STUDY OF THE REGULARITY OF THE FORMAL CR REFLECTION MAPPING

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Abstract.

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§1. Result

Let $h\colon (M,p) \to_{\mathcal{F}} (M',p')$ be a formal *invertible* mapping between two real analytic generic CR manifolds of the same CR dimension and codimension in $\mathbb{C}^n$. If (M,p) is *minimal*, then :

Theorem 1.

1. *The reflection mapping* $\mathcal{R}_h'$ (see §3) *associated to h and to a system of defining functions for (M',p') is convergent.*
2. *If (M',p') is holomorphically nondegenerate, then h is convergent.*

In §17, we evoke a standard generalization of this theorem.

Idea in the proof. In the preprint [M2], the author has observed the interest of working with some *conjugate reflection identities*. This paper is a prolongation of this idea.

Remark. While this work was under completion, the author received a preprint by Nordine Mir, in which a similar statement to Theorem 1 is given in the hypersurface case. In another preprint, Theorem 1 above is also stated by Mir for the special case where M' is an algebraic CR manifold. It would be interesting to know whether his techniques also yield Theorem 1 in higher codimension and without algebraicity.

§2. Conjugation

We follow the notation of [M1] with $d' = d$, $m' = m$ and $n' = n$.

2.1. Two identities. Let $r'(t', \tau') := (z' - \xi' - i\bar{\Theta}'(\zeta', w', \xi'))$. Hence $\bar{r}'(\tau', t') = (\xi' - z' + i\Theta'(w', \zeta', z'))$. Recall that $r'(t', \tau') = 0$ if and only if $\bar{r}'(\tau', t') = 0$. Let $h\colon (M, 0) \to_{\mathcal{F}} (M', 0)$. Then $r'(h(t), \bar{h}(\tau)) = 0$ if $\rho(t, \tau) = 0$, and also $\bar{r}'(\bar{h}(\tau), h(t)) = 0$. Applying the derivations $\underline{\mathcal{L}}^\beta$, we get :

$$\underline{\mathcal{L}}^\beta[r'(h(t), \bar{h}(\tau))] = 0 \ \text{ and also : } \ \underline{\mathcal{L}}^\beta[\bar{r}'(\bar{h}(\tau), h(t))] = 0, \qquad \forall\, \beta \in \mathbb{N}^m. \tag{2.2}$$

2.3. Equivalence. However, it is well-known that these two infinite families of analytic identities linking $h(t)$ and the jets of $\bar{h}(\tau)$ are equivalent – and even redundant :

Lemma 2.4. *If $(t, \tau) \in \mathcal{M}$, and $t' \in \mathbb{C}^n$, then*
(2.5)

$$\begin{cases} \left\langle \underline{\mathcal{L}}^\beta[r'(h(t), \bar{h}(\tau))] = 0 \quad \forall\, \beta \in \mathbb{N}^m \right\rangle \iff \left\langle \underline{\mathcal{L}}^\beta[\bar{r}'(\bar{h}(\tau), h(t))] = 0 \quad \forall\, \beta \in \mathbb{N}^m \right\rangle \\ \left\langle \underline{\mathcal{L}}^\beta[r'(t', \bar{h}(\tau))] = 0 \quad \forall\, \beta \in \mathbb{N}^m \right\rangle \iff \left\langle \underline{\mathcal{L}}^\beta[\bar{r}'(\bar{h}(\tau), t')] = 0 \quad \forall\, \beta \in \mathbb{N}^m \right\rangle. \end{cases}$$

Check. There exists a $d \times d$ matrix of analytic functions $a'(t', \tau') \in \mathbb{C}\{t', \tau'\}^{d\times d}$, such that $-r'(t', \tau') \equiv a'(t', \tau')\bar{r}'(\tau', t')$, for all small $t', \tau' \in \mathbb{C}^n$ and $a'(0, 0) = I_{d\times d}$. Thus applying all the derivations $\underline{\mathcal{L}}^\beta$ to the identity $-r'(t', h(\tau)) \equiv a'(t', h(\tau))\bar{r}'(h(\tau), t')$, we get the two implications "$\Leftarrow$", and similarly we get the two implications "$\Rightarrow$" starting with $-\bar{r}'(h(\tau), t') \equiv \bar{a}'(h(\tau), t')\, r'(t', h(\tau))$ instead. □

Assertion 2.6. *Nevertheless, in order to achieve the proof of Theorem 1.1, it will be natural to use both these two identities alternately.*

§3. Reflection mapping

Again, the notation will be that of [M1], with $d' = d$, $m' = m$ and $n' = n$. The *Reflection "mapping"* $\mathcal{R}'_h(t, \bar{\nu}')$, $t \in \mathbb{C}^n$, $\bar{\nu}' = (\bar{\lambda}', \bar{\mu}') \in \mathbb{C}^m \times \mathbb{C}^d$ will be by definition the d-vectorial *power series*

$$\mathcal{R}'_h(t, \bar{\nu}') = \mathcal{R}'_h(w, z, \bar{\lambda}', \bar{\mu}') = \bar{\mu}' - f(w, z) + i \sum_{\gamma \in \mathbb{N}^m_*} \bar{\lambda}'^\gamma \ \Theta'_\beta(g(w, z), f(w, z)), \tag{3.1}$$

in the local ring $\mathbb{C}\{\bar{\lambda}', \bar{\mu}'\}[\![w, z]\!]^d$[1].

Assertion 3.2. *The property $\mathcal{R}'_h(t, \bar{\nu}') \in \mathbb{C}\{t\}\{\bar{\nu}'\}^d$ is independent of coordinates.*

Proof. Consequence of the biholomorphic invariance of Segre varieties. □

[1] *Remark.* Let $x_1, x_2 \in \mathbb{C}$. The ring $\mathbb{C}[\![x_1]\!]\{x_2\}$ has no sense.

§4. Two different jet reflection identities

$$(4.1)\qquad \begin{cases} f = \bar f + i\bar\Theta'(\bar g, g, \bar f), & \text{(on } \mathcal{M}),\\ 0 \equiv \underline{\mathcal{L}}^\beta \bar f + i \sum_{\gamma\in\mathbb{N}^m_*} g^\gamma \underline{\mathcal{L}}^\beta(\bar\Theta'_\gamma(\bar h)), & \forall\ \beta \in \mathbb{N}^m_*. \end{cases}$$

$$(4.2)\qquad \begin{cases} \bar f = f - i\Theta'(g, \bar g, f), & \text{(on } \mathcal{M}),\\ \underline{\mathcal{L}}^\beta \bar f = -i \sum_{\gamma\in\mathbb{N}^m_*} \underline{\mathcal{L}}^\beta(\bar g^\gamma)\ \Theta'_\gamma(h), & \forall\ \beta \in \mathbb{N}^m_*. \end{cases}$$

Convention 4.3.

1. $\Theta'_\gamma(h) := f$ if $\gamma = 0$.
2. When we write in the sequel "$0 \equiv \underline{\mathcal{L}}^\beta \bar f + i\sum_{\gamma\in\mathbb{N}^m_*} g^\gamma \underline{\mathcal{L}}^\beta(\bar\Theta'_\gamma(\bar h))$, *for all* $\beta \in \mathbb{N}^m$", *we will mean eqs.* (4.1), *i.e. we incorporate* $f = \bar f + i\bar\Theta'(\bar g, g, \bar f)$.

§5. Convergence of $\mathcal{R}'_h$ on the first Segre chain $\mathcal{S}^1_0$

5.1. Differentiations. Put $(t,\tau) := (w,0,0,0)$ in (4.1). All terms $[\underline{\mathcal{L}}^\beta \bar\Theta'_\gamma(\bar h)](w, 0,0,0)$ are converging. By Artin's theorem, there exists a solution $H(w) \in \mathbb{C}\{w\}^n$:

$$(5.2)\qquad \begin{cases} F(w) \equiv \bar f(0) + i \sum_{\gamma\in\mathbb{N}^m_*} G(w)^\gamma\ [\bar\Theta'_\gamma(\bar h)](w,0,0,0) & (\beta = 0),\\ 0 \equiv [\underline{\mathcal{L}}^\beta \bar f](w,0,0,0) + i \sum_{\gamma\in\mathbb{N}^m_*} G(w)^\gamma\ [\underline{\mathcal{L}}^\beta \bar\Theta'_\gamma(\bar h)](w,0,0,0), & \forall\ \beta\in\mathbb{N}^m_*. \end{cases}$$

By Lemma 2.4, eqs. (5.2) are equivalent to :

$$(5.3)\qquad \begin{cases} \bar f(0) \equiv F(w) - i \sum_{\gamma\in\mathbb{N}^m_*} \bar h(0)^\gamma\ \Theta'_\gamma(H(w)) & (\beta = 0)\\ [\underline{\mathcal{L}}^\beta \bar f](w,0,0,0) = -i \sum_{\gamma\in\mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar g^\gamma](w,0,0,0)\ \Theta'_\gamma(H(w)), & \forall\ \beta\in\mathbb{N}^m_*. \end{cases}$$

(We specify again $(t,\tau) := (w,0,0,0)$.) By the classical Baouendi-Rothschild calculation (*see* Proposition 5.1 in [M1])[2], we have :

$$(5.4)\qquad \begin{cases} \left\langle [\underline{\mathcal{L}}^\beta \bar f](t,\tau) = -i \sum_{\gamma\in\mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar g^\gamma](t,\tau)\ \Theta'_\gamma(t'), \quad \forall\ \beta \in \mathbb{N}^m \right\rangle\\ \Longleftrightarrow\\ \left\langle \underline{\Omega}_\beta(t,\tau,\nabla^{|\beta|}\bar h(\tau)) = \Theta'_\beta(t') + \sum_{\gamma\in\mathbb{N}^m_*} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar g(\tau)^\gamma\ \Theta'_{\beta+\gamma}(t'), \quad \forall\ \beta\in\mathbb{N}^m \right\rangle. \end{cases}$$

[2] Now, we use Convention 4.3.

