

Canonical Cartan Connections on Maximally Minimal Generic Submanifolds $M^5 \subset \mathbb{C}^4$

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Abstract. On a real analytic 5-dimensional CR-generic submanifold $M^5 \subset \mathbb{C}^4$ of codimension 3 hence of CR dimension 1, which enjoys the generically satisfied nondegeneracy condition:

$$5 = \text{rank}_{\mathbb{C}}(T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] + [T^{0,1}M, [T^{1,0}M, T^{0,1}M]]),$$

a canonical Cartan connection is constructed after reduction to a certain partially explicit $\{e\}$ -structure of the concerned local biholomorphic equivalence problem.

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1. Introduction

The goal of this announcement is to study local real analytic (\mathcal{C}^ω) 5-dimensional CR-generic submanifolds:

$$M^5 \subset \mathbb{C}^4$$

of codimension 3 hence of CR dimension 1 that are *maximally minimal* in the sense that:

$$5 = \text{rank}_{\mathbb{C}}(T^{1,0}M + T^{0,1}M + [T^{1,0}M, T^{0,1}M] + [T^{1,0}M, [T^{1,0}M, T^{0,1}M]] + [T^{0,1}M, [T^{1,0}M, T^{0,1}M]]).$$

In the terminology of [4], such CR manifolds $M^5 \subset \mathbb{C}^4$ are said to belong to the *General Class III₁*. Most considerations being local, by convention, neighborhoods and their shrinkings will be unmentioned, as in Élie Cartan's original works, *cf.* also Peter Olver's monograph [6].

Therefore, if \mathcal{L} denotes a local vector field generator for $T^{1,0}M$, then the 5 vector fields:

$$\mathcal{L}, \quad \overline{\mathcal{L}}, \quad \mathcal{T} := \sqrt{-1}[\mathcal{L}, \overline{\mathcal{L}}] = \overline{\mathcal{T}}, \quad \mathcal{S} := [\mathcal{L}, \mathcal{T}], \quad \overline{\mathcal{S}} = [\overline{\mathcal{L}}, \overline{\mathcal{T}}],$$

are assumed to constitute a (local) frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, that is to say, at each point $p \in M$:

$$\mathfrak{5} = \text{rank}_{\mathbb{C}} \left(\mathcal{L}|_p, \overline{\mathcal{L}}|_p, \mathcal{T}|_p, \mathcal{S}|_p, \overline{\mathcal{S}}|_p \right).$$

In coordinates $(z, w_1, w_2, w_3) \in \mathbb{C}^4$ with $w_j = u_j + \sqrt{-1}v_j$, for Beloshapka's cubic model having equations:

$$M_c^5: \quad \begin{cases} v_1 = z\bar{z}, \\ v_2 = z^2\bar{z} + z\bar{z}^2, \\ v_3 = -\sqrt{-1}(z^2\bar{z} - z\bar{z}^2), \end{cases}$$

the five vector fields in question ([5], p. 94) are visibly everywhere linearly independent:

$$\begin{aligned} \mathcal{L}_c &= \frac{\partial}{\partial z} + \sqrt{-1}\bar{z} \frac{\partial}{\partial u_1} + \sqrt{-1}(2z\bar{z} + \bar{z}^2) \frac{\partial}{\partial u_2} + (2z\bar{z} - \bar{z}^2) \frac{\partial}{\partial u_3}, \\ \overline{\mathcal{L}}_c &= \frac{\partial}{\partial \bar{z}} - \sqrt{-1}z \frac{\partial}{\partial u_1} - \sqrt{-1}(2z\bar{z} + z^2) \frac{\partial}{\partial u_2} + (2z\bar{z} - z^2) \frac{\partial}{\partial u_3}, \\ \mathcal{T}_c &= 2 \frac{\partial}{\partial u_1} + 4(z + \bar{z}) \frac{\partial}{\partial u_2} - 4\sqrt{-1}(z - \bar{z}) \frac{\partial}{\partial u_3}, \\ \mathcal{S}_c &= 4 \frac{\partial}{\partial u_2} - 4\sqrt{-1} \frac{\partial}{\partial u_3}, \\ \overline{\mathcal{S}}_c &= 4 \frac{\partial}{\partial u_2} + 4\sqrt{-1} \frac{\partial}{\partial u_3}. \end{aligned}$$

Inspired by other similar models such as the Heisenberg quadric $M_q^3 \subset \mathbb{C}^2 \ni (z, w_1)$ the equation of which is the just the first one $v_1 = z\bar{z}$ above, inspired also by the companion Beloshapka cubic $M_c^4 \subset \mathbb{C}^3 \ni (z, w_1, w_2)$ the two equations of which are just the first two above $v_1 = z\bar{z}$, $v_2 = z^2\bar{z} + z\bar{z}^2$, knowing that canonical Cartan connections have been constructed for general geometry-preserving deformations of these two models, by, respectively, Cartan [2] and by Beloshapka-Ezhov-Schmalz [1], the objective of the present announcement is to show how to extract, from the recent republication of an extensive memoir ([5]), the construction of a canonical Cartan connection associated to local biholomorphic equivalences — or equivalently, to real analytic CR equivalences — for General Class III₁ CR manifolds.

Beloshapka's cubic model $M_c^5 \subset \mathbb{C}^4$ happens to be a *homogeneous space*, namely a quotient:

$$M_c^5 \cong G^7/H^2 \cong N_4^5,$$

of a 7-dimensional real Lie group G^7 by a 2-dimensional closed commutative Lie subgroup $N^2 \cong (\mathbb{C}^*, \times)$, the resulting quotient being the unique connected and simply connected *nilpotent* Lie group corresponding to the real *nilpotent* Lie algebra with generators x_1, x_2, x_3, x_4, x_5 named \mathfrak{n}_5^4 in the Goze-Remm classification:

$$\mathfrak{n}_5^4: \quad \begin{cases} [x_1, x_2] = x_3, \\ [x_1, x_3] = x_4, \\ [x_2, x_3] = x_5, \end{cases}$$

unwritten brackets being zero.

Indeed, it is known (see [5] for details) that the Lie algebra $\text{aut}_{\text{CR}}(M_c^5)$ of infinitesimal CR automorphisms of Beloshapka's cubic model $M_c^5 \subset \mathbb{C}^4$ is 7-dimensional,

generated by the real parts of the following 7 vector fields of type $(1, 0)$ with holomorphic coefficients:

$$\begin{aligned}
S_2 &:= \frac{\partial}{\partial w_3}, \\
S_1 &:= \frac{\partial}{\partial w_2}, \\
T &:= \frac{\partial}{\partial w_1}, \\
L_1 &:= \frac{\partial}{\partial z} + 2\sqrt{-1}z \frac{\partial}{\partial w_1} + (2\sqrt{-1}z^2 + 4w_1) \frac{\partial}{\partial w_2} + 2z^2 \frac{\partial}{\partial w_3}, \\
L_2 &:= \sqrt{-1} \frac{\partial}{\partial z} + 2z \frac{\partial}{\partial w_1} + 2z^2 \frac{\partial}{\partial w_2} - (2\sqrt{-1}z^2 - 4w_1) \frac{\partial}{\partial w_3}, \\
D &:= z \frac{\partial}{\partial z} + 2w_1 \frac{\partial}{\partial w_1} + 3w_2 \frac{\partial}{\partial w_2} + 3w_3 \frac{\partial}{\partial w_3}, \\
R &:= \sqrt{-1}z \frac{\partial}{\partial z} - w_3 \frac{\partial}{\partial w_2} + w_2 \frac{\partial}{\partial w_3}.
\end{aligned}$$

Their expressions show well that the isotropy Lie subalgebra of the origin $0 \in \mathbb{C}^4$ is generated by the last two fields D and R , the only ones all of whose coefficients vanish there. Furthermore, the complete Lie bracket commutator table:

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	*	0	0	0	0	$3S_1$	S_2
T	*	*	0	$4S_2$	$4S_1$	$2T$	0
L_2	*	*	*	0	$-4T$	L_2	$-L_1$
L_1	*	*	*	*	0	L_1	L_2
D	*	*	*	*	*	0	0
R	*	*	*	*	*	*	0

shows that the five fields S_2, S_1, T, L_2, L_1 generate a nilpotent Lie subalgebra of $\text{aut}_{\text{CR}}(M_{\mathbb{C}}^5)$ visibly isomorphic to \mathfrak{n}_5^4 .

