\big)^{q_i}$$

of weighted degree $m - a$, *namely satisfying:*

$$\mathsf{P}_a\Big(\delta\, F_i,\, \delta^3 A_i^3,\, \dots,\, \delta^{2\kappa-1} A_i^{2\kappa-1}\Big) = \delta^{m-a} \cdot \mathsf{P}_a\Big(F_i,\, A_i^3,\, \dots,\, A_i^{2\kappa-1}\Big).$$

Conversely, for every collection of such weighted homogeneous polynomials P_a *in* $\mathbb{C}\big[F_i, A_i^3, \dots, A_i^{2\kappa-1}\big]$ *of weighted degree* $m - a$ *indexed by an integer* a

running in $\big[-\frac{\kappa-1}{\kappa}\,m,\,m\big]$ *such that the reduction to the same denominator and the simplification of the finite sum:*

$$\mathsf{R}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\left(\begin{array}{ccccc} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{array}\right)$$

yields a true *jet* polynomial *in* $\mathbb{C}\big[j^\kappa f\big]$*, then* $\mathsf{R}\big(j^\kappa f\big)$ *is a polynomial invariant by reparametrization belonging to* $\mathsf{E}^n_{\kappa,m}$.

§6. DESCRIPTION OF THE ALGORITHM IN DIMENSION $n=2$ FOR JET LEVEL $\kappa=4$

Necessity of negative powers of f_1'**.** Our aim now is to prove[11] the theorem which describes the algebraic structure of E^2_4, *see* p. 27. We will thus illustrate in a concrete case the general algorithm which will be presented in Section 9 below. We hope this will make the general considerations intuitively clearer.

Proof. Compared to the initial rational representation:

$$\mathsf{P}\big(j^4 f\big) = \sum_{-\frac{3}{4}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\big(f_2',\,\Lambda^3,\,\Lambda^5_1,\,\Lambda^7_{1,1}\big)$$

of an arbitrary polynomial $\mathsf{P}\in\mathsf{E}^2_4$ that was furnished by the lemma on p. 34, the theorem on p. 27 states that 4 further invariants, namely Λ^5_2, $\Lambda^7_{1,2}$, $\Lambda^7_{2,2}$ and M^8, are necessary to generate the full algebra E^2_4. In fact, by looking at the 9 syzygies listed in the theorem in question, one may easily obtain the expression of these 4 further invariants in $\mathbb{C}\big[f_2',\,\Lambda^3,\,\Lambda^5_1,\,\Lambda^7_{1,1}\big]\big[\frac{1}{f_1'}\big]$, namely:

$$\begin{aligned}
\Lambda^5_2 &= \frac{f_2'\Lambda^5_1 - 3\,\Lambda^3\Lambda^3}{f_1'}, \\
\Lambda^7_{1,2} &= \frac{f_2'\Lambda^7_{1,1} - 5\,\Lambda^3\Lambda^5_1}{f_1'}, \\
\Lambda^7_{2,2} &= \frac{f_2'f_2'\Lambda^7_{1,1} - 10\,f_2'\Lambda^3\Lambda^5_1 + 15\,\Lambda^3\Lambda^3\Lambda^3}{f_1'f_1'f_1'}, \\
M^8 &= \frac{3\,\Lambda^3\Lambda^7_{1,1} - 5\,\Lambda^5_1\Lambda^5_1}{f_1'f_1'}.
\end{aligned}$$

Crucial observation. *So when one substitutes, in an arbitrary polynomial:*

$$\mathsf{P}\big(f_1',\,f_2',\,\Lambda^3,\,\Lambda^5_1,\,\Lambda^7_{1,1},\,\Lambda^7_{1,2},\,\Lambda^7_{2,2},\,M^8\big)$$

these rational representations of Λ^5_2, $\Lambda^7_{1,2}$, $\Lambda^7_{2,2}$, M^8, *one indeed obtains a rational expression as the one above which* necessarily and unavoidably *incorporates negative powers of* f_1'.

[11] An alternative proof was provided in [21].

Well, how then should we interpret our initial rational expression? Why are the 4 further invariants Λ_2^5, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 invisible in it?

First of all, as a preliminary, we must at least show that the 9 fundamental invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 are mutually independent.

On this purpose, we set $f_1' = 0$ in these 9 fundamental invariants, and this then leaves us with the 8 invariants:

$$f_2', \quad \Lambda^3\big|_0, \quad \Lambda_1^5\big|_0, \quad \Lambda_2^5\big|_0, \quad \Lambda_{1,1}^7\big|_0, \quad \Lambda_{1,2}^7\big|_0, \quad \Lambda_{2,2}^7\big|_0, \quad M^8\big|_0,$$

which we shall shortly call *restricted invariants*, our notation being self-evident. The following assertion is simply checked by inspecting the explicit expressions.

Fact. *The four restricted invariants:*

$$f_2', \qquad \Lambda^3\big|_0 = -f_1''f_2', \qquad \Lambda_2^5\big|_0 = 3\,f_1''f_2'f_1'' \qquad \text{and}$$
$$\Lambda_{2,2}^7\big|_0 = \big(-f_1''''f_2' + \Delta^{'','''}\big)\,f_2'f_2' + 10\,f_1'''f_2'f_2'f_2'' - 15\,f_1''f_2'f_2''f_2''$$

are mutually algebraically independent.

It follows that f_1', f_2', Λ^3, Λ_2^5 and $\Lambda_{2,2}^7$ are algebraically independent. Switching lower indices, f_1', f_2', Λ^3, Λ_1^5 and $\Lambda_{1,1}^7$ are also algebraically independent.

Next, by looking at the 9 syzygies listed in the theorem, we may express each one of the 8 restricted invariants by means of the above four algebraically independent restricted invariants, provided that one allows a division by f_2':

$$\begin{aligned}
& f_2', \\
& \Lambda^3\big|_0, \\
& \Lambda_1^5\big|_0 = 3\,\frac{\Lambda^3|_0\,\Lambda^3|_0}{f_2'}, \\
& \Lambda_2^5\big|_0,
\end{aligned}$$

$$\begin{aligned}
& \Lambda_{1,1}^7\big|_0 = 15\,\frac{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3\,|_0}{f_2'f_2'}, \\
& \Lambda_{1,2}^7\big|_0 = 5\,\frac{\Lambda^3|_0\,\Lambda_2^5|_0}{f_2'}, \\
& \Lambda_{2,2}^7\big|_0,
\end{aligned}$$

$$M^8\big|_0 = \frac{3\,\Lambda^3|_0\,\Lambda_{2,2}^7 - 5\,\Lambda_2^5|_0\,\Lambda_2^5|_0}{f_2'f_2'}.$$

In fact, all divisions by f_2' or by $f_2'f_2'$ do cancel out after simplification.

Lemma. *The 9 fundamental invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 are mutually independent. More precisely, f_1', f_2', Λ^3, Λ_1^5 and $\Lambda_{1,1}^7$ are algebraically independent and there exist no polynomial representation of either one* *of the following four forms:*

$$\begin{aligned}
\Lambda_2^5 &= \mathsf{polynomial}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7\big),\\
\Lambda_{1,2}^7 &= \mathsf{polynomial}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_2^5,\, \Lambda_{1,1}^7\big),\\
\Lambda_{2,2}^7 &= \mathsf{polynomial}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_2^5,\, \Lambda_{1,1}^7,\, \Lambda_{1,2}^7\big),\\
M^8 &= \mathsf{polynomial}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_2^5,\, \Lambda_{1,1}^7,\, \Lambda_{1,2}^7,\, \Lambda_{2,2}^7\big).
\end{aligned}$$

Proof. By setting $f_1' = 0$ in a polynomial representation such as the first one and by replacing the values of some of the restricted invariants, we get:

$$\begin{aligned}
\Lambda_2^5\big|_0 &= \sum \mathsf{coeff}\cdot \big(f_2'\big)^b \big(\Lambda^3\big)^c \big(\Lambda_1^5\big)^d \big(\Lambda_{1,1}^7\big)^e\Big|_0\\
&= \sum \mathsf{coeff}\cdot \big(f_2'\big)^b \big(\Lambda^3\big)^c \bigg(3\,\frac{\Lambda^3\,\Lambda^3}{f_2'}\bigg)^d \bigg(15\,\frac{\Lambda^3\,\Lambda^3\,\Lambda^3}{f_2'f_2'}\bigg)^e\bigg|_0\\
&= \sum \mathsf{coeff}\cdot \big(f_2'\big)^{b-d-2e} \big(\Lambda^3\big|_0\big)^{c+2d+3e},
\end{aligned}$$

where the exponents $b, c, d, e \geqslant 0$ are nonnegative integers. But this is impossible, because $\Lambda_2^5\big|_0$ is transcendental over $\mathbb{C}\big[f_2',\, \Lambda^3\big|_0\big]$.

Next, for the second hypothetical representation, the same kind of replacement yields:

$$5\,\frac{\Lambda^3\,\Lambda_2^5}{f_2'}\bigg|_0 = \sum \mathsf{coeff}\cdot \big(f_2'\big)^{b-d-2f} \big(\Lambda^3\big)^{c+2d+3f} \big(\Lambda_2^5\big)^e\Big|_0.$$

So, by identifying the powers of the restricted algebraically independent invariants f_2', $\Lambda^3\big|_0$, $\Lambda_2^5\big|_0$, we get three equations between integers:

$$-1 = b-d-2f, \qquad 1 = c+2d+3f, \qquad 1 = e,$$

which are seen to be impossible, since $b, c, d, e, f \geqslant 0$, the second one yielding $d = f = 0$, while the first one then reads $-1 = b$.

Similarly as for the first one, the third hypothetical representation is *a priori* excluded, because the right hand side does not depend upon $\Lambda_{2,2}^7\big|_0$ at all.

Finally, the fourth hypothetical representation amounts to:

$$\frac{3\,\Lambda^3\,\Lambda_{2,2}^7 - 5\,\Lambda_2^5\,\Lambda_2^5}{f_2'f_2'}\bigg|_0 = \sum \mathsf{coeff}\cdot \big(f_2'\big)^{b-d-2f-g} \big(\Lambda^3\big)^{c+2d+3f+g} \big(\Lambda_2^5\big)^{e+g} \big(\Lambda_{2,2}^7\big)^h\Big|_0,$$

hence looking at the representation of the first term $\frac{3\,\Lambda^3\,\Lambda_{2,2}^7}{f_2'f_2'}$ of the left hand side, and identifying powers, we get three equations:

$$-2 = b-d-2f-g, \qquad 1 = c+2d+3f+g, \qquad 1 = h.$$

The second one implies $d = f = 0$ and $g = 0$ or 1, whence the first one then becomes impossible. □

First loop of the algorithm. The initial expression:

$$\mathsf{P}\big(j^4f\big) = \sum_{-\frac{3}{4}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\big(f_2',\,\Lambda^3,\,\Lambda_1^5,\,\Lambda_{1,1}^7\big)$$

shows five invariants f_1', f_2', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, and four restricted invariants f_2', $\Lambda^3\big|_0$, $\Lambda_1^5\big|_0$, $\Lambda_{1,1}^7\big|_0$. To determine the structure of E_4^2, here is the first loop of our algorithm.

• Compute the ideal of relations[12] between the 4 known restricted invariants:

$$\mathsf{Ideal}-\mathsf{Rel}\Big(f_2'\big|_0,\,\Lambda^3\big|_0,\,\Lambda_1^5\big|_0,\,\Lambda_{1,1}^7\big|_0\Big),$$

namely a generating set of the ideal of all polynomials $\mathcal{Q}\big(F_2,\,A^3,\,A^5,\,A^7\big)$ in four variables that give zero, identically, after substituting these four restricted invariants.

• Get as generators of this ideal of relations the three relations, valuable for $f_1'=0$:

$$\begin{aligned} 0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5\Big|_0, \\ 0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7\Big|_0, \\ 0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7\Big|_0. \end{aligned}$$

• Consequently, without setting $f_1'=0$, there should exist remainders that are a multiple of f_1':

$$\begin{aligned} 0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\times\mathsf{something}, \\ 0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\times\mathsf{something}, \\ 0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'\times\mathsf{something}. \end{aligned}$$

• Each "something" necessarily also is an invariant belonging to E_4^2, because it is a polynomial and we can write it as $\frac{1}{f_1'}$ times a corresponding quadratic expression in the already known invariants f_2', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$.

• Find the maximal power by which f_1' factors each remaining "something".

• Get the three identically satisfied relations:

$$\begin{aligned} 0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\Lambda_2^5, \\ 0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\Lambda_{1,2}^7, \\ 0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'f_1'M^8, \end{aligned}$$

where the appearing new invariants are already known from the statement of the theorem.

[12] We will discuss in a while two ways of computing ideal of relations. The data reproduced here are obtained by means of Gröbner bases computations.

• Test whether or not the so obtained three invariants:

$$\Lambda_2^5, \qquad \Lambda_{1,2}^7, \qquad M^8,$$

belong or do not belong to the algebra generated by the previously known invariants. Here in fact, neither Λ_2^5 nor $\Lambda_{1,2}^7$, nor M^8 belongs to $\mathbb{C}\big[f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7\big]$, as we have already verified.

Second loop of the algorithm. We now restart the process with our new, extended list of 7 invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$ and M^8.

• Compute the ideal of relations between the 6 restricted invariants known at this stage:

$$\mathsf{Ideal}-\mathsf{Rel}\Big(f_2'\big|_0,\ \Lambda^3\big|_0,\ \Lambda_1^5\big|_0,\ \Lambda_2^5\big|_0,\ \Lambda_{1,1}^7\big|_0,\ \Lambda_{1,2}^7\big|_0,\ M^8\big|_0\Big).$$

• Get the 6 equations:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5\Big|_0,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7\Big|_0,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7\Big|_0,\\
0 &\equiv 5\,\Lambda^3\Lambda_2^5 - f_2'\Lambda_{1,2}^7\Big|_0,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{1,2}^7\Big|_0,\\
0 &\equiv \Lambda_{1,1}^7\Lambda_2^5 - \Lambda_1^5\Lambda_{1,2}^7\Big|_0.
\end{aligned}$$

• Compute the remainders behind a power of f_1':

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\Lambda_2^5,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\Lambda_{1,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'f_1'M^8,\\
0 &\equiv 5\,\Lambda^3\Lambda_2^5 - f_2'\Lambda_{1,2}^7 + f_1'\Lambda_{2,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{1,2}^7 + f_1'f_2'M^8,\\
0 &\equiv \Lambda_{1,1}^7\Lambda_2^5 - \Lambda_1^5\Lambda_{1,2}^7 + f_1'\Lambda^3M^8.
\end{aligned}$$

• Get only one new invariant $\Lambda_{2,2}^7$ not belonging to the algebra generated by already known invariants $\mathbb{C}\big[f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_2^5, \Lambda_{1,1}^7, \Lambda_{1,2}^7, M^8\big]$.

Third loop of the algorithm. The final list of syzygies, after filling in the remainders and testing whether new invariants appear, reads:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\Lambda_2^5,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\Lambda_{1,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'f_1'M^8,
\end{aligned}$$

$$0 \equiv 5\,\Lambda^3\Lambda_2^5 - f_2'\Lambda_{1,2}^7 + f_1'\Lambda_{2,2}^7,$$
$$0 \equiv 5\,\Lambda_1^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{1,2}^7 + f_1'f_2'M^8,$$
$$0 \equiv \Lambda_{1,1}^7\Lambda_2^5 - \Lambda_1^5\Lambda_{1,2}^7 + f_1'\Lambda^3 M^8,$$

$$0 \equiv 5\,f_2'\Lambda_1^5 M^8 + 3\,\Lambda_{1,2}^7\Lambda_{1,2}^7 - 3\,\Lambda_{1,1}^7\Lambda_{2,2}^7 + 0,$$
$$0 \equiv f_2'\Lambda^3 M^8 + \Lambda_2^5\Lambda_{1,2}^7 - \Lambda_1^5\Lambda_{2,2}^7 + 0,$$
$$0 \equiv f_2'f_2'M^8 + 5\,\Lambda_2^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{2,2}^7 + 0.$$

Three new syzygies only appear, namely the last three ones above, and for each of them, the remainders that are a multiple of f_1' are identically zero, which we specify explicitly by writing "$+0$". Importantly, *no new invariant appears at this stage*.

We then claim that the algorithm stops (*cf.* also Section 9), and that the following proposition holds true. In fact, the arguments of proof will follow from the general theorem of §9.

Proposition. *An arbitrary polynomial* $\mathsf{P} = \mathsf{P}\big(j^4 j\big)$ *in* E_4^2 *invariant by reparametrization writes uniquely under the form:*

$$\begin{aligned}\mathsf{P}\big(j^4 j\big) = \mathcal{Q}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big) + \Lambda^3\,\mathcal{R}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big) + \\ + \Lambda_1^5\,\mathcal{S}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big) + \Lambda_2^5\,\mathcal{T}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big) + \\ + \Lambda_{1,2}^7\,\mathcal{U}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big) + \Lambda^3\Lambda_{1,2}^7\,\mathcal{V}\big(f_1', f_2', \Lambda_{1,1}^7, \Lambda_{2,2}^7, M^8\big),\end{aligned}$$

where $\mathcal{Q}$*,* $\mathcal{R}$*,* $\mathcal{S}$*,* $\mathcal{T}$*,* $\mathcal{U}$ *and* $\mathcal{V}$ *are complex polynomials in five variables subjected to no restriction.*

§7. Action of $\mathsf{GL}_n(\mathbb{C})$ and unipotent reduction

Sums of irreducible Schur representations. The cohomology of Schur bundles $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T_X^*$ on a complex algebraic projective hypersurface $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ being available through Hirzebruch's Riemann-Roch formula (§13 below), we should look for a decomposition of the Demailly-Semple bundle $\mathsf{E}_{\kappa,m}^n T_X^*$ as a direct sum of Schur bundles, at least in the cases where we understand the algebraic structure of the fiber algebras $\mathsf{E}_{\kappa,m}^n$. We recall that according to a fundamental theorem of representation theory ([16]), any group action of $\mathsf{GL}_n(\mathbb{C})$ on a space of polynomials is isomorphic to a certain direct sum of irreducible Schur representations.

Action of $\mathsf{GL}_n(\mathbb{C})$ on the jet space. On this purpose, similarly as in [29], we therefore define an appropriate linear action of $\mathsf{GL}_n(\mathbb{C})$ on the κ-th jet space $J^\kappa(\mathbb{C}, \mathbb{C}^n)$. By definition, an arbitrary element w of $\mathsf{GL}_n(\mathbb{C})$ written in matrix form:

$$\mathsf{w} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix}$$

shall transform the collection $\big(f_1^{(\lambda)}, \dots, f_n^{(\lambda)}\big)$ of the n components of a κ-jet $j^\kappa f$ at each λ-th jet level, just by matrix multiplication:

$$\begin{cases} \mathsf{w} \cdot f_1^{(\lambda)} = w_{11}\, f_1^{(\lambda)} + \cdots + w_{1n}\, f_n^{(\lambda)} \\ \quad \cdots \qquad\quad \cdots \qquad\qquad\quad \cdots \\ \mathsf{w} \cdot f_n^{(\lambda)} = w_{n1}\, f_1^{(\lambda)} + \cdots + w_{nn}\, f_n^{(\lambda)}, \end{cases}$$

with the same matrix w at each jet level λ with $1 \leqslant \lambda \leqslant \kappa$.

Definition. A polynomial $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization will be called a *bi-invariant* if it is a *vector of highest weight* for this representation of $\mathsf{GL}_n(\mathbb{C})$, namely if it is invariant by the *unipotent subgroup* $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ constituted by (unipotent) matrices of the form:

$$\mathsf{u} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{21} & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix}.$$

The vector space of bi-invariant polynomials P thus satisfies:

$$\boxed{\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) \qquad \text{and} \qquad \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)}.$$

In the sequel, the vector space of bi-invariants of weight m will be denoted by $\mathsf{UE}_{\kappa,m}^n$. Also, one defines the graded algebra of bi-invariants $\mathsf{UE}_\kappa^n := \bigoplus_{m \geqslant 1} \mathsf{UE}_{\kappa,m}^n$ with of course $\mathsf{UE}_{\kappa,m_1}^n \cdot \mathsf{UE}_{\kappa,m_2}^n \subset \mathsf{UE}_{\kappa,m_1+m_2}^n$.

Without delay, we emphasize four fundamental observations.

• The full space $\mathsf{E}_{\kappa,m}^n$ is obtained as just the $\mathsf{GL}_n(\mathbb{C})$-orbit of $\mathsf{UE}_{\kappa,m}^n$.

• The algebraic structure of UE_κ^n is *always* much simpler than that of E_κ^n. For instance:

— UE_3^3 is generated by only 4 bi-invariant polynomials[13] f_1', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ and $D_{1,2,3}^6$ which are *algebraically independent* (no syzygy!), whereas, according to [28, 29] or to the description given on p. 26 here, the full algebra E_3^3 is generated by 16 invariants, submitted to the three complicated families of syzygies developed on p. 26.

— UE_4^2 is generated by the 5 bi-invariant polynomials f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ and M^8, whose ideal of relations is principal, generated by the single syzygy:

$$0 \overset{4}{\equiv} f_1' f_1' M^8 - 3\,\Lambda^3 \Lambda_{1,1}^7 + 5\,\Lambda_1^5 \Lambda_1^5,$$

while, according to the theorem on p. 27, the full algebra E_4^2 is generated by 9 invariants submitted to 9 fundamental syzygies.

— We will establish that UE_4^4 is generated by 16 mutually independent bi-invariant polynomials, while E_4^4 is generated by 2835 polynomials invariant by reparametrization. Also, we will show 41 syzygies generate the ideal of relations between (the restriction to

[13] *See* the proposition on p. 48 below, or the considerations on pp. 931–932 in [21].

$\{f_1' = 0\}$ of) these 16 generators of UE_4^4, while we ignore the structure of the (presumably out of human scale) ideal of relations between the 2835 generators of E_4^4.

— We will establish that UE_5^2 is generated by 17 mutually independent bi-invariant polynomials, while E_5^2 is generated by 56 polynomials invariant by reparametrization. We will show 66 syzygies generating the ideal of relations between (the restriction to $\{f_1' = 0\}$ of) these 17 generators of UE_5^2.

• In any case, if we can show that UE_κ^n is, for a certain n and for a certain κ, generated as an algebra by a finite number of bi-invariants, we may easily deduce as a corollary finite generation of the full algebra E_κ^n. For instance:

— For $n = \kappa = 3$, computing the $\mathsf{GL}_3(\mathbb{C})$-orbit of the 4 bi-invariants f_1', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ and $D_{1,2,3}^6$ amounts to polarize their lower indices, which yields the invariants f_i', $\Lambda_{i,j}^3$, $\Lambda_{i,j;\,k}^5$ and $D_{i,j,k}^6$ generating E_3^3.

— For $n = 2$ and $\kappa = 4$, computing the $\mathsf{GL}_2(\mathbb{C})$-orbit of the 5 bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ and M^8 again amounts to polarize their lower indices, which yields the invariants f_i', Λ^3, Λ_i^5, $\Lambda_{i,j}^7$ and M^8 generating E_4^2.

• Finally, for applications to Kobayashi hyperbolicity (which involves estimating the Euler-Poincaré characteristic of $\mathsf{E}_{\kappa,m}^n T_X^*$), it is useless to look for a complete understanding of the algebraic structure of E_κ^n, and it only suffices to possess a complete description of the algebra of bi-invariants UE_κ^n. In fact, as will be (re)explained in §12, each bi-invariant will correspond to one and to only one Schur bundle.

So from now on, we focus our attention on bi-invariants

Initial representation of bi-invariants. We now restart with the initial, rational expression of any polynomial invariant by reparametrization provided by the lemma on p. 36 and we want to determine when such a polynomial is, in addition, invariant by the unipotent action.

To begin with, we consider the subgroup $\mathsf{U}_n^*(\mathbb{C})$ of $\mathsf{U}_n(\mathbb{C})$ generated by matrices of the form:

$$\mathsf{u}^* = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{21} & 1 & 0 & \cdots & 0 \\ u_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Clearly, the components of the first order jet $j^1 f$ are modified by the action of u^*:

$$\begin{cases} \mathsf{u}^* \cdot f_1' = f_1', \\ \mathsf{u}^* \cdot f_2' = f_2' + u_{21} f_1', \\ \mathsf{u}^* \cdot f_3' = f_3' + u_{31} f_1', \\ \quad \cdots \qquad\qquad \cdots \\ \mathsf{u}^* \cdot f_n' = f_n' + u_{n1} f_1'. \end{cases}$$

On the other hand, all the $\Lambda_{1,i}^3$ are left invariant:

$$\mathsf{u}^* \cdot \Lambda_{1,i}^3 = \mathsf{u}^* \cdot \big[f_i',\, f_1'\big] = \big[f_i' + u_{i1} f_1',\, f_1'\big] = \big[f_i',\, f_1'\big] = \Lambda_{1,i}^3,$$

and in fact, more generally, one may verify that the same is true of higher Λ's:

$$\mathsf{u}^* \cdot \Lambda_{1,i;\,1}^5 = \Lambda_{1,i;\,1}^5, \qquad \mathsf{u}^* \cdot \Lambda_{1,i;\,1,1}^7 = \Lambda_{1,i;\,1,1}^7,\ \ldots\ldots,\ \mathsf{u}^* \cdot \Lambda_{1,i;\,1^{\kappa-2}}^{2\kappa-1} = \Lambda_{1,i;\,1^{\kappa-2}}^{2\kappa-1},$$

for any $i = 2, 3, \ldots, n$. Consequently, the requirement that a polynomial invariant by reparametrization $\mathsf{P}\big(j^\kappa f\big) \in \mathsf{E}_{\kappa,m}^n$ be in addition also invariant by the unipotent subgroup $\mathsf{U}_n^*(\mathbb{C}) \subset \mathsf{U}_n(\mathbb{C})$, namely $\mathsf{u}^* \cdot \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)$, shall be written in length as follows, when employing the mentioned representation given on p. 36:

$$\begin{aligned}
\sum_a (f_1')^a\, \mathsf{P}_a\Big(f_2' + u_{21} f_1',\, f_3' + u_{31} f_1',\, \ldots,\, f_n' + u_{n1} f_1',& \\
\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big) &= \\
= \sum_a (f_1')^a\, \mathsf{P}_a\Big(f_2',\, f_3',\, \ldots,\, f_n',& \\
\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big)&.
\end{aligned}$$

Because the $n+(n-1)(\kappa-1)$ invariants $f_1', \ldots, f_n'$, $\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}$, $2 \leqslant i \leqslant n$, $2 \leqslant \lambda \leqslant \kappa$, are algebraically independent, we deduce that each P_a must be independent of $f_2', f_3', \ldots, f_n'$, so that we come to the simpler rational expression:

$$\mathsf{R} = \sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big),$$

which is however not yet invariant under the full unipotent action.

Second unipotent subgroup. Next, we consider the subgroup $\mathsf{U}_n^\sharp(\mathbb{C}) \subset \mathsf{U}_n(\mathbb{C})$ constituted by matrices of the form:

$$\mathsf{u}^\sharp = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & u_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix}.$$

Since $\mathsf{U}_n^*(\mathbb{C})$ and $\mathsf{U}_n^\sharp(\mathbb{C})$ clearly generate the full unipotent group $\mathsf{U}_n(\mathbb{C})$, it now only remains to require the $\mathsf{U}_n^\sharp(\mathbb{C})$-invariance for the rational expression R obtained just above.

The requirement $\mathsf{u}^\sharp \cdot \mathsf{R} = \mathsf{R}$ can in turn be written in length as follows: :

$$\begin{aligned}
\mathsf{u}^\sharp\bigg(\sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big)\bigg) &= \sum_a (f_1')^a\, \mathsf{P}_a\Big(\mathsf{u}^\sharp \cdot \Lambda_{1,2}^3,\, \ldots,\, \mathsf{u}^\sharp \cdot \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big) \\
&= \sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big).
\end{aligned}$$

But on the other hand, for any λ with $2 \leqslant \lambda \leqslant \kappa$, one may verify that the action of $\mathsf{u}^\sharp$ on the initial Λ-invariants appearing as arguments of R is given by the triangular formulas:

$$\begin{aligned}
\mathsf{u}^\sharp \cdot \Lambda^{2\lambda-1}_{1,2;\,1^{\lambda-2}} &= \Lambda^{2\lambda-1}_{1,2;\,1^{\lambda-2}}, \\
\mathsf{u}^\sharp \cdot \Lambda^{2\lambda-1}_{1,3;\,1^{\lambda-2}} &= \Lambda^{2\lambda-1}_{1,3;\,1^{\lambda-2}} + u_{32}\,\Lambda^{2\lambda-1}_{1,2;\,1^{\lambda-2}}, \\
\mathsf{u}^\sharp \cdot \Lambda^{2\lambda-1}_{1,4;\,1^{\lambda-2}} &= \Lambda^{2\lambda-1}_{1,4;\,1^{\lambda-2}} + u_{43}\,\Lambda^{2\lambda-1}_{1,3;\,1^{\lambda-2}} + u_{42}\,\Lambda^{2\lambda-1}_{1,2;\,1^{\lambda-2}}, \\
\cdots\quad & \qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\mathsf{u}^\sharp \cdot \Lambda^{2\lambda-1}_{1,n;\,1^{\lambda-2}} &= \Lambda^{2\lambda-1}_{1,n;\,1^{\lambda-2}} + u_{n,n-1}\,\Lambda^{2\lambda-1}_{1,n-1;\,1^{\lambda-2}} + \cdots + u_{n2}\,\Lambda^{2\lambda-1}_{1,2;\,1^{\lambda-2}}.
\end{aligned}$$

The algebraic independency of f_1', $\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}$ then implies that such an R is $\mathsf{U}_n^\sharp(\mathbb{C})$-invariant if and only if every P_a is so, namely if and only if the following identity holds:

$$\begin{aligned}
\mathsf{P}_a\Big(&A^3_{1,2},\, A^3_{1,3} + u_{32}\,A^3_{1,2},\, \dots,\, A^3_{1,n} + u_{n,n-1}\,A^3_{1,n-1} + \cdots + u_{n2}\,A^3_{1,2}, \\
&A^5_{1,2},\, A^5_{1,3} + u_{32}\,A^5_{1,2},\, \dots,\, A^5_{1,n} + u_{n,n-1}\,A^5_{1,n-1} + \cdots + u_{n2}\,A^5_{1,2}, \\
&\cdots \qquad \cdots \qquad\qquad \cdots \\
&A^{2\kappa-1}_{1,2},\, A^{2\kappa-1}_{1,3} + u_{32}\,A^{2\kappa-1}_{1,2},\, \dots,\, A^{2\kappa-1}_{1,n} + u_{n,n-1}\,A^{2\kappa-1}_{1,n-1} + \cdots + u_{n2}\,A^{2\kappa-1}_{1,2}\Big) =
\end{aligned}$$

$$\begin{aligned}
= \mathsf{P}_a\Big(&A^3_{1,2},\, A^3_{1,3},\, \dots,\, A^3_{1,n}, \\
&A^5_{1,2},\, A^5_{1,3},\, \dots,\, A^5_{1,n}, \\
&\cdots \quad \cdots \qquad \cdots \\
&A^{2\kappa-1}_{1,2},\, A^{2\kappa-1}_{1,3},\, \dots,\, A^{2\kappa-1}_{1,n}\Big),
\end{aligned}$$

as polynomials in $\mathbb{C}\big[A^3_{1,2}, \dots, A^3_{1,n}, \dots\dots, A^{2\kappa-1}_{1,2}, \dots, A^{2\kappa-1}_{1,n}\big]$, for every $\mathsf{u}^\sharp$, and for every a with $-\frac{m-1}{m}\kappa \leqslant a \leqslant m$.