Thus eq. (4.2) and eq. (5.3) after equivalence (5.4) yield :

$$(5.5)\qquad \begin{cases} \underline{\Omega}_\beta = \Theta'_\beta(H) + \sum_{\gamma\in\mathbb{N}^m_*} \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}^\gamma\;\Theta'_{\beta+\gamma}(H), \qquad \forall\,\beta\in\mathbb{N}^m, \\ \underline{\Omega}_\beta = \Theta'_\beta(h) + \sum_{\gamma\in\mathbb{N}^m_*} \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}^\gamma\;\Theta'_{\beta+\gamma}(h), \qquad \forall\,\beta\in\mathbb{N}^m. \end{cases}$$

We deduce $\Theta'_\beta(h(w,0)) \equiv \Theta'_\beta(H(w)) \in \mathbb{C}\{w\}^d$, $\forall\,\beta\in\mathbb{N}^m$ thanks to uniqueness below[3].

§6. Uniqueness property

Lemma 6.1. *The solution* $(\psi_\beta(t,\tau))_{\beta\in\mathbb{N}^m} \in \mathbb{C}[\![t,\tau]\!]^{\mathbb{N}^m}$ *of the infinite trigonal matrix system*

$$(6.2)\qquad \psi_\beta(t,\tau) + \sum_{\gamma\in\mathbb{N}^m_*} \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}(\tau)^\gamma\;\psi_{\beta+\gamma}(t,\tau) = \underline{\omega}_\beta(t,\tau), \quad \forall\,\beta\in\mathbb{N}^m,$$

is unique and is given simply by

$$(6.3)\qquad \psi_\beta(t,\tau) = \underline{\omega}_\beta(t,\tau) + \sum_{\gamma\in\mathbb{N}^m_*} (-1)^\gamma \frac{(\beta+\gamma)!}{\beta!\;\gamma!}\,\bar{g}(\tau)^\gamma\;\underline{\omega}_{\beta+\gamma}(t,\tau), \quad \forall\,\beta\in\mathbb{N}^m.$$

§7. Some formal relations

Put :

$$(7.1)\qquad \begin{cases} E_\beta(t,\tau,t') := [\underline{\mathcal{L}}^\beta \bar{f}](t,\tau) + i\sum_{\gamma\in\mathbb{N}^m_*} w'^\gamma\;[\underline{\mathcal{L}}^\beta \bar{\Theta}'_\gamma(\bar{h})](t,\tau), \qquad \beta\in\mathbb{N}^m, \\ F_\beta(t,\tau,t') := [\underline{\mathcal{L}}^\beta \bar{f}](t,\tau) + i\sum_{\gamma\in\mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](t,\tau)\;\Theta'_\gamma(t'), \qquad \beta\in\mathbb{N}^m. \end{cases}$$

There exists a matrix $a'(t',\tau')\in\mathbb{C}\{t',\tau'\}^{d\times d}$ with $a'(0,0)=I_{d\times d}$ such that

$$(7.2)\qquad \begin{cases} \left(\xi'-z'+i\bar{\Theta}'(\zeta',w',\xi')\right) \equiv a'(t',\tau')\left(\xi'-z'+i\Theta'(w',\zeta',z')\right), \\ \left(\xi'-z'+i\Theta'(w',\zeta',z')\right) \equiv \bar{a}'(\tau',t')\left(\xi'-z'+i\bar{\Theta}'(\zeta',w',\xi')\right). \end{cases}$$

We have :

$$(7.3)\qquad \begin{cases} \left\langle \bar{f}(\tau) - z' + i\sum_{\gamma\in\mathbb{N}^m_*} w'^\gamma\;\bar{\Theta}'_\gamma(\bar{h}(\tau)) \right\rangle \equiv \\ \qquad\qquad \equiv a'(t',\bar{h}(\tau)) \left\langle \bar{f}(\tau) - z' + i\sum_{\gamma\in\mathbb{N}^m_*} \bar{g}(\tau)^\gamma\;\Theta'_\gamma(t') \right\rangle. \end{cases}$$

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[3] In fact, since $\bar{g}(0)=0$, we observe that this deduction is even straightforward here :

$$\Theta'_\beta(h(w,0)) + 0 \equiv \underline{\Omega}_\beta(w,0,0,0,\nabla^{|\beta|}\bar{h}(0)) \equiv \Theta'_\beta(H(w)) + 0.$$

However, Lemma 6.1 below will be really of use in the next steps as we claim here, *because the terms* $\bar{g}^\gamma$ *will not vanish on the subsequent Segre chains.*

Applying $\underline{\mathcal{L}}^\beta$ to eq. (7.3), we have furthermore :

(7.4)
$$\begin{cases} \left\langle [\underline{\mathcal{L}}^\beta \bar{f}](t,\tau) + i \sum_{\gamma\in\mathbb{N}^m_*} w'^\gamma \; [\underline{\mathcal{L}}^\beta \bar{\Theta}'_\gamma(\bar{h})](t,\tau) \right\rangle \equiv \\ \qquad \equiv a'(t',\bar{h}(\tau)) \left\langle [\underline{\mathcal{L}}^\beta \bar{f}](t,\tau) + i \sum_{\gamma\in\mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](t,\tau) \; \Theta'_\gamma(t') \right\rangle + \\ \qquad + \sum_{\delta<\beta} {a'}_\delta^\beta(t',t,\tau) \left\langle [\underline{\mathcal{L}}^\delta \bar{f}](t,\tau) + i \sum_{\gamma\in\mathbb{N}^m_*} [\underline{\mathcal{L}}^\delta \bar{g}^\gamma](t,\tau) \; \Theta'_\gamma(t') \right\rangle. \end{cases}$$

(Here, ${a'}_\delta^\beta(t',t,\tau) \in \mathbb{C}\{t',t\}[\![\tau]\!]^{d\times d}$.) Thus :

$$\begin{cases} E_\beta(t,\tau,t') \equiv a'(t',\bar{h}(\tau))\, F_\beta(t,\tau,t') + \sum_{\delta<\beta} {a'}_\delta^\beta(t',t,\tau)\, F_\delta(t,\tau,t'), \\ F_\beta(t,\tau,t') \equiv b'(\bar{h}(\tau),t')\, E_\beta(t,\tau,t') + \sum_{\delta<\beta} {b'}_\delta^\beta(t',t,\tau)\, E_\delta(t,\tau,t'). \end{cases} \tag{7.5}$$

(Here, $b' = \bar{a}'$ and ${b'}_\delta^\beta(t',t,\tau) \in \mathbb{C}\{t',t\}[\![\tau]\!]^{d\times d}$.) Consequently :

$$\text{Ideal } \langle E_\beta(t,\tau,t')\rangle_{\beta\in\mathbb{N}^m} = \text{Ideal } \langle F_\beta(t,\tau,t')\rangle_{\beta\in\mathbb{N}^m}. \tag{7.6}$$

(In $\mathbb{C}\{t,t'\}[\![\tau]\!]^d$.) Let $T' = (T'_1,\dots,T'_n) \in \mathbb{C}^n$. From (7.5) :

$$\begin{cases} \sum_{j=1}^n \frac{\partial E_\beta}{\partial t'_j}\, T'_j = \left(\sum_{j=1}^n \frac{\partial a'}{\partial t'_j}\, T'_j\right) F_\beta + a' \left(\sum_{j=1}^n \frac{\partial F_\beta}{\partial t'_j}\, T'_j\right) + \\ \qquad + \sum_{\delta<\beta} \left(\left(\sum_{j=1}^n \frac{\partial {a'}_\delta^\beta}{\partial t'_j}\, T'_j\right) F_\delta + {a'}_\delta^\beta \left(\sum_{j=1}^n \frac{\partial F_\delta}{\partial t'_j}\, T'_j\right)\right), \\ \sum_{j=1}^n \frac{\partial F_\beta}{\partial t'_j}\, T'_j = \left(\sum_{j=1}^n \frac{\partial b'}{\partial t'_j}\, T'_j\right) E_\beta + b' \left(\sum_{j=1}^n \frac{\partial E_\beta}{\partial t'_j}\, T'_j\right) + \\ \qquad + \sum_{\delta<\beta} \left(\left(\sum_{j=1}^n \frac{\partial {b'}_\delta^\beta}{\partial t'_j}\, T'_j\right) E_\delta + {b'}_\delta^\beta \left(\sum_{j=1}^n \frac{\partial E_\delta}{\partial t'_j}\, T'_j\right)\right). \end{cases} \tag{7.7}$$

Let $|\alpha| = 1$. Apply $\partial_z^\alpha|_{z=0}$ to (7.5) at the point $(t,\tau,t') := (w,0,0,0,t')$:

$$\begin{cases} [\partial_z^\alpha(E_\beta(w,z,0,z,t'))]_{z=0} = [\partial_z^\alpha(a'(t',\bar{h}(0,z)))]_{z=0}\; F_\beta(w,0,0,0,t') + \\ + a'(t',\bar{h}(0))\; [\partial_z^\alpha(F_\beta(w,z,0,z,t'))]_{z=0} + \\ + \sum_{\delta<\gamma} \Big([\partial_z^\alpha({a'}_\delta^\beta(t',\bar{h}(0,z)))]_{z=0}\, F_\beta(w,0,0,0,t') + \\ + {a'}_\delta^\beta(t',\bar{h}(0))\; [\partial_z^\alpha(F_\beta(w,z,0,z,t'))]_{z=0} \Big). \end{cases} \tag{7.8}$$

Analogously :

$$(7.9)\quad \begin{cases} [\partial_z^\alpha(F_\beta(w,z,0,z,t'))]_{z=0} = [\partial_z^\alpha(b'(\bar{h}(0,z),t'))]_{z=0}\ E_\beta(w,0,0,0,t')+ \\ +\, b'(\bar{h}(0),t')\ [\partial_z^\alpha(E_\beta(w,z,0,z,t'))]_{z=0}+ \\ +\sum_{\delta<\gamma}\Big([\partial_z^\alpha(b_\delta'^{\,\beta}(\bar{h}(0,z),t'))]_{z=0}\ E_\beta(w,0,0,0,t')+ \\ +b_\delta'^{\,\beta}(\bar{h}(0),t')\ [\partial_z^\alpha(E_\beta(w,z,0,z,t'))]_{z=0}\Big). \end{cases}$$

Conclusion (using (7.7-8-9) :

$$(7.10)\quad \begin{cases} \text{Ideal}\left\langle [\partial_z^\alpha|_{z=0}(E_\beta(w,z,0,z,t'))] + \sum_{j=1}^n \frac{\partial E_\beta}{\partial t_j'}(w,0,0,0,t')\,T_j' \right\rangle_{\beta\in\mathbb{N}^m} = \\ = \text{Ideal}\left\langle [\partial_z^\alpha|_{z=0}(F_\beta(w,z,0,z,t'))] + \sum_{j=1}^n \frac{\partial F_\beta}{\partial t_j'}(w,0,0,0,t')\,T_j' \right\rangle_{\beta\in\mathbb{N}^m}. \end{cases}$$

(In the ring $\mathbb{C}[\![w,t',T']\!]$, considering the d components of $E_\beta = (E_{\beta,1},\dots,E_{\beta,d})$ and of $F_\beta = (F_{\beta,1},\dots,F_{\beta,d})$.)