Once an expected appropriate model geometry G/H of Klein type has been set up, its curved Cartan type deformations can enter the scene.

Recall that a *Cartan geometry* on a \mathcal{C}^ω manifold M modelled on a homogeneous space G/H , where G is a connected Lie group and $H \subset G$ is a closed connected Lie subgroup, with $\text{Lie}(H) = \mathfrak{h} \subset \mathfrak{g} = \text{Lie}(G)$, is a pair (P, ϖ) consisting of an H -principal bundle $\pi: P \rightarrow M$, with right action $R_h: p \mapsto ph$ for $h \in H$ and $p \in P$, together with a \mathfrak{g} -valued differential 1-form $\varpi: TP \rightarrow \mathfrak{g}$ enjoying:

- (i) $\varpi_p: T_p P \xrightarrow{\sim} \mathfrak{g}$ is isomorphic at every point $p \in P$;
- (ii) for every $y \in \mathfrak{h}$, if Y^\dagger denotes the vector field $Y^\dagger|_p := \frac{d}{dt}\big|_0 (p \exp(ty))$, then $\varpi(Y^\dagger) = y$;
- (iii) at every $p \in P$, for every $v_p \in T_p P$:

$$\varpi_{ph}(R_{h*}(v_p)) = \text{Ad}(h^{-1})[\varpi_p(v_p)].$$

If M is a manifold endowed with a certain determined type of geometric structure, e.g. an integrable CR structure, either abstract or embedded, it is adequate to call

canonical a Cartan geometry (P, ϖ) on M when all (local or global) automorphisms $H: M \rightarrow M$ of the geometric structure *lift* as bundle automorphisms:

$$\begin{array}{ccc} P & \xrightarrow{\hat{H}} & P \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{H} & M, \end{array}$$

satisfying $\hat{H}^*(\varpi) = \varpi$, and when conversely also, all such π -fiber preserving maps \hat{H} satisfying $\hat{H}^*(\varpi) = \varpi$, *descend* as automorphisms $H: M \rightarrow M$.

The main result here, computationally less advanced than what comes out from [5], can be summarized as follows.

Theorem 1.1. *Associated to every Class III₁ local CR-generic \mathcal{C}^ω submanifold $M^5 \subset \mathbb{C}^4$, there is a canonical Cartan connection (P^7, ϖ) modelled on the nilpotent homogeneous space $G^7/H^2 \cong N_4^5 \cong M_c^5$ whose natural orbit space is Beloshapka's cubic model $M_c^5 \subset \mathbb{C}^4$.*

Generally, if (P, ϖ) is a canonical Cartan connection on a manifold M belonging to a determined general class of geometric structures, then it automatically carries an associated canonical *absolute parallelism*, or so-called $\{e\}$ -*structure*, obtained, with a basis e_1, \dots, e_r of \mathfrak{g} and with $r := \dim_{\mathbb{R}} \mathfrak{g}$, by plainly setting:

$$V_i|_p := \varpi_p^{-1}(e_i) \quad (1 \leq i \leq r),$$

the obtained vector fields V_1, \dots, V_r making up a frame on P .

The proof of the theorem consists in constructing an absolute parallelism (Section 3) and to verify that it satisfies the algebraic conditions required to constitute a Cartan connection (Section 4).

Acknowledgments. The authors would like to thank Professor Alexander Isaev and an anonymous referee, for realizing that the plain constructions of absolute parallelisms or of Cartan connections usually stays at a lower level than precise inspections of the parametric expressions of incoming invariants and of their *nonlinear* relations.

2. Initial G -structure and initial Darboux structure

The goal being to set up a Cartan procedure in order to reduce to an $\{e\}$ -structure the equivalence problem under local biholomorphisms for such $M^5 \subset \mathbb{C}^4$ belonging to Class III₁, and ultimately, to construct an associated Cartan connection, the first task is to examine how such a frame $\{\mathcal{L}, \overline{\mathcal{L}}, \mathcal{T}, \mathcal{S}, \overline{\mathcal{S}}\}$ transforms under an arbitrary local biholomorphic map:

$$H: \mathbb{C}^4 \rightarrow \mathbb{C}'^4.$$

After appropriate unmentioned shrinkings of concerned neighborhoods, the image:

$$H(M) =: M'$$

also becomes ([4], Section 3) a CR-generic 3-codimensional submanifold $M'^5 \subset \mathbb{C}^4$ of CR dimension 1, also equipped from its side with an analogous, independently constructed, frame starting from a local generator \mathcal{L}' of $T^{1,0}M'$:

$$\{\mathcal{L}', \overline{\mathcal{L}}', \mathcal{F}', \mathcal{S}', \overline{\mathcal{F}}'\}, \quad \text{with } \mathcal{F}' := \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}'], \quad \mathcal{S}' := [\mathcal{L}', \mathcal{F}'],$$

and because the restriction

$$H|_M: M \longrightarrow M'$$

is known to be a CR-diffeomorphism, namely because $H_*(T^{1,0}M) = T^{1,0}M'$, there must exist a nowhere vanishing \mathcal{C}^ω function defined on M' such that ([4], Section 4):

$$H_*(\mathcal{L}) = a' \mathcal{L}',$$

whence by plain conjugation (conventionally not bearing on the differential H_*):

$$H_*(\overline{\mathcal{L}}) = \overline{a'} \overline{\mathcal{L}}'.$$

Proposition 2.1. ([5]) *Under a local biholomorphic map $H: \mathbb{C}^4 \rightarrow \mathbb{C}^4$, two adapted frames associated to two H -equivalent CR-generic $M^5 \subset \mathbb{C}^4$ and $H(M) =: M'^5 \subset \mathbb{C}^4$ transfer, in terms of certain five \mathbb{C} -valued local \mathcal{C}^ω functions a', b', c', d', e' , defined on M' as:*

$$\begin{pmatrix} \mathcal{L} \\ \overline{\mathcal{L}} \\ \mathcal{F} \\ \mathcal{S} \\ \overline{\mathcal{F}} \end{pmatrix} = \begin{pmatrix} a' & 0 & 0 & 0 & 0 \\ 0 & \overline{a'} & 0 & 0 & 0 \\ b' & \overline{b'} & a'\overline{a'} & 0 & 0 \\ e' & d' & c' & a'a'\overline{a'} & 0 \\ \overline{d'} & \overline{e'} & \overline{c'} & 0 & a'\overline{a'}\overline{a'} \end{pmatrix} \begin{pmatrix} \mathcal{L}' \\ \overline{\mathcal{L}}' \\ \mathcal{F}' \\ \mathcal{S}' \\ \overline{\mathcal{F}}' \end{pmatrix}.$$