Here, we recognize a full unipotent action, acted by means of a general $(n-1) \times (n-1)$ unipotent matrice of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix} \in \mathsf{U}_{n-1}(\mathbb{C}),$$

on the set of the $\kappa - 1$ vectors of $\mathbb{C}^{n-1}$ defined by:

$$\big(A^{2\lambda-1}_{1,2},\, A^{2\lambda-1}_{1,3},\, \dots,\, A^{2\lambda-1}_{1,n}\big) \qquad\qquad {\scriptstyle (2 \leqslant \lambda \leqslant \kappa)}.$$

It is known ([19, 27]) that the invariants for such an action are constituted by all the minors:

$$\Pi_2^{\lambda_2} := A_{1,2}^{2\lambda_2-1}, \qquad \Pi_{2,3}^{\lambda_2,\lambda_3} := \begin{vmatrix} A_{1,2}^{2\lambda_2-1} & A_{1,3}^{2\lambda_2-1} \\ A_{1,2}^{2\lambda_3-1} & A_{1,3}^{2\lambda_3-1} \end{vmatrix},$$

$$\Pi_{2,3,4}^{\lambda_2,\lambda_3,\lambda_4} := \begin{vmatrix} A_{1,2}^{2\lambda_2-1} & A_{1,3}^{2\lambda_2-1} & A_{1,4}^{2\lambda_2-1} \\ A_{1,2}^{2\lambda_3-1} & A_{1,3}^{2\lambda_3-1} & A_{1,4}^{2\lambda_3-1} \\ A_{1,2}^{2\lambda_4-1} & A_{1,3}^{2\lambda_4-1} & A_{1,4}^{2\lambda_4-1} \end{vmatrix},$$

and generally:

$$\Pi_{2,3,4,\dots,n_1}^{\lambda_2,\lambda_3,\lambda_4,\dots,\lambda_{n_1}} := \begin{vmatrix} A_{1,2}^{2\lambda_2-1} & A_{1,3}^{2\lambda_2-1} & A_{1,4}^{2\lambda_2-1} & \cdots & A_{1,n_1}^{2\lambda_2-1} \\ A_{1,2}^{2\lambda_3-1} & A_{1,3}^{2\lambda_3-1} & A_{1,4}^{2\lambda_3-1} & \cdots & A_{1,n_1}^{2\lambda_3-1} \\ A_{1,2}^{2\lambda_4-1} & A_{1,3}^{2\lambda_4-1} & A_{1,4}^{2\lambda_4-1} & \cdots & A_{1,n_1}^{2\lambda_4-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,2}^{2\lambda_{n_1}-1} & A_{1,3}^{2\lambda_{n_1}-1} & A_{1,4}^{2\lambda_{n_1}-1} & \cdots & A_{1,n_1}^{2\lambda_{n_1}-1} \end{vmatrix},$$

for all n_1 from $n_1 = 1$ up to $n_1 = n$, and for arbitrary λ_j with $2 \leqslant \lambda_j \leqslant \kappa$. In fact, one immediately sees that these minors are obviously invariant by the unipotent action of $\mathsf{U}_{n-1}(\mathbb{C})$, thanks to the fact that column linear dependence leaves untouched any determinant.

THEOREM *In dimension $n \geqslant 1$ and for jets of arbitrary order $\kappa \geqslant 1$, every bi-invariant polynomial $\mathsf{BP} = \mathsf{BP}\big(j^\kappa f\big)$ invariant by reparametrization and invariant under the unipotent action writes under the form:*

$$\mathsf{BP}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{BP}_a\left(\begin{vmatrix} \Lambda_{1,2}^{2\lambda_2-1} & \Lambda_{1,3}^{2\lambda_2-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_2-1} \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \end{vmatrix}_{n_1=1,2\dots,n}^{2\leqslant\lambda_2,\dots,\lambda_{n_1}\leqslant\kappa} \right),$$

for certain specific polynomials BP_a which depend upon $\mathsf{BP}(j^\kappa f)$.

The case $n = \kappa = 3$. After $\mathsf{U}_3^*(\mathbb{C})$-reduction, an arbitrary element of $\mathsf{UE}_{3,m}^3$ writes:

$$\mathsf{R} = \sum_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \Lambda_{1,3}^3,\, \Lambda_{1,2;\,1}^5,\, \Lambda_{1,3;\,1}^5\Big).$$

Then the $\mathsf{U}_3^\sharp(\mathbb{C})$-reduction presented above shows that there are four initial bi-invariants, namely the three obvious ones f_1', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ together with:

$$\begin{vmatrix} \Lambda_{1,2}^3 & \Lambda_{1,3}^3 \\ \Lambda_{1,2;\,1}^5 & \Lambda_{1,3;\,1}^5 \end{vmatrix} = f_1' f_1' \cdot \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} =: f_1' f_1' \cdot D_{1,2,3}^6,$$

where the first equality, which follows from a direct calculation, gives birth to the three-dimensional Wronskian. By pluging this minor in the above rational expression of R, we obtain that any bi-invariant polynomial in $\mathsf{UE}^3_{3,m}$ writes under the form:

$$\mathsf{BP}\big(j^3f\big) = \sum_{-\frac{2}{3}m\leqslant a\leqslant m} (f_1')^a \, \widetilde{\mathsf{P}}_a\Big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\Big),$$

for certain (new) polynomials $\widetilde{\mathsf{P}}_a$. More is true, for we claim that there are no negative powers of f_1' anymore in such a rational representation.

Proposition. *Any bi-invariant polynomial* $\mathsf{BP} \in \mathsf{UE}^3_{3,m}$ *writes uniquely under the form:*

$$\mathsf{BP}\big(j^3f\big) = \sum_{0\leqslant a\leqslant m} (f_1')^a \, \mathsf{BP}_a\Big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\Big),$$

where the BP_a *are arbitrary polynomials. In fact:*

$$\boxed{\mathsf{UE}\big(j^3f\big) = \mathbb{C}\big[f_1',\, \Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\big]}\,.$$

Proof. One verifies at first sight that, after setting $f_1' = 0$, the 3 restricted invariants:

$$\Lambda^3_{1,2}\big|_0 = -f_1''f_2', \qquad \Lambda^5_{1,2;\,1}\big|_0 = 3\, f_1''f_2'f_1'' \qquad \text{and} \qquad D^6_{1,2,3}\big|_0 = \begin{vmatrix} 0 & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$$

are mutually algebraically independent. Suppose then by contradiction that the expression:

$$\mathsf{BP}\big(j^3f\big) = \sum_{-a_0\leqslant a\leqslant m} (f_1')^a \, \widetilde{\mathsf{P}}_a\Big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\Big),$$

starts with a not identically zero $\widetilde{\mathsf{P}}_{-a_0}\big(A^3, A^5, \Delta^6\big) \not\equiv 0$ for some smallest negative power $-a_0 < 0$ of f_1'. Multiplying both sides by $(f_1')^{a_0}$ and setting $f_1' = 0$ afterwards, the left term $(f_1')^{a_0}\,\mathsf{BP}\big(j^3f\big)$ then vanishes, hence one would derive an identity:

$$0 \equiv \widetilde{\mathsf{P}}_{-a_0}\Big(\Lambda^3_{1,2}\big|_0,\, \Lambda^5_{1,2;\,1}\big|_0,\, D^6_{1,2,3}\big|_0\Big)$$

between restricted bi-invariants which would then entail $\widetilde{\mathsf{P}}_{-a_0} \equiv 0$ because the arguments are algebraically independent, a contradiction.

Consequently, the rational expression for $\mathsf{BP}\big(j^3f\big)$ was already polynomial and inversely, every arbitrary polynomial in $\mathbb{C}\big[f_1',\, \Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\big]$ obviously is a bi-invariant. □

The case $n = \kappa = 4$**.** After $\mathsf{U}_4^*(\mathbb{C})$-reduction, an arbitrary element of $\mathsf{UE}^4_{4,m}$ writes under the form:

$$\mathsf{R} = \sum_{-\frac{3}{4}m\leqslant a\leqslant m} (f_1')^a \, \mathsf{P}_a\Big(\Lambda^3_{1,2},\, \Lambda^3_{1,3},\, \Lambda^3_{1,4},\, \Lambda^5_{1,2;\,1},\, \Lambda^5_{1,3;\,1},\, \Lambda^5_{1,4;\,1},\, \Lambda^7_{1,2;\,1,1},\, \Lambda^7_{1,3;\,1,1},\, \Lambda^7_{1,4;\,1,1}\Big).$$

Then the $\mathsf{U}_4^\sharp(\mathbb{C})$-reduction presented above shows that there are the 4 obvious initial bi-invariants:

$$f_1', \qquad \Lambda_{1,2}^3, \qquad \Lambda_{1,2;\,1}^5 \qquad \text{and} \qquad \Lambda_{1,2;\,1,1}^7,$$

together with the 4 further ones:

$$D^6, \qquad D^8 = \big[D^6,\, f_1'\big], \qquad N^{10} \qquad \text{and} \qquad W^{10},$$

that are obtained by dividing the 4 minors involving the Λ's by the maximal power of f_1' which appears in factor, namely:

$$\left|\begin{array}{cc} \Lambda_{1,2}^3 & \Lambda_{1,3}^3 \\ \Lambda_{1,2;1}^5 & \Lambda_{1,3;1}^5 \end{array}\right| \equiv f_1'f_1'D^6,$$
$$\left|\begin{array}{cc} \Lambda_{1,2}^3 & \Lambda_{1,3}^3 \\ \Lambda_{1,2;1,1}^7 & \Lambda_{1,3;1,1}^7 \end{array}\right| \equiv f_1'f_1'D^8,$$

$$\left|\begin{array}{cc} \Lambda_{1,2;1}^5 & \Lambda_{1,3;1}^5 \\ \Lambda_{1,2;1,1}^7 & \Lambda_{1,3;1,1}^7 \end{array}\right| \equiv f_1'f_1'N^{10},$$
$$\left|\begin{array}{ccc} \Lambda_{1,2}^3 & \Lambda_{1,3}^3 & \Lambda_{1,4}^3 \\ \Lambda_{1,2;1}^5 & \Lambda_{1,3;1}^5 & \Lambda_{1,4;1}^5 \\ \Lambda_{1,2;1,1}^7 & \Lambda_{1,3;1,1}^7 & \Lambda_{1,4;1,1}^7 \end{array}\right| \equiv f_1'f_1'f_1'f_1'f_1'\,W^{10},$$

where the last one behind $(f_1')^5$ appears to be equal to the four-dimensional *Wronskian*:

$$W^{10} := \left|\begin{array}{cccc} f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \\ f_1'''' & f_2'''' & f_3'''' & f_4'''' \end{array}\right|,$$

and where the first three ones are explicitly defined by:

$$D^6 := \left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{array}\right|,$$

$$D^8 := f_1'\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1'''' & f_2'''' & f_3'''' \end{array}\right| - 6\,f_1''\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{array}\right|,$$

$$N^{10} := f_1'f_1'\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \\ f_1'''' & f_2'''' & f_3'''' \end{array}\right| - 3\,f_1'f_1''\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1'''' & f_2'''' & f_3'''' \end{array}\right| +$$
$$+ 4\,f_1'f_1'''\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{array}\right| + 3\,f_1''f_1''\left|\begin{array}{ccc} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{array}\right|.$$

By pluging these 8 bi-invariants in the rational expression written on p. 47, we obtain that any bi-invariant polynomial in $\mathsf{UE}^4_{4,m}$ writes under the form:

$$\boxed{\mathsf{BP}\big(j^4j\big) = \sum_{-\frac{3}{4}m\leqslant a\leqslant m} (f_1')^a\,\widetilde{\mathsf{P}}_a\Big(\Lambda^3,\,\Lambda^5,\,\Lambda^7,\,D^6,\,D^8,\,N^{10},\,W^{10}\Big)}\,.$$

This expression will be the very starting point for the application of our general algorithm, to be presented in Section 9 below. In fact, as in the case $n=2$, $\kappa=4$ of Section 6, there will appear further independent *ghost bi-invariants hidden behind powers of* f_1'.

§8. COUNTEREXPECTATION: INSUFFICIENCY OF BRACKET INVARIANTS

According to the unexpected, main outcome of [21], the theorem for $n=2$ and $\kappa=5$ on p. 28 about bracket invariants does *not* capture all Demailly-Semple (bi-)invariants. This was striking, because brackets were sufficient to capture all invariants in all previously known studies[14], namely for E^n_2, for E^2_3, for E^3_3 and for E^2_4.

Aside from the 11 bi-invariants f_1', Λ^3, Λ^5_1, $\Lambda^7_{1,1}$, M^8, $\Lambda^9_{1,1,1}$, M^{10}_1, N^{12}, $K^{12}_{1,1}$, H^{14}_1 and $F^{16}_{1,1}$, there are yet the following 6 bi-invariants X^{18}, X^{19}, X^{21}, X^{23}, X^{25} and Y^{27} that are defined by dividing by f_1' some appropriate quadratic combinations between already known bi-invariants. We provide here the complete explicit expressions. It is shown in [21] that the 16 first bi-invariants are mutually independent and it would be easy, by using the same method, to verify that when one adds the last, 17-th bi-invariant Y^{27}, one still gets a list of 17 mutually independent bi-invariants.

Importantly, we emphasize that by no means any of these 6 further bi-invariants can come from inspecting the bracket invariants, by dividing them either by f_1', or by Λ^3 or by anything based in brackets, because in [21], *all the possible bracket invariants* were computed thoroughly, were simplified and were analyzed at a piece. *The existence of* X^{18}, X^{19}, X^{21}, X^{23}, X^{25}, Y^{27} *really shows that bracketing does not generate the algebra of bi-invariants* UE^2_5. A similar phenomenon will appear to take place in dimension $n=3$ for jet order $\kappa=4$.

Before reading the formulas, we would like to mention that the invariant X^{21} of UE^2_5 below is not the same as the invariant X^{21} of UE^4_3 appearing in §11. Our manuscript sheets used the same notation, and we hope this should not cause any

[14] On observes that UE^3_3 is *not* obtained by bracketing bi-invariants in UE^3_2 (think of D^6), but nevertheless UE^3_3 is the unipotent-invariant subalgebra of E^3_3, and E^3_3 itself is obtained by bracketing invariants from the preceding jet level.

confusion.

$$X^{18} := \frac{-5\,\Lambda^{9}_{1,1,1}\,M_1^{10} + 56\,\Lambda^{7}_{1,1}\,K^{12}_{1,1}}{f_1'}$$

$$= f_1' f_1' f_1' \Big(- 18816\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2 - 25088\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^3 - 15\,\big[\Delta^{\prime,\prime\prime\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} - 150\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$+ 315\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime} + 960\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime} - 375\,\big[\Delta^{\prime\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} + 1575\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}$$

$$+ 4800\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime} - 392\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^3 - 4704\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime\prime,\prime\prime\prime}\Big) - f_1' f_1' f_1''\,\Big(- 2475\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 9900\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} - 2850\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 + 51330\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}$$

$$+ 92760\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime\prime} - 14250\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 + 7035\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime\prime} - 495\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 1980\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\Big) - f_1' f_1' f_1'''\,\Big(- 11100\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 - 3150\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\Big)$$

$$+ f_1' f_1'' f_1''\,\Big(- 109440\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} - 19050\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 - 32325\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2$$

$$+ 11025\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} + 55125\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} - 6840\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime}$$

$$- 54720\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\Big) - f_1' f_1'' f_1'''\,\Big(+ 30000\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3\Big) - f_1'' f_1'' f_1''\,\Big(11025\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2$$

$$- 55125\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 55125\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} + 110250\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 49000\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3\Big).$$

$$X^{19} := \frac{-5\,M_1^{10}\,M_1^{10} + 64\,M^{8}\,K^{12}_{1,1}}{f_1'}$$

$$= f_1'\,\Big(1170\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} - 45\,\big[\Delta^{\prime,\prime\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 450\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2$$

$$+ 74220\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 + 3780\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} - 1600\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3$$

$$- 1125\,\big[\Delta^{\prime\prime,\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 5850\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} + 18900\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 8000\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3 - 1344\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^3\,\Delta^{\prime,\prime\prime} - 16128\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime} + 1995\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2$$

$$- 64512\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} + 27660\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2 - 86016\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^3\,\Delta^{\prime,\prime\prime}\Big)$$

$$+ f_1''\,\Big(- 74400\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3 - 10800\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 2160\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2$$

$$- 8640\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 3600\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} + 64800\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 43200\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 18000\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} + 10800\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}$$

$$- 27600\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3 + 86400\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime}\Big) + f_1'''\,\Big(16000\,\big[\Delta^{\prime,\prime\prime\prime}\big]^4\Big).$$

$$X^{21} := \frac{-5\,M_1^{10}\,N^{12} + 8\,M^{8}\,H_1^{14}}{f_1'}$$

$$= - 135\,\big[\Delta^{\prime,\prime\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime}\big]^3 - 1350\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^3 + 1350\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2$$

$$+ 2700\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 1200\,\Delta^{\prime,\prime\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3\,\Delta^{\prime,\prime\prime} - 3375\,\big[\Delta^{\prime\prime,\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime}\big]^3$$

$$+ 6750\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 13500\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\Delta^{\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 6000\,\Delta^{\prime\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^3\,\Delta^{\prime,\prime\prime}$$

$$- 576\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^3\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 6912\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime}\big]^2 - 495\,\big[\Delta^{\prime,\prime\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime}$$

$$- 27648\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 9540\,\Delta^{\prime,\prime\prime\prime\prime}\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} + 1200\,\Delta^{\prime,\prime\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^4$$

$$- 36864\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^3\,\big[\Delta^{\prime,\prime\prime}\big]^2 + 32580\,\big[\Delta^{\prime\prime,\prime\prime\prime}\big]^2\,\big[\Delta^{\prime,\prime\prime\prime}\big]^2\,\Delta^{\prime,\prime\prime} - 7200\,\Delta^{\prime\prime,\prime\prime\prime}\,\big[\Delta^{\prime,\prime\prime\prime}\big]^4.$$

$$
\begin{aligned}
X^{23} &:= \frac{-7\,N^{12}\,K_{1,1}^{12} + M^8\,F_{1,1}^{16}}{f_1'} \\
&= f_1' \Big(432\,\Delta^{','''''}\,\big[\Delta^{','''}\big]^2\,\big[\Delta^{','' }\big]^2 + 3456\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{','' }\big]^2 + 1710\,\Delta^{','''''}\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} \\
&\quad - 3150\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 + 540\,\Delta^{','''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} - 1600\,\Delta^{','''''}\,\big[\Delta^{','''}\big]^4 \\
&\quad - 7875\,\big[\Delta^{'',''''}\big]^2\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 + 6912\,\Delta^{','''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{',''}\big]^2 - 8000\,\Delta^{'',''''}\,\big[\Delta^{','''}\big]^4 \\
&\quad - 2352\,\big[\Delta^{',''''}\big]^3\,\Delta^{','''}\,\Delta^{',''} - 23904\,\big[\Delta^{',''''}\big]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} + 2205\,\big[\Delta^{',''''}\big]^2\,\big[\Delta^{','''}\big]^3
\end{aligned}
$$

$$
\begin{aligned}
&- 78336\,\Delta^{',''''}\,\big[\Delta^{'','''}\big]^2\,\Delta^{','''}\,\Delta^{',''} + 34740\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^3 - 81408\,\big[\Delta^{'','''}\big]^3\,\Delta^{','''}\,\Delta^{',''} \\
&+ 72180\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{','''}\big]^3 + 2160\,\Delta^{'',''''}\,\big[\Delta^{',''''}\big]^2\,\big[\Delta^{',''}\big]^2 + 17280\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2 \\
&+ 8550\,\Delta^{'',''''}\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} + 34560\,\Delta^{'',''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{',''}\big]^2 + 2700\,\Delta^{'',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} \\
&- 315\,\big[\Delta^{','''''}\big]^2\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2\Big) + f_1''\Big(23625\,\big[\Delta^{'',''''}\big]^2\,\big[\Delta^{',''}\big]^3 - 47250\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 \\
&- 94500\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 + 42000\,\Delta^{'',''''}\,\big[\Delta^{','''}\big]^3\,\Delta^{',''} + 576\,\big[\Delta^{',''''}\big]^3\,\big[\Delta^{',''}\big]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 6912\,\big[\Delta^{',''''}\big]^2\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2 + 20745\,\big[\Delta^{',''''}\big]^2\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} + 27648\,\Delta^{',''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{',''}\big]^2 \\
&+ 945\,\big[\Delta^{','''''}\big]^2\,\big[\Delta^{',''}\big]^3 + 9450\,\Delta^{','''''}\,\Delta^{'',''''}\,\big[\Delta^{',''}\big]^3 - 9450\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 \\
&- 18900\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{','''}\,\big[\Delta^{',''}\big]^2 + 8400\,\Delta^{','''''}\,\big[\Delta^{','''}\big]^3\,\Delta^{',''} + 71460\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} \\
&- 37200\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^4 + 36864\,\big[\Delta^{'','''}\big]^3\,\big[\Delta^{',''}\big]^2 + 48420\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} \\
&- 64800\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^4\Big) + f_1'''\Big(16000\,\big[\Delta^{','''}\big]^5\Big).
\end{aligned}
$$

$$
\begin{aligned}
X^{25} &:= \frac{-56\,K_{1,1}^{12}\,H_1^{14} + 5\,M_1^{10}\,F_{1,1}^{16}}{f_1'} \\
&= f_1'\,f_1'\Big(- 45\,\big[\Delta^{','''''}\big]^2\,\Delta^{',''''}\,\big[\Delta^{',''}\big]^2 - 180\,\big[\Delta^{','''''}\big]^2\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2 - 3600\,\big[\Delta^{','''''}\big]^2\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} \\
&\quad - 2800\,\Delta^{','''''}\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^3 - 83200\,\Delta^{','''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^3 - 1125\,\big[\Delta^{'',''''}\big]^2\,\Delta^{',''''}\,\big[\Delta^{',''}\big]^2 \\
&\quad - 4500\,\big[\Delta^{'',''''}\big]^2\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2 - 90000\,\big[\Delta^{'',''''}\big]^2\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} - 14000\,\Delta^{'',''''}\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^3 \\
&\quad - 416000\,\Delta^{'',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^3 - 150528\,\big[\Delta^{',''''}\big]^3\,\Delta^{'','''}\,\Delta^{',''} - 903168\,\big[\Delta^{',''''}\big]^2\,\big[\Delta^{'','''}\big]^2\,\Delta^{',''} \\
&\quad + 163800\,\big[\Delta^{',''''}\big]^2\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^2 - 2408448\,\Delta^{',''''}\,\big[\Delta^{'','''}\big]^3\,\Delta^{',''} + 1129500\,\Delta^{',''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{','''}\big]^2 \\
&\quad - 9408\,\big[\Delta^{',''''}\big]^4\,\Delta^{',''} + 3675\,\big[\Delta^{',''''}\big]^3\,\big[\Delta^{','''}\big]^2 - 2408448\,\big[\Delta^{'','''}\big]^4\,\Delta^{',''} + 2132400\,\big[\Delta^{'','''}\big]^3\,\big[\Delta^{','''}\big]^2 \\
&\quad - 450\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,\big[\Delta^{',''}\big]^2 - 1800\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2
\end{aligned}
$$

$$
\begin{aligned}
&- 36000\,\Delta^{','''''}\,\Delta^{'',''''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} + 11970\,\Delta^{','''''}\,\big[\Delta^{',''''}\big]^2\,\Delta^{','''}\,\Delta^{',''} \\
&+ 187920\,\Delta^{','''''}\,\big[\Delta^{'','''}\big]^2\,\Delta^{','''}\,\Delta^{',''} + 59850\,\Delta^{'',''''}\,\big[\Delta^{',''''}\big]^2\,\Delta^{','''}\,\Delta^{',''} \\
&+ 939600\,\Delta^{'',''''}\,\big[\Delta^{'','''}\big]^2\,\Delta^{','''}\,\Delta^{',''} + 474300\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} \\
&+ 94860\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''}\Big) + f_1' f_1''\Big(- 2556600\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^3
\end{aligned}
$$

$$
\begin{aligned}
&- 5014200\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{','''}\big]^3 - 187950\,\big[\Delta^{',''''}\big]^2\,\big[\Delta^{','''}\big]^3 + 5621760\,\Delta^{',''''}\,\big[\Delta^{'','''}\big]^2\,\Delta^{','''}\,\Delta^{',''} \\
&+ 5652480\,\big[\Delta^{'','''}\big]^3\,\Delta^{','''}\,\Delta^{',''} - 2764800\,\Delta^{'',''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{',''}\big]^2 \\
&+ 99000\,\Delta^{'',''''}\,\Delta^{'','''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} + 500000\,\Delta^{'',''''}\,\big[\Delta^{','''}\big]^4 + 174720\,\big[\Delta^{',''''}\big]^3\,\Delta^{','''}\,\Delta^{',''} \\
&+ 1751040\,\big[\Delta^{',''''}\big]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 276480\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\big[\Delta^{',''}\big]^2 \\
&- 105300\,\Delta^{','''''}\,\Delta^{',''''}\,\big[\Delta^{','''}\big]^2\,\Delta^{',''} - 552960\,\Delta^{','''''}\,\big[\Delta^{'','''}\big]^2\,\big[\Delta^{',''}\big]^2
\end{aligned}
$$

$$+ 19800\,\Delta^{','''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''} + 100000\,\Delta^{','''''}\,[\Delta^{',\prime\prime\prime}]^4 + 551250\,[\Delta^{'',''''}]^2\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2$$
$$- 172800\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 - 1382400\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',''}]^2$$
$$- 526500\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''} - 34560\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 22050\,[\Delta^{','''''}]^2\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2$$
$$+ 220500\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2\Big)$$
$$+ f_1' f_1''' \Big(28000\,\Delta^{',''''}\,[\Delta^{',\prime\prime\prime}]^4 + 472000\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^4\Big)$$

$$+ f_1'' f_1'' \Big(330750\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2 + 661500\,\Delta^{','''''}\,\Delta^{'',\prime\prime\prime}\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2$$
$$- 294000\,\Delta^{','''''}\,[\Delta^{',\prime\prime\prime}]^3\,\Delta^{',''} - 330750\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^3$$
$$+ 1653750\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2 + 3307500\,\Delta^{'',''''}\,\Delta^{'',\prime\prime\prime}\,\Delta^{',\prime\prime\prime}\,[\Delta^{',''}]^2$$
$$- 1470000\,\Delta^{'',''''}\,[\Delta^{',\prime\prime\prime}]^3\,\Delta^{',''} - 2880\,[\Delta^{',''''}]^3\,[\Delta^{',''}]^2 - 34560\,[\Delta^{',''''}]^2\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',''}]^2$$

$$- 812475\,[\Delta^{',''''}]^2\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''} - 138240\,\Delta^{',''''}\,[\Delta^{'',\prime\prime\prime}]^2\,[\Delta^{',''}]^2 - 33075\,[\Delta^{','''''}]^2\,[\Delta^{',''}]^3$$
$$+ 1446000\,\Delta^{',''''}\,[\Delta^{',\prime\prime\prime}]^4 - 184320\,[\Delta^{'',\prime\prime\prime}]^3\,\Delta^{',''}]^2 - 3077100\,[\Delta^{'',\prime\prime\prime}]^2\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''}$$
$$+ 2844000\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^4 - 826875\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^3 - 3192300\,\Delta^{',''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''}\Big)$$
$$+ f_1'' f_1''' \Big(-640000\,[\Delta^{',\prime\prime\prime}]^5\Big).$$