§8. Convergence of the first order jets of $\mathcal{R}_h'$ on the first Segre $\mathcal{S}_0^1$ chain

Put $(t,\tau) := (w,z,0,z)$ in eqs. (4.1) (recall $\Theta(w,0,z)\equiv 0$), hence $(w,z,0,z)\in\mathcal{M}$. Set $\partial_z^\alpha = (\partial_{z_1}^{\alpha_1},\dots,\partial_{z_d}^{\alpha_d})$. Differentiate eqs. (4.1) by $\partial_z^\alpha|_{z=0}$. Treat first $|\alpha|=1$ before the general case. Thus writing eqs. (4.1) at the point $(t,\tau) := (w,z,0,z)$:

$$(8.1)\quad \begin{cases} \left\langle 0\equiv [\underline{\mathcal{L}}^\beta \bar{f}](w,z,0,z) + i\sum_{\gamma\in\mathbb{N}_*^m}[g^\gamma \underline{\mathcal{L}}^\beta(\bar{\Theta}_\gamma'(\bar{h}))](w,z,0,z), \qquad \forall\,\beta\in\mathbb{N}^m \right\rangle \\ \Longleftrightarrow \\ \langle\, E_\beta(w,z,0,z,h(w,z)) = 0, \qquad \forall\,\beta\in\mathbb{N}^m\,\rangle. \end{cases}$$

Consider the system :

$$(8.2)\quad \begin{cases} 0\equiv [E_\beta(w,0,0,0,t')]^{t':=h(w,0)}, \qquad \forall\,\beta\in\mathbb{N}^m, \\ 0\equiv [\partial_z^\alpha|_{z=0}(E_\beta(w,z,0,z,t'))]^{t':=h(w,0)} + \left[\sum_{j=1}^n \frac{\partial E_\beta}{\partial t_j'}(w,0,0,0,t')T_j'\right]_{T':=\partial_z^\alpha h(w,0)}^{t':=h(w,0)}, \end{cases}$$

of which a formal solution $(h(w,0),\partial_z^\alpha h(w,0))\in\mathbb{C}[\![w]\!]^{n+n}$ exists, by assumption (recall $\alpha\in\mathbb{N}^d$ with $|\alpha|=1$ is *fixed*). Important observation : we have :

$$(8.3)\quad [\partial_z^\alpha|_{z=0}(E_\beta(w,z,0,z,t'))]+\left[\sum_{j=1}^n \frac{\partial E_\beta}{\partial t_j'}(w,0,0,0,t')T_j'\right]\in\mathbb{C}\{w,t',T'\}^d, \quad \forall\,\beta\in\mathbb{N}^m.$$

(Because $[\nabla_\tau^\kappa(\bar{\Theta}'_\gamma(\bar{h}))](0)$ is constant for all $\kappa \in \mathbb{N}$ and because the coefficients of $\underline{\mathcal{L}}$ are analytic in (t,τ).) Therefore (by [A], Theorem 1), there exists $(H(w), H^1(w)) \in \mathbb{C}\{w\}^{n+n}$ such that :

$$(8.4)\qquad \begin{cases} 0 \equiv E_\beta(w,0,0,0,H(w)), \qquad \forall\, \beta \in \mathbb{N}^m, \\ 0 \equiv \partial_z^\alpha|_{z=0}(E_\beta(w,z,0,z,H(w)) + \displaystyle\sum_{j=1}^n \frac{\partial E_\beta}{\partial t'_j}(w,0,0,0,H(w))\; H_j^1(w). \end{cases}$$

From property (7.10), we deduce :

$$(8.5)\qquad \begin{cases} 0 \equiv F_\beta(w,0,0,0,H(w)), \qquad \forall\, \beta \in \mathbb{N}^m, \\ 0 \equiv \partial_z^\alpha|_{z=0}(F_\beta(w,z,0,z,H(w)) + \displaystyle\sum_{j=1}^n \frac{\partial F_\beta}{\partial t'_j}(w,0,0,0,H(w))\; H_j^1(w). \end{cases}$$

As in §5, we deduce first from the first family of equations in (8.5) : $\Theta'_\beta(h(w,0)) \equiv \Theta'_\beta(H(w))$, $\forall\, \beta \in \mathbb{N}^m$. I claim that these equalities all imply :

$$(8.6)\qquad \partial_z^\alpha|_{z=0}(F_\beta(w,z,0,z,H(w)) \equiv \partial_z^\alpha|_{z=0}(F_\beta(w,z,0,z,h(w,0)).$$

Indeed, by definition :

$$(8.7)\qquad \begin{cases} \partial_z^\alpha|_{z=0}[F_\beta(w,z,0,z,t')] = \\ \qquad = \partial_z^\alpha|_{z=0}\left[[\underline{\mathcal{L}}^\beta \bar{f}](w,z,0,z) + i \displaystyle\sum_{\gamma\in\mathbb{N}_*^m} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,z,0,z)\; \Theta'_\gamma(t') \right]. \end{cases}$$

Now put $\chi(w) := -\partial_z^\alpha|_{z=0}(F_\beta(w,z,0,z,h(w,0))$. Comparing (8.5) with (8.2) :

$$(8.8)\quad \sum_{j=1}^n \frac{\partial F_\beta}{\partial t'_j}(w,0,0,0,h(w,0)))\; \partial_z^\alpha h_j(w,0) \equiv \sum_{j=1}^n \frac{\partial F_\beta}{\partial t'_j}(w,0,0,0,H(w))\; H_j^1(w),$$

In other words :

$$(8.9)\qquad \begin{cases} \chi(w) \equiv -i \displaystyle\sum_{\gamma\in\mathbb{N}_*^m} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,0,0,0) \left(\sum_{j=1}^n \frac{\partial \Theta'_\gamma}{\partial t'_j}(h(w,0))\; \partial_z^\alpha h_j(w,0) \right), \\ \chi(w) \equiv -i \displaystyle\sum_{\gamma\in\mathbb{N}_*^m} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,0,0,0) \left(\sum_{j=1}^n \frac{\partial \Theta'_\gamma}{\partial t'_j}(H(w))\; H_j^1(w) \right). \end{cases}$$

By calculation (5.4) and Lemma 6.1 :

$$(8.10)\qquad \left(\sum_{j=1}^n \frac{\partial \Theta'_\beta}{\partial t'_j}(h(w,0))\; \partial_z^\alpha h_j(w,0) \right) \equiv \left(\sum_{j=1}^n \frac{\partial \Theta'_\beta}{\partial t'_j}(H(w))\; H_j^1(w) \right),$$

for all $\beta \in \mathbb{N}^m$, which proves that $\partial_z^\alpha|_{z=0}[\Theta'_\beta(h(w,z))] \in \mathbb{C}\{w\}^d$, $\forall\, \beta \in \mathbb{N}^m$.

§9. Cauchy-type growth estimates

Up to now, we only know that $\partial_z^\alpha|_{z=0}[\Theta'_\beta(h(w,z))] \in \mathbb{C}\{w\}^d$, $\forall\, \beta \in \mathbb{N}^m$. Notice that a series $\sum_{\gamma\in\mathbb{N}^m} \bar{\lambda}'^\gamma \theta_\gamma(u)$ is convergent *iff* $\theta_\gamma(u) \in \mathbb{C}\{u\}^d, \forall\, \gamma \in \mathbb{N}^m$ and there exist two positive constants $C > 0$, $\varepsilon > 0$ such that $|\theta_\gamma(u)| \leq C^{|\gamma|+1}$ if $|u| \leq \varepsilon$. Thus, to finish the proof of the convergence property $\partial_z^\alpha|_{z=0}[\mathcal{R}'_h(w,z,\bar{\nu}')] \in \mathbb{C}\{w,\bar{\nu}'\}^d$, we have to estimate the rate of growth of this collection of convergent power series as $|\gamma|$ tends to infinity. This is done in the following statement:

Proposition 9.1. *Let $m \in \mathbb{N}_*$, $d \in \mathbb{N}_*$, $k \in \mathbb{N}^*$, let $x \in \mathbb{C}^k$, let $(\Xi_\gamma(x))_{\gamma\in\mathbb{N}^m}$ be a collection of holomorphic d-vectorial functions satisfying : $\exists\, C > 0$, $\exists\, \varepsilon > 0$ such that : $(|x| \leq \varepsilon \Rightarrow |\Xi_\gamma(x)| \leq C^{|\gamma|+1})$. Let $\nu \in \mathbb{N}_*$, let $u \in \mathbb{C}^\nu$, $h(u) \in \mathbb{C}[\![u]\!]^n$, let $\alpha \in \mathbb{N}^n$, and suppose that $\partial_u^\alpha[\Xi_\gamma(h(u))] \in \mathbb{C}\{u\}^d$, $\forall\, \gamma \in \mathbb{N}^m$. Then there exist positive constants $C' > 0$, $\varepsilon' > 0$ such that : $(|u| \leq \varepsilon' \Rightarrow |\partial_u^\alpha[\Xi_\gamma(h(u))]| \leq C'^{|\gamma|+1})$.*

Idea of proof. Application of Artin's approximation theorem in the suitable space of jets. See [M2], §7 for details. □

Remark. In the main applications, the integer $\nu \in \mathbb{N}_*$ will correspond to the number of variables in a Segre chain : $\nu = m$, $2m$, $3m$, *etc.*

Thus we have proved :

Proposition 9.2. *For all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 1$, $\partial_z^\alpha|_{z=0}[\mathcal{R}'_h(w,z,\bar{\nu}')] \in \mathbb{C}\{w,\bar{\nu}'\}^d$.*

Recall the 1-jet : $\nabla_t^1 u = ((\partial_w^\gamma u)_{|\gamma|\leq 1}, (\partial_z^\alpha u)_{|\alpha|\leq 1})$. We already know :

$$\left\langle\ \mathcal{R}'_h(w,0,\bar{\nu}') \in \mathbb{C}\{w,\nu'\}^d\ \right\rangle \Rightarrow \left\langle\ \partial_w^\gamma[\mathcal{R}'_h(w,0,\bar{\nu}')] \in \mathbb{C}\{w,\bar{\nu}'\}^d\ \right\rangle. \tag{9.3}$$

Corollary 9.4. $[\nabla_t^1\mathcal{R}'_h](w,0,\bar{\nu}') \in \mathbb{C}\{w,\bar{\nu}'\}^d$.