Proof. Computing further the bracket shows:

$$\begin{aligned} H_*(\mathcal{F}) &= H_*(\sqrt{-1} [\mathcal{L}, \overline{\mathcal{L}}]) \\ &= \sqrt{-1} [H_*(\mathcal{L}), H_*(\overline{\mathcal{L}})] \\ &= \sqrt{-1} [a' \mathcal{L}', \overline{a'} \overline{\mathcal{L}}'] \\ &= a'\overline{a'} \sqrt{-1} [\mathcal{L}', \overline{\mathcal{L}}'] - \underbrace{\sqrt{-1} \overline{a'} \mathcal{L}'(a')}_{=: b'} \cdot \mathcal{L}' + \sqrt{-1} a' \mathcal{L}'(\overline{a'}) \cdot \overline{\mathcal{L}}' \\ &=: a'\overline{a'} \mathcal{F}' + b' \mathcal{L}' + \overline{b'} \overline{\mathcal{L}}', \end{aligned}$$

in terms of some new function b' to which an independent name is given. Then quite similarly ([5], p. 98):

$$\begin{aligned} H_*(\mathcal{S}) &= a'a'\overline{a'} \mathcal{S}' + c' \mathcal{F}' + e' \mathcal{L}' + d' \overline{\mathcal{L}}', \\ H_*(\overline{\mathcal{F}}) &= a'\overline{a'}\overline{a'} \overline{\mathcal{F}}' + \overline{c'} \mathcal{F}' + \overline{d'} \overline{\mathcal{L}}' + \overline{e'} \mathcal{L}', \end{aligned}$$

which completes the proof. \square

The 10 Lie brackets between the 5 local vector fields $\mathcal{L}, \overline{\mathcal{L}}, \mathcal{I}, \mathcal{J}, \overline{\mathcal{I}}$ involve a first set of 5 local real analytic local functions A, B, P, Q, R appearing in:

$$\begin{aligned} [\mathcal{L}, \mathcal{I}] &= P \cdot \mathcal{I} + Q \cdot \mathcal{J} + R \cdot \overline{\mathcal{I}}, \\ [\overline{\mathcal{L}}, \mathcal{I}] &= A \cdot \mathcal{I} + B \cdot \mathcal{J} + \overline{B} \cdot \overline{\mathcal{I}}, \\ [\mathcal{L}, \overline{\mathcal{I}}] &= A \cdot \mathcal{I} + B \cdot \mathcal{J} + \overline{B} \cdot \overline{\mathcal{I}}, \\ [\overline{\mathcal{L}}, \overline{\mathcal{I}}] &= \overline{P} \cdot \mathcal{I} + \overline{R} \cdot \mathcal{J} + \overline{Q} \cdot \overline{\mathcal{I}}, \end{aligned}$$

with A being real-valued, as follows from an inspection of Jacobi identities ([5], Lemma 13.3).

The 5 further real analytic functions E, F, G, J, K , with J also real-valued, appearing in the remaining 3 Lie brackets:

$$\begin{aligned} [\mathcal{I}, \mathcal{J}] &= E \cdot \mathcal{I} + F \cdot \mathcal{J} + G \cdot \overline{\mathcal{I}}, \\ [\mathcal{I}, \overline{\mathcal{I}}] &= \overline{E} \cdot \mathcal{I} + \overline{G} \cdot \mathcal{J} + \overline{F} \cdot \overline{\mathcal{I}}, \\ [\mathcal{J}, \overline{\mathcal{I}}] &= \sqrt{-1} J \cdot \mathcal{I} + K \cdot \mathcal{J} - \overline{K} \cdot \overline{\mathcal{I}}, \end{aligned}$$

all express in terms of A, B, P, Q, R , for instance:

$$\begin{aligned} E &= \sqrt{-1} \left(\mathcal{L}(A) - \overline{\mathcal{L}}(P) - A Q - \overline{P} R + B P + A \overline{B} \right), \\ F &= \sqrt{-1} \left(\mathcal{L}(B) - \overline{\mathcal{L}}(Q) - R \overline{R} + B \overline{B} + A \right), \\ G &= \sqrt{-1} \left(\mathcal{L}(\overline{B}) - \overline{\mathcal{L}}(R) + B R + \overline{B} \overline{B} - R \overline{Q} - \overline{B} Q - P \right), \end{aligned}$$

with similar longer expressions for J and K unwritten here ([5], Lemma 13.5).

Introduce then the coframe of \mathbb{C} -valued 1-forms on M :

$$\{\overline{\sigma}_0, \sigma_0, \rho_0, \overline{\zeta}_0, \zeta_0\},$$

which is *dual* to the frame $\{\overline{\mathcal{I}}, \mathcal{I}, \mathcal{J}, \overline{\mathcal{L}}, \mathcal{L}\}$, in this order. The 10 two-forms making up a basis of $\mathbb{C} \otimes_{\mathbb{R}} \Lambda^2 T^* M$ will be constantly ordered as:

$$\begin{array}{cccc} \overline{\sigma}_0 \wedge \sigma_0, & \overline{\sigma}_0 \wedge \rho_0, & \overline{\sigma}_0 \wedge \overline{\zeta}_0, & \overline{\sigma}_0 \wedge \zeta_0, \\ & \sigma_0 \wedge \rho_0, & \sigma_0 \wedge \overline{\zeta}_0, & \sigma_0 \wedge \zeta_0, \\ & & \rho_0 \wedge \overline{\zeta}_0, & \rho_0 \wedge \zeta_0, \\ & & & \overline{\zeta}_0 \wedge \zeta_0. \end{array}$$

Then the above initial Lie bracket structure translates as the following initial Darboux structure ([5], p. 92) for the exterior differentials of the basis initial 1-forms:

$$\begin{aligned}
d\bar{\sigma}_0 &= -\bar{K} \cdot \bar{\sigma}_0 \wedge \sigma_0 + \bar{F} \cdot \bar{\sigma}_0 \wedge \rho_0 + \bar{Q} \cdot \bar{\sigma}_0 \wedge \bar{\zeta}_0 + \bar{B} \cdot \bar{\sigma}_0 \wedge \zeta_0 + \\
&\quad + G \cdot \sigma_0 \wedge \rho_0 + \bar{B} \cdot \sigma_0 \wedge \bar{\zeta}_0 + R \cdot \sigma_0 \wedge \zeta_0 + \\
&\quad + \rho_0 \wedge \bar{\zeta}_0, \\
d\sigma_0 &= K \cdot \bar{\sigma}_0 \wedge \sigma_0 + \bar{G} \cdot \bar{\sigma}_0 \wedge \rho_0 + \bar{R} \cdot \bar{\sigma}_0 \wedge \bar{\zeta}_0 + B \cdot \bar{\sigma}_0 \wedge \zeta_0 + \\
&\quad + F \cdot \sigma_0 \wedge \rho_0 + B \cdot \sigma_0 \wedge \bar{\zeta}_0 + Q \cdot \sigma_0 \wedge \zeta_0 + \\
&\quad + \rho_0 \wedge \zeta_0, \\
d\rho_0 &= \sqrt{-1} J \cdot \bar{\sigma}_0 \wedge \sigma_0 + \bar{E} \cdot \bar{\sigma}_0 \wedge \rho_0 + \bar{P} \cdot \bar{\sigma}_0 \wedge \bar{\zeta}_0 + A \cdot \bar{\sigma}_0 \wedge \zeta_0 + \\
&\quad + E \cdot \sigma_0 \wedge \rho_0 + A \cdot \sigma_0 \wedge \bar{\zeta}_0 + P \cdot \sigma_0 \wedge \zeta_0 - \\
&\quad - \sqrt{-1} \bar{\zeta}_0 \wedge \zeta_0,
\end{aligned}$$

while $d\zeta_0 = 0$ and $d\bar{\zeta}_0 = 0$.