$$Y^{27} := \frac{-56\,K_{1,1}^{12} F_{1,1}^{16} + M_1^{10} X^{18}}{f_1'}$$
$$= f_1' f_1' f_1' \Big(572820\,\Delta^{','''''}\,[\Delta^{'',\prime\prime\prime}]^2\,[\Delta^{',\prime\prime\prime}]^2 - 5343744\,\Delta^{',''''}\,[\Delta^{'',\prime\prime\prime}]^3\,\Delta^{',\prime\prime\prime} - 752640\,\Delta^{'',''''}\,[\Delta^{'',\prime\prime\prime}]^3\,\Delta^{',''} -$$
$$- 286944\,[\Delta^{',''''}]^3\,\Delta^{'',\prime\prime\prime}\,\Delta^{',\prime\prime\prime} + 2864100\,\Delta^{'',''''}\,[\Delta^{'',\prime\prime\prime}]^2\,[\Delta^{',\prime\prime\prime}]^2 - 1862784\,[\Delta^{',''''}]^2\,[\Delta^{'',\prime\prime\prime}]^2\,\Delta^{',\prime\prime\prime} +$$
$$+ 27195\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,[\Delta^{',\prime\prime\prime}]^2 - 112000\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{',\prime\prime\prime}]^3 - 150528\,\Delta^{','''''}\,[\Delta^{'',\prime\prime\prime}]^3\,\Delta^{',''} -$$
$$- 2352\,\Delta^{','''''}\,[\Delta^{',''''}]^3\,\Delta^{',''} + 135975\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{',\prime\prime\prime}]^2 - 11760\,\Delta^{'',''''}\,[\Delta^{',''''}]^3\,\Delta^{',''} -$$
$$- 3375\,\Delta^{','''''}\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^2 - 675\,[\Delta^{','''''}]^2\,\Delta^{'',''''}\,[\Delta^{',''}]^2 - 45\,[\Delta^{','''''}]^3\,[\Delta^{',''}]^2 -$$

$$- 11200\,[\Delta^{','''''}]^2\,[\Delta^{',\prime\prime\prime}]^3 - 5625\,[\Delta^{'',''''}]^3\,[\Delta^{',''}]^2 - 280000\,[\Delta^{'',''''}]^2\,[\Delta^{',\prime\prime\prime}]^3 -$$
$$- 16464\,[\Delta^{',''''}]^4\,\Delta^{',\prime\prime\prime} - 5720064\,[\Delta^{'',\prime\prime\prime}]^4\,\Delta^{',\prime\prime\prime} + 1890\,[\Delta^{','''''}]^2\,\Delta^{',''''}\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} +$$
$$+ 6210\,[\Delta^{','''''}]^2\,\Delta^{'',\prime\prime\prime}\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} - 28224\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,\Delta^{'',\prime\prime\prime}\,\Delta^{',''} - 112896\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{'',\prime\prime\prime}]^2\,\Delta^{',''} +$$
$$+ 255360\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^2 + 47250\,[\Delta^{'',''''}]^2\,\Delta^{',''''}\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} + 155250\,[\Delta^{'',''''}]^2\,\Delta^{'',\prime\prime\prime}\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} -$$
$$- 141120\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,\Delta^{'',\prime\prime\prime}\,\Delta^{',''} - 564480\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{'',\prime\prime\prime}]^2\,\Delta^{',''} + 1276800\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^2 +$$
$$+ 18900\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} + 62100\,\Delta^{','''''}\Big) +$$

$$+ \Big(\Big(-36450\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 145800\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',''}]^2 + 832500\,\Delta^{','''''}\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''} -$$
$$- 149310\,\Delta^{','''''}\,[\Delta^{',''''}]^2 - 2680560\,\Delta^{','''''}\,[\Delta^{'',\prime\prime\prime}]^2\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} - 746550\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} -$$
$$- 13402800\,\Delta^{'',''''}\,[\Delta^{'',\prime\prime\prime}]^2\,\Delta^{',\prime\prime\prime}\,\Delta^{',''} - 3645\,[\Delta^{','''''}]^2\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 14580\,[\Delta^{','''''}]^2\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',''}]^2 +$$
$$+ 83250\,[\Delta^{','''''}]^2\,[\Delta^{',\prime\prime\prime}]^2\,\Delta^{',''} - 245700\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{',\prime\prime\prime}]^3 + 682200\,\Delta^{','''''}\,\Delta^{'',\prime\prime\prime}\,[\Delta^{',\prime\prime\prime}]^3 -$$

$$- 91125 \left[\Delta^{'','''''}\right]^2 \Delta^{','''''} \left[\Delta^{','''}\right]^2 - 364500 \left[\Delta^{'','''''}\right]^2 \Delta^{'',''''} \left[\Delta^{','''}\right]^2 + 2081250 \left[\Delta^{'','''''}\right]^2 \left[\Delta^{',''''}\right]^2 \Delta^{','''} -$$

$$- 1228500 \,\Delta^{'','''''} \Delta^{','''''} \left[\Delta^{',''''}\right]^3 + 3411000 \,\Delta^{'','''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^3 + 1354752 \left[\Delta^{','''''}\right]^3 \Delta^{'',''''} \Delta^{','''} +$$

$$+ 8128512 \left[\Delta^{','''''}\right]^2 \left[\Delta^{'',''''}\right]^2 \Delta^{','''} + 1796760 \left[\Delta^{','''''}\right]^2 \Delta^{'',''''} \left[\Delta^{',''''}\right]^2 + 21676032 \,\Delta^{','''''} \left[\Delta^{'',''''}\right]^3 \Delta^{','''} +$$

$$+ 850140 \,\Delta^{','''''} \left[\Delta^{'',''''}\right]^2 \left[\Delta^{',''''}\right]^2 + 84672 \left[\Delta^{','''''}\right]^4 \Delta^{','''} + 274155 \left[\Delta^{','''''}\right]^3 \left[\Delta^{',''''}\right]^2 +$$

$$+ 21676032 \left[\Delta^{'',''''}\right]^4 \Delta^{','''} - 7801680 \left[\Delta^{'',''''}\right]^3 \left[\Delta^{',''''}\right]^2 - 6336900 \,\Delta^{'','''''} \Delta^{','''''} \Delta^{'',''''} \Delta^{',''''} \Delta^{','''} -$$

$$- 1267380 \,\Delta^{',''''''} \Delta^{','''''} \Delta^{'',''''} \Delta^{',''''} \Delta^{','''}\Big) f_1'' +$$

$$+ \Big(1120000 \,\Delta^{'','''''} \left[\Delta^{',''''}\right]^4 - 5062200 \left[\Delta^{'',''''}\right]^2 \left[\Delta^{',''''}\right]^3 + 224000 \,\Delta^{',''''''} \left[\Delta^{',''''}\right]^4 -$$

$$- 271950 \left[\Delta^{','''''}\right]^2 \left[\Delta^{',''''}\right]^3 - 2364600 \,\Delta^{','''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^3\Big) f_1'''\Big) f_1' f_1' +$$

$$+ \Big(\Big(34044300 \left[\Delta^{'',''''}\right]^2 \left[\Delta^{',''''}\right]^3 + 108675 \left[\Delta^{','''''}\right]^2 \left[\Delta^{',''''}\right]^3 - 231525 \left[\Delta^{',''''''}\right]^2 \Delta^{',''''} \left[\Delta^{','''}\right]^2 +$$

$$+ 278640 \,\Delta^{',''''''} \left[\Delta^{','''''}\right]^2 \left[\Delta^{','''}\right]^2 + 4458240 \,\Delta^{',''''''} \left[\Delta^{'',''''}\right]^2 \left[\Delta^{','''}\right]^2 - 5788125 \left[\Delta^{'','''''}\right]^2 \Delta^{',''''} \left[\Delta^{','''}\right]^2 +$$

$$+ 1393200 \,\Delta^{'','''''} \left[\Delta^{','''''}\right]^2 \left[\Delta^{','''}\right]^2 + 22291200 \,\Delta^{'','''''} \left[\Delta^{'',''''}\right]^2 \left[\Delta^{','''}\right]^2 - 1396080 \left[\Delta^{','''''}\right]^3 \Delta^{',''''} \Delta^{','''} +$$

$$+ 14733900 \,\Delta^{','''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^3 - 44766720 \left[\Delta^{'',''''}\right]^3 \Delta^{',''''} \Delta^{','''} - 1284000 \,\Delta^{',''''''} \left[\Delta^{',''''}\right]^4 -$$

$$- 6420000 \,\Delta^{'','''''} \left[\Delta^{',''''}\right]^4 - 2315250 \,\Delta^{',''''''} \Delta^{'','''''} \Delta^{',''''} \left[\Delta^{','''}\right]^2 + 2229120 \,\Delta^{',''''''} \Delta^{','''''} \Delta^{'',''''} \left[\Delta^{','''}\right]^2 +$$

$$+ 1386450 \,\Delta^{',''''''} \Delta^{','''''} \left[\Delta^{',''''}\right]^2 \Delta^{','''} + 915300 \,\Delta^{',''''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^2 \Delta^{','''} + 11145600 \,\Delta^{'','''''} \Delta^{','''''} \Delta^{'',''''} \left[\Delta^{','''}\right]^2 +$$

$$+ 6932250 \,\Delta^{'','''''} \Delta^{','''''} \left[\Delta^{',''''}\right]^2 \Delta^{','''} + 4576500 \,\Delta^{'','''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^2 \Delta^{','''} - 13966560 \left[\Delta^{','''''}\right]^2 \Delta^{'',''''} \Delta^{',''''} \Delta^{','''} -$$

$$- 44720640 \,\Delta^{','''''} \left[\Delta^{'',''''}\right]^2 \Delta^{',''''} \Delta^{','''}\Big) f_1'' f_1'' + \Big(2268000 \,\Delta^{','''''} \left[\Delta^{',''''}\right]^4 + 792000 \,\Delta^{'',''''} \left[\Delta^{',''''}\right]^4\Big) f_1'' f_1''' -$$

$$- 1120000 \left[\Delta^{',''''}\right]^5 f_1''' f_1'''\Big) f_1' +$$

$$+ \Big(- 4630500 \,\Delta^{',''''''} \Delta^{'',''''} \Delta^{',''''} \left[\Delta^{','''}\right]^2 + 2058000 \,\Delta^{',''''''} \left[\Delta^{',''''}\right]^3 \Delta^{','''} - 11576250 \,\Delta^{'','''''} \Delta^{','''''} \Delta^{',''''} \left[\Delta^{','''}\right]^2 -$$

$$- 23152500 \,\Delta^{'','''''} \Delta^{'',''''} \Delta^{',''''} \left[\Delta^{','''}\right]^2 + 10290000 \,\Delta^{'','''''} \left[\Delta^{',''''}\right]^3 \Delta^{','''} + 2880 \left[\Delta^{','''''}\right]^3 \left[\Delta^{','''}\right]^2 +$$

$$+ 34560 \left[\Delta^{','''''}\right]^2 \Delta^{'',''''} \left[\Delta^{','''}\right]^2 + 5773725 \left[\Delta^{','''''}\right]^2 \left[\Delta^{',''''}\right]^2 \Delta^{','''} + 138240 \,\Delta^{','''''} \left[\Delta^{'',''''}\right]^2 \left[\Delta^{','''}\right]^2 +$$

$$+ 231525 \left[\Delta^{',''''''}\right]^2 \left[\Delta^{','''}\right]^3 + 2315250 \,\Delta^{',''''''} \Delta^{'','''''} \left[\Delta^{','''}\right]^3 - 2315250 \,\Delta^{',''''''} \Delta^{','''''} \Delta^{',''''} \left[\Delta^{','''}\right]^2 +$$

$$+ 22922100 \left[\Delta^{'',''''}\right]^2 \left[\Delta^{',''''}\right]^2 \Delta^{','''} - 20484000 \,\Delta^{'',''''} \left[\Delta^{',''''}\right]^4 + 5788125 \left[\Delta^{'','''''}\right]^2 \left[\Delta^{','''}\right]^3 -$$

$$- 10266000 \,\Delta^{','''''} \left[\Delta^{',''''}\right]^4 + 184320 \left[\Delta^{'',''''}\right]^3 \left[\Delta^{','''}\right]^2 + 23037300 \,\Delta^{','''''} \Delta^{'',''''} \left[\Delta^{',''''}\right]^2 \Delta^{','''}\Big) f_1'' f_1'' f_1'' +$$

$$+ 4560000 \left[\Delta^{',''''}\right]^5 f_1'' f_1'' f_1'''.$$

It will be a theorem, to be established in §10 below, that the 17 mutually independent bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14}, $F_{1,1}^{16}$, X^{18}, X^{19}, X^{21}, X^{23}, X^{25} and Y^{27} generate the algebra UE_5^2.

§9. Principle of the general algorithm

Initializing the algorithm. We now explain a general algorithm which generates all bi-invariants, which stops after a finite number of steps if and only if the algebra of bi-invariants is finitely generated and which, in such a circumstance, yields a complete generating family of mutually independent bi-invariants together with a complete generating family of syzygies between these bi-invariants. The same algorithm would work equally well for Demailly-Semple invariants, but as we already observed, in the desired applications, the complexity and the cardinality of generators and of syzygies being much higher, only the exploration of bi-invariants seems accessible.

Fix the dimension n and the jet order κ, both arbitrary. Start from the representation of an arbitrary bi-invariant of weight m gained previously thanks to the proposition on p. 47:

$$\mathsf{P} = \mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}\big(L^{l_1}, \ldots, L^{l_{k_1}}\big),$$

where the L^{l_i}, $i = 1, \ldots, k_1$, have weight l_i and come from the Λ-minors written there, after a division by an appropriate maximal factoring power of f_1', *cf.* the two special cases analyzed after the general proposition. Call $f_1', L^{l_1}, \ldots, L^{l_{k_1}}$ the *initial bi-invariants*.

First loop of the algorithm. The first step of the algorithm consists in computing a reduced Gröbner basis (for a certain monomial order) of the ideal of relations of the restrictions to $\{f_1' = 0\}$ of these initial bi-invariants:

$$\textsf{Ideal-Rel}\Big(L^{l_1}\big|_0, \, \ldots, \, L^{l_{k_1}}\big|_0\Big).$$

In some favorables circumstances, this task may be done by symbolic Gröbner bases packages, although it is well known that due to exponentiality of time computation and to expression swelling, Gröbner bases often appear to be frustratingly unusable. Write as follows the so obtained gröbnerized syzygies:

$$0 \equiv \mathsf{S}_i\Big(L^{l_1}(j^\kappa f)\big|_0, \, \ldots, \, L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad (i = 1 \cdots N_1).$$

At first, we claim that, without loss of generality, one may assume that each syzygy polynomial S_i is *weighted homogeneous*, say of weight μ_i, namely satisfies:

$$\mathsf{S}_i\big(\delta^{l_1} A_1, \, \ldots, \, \delta^{l_{k_1}} A_{k_1}\big) = \delta^{\mu_i} \, \mathsf{S}_i\big(A_1, \, \ldots, \, A_{k_1}\big),$$

in $\mathbb{C}\big[A_1, \ldots, A_{k_1}\big]$ for every weighted dilation factor $\delta \in \mathbb{C}$. Indeed, dilating $j^\kappa f$ as usual:

$$\delta \cdot j^\kappa f := \big(\delta^\lambda f_i^{(\lambda)}\big)_{1 \leqslant i \leqslant n}^{1 \leqslant \lambda \leqslant \kappa},$$

since the syzygies hold for any collection of $n\kappa$ components $\big(f_i^{(\lambda)}\big)_{1 \leqslant i \leqslant n}^{1 \leqslant \lambda \leqslant \kappa}$ in the jet space, they must hold too with $\big(\delta^\lambda f_i^{(\lambda)}\big)_{1 \leqslant i \leqslant n}^{1 \leqslant \lambda \leqslant \kappa}$, namely:

$$\begin{aligned} 0 &\equiv \mathsf{S}_i\Big(L^{l_1}(\delta \cdot j^\kappa f)\big|_0, \, \ldots, \, L^{l_{k_1}}(\delta \cdot j^\kappa f)\big|_0\Big) \\ &= \mathsf{S}_i\Big(\delta^{l_1} \, L^{l_1}(j^\kappa f)\big|_0, \, \ldots, \, \delta^{l_{k_1}} \, L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad (i = 1 \cdots N_1), \end{aligned}$$

and we may use the fact that the L^{l_i} are invariant under reparametrization. Therefore, if we gather together, in each syzygy polynomial S_i, all terms which have equal, constant weight μ:

$$\mathsf{S}_i = \sum_\mu \mathsf{S}_i^\mu, \qquad \text{with} \qquad \mathsf{S}_i^\mu\big(\delta^{l_1} A_1, \, \ldots, \, \delta^{l_{k_1}} A_{k_1}\big) = \delta^\mu \, \mathsf{S}_i^\mu\big(A_1, \, \ldots, \, A_{k_1}\big),$$

we may expand according to weight the obtained relations under the specific form:

$$0 \equiv \sum_\mu \delta^\mu \, \mathsf{S}_i^\mu\Big(L^{l_1}(j^\kappa f)\big|_0, \, \dots, \, L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad\qquad {\scriptstyle (i=1\cdots N_1)}.$$

Because these identities then hold in $\mathbb{C}\big[\delta, \, j^\kappa f\big]$, they are equivalent to the (possibly larger) collection of *constantly weighted* syzygies:

$$0 \equiv \mathsf{S}_i^\mu\Big(L^{l_1}(j^\kappa f)\big|_0, \, \dots, \, L^{l_{k_1}}(j^\kappa f)\Big) \qquad\qquad {\scriptstyle (i=1\cdots N_1\,;\,\forall\,\mu)},$$

and this justifies the claim.

So let μ_i be the weight of the (homogeneous) syzygy S_i, for $i = 1, \dots, N_1$. Because by assumption each polynomial $\mathsf{S}_i\big(L^{l_1}(j^\kappa f), \dots, L^{l_{k_1}}(j^\kappa f)\big)$ vanishes identically in $\mathbb{C}\big[j^\kappa f\big]$ after setting $f_1' = 0$, there are maximal factoring powers $(f_1')^{\nu_i}$ of f_1', with $1 \leqslant \nu_i \leqslant \infty$, and there are certain (possibly zero) polynomial remainders $\mathsf{R}_i\big(j^\kappa f\big)$ such that we may write in $\mathbb{C}\big[j^\kappa f\big]$:

$$\mathsf{S}_i\big(L^{l_1}, \, \dots, \, L^{l_{k_1}}\big) = (f_1')^{\nu_i} \, \mathsf{R}_i\big(j^\kappa f\big) \qquad\qquad {\scriptstyle (i=1\cdots N_1)},$$

with $\mathsf{R}_i \not\equiv 0$ when $1 \leqslant \nu_i < \infty$ and with $\mathsf{R}_i = 0$ by convention when $\nu_i = \infty$.

We claim that each such $\mathsf{R}_i\big(j^\kappa f\big)$ is then a bi-invariant. In fact, it is a polynomial by definition, and its representation as a quotient:

$$\mathsf{R}_i\big(j^\kappa f\big) = \frac{\mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big)}{(f_1')^{\nu_i}}$$

of two polynomials invariant by reparametrizations and invariant under the unipotent action shows at once that R_i too enjoys bi-invariancy.

The second step of the algorithm consists in testing, for each i, whether or not R_i belongs to the algebra $\mathbb{C}\big[f_1', L^{l_1}, \dots, L^{l_{k_1}}\big]$ generated by the initial bi-invariants. In the case where no new bi-invariant appears, the algorithm will be shown to terminate, so let us assume that at least one R_i provides a new bi-invariant, independent of $f_1', L^{l_1}, \dots, L^{l_{k_1}}$. It is then clear that after renumbering the R_i if necessary, one may assume that:

$$\begin{cases} \mathsf{R}_1 \quad \text{is independent of} \;\; f_1', L^{l_1}, \dots, L^{l_{k_1}}, \\ \mathsf{R}_2 \quad \text{is independent of} \;\; f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \\ \cdots\cdots\cdots \\ \mathsf{R}_{k_2} \quad \text{is independent of} \;\; f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \dots, \mathsf{R}_{k_2-1}, \end{cases}$$

while for the next indices $i = k_2 + 1, \dots, N_1$:

$$\Big\{ \mathsf{R}_i \quad \text{belongs to the algebra} \;\; \mathbb{C}\big[f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \dots, \mathsf{R}_{k_2}\big].$$

Denoting instead by $M^{m_1}, \dots, M^{m_{k_2}}$ these R_i for $i = 1, \dots, k_2$ which provide new mutually independent bi-invariants, where as usual the weights $m_i := \mu_i - \nu_i$, for $i = 1, \dots, k_2$ are put in exponent place, we can therefore write down in more explicit form the filled syzygy polynomials (without setting $f_1' = 0$):

$$\begin{cases} 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, M^{m_i} & {\scriptstyle (i=1\cdots k_2)}, \\ 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, \mathsf{R}_i\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}\big) & {\scriptstyle (i=k_2+1\cdots N_1)}, \end{cases}$$

from which we recover at once, by setting f_1', the original syzygies:

$$0 \equiv \mathsf{S}_i\big(L^{l_1}\big|_0, \, \dots, \, L^{l_{k_1}}\big|_0\big) \qquad {\scriptstyle (i=1\cdots N_1)}.$$

So the equations above, when written explicitly in specific applications below, shall show both the collection of new appearing bi-invariants $M^{m_1}, \dots, M^{m_{k_2}}$ (without setting $f_1' = 0$) and (after setting f_1') a reduced Gröbner basis for the ideal of relations between the initial bi-invariants $L^{l_1}\big|_0, \dots, L^{L_{k_1}}\big|_0$.

Second and further loops of the algorithm. Next, we restart the process with the new, larger collection of bi-invariants, namely we compute a reduced Gröbner basis (for a certain monomial order compatible with the preceding loop):

$$\textsf{Ideal-Rel}\Big(L^{l_1}\big|_0, \, \dots, \, L^{l_{k_1}}\big|_0, \; M^{m_1}\big|_0, \, \dots, \, M^{m_{k_2}}\big|_0\Big).$$

Write as follows the so obtained gröbnerized syzygies, after filling the remainders behind a power of f_1' and after testing whether these remainders provide new bi-invariants:

$$\begin{cases} 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, M^{m_i} & {\scriptstyle (i=1\cdots k_2)}, \\ 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, \mathsf{R}_i\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}\big) & {\scriptstyle (i=k_2+1\cdots N_1)}, \\ 0 \equiv \mathsf{T}_j\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}\big) + (f_1')^{\nu_j}\, N^{n_j} & {\scriptstyle (j=1\cdots k_3)}, \\ 0 \equiv \mathsf{T}_j\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}\big) + \\ \qquad + (f_1')^{\nu_j}\, R_j\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}, N^{n_1}, \dots, N^{n_{k_3}}\big) & {\scriptstyle (j=k_3+1\cdots N_2)}. \end{cases}$$

with $N^{n_1}, \dots, N^{n_{k_3}}$ denoting the new appearing bi-invariants, of weight $n_1, \dots, n_{k_3}$.

Successively, continue to perform further loops as long as new bi-invariants appear which do not belong to the algebra generated by already known bi-invariants.

Termination of the algorithm. Either there always appear new bi-invariants or, after a finite number of loops, we come to a situation which falls under the scope of the following important statement.

THEOREM *For a certain dimension n and for a certain jet order κ, suppose that, after performing a finite number of loops of the algorithm, one possesses a finite number $1 + M$ of mutually independent bi-invariants f_1', $\Lambda^{\ell_1}, \dots, \Lambda^{\ell_M} \in \mathbb{C}\big[j^\kappa f_1, \dots, j^\kappa f_n\big]$ of weights $1, \ell_1, \dots, \ell_M$ belonging to UE_κ^n, whose restrictions to $\{f_1' = 0\}$ share an ideal of relations:*

$$\textsf{Ideal-Rel}\Big(\Lambda^{\ell_1}\big|_0, \;\; \dots\dots, \, \Lambda^{\ell_M}\big|_0 \Big)$$

generated by a finite number N (often large) of homogeneous syzygies:

$$0 \equiv \mathsf{S}_i\big(\Lambda^{\ell_1}\big|_0, \, \dots, \, \Lambda^{\ell_M}\big|_0\big), \qquad {\scriptstyle (i=1\cdots N)}$$

of weight μ_i assumed to be represented by a certain reduced Gröbner basis $\big\langle \mathsf{S}_i \big\rangle_{1\leqslant i\leqslant N}$ for a certain monomial order, with the crucial property that no *new bi-invariant appears behind f_1', namely with the property that, without setting $f_1'=0$, one has N identically satisfied relations:*

$$0 \equiv \mathsf{S}_i\big(\Lambda^{\ell_1},\,\dots,\,\Lambda^{\ell_M}\big) - f_1'\,\mathsf{R}_i\big(f_1',\,\Lambda^{\ell_1},\,\dots,\,\Lambda^{\ell_M}\big) \qquad\qquad {\scriptstyle (i=1\,\cdots\,N)},$$

for some remainders R_i which all depend polynomially upon the same collection of invariants $f_1', \Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}$*, so that no new bi-invariant appears at this stage.*

Then the algorithm terminates and the algebra of bi-invariants coincides with:

$$\boxed{\mathsf{UE}_\kappa^n = \mathbb{C}\big[f_1',\,\Lambda^{\ell_1},\,\dots\dots,\,\Lambda^{\ell_M}\big] \quad \text{modulo syzygies}}\,.$$

In addition, for these values of n and of κ, if one denotes the leading terms of the syzygies by:

$$\mathsf{LT}\big(\mathsf{S}_i(\Lambda)\big) = \big(\Lambda^{\ell_1}\big)^{\alpha_1^i}\cdots\big(\Lambda^{\ell_M}\big)^{\alpha_M^i} \qquad\qquad {\scriptstyle (i=1\,\cdots\,N)},$$

for certain specific multiindices $\big(\alpha_1^i,\dots,\alpha_M^i\big)\in\mathbb{N}^M$, and if for $i=1,\dots,N$ one denotes by:

$$\square_i := \alpha^i + \mathbb{N}^M = \big\{\big(\alpha_1^i+b_1,\,\dots,\,\alpha_M^i+b_M\big) :\, b_1,\dots,b_M\in\mathbb{N}^M\big\}$$

the positive quadrant of $\mathbb{N}^M$ having vertex at α^i, then a general, arbitrary bi-invariant in $\mathsf{UE}_{\kappa,m}^n$ of weight m writes uniquely *under the* normal form*:*

$$\sum_{0\leqslant a\leqslant m}(f_1')^a\,\widetilde{\mathsf{P}}_a\big(\Lambda^{\ell_1},\,\dots,\,\Lambda^{\ell_M}\big),$$

with summation containing only positive powers *of f_1', where each $\widetilde{P}_a$ is of weight $m-a$ and is put under* Gröbner-normalized form*:*

$$\boxed{\widetilde{\mathsf{P}}_a = \sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\setminus(\square_1\cup\cdots\cup\square_N)\\ \ell_1b_1+\cdots+\ell_Mb_M=m-a}} \mathsf{coeff}_{a;\,b_1,\dots,b_M}\cdot\big(\Lambda^{\ell_1}\big)^{b_1}\cdots\big(\Lambda^{\ell_M}\big)^{b_M}},$$

with complex coefficients $\mathsf{coeff}_{a;\,b_1,\dots,b_M}$ subjected to no restriction at all.