Important remark 9.5. In the sequel, it will be thus sufficient only to prove that the jets of the reflection mapping are convergent on the subsequent Segre chains – and then the right estimation as $|\gamma| \to \infty$ will follow from Proposition 9.1.

§10. Convergence of all jets of $\mathcal{R}'_h$ on the first Segre chain $\mathcal{S}_0^1$

The purpose of this paragraph is to prove that for all $\alpha \in \mathbb{N}^d$ and all $\gamma \in \mathbb{N}^m$, then $[\partial_z^\alpha\partial_w^\gamma]\mathcal{R}'_h(w,0,\bar{\nu}') \in \mathbb{C}\{w,\bar{\nu}'\}^d$.

Clearly again :

Lemma 10.1. *If $\alpha \in \mathbb{N}^d$, we have :*

$$\left\langle\ [\partial_z^\alpha\mathcal{R}'_h](w,0,\bar{\nu}') \in \mathbb{C}\{w,\bar{\nu}'\}^d\ \right\rangle\ \Rightarrow\ \left\langle\ [\partial_w^\gamma\partial_z^\alpha\mathcal{R}'_h](w,0,\bar{\nu}')\ \ \forall\, \gamma \in \mathbb{N}^m\ \right\rangle. \tag{10.2}$$

Thanks to this observation, it suffices now to show :

Assertion 10.3. $\partial_z^\alpha\mathcal{R}'_h(w,0,\bar{\nu}') \in \mathbb{C}\{w,\bar{\nu}'\}^d$, $\forall\, \alpha \in \mathbb{N}^d$.

Proof. Write shortly $E_\beta(w,z,t')$ and $F_\beta(w,z,t')$. Let us define by induction on $\alpha \in \mathbb{N}^d$ a collection $E_\beta^{(\alpha)}$ of d-vectorial functions as follows. Let $\alpha^1 \in \mathbb{N}^d$ with

$|\alpha^1| = 1$.

$$(10.4)\qquad \begin{cases} E_\beta^{(0)}(w,z,t') := E_\beta(w,z,t'), \qquad T_0' = t', \\ E_\beta^{(\alpha+\alpha^1)}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha+\alpha^1}) := \\ := \dfrac{\partial E_\beta^{(\alpha)}}{\partial z^{\alpha^1}}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha}) + \displaystyle\sum_{\alpha'\leq\alpha} \dfrac{\partial E_\beta^{(\alpha)}}{\partial T'_{\alpha'}}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha})\, T'_{\alpha'+\alpha^1}. \end{cases}$$

Similarly, we also define the collection $(F_\beta^{(\alpha)})_{\alpha\in\mathbb{N}^d,\beta\in\mathbb{N}^m}$. Clearly, this collection satisfies by construction :

$$(10.5)\qquad \Big[E_\beta^{(\alpha)}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha})\Big]_{T'_{\alpha'}:=\partial_z^{\alpha'}h(w,z),\forall\ \alpha'\leq\alpha} = \partial_z^\alpha[E_\beta(w,z,h(w,z))].$$

Assertion 10.6. *In* $\mathbb{C}[\![w,z,(T'_{\alpha'})_{\alpha'\leq\alpha}]\!]$ *we have*

$$(10.7)\qquad \text{Ideal}\left\langle (E_\beta^{(\alpha)}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha}))_{\beta\in\mathbb{N}^m}\right\rangle = \text{Ideal}\left\langle (F_\beta^{(\alpha)}(w,z,(T'_{\alpha'})_{\alpha'\leq\alpha}))_{\beta\in\mathbb{N}^m}\right\rangle.$$

Check. Using (10.4), this is proved by generalizing (7.5-10). □

Now, $h(w,z)$ is a solution of the system of formal equations $E_\beta(w,z,h(w,z)) \equiv 0$, $\forall\ \beta\in\mathbb{N}^m$, hence

$$(10.8)\quad 0\equiv \partial_z^\alpha|_{z=0}[E_\beta(w,z,h(w,z))] = E_\beta^{(\alpha)}(w,0,(\partial_z^{\alpha'}h(w,0))_{\alpha'\leq\alpha}), \qquad \forall\ \alpha\in\mathbb{N}^d.$$

Fix $\alpha\in\mathbb{N}^d$ and consider the *finite* subsystem :

$$(10.9)\qquad E_\beta^{(\alpha')}(w,0,(\partial_z^{\alpha''}h(w,0))_{\alpha''\leq\alpha'}) = 0, \qquad \forall\ \alpha'\leq\alpha.$$

Here, the equations (10.9) are *analytic* :

$$(10.10)\qquad E_\beta^{(\alpha')}(w,0,(T_{\alpha''})_{\alpha''\leq\alpha'}) \in \mathbb{C}\{w,(T_{\alpha''})_{\alpha''\leq\alpha'}\}^d, \qquad \forall\ \alpha'\leq\alpha,$$

because every term $\partial_z|_{z=0}[[\underline{\mathcal{L}}^\beta\Theta'_\gamma(\bar{h})](w,z,0,z)] \in \mathbb{C}\{w\}^d$. Therefore we can find *analytic* solutions $(H_{\alpha'}(w))_{\alpha'\leq\alpha}$, $H_{\alpha'}(w)\in\mathbb{C}\{w\}^n$, satisfying :

$$(10.11)\qquad E_\beta^{(\alpha')}(w,0,(H_{\alpha''}(w))_{\alpha''\leq\alpha'}) \equiv 0, \qquad \forall\ \alpha'\leq\alpha.$$

Thanks to property (10.7), we deduce :

$$(10.12)\qquad F_\beta^{(\alpha')}(w,0,(H_{\alpha''}(w))_{\alpha''\leq\alpha'}) \equiv 0, \qquad \forall\ \alpha'\leq\alpha.$$

But we clearly have also :

$$(10.13)\qquad F_\beta^{(\alpha')}(w,0,(\partial_z^{\alpha''}h(w,0))_{\alpha''\leq\alpha'}) \equiv 0, \qquad \forall\ \alpha'\leq\alpha,$$

an identity which we can write more explicitely :

$$\text{(10.14)}\qquad \begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](w,z,0,z)] + i \sum_{\gamma\in\mathbb{N}_*^m} \Bigg(\sum_{\alpha''\leq\alpha'} \frac{\alpha'!}{\alpha''!\,(\alpha'-\alpha'')!} \\ \qquad \partial_z^{\alpha'-\alpha''}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,z,0,z)]\ \partial_z^{\alpha''}|_{z=0}[\Theta'_\gamma(h(w,z))] \Bigg), \end{cases}$$

for all $\alpha' \leq \alpha$. Denote :

$$\text{(10.15)}\qquad {\Theta'_\gamma}^{\alpha'}((\partial_z^{\alpha''} h(w,0))_{\alpha''\leq\alpha'}) := \partial_z^{\alpha'}|_{z=0}[\Theta'_\gamma(h(w,z))].$$

Then we can rewrite eqs. (10.12) and (10.14) respectively as :

$$\text{(10.16)}\qquad \begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](w,z,0,z)] + i \sum_{\gamma\in\mathbb{N}_*^m} \Bigg(\sum_{\alpha''\leq\alpha'} \frac{\alpha'!}{\alpha''!\,(\alpha'-\alpha'')!} \\ \partial_z^{\alpha'-\alpha''}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,z,0,z)]\ {\Theta'_\gamma}^{\alpha'}(H_{\alpha''}(w))_{\alpha''\leq\alpha'}) \Bigg) \equiv 0, \qquad \forall\ \alpha' \leq \alpha, \end{cases}$$

and :

$$\text{(10.17)}\qquad \begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](w,z,0,z)] + i \sum_{\gamma\in\mathbb{N}_*^m} \Bigg(\sum_{\alpha''\leq\alpha'} \frac{\alpha'!}{\alpha''!\,(\alpha'-\alpha'')!} \\ \partial_z^{\alpha'-\alpha''}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{g}^\gamma]g(w,z,0,z)]\ {\Theta'_\gamma}^{\alpha'}((\partial_z^{\alpha''} h(w,0))_{\alpha''\leq\alpha'}) \Bigg) \equiv 0, \qquad \forall\ \alpha' \leq \alpha. \end{cases}$$

It suffices now to apply Lemma 6.1 to deduce from eqs. (10.16-17) :

Assertion 10.18. *For all $\gamma \in \mathbb{N}^m$ and all $\alpha' \leq \alpha$,*

$$\text{(10.19)}\qquad {\Theta'_\gamma}^{\alpha'}((\partial_z^{\alpha''} h(w,0))_{\alpha''\leq\alpha'}) \equiv {\Theta'_\gamma}^{\alpha'}(H_{\alpha''}(w))_{\alpha''\leq\alpha'}) \in \mathbb{C}\{w\}^d.$$