3. Reduction of the initial G -structure to an $\{e\}$ -structure

In accordance with Proposition 2.1, the initial G -structure encoding local biholomorphic equivalences of manifolds $M^5 \subset \mathbb{C}^4$ belonging to Class III₁ is, after reordering the frame as $\{\bar{\mathcal{S}}, \mathcal{S}, \mathcal{T}, \bar{\mathcal{L}}, \mathcal{L}\}$, the following closed Lie subgroup of $GL_5(\mathbb{C})$:

$$G_{III_1} := \left\{ g := \begin{pmatrix} a\bar{a}\bar{a} & 0 & \bar{c} & \bar{c} & \bar{d} \\ 0 & a\bar{a}\bar{a} & c & d & e \\ 0 & 0 & a\bar{a} & \bar{b} & b \\ 0 & 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} : a \in \mathbb{C} \setminus \{0\}, b, c, d, e \in \mathbb{C} \right\}.$$

Following Cartan and Olver ([6]), transposing then this matrix in order to express how coframes do transfer, introduce the so-called *lifted coframe*:

$$\begin{pmatrix} \bar{\sigma} \\ \sigma \\ \rho \\ \bar{\zeta} \\ \zeta \end{pmatrix} = \begin{pmatrix} a\bar{a}\bar{a} & 0 & 0 & 0 & 0 \\ 0 & a\bar{a}\bar{a} & 0 & 0 & 0 \\ \bar{c} & c & a\bar{a} & 0 & 0 \\ \bar{e} & d & \bar{b} & \bar{a} & 0 \\ \bar{d} & e & b & 0 & a \end{pmatrix} \begin{pmatrix} \bar{\sigma}_0 \\ \sigma_0 \\ \rho_0 \\ \bar{\zeta}_0 \\ \zeta_0 \end{pmatrix},$$

with a, b, c, d, e here being *independent* variables, replacing the (unknown) functions from Proposition 2.1. The inverse matrix is:

$$g^{-1} = \begin{pmatrix} \frac{1}{a\bar{a}^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2\bar{a}} & 0 & 0 & 0 \\ -\frac{\bar{c}}{a^2\bar{a}^3} & -\frac{c}{a^3\bar{a}^2} & \frac{1}{a\bar{a}} & 0 & 0 \\ \frac{\bar{b}\bar{c}-e\bar{a}\bar{a}}{a^2\bar{a}^4} & \frac{\bar{b}c-a\bar{a}d}{a^3\bar{a}^3} & -\frac{b}{a\bar{a}^2} & \frac{1}{\bar{a}} & 0 \\ \frac{\bar{b}c-a\bar{a}d}{a^3\bar{a}^3} & \frac{\bar{b}c-e\bar{a}\bar{a}}{a^4\bar{a}^2} & -\frac{b}{a^2\bar{a}} & 0 & \frac{1}{a} \end{pmatrix}.$$

Then the Maurer-Cartan forms of this matrix group G_{III_1} appear in the full expression of $dg \cdot g^{-1}$, a new 5×5 matrix which happens to be of the form:

$$dg \cdot g^{-1} = \begin{pmatrix} \alpha^1 + 2\bar{\alpha}^1 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha^1 + \bar{\alpha}^1 & 0 & 0 & 0 \\ \bar{\alpha}^2 & \alpha^2 & \alpha^1 + \bar{\alpha}^1 & 0 & 0 \\ \bar{\alpha}^3 & \bar{\alpha}^4 & \bar{\alpha}^5 & \bar{\alpha}^1 & 0 \\ \alpha^4 & \alpha^3 & \alpha^5 & 0 & \alpha^1 \end{pmatrix},$$

where (the expressions of $\alpha^3, \alpha^4, \alpha^5$ will not be used):

$$\alpha^1 := \frac{da}{a}, \quad \alpha^2 := -\frac{c da}{aaaa} - \frac{c d\bar{a}}{aaa\bar{a}} + \frac{dc}{aa\bar{a}}.$$

In order to compute *at least partly* the exterior derivative $d\sigma$ and to determine the explicit expressions of *at least some of* the torsion coefficients X_\bullet which will appear:

$$\begin{aligned} d\sigma = & (2\alpha_1 + \bar{\alpha}_1) \wedge \sigma + \\ & + X_1 \bar{\sigma} \wedge \sigma + X_2 \bar{\sigma} \wedge \rho + X_3 \bar{\sigma} \wedge \bar{\zeta} + X_4 \bar{\sigma} \wedge \zeta + \\ & + X_5 \sigma \wedge \rho + X_6 \sigma \wedge \bar{\zeta} + X_7 \sigma \wedge \zeta + \\ & + \rho \wedge \zeta, \end{aligned}$$

differentiate $\sigma = a^2 \bar{a} \sigma_0$, which gives:

$$d\sigma = (2 a \bar{a} da + a^2 d\bar{a}) \wedge \sigma_0 + a^2 \bar{a} d\sigma_0,$$

obtain as an intermediate result:

$$\begin{aligned} d\sigma = & (2\alpha^1 + d\bar{\alpha}^1) \wedge \sigma + \\ & + a^2 \bar{a} K \bar{\sigma}_0 \wedge \sigma_0 + a^2 \bar{a} \bar{G} \bar{\sigma}_0 \wedge \rho_0 + a^2 \bar{a} \bar{R} \bar{\sigma}_0 \wedge \bar{\zeta}_0 + a^2 \bar{a} B \bar{\sigma}_0 \wedge \zeta_0 + \\ & + a^2 \bar{a} F \sigma_0 \wedge \bar{\zeta}_0 + a^2 \bar{a} Q \sigma_0 \wedge \zeta_0 + \\ & + a^2 \bar{a} \rho_0 \wedge \zeta_0, \end{aligned}$$

and replace:

$$\begin{aligned} \sigma_0 &= \frac{1}{a^2 \bar{a}} \sigma, \\ \rho_0 &= -\frac{\bar{c}}{a^2 \bar{a}^3} - \frac{c}{a^3 \bar{a}^2} \sigma + \frac{1}{a \bar{a}} \rho, \\ \zeta_0 &= \frac{b\bar{c} - a\bar{a}\bar{d}}{a^3 \bar{a}^3} + \frac{bc - a\bar{a}e}{a^4 \bar{a}^2} \sigma - \frac{b}{a^2 \bar{a}} \rho + \frac{1}{a} \zeta, \end{aligned}$$

which, notably, provides:

$$\begin{aligned} X_2 &= \frac{1}{\bar{a}^2} \bar{G} - \frac{b}{a\bar{a}^2} B - \frac{\bar{b}}{\bar{a}^3} \bar{R} + \frac{\bar{d}}{a\bar{a}^2}, \\ X_3 &= \frac{a}{\bar{a}^2} \bar{R}, \\ X_4 &= \frac{1}{\bar{a}} B - \frac{\bar{c}}{a\bar{a}^2}, \\ X_6 &= \frac{1}{\bar{a}} B, \\ X_7 &= \frac{1}{a} Q - \frac{c}{a^2\bar{a}}, \end{aligned}$$

other torsion coefficients being useless in what follows.

