A MORSE-THEORETICAL PROOF OF THE HARTOGS EXTENSION THEOREM

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ABSTRACT. 100 years ago exactly, in 1906, Hartogs published a celebrated extension phenomenon (birth of *Several Complex Variables*), whose global counterpart was understood later: *holomorphic functions in a connected neighborhood* $\mathcal{V}(\partial\Omega)$ *of a connected boundary* $\partial\Omega \Subset \mathbb{C}^n$ $(n \geqslant 2)$ *do extend holomorphically and uniquely to the domain* Ω. Martinelli in the early 1940's and Ehrenpreis in 1961 obtained a rigorous proof, using a new multidimensional integral kernel or a short $\overline{\partial}$ argument, but it remained unclear how to derive a proof using only analytic discs, as did Hurwitz (1897), Hartogs (1906) and E.E. Levi (1911) in some special, model cases. In fact, known attempts (*e.g.* Osgood 1929, Brown 1936) struggled for monodromy against multivaluations, but failed to get the general global theorem.

Moreover, quite unexpectedly, Fornæss in 1998 exhibited a topologically strange (nonpseudoconvex) domain $\Omega^{\mathsf{F}} \subset \mathbb{C}^2$ that cannot be filled in by holomorphic discs, when one makes the additional requirement that discs must all lie entirely inside Ω^{F}. However, one should point out that the standard, unrestricted disc method usually allows discs to go outsise the domain (just think of Levi pseudoconcavity).

Using the method of analytic discs for local extensional steps and Morse-theoretical tools for the global topological control of monodromy, we show that the Hartogs extension theorem can be established in such a way.

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[22 colored illustrations]

§1. THE HARTOGS EXTENSION THEOREM AND THE METHOD OF ANALYTIC DISCS

100 years ago exactly, in 1906, the publication of Hartogs's thesis ([14] under the direction of Hurwitz) revealed what is now considered to be the most striking fact of multidimensional complex analysis: the automatic, compulsory holomorphic extension of functions of several complex variables to larger domains, especially for a class of "pot-looking" domains, nowadays called *Hartogs figures*, that may be filled in up to their top. Soon after, E.E. Levi [25] applied the Hurwitz-Hartogs argument of Cauchy integration on complex affine

Date: 2014-10-31.

circles moving in the domain (firstly discovered in [21]), in order to perform local holomorphic extension across strictly (Levi) pseudoconcave boundaries. The so-called *method of analytic discs* was born, historically.

Hartogs extension theorem. *Let* $\Omega \Subset \mathbb{C}^n$ *be a bounded domain having* connected *boundary. If* $n \geqslant 2$*, every function holomorphic in some connected open neighborhood* $\mathcal{V}(\partial\Omega)$ *of* $\partial\Omega$ *extend holomorphically and uniquely inside* Ω*, i.e.*:

$$\forall\, f \in \mathcal{O}\big(\mathcal{V}(\partial\Omega)\big), \quad \exists\,!\, F \in \mathcal{O}\big(\Omega \cup \mathcal{V}(\partial\Omega)\big) \quad \text{s.t.} \quad F\big|_{\mathcal{V}(\partial\Omega)} = f.$$

Classically, one also presents an alternative formulation, which is checked to be equivalent — think that $K = \Omega \backslash \mathcal{V}(\partial\Omega)$.

Hartogs theorem$^{\text{bis}}$. *If* $\Omega \Subset \mathbb{C}^n$ $(n \geqslant 2)$ *is a domain and if* $K \subset \Omega$ *is any compact such that* $\Omega \backslash K$ connected*, then* $\mathcal{O}(\Omega \backslash K) = \mathcal{O}(\Omega)\big|_{\Omega \backslash K}$.

Already in [14] (p. 231), Hartogs stated such a global theorem in the typical language of those days, without claiming single-valuedness however — something that he consistently mentions in other places. Later in [32], Osgood (who gives the reference to Hartogs) "proves" unique holomorphic extension with discs, but what is written there is seriously erroneous, even when applied to a ball. In 1936, well before Milnor ([31]) had popularized Morse theory, using topological concepts and a language which are nowadays difficult to grasp, Brown ([5]) fixed somehow single-valuedness of the extension[1]: discretizing $\Omega \backslash K$ to tame the topology, he exhausts $\mathbb{C}^n$ by spheres of decreasing radius (as we will do in this paper), but we believe that his proof still contains imprecisions, because the subtracting process that we encounter unavoidably when applying Morse theory does not appear in [5].

Since the 1940's, few complex analysts have seriously thought about testing the limit of the disc method probably because the motivation was gone, and in fact, the possible existence of an elementary *rigorous proof* of the global Hartogs extension theorem using only a finite number of Hartogs figures remained a folklore belief; for instance, in [35], p. 133, it is just left as an "exercise". But to the authors' knowledge, no reliable mathematical publication shows fully how to perform a rigorous proof of the global theorem, using only the original Hurwitz-Hartogs-Levi analytic discs as a tool.