Proof. Thanks to §4-9, Assertion 10.18 is known for $|\alpha| = 0, 1$. Assume therefore by induction that identities 10.19 hold for all $|\alpha'| \leq \kappa$, where κ is a positive integer with $1 \leq \kappa < |\alpha|$. Let $|\alpha'_0| = \kappa + 1$. Write eqs. (10.16) and (10.17) for $\alpha' := \alpha'_0$ and substract them together. Using the induction assumption, we get

$$\text{(10.20)}\qquad i \sum_{\gamma\in\mathbb{N}_*^m} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w,0,0,0)\ \left[{\Theta'_\gamma}^{\alpha'_0}(H_{\alpha''}(w))_{\alpha''\leq\alpha'_0}) - {\Theta'_\gamma}^{\alpha'_0}((\partial_z^{\alpha''} h(w,0))_{\alpha''\leq\alpha'_0}) \right] \equiv 0,$$

for all $\beta \in \mathbb{N}^m$. By (5.4) and Lemma 6.1, this yields (10.19) for $\alpha' = \alpha'_0$. □

Proposition 10.21. *We have*

1. $\partial_z^\alpha \partial_w^\delta|_{z=0}[\Theta'_\gamma(h(w,z))] \in \mathbb{C}\{w\}^d$ *for all* $\alpha \in \mathbb{N}^d$, $\delta \in \mathbb{N}^m$, $\gamma \in \mathbb{N}^m$.
2. *Applying Proposition 9.1,* $\partial_z^\alpha \partial_w^\delta|_{z=0}[\mathcal{R}'_h(w,z,\bar{\nu}')] \in \mathbb{C}\{w,\bar{\nu}'\}^d$, *for all* $\alpha \in \mathbb{N}^d$, $\delta \in \mathbb{N}^m$.

This completes the proof of Assertion 10.3. □

§11. Convergence of $\mathcal{R}'_h$ on the second Segre chain $\underline{\mathcal{S}}_0^2$

Thanks to (10.21), we can now apply Artin's theorem to the following family of analytic equations with the formal solutions $h(w, i\bar{\Theta}(\zeta, w, 0)) \in \mathbb{C}[\![w,\zeta]\!]^n$:

$$
(11.1)\quad \begin{cases} f(w, i\bar{\Theta}(\zeta, w, 0)) \equiv \bar{f}(\zeta, 0) + i \sum_{\gamma\in\mathbb{N}_*^m} g(w, i\bar{\Theta}(\zeta, w, 0))^\gamma \; \bar{\Theta}'_\gamma(\bar{h}(\zeta, 0)), \\ 0 \equiv [\underline{\mathcal{L}}^\beta \bar{f}](w, i\bar{\Theta}(\zeta, w, 0), \zeta, 0) + \\ + i \sum_{\gamma\in\mathbb{N}_*^m} g(w, i\bar{\Theta}(\zeta, w, 0))^\gamma \; [\underline{\mathcal{L}}^\beta \bar{\Theta}'_\gamma(\bar{h})](w, i\bar{\Theta}(\zeta, w, 0), \zeta, 0), \quad \forall\, \beta\in\mathbb{N}_*^m. \end{cases}
$$

This application yields a *convergent* solution $H(w,\zeta) \in \mathbb{C}\{w,\zeta\}^n$ with $H(0,0) = 0$. By eq. (7.5), this solution also satisfies

$$
(11.2)\quad \begin{cases} \bar{f}(\zeta, 0) \equiv F(w,\zeta) - i \sum_{\gamma\in\mathbb{N}_*^m} \bar{g}(\zeta, 0)^\gamma \; \Theta'_\gamma(H(w,\zeta)), \\ [\underline{\mathcal{L}}^\beta \bar{f}](w, i\bar{\Theta}(\zeta, w, 0), \zeta, 0) \equiv - \\ - i \sum_{\gamma\in\mathbb{N}_*^m} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](w, i\bar{\Theta}(\zeta, w, 0), \zeta, 0) \; \Theta'_\gamma(H(w,\zeta)), \quad \forall\, \beta\in\mathbb{N}_*^m. \end{cases}
$$

By equivalence (5.4), we deduce that $H(w,\zeta)$ satisfies :

$$
(11.3)\quad \begin{cases} \Theta'_\beta(H(w,\zeta)) + \sum_{\gamma\in\mathbb{N}_*^m} \frac{(\beta+\gamma)!}{\beta!\;\gamma!} \bar{g}(\zeta, 0)^\gamma \; \Theta'_{\beta+\gamma}(H(w,\zeta)) = \\ \qquad = \underline{\Omega}_\beta(w, \zeta, 0, \nabla^{|\beta|}\bar{h}(\zeta, 0)), \qquad \forall\, \beta\in\mathbb{N}^m. \end{cases}
$$

We recall that $h(w, i\bar{\Theta}(w, \zeta, 0))$ also satisfies the similar system :

$$
(11.4)\quad \begin{cases} \Theta'_\beta(h(w, i\bar{\Theta}(w,\zeta,0))) + \sum_{\gamma\in\mathbb{N}_*^m} \frac{(\beta+\gamma)!}{\beta!\;\gamma!} \bar{g}(\zeta, 0)^\gamma \; \Theta'_{\beta+\gamma}(h(w, i\bar{\Theta}(w,\zeta,0)) = \\ \qquad = \underline{\Omega}_\beta(w, \zeta, 0, \nabla^{|\beta|}\bar{h}(\zeta, 0)), \qquad \forall\, \beta\in\mathbb{N}^m. \end{cases}
$$

Finally, using the uniqueness property Lemma 6.1, we deduce that

$$
(11.5)\quad \Theta'_\beta(h(w, i\bar{\Theta}(w,\zeta,0)) \equiv \Theta'_\beta(H(w,\zeta)) \in \mathbb{C}\{w,\zeta\}^d, \quad \forall\, \beta\in\mathbb{N}^m,
$$

and by a final application of Proposition 9.1, we can conclude :

Proposition 11.6. *The reflection function converges on the second Segre chain :*

$$
(11.7)\quad \bar{\mu}' - i \sum_{\gamma\in\mathbb{N}_*^m} \bar{\lambda}'^\gamma \; \Theta'_\gamma(h(w, i\bar{\Theta}(\zeta, w, 0)) \in \mathbb{C}\{w, \zeta, \bar{\nu}'\}^d. \quad \square
$$

§12. Discussion of the general induction

Using only the family of eqs. (4.2), we have provided in [M2] a noticeably more direct proof of Proposition 10.21 (this proof, written in the hypersurface case, works in fact for arbitrary codimension). However, *the general induction based only on eqs.* (4.2) *would block while trying to establish Proposition 11.6 above.* Fortunately, it will appear in §13-14-15 below that the formal computations in §4-11 above can be easily generalized to achieve the general induction. But let us first explain which difficulties one encounters in these matters.

§**12.1. Explanation.** The general induction would block for the following reason. Suppose that we have established Proposition 10.21. In general, in classical published previous works, *only the family of equations* (4.2) – *or equivalently* (5.4) – *is considered*. While trying to "jump to" the second Segre chain, one encounters the following obstructing facts :

1. One only knows that the jets of all the functions Θ'_γ restricted to the first Segre chain are converging, *and not at all that the jets of the mapping h all converge upon this chain*[4]. Unfortunately, they appear in (5.4) ! Let us write again (5.4) and try to get Proposition 11.6 from 10.21. On the chain $\underline{\mathcal{S}}_0^2 = \{\mathcal{L}_w \circ \underline{\mathcal{L}}_\zeta(0) \colon w, \zeta \in \mathbb{C}^m \text{ small}\}$, we have :

(12.2)
$$\begin{cases} \underline{\Omega}_\beta(w, i\bar{\Theta}(\zeta, w, 0), \zeta, 0, \nabla^{|\beta|}\bar{h}(\zeta, 0)) \equiv \\ \equiv \Theta'_\beta(h(w, i\bar{\Theta}(\zeta, w, 0))) + \sum_{\gamma \in \mathbb{N}_*^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar{g}(\zeta, 0)^\gamma\ \Theta'_{\beta+\gamma}(h(w, i\bar{\Theta}(\zeta, w, 0))), \end{cases}$$

for all $\beta \in \mathbb{N}^m$.

2. One could think about approximating all the jets $\nabla^{|\beta|}\bar{h}(\zeta, 0)$ by converging power series. However, there is an infinite family of such equations, hence an *infinite number of variables* $(T'_\beta)_{\beta \in \mathbb{N}^m}$ for these jets $(\nabla^{|\beta|}\bar{h}(\zeta, 0))_{\beta \in \mathbb{N}^m}$! Unfortunately, Artin's theorem then fails to apply...
3. To bypass this difficulty, Nordine Mir has devised some astuteness. To begin with, he writes (12.2) not on the second chain, but on the *third* Segre chain :

$$\text{(12.3)} \qquad \begin{cases} \underline{\Omega}_\beta(\mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0), (\nabla^{|\beta|}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)) \equiv \\ \equiv \Theta'_\beta(h) \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) + \sum_{\gamma \in \mathbb{N}_*^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!} \\ \bar{g}^\gamma \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)\ \Theta'_{\beta+\gamma}(h) \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0). \end{cases}$$

He tries to apply the following direct corollary of Artin's theorem, in order to replace all the formal terms $(\nabla^{|\beta|}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$ by some *converging power series*. Let $\mathfrak{m}$ denote the maximal ideal in a local ring.

[4] This is why strong assumptions like essential finitess or Segre nondegeneracy of (M', p') are made in some articles. In these cases, one can deduce at once that the jets of h converge on the first Segre chain. We refer to [M2] which contains an example of Baouendi-Ebenfelt-Rothschild showing that not all holomorphically nondegenerate hypersurfaces are Segre nondegenerate.

Lemma 12.4. *Let* $\mu \in \mathbb{N}_*$, $\nu \in \mathbb{N}_*$, $v \in \mathbb{C}^\mu$, $u \in \mathbb{C}^\nu$. *Let* $R(v, u, T(v)) \equiv 0$, *where* $R(v, u, T) \in \mathbb{C}[\![v, u, T]\!]^\lambda$, *where* $T(v) \in \mathbb{C}[\![v]\!]^\iota$, $\lambda \in \mathbb{N}_*$, $\iota \in \mathbb{N}_*$. *If* $\partial_u^\beta|_{u=0}[R(v, u, T)] \in \mathbb{C}\{v, T\}$ *for all* $\beta \in \mathbb{N}^\nu$, *then for each* $N \in \mathbb{N}_*$, *there exists* $T_N(v) \in \mathbb{C}\{v\}^\iota$ *with* $T_N \equiv T \pmod{\mathfrak{m}_v^N}$ *such that* $R(v, u, T_N(v)) \equiv 0$.