Proceeding similarly, the remaining two expressions of $d\rho$ and of $d\zeta$:

$$\begin{aligned} d\rho &= \alpha^2 \wedge \sigma + \bar{\alpha}^2 \wedge \bar{\sigma} + \alpha^1 \wedge \rho + \bar{\alpha}^1 \wedge \rho + \\ &+ Y_1 \bar{\sigma} \wedge \sigma + Y_2 \bar{\sigma} \wedge \rho + Y_3 \bar{\sigma} \wedge \bar{\zeta} + Y_4 \bar{\sigma} \wedge \zeta + \\ &+ \bar{Y}_2 \sigma \wedge \rho + \bar{Y}_4 \sigma \wedge \bar{\zeta} + \bar{Y}_3 \sigma \wedge \zeta + \\ &+ Y_8 \rho \wedge \bar{\zeta} + \bar{Y}_8 \rho \wedge \zeta + \\ &+ \sqrt{-1} \zeta \wedge \bar{\zeta}, \end{aligned}$$

$$\begin{aligned} d\zeta &= \alpha^3 \wedge \sigma + \alpha^4 \wedge \bar{\sigma} + \alpha^5 \wedge \rho + \alpha^1 \wedge \zeta + \\ &+ Z_1 \bar{\sigma} \wedge \sigma + Z_2 \bar{\sigma} \wedge \rho + Z_3 \bar{\sigma} \wedge \bar{\zeta} + Z_4 \bar{\sigma} \wedge \zeta + \\ &+ Z_5 \sigma \wedge \rho + Z_6 \sigma \wedge \bar{\zeta} + Z_7 \sigma \wedge \zeta + \\ &+ Z_8 \rho \wedge \bar{\zeta} + Z_9 \rho \wedge \zeta + \\ &+ Z_{10} \zeta \wedge \bar{\zeta}, \end{aligned}$$

can in principle be complemented by *explicit* expressions of all the appearing torsion coefficients Y_\bullet and Z_\bullet , but only the single:

$$\bar{Y}_8 = \frac{c}{a^2\bar{a}} + \sqrt{-1} \frac{\bar{b}}{a\bar{a}}$$

will be useful at this stage, because of absorption facts.

Indeed, following Cartan's procedure of determining the linear subspace of torsion coefficients that are *absorbable into modified Maurer-Cartan forms* ([6]), introduce modifications:

$$\begin{aligned} \tilde{\alpha}^1 &:= \alpha^1 - A_1 \cdot \bar{\sigma} - B_1 \cdot \sigma - C_1 \cdot \rho - D_1 \cdot \bar{\zeta} - E_1 \cdot \zeta, \\ \tilde{\alpha}^2 &:= \alpha^2 - A_2 \cdot \bar{\sigma} - B_2 \cdot \sigma - C_2 \cdot \rho - D_2 \cdot \bar{\zeta} - E_2 \cdot \zeta, \end{aligned}$$

while the remaining three 1-forms $\alpha^3, \alpha^4, \alpha^5$ are kept untouched, where $A_\bullet, B_\bullet, C_\bullet, D_\bullet, E_\bullet$ are functions defined on M , *i.e.* assumed to be *independent of the group variables*

a, b, c, d, e. Replacing then α^1 and α^2 gives, after collecting appropriately:

$$\begin{aligned} d\sigma &= (2\tilde{\alpha}^1 + \bar{\alpha}^1) \wedge \sigma + \\ &+ \bar{\sigma} \wedge \sigma [X_1 - 2A_1 - \bar{B}_1] + \bar{\sigma} \wedge \rho [X_2] + \bar{\sigma} \wedge \bar{\zeta} [X_3] + \bar{\sigma} \wedge \zeta [X_4] + \\ &+ \sigma \wedge \rho [X_5 - 2C_1 - \bar{C}_1] + \sigma \wedge \bar{\zeta} [X_6 - 2D_1 - \bar{E}_1] + \sigma \wedge \zeta [X_7 - 2E_1 - \bar{D}_1] + \\ &+ \rho \wedge \zeta, \end{aligned}$$

$$\begin{aligned} d\rho &= \tilde{\alpha}^2 \wedge \sigma + \bar{\alpha}^2 \wedge \bar{\sigma} + \tilde{\alpha}^1 \wedge \rho + \bar{\alpha}^1 \wedge \rho + \\ &+ \bar{\sigma} \wedge \sigma [Y_1 + A_2 - \bar{A}_2] + \bar{\sigma} \wedge \rho [Y_2 - \bar{C}_2 + A_1 + \bar{B}_1] + \bar{\sigma} \wedge \bar{\zeta} [Y_3 - \bar{E}_2] + \bar{\sigma} \wedge \zeta [Y_4 - \bar{D}_2] + \\ &+ \sigma \wedge \rho [\bar{Y}_2 - C_2 + \bar{A}_1 + B_1] + \sigma \wedge \bar{\zeta} [\bar{Y}_4 - D_2] + \sigma \wedge \zeta [\bar{Y}_3 - E_2] + \\ &+ \rho \wedge \bar{\zeta} [Y_8 - D_1 - \bar{E}_1] + \rho \wedge \zeta [\bar{Y}_8 - \bar{D}_1 - E_1] + \\ &+ \sqrt{-1} \zeta \wedge \bar{\zeta}. \end{aligned}$$

At first, the fact that in $d\rho$ the coefficient X_2 of $\bar{\sigma} \wedge \rho$, the coefficient X_3 of $\bar{\sigma} \wedge \bar{\zeta}$ and the coefficient X_4 of $\bar{\sigma} \wedge \zeta$ are left invariant, namely do not incorporate any A_\bullet , B_\bullet , C_\bullet , D_\bullet , E_\bullet , immediately explains the first steps of the following:

Lemma 3.1. *The three torsion coefficients X_2 , X_3 , X_4 are essential, and the same also holds true for $\bar{X}_6 + X_7 - 3\bar{Y}_8$.*

Indeed, for this last, not immediately seen, linear combination, compute:

$$\begin{aligned} \bar{X}_6 + \tilde{X}_7 - 3\bar{Y}_8 &= \bar{X}_6 - 2\bar{D}_1 - E_1 + \\ &+ X_7 - 2E_1 - \bar{D}_1 + \\ &- 3\bar{Y}_8 + 3\bar{D}_1 + 3E_1 = \bar{X}_6 + X_7 - 3\bar{Y}_8. \quad \square \end{aligned}$$

Observe here that when $R \neq 0$, the essential torsion coefficient $X_3 = \frac{a}{\bar{a}} \bar{R}$ can be assigned the value $X_3 := 1$, which, after relocalization to an open set on which R is nowhere vanishing, conducts to the normalization of the variable a . This observation led the first and the third authors in [5] to set up a natural *bifurcation* in the concerned equivalence problem, according to whether $R \equiv 0$ or $R \neq 0$. Several other potentially normalizable essential torsion coefficients also appeared in the advanced computational explorations performed in [5] which also concerned the diagonal group parameter a .

However, on the (simpler) way to construct just a canonical Cartan connection, it is advisable to ignore all such possible bifurcations, and to only look at normalizations of group variables which come from X_2 , X_4 , $\bar{X}_6 + X_7 - 3\bar{Y}_8$, *disregarding therefore* X_3 .

In fact, setting equal to zero these three essential torsion coefficients provides, after elementary resolution, the following three normalizations:

$$\begin{aligned} b &= a \left(\frac{\sqrt{-1}}{3} \bar{Q} - \sqrt{-1} B \right), \\ c &= a \bar{a} \bar{B}, \\ d &= \bar{a} \left(-G - \frac{\sqrt{-1}}{3} Q + \sqrt{-1} \bar{B} + \frac{\sqrt{-1}}{3} \bar{Q} R - \sqrt{-1} B R \right), \end{aligned}$$

which can be abbreviated as:

$$\mathbf{b} = a \mathbf{B}_0, \quad \mathbf{c} = a\bar{a} \mathbf{C}_0, \quad \mathbf{d} = \bar{a} \mathbf{D}_0,$$

the three functions $\mathbf{B}_0, \mathbf{C}_0, \mathbf{D}_0$ being functions defined on the basis M , independent of the group variables.