Proof. We start with the list of initial bi-invariants $f_1', L^{l_1},\dots,L^{l_{k_1}}$ and with the initial, rational representation of an arbitrary bi-invariant $\mathsf{P}\big(j^\kappa f\big)\in\mathsf{UE}_{\kappa,m}^n$ which was obtained previously:

$$\begin{aligned}\mathsf{P}\big(j^\kappa f\big) &= \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m}(f_1')^a\,\mathsf{P}_a\big(L^{l_1},\dots,L^{l_{k_1}}\big)\\ &= (f_1')^{a_0}\,\mathsf{P}_{a_0} + \sum_{a_0+1\leqslant a\leqslant m}(f_1')^a\,\mathsf{P}_a,\end{aligned}$$

and we denote by a_0 the smallest appearing exponent of f_1'. Clearly, the final list of bi-invariants $\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}$ stabilized after a finite number of loops of the algorithm contains $L^{l_1},\dots,L^{l_{k_1}}$ as its first k_1 terms. Working in the polynomial ring $\mathbb{C}\big[A^1, \dots, A^{k_1}, \dots, A^M\big]$, we may then divide P_{a_0} by the ideal of relations $\big\langle \mathsf{S}_i(A)\big\rangle_{1 \leqslant i \leqslant N}$:

$$\mathsf{P}_{a_0}\big(A^1, \dots, A^{k_1}\big) = \widetilde{\mathsf{P}}_{a_0}\big(A^1, \dots, A^{k_1}, \dots, A^M\big) + \sum_{i=1}^N q_i(A) \cdot \mathsf{S}_i(A),$$

with multiplicands $q_i(A)$ of weight $m - a_0 - \mu_i$, getting a remainder $\widetilde{\mathsf{P}}_{a_0}$ of weight $m - a_0$ which in general will depend upon all the variables $A^1, \dots, A^{k_1}, \dots, A^M$ and which is *unique* (while the multiplicands q_i cannot be unique, as soon as $N \geqslant 2$), by virtue of a classical feature of Gröbner bases. Consequently, replacing the independent variables A^l by the bi-invariants in the arguments and then substituting each $\mathsf{S}_i(\Lambda)$ by $f_1' \, \mathsf{R}_i(f_1', \Lambda)$ — thanks to the main assumption that in filled syzygies, all the remainders behind f_1' depend polynomially upon the same bi-invariants $f_1', \Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}$ —, we then get a normalized representation of $\widetilde{\mathsf{P}}_{a_0}$:

$$\begin{aligned}
\mathsf{P}_{a_0}\big(L^{l_1}, \dots, L^{l_{k_1}}\big) &= \widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}\big) + \sum_{i=1}^N q_i(\Lambda) \cdot \mathsf{S}_i(\Lambda) \\
&= \widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}\big) + \sum_{i=1}^N q_i(\Lambda) \cdot f_1' \, \mathsf{R}_i\big(f_1', \Lambda\big) \\
&= \widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}\big) + f_1' \, \widetilde{\mathsf{R}}_{a_0}\big(f_1', \, \Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}\big),
\end{aligned}$$

(modulo an uncontrolled remainder $\widetilde{\mathsf{R}}_{a_0}$ which hopefully, lies behind f_1') which we may therefore inject in our rational representation:

$$\mathsf{P}\big(j^\kappa f\big) = (f_1')^{a_0} \, \widetilde{\mathsf{P}}_{a_0}(\Lambda) + (f_1')^{a_0+1} \, \widetilde{\mathsf{R}}_{a_0}\big(f_1', \, \Lambda\big) + \sum_{a_0+1 \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a(L).$$

But both $\widetilde{\mathsf{P}}_{a_0}$ and $f_1' \, \widetilde{\mathsf{R}}_{a_0}$ being of weight $m - a_0$ as was P_{a_0}, it follows that, when developping the perturbing term $(f_1')^{a_0+1} \, \widetilde{\mathsf{R}}_{a_0}\big(f_1', \Lambda\big)$ in powers of f_1', the fact that this remainder is of weight m guarantees that the sum does not go beyond $(f_1')^m$, and thus, we come to an expression:

$$\mathsf{P}\big(j^\kappa f\big) = (f_1')^{a_0} \, \widetilde{\mathsf{P}}_{a_0}(\Lambda) + \sum_{a_0+1 \leqslant a \leqslant m} (f_1')^a \, \mathsf{Q}_a(\Lambda)$$

entirely similar to the one we started with, whose first term:

$$\widetilde{\mathsf{P}}_{a_0} = \sum_{\substack{(b_1, \dots, b_M) \in \mathbb{N}^M \backslash (\square_1 \cup \dots \cup \square_N) \\ \ell_1 b_1 + \dots + \ell_M b_M = m - a_0}} \mathsf{coeff}_{a_0; \, b_1, \dots, b_M} \cdot \big(\Lambda^{\ell_1}\big)^{b_1} \cdots \big(\Lambda^{\ell_M}\big)^{b_M},$$

is normalized modulo the syzygies. But we can then subject the next term $\mathsf{Q}_{a_0+1}(\Lambda)$ to the same process, and consequently by induction, after a finite number of steps, we come to an expression in which *all* multiplicands of a power of f_1' have been normalized:

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{a_0' \leqslant a \leqslant m} (f_1')^a \sum_{\substack{(b_1, \dots, b_M) \in \mathbb{N}^M \backslash (\square_1 \cup \dots \cup \square_N) \\ \ell_1 b_1 + \dots + \ell_M b_M = m - a}} \mathsf{coeff}_{a; \, b_1, \dots, b_M} \cdot \big(\Lambda^{\ell_1}\big)^{b_1} \cdots \big(\Lambda^{\ell_M}\big)^{b_M},$$

with a possibly larger a_0' (in case $\widetilde{P}_{a_0}$ vanishes identically). However, the smallest a_0 in the initial expression for $\mathsf{P}\big(j^\kappa f\big)$ was possibly negative and hence our a_0' here can still be negative too, and our gained representation of $\mathsf{P}\big(j^\kappa f\big)$ can still be not polynomial.

Hopefully, we may now claim that there are no negative powers of f_1' anymore in such a normalized expression, so that the right hand side is a true polynomial.

Indeed, suppose that $a_0' < 0$ with $\widetilde{P}_{a_0'} \not\equiv 0$. Multiply both sides by $(f_1')^{-a_0'}$, set afterwards $f_1' = 0$ and then get in such a way a nontrivial identity:

$$0 \equiv \sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\backslash(\square_1\cup\cdots\cup\square_N)\\ \ell_1 b_1+\cdots+\ell_M b_M = m-a_0'}} \mathsf{coeff}_{a_0';\,b_1,\dots,b_M} \cdot \big(\Lambda^{\ell_1}\big|_0\big)^{b_1}\cdots\big(\Lambda^{\ell_M}\big|_0\big)^{b_M}.$$

This equation would then represent a syzygy between bi-invariants restricted to $\{f_1' = 0\}$ whose leading term is strictly smaller than the leadings terms of the syzygies S_i. This would contradict the assumption that the collection $\big\langle \mathsf{S}_i\big\rangle_{1\leqslant i\leqslant N}$ is a Gröbner basis for the ideal of relations between $\Lambda^{\ell_1}\big|_0,\dots,\Lambda^{\ell_M}\big|_0$. So $a_0' \geqslant 0$, namely the normalized representation is polynomial.

The same argument shows that the normalized representation is unique.

Finally, it suffices to say, if not remarked stealthily before, that any polynomial in $f_1',\ \Lambda^{\ell_1},\dots,\Lambda^{\ell_M}$ obviously is a bi-invariant. The proof is now complete. $\square$

§10. Seventeen bi-invariant generators in dimension $n = 2$ for jet level $\kappa = 5$

First loop of the algorithm. According to these general principles, in the case $n = 2$, $\kappa = 5$, we should therefore start with the initial rational representation:

$$\mathsf{P}\big(j^5 f\big) = \sum_{-\frac{4}{5}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\big(\Lambda^3,\,\Lambda^5,\,\Lambda^7,\,\Lambda^9\big)$$

of an arbitrary bi-invariant $\mathsf{P}\big(j^5 f\big) \in \mathsf{UE}_5^2$. Here for simplicity, we shall denote without any lower index each one of the appearing bi-invariants. In fact, among all the invariants explicitly defined in the theorem on p. 28, bi-invariants correspond to lower indices being constantly equal to 1, and one has also to consider the *non-bracket* bi-invariants introduced in §8.

So according to the general algorithm, we have to start by computing the ideal of relations:

$$\mathsf{Ideal\text{-}Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ \Lambda^9\big|_0\Big).$$

For this easy first step, we may use any Gröbner bases package[15]. For the Reverse Degree Lexicographic Ordering, the result provided is:

$$\begin{aligned}
0 &\equiv -7\,\Lambda^7\big|_0\Lambda^7\big|_0 + 5\,\Lambda^5\big|_0\Lambda^9\big|_0,\\
0 &\equiv -7\,\Lambda^5\big|_0\Lambda^7\big|_0 + 3\,\Lambda^3\big|_0\Lambda^9\big|_0,\\
0 &\equiv -5\,\Lambda^5\big|_0\Lambda^5\big|_0 + 3\,\Lambda^3\big|_0\Lambda^7\big|_0.
\end{aligned}$$

[15] *See* dim-2-order-5-step-1-with-FGb.mw at [23].

Then we compute the remainder bi-invariants appearing behind a power of f_1'. Here, for the three syzygies, the maximal factoring power of f_1' is the same, equal to 2, and three new bi-invariants appear:

$$
\begin{aligned}
0 &\equiv -7\,\Lambda^7\Lambda^7 + 5\,\Lambda^5\Lambda^9 - f_1'f_1'K^{12},\\
0 &\equiv -7\,\Lambda^5\Lambda^7 + 3\,\Lambda^3\Lambda^9 - f_1'f_1'M^{10},\\
0 &\equiv -5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7 - f_1'f_1'M^{8},
\end{aligned}
$$

namely: M^8, M^{10} and K^{12}. Either looking at the syzygies of the second loop below, or computing directly by hand, or playing a bit with Maple, we find the values of the restrictions to $\{f_1' = 0\}$ of all the bi-invariants obtained so far, expressed in (rational) terms of the three restricted bi-invariants, $\Lambda^3|_0$, $\Lambda^5|_0$ and $M^8|_0$ which are easily checked to be algebraically independent:

$$
\begin{aligned}
&\underline{\Lambda^3}\big|_0\\
&\underline{\Lambda^5}\big|_0\\
&\Lambda^7\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0}{\Lambda^3|_0},\\
&\Lambda^9\big|_0 = \tfrac{35}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0}{\Lambda^3|_0\,\Lambda^3|_0},\\
&M^8\big|_0\\
&M^{10}\big|_0 = \tfrac{8}{3}\,\tfrac{\Lambda^5|_0\,M^8|_0}{\Lambda^3|_0},\\
&K^{12}\big|_0 = \tfrac{5}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}
$$

Proceeding then as in the lemma on p. 38 and using these rational expressions, one may establish that the 8 bi-invariants known so far, namely f_1', Λ^3, Λ^5, Λ^7, Λ^9, M^8, M^{10} and K^{12}, are mutually independent.

Second loop of the algorithm. Afterwards, we must compute the ideal of relations between the 7 restricted bi-invariants in question:

$$
\mathsf{Ideal\text{-}Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ \Lambda^9\big|_0,\ M^8\big|_0,\ M^{10}\big|_0,\ K^{12}\big|_0\Big).
$$

For the Degree Reverse Lexicographic Ordering, a Gröbner basis for this ideal of relations consists of the following 10 polynomials[16] (in which the remainders behind a power of f_1' have already been filled):

$$
\begin{aligned}
0 &\equiv -5\,M^{10}M^{10} + 64\,M^8K^{12} - f_1'X^{19},\\
0 &\equiv -5\,\Lambda^9M^{10} + 56\,\Lambda^7K^{12} - f_1'X^{18},\\
0 &\equiv -8\,\Lambda^9M^8 + 7\,\Lambda^7M^{10} - f_1'F^{16},\\
0 &\equiv -\Lambda^9M^8 + 7\,\Lambda^5K^{12} - f_1'F^{16},\\
0 &\equiv -8\,\Lambda^7M^8 + 5\,\Lambda^5M^{10} - f_1'H^{14},
\end{aligned}
$$

16 *See* dim-2-order-5-step-2-with-FGb.mw at [23].

$$0 \equiv -\Lambda^7 M^8 + 3\,\Lambda^3 K^{12} - f_1' H^{14},$$
$$0 \equiv -8\,\Lambda^5 M^8 + 3\,\Lambda^3 M^{10} - f_1' N^{12},$$
$$0 \equiv -7\,\Lambda^7 \Lambda^7 + 5\,\Lambda^5 \Lambda^9 - f_1' f_1' K^{12},$$
$$0 \equiv -7\,\Lambda^5 \Lambda^7 + 3\,\Lambda^3 \Lambda^9 - f_1' f_1' M^{10},$$
$$0 \equiv -5\,\Lambda^5 \Lambda^5 + 3\,\Lambda^3 \Lambda^7 - f_1' f_1' M^8.$$

How exactly do we manage to fill in what appears at the end of each syzygy behind any power of f_1'?

A standard obstacle: unavailability because of size computations. A natural idea would be to automatically apply the *Algebra Membership Algorithm* based on Gröbner bases ([14], p. 289), but this would be (at least for us) impossible, because this test would rely upon the (unavalaible to us) knowledge of a full Gröbner basis for the ideal generated by the 8 equations:

$$t_1 - f_1',\ l_3 - \Lambda^3,\ l_5 - \Lambda^5,\ l_7 - \Lambda^7,\ l_9 - \Lambda^9,\ m_8 - M^8,\ m_{10} - M^{10},\ k_{12} - K^{12},$$

in the ring of 18 variables:

$$\mathbb{C}\big[j^5 f_1, j^5 f_2,\ t_1, l_3, l_5, l_7, l_9, m_8, m_{10}, k_{12}\big]$$

with any monomial ordering having the only property that each jet variable $f_i^{(\lambda)}$ is bigger than any monomial written with only the 8 auxiliary variables $t_1, l_3, l_5, l_7, l_9, m_8, m_{10}, k_{12}$. Indeed, according to Proposition C.2.3 in the reference cited, any remainder behind a power of f_1', for instance the one appearing in the sixth syzygy above:

$$\mathsf{rem}_6 := \frac{1}{f_1'}\big(8\,\Lambda^5 M^8 - 3\,\Lambda^3 M^{10}\big),$$

would then belong to the algebra generated by the 8 already known bi-invariants: f_1', Λ^3, Λ^5, Λ^7, Λ^9, M^8, M^{10}, K^{12}, if and only if the *normal form* of rem_6 with respect to such a Gröbner basis would belong to $\mathbb{C}\big[t_1, \dots, k_{12}\big]$, and in such a case, the (unique) normal form in question rem_6 would provide without any further effort the corresponding polynomial.

However, Gröbner bases here are blocked due to oversizeness

Hence to bypass such a (usual, forseeable) drawback of Gröbner bases, we have to proceed differently.

What we do using Maple is a little bit tricky, and it works well. After division by f_1' (most often, and sometimes also by $(f_1')^2$, but never by $(f_1')^3$), we start by computing each one of the 10 remainder; in fact, since 3 of them were already treated in the first loop, only 7 remainders have to be studied here. On the other hand and as an independent preparation, we may check by inspecting the explicit expressions given at the end of §4, that $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $M^8\big|_0$ and $N^{12}\big|_0$ (our rem_6 itself!) are mutually algebraically independent. Subsequently, we compute a Gröbner basis for the four polynomial:

$$l_{30} - \Lambda^3\big|_0,\ l_{50} - \Lambda^5\big|_0,\ m_{80} - M^8\big|_0,\ n_{120} - N^{12}\big|_0,$$

in the ring $\mathbb{C}\big[j^5 f_1|_0, j^5 f_2, l_{30}, l_{50}, m_{80}, n_{120}\big]$, where l_{30}, l_{50}, m_{80} and n_{120} denote auxiliary, supplementary variables, with any monomial order having the property that each jet variable $f_i^{(\lambda)}$ is bigger than any monomial written with only the 4 auxiliary variables l_{30}, l_{50}, m_{80} and n_{120}. This then is available to the computer: size is reasonable and it costs less than 5 minutes on any computer. Then we set $f_1' = 0$ in each remainder rem_k, getting $\mathsf{rem}_k\big|_0$. We then multiply each restricted remainder $\mathsf{rem}_k\big|_0$ for $k = 1, 2, \ldots, 10$ by a suitable power of $\Lambda^3\big|_0$ choosen by head, for instance if one looks at the third remainder:

$$\Lambda^3\big|_0\Lambda^3\big|_0 \cdot \mathsf{rem}_3\big|_0 = \Lambda^3\big|_0\Lambda^3\big|_0 \cdot \left[\frac{1}{f_1'}\big(8\,\Lambda^9 M^8 - 7\,\Lambda^7 M^{10}\big)\right]_{f_1'=0}.$$

Then we compute the normal form of this latter polynomial with respect to the mentioned auxiliary Gröbner basis. For instance, our computer yields for the third remainder the normal form:

$$\tfrac{35}{9}\, l_{50} l_{50} n_{120}.$$

This result therefore means that the third unknown remainder rem_3 (appearing in the third syzygy) which we denoted in advance by F^{16}, has the following value after setting $f_1' = 0$:

$$F^{16}\big|_0 = \tfrac{35}{9}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, N^{12}|_0}{\Lambda^3|_0\, \Lambda^3|_0}.$$

Then we test by hand and by head whether such a value for $f_1' = 0$ can be obtained as a polynomial in terms of the 7 previously known restricted bi-invariants $\Lambda^3|_0, \ldots, K^{12}|_0$. Here, it is easy to convince oneself that this cannot be the case, so that F^{16} really is a new bi-invariant.

On the other hand, we should do the same work for the fourth remainder rem_4. It then happens that we find the *same* value at $f_1' = 0$ in terms of $\Lambda^3|_0, \Lambda^5|_0, M^8|_0, N^{12}|_0$. So we suspect that *without setting* $f_1' = 0$, the two remainders rem_3 and rem_4 could be identical and finally, a simple computation with Maple verifies that this is indeed the case. Other remainders are computed similary, and we thus have fully explained all our trick to bypass the unavailability of full Gröbner bases due to oversizeness in this problem.

However, we would like to mention that achieving such a kind of task took hours and days of patience. Hopefully, checking *a posteriori* with Maple that a syzygy effectively holds is much, much more rapid and the reader will find in the Maple worksheets referenced here the declaration of new bi-invariants at each step and the checking (at a piece) of all syzygies by means of the basic “simplify” command of Maple.

Finally, to finish with the second loop, we give the values, restricted to $\{f_1' = 0\}$, of the 5 appearing new bi-invariants at this stage:

$$\begin{aligned}
&N^{12}\big|_0 \\
&H^{14}\big|_0 = \tfrac{5}{3}\,\frac{\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0}, \\
&F^{16}\big|_0 = \tfrac{35}{9}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \\
&X^{18}\big|_0 = \tfrac{1225}{27}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0}, \\
&X^{19}\big|_0 = \tfrac{80}{3}\,\frac{\Lambda^5|_0\,M^8|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}$$

Third loop of the algorithm. Now that we have explained how we proceed, we can offer directly the 32 filled syzygies appearing at the next step[17], again for the Degree Reverse Lexicographic Ordering.

$$\begin{aligned}
0 &\equiv -5\,F^{16}F^{16} + H^{14}X^{18} - f_1'K^{12}X^{19}, \\
0 &\equiv -7\,H^{14}F^{16} + N^{12}X^{18} - f_1'M^{10}X^{19}, \\
0 &\equiv -7\,H^{14}H^{14} + 5\,N^{12}F^{16} - f_1'M^8X^{19}, \\
0 &\equiv -56\,K^{12}F^{16} + M^{10}X^{18} - f_1'Y^{27},
\end{aligned}$$

$$\begin{aligned}
0 &\equiv -56K^{12}H^{14} + 5\,M^{10}F^{16} - f_1'X^{25}, \\
0 &\equiv -8\,K^{12}N^{12} + M^{10}H^{14} - f_1'X^{23}, \\
0 &\equiv -49\,K^{12}H^{14} + M^8X^{18} - f_1'X^{25}, \\
0 &\equiv -7\,K^{12}N^{12} + M^8F^{16} - f_1'X^{23}, \\
0 &\equiv -5\,M^{10}N^{12} + 8\,M^8H^{14} - f_1'X^{21},
\end{aligned}$$

$$\begin{aligned}
0 &\equiv -48\,K^{12}F^{16} + \Lambda^9X^{19} - f_1'Y^{27}, \\
0 &\equiv -48\,K^{12}H^{14} + \Lambda^7X^{19} - f_1'X^{25} \\
0 &\equiv -5\,\Lambda^9F^{16} + \Lambda^7X^{18} + 8f_1'K^{12}K^{12}, \\
0 &\equiv -\Lambda^9H^{14} + \Lambda^7F^{16} + f_1'M^{10}K^{12}, \\
0 &\equiv -5\,\Lambda^9N^{12} + 7\,\Lambda^7H^{14} + 56f_1'M^8K^{12} - f_1'f_1'X^{19}, \\
0 &\equiv -48K^{12}N^{12} + \Lambda^5X^{19} - 7\,f_1'X^{23},
\end{aligned}$$

[17] *See* dim-2-order-5-step-3-with-FGb.mw at [23].

$$
\begin{aligned}
0 &\equiv -7\,\Lambda^9 H^{14} + \Lambda^5 X^{18} + 8\, f_1' M^{10} K^{12},\\
0 &\equiv -\Lambda^9 N^{12} + \Lambda^5 F^{16} + f_1' M^{10} M^{10},\\
0 &\equiv -\Lambda^7 N^{12} + \Lambda^5 H^{14} + f_1' M^8 M^{10},\\
0 &\equiv -10\, M^{10} N^{12} + \Lambda^3 X^{19} - \tfrac{7}{3}\, f_1' X^{21},\\
0 &\equiv -35\,\Lambda^9 N^{12} + 3\,\Lambda^3 X^{18} + \tfrac{285}{8}\, f_1' M^{10} M^{10} - \tfrac{7}{8}\, f_1' f_1' X^{19},\\
0 &\equiv -7\,\Lambda^7 N^{12} + 3\,\Lambda^3 F^{16} + 8 f_1' M^8 M^{10},\\
0 &\equiv -5\,\Lambda^5 N^{12} + 3\,\Lambda^3 H^{14} + 8\, f_1' M^8 M^8,
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -5\, M^{10} M^{10} + 64\, M^8 K^{12} - f_1' X^{19},\\
0 &\equiv -5\,\Lambda^9 M^{10} + 56\,\Lambda^7 K^{12} - f_1' X^{18},\\
0 &\equiv -8\,\Lambda^9 M^8 + 7\,\Lambda^7 M^{10} - f_1' F^{16},\\
0 &\equiv -\Lambda^9 M^8 + 7\,\Lambda^5 K^{12} - f_1' F^{16},\\
0 &\equiv -8\,\Lambda^7 M^8 + 5\,\Lambda^5 M^{10} - f_1' H^{14},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -\Lambda^7 M^8 + 3\,\Lambda^3 K^{12} - f_1' H^{14},\\
0 &\equiv -8\,\Lambda^5 M^8 + 3\,\Lambda^3 M^{10} - f_1' N^{12},\\
0 &\equiv -7\,\Lambda^7 \Lambda^7 + 5\,\Lambda^5 \Lambda^9 - f_1' f_1' K^{12},\\
0 &\equiv -7\,\Lambda^5 \Lambda^7 + 3\,\Lambda^3 \Lambda^9 - f_1' f_1' M^{10},\\
0 &\equiv -5\,\Lambda^5 \Lambda^5 + 3\,\Lambda^3 \Lambda^7 - f_1' f_1' M^8.
\end{aligned}
$$

Here, 4 new bi-invariants appear:

$$
X^{21}, \qquad X^{23}, \qquad X^{25}, \qquad Y^{17}.
$$

Their values restricted to $\{f_1' = 0\}$ are:

$$
\begin{aligned}
X^{21}\big|_0 &= -\tfrac{5}{3}\, \tfrac{N^{12}|_0\, N^{12}|_0}{\Lambda^3|_0} - \tfrac{64}{3}\, \tfrac{M^8|_0\, M^8|_0\, M^8|_0}{\Lambda^3|_0},\\
X^{23}\big|_0 &= -\tfrac{35}{3}\, \tfrac{\Lambda^5|_0\, N^{12}|_0\, N^{12}|_0}{\Lambda^3|_0\, \Lambda^3|_0} - \tfrac{64}{9}\, \tfrac{\Lambda^5|_0\, M^8|_0\, M^8|_0\, M^8|_0}{\Lambda^3|_0\, \Lambda^3|_0},\\
X^{25}\big|_0 &= -\tfrac{1225}{27}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, N^{12}|_0\, N^{12}|_0}{\Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0} - \tfrac{320}{27}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, M^8|_0\, M^8|_0\, M^8|_0}{\Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0},\\
Y^{27}\big|_0 &= -\tfrac{8575}{81}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, \Lambda^5|_0\, N^{12}|_0\, N^{12}|_0}{\Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0} - \tfrac{320}{81}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, \Lambda^5|_0\, M^8|_0\, M^8|_0\, M^8|_0}{\Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0\, \Lambda^3|_0}.
\end{aligned}
$$

Fourth loop of the algorithm. The Gröbner basis of syzygies between the restriction to $\{f_1' = 0\}$ of the 17 bi-invariants known so far consists here of 105 equations. By an independent calculation, we checked that 39 among these 105 generators belong to the ideal of the 66 remaining ones. We could fill in the remainders behind a power of f_1'. To test whether there appear new bi-invariants, it is in fact useless to fill in the 39 left out remainders. Here are the 66 syzygies[18] in

[18] *See* dim-2-order-5-step-4-with-FGb.mw at [23].

question:

$$
\begin{aligned}
0 &\equiv X^{18}X^{23} - 8\,F^{16}X^{25} + 7\,H^{14}Y^{27} {+} 0,\\
0 &\equiv 5\,F^{16}X^{23} - 8\,H^{14}X^{25} + 5\,N^{12}Y^{27} {+} f_1'\,X^{19}X^{19},\\
0 &\equiv 7\,K^{12}X^{23} - M^{10}X^{25} + M^{8}Y^{27} {+} 0,\\
0 &\equiv 5\,\Lambda^{9}X^{23} - 8\,\Lambda^{7}X^{25} + 5\,\Lambda^{5}Y^{27} {-} 8\,f_1'\,K^{12}X^{19},\\
0 &\equiv 7\,\Lambda^{7}X^{23} - 8\,\Lambda^{5}X^{25} + 3\,\Lambda^{3}Y^{27} {-} f_1'\,M^{10}X^{19},\\
0 &\equiv X^{18}X^{21} - 57\,H^{14}X^{25} + 40\,N^{12}Y^{27} {+} 7\,f_1'\,X^{19}X^{19},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv F^{16}X^{21} - 8\,H^{14}X^{23} + N^{12}X^{25} {+} 0,\\
0 &\equiv 7\,K^{12}X^{21} - 5\,M^{10}X^{23} + M^{8}X^{25} {+} 0,\\
0 &\equiv 7\,\Lambda^{9}X^{21} - 57\,\Lambda^{5}X^{25} + 24\,\Lambda^{3}Y^{27} {-} 15\,f_1'\,M^{10}X^{19},\\
0 &\equiv 7\,\Lambda^{7}X^{21} - 40\,\Lambda^{5}X^{23} + 3\,\Lambda^{3}X^{25} {-} 8\,f_1'\,M^{8}X^{19},\\
0 &\equiv X^{18}X^{19} - 8\,K^{12}X^{25} + 5\,M^{10}Y^{27} {+} 0,\\
0 &\equiv 7\,F^{16}X^{19} - M^{10}X^{25} + 8\,M^{8}Y^{27} {+} 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv 7\,H^{14}X^{19} - 5\,M^{10}X^{23} + 8\,M^{8}X^{25} {+} 0,\\
0 &\equiv N^{12}X^{19} - M^{10}X^{21} + 8\,M^{8}X^{23} {+} 0,\\
0 &\equiv 6\,F^{16}X^{18} - \Lambda^{9}X^{25} + 7\,\Lambda^{7}Y^{27} {+} 0,\\
0 &\equiv 6\,H^{14}X^{18} - \Lambda^{7}X^{25} + 5\,\Lambda^{5}Y^{27} {-} 7\,f_1'\,K^{12}X^{19},\\
0 &\equiv 6\,N^{12}X^{18} - \Lambda^{5}X^{25} + 3\,\Lambda^{3}Y^{27} {-} 7\,f_1'\,M^{10}X^{19},\\
0 &\equiv 6\,M^{10}X^{18} - 7\,\Lambda^{9}X^{19} {+} f_1'Y^{27},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv 48\,M^{8}X^{18} - 49\,\Lambda^{7}X^{19} {+} f_1'X^{25},\\
0 &\equiv 30\,F^{16}F^{16} - \Lambda^{7}X^{25} + 5\,\Lambda^{5}Y^{27} {-} f_1'\,K^{12}X^{19},\\
0 &\equiv 42\,H^{14}F^{16} - \Lambda^{5}X^{25} + 3\,\Lambda^{3}Y^{27} {-} f_1'\,M^{10}X^{19},\\
0 &\equiv 30\,N^{12}F^{16} - 5\,\Lambda^{5}X^{23} + 3\,\Lambda^{3}X^{25} {-} f_1'\,M^{8}X^{19},\\
0 &\equiv 48\,K^{12}F^{16} - \Lambda^{9}X^{19} {+} f_1'Y^{27},\\
0 &\equiv 30\,M^{10}F^{16} - 7\,\Lambda^{7}X^{19} {+} f_1'X^{25},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv 48\,M^{8}F^{16} - 7\,\Lambda^{5}X^{19} {+} f_1'X^{23},\\
0 &\equiv 5\,\Lambda^{9}F^{16} - \Lambda^{7}X^{18} {-} 8\,f_1'\,K^{12}K^{12},\\
0 &\equiv 7\,\Lambda^{7}F^{16} - \Lambda^{5}X^{18} {-} f_1'\,M^{10}K^{12},\\
0 &\equiv 35\,\Lambda^{5}F^{16} - 3\,\Lambda^{3}X^{18} {-} 8\,f_1'\,M^{8}K^{12} + f_1'f_1'\,X^{19},\\
0 &\equiv 42\,H^{14}H^{14} - 5\,\Lambda^{5}X^{23} + 3\,\Lambda^{3}X^{25} {-} f_1'M^{8}X^{19},\\
0 &\equiv 6\,N^{12}H^{14} - \Lambda^{5}X^{21} + 3\,\Lambda^{3}X^{23} {+} 0,
\end{aligned}
$$

$$0 \equiv 48\, K^{12} H^{14} - \Lambda^7 X^{19} + f_1' X^{25},$$
$$0 \equiv 6\, M^{10} H^{14} - \Lambda^5 X^{19} + f_1' X^{23},$$
$$0 \equiv 16\, M^8 H^{14} - \Lambda^3 X^{19} + \tfrac{1}{3}\, f_1'\, X^{21},$$
$$0 \equiv 7\, \Lambda^9 H^{14} - \Lambda^5 X^{18} - 8\, f_1'\, M^{10} K^{12},$$
$$0 \equiv 49\, \Lambda^7 H^{14} - 3\, \Lambda^3 X^{18} - 5\, f_1'\, M^{10} M^{10},$$
$$0 \equiv 7\, \Lambda^5 H^{14} - 3\, \Lambda^3 F^{16} - f_1' M^8 M^{10},$$

$$0 \equiv 48\, K^{12} N^{12} - \Lambda^5 X^{19} + 7\, f_1' X^{23},$$
$$0 \equiv 10\, M^{10} N^{12} - \Lambda^3 X^{19} + \tfrac{7}{3}\, f_1'\, X^{21},$$
$$0 \equiv 35\, \Lambda^9 N^{12} - 3\, \Lambda^3 X^{18} - \tfrac{285}{8}\, f_1'\, M^{10} M^{10} + \tfrac{7}{8}\, f_1' f_1'\, X^{19},$$
$$0 \equiv 7\, \Lambda^7 N^{12} - 3\, \Lambda^3 F^{16} - 8\, f_1'\, M^8 M^{10},$$
$$0 \equiv 5\, \Lambda^5 N^{12} - 3\, \Lambda^3 H^{14} - 8\, f_1'\, M^8 M^8,$$
$$0 \equiv 5\, M^{10} M^{10} - 64\, M^8 K^{12} + f_1' X^{19},$$

$$0 \equiv 5\, \Lambda^9 M^{10} - 56\, \Lambda^7 K^{12} + f_1' X^{18},$$
$$0 \equiv \Lambda^7 M^{10} - 8\, \Lambda^5 K^{12} + f_1' F^{16},$$
$$0 \equiv 5\, \Lambda^5 M^{10} - 24\, \Lambda^3 K^{12} + f_1' H^{14},$$
$$0 \equiv \Lambda^9 M^8 - 7\, \Lambda^5 K^{12} + f_1' F^{16},$$
$$0 \equiv \Lambda^7 M^8 - 3\, \Lambda^3 K^{12} + f_1' H^{14},$$
$$0 \equiv 8\, \Lambda^5 M^8 - 3\, \Lambda^3 M^{10} + f_1' N^{12},$$

$$0 \equiv 7\, \Lambda^7 \Lambda^7 - 5\, \Lambda^5 \Lambda^9 + f_1' f_1'\, K^{12},$$
$$0 \equiv 7\, \Lambda^5 \Lambda^7 - 3\, \Lambda^3 \Lambda^9 + f_1' f_1' M^{10},$$
$$0 \equiv 5\, \Lambda^5 \Lambda^5 - 3\, \Lambda^3 \Lambda^7 + f_1' f_1'\, M^8,$$
$$0 \equiv 7\, K^{12} X^{19} X^{19} + X^{25} X^{25} - 5\, X^{23} Y^{27} + 0,$$
$$0 \equiv M^{10} X^{19} X^{19} + X^{23} X^{25} - X^{21} Y^{27} + 0,$$
$$0 \equiv M^8 X^{19} X^{19} + 5\, X^{23} X^{23} - X^{21} X^{25} + 0,$$

$$0 \equiv 56\, K^{12} K^{12} X^{19} + X^{18} X^{25} - 5\, F^{16} Y^{27} + 0,$$
$$0 \equiv M^{10} K^{12} X^{19} + F^{16} X^{25} - H^{14} Y^{27} + 0,$$
$$0 \equiv 8\, M^8 K^{12} X^{19} + 7\, H^{14} X^{25} - 5\, N^{12} Y^{27} - f_1' X^{19} X^{19},$$
$$0 \equiv M^8 M^{10} X^{19} + 7\, H^{14} X^{23} - N^{12} X^{25} + 0,$$
$$0 \equiv 8\, M^8 M^8 X^{19} + 7\, H^{14} X^{21} - 5\, N^{12} X^{23} + 0,$$
$$0 \equiv 448\, K^{12} K^{12} K^{12} + X^{18} X^{18} + 5\, \Lambda^9 Y^{27} + 0,$$

$$0 \equiv 48\, M^{10} K^{12} K^{12} + \Lambda^9 X^{25} - \Lambda^7 Y^{27} + 0,$$
$$0 \equiv 384\, M^8 K^{12} K^{12} + 7\, \Lambda^7 X^{25} - 5\, \Lambda^5 Y^{27} + f_1' K^{12} X^{19},$$
$$0 \equiv 48\, M^8 M^{10} K^{12} + 7\, \Lambda^5 X^{25} - 3\, \Lambda^3 Y^{27} + f_1'\, M^{10} X^{19},$$
$$0 \equiv 384\, M^8 M^8 K^{12} + 35\, \Lambda^5 X^{23} - 3\, \Lambda^3 X^{25} + f_1' M^8 X^{19},$$
$$0 \equiv 48\, M^8 M^8 M^{10} + 7\, \Lambda^5 X^{21} - 3\, \Lambda^3 X^{23} + 0,$$
$$0 \equiv 64\, M^8 M^8 M^8 + 5\, N^{12} N^{12} + 3\, \Lambda^3 X^{21} + 0.$$

Remarkably, no new bi-invariant appears at this fourth stage. According to the general principle, we may therefore conclude that the algorithm stops.