Check. Since $\partial_u^\beta|_{u=0}[R(v, u, T)] \in \mathbb{C}\{v, T\}$ for all $\beta \in \mathbb{N}^\nu$, a direct application of Artin's theorem yields a solution $T_N(v) \in \mathbb{C}\{v\}^\iota$ with $T_N \equiv T \pmod{\mathfrak{m}_v^N}$ such that $\partial_u^\beta|_{u=0}[R(v, u, T_N(v))] \equiv 0$ for all $\beta \in \mathbb{N}^\nu$. Then obviously $R(v, u, T_N(v)) \equiv 0$. □

Here, the main assumption of this lemma is satisfied ; Indeed, let us check :

(12.5)
$$\partial_{w_2}^\beta|_{w_2=0}[\Theta_\gamma'(h) \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)] = [\mathcal{L}^\beta \Theta_\gamma'(h)] \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \in \mathbb{C}\{w_1, \zeta_1\}^d,$$

for all γ and all β. Indeed, we assume Proposition 10.21 to hold and we have $(\nabla_t^\kappa \Theta_\gamma'(h)) \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) \equiv (\nabla_t^\kappa \Theta_\gamma'(h)) \circ \mathcal{L}_{w_1}(0)$ for all $\kappa \in \mathbb{N}$. *However again, there is an infinite number of variables* $T = (T_\beta')_{\beta \in \mathbb{N}^m}$ *for the formal solutions* $(\nabla^{|\beta|}\bar{h}) \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$ *in eqs.* (12.3) *and one cannot apply Lemma 12.4 in this form.* It'a pitty, since if we could have approximated the left hand side of eq. (12.3), and also the term in $\bar{g}^\gamma$, we could have easily deduced the convergence of all $\Theta_\gamma'(h) \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0)$, after applying correctly the inversion system given by equivalence (6.2) and after checking the Cauchy-type growth estimates.

4. However, in codimension one, Mir applies Lemma 12.4 *only to eq.* (12.3) *for* $\beta = 0$, *i.e.* to eq. (4.2) with these arguments, which yields a convergent solution $h_N(w_1, \zeta_1) \in \mathbb{C}\{w_1, \zeta_1\}^n$ satisfying :

(12.6)
$$\bar{f}_N(w_1, \zeta_1) \equiv f \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0) - i \sum_{\gamma \in \mathbb{N}_*^m} \bar{g}_N^\gamma(w_1, \zeta_1)\ \Theta_\gamma'(h) \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(0).$$

Then, eq. (12.6) shows that the left-hand side is convergent, since f_N is ! Finally, Mir deduces easily from eq. (12.6) that the reflection function is convergent, *only in the hypersurface case and for dimensional reasons.*

Remark. Similarly, the author deduces in [M2] the convergence of h in his main theorem, *only for dimensional reasons.*

5. It could seem that the above tricks can provide the convergence of the reflection mapping on the second Segre chain. *However, we shall see below that a clever use of Lemma 12.4 together with Lemma 6.1 and with properties* (7.6) *and* (10.7) *can nevertheless gives the general result.* This will give a slightly different "second demonstration" of the main theorem.

Interpretation 12.7.

1. The alternate use of both identities (4.1) and (4.2) seems natural : it reflects an important **symmetry property of the CR mapping problem**.

2. The proof given here seems to deal with the adequate structures of the CR mapping problem itself.

Analogy 12.8. The analogy with the interpretation of Segre chain as being constructed from the pair of conjugate CR m-vector fields (*see* [M1]) is evident :

* * * * *	"*Holomorphic*"	"*Conjugate*"
CR fields	$\mathcal{L}$	$\underline{\mathcal{L}}$
Equations	$z' = \xi' + i\bar{\Theta}'(\zeta', w', \xi')$	$\xi' = z' - i\Theta'(w', \zeta', z')$

§13. Execution of the general induction

We denote $\mathcal{L} = (\mathcal{L}^1, \dots, \mathcal{L}^m)$ and $\underline{\mathcal{L}} = (\underline{\mathcal{L}}_1, \dots, \underline{\mathcal{L}}_m)$ two bases of $T^{1,0}\mathcal{M}$ and of $T^{0,1}\mathcal{M}$ which commute, we denote by $\mathcal{L}_w(p) := \mathcal{L}^1_{w_1} \circ \cdots \circ \mathcal{L}_{w_m}(p)$ the m-flow of $\mathcal{L}$ (*idem* for $\underline{\mathcal{L}}_\zeta(p)$), $w \in \mathbb{N}^m$, $\zeta \in \mathbb{N}^m$. The concatenations of such flows will be called *Segre k-chains*. For instance, for $k = 2j$ (and similarly for $k = 2j+1$), the map $(w_1, w_2, \dots, w_{2j-1}, w_{2j}) \mapsto \underline{\mathcal{L}}_{w_{2j}} \circ \mathcal{L}_{w_{2j-1}} \circ \cdots \circ \underline{\mathcal{L}}_{w_2} \circ \mathcal{L}_{w_1}(0) \in \mathcal{M}$. If we agree to denote $w_{(k)} := (w_1, \dots, w_k)$, where $w_1, \dots, w_k \in \mathbb{C}^m$ are close to 0, we can abbreviate this map as $w_{(k)} \mapsto \Gamma_k(w_{(k)})$. We can also consider the maps $\bar{\Gamma}_k(w_{(k)})$ given by $(w_1, w_2, \dots, w_{2j-1}, w_{2j}) \mapsto \mathcal{L}_{w_{2j}} \circ \underline{\mathcal{L}}_{w_{2j-1}} \circ \cdots \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{w_1}(0) \in \mathcal{M}$ for $k = 2j$ (and similarly for $k = 2j+1$). We also denote by Υ the vector field $\Upsilon = \frac{\partial}{\partial z} + (1 - i\Theta_z(w, \zeta, z))\frac{\partial}{\partial \xi}$ and $\underline{\Upsilon} = \frac{\partial}{\partial \xi} + (1 + i\bar{\Theta}_\xi(\zeta, w, \xi))\frac{\partial}{\partial z}$. Similarly, we denote by $(z, p) \mapsto \Upsilon_z(p)$ the d-flow of Υ.

The following properties hold :

Assertion 13.1.

1. $\mathcal{R}'_h(\Gamma_k(w_{(k)}), \bar{\nu}') \in \mathbb{C}\{w_{(k)}, \bar{\nu}'\}^d$, $\forall\ k \in \mathbb{N}$.
$\iff \Theta'_\gamma(h(\Gamma_k(w_{(k)}))) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall\ \gamma \in \mathbb{N}^m$.

2. $[\partial_z^\alpha \partial_w^\delta [\Theta'_\gamma(h)]](\Gamma_k(w_{(k)}) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall\ \gamma \in \mathbb{N}^m$, $\forall\ \alpha \in \mathbb{N}^d$, $\forall\ \delta \in \mathbb{N}^m$.
$\iff [\Upsilon^\alpha \Theta'_\gamma(h)](\Gamma_k(w_{(k)})) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall \gamma \in \mathbb{N}^m$, $\forall\ \alpha \in \mathbb{N}^d$.

3. $\mathcal{R}'_h(\Gamma_{2d}(w_{(2d)}), \bar{\nu}') \in \mathbb{C}\{w_{(2d)}, \bar{\nu}'\}^d$
$\Rightarrow \mathcal{R}'_h(t, \bar{\nu}') \in \mathbb{C}\{t, \bar{\nu}'\}^d$.

Proof.

1 : by Proposition 9.1.
2 : by the chain rule, knowing span $(\Upsilon, \mathcal{L}, \underline{\mathcal{L}}) = T\mathcal{M}$ (see [M1], §8 for details).
3 : by minimality of (M, p) (*see* [M1], §3 for details). □

We conduct the main induction in essentially two steps, as in all known cases, where (M', p') is either finitely nondegenerate, or essentially finite, or Segre nondegenerate.

Assertion 13.2.

I. $\Theta'_\gamma(h) \circ \Gamma_k(w_{(k)}) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall\ \gamma \in \mathbb{N}^m$.
$\Rightarrow [\nabla_t^\kappa(\Theta'_\gamma(h))] \circ \Gamma_k(w_{(k)}) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall\ \kappa \in \mathbb{N}$.

II. $[\nabla_t^\kappa(\Theta'_\gamma(h))] \circ \Gamma_k(w_{(k)}) \in \mathbb{C}\{w_{(k)}\}^d$, $\forall\ \kappa \in \mathbb{N}$
$\Rightarrow \Theta'_\gamma(h) \circ \Gamma_k(w_{(k+1)}) \in \mathbb{C}\{w_{(k+1)}\}^d$, $\forall\ \gamma \in \mathbb{N}^m$.