Then following Cartan and Olver ([6]), replace these group normalizations in:

$$\begin{aligned} \sigma &= a^2 \bar{a} \sigma_0, \\ \rho &= \bar{c} \bar{\sigma}_0 + c \sigma_0 + a\bar{a} \rho_0, \\ \zeta &= \bar{d} \bar{\sigma}_0 + e \sigma_0 + \mathbf{b} \rho_0 + a \zeta_0, \end{aligned}$$

which gives:

$$\begin{aligned} \sigma &= a^2 \bar{a} \sigma_0 =: a^2 \bar{a} \sigma_1, \\ \rho &= a\bar{a} (\bar{\mathbf{C}}_0 \bar{\sigma}_0 + \mathbf{C}_0 \sigma_0 + \rho_0) =: a\bar{a} \rho_1, \\ \zeta &= e \sigma_0 + a (\bar{\mathbf{D}}_0 \bar{\sigma}_0 + \mathbf{B}_0 \rho_0 + \zeta_0) =: e \sigma_1 + a \zeta_1, \end{aligned}$$

and restart the procedure of determining whether some group variables are normalizable. In the present case, the second loop of Cartan's procedure will happen to be the last one.

Of course, the new lifted coframe is:

$$\begin{pmatrix} \bar{\sigma} \\ \sigma \\ \rho \\ \bar{\zeta} \\ \zeta \end{pmatrix} = \begin{pmatrix} a\bar{a}\bar{a} & 0 & 0 & 0 & 0 \\ 0 & a\bar{a}\bar{a} & 0 & 0 & 0 \\ 0 & 0 & a\bar{a} & 0 & 0 \\ \bar{e} & 0 & 0 & \bar{a} & 0 \\ 0 & e & 0 & 0 & a \end{pmatrix} \begin{pmatrix} \bar{\sigma}_1 \\ \sigma_1 \\ \rho_1 \\ \bar{\zeta}_1 \\ \zeta_1 \end{pmatrix},$$

with two Maurer-Cartan 1-forms (and their conjugates):

$$\beta^1 := \frac{da}{a}, \quad \beta^2 := \frac{de}{a^2 \bar{a}} - \frac{e da}{a^3 \bar{a}}.$$

Some computations achieved in [5] gave the explicit expressions of the new torsion coefficients in:

$$\begin{aligned} d\sigma &= (2\beta^1 + \bar{\beta}^1) \wedge \sigma + \\ &+ X'_1 \bar{\sigma} \wedge \sigma + 0 + X'_3 \bar{\sigma} \wedge \bar{\zeta} + 0 + \\ &+ X'_5 \sigma \wedge \rho + X'_6 \sigma \wedge \bar{\zeta} + X'_7 \sigma \wedge \zeta + \rho \wedge \zeta, \end{aligned}$$

the new $X'_2 = 0$ and $X'_4 = 0$ being zero thanks to the preceding normalizations, and also the explicit expressions of the new torsion coefficients appearing in:

$$\begin{aligned} d\rho &= (\beta^1 + \bar{\beta}^1) \wedge \rho + \\ &+ Y'_1 \bar{\sigma} \wedge \sigma + Y'_2 \bar{\sigma} \wedge \rho + Y'_3 \bar{\sigma} \wedge \bar{\zeta} + Y'_4 \bar{\sigma} \wedge \zeta + \\ &+ \bar{Y}'_2 \sigma \wedge \rho + \bar{Y}'_4 \sigma \wedge \bar{\zeta} + \bar{Y}'_3 \sigma \wedge \zeta + \\ &+ \left(\frac{1}{3} X'_6 + \frac{1}{3} \bar{X}'_7\right) \rho \wedge \bar{\zeta} + \left(\frac{1}{3} \bar{X}'_6 + \frac{1}{3} X'_7\right) \rho \wedge \zeta + \sqrt{-1} \zeta \wedge \bar{\zeta}, \end{aligned}$$

but if just a Cartan connection is searched for as is admitted in the present article, less computational efforts are demanded.

Proposition 3.2. *The appearing new torsion coefficient Y'_4 in $d\rho$ is essential, and assigned to 0, it leads to a normalization of the last non-diagonal group parameter e under the form:*

$$e = a \mathbf{E}_0.$$

Proof. Indeed, the coefficient Y'_4 of the 2-form $\bar{\sigma} \wedge \zeta$ can visibly *not* be absorbed in $(\beta^1 + \bar{\beta}^1) \wedge \rho$ by modifying:

$$\tilde{\beta}^1 := \beta^1 - A_1 \bar{\sigma} - B_1 \sigma - C_1 \rho - D_1 \bar{\zeta} - E_1 \zeta,$$

which shows that Y'_4 is essential. It therefore only remains to explain how to shortly get at Y'_4 .

Computing $d\rho_1$ and re-expressing it in terms of the new coframe $\{\bar{\sigma}_1, \sigma_1, \rho_1, \bar{\zeta}_1, \zeta_1\}$ conducts to certain functions T_{\bullet}^{\bullet} defined on M that are independent of the group parameters:

$$\begin{aligned} d\rho_1 = & T_{\bar{\sigma}_1 \sigma_1}^{\rho_1} \bar{\sigma}_1 \wedge \sigma_1 + T_{\bar{\sigma}_1 \rho_1}^{\rho_1} \bar{\sigma}_1 \wedge \rho_1 + T_{\bar{\sigma}_1 \bar{\zeta}_1}^{\rho_1} \bar{\sigma}_1 \wedge \bar{\zeta}_1 + T_{\bar{\sigma}_1 \zeta_1}^{\rho_1} \bar{\sigma}_1 \wedge \zeta_1 + \\ & + T_{\sigma_1 \rho_1}^{\rho_1} \sigma_1 \wedge \rho_1 + T_{\sigma_1 \bar{\zeta}_1}^{\rho_1} \sigma_1 \wedge \bar{\zeta}_1 + T_{\sigma_1 \zeta_1}^{\rho_1} \sigma_1 \wedge \zeta_1 + \\ & + T_{\rho_1 \bar{\zeta}_1}^{\rho_1} \rho_1 \wedge \bar{\zeta}_1 + T_{\rho_1 \zeta_1}^{\rho_1} \rho_1 \wedge \zeta_1 - \\ & - \sqrt{-1} \bar{\zeta}_1 \wedge \zeta_1. \end{aligned}$$

Differentiating $\rho = a\bar{a} \rho_1$ leads firstly to:

$$\begin{aligned} d\rho = & \left(\frac{da}{a} + \frac{d\bar{a}}{\bar{a}} \right) \wedge \rho + \\ & + a\bar{a} T_{\bar{\sigma}_1 \sigma_1}^{\rho_1} \bar{\sigma}_1 \wedge \sigma_1 + a\bar{a} T_{\bar{\sigma}_1 \rho_1}^{\rho_1} \bar{\sigma}_1 \wedge \rho_1 + a\bar{a} T_{\bar{\sigma}_1 \bar{\zeta}_1}^{\rho_1} \bar{\sigma}_1 \wedge \bar{\zeta}_1 + a\bar{a} T_{\bar{\sigma}_1 \zeta_1}^{\rho_1} \bar{\sigma}_1 \wedge \zeta_1 + \\ & + a\bar{a} T_{\sigma_1 \rho_1}^{\rho_1} \sigma_1 \wedge \rho_1 + a\bar{a} T_{\sigma_1 \bar{\zeta}_1}^{\rho_1} \sigma_1 \wedge \bar{\zeta}_1 + a\bar{a} T_{\sigma_1 \zeta_1}^{\rho_1} \sigma_1 \wedge \zeta_1 + \\ & + a\bar{a} T_{\rho_1 \bar{\zeta}_1}^{\rho_1} \rho_1 \wedge \bar{\zeta}_1 + a\bar{a} T_{\rho_1 \zeta_1}^{\rho_1} \rho_1 \wedge \zeta_1 - \\ & - \sqrt{-1} a\bar{a} \bar{\zeta}_1 \wedge \zeta_1, \end{aligned}$$

then replacing:

$$\bar{\sigma}_1 = \frac{1}{a^2 \bar{a}} \sigma, \quad \zeta_1 = -\frac{e}{a^3 \bar{a}} \sigma + \frac{1}{a} \zeta, \quad \bar{\zeta}_1 = -\frac{\bar{e}}{a \bar{a}^3} \sigma + \frac{1}{\bar{a}} \bar{\zeta},$$

only the two underlined 2-forms above contribute to $\bar{\sigma} \wedge \zeta$ in the final expression of $d\rho$, which gives:

$$Y'_4 = \frac{1}{a\bar{a}} T_{\bar{\sigma}_1 \zeta_1}^{\rho_1} + \sqrt{-1} \frac{1}{a\bar{a}^2} \bar{e}.$$