THEOREM *In dimension $n = 2$ for jet order $\kappa = 5$, the algebra UE_5^2 of jet polynomials $\mathsf{P}\big(j^5 f_1, j^5 f_2\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the 17 mutually independent bi-invariants explicitly defined above:*

$$\boxed{\begin{array}{ccccccccc} f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & \Lambda^9, & M^8, & M^{10}, & K^{12}, & \\ N^{12}, & H^{14}, & F^{16}, & X^{18}, & X^{19}, & X^{21}, & X^{23}, & X^{25}, & Y^{27} \end{array}}$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations for the Degree Reverse Lexicographic ordering which consists of 105 equations, 66 of which generate the ideal in question and whose remainders behind a power of f_1' have been filled just above.

As a consequence, the full algebra E_5^2 of jet polynomials $\mathsf{P}\big(j^5 f\big)$ invariant by reparametrization is generated by the polarizations:

$$\boxed{\begin{array}{ccccccccc} f_i', & \Lambda^3, & \Lambda_i^5, & \Lambda_{i,j}^7, & \Lambda_{i,j,k}^9, & M^8, & M_i^{10}, & K_{i,j}^{12}, & \\ N^{12}, & H_i^{14}, & F_{i,j}^{16}, & X_{i,j,k}^{18}, & X_i^{19}, & X^{21}, & X_i^{23}, & X_{i,j}^{25}, & Y_{i,j,k}^{27} \end{array}}$$

of these 17 bi-invariants, where the indices i, j, k vary in $\{1, 2\}$, whence the **total number** *of these invariants equals:*

$$2+1+2+4+8+1+2+4+1+2+4+8+2+1+2+4+8 = \boxed{\mathbf{56}}.$$

§11. Sixteen (fifteen) bi-invariant in dimension $n = 4$ $(n = 3)$ for jet level $\kappa = 4$

First loop of the algorithm. Coming back to the end of §7, we start with the seven initial bi-invariants:

$$\Lambda^3 = \Delta_{1,2}^{','''},$$
$$\Lambda^5 = \Delta_{1,2}^{','''''}\, f_1' - 3\, \Delta_{1,2}^{','''}\, f_1'',$$
$$\Lambda^7 = \Delta_{1,2}^{',''''''}\, f_1' f_1' + \Delta_{1,2}^{'','''''}\, f_1' f_1' - 10\, \Delta_{1,2}^{','''''}\, f_1' f_1'' + 15\, \Delta_{1,2}^{','''}\, f_1'' f_1'',$$

$$\begin{aligned}
D^6 &= \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}, \\
D^8 &= \Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\, f_1' - 3\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\, f_1'', \\
N^{10} &= \Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\, f_1' f_1' - 3\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\, f_1' f_1'' + 4\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\, f_1' f_1''' + 3\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\, f_1'' f_1'', \\
W^{10} &= \Delta_{1,2,3,4}^{\prime,\prime\prime,\prime\prime\prime,\prime\prime\prime\prime}.
\end{aligned}$$

Then we compute the ideal of relations between these bi-invariants, after setting $f_1' = 0$ in them:

$$\mathsf{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0,\ W^{10}\big|_0\Big).$$

We should observe that the first six initial bi-invariants Λ^3, Λ^5, Λ^7, D^6, D^8 and N^{10} depend only upon the first three jet components $\big(j^4 f_1,\, j^4 f_2,\, j^4 f_3\big)$ of $j^4 f$, while W^{10} and $W^{10}\big|_0$ — which both contain the monomial $-f_4'''' f_3' f_2'' f_1'''$ — really depend upon the fourth jet component $j^4 f_4$. It follows that $W^{10}\big|_0$ is algebraically independent of $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $\Lambda^7\big|_0$, $D^6\big|_0$, $D^8\big|_0$, $N^{10}\big|_0$, so it cannot intervene in the ideal of relations. Without loss of generality, we therefore have to consider:

$$\mathsf{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0\Big).$$

A Maple computation with the Degree Reverse Lexicographic ordering yields a reduced Gröbner basis for this ideal consisting of the following 6 generators[19]:

$$\begin{aligned}
0 &\overset{a}{\equiv} 5\, \Lambda^5\Lambda^5 - 3\, \Lambda^3\Lambda^7 + f_1' f_1' M^8, \\
0 &\overset{b}{\equiv} 2\, \Lambda^5 D^6 - \Lambda^3 D^8 + \tfrac{1}{3}\, f_1' E^{10}, \\
0 &\overset{c}{\equiv} \Lambda^7 D^6 - 5\, \Lambda^3 N^{10} + f_1' L^{12}, \\
0 &\overset{d}{\equiv} \Lambda^5 D^8 - 6\, \Lambda^3 N^{10} + f_1' L^{12}, \\
0 &\overset{e}{\equiv} \Lambda^7 D^8 - 10\, \Lambda^5 N^{10} - f_1' Q^{14}, \\
0 &\overset{f}{\equiv} D^8 D^8 - 12\, D^6 N^{10} - f_1' R^{15}.
\end{aligned}$$

To read these equations (*cf.* §9), one should at first set $f_1' = 0$ virtually in one's head and then consider that further computations show what are the remainders behind a power of f_1'. Five new bi-invariants appear which are implicitly defined by five among these six sizygies and we provide their explicit expression in terms of Δ determinants, after mild simplifications:

$$\begin{aligned}
M^8 &:= \frac{-5\, \Lambda^5\Lambda^5 + 3\, \Lambda^3\Lambda^7}{f_1' f_1'} \\
&= 3\, \Delta_{1,2}^{\prime,\prime\prime\prime\prime}\, \Delta_{1,2}^{\prime,\prime\prime} + 12\, \Delta_{1,2}^{\prime\prime,\prime\prime\prime}\, \Delta_{1,2}^{\prime,\prime\prime} - 5\, \Delta_{1,2}^{\prime,\prime\prime\prime}\, \Delta_{1,2}^{\prime,\prime\prime\prime}, \\
E^{10} &:= \frac{-6\, \Lambda^5\, D^6 + 3\, \Lambda^3\, D^8}{f_1'} \\
&= 3\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\, \Delta_{1,2}^{\prime,\prime\prime} - 6\, \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\, \Delta_{1,2}^{\prime,\prime\prime\prime},
\end{aligned}$$

[19] *See* dim-3-order-4-step-1-with-FGb.mw at [23].

$$
\begin{aligned}
L^{12} &:= \frac{-\Lambda^7 D^6 + 5\,\Lambda^3 N^{10}}{f_1'} \\
&= -\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime\prime}\,f_1' - 4\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime\prime,\prime\prime\prime}\,f_1' + 5\,\Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1' + 10\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime}\,f_1''- \\
&\qquad - 15\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1'' + 20\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1''', \\
Q^{14} &:= \frac{\Lambda^7 D^8 - 10\,\Lambda^5 N^{10}}{f_1'} \\
&= -10\,\Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime}\,f_1'f_1' + \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime\prime}\,f_1'f_1' + 4\,\Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime\prime,\prime\prime\prime}\,f_1'f_1'+ \\
&\quad + 20\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime}\,f_1'f_1'' + 30\,\Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1'f_1'' - 6\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime\prime}\,f_1'f_1''- \\
&\quad - 24\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime\prime,\prime\prime\prime}\,f_1'f_1'' - 40\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime}\,f_1'f_1''' - 75\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1''f_1''+ \\
&\quad + 30\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime\prime}\,f_1''f_1'' + 120\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2}^{\prime,\prime\prime}\,f_1''f_1''', \\
R^{15} &:= \Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,f_1' - 12\,\Delta_{1,2,3}^{\prime,\prime\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,f_1' + 24\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime\prime}\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,f_1''- \\
&\qquad - 48\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,\Delta_{1,2,3}^{\prime,\prime\prime,\prime\prime\prime}\,f_1''',
\end{aligned}
$$

and as usual, the weights are denoted by an upper index. Setting W^{10} apart, in order to verify that these 11 bi-invariants are mutually independent, one computes at first which value they have after setting $f_1' = 0$:

$$
\begin{aligned}
&\underline{\Lambda^3}\big|_0 \\
&\underline{\Lambda^5}\big|_0 \\
&\Lambda^7\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0}{\Lambda^3|_0} \\
&\underline{D^6}\big|_0 \\
&D^8\big|_0 = 2\,\tfrac{\Lambda^5|_0\,D^6|_0}{\Lambda^3|_0} \\
&N^{10}\big|_0 = \tfrac{1}{3}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,D^6|_0}{\Lambda^3|_0\,\Lambda^3|_0} \\
&\underline{M^8}\big|_0 \\
&\underline{E^{10}}\big|_0 \\
&L^{12}\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5|_0\,E^{10}|_0}{\Lambda^3|_0} \\
&Q^{14}\big|_0 = -\tfrac{25}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,E^{10}|_0}{\Lambda^3|_0\Lambda^3|_0} \\
&R^{15}\big|_0 = -\tfrac{8}{3}\,\tfrac{\Lambda^5|_0\,D^6|_0\,E^{10}|_0}{\Lambda^3|_0\,\Lambda^3|_0},
\end{aligned}
$$

with the 5 underlined bi-invariants being algebraically independent and being considered as a transcendence basis, while the value of $\Lambda^7\big|_0$ comes from "$\overset{a}{\equiv}$" above; the value of $D^8\big|_0$ comes from "$\overset{b}{\equiv}$" above; the value of $N^{10}\big|_0$ comes from "$\overset{d}{\equiv}$" above; the value of $L^{12}\big|_0$ comes from "$\overset{r}{\equiv}$" below; the value of $Q^{14}\big|_0$ comes from "$\overset{q}{\equiv}$" below; and the value of $R^{15}\big|_0$ comes from "$\overset{p}{\equiv}$" below. Then one proceeds as in the proof of the lemma on p. 38 to show mutual independence (details will not be provided).

Importantly, the five new bi-invariants M^8, E^{10}, L^{12}, Q^{14} and R^{15} again depend only upon the first three jet components $\big(j^4 f_1, j^4 f_2, j^4 f_3\big)$, so that $W^{10}\big|_0$ again will not intervene in the next ideal of relations. In fact, all bi-invariants except W^{10} live in dimension $n = 3$, and hence it is enough to explore the structure of UE_4^3.

Second loop of the algorithm. Setting therefore W^{10} apart, a Maple computation with the Degree Reverse Lexicographic Ordering offers a reduced Gröbner basis for the ideal of relations:

$$\mathsf{Ideal} - \mathsf{Rel}\left(\begin{array}{c}\Lambda^3\big|_0,\ \ \Lambda^5\big|_0,\ \ \Lambda^7\big|_0,\ \ D^6\big|_0,\ \ D^8\big|_0,\ \ N^{10}\big|_0, \\ M^8\big|_0,\ \ E^{10}\big|_0,\ \ L^{12}\big|_0,\ \ Q^{14}\big|_0,\ \ R^{15}\big|_0\end{array}\right)$$

between our 11 bi-invariants restricted to $\{f_1' = 0\}$, and this basis consists of the 6 generators above together with the following 14 generators [20]:

$$\begin{aligned}
0 &\overset{g}{\equiv} 4\, D^8Q^{14} - 5\,\Lambda^7R^{15} - f_1'X^{21}, \\
0 &\overset{h}{\equiv} 24\, D^6Q^{14} - 25\,\Lambda^5R^{15} + f_1'V^{19}, \\
0 &\overset{i}{\equiv} L^{12}L^{12} + E^{10}Q^{14} - f_1'M^8R^{15}, \\
0 &\overset{j}{\equiv} 8\, N^{10}L^{12} + \Lambda^7R^{15} + f_1'X^{21}, \\
0 &\overset{k}{\equiv} 4\, D^8L^{12} + 5\,\Lambda^5R^{15} - f_1'V^{19},
\end{aligned}$$

$$\begin{aligned}
0 &\overset{l}{\equiv} 8\, D^6L^{12} + 5\,\Lambda^3R^{15} - \tfrac{1}{3}\, f_1'U^{17}, \\
0 &\overset{m}{\equiv} \Lambda^7L^{12} + \Lambda^5Q^{14} - 2\, f_1'M^8N^{10}, \\
0 &\overset{n}{\equiv} 5\,\Lambda^5L^{12} + 3\,\Lambda^3Q^{14} - f_1'D^8M^8, \\
0 &\overset{o}{\equiv} 8\, N^{10}E^{10} + \Lambda^5\, R^{15} - f_1'V^{19}, \\
0 &\overset{p}{\equiv} 4\, D^8E^{10} + 3\,\Lambda^3R^{15} - f_1'U^{17},
\end{aligned}$$

$$\begin{aligned}
0 &\overset{q}{\equiv} 5\,\Lambda^7E^{10} + 3\,\Lambda^3Q^{14} - 6f_1'D^8M^8, \\
0 &\overset{r}{\equiv} 5\,\Lambda^5E^{10} - 3\,\Lambda^3L^{12} - 6\, f_1'D^6M^8, \\
0 &\overset{s}{\equiv} 8\,\Lambda^5N^{10}Q^{14} - \Lambda^7\Lambda^7R^{15} + f_1'Q^{14}Q^{14} + 4\, f_1'N^{10}N^{10}M^8, \\
0 &\overset{t}{\equiv} 24\,\Lambda^3N^{10}Q^{14} - 5\,\Lambda^5\Lambda^7R^{15} - 5\, f_1'L^{12}Q^{14} + 2\, f_1'M^8D^8N^{10}.
\end{aligned}$$

Here, three new bi-invariants appear: U^{17}, V^{19} and X^{21}, which are implicitly defined by the syzygies "$\overset{p}{\equiv}$", "$\overset{o}{\equiv}$", and "$\overset{g}{\equiv}$", and we provide their explicit expression

[20] *See* dim-3-order-4-step-2-with-FGb.mw at [23]. Here again, the remainders behind a power of f_1' have all been computed and tested to know whether they belong to the algebra of the already known 11 bi-invariants.

in terms of Δ determinants[21]:

$$
\begin{aligned}
U^{17} &= \frac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'} \\
&= 15\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''} - 36\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''} - \\
&\quad - 24\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''} + 144\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''},
\end{aligned}
$$

$$
\begin{aligned}
V^{19} &= \frac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'} \\
&= 24\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1' - 60\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1' + \\
&\quad + \Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1' - 75\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1'' + \\
&\quad + 36\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1'' + 168\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'' - \\
&\quad - 144\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'' + 96\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''' - \\
&\quad - 240\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''',
\end{aligned}
$$

$$
\begin{aligned}
X^{21} &= \frac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'} \\
&= -40\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1' - 4\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1' - \\
&\quad - 4\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1' + 60\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1' + \\
&\quad + 240\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1' + 130\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1'' + \\
&\quad + 120\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1'f_1'' - 168\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1'' - \\
&\quad - 668\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1'' - 360\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1'' - \\
&\quad - 160\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1''' + 240\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1''' + \\
&\quad + 960\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1''' - 375\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1'' + \\
&\quad + 840\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''f_1'' + 180\,\Delta_{1,2,3}^{',\,''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1'' + \\
&\quad + 144\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1''f_1'' + 144\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1''f_1'' - \\
&\quad - 1440\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''f_1''' + 480\,\Delta_{1,2,3}^{',\,'',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1'''.
\end{aligned}
$$

Either a Maple computation or a glance at the syzygies "$\overset{7}{\equiv}$", "$\overset{8}{\equiv}$", "$\overset{9}{\equiv}$" below arriving in the third loop provides the values of these two bi-invariants after setting $f_1'=0$:

$$
\begin{aligned}
U^{17}\big|_0 &= 12\,\tfrac{D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0} + \tfrac{5}{3}\,\tfrac{E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0}, \\
V^{19}\big|_0 &= \tfrac{25}{9}\,\tfrac{\Lambda^5|_0\,E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0\,\Lambda^3|_0} + 4\,\tfrac{\Lambda^5|_0\,D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \\
X^{21}\big|_0 &= -\tfrac{4}{3}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0} - \tfrac{125}{27}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}
$$

---

[21] To be able do divide by f_1', as in [21], we sometimes need to replace $\Delta_{1,2}^{',\,''}\,f_1'''$ by $-\Delta_{1,2}^{'',\,'''}\,f_1' + \Delta_{1,2}^{',\,'''}\,f_1''$, using the immediately checked syzygy: $0 \equiv \Delta_{1,2}^{'',\,'''}\,f_1' - \Delta_{1,2}^{',\,'''}\,f_1'' + \Delta_{1,2}^{',\,''}\,f_1'''$.

Proceeding as in the lemma on p. 38, one checks patiently by hand that the 16 bi-invariants known so far:

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21} \end{array}}$$

are mutually independent.

Third loop of the algorithm. Again for the Degree Reverse Lexicographic ordering, setting W^{10} apart, a Maple computation offers a reduced Gröbner basis for the ideal of relations between the $14 = 15 - 1$ (f_1' goes to zero) restricted bi-invariants. The result consists of 50 generators [22]. Taking the Lexicographic ordering instead:

$$\Lambda^3 > \Lambda^5 > \Lambda^7 > D^6 > D^8 > N^{10} > M^8 > E^{10} > L^{12} > \\ > Q^{14} > R^{15} > U^{17} > V^{19} > X^{21},$$

one shows that the ideal of relations, in Gröbnerized form, contains less equations — which is convenient —, namely the following 41 equations[23], where we underline the Leading Term of each syzygy with the acronym "LT" appended[24]:

$$0 \overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\underline{\Lambda^3\Lambda^7}_{\mathsf{LT}} - f_1'f_1'M^8,$$

$$0 \overset{2}{\equiv} -2\,\Lambda^5D^6 + \underline{\Lambda^3D^8}_{\mathsf{LT}} - \tfrac{1}{3}f_1'\,E^{10},$$

$$0 \overset{3}{\equiv} -\Lambda^7D^6 + 5\,\underline{\Lambda^3N^{10}}_{\mathsf{LT}} - f_1'L^{12},$$

$$0 \overset{4}{\equiv} -5\,\Lambda^5E^{10} + 3\,\underline{\Lambda^3L^{12}}_{\mathsf{LT}} + 6\,f_1'D^6M^8,$$

$$0 \overset{5}{\equiv} 5\,\Lambda^7E^{10} + 3\,\underline{\Lambda^3Q^{14}}_{\mathsf{LT}} - 6\,f_1'D^8M^8,$$

$$0 \overset{6}{\equiv} 4\,D^8E^{10} + 3\,\underline{\Lambda^3R^{15}}_{\mathsf{LT}} - f_1'U^{17},$$

$$0 \overset{7}{\equiv} -36\,D^6D^6M^8 - 5\,E^{10}E^{10} + 3\,\underline{\Lambda^3U^{17}}_{\mathsf{LT}} + 0,$$

$$0 \overset{8}{\equiv} -5\,E^{10}L^{12} - 6\,D^6D^8M^8 + 3\,\underline{\Lambda^3V^{19}}_{\mathsf{LT}} + 0,$$

$$0 \overset{9}{\equiv} 5\,L^{12}L^{12} + 3\,\underline{\Lambda^3X^{21}}_{\mathsf{LT}} + M^8D^8D^8 + 0,$$

$$0 \overset{10}{\equiv} -6\,\Lambda^7D^6 + 5\,\underline{\Lambda^5D^8}_{\mathsf{LT}} - f_1'L^{12},$$

$$0 \overset{11}{\equiv} -\Lambda^7D^8 + 10\,\underline{\Lambda^5N^{10}}_{\mathsf{LT}} + f_1'Q^{14},$$

$$0 \overset{12}{\equiv} \underline{\Lambda^5L^{12}}_{\mathsf{LT}} - \Lambda^7E^{10} + f_1'D^8M^8,$$

[22] *See* dim-3-order-4-step-3-with-FGb.mws at [23].

[23] *See* 41-syzygies-dim-3-order-4.mw at [23].

[24] We recall that, in order to appropriately read the ideal of relations between restricted bi-invariants, one should set $f_1' = 0$, namely disregard the last term(s) of each equation. We specify "$+0$" when the remainder behing a power of f_1' vanishes identically.

$$0\overset{13}{\equiv}\Lambda^7L^{12}+\underline{\Lambda^5Q^{14}}_{\text{LT}}-2\,f_1'M^8N^{10},$$
$$0\overset{14}{\equiv}8\,N^{10}E^{10}+\underline{\Lambda^5R^{15}}_{\text{LT}}-f_1'V^{19},$$
$$0\overset{15}{\equiv}\underline{\Lambda^5U^{17}}_{\text{LT}}-E^{10}L^{12}-6\,D^6D^8M^8+0,$$
$$0\overset{16}{\equiv}\underline{\Lambda^5V^{19}}_{\text{LT}}-M^8D^8D^8-L^{12}L^{12}+f_1'M^8R^{15},$$
$$0\overset{17}{\equiv}\underline{\Lambda^5X^{21}}_{\text{LT}}-L^{12}Q^{14}+2\,D^8N^{10}M^8+0,$$
$$0\overset{18}{\equiv}8\,N^{10}L^{12}+\underline{\Lambda^7R^{15}}_{\text{LT}}+f_1'X^{21},$$

$$0\overset{19}{\equiv}-L^{12}L^{12}+\underline{\Lambda^7U^{17}}_{\text{LT}}+f_1',-5\,M^8D^8D^8+0,$$
$$0\overset{20}{\equiv}L^{12}Q^{14}+\underline{\Lambda^7V^{19}}_{\text{LT}}-10\,D^8M^8N^{10}+0,$$

$$0\overset{21}{\equiv}20\,N^{10}N^{10}M^8+Q^{14}Q^{14}+\underline{\Lambda^7X^{21}}_{\text{LT}}+0,$$
$$0\overset{22}{\equiv}6\,\underline{D^6M^8R^{15}}_{\text{LT}}+L^{12}U^{17}-E^{10}V^{19}+0,$$
$$0\overset{23}{\equiv}5\,\underline{D^8M^8R^{15}}_{\text{LT}}-Q^{14}U^{17}-L^{12}V^{19}+0,$$
$$0\overset{24}{\equiv}10\,\underline{N^{10}M^8R^{15}}_{\text{LT}}-Q^{14}V^{19}+L^{12}X^{21}+0,$$

$$0\overset{25}{\equiv}5\,\underline{M^8R^{15}R^{15}}_{\text{LT}}+V^{19}V^{19}+U^{17}X^{21}+0,$$
$$0\overset{26}{\equiv}-D^8D^8+12\,\underline{D^6N^{10}}_{\text{LT}}+f_1'R^{15},$$
$$0\overset{27}{\equiv}-5\,D^8E^{10}+6\,\underline{D^6L^{12}}_{\text{LT}}+f_1'U^{17},$$
$$0\overset{28}{\equiv}3\,\underline{D^6Q^{14}}_{\text{LT}}+25\,N^{10}E^{10}-3\,f_1'V^{19},$$
$$0\overset{29}{\equiv}5\,E^{10}R^{15}-D^8U^{17}+6\,\underline{D^6V^{19}}_{\text{LT}}+0,$$
$$0\overset{30}{\equiv}-3\,L^{12}R^{15}+N^{10}U^{17}+3\,\underline{D^6X^{21}}_{\text{LT}}+0,$$

$$0\overset{31}{\equiv}-10\,N^{10}E^{10}+\underline{D^8L^{12}}_{\text{LT}}+f_1'V^{19},$$
$$0\overset{32}{\equiv}\underline{D^8Q^{14}}_{\text{LT}}+10\,N^{10}L^{12}+f_1'X^{21},$$
$$0\overset{33}{\equiv}-2\,N^{10}U^{17}+\underline{D^8V^{19}}_{\text{LT}}+L^{12}R^{15}+0,$$
$$0\overset{34}{\equiv}Q^{14}R^{15}+2\,N^{10}V^{19}+\underline{D^8X^{21}}_{\text{LT}}+0,$$
$$0\overset{35}{\equiv}-2\,L^{12}N^{10}U^{17}+R^{15}L^{12}L^{12}+10\,\underline{V^{19}N^{10}E^{10}}_{\text{LT}}-f_1'V^{19}V^{19},$$
$$0\overset{36}{\equiv}2\,N^{10}U^{17}Q^{14}-R^{15}L^{12}Q^{14}+10\,\underline{V^{19}N^{10}L^{12}}_{\text{LT}}+f_1'V^{19}X^{21},$$

$$0 \overset{37}{\equiv} 10\, \underline{N^{10} L^{12} X^{21}}_{\mathsf{LT}} - R^{15} Q^{14} Q^{14} - 2\, Q^{14} N^{10} V^{19} + f_1' X^{21} X^{21},$$
$$0 \overset{38}{\equiv} 2\, \underline{N^{10} U^{17} X^{21}}_{\mathsf{LT}} - X^{21} L^{12} R^{15} + V^{19} Q^{14} R^{15} + 2\, N^{10} V^{19} V^{19} + 0,$$
$$0 \overset{39}{\equiv} \underline{E^{10} Q^{14}}_{\mathsf{LT}} + L^{12} L^{12} - f_1' M^8 R^{15},$$
$$0 \overset{40}{\equiv} Q^{14} U^{17} + 6\, L^{12} V^{19} + 5\, \underline{E^{10} X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{41}{\equiv} -6\, Q^{14} L^{12} V^{19} - Q^{14} Q^{14} U^{17} + 5\, \underline{X^{21} L^{12} L^{12}}_{\mathsf{LT}} - 5\, f_1' M^8 R^{15} X^{21}.$$

Remarkably, each one of the 41 remainders behind a power of f_1' belongs to the algebra of already known bi-invariants. No new bi-invariant appears at this stage. In such a circumstance, according to the general theorem on p. 57, we know that our algorithm stops, so that we have gained the following complete, quite nontrivial result.