§14. Proof of Assertion 13.2, I

We choose k odd and prove 13.2,I in this case. The even case is similar (as in [M1]). Thus, let k be odd. We assume by induction that Assertion 13.2, I holds for $k-1$. Let $z \in \mathbb{C}^d$, $w_{(k)} \in \mathbb{C}^{mk}$. We consider everything at the point $\Upsilon_z \circ \Gamma_k(w_{(k)})$. Set :

(14.1)
$$\begin{cases} E_\beta(w_{(k)}, z, t') := [\underline{\mathcal{L}}^\beta \bar{f}](\Upsilon_z \circ \Gamma_k(w_{(k)})) + i \sum_{\gamma \in \mathbb{N}^m_*} w'^\gamma \ [\underline{\mathcal{L}}^\beta \bar{\Theta}'_\gamma(\bar{h})](\Upsilon_z \circ \Gamma_k(w_{(k)})), \\ F_\beta(w_{(k)}, z, t') := [\underline{\mathcal{L}}^\beta \bar{f}](\Upsilon_z \circ \Gamma_k(w_{(k)})) + i \sum_{\gamma \in \mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](\Upsilon_z \circ \Gamma_k(w_{(k)})) \ \Theta'_\gamma(t'), \end{cases}$$

for all $\beta \in \mathbb{N}^m$. We have :

$$\text{(14.2)} \qquad \begin{cases} E_0(w_{(k)}, z, t') = a'(t', \bar{h}(\Upsilon_z \circ \Gamma_k(w_{(k)}))) \ F_0(w_{(k)}, z, t'), \\ F_0(w_{(k)}, z, t') = \bar{a}'(\bar{h}(\Upsilon_z \circ \Gamma_k(w_{(k)})), t') \ E_0(w_{(k)}, z, t'). \end{cases}$$

Applying all the derivations $\underline{\mathcal{L}}^\beta$ to eqs. (14.2) :

(14.3)
$$\begin{cases} E_\beta(w_{(k)}, z, t') \equiv a'(t', w_{(k)}, z) \ F_\beta(w_{(k)}, z, t') + \sum_{\delta < \beta} {a'}_\delta^\beta(t', w_{(k)}, z) \ F_\delta(w_{(k)}, z, t'), \\ F_\beta(w_{(k)}, z, t') \equiv b'(t', w_{(k)}, z) \ E_\beta(w_{(k)}, z, t') + \sum_{\delta < \beta} {b'}_\delta^\beta(t', w_{(k)}, z) \ E_\delta(w_{(k)}, z, t'). \end{cases}$$

Here, ${a'}_\delta^\beta(w_{(k)}, z, t')$, ${b'}_\delta^\beta(w_{(k)}, z, t') \in \mathbb{C}\{t'\}[\![w_{(k)}, z]\!]^{d \times d}$. Let us define by induction on $\alpha \in \mathbb{N}^d$ a collection $E_\beta^{(\alpha)}$ of d-vectorial functions as follows. Let $\alpha^1 \in \mathbb{N}^d$ with $|\alpha^1| = 1$.

(14.4)
$$\begin{cases} E_\beta^{(0)}(w_{(k)}, z, t') := E_\beta(w_{(k)}, z, t'), \qquad T'_0 = t', \\ E_\beta^{(\alpha+\alpha^1)}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha+\alpha^1}) := \\ := \frac{\partial E_\beta^{(\alpha)}}{\partial z^{\alpha^1}}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha}) + \sum_{\alpha' \leq \alpha} \frac{\partial E_\beta^{(\alpha)}}{\partial T'_{\alpha'}}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha}) \ T'_{\alpha'+\alpha^1}. \end{cases}$$

Similarly, we also define the collection $(F_\beta^{(\alpha)})_{\alpha \in \mathbb{N}^d, \beta \in \mathbb{N}^m}$. Clearly, this collection satisfies by construction :

$$\text{(14.5)} \qquad \begin{cases} \left[E_\beta^{(\alpha)}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha})\right]_{T'_{\alpha'} := [\Upsilon_z^{\alpha'} h](\Upsilon_z \circ \Gamma_k(w_{(k)})), \ \forall \ \alpha' \leq \alpha} = \\ = \partial_z^\alpha [E_\beta(w_{(k)}, z, h(\Upsilon_z \circ \Gamma_k(w_{(k)})))]. \end{cases}$$

Assertion 14.6. *In $\mathbb{C}[\![w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha}]\!]$ we have for every $\alpha \in \mathbb{N}^d$:*

(14.7)
$$\text{Ideal}\left\langle (E_\beta^{(\alpha)}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha}))_{\beta \in \mathbb{N}^m} \right\rangle = \text{Ideal}\left\langle (F_\beta^{(\alpha)}(w_{(k)}, z, (T'_{\alpha'})_{\alpha' \leq \alpha}))_{\beta \in \mathbb{N}^m} \right\rangle.$$

Check. Using (14.4), this is proved by generalizing (7.5-10). □

Now, $h(\Upsilon_z \circ \Gamma_k(w_{(k)}))$ is a solution of the system of formal equations $E_\beta(w_{(k)}, z, h(\Upsilon_z \circ \Gamma_k(w_{(k)}))) \equiv 0$, $\forall \beta \in \mathbb{N}^m$, hence

$$(14.8)\qquad \begin{cases} 0 \equiv \partial_z^\alpha|_{z=0}[E_\beta(w_{(k)}, z, h(\Upsilon_z \circ \Gamma_k(w_{(k)})))] = \\ = E_\beta^{(\alpha)}(w_{(k)}, 0, ([\Upsilon^{\alpha'} h](\Gamma_k(w_{(k)}))_{\alpha' \leq \alpha}), \qquad \forall \alpha \in \mathbb{N}^d, \ \forall \beta \in \mathbb{N}^m. \end{cases}$$

Now, *we fix* $\alpha \in \mathbb{N}^d$ and we consider the *finite* subsystem :

$$(14.9)\qquad E_\beta^{(\alpha')}(w_{(k)}, 0, ([\Upsilon^{\alpha''} h](\Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'}) = 0, \qquad \forall \alpha' \leq \alpha.$$

Here, the equations (14.9) are *analytic* :

$$(14.10)\qquad E_\beta^{(\alpha')}(w_{(k)}, 0, (T_{\alpha''})_{\alpha'' \leq \alpha'}) \in \mathbb{C}\{w_{(k)}, (T_{\alpha''})_{\alpha'' \leq \alpha'}\}^d, \qquad \forall \alpha' \leq \alpha,$$

because every term $\partial_z|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{\Theta}_\gamma'(\bar{h})](\Upsilon_z \circ \Gamma_k(w_{(k)}))] \in \mathbb{C}\{w_{(k)}\}^d$ by the induction assumption, because $\nabla_t^\kappa \bar{\Theta}_\gamma'(\bar{h}(\Gamma_k(w_{(k)}))) \equiv \nabla_t^\kappa \bar{\Theta}_\gamma'(\bar{h}(\Gamma_k(w_{(k-1)})))$ if k is odd (recall $\Gamma_k(w_{(k)}) = \mathcal{L}_{w_k} \circ \Gamma_{k-1}(w_{(k-1)})$ and that $\bar{h}(\mathcal{L}_w(p)) \equiv \bar{h}(p)$). Therefore we can find *analytic* solutions $(H_{\alpha'}(w_{(k)}))_{\alpha' \leq \alpha}$: $H_{\alpha'}(w_{(k)}) \in \mathbb{C}\{w_{(k)}\}^n$, satisfying :

$$(14.11)\qquad E_\beta^{(\alpha')}(w_{(k)}, 0, (H_{\alpha''}(w_{(k)}))_{\alpha'' \leq \alpha'}) \equiv 0, \qquad \forall \alpha' \leq \alpha.$$

Thanks to property (14.7), we deduce :

$$(14.12)\qquad F_\beta^{(\alpha')}(w_{(k)}, 0, (H_{\alpha''}(w_{(k)}))_{\alpha'' \leq \alpha'}) \equiv 0, \qquad \forall \alpha' \leq \alpha.$$

But we also clearly have :

$$(14.13)\qquad F_\beta^{(\alpha')}(w_{(k)}, 0, [\Upsilon_z^{\alpha''} h](\Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'}) \equiv 0, \qquad \forall \alpha' \leq \alpha,$$

an identity which we can write more explicitely :

$$(14.14)\qquad \begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](\Upsilon_z \circ \Gamma_k(w_{(k)}))] + i \displaystyle\sum_{\gamma \in \mathbb{N}_*^m} \left(\sum_{\alpha'' \leq \alpha'} \frac{\alpha'!}{\alpha''! \, (\alpha' - \alpha'')!} \right. \\ \left. \partial_z^{\alpha' - \alpha''}[[\underline{\mathcal{L}}^\beta \bar{g}](\Upsilon_z \circ \Gamma_k(w_{(k)}))^\gamma)] \ \partial_z^{\alpha''}[\Theta_\gamma'(h(\Upsilon_z \circ \Gamma_k(w_{(k)})))] \right)_{z=0} \equiv 0. \end{cases}$$

Moreover, if we denote :

$$(14.15)\qquad {\Theta_\gamma'}^{\alpha'}(([\Upsilon_z^{\alpha''} h] \circ \Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'}) := \partial_z^{\alpha'}|_{z=0}[\Theta_\gamma'(h(\Upsilon_z \circ \Gamma_k(w_{(k)})))],$$

then we can rewrite eqs. (14.12) and (14.14) respectively as

$$(14.16)\qquad \begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](\Upsilon_z \circ \Gamma_k(w_{(k)}))] + i \displaystyle\sum_{\gamma \in \mathbb{N}_*^m} \left(\sum_{\alpha'' \leq \alpha'} \frac{\alpha'!}{\alpha''! \, (\alpha' - \alpha'')!} \right. \\ \left. \partial_z^{\alpha' - \alpha''}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{g}^\gamma](\Upsilon_z \circ \Gamma_k(w_{(k)})))] \ {\Theta_\gamma'}^{\alpha'}((H_{\alpha''}(w_{(k)}))_{\alpha'' \leq \alpha'}) \right) \equiv 0. \end{cases}$$

(14.17)

$$\begin{cases} 0 \equiv \partial_z^{\alpha'}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{f}](\Upsilon_z \circ \Gamma_k(w_{(k)}))] + i \displaystyle\sum_{\gamma \in \mathbb{N}_*^m} \left(\sum_{\alpha'' \leq \alpha'} \frac{\alpha'!}{\alpha''! \, (\alpha' - \alpha'')!} \right. \\ \left. \partial_z^{\alpha' - \alpha''}|_{z=0}[[\underline{\mathcal{L}}^\beta \bar{g}^\gamma](\Upsilon_z \circ \Gamma_k(w_{(k)})))] \ {\Theta_\gamma'}^{\alpha'}(([\Upsilon_z^{\alpha''} h] \circ \Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'}) \right) \equiv 0. \end{cases}$$

It will suffice now to apply the uniqueness principle Lemma 6.1 to deduce from eqs. (14.17-18) :