Setting $\bar{Y}'_4 = 0$ normalizes e under the form claimed. \square

Performing therefore the obtained normalizations of all the nondiagonal group parameters:

$$b = a \mathbf{B}_0, \quad c = a\bar{a} \mathbf{C}_0, \quad d = \bar{a} \mathbf{D}_0, \quad e = a \mathbf{E}_0,$$

the reduced G -structure now involves only $a \in \mathbb{C}^*$ and the new lifted coframe becomes:

$$\begin{aligned}\sigma &= a^2 \bar{a} \sigma_0 =: a^2 \bar{a} \sigma_2, \\ \rho &= a \bar{a} (\bar{\mathbf{C}}_0 \bar{\sigma}_0 + \mathbf{C}_0 \sigma_0 + \rho_0) =: a \bar{a} \rho_2 \\ \zeta &= a (\bar{\mathbf{D}}_0 \bar{\sigma}_0 + \mathbf{E}_0 \sigma_0 + \mathbf{B}_0 \rho_0 + \zeta_0) =: a \zeta_2,\end{aligned}$$

the single Maurer-Cartan 1-form being (with its conjugate):

$$\gamma^1 := \frac{da}{a}.$$

Inversion yields:

$$\sigma_2 = \frac{1}{a^2 \bar{a}} \sigma, \quad \rho_2 = \frac{1}{a \bar{a}} \rho, \quad \zeta_2 = \frac{1}{a} \zeta,$$

whence immediately:

$$\bar{\sigma}_2 \wedge \sigma_2 = \frac{1}{a^3 \bar{a}^3} \bar{\sigma} \wedge \sigma, \quad \bar{\sigma}_2 \wedge \rho_2 = \frac{1}{a^2 \bar{a}^3} \bar{\sigma} \wedge \rho, \quad \dots, \quad \bar{\zeta}_2 \wedge \zeta_2 = \frac{1}{a \bar{a}} \bar{\zeta} \wedge \zeta.$$

Then performing a last absorption thanks to the computer programs of the second author:

$$\lambda := \frac{da}{a} + \text{linear combination of } (\bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta),$$

the structure equations receive the final form:

(1)

$$\begin{aligned}d\sigma &= (2\lambda + \bar{\lambda}) \wedge \sigma + \frac{a}{\bar{a}^2} \bar{R} \bar{\sigma} \wedge \bar{\zeta} + \rho \wedge \zeta, \\ d\rho &= (\lambda + \bar{\lambda}) \wedge \rho + \frac{1}{a^2 \bar{a}^2} V_1 \bar{\sigma} \wedge \sigma + \frac{1}{\bar{a}^2} V_3 \bar{\sigma} \wedge \bar{\zeta} + \frac{1}{a^2 \bar{a}} V_3 \sigma \wedge \rho + \sqrt{-1} \zeta \wedge \bar{\zeta}, \\ d\zeta &= \lambda \wedge \zeta + \frac{1}{a^2 \bar{a}^3} W_1 \bar{\sigma} \wedge \sigma + \frac{1}{a \bar{a}^3} W_2 \bar{\sigma} \wedge \rho + \frac{1}{\bar{a}^3} W_3 \bar{\sigma} \wedge \bar{\zeta} + \frac{1}{a \bar{a}^2} W_4 \bar{\sigma} \wedge \zeta + \\ &\quad + \frac{1}{a^2 \bar{a}^2} W_5 \sigma \wedge \rho + \frac{1}{a \bar{a}^2} W_6 \sigma \wedge \bar{\zeta} + \frac{1}{a^2 \bar{a}} W_7 \sigma \wedge \zeta + \\ &\quad + \frac{1}{\bar{a}^2} W_8 \rho \wedge \bar{\zeta} + \frac{1}{a \bar{a}} W_9 \rho \wedge \zeta + \\ &\quad + \frac{1}{a} W_{10} \bar{\zeta} \wedge \zeta,\end{aligned}$$

in which all functions $\bar{R}, V_\bullet, W_\bullet$ depend only on the variables of M , not on the group parameter a . But since $\frac{da}{a}$ occurring in λ is *closed*, $d\lambda$ is a linear combination of 2-forms on M , namely there are certain functions $I_{\nu\mu}$ for $\nu, \mu = \bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta$ so that:

$$(2) \quad d\lambda = \sum_{\nu, \mu} I_{\nu\mu} \nu \wedge \mu,$$

and Cartan's method then naturally stops ([6]).

Theorem 3.1. *The 7 differential 1-forms $\bar{\lambda}, \lambda, \bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta$ define an absolute parallelism on the principal bundle $P^7 := M \times \mathbb{C}^*$ satisfying the structure equations (1) and (2) which reduces the local biholomorphic equivalence problem for Class III₁*

CR-generic submanifolds $M^5 \subset \mathbb{C}^4$ with initial structure group G_{III_1} to an $\{e\}$ -structure. \square

Lastly, elementary linear algebra shows ([5], end of Section 12) that the Maurer-Cartan structure equations corresponding to the Lie algebra spanned by the above 7 infinitesimal CR automorphisms $S_2, S_1, T, L_2, L_1, D, R$ can be represented as:

$$(3) \quad \begin{aligned} d\alpha_c &= 0, \\ d\sigma_c &= (2\alpha_c + \bar{\alpha}_c) \wedge \sigma_c + \rho_c \wedge \zeta_c, \\ d\rho_c &= (\alpha_c + \bar{\alpha}_c) \wedge \rho_c + \sqrt{-1}\zeta_c \wedge \bar{\zeta}_c, \\ d\zeta_c &= \alpha_c \wedge \zeta_c, \end{aligned}$$

where $\alpha_c, \sigma_c, \zeta_c$ are \mathbb{C} -valued 1-forms on the tangent bundle TM_c to Beloshapka's cubic, and where ρ_c is \mathbb{R} -valued 1-form on TM_c .

Notably, when all invariants $\bar{R}, V_\bullet, W_\bullet, I_{\bullet\bullet}$, vanish in (1) and in (2), renaming $\lambda \mapsto \alpha_c$ makes recover (3).