THEOREM *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UE_4^4 of jet polynomials $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the 16 mutually independent bi-invariants defined above:*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the 41 syzygies written above.

Furthermore, any bi-invariant of weight m writes uniquely in the finite polynomial form:

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,\dots,n) \in \mathbb{N}^{14} \backslash (\square_1 \cup \cdots \cup \square_{41}) \\ 3a + \cdots + 21n = m - o - 10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot$$
$$\cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h$$
$$\big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ subjected to no restriction, where $\square_1, \dots, \square_{41}$ denote the quadrants in $\mathbb{N}^{14}$ having vertex at the leading terms of the 41 syzygies in question.

Finally, in the preceding dimension $n = 3$ for jets of the same order $\kappa = 4$, the algebra UE_4^3 is generated by the same list from which one removes only the four-dimensional Wronskian W^{10}, the ideal of relations for the 15 restricted bi-invariants being exactly the same, with an entirely similar normal form for a general bi-invariant of weight m.

As a consequence, by looking at the $\mathsf{GL}_4(\mathbb{C})$-orbit of each one of these 16 bi-invariants, we deduce a system of **2835** generators for the algebra E_4^4 of polynomials which are invariant (only) by reparametrization.

THEOREM *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra E_4^4 of jet polynomials $\mathsf{P}\big(j^4 f\big)$ invariant by reparametrization is generated by the polarizations:*

$$\boxed{\begin{array}{c} W^{10}, \quad f_i', \quad \Lambda^3_{[i,j]}, \quad \Lambda^5_{[i,j];\,\alpha}, \quad \Lambda^7_{[i,j];\,\alpha,\beta}, \quad D^6_{[i,j,k]}, \\ D^8_{[i,j,k];\,\alpha}, \quad N^{10}_{[i,j,k];\,\alpha,\beta}, \quad M^8_{[i,j],[k,l]}, \quad E^{10}_{[i,j,k],[p,q]}, \quad L^{12}_{[i,j,k],[p,q];\,\alpha}, \\ Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta}, \quad R^{15}_{[i,j,k],[p,q,r];\,\alpha}, \quad U^{17}_{[i,j,k],[p,q,r],[s,t]}, \\ V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha}, \quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta}, \end{array}}$$

of the 16 bi-invariants W^{10}, f_1', Λ^3, Λ^5, Λ^7, D^6, D^8, N^{10}, M^8, E^{10}, L^{12}, Q^{14}, R^{15}, U^{17}, V^{19}, X^{21} generating the algebra UE_4^4 of bi-invariants; these polarized invariants are skew-symmetric with respect to each collection of bracketed indices $[i,j,k]$, $[p,q,r]$, $[s,t]$, and they are explicitly *represented in terms of Δ-determinants by the following complete formulas:*

$$\begin{aligned} & W^{10}_{1,2,3,4}, \\ & f_i', \\ \Lambda^3_{[i,j]} &:= \Delta^{',''}_{i,j}, \end{aligned}$$

$$\Lambda^5_{[i,j];\,\alpha} := \Delta^{',''' }_{i,j}\, f_\alpha' - 3\,\Delta^{',''}_{i,j}\, f_\alpha'',$$

$$\begin{aligned} \Lambda^7_{[i,j];\,\alpha,\beta} &:= \Delta^{','''' }_{i,j}\, f_\alpha' f_\beta' + 4\,\Delta^{'',''' }_{i,j}\, f_\alpha' f_\beta' - 5\Delta^{',''' }_{i,j}\,\big(f_\alpha' f_\beta'' + f_\alpha'' f_\beta'\big) + \\ &\qquad + 15\,\Delta^{',''}_{i,j}\, f_\alpha'' f_\beta'', \\ D^6_{[i,j,k]} &:= \Delta^{',\,'',\,'''}_{i,j,k}, \end{aligned}$$

$$\begin{aligned} D^8_{[i,j,k];\,\alpha} &:= \Delta^{',\,'',\,''''}_{i,j,k}\, f_\alpha' - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\, f_\alpha'', \\ N^{10}_{[i,j,k];\,\alpha,\beta} &:= \Delta^{',\,''',\,''''}_{i,j,k}\, f_\alpha' f_\beta' - \frac{3}{2}\,\Delta^{',\,'',\,''''}_{i,j,k}\,\big(f_\alpha' f_\beta'' + f_\alpha'' f_\beta'\big) + \\ &\qquad + 2\,\Delta^{',\,'',\,'''}_{i,j,k}\,\big(f_\alpha' f_\beta''' + f_\alpha''' f_\beta'\big) + 3\,\Delta^{',\,'',\,'''}_{i,j,k}\, f_\alpha'' f_\beta'', \end{aligned}$$

$$\begin{aligned} M^8_{[i,j],[k,l]} &:= 3\,\Delta^{','''' }_{i,j}\,\Delta^{',''}_{k,l} + 12\,\Delta^{'',''' }_{i,j}\,\Delta^{',''}_{k,l} - \\ &\quad - 5\,\Delta^{',''' }_{i,j}\,\Delta^{',''' }_{k,l}, \\ E^{10}_{[i,j,k],[p,q]} &:= 3\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',''}_{l,m} - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',''' }_{l,m}, \\ L^{12}_{[i,j,k],[l,m];\,\alpha} &:= 5\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',''' }_{p,q}\, f_\alpha' - 15\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',''}_{p,q}\, f_\alpha'' - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{','''' }_{p,q}\, f_\alpha' - \\ &\quad - 24\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{'',''' }_{p,q}\, f_\alpha' + 30\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',''' }_{p,q}\, f_\alpha', \end{aligned}$$

$$
\begin{aligned}
Q^{14}_{[i,j,k],[p,q];\alpha,\beta} := & -10\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''''}_{p,q}\,f'_\alpha f'_\beta + \Delta^{','',''''}_{i,j,k}\,\Delta^{',''''}_{p,q}\,f'_\alpha f'_\beta + \\
& + 4\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{'',''''}_{p,q}\,f'_\alpha f'_\beta + +20\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','''}_{p,q}\,f'_\alpha f''_\beta + \\
& + 30\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''}_{p,q}\,f'_\alpha f''_\beta - 6\,\Delta^{','','''}_{i,j,k}\,\Delta^{',''''}_{p,q}\,f'_\alpha f''_\beta - \\
& - 24\,\Delta^{','','''}_{i,j,k}\,\Delta^{'','''}_{p,q}\,f'_\alpha f''_\beta - 40\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{p,q}\,f'_\alpha f'''_\beta - \\
& - 75\,\Delta^{','',''''}_{i,j,k}\,\Delta^{',''}_{p,q}\,f''_\alpha f''_\beta + 30\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{p,q}\,f''_\alpha f''_\beta + \\
& + 120\,\Delta^{','','''}_{i,j,k}\,\Delta^{',''}_{p,q}\,f''_\alpha f'''_\beta,
\end{aligned}
$$

$$
\begin{aligned}
R^{15}_{[i,j,k],[p,q,r];\alpha} := & \,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,f'_\alpha - 12\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,f'_\alpha + \\
& + 24\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,f''_\alpha - 48\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,f'''_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
U^{17}_{[i,j,k],[p,q,r],[s,t]} := & \,15\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t} - 36\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t} - \\
& - 24\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t} + 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t},
\end{aligned}
$$

$$
\begin{aligned}
V^{19}_{[i,j,k],[p,q,r],[s,t];\alpha} := & \,24\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f'_\alpha - 60\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha + \\
& + \Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha - 75\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha + \\
& + 36\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha + 168\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f''_\alpha -
\end{aligned}
$$

$$
\begin{aligned}
& - 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f''_\alpha + 96\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f'''_\alpha - \\
& - 240\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'''_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
& X^{21}_{[i,j,k],[p,q,r],[s,t];\alpha,\beta} := \\
& := -40\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f'_\beta - 4\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'_\beta - \\
& - 4\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'_\beta + 60\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'_\beta + \\
& + 240\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'_\beta + 130\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f''_\beta + \\
& + 120\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f'_\alpha f''_\beta - 168\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f''_\beta - \\
& - 668\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f''_\beta - 360\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f''_\beta -
\end{aligned}
$$

$$
\begin{aligned}
& - 160\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f'''_\beta + 240\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'''_\beta + \\
& + 960\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'''_\beta - 375\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f''_\beta + \\
& + 840\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f''_\alpha f''_\beta + 180\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f''_\beta + \\
& + 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f''_\alpha f''_\beta + 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f''_\alpha f''_\beta - \\
& - 1440\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f''_\alpha f'''_\beta + 480\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f'''_\beta,
\end{aligned}
$$

where the roman indices satisfy $1 \leqslant i < j < k \leqslant 4$*, where* $1 \leqslant p < q < r \leqslant 4$*, where* $1 \leqslant s < r \leqslant 4$ *and where the two greek indices* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 4$ *without restriction and finally the* **total number** *of these* *invariants generating the Demailly-Semple algebra* E_4^4 *equals:*

$$1+4+6+24+96+4+16+64+ \\ +36+24+96+384+64+96+384+1536 = \boxed{\mathbf{2835}}.$$

Furthermore, in the preceding dimension $n=3$ *for jets of the same order* $\kappa=4$*, the Demailly-Semple algebra* E_4^3 *is generated by the analogous list from which one removes the four-dimensional Wronskian* $W_{1,2,3,4}^{10}$ *and in which the triples of skew-symmetric indices* $[i,j,k]$ *and* $[p,q,r]$ *are set to* $[1,2,3]$ *while* $[p,q]$ *satisfy* $1\leqslant p<q\leqslant 3$ *and* α,β *satisfy* $1\leqslant \alpha,\beta\leqslant 3$ *without restriction, whence the total number of generators of* E_4^3 *equals:*

$$3+3+9+27+1+3+9+9+3+9+27+3+3+9+27 = \boxed{\mathbf{145}}.$$

§12. Approximate Schur bundle decomposition of $\mathsf{E}_{4,m}^4 T_X^*$

Finite generation. Thus, we know from the preceding section that UE_4^4 is generated by the sixteen bi-invariant polynomials:

$$\Lambda^3,\ \ \Lambda^5,\ \ \Lambda^7,\ \ D^6,\ \ D^8,\ \ N^{10},\ \ M^8,\ \ E^{10},\ \ L^{12},\ \ Q^{14},\ \ R^{15},\ \ U^{17},\ \ V^{19},\ \ X^{21},\ \ f_1',\ \ W^{10},$$

whose weight appears as an exponent. A general polynomial in these 16 invariants writes:

$$\sum \mathsf{coeff}\cdot (\Lambda^3)^a\,(\Lambda^5)^b\,(\Lambda^7)^c\,(D^6)^d\,(D^8)^e\,(N^{10})^f\,(M^8)^g\,(E^{10})^h\,(L^{12})^i \\ (Q^{14})^j\,(R^{15})^k\,(U^{17})^l\,(V^{19})^m\,(X^{21})^n\,(f_1')^o\,(W^{10})^p,$$

where $a,b,c,d,e,f,g,h,i,j,k,l,m,n,o$ and p are nonnegative integer exponents. We temporarily use the letter m which should not make confusion with the weighting m appearing in $\mathsf{UE}_{\kappa,m}^n$. When one requires that such a polynomial has weight m, the sum should be restricted to exponents satisfying:

$$m = 3a+5b+7c+6d+8e+10f+8g+10h+12i+14j+15k+17l+19m+21n+o+10p.$$

When one furthermore restricts such a general polynomial to $\{f_1'=0\}$, one gets:

$$\sum_{3a+5b+\cdots+21n+10p=m} \mathsf{coeff}\cdot (\Lambda^3|_0)^a\,(\Lambda^5|_0)^b\,(\Lambda^7|_0)^c\,(D^6|_0)^d\,(D^8|_0)^e\,(N^{10}|_0)^f\,(M^8|_0)^g\,(E^{10}|_0)^h \\ (L^{12}|_0)^i\,(Q^{14}|_0)^j\,(R^{15}|_0)^k\,(U^{17}|_0)^l\,(V^{19}|_0)^m\,(X^{21}|_0)^n\,(W^{10}|_0)^p.$$

Next, let Syz_{41} denote the ideal of $\mathbb{C}\big[\Lambda^3|_0,\dots,X^{21}|_0\big]$ generated by the 41 lexicographic syzygies written on p. 73 (in which one sets $f_1'=0$) holding between the ordered variables:

$$\Lambda^3|_0 > \Lambda^5|_0 > \Lambda^7|_0 > D^6|_0 > D^8|_0 > N^{10}|_0 > M^8|_0 > E^{10}|_0 > \\ L^{12}|_0 > Q^{14}|_0 > R^{15}|_0 > U^{17}|_0 > V^{19}|_0 > X^{21}|_0.$$

We list in columns the 41 Leading Terms of these 41 syzygies:

$\underline{\Lambda^3|_0\Lambda^7|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad c \geqslant 1$

$\underline{\Lambda^3|_0 D^8|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad e \geqslant 1$

$\underline{\Lambda^3|_0 N^{10}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad f \geqslant 1$

$\underline{\Lambda^3|_0 L^{12}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad i \geqslant 1$

$\underline{\Lambda^3|_0 Q^{14}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad j \geqslant 1$

$\underline{\Lambda^3|_0 R^{15}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^3|_0 U^{17}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^3|_0 V^{19}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^3|_0 X^{21}|_0}_{\mathsf{LT}}: \quad a \geqslant 1, \quad n \geqslant 1$

$\underline{\Lambda^5|_0 D^8|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad e \geqslant 1$

$\underline{\Lambda^5|_0 N^{10}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad f \geqslant 1$

$\underline{\Lambda^5|_0 L^{12}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad i \geqslant 1$

$\underline{\Lambda^5|_0 Q^{14}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad j \geqslant 1$

$\underline{\Lambda^5|_0 R^{15}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^5|_0 U^{17}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^5|_0 V^{19}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^5|_0 X^{21}|_0}_{\mathsf{LT}}: \quad b \geqslant 1, \quad n \geqslant 1$

$\underline{\Lambda^7|_0 R^{15}|_0}_{\mathsf{LT}}: \quad c \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^7|_0 U^{17}|_0}_{\mathsf{LT}}: \quad c \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^7|_0 V^{19}|_0}_{\mathsf{LT}}: \quad c \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^7|_0 X^{21}|_0}_{\mathsf{LT}}: \quad c \geqslant 1, \quad n \geqslant 1$

$\underline{D^6|_0 N^{10}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad f \geqslant 1$

$\underline{D^6|_0 L^{12}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad i \geqslant 1$

$\underline{D^6|_0 Q^{14}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad j \geqslant 1$

$\underline{D^6|_0 V^{19}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad m \geqslant 1$

$\underline{D^6|_0 X^{21}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad n \geqslant 1$

$\underline{D^8|_0 L^{12}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad i \geqslant 1$

$\underline{D^8|_0 Q^{14}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad j \geqslant 1$

$\underline{D^8|_0 V^{19}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad m \geqslant 1$

$\underline{D^8|_0 X^{21}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad n \geqslant 1$

$\underline{D^6|_0 M^8|_0 R^{15}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$

$\underline{D^8|_0 M^8|_0 R^{15}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$

$\underline{N^{10}|_0 M^8|_0 R^{15}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$

$\underline{M^8|_0 R^{15}|_0 R^{15}|_0}_{\mathsf{LT}}: \quad g \geqslant 1, \quad k \geqslant 2$

$\underline{E^{10}|_0 Q^{14}|_0}_{\mathsf{LT}}: \quad h \geqslant 1, \quad j \geqslant 1$

$\underline{E^{10}|_0 X^{21}|_0}_{\mathsf{LT}}: \quad h \geqslant 1, \quad n \geqslant 1$

$\underline{L^{12}|_0 L^{12}|_0 X^{21}|_0}_{\mathsf{LT}}: \quad i \geqslant 2, \quad n \geqslant 1$

$\underline{N^{10}|_0 E^{10}|_0 V^{19}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad h \geqslant 1, \quad m \geqslant 1$

$\underline{N^{10}|_0 L^{12}|_0 V^{19}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad i \geqslant 1, \quad m \geqslant 1$

$\underline{N^{10}|_0 L^{12}|_0 X^{21}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad i \geqslant 1, \quad n \geqslant 1$

$\underline{N^{10}|_0 U^{17}|_0 X^{21}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad l \geqslant 1, \quad n \geqslant 1$

If, by $\mathsf{LT}(\mathsf{Syz}_{41})$, we denote the monomial ideal of $\mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]$ generated by these 41 Leading Terms, a known elementary property of reduced Gröbner bases shows that:

$$\mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]\big/\mathsf{Syz}_{41} \simeq \mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]\big/\mathsf{LT}\big(\mathsf{Syz}_{41}\big).$$

More suitably for our purposes, the theorem on p. 75 states that any bi-invariant of weight m writes uniquely under the form:

$$\begin{aligned}\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\backslash(\square_1\cup\cdots\cup\square_{41})\\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p}\,\cdot \\ \cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h \\ \big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,\end{aligned}$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ subjected to no restriction, where $\square_1, \dots, \square_{41}$ denote the quadrants in $\mathbb{N}^{14}$ having vertex at the leading terms of our 41 syzygies.

Our goal now is to compute an approximation of this general sum of monomials which will suffice for our Euler-Poincaré characteristic computations below.

A general monomial in $\mathbb{C}\big[\Lambda^3, \dots, X^{21}\big]$ writes:

$$\begin{aligned}\mathsf{Monomial} = \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h \\ \big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n.\end{aligned}$$

Such a monomial *belongs* to the monomial ideal $\mathsf{LT}\big(\mathsf{Syz}_{41}\big)$ *if and only if* it is a multiple of at least one of the 41 Leading Terms. Equivalently, the 14-tuple of integers $(a, \dots, n)$ belongs to at least one quadrant $\square_i$ with vertex the exponent of the leading term of the i-th syzygy. For instance, being a multiple of $\Lambda^3\Lambda^7$ occurs when and only when $a \geqslant 1$ and $c \geqslant 1$. In fact, in our complete list of the 41 leading terms above, just after each leading Term, we have in advance written the condition that such a Monomial be a multiple of it.

On the contrary, for Monomial *not to be a multiple* of $\Lambda^3\Lambda^7$, it is necessary and sufficient that $a = 0$ or $c = 0$, and more generally, for it to belong to the relevant quotient ideal:

$$\mathbb{C}\big[\Lambda^3, \dots, X^{21}\big]\big/\mathsf{LT}\big(\mathsf{Syz}_{41}\big),$$

it is necessary and sufficient that its 14-tuple exponent $\big(a, b, c, d, e, f, g, h, i, j, k, l, m, n\big) \in \mathbb{N}^{14}$ belongs to the following *intersection* of 41 subsets of $\mathbb{N}^{14}$:

$$\big\{a=0\big\}\cup\big\{c=0\big\}\bigcap\big\{a=0\big\}\cup\big\{e=0\big\}\bigcap\cdots\bigcap\big\{f=0\big\}\cup\big\{l=0\big\}\cup\big\{n=0\big\}.$$

To compute this intersection, we shall abbreviate for instance $\big\{a = 0\big\} \cup \big\{c = 0\big\}$ by $(a + c)$ with the symbol "$+$" denoting union, and with the intersection being denoted by an unwritten multiplication symbol, so that we may develope for instance the product of the first two terms as follows:

$$\begin{aligned}\big\{a=0\big\}\cup\big\{c=0\big\}\bigcap\big\{a=0\big\}\cup\big\{e=0\big\} &\equiv (a+c)(a+e) \\ &= aa + ae + ca + ce \\ &= a + ce,\end{aligned}$$

and simplify it immediately, on understanding that the symbol a represents $\big\{a = 0\big\}$, hence contains both $ae \equiv \big\{a = e = 0\big\}$ and $ca \equiv \big\{c = a = 0\big\}$.

With such a convention, grouping by packages, we may compute the intersections colum by column, starting with the first column containing $\Lambda^3|_0$:

$$(a+c)(a+e)(a+f)(a+i)(a+j)(a+k)(a+l)(a+m)(a+n) = a+cefijklmn,$$

and getting in sum nine "words" that we should further "intersect":

$$\begin{aligned}
&a + cefijklmn,\\
&b + efijklmn,\\
&c + klmn,\\
&d + fijmn,\\
&e + ijmn,\\
&(d+g+k)(e+g+k)(f+g+k)(g+k+k^1),\\
&h + jn,\\
&i + i^1 + n,\\
&(f+h+m)(f+i+m)(f+i+n)(f+l+n).
\end{aligned}$$

Here, the "letter" k^1 appearing at the end of the sixth line means the subset $\{k = 1\}$ of $\mathbb{N}^{14}$, not to be confused with $k \equiv \{k = 0\}$. Let us develope step by step the sixth and the ninth lines:

$$\begin{aligned}
&(d+g+k)(e+g+k)(f+g+k)(g+k+k^1) =\\
&(d+g+k)(e+g+k)(g+k+fk^1) =\\
&(d+g+k)(g+k+efk^1) =\\
&g+k+defk^1
\end{aligned}$$

$$\begin{aligned}
&(f+h+m)(f+i+m)(f+i+n)(f+l+n) =\\
&(f+h+m)(f+i+m)(f+n+il) =\\
&(f+h+m)(f+il+in+mn) =\\
&f+hil+hin+mn+ilm.
\end{aligned}$$

Now we compute the product of the lines 3, 4, 5, 7:

$$\begin{aligned}
&(c+klmn)(d+fijmn)(e+ijmn)(h+jn) =\\
&(c+klmn)(d+fijmn)(eh+ejn+ijmn) =\\
&(c+klmn)(deh+dejn+dijmn+fijmn) =\\
&cdeh+cdejn+cdijmn+cfijmn+dehklmn+dejklmn+dijklmn+fijklmn
\end{aligned}$$

and the product of the lines 1 and 2:

$$ab + aefijklmn + cefijklmn,$$

whence the product of the lines 1, 2, 3, 4, 5, 7 is:

$$\begin{aligned}
&abcdeh + abcdejn + abcdijmn + abcfijmn + abdehklmn + abdejklmn+\\
&+ abdijklmn + abfijklmn + aefijklmn + cefijklmn.
\end{aligned}$$

On the other hand, the product of the lines 9, 6, 8 is:

$$
\begin{aligned}
&(f+hil+hin+mn+ilm)(g+k+defk^1)(i+i^1+n) = \\
&(f+hil+hin+mn+ilm)(gi+gi^1+gn+ik+i^1k+kn+defik^1+defi^1k^1+defk^1n) =
\end{aligned}
$$

When developing the latter product, sometimes words containing the product ii^1 (or kk^1) might appear. But they denote the empty set $\{i=0\}\cap\{i=1\}$, so they should be left out. The direct result of the product, before any simplification, is:

$$
\begin{aligned}
= {} & fgi+fgi^1+fgn+fik+fi^1k+fkn+defik^1+defi^1k^1+defk^1n+ \\
&+ghil+\emptyset+ghiln+hikl+\emptyset+hikln+defik^1l+\emptyset+defhik^1ln+ \\
&+ghin+\emptyset+ghin+hikn+\emptyset+hikn+defhik^1n+\emptyset+defhik^1n+ \\
&+gimn+gi^1mn+gmn+ikmn+i^1kmn+kmn+defik^1mn+defi^1k^1mn+defk^1mn+ \\
&+gilm+\emptyset+gilmn+iklm+\emptyset+iklmn+defik^1lm+\emptyset+defik^1lmn,
\end{aligned}
$$

and after simplification:

$$
\begin{aligned}
= {} & fgi+fgi^1+fgn+fik+fi^1k+fkn+defik^1+defi^1k^1+defk^1n+ \\
&+ghil+hikl+ghin+hikn+gmn+kmn+gilm+iklm.
\end{aligned}
$$

The final multiplication shall be:

$$
\begin{aligned}
&\Big(abcdeh+abcdejn+abcdijmn+abcfijmn+abdehklmn+abdejklmn+ \\
&\quad +abdijklmn+abfijklmnaefijklmn+cefijklmn\Big)\cdot \\
&\cdot\Big(fgi+fgi^1+fgn+fik+fi^1k+fkn+defik^1+defi^1k^1+defk^1n+ \\
&\quad +ghil+hikl+ghin+hikn+gmn+kmn+gilm+iklm\Big),
\end{aligned}
$$

but we will not expand it completely.

Twenty-four families of monomials. Instead, we will compute the product modulo words which contain more than 9 letters. The reason why we do so will be appearent later. The result then consists of 30 words of 9 letters:

A :	$abcdefghi$	J :	$abcdegjmn$
A′ :	$abcdefghi^1$	K :	$abcdehikl$
B :	$abcdefghn$	L :	$abcdehikn$
C :	$abcdefgjn$	M :	$abcdehkmn$
D :	$abcdefhik$	N :	$abcdejkmn$
D′ :	$abcdefhi^1k$	O :	$abcdgijmn$
D″ :	$abcdefhik^1$	P :	$abcdijkmn$
D‴ :	$abcdefhi^1k^1$	Q :	$abcfgijmn$
E :	$abcdefhkn$	R :	$abcfijkmn$
E′ :	$abcdefhk^1n$	S :	$abdehklmn$
F :	$abcdefjkn$	T :	$abdejklmn$
F′ :	$abcdefjk^1n$	U :	$abdijklmn$
G :	$abcdeghil$	V :	$abfijklmn$
H :	$abcdeghin$	W :	$aefijklmn$
I :	$abcdeghimn$	X :	$cefijklmn$

Recalling that the first word $abcdefghi$ for instance means the condition $\{a = b = c = d = e = f = g = h = i = 0\}$ on the exponents of a general monomial, we may therefore list in an extensive array the 24 families A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X of corresponding monomials, the subsidiary families A′; D′, D″, D‴; E′; F′ being considered as similar to A; D; E; F:

A :	•	•	•	•	•	•	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
B :	•	•	•	•	•	•	•	•	$(L^{12})^i$	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
C :	•	•	•	•	•	•	•	$(E^{10})^h$	$(L^{12})^i$	•	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
D :	•	•	•	•	•	•	$(M^8)^g$	•	•	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
E :	•	•	•	•	•	•	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
F :	•	•	•	•	•	•	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
G :	•	•	•	•	•	$(N^{10})^f$	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	•	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
H :	•	•	•	•	•	$(N^{10})^f$	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
I :	•	•	•	•	•	$(N^{10})^f$	•	•	$(L^{12})^i$	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
J :	•	•	•	•	•	$(N^{10})^f$	•	$(E^{10})^h$	$(L^{12})^i$	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
K :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	•	$(Q^{14})^j$	•	•	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
L :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	•	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
M :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
N :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
O :	•	•	•	•	$(D^8)^e$	$(N^{10})^f$	•	$(E^{10})^h$	•	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
P :	•	•	•	•	$(D^8)^e$	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	•	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
Q :	•	•	•	$(D^6)^d$	$(D^8)^e$	•	•	$(E^{10})^h$	•	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
R :	•	•	•	$(D^6)^d$	$(D^8)^e$	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
S :	•	•	$(\Lambda^7)^c$	•	•	$(N^{10})^f$	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
T :	•	•	$(\Lambda^7)^c$	•	•	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
U :	•	•	$(\Lambda^7)^c$	•	$(D^8)^e$	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
V :	•	•	$(\Lambda^7)^c$	$(D^6)^d$	$(D^8)^e$	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
W :	•	$(\Lambda^5)^b$	$(\Lambda^7)^c$	$(D^6)^d$	•	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
X :	$(\Lambda^3)^a$	$(\Lambda^5)^b$	•	$(D^6)^d$	•	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$

General Schur bundle decomposition of $\mathsf{E}^4_{4,m}T^*_X$. By general representation theory, the polynomial action of $\mathsf{GL}_4(\mathbb{C})$ decomposes in a certain direct sum of irreducible Schur representations. What we call bi-invariants correspond to vectors of highest weight for the $\mathsf{GL}_4(\mathbb{C})$-representation. To each vector of highest weight corresponds one and only one irreducible Schur representation. Such a vector of highest weight is nothing else but a monomial:

$$\begin{gathered}\big(\Lambda^3\big)^a\,\big(\Lambda^5\big)^b\,\big(\Lambda^7\big)^c\,\big(D^6\big)^d\,\big(D^8\big)^e\,\big(N^{10}\big)^f\,\big(M^8\big)^g\,\big(E^{10}\big)^h\\ \big(L^{12}\big)^i\,\big(Q^{14}\big)^j\,\big(R^{15}\big)^k\,\big(U^{17}\big)^l\,\big(V^{19}\big)^{m'}\,\big(X^{21}\big)^n\,(f_1')^o\,\big(W^{10}\big),\end{gathered}$$

with the usual condition on exponents: $3a+\cdots+21n+o+10p=m$ and $(a,\dots,n)$ belonging to the complement $\mathbb{N}^{14}\backslash\big(\square_1\cup\cdots\cup\square_{41}\big)$ of the 41 quadrants. From now on, we denote by m' the exponent of V^{19} to distinguish it from the weight m of the bi-invariant.

To know what are the four integers $\ell_1,\ell_2,\ell_3,\ell_4$ of the corresponding Schur representations $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\mathbb{C}^4$, it suffices to consider the diagonal matrices of $\mathsf{GL}_4(\mathbb{C})$ of the form:

$$\mathsf{x}:=\begin{pmatrix} x_1 & 0 & 0 & 0\\ 0 & x_2 & 0 & 0\\ 0 & 0 & x_3 & 0\\ 0 & 0 & 0 & x_4\end{pmatrix},$$

for which all vectors of highest weight are then just eigenvectors having eigenvalue of the form $x_1^{\ell_1}x_2^{\ell_2}x_3^{\ell_3}x_4^{\ell_4}$.