Assertion 14.18. *For all $\gamma \in \mathbb{N}^m$ and all $\alpha' \leq \alpha$,*

$$\Theta'^{\alpha'}_{\gamma}(([\Upsilon^{\alpha''}_z h] \circ \Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'}) \equiv \Theta'^{\alpha'}_{\gamma}((H_{\alpha''}(w_{(k)}))_{\alpha'' \leq \alpha'}) \in \mathbb{C}\{w_{(k)}\}^d. \tag{14.19}$$

Proof. Assertion 14.18 is known for $|\alpha| = 0$. Assume therefore by induction that identities 14.19 hold for all $|\alpha'| \leq \kappa$, where κ is a positive integer with $0 \leq \kappa < |\alpha|$. Let $|\alpha'_0| = \kappa + 1$. We write eqs. (14.16) and (14.17) for $\alpha' := \alpha'_0$ and we substract them together. Using the induction assumption, we get

$$\begin{cases} i \sum_{\gamma \in \mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](\Upsilon_z \circ \Gamma_{(k)}(w_{(k)})) \ \Big[\Theta'^{\alpha'_0}_{\gamma}((H_{\alpha''}(w_{(k)}))_{\alpha'' \leq \alpha'_0}) - \\ -\Theta'^{\alpha'_0}_{\gamma}(([\Upsilon^{\alpha''}_z h] \circ \Gamma_k(w_{(k)}))_{\alpha'' \leq \alpha'_0})\Big] \equiv 0, \end{cases} \tag{14.20}$$

for all $\beta \in \mathbb{N}^m$. By (5.4) and Lemma 6.1, this yields (14.19) for $\alpha' = \alpha'_0$. □

Conclusion 14.21. *We have for such k odd by induction :*

1. $\partial^\alpha_z \partial^\delta_w|_{z=0}[\Theta'_\gamma(h(\Upsilon_z \circ \Gamma_k(w_{(k)})))] \in \mathbb{C}\{w_{(k)}\}^d$ *for all $\alpha \in \mathbb{N}^d$, $\delta \in \mathbb{N}^m$, $\gamma \in \mathbb{N}^m$.*
2. *Applying Proposition 9.1,* $\partial^\alpha_z \partial^\delta_w|_{z=0}[\mathcal{R}'_h(\Upsilon_z \circ \Gamma_k(w_{(k)}), \bar{\nu}')] \in \mathbb{C}\{w_{(k)}, \bar{\nu}'\}^d$, *for all $\alpha \in \mathbb{N}^d$, $\delta \in \mathbb{N}^m$.*

§15. Proof of Assertion 13.2, II

We now choose $k+1$ even and prove part II of Assertion 13.2. Thanks to (14.21), we can now apply Artin's theorem to the following family of analytic equations with the formal solutions $h(\bar{\Gamma}_{(k+1)}(w_{(k+1)})) \in \mathbb{C}[\![w_{(k+1)}]\!]^n$:

$$\begin{cases} f(\bar{\Gamma}_{(k+1)}(w_{(k+1)})) \equiv \bar{f}(\bar{\Gamma}_k(w_{(k)})) + i \sum_{\gamma \in \mathbb{N}^m_*} g^\gamma(\bar{\Gamma}_{k+1}(w_{(k+1)})) \ \bar{\Theta}'_\gamma(\bar{h}(\bar{\Gamma}_k(w_{(k)}))), \\ 0 \equiv [\underline{\mathcal{L}}^\beta \bar{f}](\bar{\Gamma}_{k+1} w_{(k+1)})) + i \sum_{\gamma \in \mathbb{N}^m_*} g^\gamma(\bar{\Gamma}_{k+1}(w_{(k+1)})) \ [\underline{\mathcal{L}}^\beta \bar{\Theta}'_\gamma(\bar{h})](\bar{\Gamma}_{k+1}(w_{(k+1)})), \end{cases} \tag{15.1}$$

for all $\beta \in \mathbb{N}_*$. We get a *convergent* solution $H(w_{(k+1)}) \in \mathbb{C}\{w_{(k+1)}\}^n$ with $H(0) = 0$. By eq. (7.5) written for $(t, \tau) := \bar{\Gamma}_{(k+1)}(w_{(k+1)})$, this solution also satisfies :

$$\begin{cases} \bar{f}(\bar{\Gamma}_k(w_{(k)})) \equiv F(w_{(k+1)}) - i \sum_{\gamma \in \mathbb{N}^m_*} \bar{g}^\gamma(\bar{\Gamma}_k(w_{(k)})) \ \Theta'_\gamma(H(w_{(k+1)})) \\ [\underline{\mathcal{L}}^\beta \bar{f}](\bar{\Gamma}_{k+1}(w_{(k+1)})) \equiv -i \sum_{\gamma \in \mathbb{N}^m_*} [\underline{\mathcal{L}}^\beta \bar{g}^\gamma](\bar{\Gamma}_{k+1}(w_{(k+1)})) \ \Theta'_\gamma(H(w_{(k+1)})), \end{cases} \tag{15.2}$$

for all $\beta \in \mathbb{N}_*$. By equivalence (5.4), we deduce that $H(w_{(k+1)})$ satisfies :

$$\begin{cases} \Theta'_\beta(H(w_{(k+1)})) + \sum_{\gamma \in \mathbb{N}^m_*} \frac{(\beta+\gamma)!}{\beta! \ \gamma!} \ \bar{g}^\gamma \circ \bar{\Gamma}_k(w_{(k)}) \ \Theta'_{\beta+\gamma}(H(w_{(k+1)})) = \\ \qquad = \underline{\Omega}_\beta(w_{(k+1)}, (\nabla^{|\beta|}\bar{h}) \circ \bar{\Gamma}_k(w_{(k)})), \qquad \forall \ \beta \in \mathbb{N}^m. \end{cases} \tag{15.3}$$

We recall that $h(\bar\Gamma_{k+1}(w_{(k+1)}))$ also satisfies the similar system :

(15.4)
$$\begin{cases} \Theta'_\beta(h(\bar\Gamma_{k+1}(w_{(k+1)})))+\sum_{\gamma\in\mathbb{N}^m_*}\frac{(\beta+\gamma)!}{\beta!\ \gamma!}\ \bar g^\gamma\circ\bar\Gamma_k(w_{(k)})\ \Theta'_{\beta+\gamma}(h(\bar\Gamma_{k+1}(w_{(k+1)})))= \\ \qquad\qquad = \underline{\Omega}_\beta(w_{(k+1)},(\nabla^{|\beta|}\bar h)\circ\bar\Gamma_k(w_{(k)}))), \qquad \forall\ \beta\in\mathbb{N}^m. \end{cases}$$

Finally, using the uniqueness property Lemma 6.1, we deduce that

$$\Theta'_\beta(h(\bar\Gamma_{k+1}(w_{(k+1)})))\equiv\Theta'_\beta(H(w_{(k+1)}))\in\mathbb{C}\{w_{(k+1)}\}^d, \qquad \forall\ \beta\in\mathbb{N}^m. \tag{15.5}$$

This completes the proof of Assertion 13.2. □

§16. Conclusion

We have proved that the reflection function converges on all Segre chains, *i.e.* that $\mathcal{R}'_h(\Gamma_k(w_{(k)}),\bar\nu')\in\mathbb{C}\{w_{(k)}\}^d$ for all $k\in\mathbb{N}$. For $k=2d$ (or = the Segre number of M at p), we deduce that $\mathcal{R}'_h(t,\bar\nu')\in\mathbb{C}\{t,\bar\nu'\}^d$, as in [M1].

Part 2 of Theorem 1 is standard.

The proof of Theorem 1 is complete. □

§17. Refinement

Theorem 17.1. *The reflection mapping also converges in the following circumstance : If $(M,p)\subset\mathbb{C}^n$ is minimal, $(M',p')\subset\mathbb{C}^{n'}$, $m\geq m'$ and h induces a formal map $(S_{\bar p},p)\to_{\mathcal{F}}(S_{\bar p'},p')$ of generic rank equal to $m'=\dim_{\mathbb{C}}S_{\bar p'}$.*

Hints for the proof. Mix §6 of [M1] together with the computations of §2-15 above. The proof is rather lengthy and technical. □

§18. Variation for a second proof

The goal is to obtain Proposition 11.6 differently. As we have discussed, Lemma 12.4 cannot be applied to eqs. (12.3), because there is an infinite number of variables. *Our idea here is to approximate the h-terms before applying the derivations $\underline{\mathcal{L}}^\beta$.* To this aim, let us start with :

$$\begin{cases} f\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\equiv\bar f\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)+ \\ +i\sum_{\gamma\in\mathbb{N}^m_*}g^\gamma\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\ \bar\Theta'_\gamma(\bar h\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)). \end{cases}\tag{18.1}$$

By Lemma 12.4, there exist $H(\zeta_1,w_1)\in\mathbb{C}\{\zeta_1,w_1\}^n$ such that

$$\begin{cases} F\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\equiv\bar f\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)+ \\ +i\sum_{\gamma\in\mathbb{N}^m_*}G^\gamma\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\ \bar\Theta'_\gamma(\bar h\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)). \end{cases}\tag{18.2}$$

By (7.3),

$$\begin{cases} \bar f\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\equiv F\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)- \\ -i\sum_{\gamma\in\mathbb{N}^m_*}\bar g^\gamma\circ\underline{\mathcal{L}}_{\zeta_2}\circ\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0)\ \Theta'_\gamma(H(\zeta_1,w_1)). \end{cases}\tag{18.3}$$

Now, we apply $\partial_{\zeta_2}^{\beta}|_{\zeta_2=0}$ *to eq.* (18.3). This yields :

$$[\underline{\mathcal{L}}^{\beta}\bar{f}](\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0))\equiv -i\sum_{\gamma\in\mathbb{N}_*^m}[\underline{\mathcal{L}}^{\beta}\bar{g}^{\gamma}](\mathcal{L}_{w_1}\circ\underline{\mathcal{L}}_{\zeta_1}(0))\ \Theta_{\gamma}'(H(\zeta_1,w_1)). \tag{18.4}$$

At this point, we have reached (11.2) and we can finish the proof of Proposition 11.6 exactly as in §11. The jets are treated similarly.

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