4. Construction of a Cartan connection (Proof of Theorem 1.1)

Introduce then the set of vector fields:

$$\{e_{\bar{\alpha}}, e_\alpha, e_{\bar{\sigma}}, e_\sigma, e_\rho, e_{\bar{\zeta}}, e_\zeta\}$$

that is dual to $\{\bar{\alpha}_c, \alpha_c, \bar{\sigma}_c, \sigma_c, \rho_c, \bar{\zeta}_c, \zeta_c\}$, hence defines the structure of a 7-dimensional real Lie algebra:

$$\mathfrak{g}^7 \cong \mathbf{aut}_{\text{CR}}(M_c^5).$$

From (3), the Lie bracket structure of \mathfrak{g}^7 is:

$$\begin{aligned} [e_{\bar{\alpha}}, e_{\bar{\sigma}}] &= -2e_{\bar{\sigma}}, & [e_\alpha, e_{\bar{\sigma}}] &= -e_{\bar{\sigma}}, \\ [e_{\bar{\alpha}}, e_\sigma] &= -e_\sigma, & [e_\alpha, e_\sigma] &= -2e_\sigma, & [e_\rho, e_{\bar{\zeta}}] &= -e_{\bar{\sigma}}, \\ [e_{\bar{\alpha}}, e_\rho] &= -e_\rho, & [e_\alpha, e_\rho] &= -e_\rho, & [e_\rho, e_\sigma] &= -e_\sigma, & [e_{\bar{\zeta}}, e_\zeta] &= -\sqrt{-1}e_\rho, \\ [e_{\bar{\alpha}}, e_{\bar{\zeta}}] &= -e_{\bar{\zeta}}, & [e_\alpha, e_\zeta] &= -e_\zeta, \end{aligned}$$

unwritten brackets being zero.

Next, let $\mathfrak{h} \cong \mathbb{R}^2$ be the real Lie algebra spanned by $\{e_{\bar{\alpha}}, e_\alpha\}$. Let P^7 be $M \times \mathbb{C}^*$ equipped with some local coordinates on M and with the fiber coordinates (a, \bar{a}) .

Theorem 4.1. *In terms of the 7 differential 1-forms $\bar{\lambda}, \lambda, \bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta$, obtained by reducing to an $\{e\}$ -structure the biholomorphic equivalence problem for Class III_1 CR-generic submanifolds $M^5 \subset \mathbb{C}^4$, the 1-form ϖ with value in \mathfrak{g}^7 defined at an arbitrary point $p \in P$ for every tangent vector $v_p \in T_p P^7$ by:*

$$\varpi_p(v_p) := \bar{\lambda}_p(v_p) \cdot e_{\bar{\alpha}} + \lambda_p(v_p) \cdot e_\alpha + \bar{\sigma}_p(v_p) \cdot e_{\bar{\sigma}} + \sigma_p(v_p) \cdot e_\sigma + \rho_p(v_p) \cdot e_\rho + \bar{\zeta}_p(v_p) \cdot e_{\bar{\zeta}} + \zeta_p(v_p) \cdot e_\zeta$$

defines a canonical Cartan connection on $P^7 = M^5 \times \mathbb{C}^*$.

Proof. Since by construction $\bar{\lambda}, \lambda, \bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta$ span the cotangent bundle to P^7 at every point, condition (i) for a Cartan connection that $\varpi_p: T_p P^7 \rightarrow \mathfrak{g}^7$ be an isomorphism at every point $p \in P$ is automatically satisfied.

As well, the condition (ii) that $\varpi(Y^\dagger) = y$ comes directly from the fact that:

$$\lambda = \frac{da}{a} + \text{linear combinations of } (\bar{\sigma}, \sigma, \rho, \bar{\zeta}, \zeta).$$

Lastly, condition (iii) is known ([7]) to be equivalent to its infinitesimal counterpart:

$$\text{Lie}_{e_\alpha^\dagger}(\varpi) = -\text{ad}_{e_\alpha} \circ \varpi, \quad \text{and} \quad \text{Lie}_{e_{\bar{\alpha}}^\dagger}(\varpi) = -\text{ad}_{e_{\bar{\alpha}}} \circ \varpi,$$

where $\text{Lie}_X(\varpi)$ denotes the Lie derivative of ϖ with respect to a vector field X on P , and where ad_k is the linear map $\mathfrak{k} \rightarrow \mathfrak{k}$ defined on a Lie algebra \mathfrak{k} by $\text{ad}_k(l) := [k, l]$.

Now, condition (ii) verified at the moment shows — using \lrcorner to denote interior product — that in Cartan's formula:

$$\text{Lie}_{e_\alpha^\dagger}(\varpi) = e_{\alpha^\dagger} \lrcorner d\varpi + \underbrace{d(e_\alpha^\dagger \lrcorner \varpi)}_{\circ}, \quad \text{and in:} \quad \text{Lie}_{e_{\bar{\alpha}}^\dagger}(\varpi) = e_{\bar{\alpha}^\dagger} \lrcorner d\varpi + \underbrace{d(e_{\bar{\alpha}}^\dagger \lrcorner \varpi)}_{\circ},$$

the second terms, differentiating a constant, drop, so that verifying (iii) amounts to checking the two coincidences:

$$e_{\alpha^\dagger} \lrcorner d\varpi = -\text{ad}_{e_\alpha}(\varpi), \quad \text{and} \quad e_{\bar{\alpha}^\dagger} \lrcorner d\varpi = -\text{ad}_{e_{\bar{\alpha}}}(\varpi).$$

But from the structure equations (1) and (2) satisfied by the 7 two-forms occurring in $d\varpi$, it is clear that:

$$\begin{aligned} e_{\alpha^\dagger} \lrcorner d\bar{\lambda} &= 0, & e_{\alpha^\dagger} \lrcorner d\lambda &= 0, & e_{\alpha^\dagger} \lrcorner d\bar{\sigma} &= \bar{\sigma}, & e_{\alpha^\dagger} \lrcorner d\sigma &= 2\sigma, \\ e_{\alpha^\dagger} \lrcorner d\rho &= \rho, & e_{\alpha^\dagger} \lrcorner d\bar{\zeta} &= 0, & e_{\alpha^\dagger} \lrcorner d\zeta &= \zeta, \end{aligned}$$

while the Lie bracket structure shows on the other hand that:

$$\begin{aligned} -\text{ad}_{e_\alpha} \circ \varpi &= -\text{ad}_{e_\alpha} \circ (\bar{\lambda} \cdot e_{\bar{\alpha}} + \lambda \cdot e_\alpha + \bar{\sigma} \cdot e_{\bar{\sigma}} + \sigma \cdot e_\sigma + \rho \cdot e_\rho + \bar{\zeta} \cdot e_{\bar{\zeta}} + \zeta \cdot e_\zeta) \\ &= -\bar{\lambda} [e_\alpha, e_{\bar{\alpha}}] - \lambda [e_\alpha, e_\alpha] - \bar{\sigma} [e_\alpha, e_{\bar{\sigma}}] - \sigma [e_\alpha, e_\sigma] - \rho [e_\alpha, e_\rho] - \bar{\zeta} [e_\alpha, e_{\bar{\zeta}}] - \zeta [e_\alpha, e_\zeta] \\ &= 0 + 0 + \bar{\sigma} \cdot e_{\bar{\sigma}} + 2\sigma \cdot e_\sigma + \rho \cdot e_\rho + 0 + \zeta \cdot e_\zeta, \end{aligned}$$

so that the first coincidence holds; the second one is treated similarly. \square

5. Beyond Cartan connections

This article announces that, beyond plain linear algebra considerations that are sufficient to construct $\{e\}$ -structures and Cartan connections associated to the local biholomorphic equivalence problem for CR-generic \mathcal{C}^ω submanifolds $M^5 \subset \mathbb{C}^4$ belonging to Class III₁, the memoir [5] presents *explicit* expressions of the incoming invariants $\bar{R}, V_\bullet, W_\bullet, I_{\bullet\bullet}$, it examines the *nonlinear* relations these invariants may share, and it conducts a ramified analysis of the *bifurcation tree* of possible further normalizations for the diagonal last group parameter a which entails non-uniqueness of the concerned Cartan connections or $\{e\}$ -structures.

A unified presentation of recent results on Cartan equivalences for all six general classes I, II, III₁, III₂, IV₁, IV₂ of CR manifolds of dimension ≤ 5 is upcoming ([3]).

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