Here in our situation, coming back to the theorem which describes the 2835 generators of E_4^4, we should at first write down our 16 bi-invariants under a form in which we emphasize the lower indices as we did for the general invariants. This gives us the following more informative list:

$$\boxed{\begin{array}{l} \ell^3_{[1,2]}, \quad \ell^5_{[1,2];\,1}, \quad \ell^7_{[1,2];\,1,1}, \quad D^6_{[1,2,3]}, \quad D^8_{[1,2,3];\,1}, \quad N^{10}_{[1,2,3];\,1,1}, \quad M^8_{[1,2],[1,2]}, \\ \qquad E^{10}_{[1,2,3],[1,2]}, \quad L^{12}_{[1,2,3],[1,2];\,1}, \quad Q^{14}_{[1,2,3],[1,2];\,1,1}, \quad R^{15}_{[1,2,3],[1,2,3];\,1}, \\ U^{17}_{[1,2,3],[1,2,3],[1,2]}, \quad V^{19}_{[1,2,3],[1,2,3],[1,2];\,1}, \quad X^{21}_{[1,2,3],[1,2,3],[1,2];\,1,1}, \quad f'_1, \quad W^{10}_{[1,2,3,4]} \end{array}}\,.$$

Then it is easy to realize that ℓ_1, ℓ_2, ℓ_3, ℓ_4 just count the number of indices 1, 2, 3, 4 respectively at the bottom of each invariant. Consequently, we have the sixteen correspondences:

$$\begin{aligned}
\big(\ell^3\big)^a &: \qquad \Gamma^{(a,a,0,0)}\mathbb{C}^4 \\
\big(\ell^5\big)^b &: \qquad \Gamma^{(2b,b,0,0)}\mathbb{C}^4 \\
\big(\ell^7\big)^c &: \qquad \Gamma^{(3c,c,0,0)}\mathbb{C}^4 \\
\big(D^6\big)^d &: \qquad \Gamma^{(d,d,d,0)}\mathbb{C}^4 \\
\big(D^8\big)^e &: \qquad \Gamma^{(2e,e,e,0)}\mathbb{C}^4 \\
\big(N^{10}\big)^f &: \qquad \Gamma^{(3f,f,f,0)}\mathbb{C}^4 \\
\big(M^8\big)^g &: \qquad \Gamma^{(2g,2g,0,0)}\mathbb{C}^4 \\
\big(E^{10}\big)^h &: \qquad \Gamma^{(2h,2h,h,0)}\mathbb{C}^4 \\
\big(L^{12}\big)^i &: \qquad \Gamma^{(3i,2i,i,0)}\mathbb{C}^4 \\
\big(Q^{14}\big)^j &: \qquad \Gamma^{(4j,2j,j,0)}\mathbb{C}^4 \\
\big(R^{15}\big)^k &: \qquad \Gamma^{(3k,2k,2k,0)}\mathbb{C}^4 \\
\big(U^{17}\big)^l &: \qquad \Gamma^{(3l,3l,2l,0)}\mathbb{C}^4 \\
\big(V^{19}\big)^{m'} &: \qquad \Gamma^{(4m',3m',2m',0)}\mathbb{C}^4 \\
\big(X^{21}\big)^n &: \qquad \Gamma^{(5n,3n,2n,0)}\mathbb{C}^4 \\
\big(f'_1\big)^o &: \qquad \Gamma^{(o,0,0,0)}\mathbb{C}^4 \\
\big(W^{10}\big)^p &: \qquad \Gamma^{(p,p,p,p)}\mathbb{C}^4
\end{aligned}$$

and it immediately follows that the Schur representation $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\mathbb{C}^4$ which corresponds to the general monomial written above has integers ℓ_i given by:

$$\begin{cases}
\ell_1 = o + a + 2b + 3c + d + 2e + 3f + 2g + 2h + 3i + 4j + 3k + 3l + 4m' + 5n + p, \\
\ell_2 = a + b + c + d + e + f + 2g + 2h + 2i + 2j + 2k + 3l + 3m' + 3n + p, \\
\ell_3 = d + e + f + h + i + j + 2k + 2l + 2m' + 2n + p, \\
\ell_4 = p.
\end{cases}$$

By a direct application of the theorem on p. 75 of §11, we obtain an exact Schur bundle decompositition of the graduate m-th part $\mathsf{E}_{4,m}^4 T_X^*$ of the Demailly-Semple bundle $\mathsf{E}_4^4 T_X^*$ on a complex algebraic hypersurface $X \subset \mathbb{P}^5(\mathbb{C})$.

THEOREM *In dimension $n = 4$ for jet order $\kappa = 4$, graduate m-th part $\mathsf{E}_{4,m}^4 T_X^*$ of the Demailly-Semple bundle $\mathsf{E}_4^4 T_X^* = \oplus_m \mathsf{E}_{4,m}^4 T_X^*$ on a complex algebraic hypersurface $X \subset \mathbb{P}^5(\mathbb{C})$ has the following decomposition in direct sums of Schur bundles:*

$$\mathsf{E}_{4,m}^4 T_X^* = \bigoplus_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\backslash(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n+10p=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T_X^*,$$

where the 41 subsets $\square_i$ of $\mathbb{N}^{14}$ are precisely defined by:

$$\begin{gathered}
\{a\geqslant 1,\, c\geqslant 1\},\quad \{a\geqslant 1,\, e\geqslant 1\},\quad \{a\geqslant 1,\, f\geqslant 1\},\quad \{a\geqslant 1,\, i\geqslant 1\},\\
\{a\geqslant 1,\, j\geqslant 1\},\quad \{a\geqslant 1,\, k\geqslant 1\},\quad \{a\geqslant 1,\, l\geqslant 1\},\quad \{a\geqslant 1,\, m'\geqslant 1\},\\
\{a\geqslant 1,\, n\geqslant 1\},\quad \{b\geqslant 1,\, e\geqslant 1\},\quad \{b\geqslant 1,\, f\geqslant 1\},\quad \{b\geqslant 1,\, i\geqslant 1\},\\
\{b\geqslant 1,\, j\geqslant 1\},\quad \{b\geqslant 1,\, k\geqslant 1\},\quad \{b\geqslant 1,\, l\geqslant 1\},\quad \{b\geqslant 1,\, m'\geqslant 1\},\\
\{b\geqslant 1,\, n\geqslant 1\},\quad \{c\geqslant 1,\, k\geqslant 1\},\quad \{c\geqslant 1,\, l\geqslant 1\},\quad \{c\geqslant 1,\, m'\geqslant 1\},\\
\{c\geqslant 1,\, n\geqslant 1\},\quad \{d\geqslant 1,\, f\geqslant 1\},\quad \{d\geqslant 1,\, i\geqslant 1\},\quad \{d\geqslant 1,\, j\geqslant 1\},\\
\{d\geqslant 1,\, m\geqslant 1\},\quad \{d\geqslant 1,\, n\geqslant 1\},\quad \{e\geqslant 1,\, i\geqslant 1\},\quad \{e\geqslant 1,\, j\geqslant 1\},\\
\{e\geqslant 1,\, m'\geqslant 1\},\quad \{e\geqslant 1,\, n\geqslant 1\},\quad \{d\geqslant 1,\, g\geqslant 1,\, k\geqslant 1\},\\
\{e\geqslant 1,\, g\geqslant 1,\, k\geqslant 1\},\quad \{f\geqslant 1,\, g\geqslant 1,\, k\geqslant 1\},\quad \{g\geqslant 1,\, k\geqslant 2\},\\
\{h\geqslant 1,\, j\geqslant 1\},\quad \{h\geqslant 1,\, n\geqslant 1\},\quad \{i\geqslant 2,\, n\geqslant 1\},\\
\{f\geqslant 1,\, h\geqslant 1,\, m'\geqslant 1\},\quad \{f\geqslant 1,\, i\geqslant 1,\, m'\geqslant 1\},\quad \{f\geqslant 1,\, i\geqslant 1,\, n\geqslant 1\},\\
\{f\geqslant 1,\, l\geqslant 1,\, n\geqslant 1\}.
\end{gathered}$$

In addition, in the preceding dimension $n = 3$ for jets of the same order $\kappa = 4$, one has an entirely similar Schur bundle decomposition of $\mathsf{E}_{4,m}^3 T_X^*$ for any m in which one removes W^{10}, one sets $p = 0$ and one removes the fourth component ℓ_4 of $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}$:

$$\mathsf{E}_{4,m}^3 T_X^* = \bigoplus_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\backslash(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n \\ d+e+f+h+i+j+2k+2l+2m'+2n \end{pmatrix} T_X^*.$$

Approximate Schur bundle decomposition. We now come back to our 24 words of 9 letters and we make three remarks which will simplify a bit the further computations.

• The full complement $\mathbb{N}^{14}\big\backslash\big(\square_1\cup\cdots\cup\square_{41}\big)$ is slightly larger than the union of the 30 subsets of $\mathbb{N}^{14}$ defined by A, A', B, ..., W X, in the sense that it contains also a finite number of subsets defined by equating to 0 (or to 1) more than 9 exponents. These subsets will not contribute to the dominant term m^{16} when calculating the Euler-Poincaré characteristic of $\mathsf{E}_{4,m}^4 T_X^*$ and hence, they will at once be left out.

• The first family A corresponds to a general polynomial of the form:

$$\sum_{o+14j+15k+17l+19m'+21n+10p=m} \mathsf{A}_{j,k,l,m',n,o,p}\cdot\big(Q^{14}\big)^j\big(R^{15}\big)^k\big(U^{17}\big)^l\big(V^{19}\big)^{m'} \big(X^{21}\big)^n\big(f_1'\big)^o\big(W^{10}\big)^p.$$

The second family A' corresponds to a general polynomial of the form:

$$L^{12}\sum_{o+14j+15k+17l+19m'+21n+10p=m-12} \mathsf{A}'_{j,k,l,m',n,o,p}\cdot\big(Q^{14}\big)^j\big(R^{15}\big)^k\big(U^{17}\big)^l\big(V^{19}\big)^{m'} \big(X^{21}\big)^n\big(f_1'\big)^o\big(W^{10}\big)^p.$$

It is entirely of the same type as A, except that the weight m is replaced by $m-12$. We will see that its contribution to the dominant m^{16}-term of the Euler-Poincaré characteristic is exactly the same[25], hence we will remove A' and provide the family A with the multiplicity 2. Similarly, D, E and F will have multiplicity 4, 2 and 2.

• The third (now second) family B corresponds to a general polynomial of the form:

$$\sum_{o+12i+14j+15k+17l+19m'+10p=m} \mathsf{B}_{i,j,k,l,m',o,p}\cdot\big(L^{12}\big)^i\big(Q^{14}\big)^j\big(R^{15}\big)^k \big(U^{17}\big)^l\big(V^{19}\big)^{m'}\big(f_1'\big)^o\big(W^{10}\big)^p,$$

hence its intersection with the family A is nontrivial, consisting of polynomials of the form:

$$\sum_{o+14j+15k+17l+19m'+10p=m} \widetilde{\mathsf{B}}_{j,k,l,m',o,p}\cdot\big(Q^{14}\big)^j\big(R^{15}\big)^k\big(U^{17}\big)^l\big(V^{19}\big)^{m'}\big(f_1'\big)^o\big(W^{10}\big)^p.$$

In principle, we should write the union of two overlapping families $\mathsf{A}\cup\mathsf{B}$ in the form of two non-intersecting families: $\mathsf{A}\cup\big(\mathsf{B}\backslash\mathsf{A}\big)$, but here again, because the intersection $\mathsf{A}\cap\mathsf{B}$ is represented by the word $abcdefghin$ which has $10>9$ letters, this intersection will only contribute the Euler-Poincaré characteristic as an $\mathrm{O}(m^{15})$, which will not perturb the dominant term m^{16}, as $m\to\infty$. So we can consider the 24 remaining families (a bit of which have multiplicities) without caring about overlappings.

[25] The argument will simply be that $(m-\mathrm{cst.})^{16}=m^{16}+\mathrm{O}(m^{15})$ as $m\to\infty$.

In summary, up to certain negligible sums of Schur bundles which will not contribute to the dominant m^{16}-term while calculating the Euler-Poincaré characteristic of $\mathsf{E}_{4,m}^4 T_X^*$, we have to consider **24** direct sums of Schur bundles with multiplicities, indexed from A up to X in the roman alphabet:

$$\underline{\mathsf{A:}}\quad \mathbf{2}\cdot \bigoplus_{m=o+14j+15k+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+4j+3k+3l+4m+5n+p \\ 2j+2k+3l+3m+3n+p \\ j+2k+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{B:}}\quad \bigoplus_{m=o+12i+14j+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+3i+4j+3k+3l+4m+p \\ 2i+2j+2k+3l+3m+p \\ i+j+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{C:}}\quad \bigoplus_{m=o+10h+12i+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+2h+3i+3k+3l+4m+p \\ 2h+2i+2k+3l+3m+p \\ h+i+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{D:}}\quad \mathbf{4}\cdot \bigoplus_{m=o+8g+14j+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+2g+4j+3l+4m+5n+p \\ 2g+2j+3l+3m+3n+p \\ j+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{E:}}\quad \mathbf{2}\cdot \bigoplus_{m=o+8g+12i+14j+17l+19m+10p} \Gamma\begin{pmatrix} o+2g+3i+4j+3l+4m+p \\ 2g+2i+2j+3l+3m+p \\ i+j+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{F:}}\quad \mathbf{2}\cdot \bigoplus_{m=o+8g+10h+12i+17l+19m+10p} \Gamma\begin{pmatrix} o+2g+2h+3i+3l+4m+p \\ 2g+2h+2i+3l+3m+p \\ h+i+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{G:}}\quad \bigoplus_{m=o+10f+14j+15k+19m+21n+10p} \Gamma\begin{pmatrix} o+3f+4j+3k+4m+5n+p \\ f+2j+2k+3m+3n+p \\ f+j+2k+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{H:}}\quad \bigoplus_{m=o+10f+14j+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+3f+4j+3k+3l+4m+p \\ f+2j+2k+3l+3m+p \\ f+j+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{I:}}\quad \bigoplus_{m=o+10f+12i+14j+15k+17l+10p} \Gamma\begin{pmatrix} o+3f+3i+4j+3k+3l+p \\ f+2i+2j+2k+3l+p \\ f+i+j+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

J : $$\bigoplus_{m=o+10f+10h+12i+15k+17l+10p} \Gamma \begin{pmatrix} o+3f+2h+3i+3k+3l+p \\ f+2h+2i+2k+3l+p \\ f+h+i+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

K : $$\bigoplus_{m=o+10f+8g+14j+19m+21n+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+4m+5n+p \\ f+2g+2j+3m+3n+p \\ f+j+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

L : $$\bigoplus_{m=o+10f+8g+14j+17l+19m+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+3l+4m+p \\ f+2g+2j+3l+3m+p \\ f+j+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

M : $$\bigoplus_{m=o+10f+8g+12i+14j+17l+10p} \Gamma \begin{pmatrix} o+3f+2g+3i+4j+3l+p \\ f+2g+2i+2j+3l+p \\ f+i+j+2l+p \\ p \end{pmatrix} T_X^*,$$

N : $$\bigoplus_{m=o+10f+8g+10h+12i+17l+10p} \Gamma \begin{pmatrix} o+3f+2g+2h+3i+3l+p \\ f+2g+2h+2i+3l+p \\ f+h+i+2l+p \\ p \end{pmatrix} T_X^*,$$

O : $$\bigoplus_{m=o+8e+10f+10h+15k+17l+10p} \Gamma \begin{pmatrix} o+2e+3f+2h+3k+3l+p \\ e+f+2h+2k+3l+p \\ e+f+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

P : $$\bigoplus_{m=o+8e+10f+8g+10h+17l+10p} \Gamma \begin{pmatrix} o+2e+3f+2g+2h+3l+p \\ e+f+2g+2h+3l+p \\ e+f+h+2l+p \\ p \end{pmatrix} T_X^*,$$

Q : $$\bigoplus_{m=o+6d+8e+10h+15k+17l+10p} \Gamma \begin{pmatrix} o+d+2e+2h+3k+3l+p \\ d+e+2h+2k+3l+p \\ d+e+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

R : $$\bigoplus_{m=o+6d+8e+8g+10h+17l+10p} \Gamma \begin{pmatrix} o+d+2e+2g+2h+3l+p \\ d+e+2g+2h+3l+p \\ d+e+h+2l+p \\ p \end{pmatrix} T_X^*,$$

S : $$\bigoplus_{m=o+7c+10f+8g+12i+14j+10p} \Gamma \begin{pmatrix} o+3c+3f+2g+3i+4j+p \\ c+f+2g+2i+2j+p \\ f+i+j+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{T}:}\qquad \bigoplus_{m=o+7c+10f+8g+10h+12i+10p} \Gamma\begin{pmatrix} o+3c+3f+2g+2h+3i+p \\ c+f+2g+2h+2i+p \\ f+h+i+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{U}:}\qquad \bigoplus_{m=o+7c+8e+10f+8g+10h+10p} \Gamma\begin{pmatrix} o+3c+2e+3f+2g+2h+p \\ c+e+f+2g+2h+p \\ e+f+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{V}:}\qquad \bigoplus_{m=o+7c+6d+8e+8g+10h+10p} \Gamma\begin{pmatrix} o+3c+d+2e+2g+2h+p \\ c+d+e+2g+2h+p \\ d+e+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{W}:}\qquad \bigoplus_{m=o+5b+7c+6d+8g+10h+10p} \Gamma\begin{pmatrix} o+2b+3c+d+2g+2h+p \\ b+c+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{X}:}\qquad \bigoplus_{m=o+3a+5b+6d+8g+10h+10p} \Gamma\begin{pmatrix} o+a+2b+d+2g+2h+p \\ a+b+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*.$$

It is now time to speak of the asymptotic of the Euler characteristic of a single Schur bundle.

§13. Asymptotic expansion of the Euler characteristic $\chi\big(\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big)$

Euler-Poincaré characteristic of Schur bundles. Let $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a complex algebraic hypersurface and denote by $\mathsf{c}_1, \mathsf{c}_2, \dots, \mathsf{c}_n$ be the Chern classes $\mathsf{c}_k\big(T_X\big)$ of the tangent bundle T_X. Each c_k may be represented by a smooth differential form of bidegree (k,k) on X. One thus assigns the weight k to c_k. Because the total degrees of these forms are all even, the commutation relations $\mathsf{c}_{k_1}\mathsf{c}_{k_2} = \mathsf{c}_{k_2}\mathsf{c}_{k_1}$ hold for the cup product.

Every polynomial in the Chern classes:

$$\sum_{k_1+\cdots+k_n=n} \mathsf{coeff}\cdot \mathsf{c}_{k_1}\,\mathsf{c}_{k_2}\cdots \mathsf{c}_{k_n}$$

which is homogeneous of degree $n = \dim X$ is represented by an (n,n)-form on X, hence may be integrated. By a standard abuse of language, such a polynomial is usually considered both as an (n,n)-form and as the purely numerical quantity:

$$\int_X \sum_{k_1+\cdots+k_n=n} \mathsf{coeff}\,\mathsf{c}_{k_1}\cdot \mathsf{c}_{k_2}\cdots \mathsf{c}_{k_n}.$$

For instance, if d denotes the degree of X, one shows $\int_X \mathsf{c}_1^n = d^{n+1}$, a kind of relation often abbreviated $\mathsf{c}_1^n = d^{n+1}$.

To speak in full generality ([4, 29, 9]), the short exact sequence:

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^{n+1}}\big|_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

gives the relation $c_\bullet\big(T_{\mathbb{P}^{n+1}}\big|_X\big) = c_\bullet\big(T_X\big) \cdot c_\bullet\big(\mathcal{O}_X(d)\big)$ between total Chern classes of the middle term and of the two extreme ones, or more explicitly:

$$(1+h)^{n+2} = \big[1 + \mathsf{c}_1 + \cdots + \mathsf{c}_n\big](1+dh),$$

where $(1+h)^{n+2}$ is the total Chern class of $\mathbb{P}^{n+1}$ with $h = \mathsf{c}_1\big(\mathcal{O}_{\mathbb{P}^{n+1}}(1)\big)$ being a $(1,1)$-form. Consequently, by expanding both the left-hand and the right-hand sides and by identifying terms of the same bidegree, we get closed expressions for all the Chern classes.

Lemma. *In terms of the hyperplane divisor $h = \mathsf{c}_1\big(\mathcal{O}_{\mathbb{P}^{n+1}}(1)\big)$ which satisfies $\int_X h^n = d = \deg X$, the Chern classes c_k of T_X are given by:*

$$\mathsf{c}_k = (-1)^k\, h^k \left(d^k - \tfrac{(n+2)!}{1!\,(n+1)!}\, d^{k-1} + \cdots + (-1)^k \tfrac{(n+2)!}{k!\,(n+2-k)!}\right).$$

Proof. We indeed expand the two sides of the above relation between total Chern classes:

$$1 + \tfrac{(n+2)!}{1!\,(n+1)!}\, h + \cdots + \tfrac{(n+2)!}{n!\,2!}\, h^n = 1 + (c_1 + dh) + (\mathsf{c}_2 + d\mathsf{c}_1 h) + \cdots + (\mathsf{c}_n + d\mathsf{c}_{n-1} h),$$

on understanding that the forms h^{n+1}, h^{n+2} and $\mathsf{c}_n h$ of degree $> 2n$ vanish identically. Identifying forms of the same bidegree yields the binomial-type recurrence relations: $\mathsf{c}_k = \frac{(n+2)!}{k!\,(n+2-k)!}\, h^k - d\mathsf{c}_{k-1} h$. □

It follows for instance as we said that $\mathsf{c}_1^n = (-1)^n d^{n+1}$ and that $\mathsf{c}_1^{n-2}\mathsf{c}_2 = (-1)^{n-2} d\big(d - \frac{(n+2)!}{(n+1)!\,1!}\big)^{n-2}\big(d^2 - \frac{(n+2)!}{(n+1)!\,1!}\, d + \frac{(n+2)!}{n!\,2!}\big)$ are numerical quantities.

Following [18], one introduces the formal factorization:

$$1 + \mathsf{c}_1\, x + \mathsf{c}_2\, x^2 + \cdots + \mathsf{c}_n\, x^n = \prod_{0 \leqslant i \leqslant n} \big(1 + \mathsf{a}_i\, x\big),$$

using new formal symbols a_i whose elementary symmetric functions regive the Chern classes c_k:

$$\mathsf{c}_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \mathsf{a}_{i_1}\, \mathsf{a}_{i_2} \cdots \mathsf{a}_{i_k},$$

so that any polynomial $\mathsf{P}\big(\mathsf{a}_1, \ldots, \mathsf{a}_n\big)$ in the a_i which is invariant under all permutations of its arguments may in fact be expressed in terms of the c_k. Every such a symmetric $\mathsf{P}\big(\mathsf{a}_1, \ldots, \mathsf{a}_n\big)$ which is homogeneous of degree n may thus be considered as a numerical quantity, after integration.

Proposition. ([18, 28]) *The Euler-Poincaré characteristic:*

$$\chi\Big(X,\, \Gamma^{(\ell_1,\ldots,\ell_n)}\, T_X\Big) = \sum_{i=0}^{n} (-1)^i\, \dim H^i\big(X,\, \Gamma^{(\ell_1,\ldots,\ell_n)}\, T_X\big)$$

of an arbitrary Schur bundle $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}T_X$ *with* $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n$ *is given as* (*the integral over* X *of*) *the rewriting by means of the* c_k *of all the terms which are homogeneous of degree* n *with respect to* $\mathsf{a}_1,\dots,\mathsf{a}_n$ *in the expansion of the* (*symmetric*) *quotient:*

$$\begin{vmatrix} e^{\mathsf{a}_1\ell_1'} & \cdots & e^{\mathsf{a}_1\ell_n'} \\ \vdots & \ddots & \vdots \\ e^{\mathsf{a}_n\ell_1'} & \cdots & e^{\mathsf{a}_n\ell_n'} \end{vmatrix} \Bigg/ \begin{vmatrix} e^{(n-1)\mathsf{a}_1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ e^{(n-1)\mathsf{a}_n} & \cdots & 1 \end{vmatrix},$$

in which one has abbreviated for notational condensation:

$$\ell_1' := \ell_1 + n - 1, \qquad \ell_2' := \ell_2 + n - 2, \;\dots\dots, \quad \ell_n' := \ell_n.$$

We shall admit this result. In fact, the well known Van der Monde determinant yields an approximate expression of the denominator:

$$\begin{aligned} \begin{vmatrix} e^{(n-1)\mathsf{a}_1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ e^{(n-1)\mathsf{a}_n} & \cdots & 1 \end{vmatrix} &= \prod_{1\leqslant i<j\leqslant n} \big(e^{\mathsf{a}_i} - e^{\mathsf{a}_j}\big) \\ &= \prod_{1\leqslant i<j\leqslant n} \big(\mathsf{a}_i - \mathsf{a}_j\big) \cdot \big[1 + R(\mathsf{a}_1,\dots,\mathsf{a}_n)\big], \end{aligned}$$

where the remainder $R(\mathsf{a})$ denotes a local holomorphic function which vanishes at the origin. Because the determinant at the numerator also visibly vanishes whenever one a_{i_1} is equal to another a_{i_2}, for some two distinct indices i_1 and i_2, this numerator also is a multiple, as a holomorphic function, of the same product $\prod_{1\leqslant i<j\leqslant n}\big(\mathsf{a}_i - \mathsf{a}_j\big)$. Consequently, when one expands simultaneously the numerator and the denominator, the two products should cancel out:

$$\frac{\prod_{i<j}(\mathsf{a}_i - \mathsf{a}_j)\,\big[S(\mathsf{a},\,\ell')\big]}{\prod_{i<j}(\mathsf{a}_i - \mathsf{a}_j)\big[1 + R(\mathsf{a})\big]} = S(\mathsf{a},\,\ell')\Big[1 - R(\mathsf{a}) + R(\mathsf{a})^2 - R(\mathsf{a})^3 + \cdots\Big]$$

and one should obtain a power series in which only the homogeneous terms of degree n in the a_i are relevant. Getting a partial *explicit* expression of the result is our next goal.

Asymptotic expansion of the Euler-Poincaré characteristic of $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}T_X$. A *partition* of n is any sequence:

$$\lambda = \big(\lambda_1,\,\lambda_2,\,\dots,\,\lambda_n\big)$$

of non-negative integers listed in decreasing order:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n,$$

whose total sum equals n:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n.$$

The *diagram* of a partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_n)$ in the real plane consists of λ_1 squares of length one placed above λ_2 squares of length one, *etc.*, placed above

λ_n squares of length one, all horizontal series of squares being justified to the left along a fixed vertical line; some figures appear below. The *conjugate* of a partition λ is the partition $\lambda^c = (\lambda_1^c, \lambda_2^c, \dots, \lambda_n^c)$ whose diagram is obtained from the diagram of λ by reflecting it across its main diagonal. Hence λ_i^c is the number of squares in the i-th column of λ, or equivalently $\lambda_i^c = \text{Card}\,\big\{j :\ \lambda_j \geqslant i\big\}$.

THEOREM *The terms of highest order with respect to $|\ell| = \max_{1\leqslant i\leqslant n} \ell_i$ in the Euler-Poincaré characteristic of the Schur bundle $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T_X$ are homogeneous of order $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ and they are given by a sum of ℓ_i'-determinants indexed by all the partitions $(\lambda_1,\dots,\lambda_n)$ of n:*

$$\begin{aligned}
&\chi\Big(X,\ \Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T_X\Big) = \\
&= \sum_{\lambda \text{ partition of } n} \frac{\mathsf{C}_{\lambda^c}}{(\lambda_1+n-1)!\cdots\lambda_n!}
\begin{vmatrix}
{\ell_1'}^{\lambda_1+n-1} & {\ell_2'}^{\lambda_1+n-1} & \cdots & {\ell_n'}^{\lambda_1+n-1} \\
{\ell_1'}^{\lambda_2+n-2} & {\ell_2'}^{\lambda_2+n-2} & \cdots & {\ell_n'}^{\lambda_2+n-2} \\
\vdots & \vdots & \ddots & \vdots \\
{\ell_1'}^{\lambda_n} & {\ell_2'}^{\lambda_n} & \cdots & {\ell_n'}^{\lambda_n}
\end{vmatrix} + \\
&\qquad + \mathrm{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big),
\end{aligned}$$

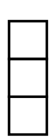
where $\ell_i' := \ell_i + n - i$ for notational brevity, with coefficients C_{λ^c} being expressed in terms of the Chern classes $\mathsf{c}_k\big(T_X\big) = \mathsf{c}_k$ of T_X by means of Giambelli's determinantal expression *depending upon the* conjugate *partition λ^c:*

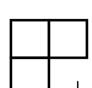
$$\mathsf{C}_{\lambda^c} = \mathsf{C}_{(\lambda_1^c,\dots,\lambda_n^c)} = \begin{vmatrix}
\mathsf{c}_{\lambda_1^c} & \mathsf{c}_{\lambda_1^c+1} & \mathsf{c}_{\lambda_1^c+2} & \cdots & \mathsf{c}_{\lambda_1^c+n-1} \\
\mathsf{c}_{\lambda_2^c-1} & \mathsf{c}_{\lambda_2^c} & \mathsf{c}_{\lambda_2^c+1} & \cdots & \mathsf{c}_{\lambda_2^c+n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathsf{c}_{\lambda_n^c-n+1} & \mathsf{c}_{\lambda_n^c-n+2} & \mathsf{c}_{\lambda_n^c-n+3} & \cdots & \mathsf{c}_{\lambda_n^c}
\end{vmatrix},$$

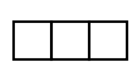
on understanding by convention that $\mathsf{c}_k := 0$ for $k < 0$ or $k > n$, and that $\mathsf{c}_0 := 1$.

In fact, replacing the ℓ_i' by the ℓ_i everywhere in the framed formula would be harmless, because the difference between any two corresponding determinants is easily seen to be an $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$, neglected in the remainder.

We give two expanded instances of this general formula. Firstly, in dimension $n = 3$, there are only three partitions of 3, namely $3+0+0$, $2+1+0$ and $1+1+1$, along which we draw the diagram of the conjugate partitions $1+1+1$, $2+1+1$ and $3+0+0$ together with the corresponding Giambelli determinants:


$$\begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 1 & \mathsf{c}_1 \end{vmatrix}$$

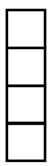

$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 \end{vmatrix}$$


$$\begin{vmatrix} \mathsf{c}_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

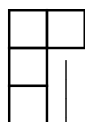
so that we can write down in great details the leading terms, for $|\ell| \to \infty$, of the Euler-Poincaré characteristic:

$$\begin{aligned}\chi\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3)}\,T_X\big) &= \\ &= \frac{\mathsf{c}_1^3 - 2\,\mathsf{c}_1\mathsf{c}_2 + \mathsf{c}_3}{0!\;1!\;5!}\begin{vmatrix} 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 \end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_2 - \mathsf{c}_3}{0!\;2!\;4!}\begin{vmatrix} 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 \end{vmatrix} + \\ &\quad + \frac{\mathsf{c}_3}{1!\;2!\;3!}\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 \end{vmatrix} + \mathrm{O}\big(|\ell|^5\big).\end{aligned}$$

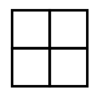
Secondly, in dimension $n = 4$, there are five partitions of 4, namely 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$ along which we again draw the diagram of the conjugate partition together with the corresponding Giambelli determinants:



$$\begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 & \mathsf{c}_1 \end{vmatrix}$$

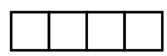


$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 & 0 \\ \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} \mathsf{c}_3 & \mathsf{c}_4 & 0 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} \mathsf{c}_4 & 0 & 0 & 0 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

so that we can write down in length the asymptotic of the Euler-Poincaré characteristic also in this case, of major interest to us:

$$\boxed{\begin{aligned}&\chi\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\,T_X\big) = \\ &= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\;1!\;2!\;7!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \end{vmatrix} + \\ &+ \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\;1!\;3!\;6!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\;1!\;4!\;5!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \\ &+ \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\;2!\;3!\;5!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \frac{\mathsf{c}_4}{1!\;2!\;3!\;4!}\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \end{vmatrix} + \mathrm{O}\big(|\ell|^9\big).\end{aligned}}$$

Proof of the general theorem. Taking the proposition for granted, we start by expanding plainly in Taylor series the exponentials of the numerator determinant:

$$\begin{vmatrix} e^{\mathsf{a}_1\ell'_1} & \cdots & e^{\mathsf{a}_1\ell'_n} \\ \vdots & \ddots & \vdots \\ e^{\mathsf{a}_n\ell'_1} & \cdots & e^{\mathsf{a}_n\ell'_n} \end{vmatrix} = \begin{vmatrix} \sum_{\mu\geqslant 0} \frac{(\ell'_1)^\mu}{\mu!}\,\mathsf{a}_1^\mu & \cdots & \sum_{\mu\geqslant 0} \frac{(\ell'_n)^\mu}{\mu!}\,\mathsf{a}_1^\mu \\ \vdots & \ddots & \vdots \\ \sum_{\mu\geqslant 0} \frac{(\ell'_1)^\mu}{\mu!}\,\mathsf{a}_n^\mu & \cdots & \sum_{\mu\geqslant 0} \frac{(\ell'_n)^\mu}{\mu!}\,\mathsf{a}_n^\mu \end{vmatrix}$$

$$= \sum_{\mu_1,\mu_2,\dots,\mu_n\geqslant 0} \frac{(\ell'_1)^{\mu_1}}{\mu_1!}\,\frac{(\ell'_2)^{\mu_2}}{\mu_2!}\cdots\frac{(\ell'_n)^{\mu_n}}{\mu_n!} \begin{vmatrix} \mathsf{a}_1^{\mu_1} & \cdots & \mathsf{a}_1^{\mu_n} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\mu_1} & \cdots & \mathsf{a}_n^{\mu_n} \end{vmatrix}$$

and we then develope the result by multilinearity. According to what has already been noticed after the proposition, dividing this last sum by the determinant at the denominator amounts to multiplying it by $\big[1/\prod_{i<j}\,(\mathsf{a}_i-\mathsf{a}_j)\big]\cdot\big[1+\sum_{k\geqslant 1}(-1)^k\,R(\mathsf{a})^k\big]$, so we obtain:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\dots,\ell_n)}\,T_X\Big) = \sum_{\mu_1,\mu_2,\dots,\mu_n\geqslant 0} \frac{(\ell'_1)^{\mu_1}}{\mu_1!}\,\frac{(\ell'_2)^{\mu_2}}{\mu_2!}\cdots\frac{(\ell'_n)^{\mu_n}}{\mu_n!}\cdot$$

$$\cdot\,\text{homogeneous n-th part of}\left(\frac{1}{\prod_{i<j}\,(\mathsf{a}_i-\mathsf{a}_j)}\begin{vmatrix} \mathsf{a}_1^{\mu_1} & \cdots & \mathsf{a}_1^{\mu_n} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\mu_1} & \cdots & \mathsf{a}_n^{\mu_n} \end{vmatrix}\Big[1+\mathrm{O}_1(\mathsf{a})\Big]\right),$$

where we have gathered all terms $-R(\mathsf{a})+R(\mathsf{a})^2-\cdots$ simply as a remainder $\mathrm{O}_1(\mathsf{a})$ vanishing at $\mathsf{a}=0$. The order at $\mathsf{a}=0$ of the Van der Monde denominator $\prod_{i<j}\,(\mathsf{a}_i-\mathsf{a}_j)$ is equal to $\frac{n(n-1)}{2}$, while the order of the determinant $\big|\mathsf{a}_i^{\mu_j}\big|$ equals $\mu_1+\cdots+\mu_n$. Consequently, when selecting in the sum $\sum_{\mu_1,\dots,\mu_n\geqslant 0}$ only homogeneous terms of order n with respect to a, one must consider:

- all terms with $\mu_1+\cdots+\mu_n = n+\frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ if the determinant is multiplied by the term 1 inside the last brackets; with respect to the ℓ'_i, this then gives terms which are homogeneous of degree $\frac{n(n+1)}{2}$;
- some appropriate terms with $\mu_1+\cdots+\mu_n < \frac{n(n+1)}{2}$ if the determinant is multiplied by some nonzero monomial belonging to the remainder $\mathrm{O}_1(\mathsf{a})$; with respect to the ℓ'_i, this then gives terms in $\mathrm{O}\big(|\ell'|\big)^{\frac{n(n+1)}{2}-1}$, and we announced in the theorem that we should neglect them.

As a result, we may therefore represent as follows the principal terms of the Euler-Poincaré characteristic, considered asymptotically for $|\ell|\to\infty$:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\dots,\ell_n)}\,T_X\Big) = \sum_{\substack{\mu_1+\cdots+\mu_n=\frac{n(n+1)}{2}\\ \mu_1,\dots,\mu_n\geqslant 0}} \frac{(\ell'_1)^{\mu_1}}{\mu_1!}\,\frac{(\ell'_2)^{\mu_2}}{\mu_2!}\cdots\frac{(\ell'_n)^{\mu_n}}{\mu_n!}\cdot$$

$$\cdot\,\frac{1}{\prod_{i<j}\,(\mathsf{a}_i-\mathsf{a}_j)}\begin{vmatrix} \mathsf{a}_1^{\mu_1} & \cdots & \mathsf{a}_1^{\mu_n} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\mu_1} & \cdots & \mathsf{a}_n^{\mu_n} \end{vmatrix} + \mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

Whenever there exist two equal exponents $\mu_{i_1} = \mu_{i_2}$ for two distinct indices $i_1 \neq i_2$, the determinant obviously vanishes. So in the sum, one may assume the μ_i to be pairwise distinct. Furthermore, for any n-tuple $(\mu_1, \dots, \mu_n)$ of pairwise distinct μ_i, there exists a unique permutation $\sigma \in \mathfrak{S}_n$ rearranging them in decreasing order: $\mu_{\sigma(1)} > \mu_{\sigma(2)} > \cdots > \mu_{\sigma(n)}$. Consequently, we can split as follows the sum to be considered:

$$\chi\Big(X,\, \Gamma^{(\ell_1,\dots,\ell_n)}\, T_X\Big) = \sum_{\sigma\in\mathfrak{S}_n} \sum_{\substack{\mu_1+\cdots+\mu_n=\frac{n(n+1)}{2}\\ \mu_1>\cdots>\mu_n\geqslant 0}} \frac{(\ell_1')^{\mu_{\sigma(1)}}}{\mu_{\sigma(1)}!}\cdots\frac{(\ell_n')^{\mu_{\sigma(n)}}}{\mu_{\sigma(n)}!}\cdot$$

$$\cdot\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix} \mathsf{a}_1^{\mu_{\sigma(1)}} & \cdots & \mathsf{a}_1^{\mu_{\sigma(n)}} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\mu_{\sigma(1)}} & \cdots & \mathsf{a}_n^{\mu_{\sigma(n)}} \end{vmatrix} + \mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

Finally, one easily convinces oneself that there is a one-to-one correspondence between the n-tuples $\mu = (\mu_1, \dots, \mu_n)$ as above with $\mu_1 > \cdots > \mu_n \geqslant 0$ and $\mu_1 + \cdots + \mu_n = \frac{n(n+1)}{2}$ on the one hand, and on the other hand, the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of n, namely with $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ and $\lambda_1 + \cdots + \lambda_n = n$, a correspondence which is simply given by:

$$\mu_i \longmapsto \lambda_i := \mu_i - n + i \qquad \text{and has obvious inverse} \qquad \lambda_i \longmapsto \mu_i := \lambda_i + n - i.$$

Taking account of the skew-symmetry $\big|\mathsf{a}_i^{\mu_{\sigma(j)}}\big| = \mathrm{sgn}(\sigma)\,\big|\mathsf{a}_i^{\mu_j}\big|$, we thus obtain an almost final asymptotic representation of the Euler-Poincaré characteristic:

$$\chi\Big(X,\, \Gamma^{(\ell_1,\dots,\ell_n)}\, T_X\Big) = \sum_{\sigma\in\mathfrak{S}_n} \sum_{\substack{\lambda_1+\cdots+\lambda_n=n\\ \lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0}} \frac{\big(\ell'_{\sigma^{-1}(1)}\big)^{\lambda_1+n-1}}{(\lambda_1+n-1)!}\cdots\frac{\big(\ell'_{\sigma^{-1}(n)}\big)^{\lambda_n}}{\lambda_n!}\cdot\mathrm{sgn}(\sigma)\cdot$$

$$\cdot\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix} \mathsf{a}_1^{\lambda_1+n-1} & \cdots & \mathsf{a}_1^{\lambda_n} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\lambda_1+n-1} & \cdots & \mathsf{a}_n^{\lambda_n} \end{vmatrix} + \mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

To conclude the proof of the theorem, using $\mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\sigma)$, it now suffices only to observe the compulsory reconstitution of ℓ'-determinants:

$$\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}\,(\sigma^{-1})\cdot\frac{\big(\ell'_{\sigma^{-1}(1)}\big)^{\lambda_1+n-1}}{(\lambda_1+n-1)!}\cdots\frac{\big(\ell'_{\sigma^{-1}(n)}\big)^{\lambda_n}}{\lambda_n!} =$$

$$= \frac{1}{(\lambda_1+n-1)!\cdots\lambda_n!}\cdot\begin{vmatrix} {\ell_1'}^{\lambda_1+n-1} & \cdots & {\ell_n'}^{\lambda_1+n-1} \\ \vdots & \ddots & \vdots \\ {\ell_1'}^{\lambda_n} & \cdots & {\ell_n'}^{\lambda_n}, \end{vmatrix},$$

and also to recognize the Schur polynomials:

$$S_\lambda(\mathsf{a}) = S_{(\lambda_1,\dots,\lambda_n)}(\mathsf{a}) = \frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix} \mathsf{a}_1^{\lambda_1+n-1} & \mathsf{a}_1^{\lambda_2+n-2} & \cdots & \mathsf{a}_1^{\lambda_n} \\ \mathsf{a}_2^{\lambda_1+n-1} & \mathsf{a}_2^{\lambda_2+n-2} & \cdots & \mathsf{a}_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\lambda_1+n-1} & \mathsf{a}_n^{\lambda_2+n-2} & \cdots & \mathsf{a}_n^{\lambda_n} \end{vmatrix}$$

indexed by the partitions of n, which according to Giambelli's formulas (Appendix A of [16]), are expressed in terms of the elementary symmetric functions $\mathsf{c}_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \mathsf{a}_{i_1} \cdots \mathsf{a}_{i_k}$ of the a_i by means of the specific determinants written and exemplified above. Thus, the proof is achieved. □

Computation of the Euler-Poincaré characteristic of $\mathsf{E}^4_{4,m} T^*_X$. As is known, duality shows that the cotangent bundle T^*_X has Chern classes $\mathsf{c}_k\big(T^*_X\big)$ related to those of T_X by the relations:

$$\mathsf{c}^*_k := \mathsf{c}_k\big(T^*_X\big) = (-1)^k\, \mathsf{c}_k\big(T_X\big) = (-1)^k\, \mathsf{c}_k.$$

Consequently, the dual Giambelli determinants satisfy $\mathsf{C}^*_{\lambda^c} = (-1)^n \mathsf{C}_{\lambda^c}$, because all monomials $\mathsf{c}^*_{\mu_1} \cdots \mathsf{c}^*_{\mu_n}$ have total weight $\mu_1 + \cdots + \mu_n = n$ and we therefore deduce:

$$\chi\big(X,\, \Gamma^{(\ell_1,\dots,\ell_n)} T^*_X\big) = (-1)^n\, \chi\big(X,\, \Gamma^{(\ell_1,\dots,\ell_n)} T_X\big).$$

When considering Demailly-Semple and Schur bundles, everything shall be expressed in terms of Chern classes of T_X (not of T^*_X).

§14. Euler characteristic calculations

Explaining the final calculations on an example. We may now come back to our 24 sums of Schur bundles (with multiplicities). Consider for instance the family A. In it, we have:

$$\begin{cases} \ell_1 = o + 4j + 3k + 3l + 4m' + 5n + p, \\ \ell_2 = 2j + 2k + 3l + 3m' + 3n + p, \\ \ell_3 = j + 2k + 2l + 2m' + 2n + p, \\ \ell_4 = p. \end{cases}$$

But since sums of weight should be equal to m:

$$o + 14j + 15k + 17l + 19m' + 21n + 10p = m,$$

we may eliminate o and this provides ℓ_1 with the value:

$$\ell_1 = m - 10j - 12k - 14l - 15m' - 16n - 9p,$$

while ℓ_2, ℓ_3 and ℓ_4 where at the beginning independent of o. The Euler-Poincaré characteristic being additive, we have:

$$\chi\Big(X,\, \oplus_{\mathsf{A}}\, \Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)} T^*_X\Big) = \sum_{o+14j+15k+17l+19m'+21n+10p=m} \chi\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)} T^*_X\big).$$

Furthermore, according to the formula written on p. 94, the dominant term of the Euler-Poincaré characteristic, as $|\ell| \to \infty$, of a single Schur bundle in such a sum is given, in terms of the Chern classes c_k of T_X, by:

$$
\begin{aligned}
\chi\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}T_X^*\big) = {} & \frac{\mathsf{c}_1^4-3\,\mathsf{c}_1^2\mathsf{c}_2+\mathsf{c}_2^2+2\,\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_4}{0!\,1!\,2!\,7!}\,\Delta_{0127}+ \\
& +\frac{\mathsf{c}_1^2\mathsf{c}_2-\mathsf{c}_2^2-\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_4}{0!\,1!\,3!\,6!}\,\Delta_{0136}+\frac{-\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_2^2}{0!\,1!\,4!\,5!}\,\Delta_{0145}+ \\
& +\frac{\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_4}{0!\,2!\,3!\,5!}\,\Delta_{0235}+\frac{\mathsf{c}_4}{1!\,2!\,3!\,4!}\,\Delta_{1234} \\
& +\mathrm{O}\big(|\ell|^9\big),
\end{aligned}
$$

on understanding that, in the five determinants:

$$
\Delta_{0137} := \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \end{vmatrix}, \qquad
\Delta_{0136} := \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \end{vmatrix},
$$

$$
\Delta_{0145} := \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix}, \qquad
\Delta_{0235} := \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix},
$$

$$
\Delta_{1234} := \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \end{vmatrix},
$$

one should substitute the above values for ℓ_1, ℓ_2, ℓ_3 and ℓ_4 in terms of j, k, l, m', n and p.

On the other hand, it is well known that the dominant term of a multiple sum is given by an integral, so that we have to compute[26]:

$$
\int_0^{\frac{(m-15k-17l-19m'-21n-10p)}{14}} dj \int_0^{\frac{(m-17l-19m'-21n-10p)}{15}} dk \int_0^{\frac{(m-19m'-21n-10p)}{17}} dl
$$

$$
\int_0^{\frac{(m-21n-10p)}{19}} dm' \int_0^{\frac{(m-10p)}{21}} dn \int_0^{\frac{m}{10}} dp \;\left\{ \begin{array}{l} \Delta_{0127} \\ \Delta_{0136} \\ \Delta_{0145} \\ \Delta_{0235} \\ \Delta_{1234} \end{array} \right.
$$

It happens that all the five integrals are equal to m^{16} times a fractional number. A computation with the help of Maple yields the values of these five fractional

---

[26] The Δ determinants being of degree 10 in the ℓ_i, the presence of six integrals entails that the result is m^{16} times a fractional constant plus an $\mathrm{O}\big(m^{15}\big)$. If there would be 5 or less integrals, this would leave us with an $\mathrm{O}\big(m^{15}\big)$, negligible in comparison with m^{16} as $m\to\infty$. By this remark we therefore justify why we considered only the approximate Schur bundle decomposition of $\mathsf{E}_{4,m}^4 T_X^*$ in the §12.

numbers, which, we guess, would be quite uneasy to get by hand:

$$\mathsf{A}_{0127} = \frac{15742375476686365148211006393963161771361426 7}{74701304405498490709957626608227816855454124187200000000000000},$$

$$\mathsf{A}_{0136} = \frac{2852246112539025445894910116384578085373150 47}{348606087225659623313135590838396478658785912873600000000000000}$$

$$\mathsf{A}_{0145} = \frac{10306128852122999807705628256770676631371801}{52290913083848943496970338625759471798817886931040000000000000 0}$$

$$\mathsf{A}_{0235} = \frac{20975222336265133050996115522925065371392 47}{23768596856294974316804699375345214454008130423200000000000000 0}$$

$$\mathsf{A}_{1234} = \frac{200513595153718202861975082479028443 53}{2102485347748339169995992868230447983547822240000000000000 0}$$

End of the computation. Similarly, for the other 23 families, we compute these 5-tuples of rational numbers and at the end, we make the addition[27]:

$$\begin{aligned}
\mathsf{Coeff}_{0127} &= 2\,\mathsf{A}_{0127} + \mathsf{B}_{0127} + \mathsf{C}_{0127} + 4\,\mathsf{D}_{0127} + 2\,\mathsf{E}_{0127} + 2\,\mathsf{F}_{0127} + \mathsf{G}_{0127} + \mathsf{H}_{0127} + \\
&\quad + \mathsf{I}_{0127} + \mathsf{J}_{0127} + \mathsf{K}_{0127} + \mathsf{L}_{0127} + \mathsf{M}_{0127} + \mathsf{N}_{0127} + \mathsf{O}_{0127} + \mathsf{P}_{0127} + \\
&\quad + \mathsf{Q}_{0127} + \mathsf{R}_{0127} + \mathsf{S}_{0127} + \mathsf{T}_{0127} + \mathsf{U}_{0127} + \mathsf{V}_{0127} + \mathsf{W}_{0127} + \mathsf{X}_{0127} \\
&= \frac{2127566277536547206644157}{651447337452328538298777600000000000000}, \\
\mathsf{Coeff}_{0136} &= 2\,\mathsf{A}_{0136} + \mathsf{B}_{0136} + \mathsf{C}_{0136} + 4\,\mathsf{D}_{0136} + 2\,\mathsf{E}_{0136} + 2\,\mathsf{F}_{0136} + \mathsf{G}_{0136} + \mathsf{H}_{0136} + \\
&\quad + \mathsf{I}_{0136} + \mathsf{J}_{0136} + \mathsf{K}_{0136} + \mathsf{L}_{0136} + \mathsf{M}_{0136} + \mathsf{N}_{0136} + \mathsf{O}_{0136} + \mathsf{P}_{0136} + \\
&\quad + \mathsf{Q}_{0136} + \mathsf{R}_{0136} + \mathsf{S}_{0136} + \mathsf{T}_{0136} + \mathsf{U}_{0136} + \mathsf{V}_{0136} + \mathsf{W}_{0136} + \mathsf{X}_{0136} \\
&= \frac{5267640708714311654799 7}{40534500997033775716368384000000000000 0}, \\
\mathsf{Coeff}_{0145} &= 2\,\mathsf{A}_{0145} + \mathsf{B}_{0145} + \mathsf{C}_{0145} + 4\,\mathsf{D}_{0145} + 2\,\mathsf{E}_{0145} + 2\,\mathsf{F}_{0145} + \mathsf{G}_{0145} + \mathsf{H}_{0145} + \\
&\quad + \mathsf{I}_{0145} + \mathsf{J}_{0145} + \mathsf{K}_{0145} + \mathsf{L}_{0145} + \mathsf{M}_{0145} + \mathsf{N}_{0145} + \mathsf{O}_{0145} + \mathsf{P}_{0145} + \\
&\quad + \mathsf{Q}_{0145} + \mathsf{R}_{0145} + \mathsf{S}_{0145} + \mathsf{T}_{0145} + \mathsf{U}_{0145} + \mathsf{V}_{0145} + \mathsf{W}_{0145} + \mathsf{X}_{0145} \\
&= \frac{1646852821245426649460 51}{5066812624629221964546048000000000000000}, \\
\mathsf{Coeff}_{0235} &= 2\,\mathsf{A}_{0235} + \mathsf{B}_{0235} + \mathsf{C}_{0235} + 4\,\mathsf{D}_{0235} + 2\,\mathsf{E}_{0235} + 2\,\mathsf{F}_{0235} + \mathsf{G}_{0235} + \mathsf{H}_{0235} + \\
&\quad + \mathsf{I}_{0235} + \mathsf{J}_{0235} + \mathsf{K}_{0235} + \mathsf{L}_{0235} + \mathsf{M}_{0235} + \mathsf{N}_{0235} + \mathsf{O}_{0235} + \mathsf{P}_{0235} + \mathsf{Q}_{0235} + \\
&\quad + \mathsf{R}_{0235} + \mathsf{S}_{0235} + \mathsf{T}_{0235} + \mathsf{U}_{0235} + \mathsf{V}_{0235} + \mathsf{W}_{0235} + \mathsf{X}_{0235} \\
&= \frac{1222982407435661052177 37}{11400328405415749420228608000000000000000}, \\
\mathsf{Coeff}_{1234} &= 2\,\mathsf{A}_{1234} + \mathsf{B}_{1234} + \mathsf{C}_{1234} + 4\,\mathsf{D}_{1234} + 2\,\mathsf{E}_{1234} + 2\,\mathsf{F}_{1234} + \mathsf{G}_{1234} + \mathsf{H}_{1234} + \\
&\quad + \mathsf{I}_{1234} + \mathsf{J}_{1234} + \mathsf{K}_{1234} + \mathsf{L}_{1234} + \mathsf{M}_{1234} + \mathsf{N}_{1234} + \mathsf{O}_{1234} + \mathsf{P}_{1234} + \mathsf{Q}_{1234} + \\
&\quad + \mathsf{R}_{1234} + \mathsf{S}_{1234} + \mathsf{T}_{1234} + \mathsf{U}_{1234} + \mathsf{V}_{1234} + \mathsf{W}_{1234} + \mathsf{X}_{1234} \\
&= \frac{1429957461022772407321}{13028946749046570765975552000000000000000}.
\end{aligned}$$

[27] *See* new-riemann-roch-4-4.mws at [23].

Coming back to the Euler-Poincaré characteristic we therefore get:

$$
\begin{aligned}
\chi\big(X,\,\mathsf{E}^4_{4,m}T^*_X\big) &= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\,1!\,2!\,7!}\,\mathsf{Coeff}_{0127} + \\
&\quad + \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\,1!\,3!\,6!}\,\mathsf{Coeff}_{0136} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\,1!\,4!\,5!}\,\mathsf{Coeff}_{0145} + \\
&\quad + \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\,2!\,3!\,5!}\,\mathsf{Coeff}_{0235} + \frac{\mathsf{c}_4}{1!\,2!\,3!\,4!}\,\mathsf{Coeff}_{1234} + \mathrm{O}\big(m^{15}\big) \\
&= m^{16}\bigg(\frac{2127566277536547206644157}{6566589161519471666051678208000000000000000}\,\mathsf{c}_1^4 - \\
&\qquad - \frac{139915351328310309504209}{20846314798474513225560883200000000000000}\,\mathsf{c}_1^2\mathsf{c}_2 + \\
&\qquad + \frac{18230301659778006701051}{13401202370447901359289139200000000000000}\,\mathsf{c}_2^2 + \\
&\qquad + \frac{405575296543809994270429}{131331783230389433321033564160000000000000}\,\mathsf{c}_1\mathsf{c}_3 - \\
&\qquad - \frac{6163697191750462398371}{6566589161519471666051678208000000000000000}\,\mathsf{c}_4\bigg) + \\
&\quad + \mathrm{O}\big(m^{15}\big).
\end{aligned}
$$

In terms of the degree:

$$
\boxed{
\begin{aligned}
\chi\big(X,\,\mathsf{E}^4_{4,m}T^*_X\big) = \frac{m^{16}}{131331783230389433321033564160000000000000}\cdot d\cdot \\
\cdot\big(50048511135797034256235\,d^4 - \\
- 6170606622505955255988786\,d^3 - \\
- 928886901354141153880624704\,d + \\
+ 1411170475250247662147363941\,d^2 + \\
+ 1624908955061039283976041114\big) \\
+ \mathrm{O}\big(m^{15}\big).
\end{aligned}
}
$$

The four roots of the 4-th degree numerator in parentheses are:

$$
2.794353346\cdots, \qquad 6.784939538\cdots, \qquad 17.86618823\cdots, \qquad 95.84703014\cdots,
$$

hence in conclusion, the characteristic is positive for all degrees $d \geqslant 96$.

Jets of order $\kappa = 4$ **in dimension** $n = 3$**.** For a hypersurface $X^3 \subset \mathbb{P}_4(\mathbb{C})$ of degree d, thanks to a similar but quicker Maple computation[28], one obtains the

[28] *See* new-riemann-roch-3-4.mws at [23].

asymptotic:

$$\chi\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big) = m^{11}\bigg(-\frac{78181453985171}{20130233500548864000000000}\,\mathsf{c}_1^3 + \\ + \frac{3780346214152789}{343555985076033945600000000}\,\mathsf{c}_3 - \\ - \frac{46223512567695359}{1030667955228101836800000000}\,\mathsf{c}_1\mathsf{c}_2\bigg) + \\ + \mathrm{O}\big(m^{10}\big),$$

and then in terms of the degree d of the hypersurface X:

$$\boxed{\begin{aligned}\chi\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big) = \frac{m^{11}}{2061335910456203673600000000}\cdot d\cdot \\ \cdot\big(1029286103034112\,d^3 - 38980726828290305\,d^2 + \\ + 299551055917162501\,d - 561169562618151944\big)\end{aligned}}\,.$$

The three roots of the third degree numerator in parentheses are:

$$2.852373090\cdots,\qquad 6.765004304\cdots,\qquad 28.25423742,$$

hence in conclusion, the Euler-Poincaré characteristic of $\mathsf{E}_{4,m}^3T_X^*$ is positive in all degrees $d\geqslant 29$ as $m\to\infty$. This condition improves the condition $d\geqslant 43$ obtained in [29] for the positivity of $\chi\big(X,\,\mathsf{E}_{3,m}^3T_X^*\big)$ as $m\to\infty$.

Existence of sections. Finally, in order to get positivity of the dimension h^0 of the vector space of sections of $\mathsf{E}_{4,m}^3T_X^*$, it would suffice, in the trivial minoration:

$$h^0\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)\geqslant\chi\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big)-h^2\big(X,\,\mathsf{E}_{4,m}^3T_X^*\big),$$

stemming from the definition $\chi=h^0-h^1+h^2-h^3$, to possess a good majoration of h^2. This main task is achieved in [30, 32]: for each Schur bundle, one has:

$$h^2\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3)}T_X^*\big)\leqslant d(d{+}13)\,\frac{3(\ell_1+\ell_2+\ell_3)^3}{2}\,(\ell_1{-}\ell_2)(\ell_1{-}\ell_3)(\ell_2{-}\ell_3){+}\mathrm{O}\big(|\ell|^5\big).$$

When summing up our 24 sums of Schur bundles (with multiplicities), a Maple computation provides:

$$h^2\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3)}T_X^*\big)\leqslant d(d+13)\,\frac{342988705758851}{29822568148961280000000}\,m^{11}+\mathrm{O}\big(m^{10}\big).$$

Finally, one sees that χ minus this upper bound for h^2 is positive, for $m\to\infty$, in all degrees $d\geqslant 72$. This last condition on the degre insuring the existence of invariant jet differentials improves the condition $d\geqslant 97$ obtained in [30] and appears to be slightly better than the condition $d\geqslant 74$ obtained recently in [9].

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