On the other hand, thanks to the contributions of Kneser ([24]), of Fueter ([11]), of Martinelli ([27, 28]), of Bochner ([4]) and of Fichera ([9]), powerful multidimensional integral kernels were discovered that provided a complete proof, from the side of Analysis. Soon after, Ehrenpreis ([8]) found what is known to be the most concise proof, based on the vanishing of $\overline{\partial}$-cohomology

[1] We thank an anonymous referee for pointing historical incorrections in the preliminary version of this paper and for providing us with exact informations.

with compact support. This proof was learnt by generations of complex analysts, thanks to Hörmander's book [20]. Range's *Correction of the Historical Record* [34] provides an excellent account of the very birth of integral formulas in $\mathbb{C}^n$. Since the 1960's, $\overline{\partial}$ techniques, L^2 methods and integral kernels developed into a vast field of research in Several Complex Variables, *c.f.* [20, 2, 16, 15, 33, 6, 7, 22, 23, 26, 18].

A decade ago, Fornæss [10] produced a topologically strange domain Ω^{F} that cannot be filled in by means of analytic discs, when one makes the additional requirement that discs must all lie *entirely inside* the domain. Possibly, one could interpret this example as a "defeat" of geometrical methods.

But in absence of pseudoconvexity, it is much more natural to allow discs to *go outside* the domain, because the local E.E. Levi extension theorem already needs that. In fact, as remarked by Bedford in his review [3] of [10], Hartogs' phenomenon for Fornæss' domain Ω^{F} may be shown to hold straightforwardly by means of the usual, unrestricted disk method.

Furthermore, the study of envelopes of holomorphy (*see* the monograph of Jarnicki and Pflug [22] for an introduction to Riemann domains spread over $\mathbb{C}^n$ and [29] for applications in a CR context) shows well how natural it is to deal with sucessively enlarged (Riemann) domains. Bishop's constructive approach, especially his famous idea of gluing discs to real submanifolds, reveals to be adequate in such a widely open field of research. We hence may hope that, after the very grounding historical theorem of Hartogs has enjoyed a renewed proof, geometrical methods will undergo further developments, especially to devise fine holomorphic extension theorems that are unreachable by means of contemporary $\overline{\partial}$ techniques.

In this paper, we establish rigorously that the Hartogs extension theorem can be proved by means of a *finite number* of parameterized families of analytic discs (Theorems 2.7 and 5.4). The discs we use are all (tiny) pieces of complex lines in $\mathbb{C}^n$. The main difficulty is topological and we use the Morse machinery to tame multisheetedness.

At first, we shall replace the boundary $\partial\Omega$ by a $\mathcal{C}^\infty$ connected oriented hypersurface $M \Subset \mathbb{C}^n$ ($n \geqslant 2$) for which the restriction to M of the Euclidean norm function $z \mapsto \|z\|$ is a good Morse function (Lemma 3.3), namely there exist only finitely many points $\widehat{p}_\lambda \in M$, $1 \leqslant \lambda \leqslant \kappa$, with $\|\widehat{p}_1\| < \cdots < \|\widehat{p}_\kappa\|$ at which $z \mapsto \|z\|$ restricted to M has vanishing differential. We also replace $\mathcal{V}(\partial\Omega)$ by a very thin tubular neighborhood $\mathcal{V}_\delta(M)$, $0 < \delta \ll 1$, and Ω by a domain $\Omega_M \Subset \mathbb{C}^n$ bounded by M. Next, we will introduce a modification of the Hartogs figure, called a *Levi-Hartogs figure*, which is more appropriate to produce holomorphic extension from the cut out domains $\big\{\|z\| > r\big\} \cap \Omega_M$, where the radius r will decrease, inductively. Local Levi pseudoconcavity of the exterior of a ball then enables us to prolong the holomorphic functions to $\big\{\|z\| > r - \eta\big\} \cap \Omega_M$, for some uniform η with $0 < \eta \ll 1$, which depends on the dimension $n \geqslant 2$, on δ, and on the diameter of $\overline{\Omega}$. We hence descend stepwise to lower radii until the domain is fully filled in.

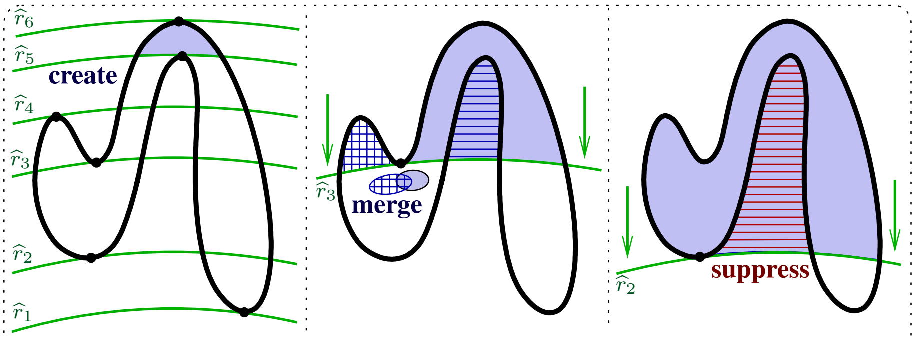


Fig. 1: Filling the domain, creating, merging and suppressing components

However, this naive conclusion fails because of multivaluations and a crucial three-piece topological device is required. We begin by filling the top of the domain, which is simply diffeomorphic to a cut out piece of ball. Geometrically speaking, Morse points $\widehat{p}_\lambda$, $1 \leqslant \lambda \leqslant \kappa$, are the only points of M at which the family of spheres $\big(\{\|z\| = r\}\big)_{0<r<\infty}$ are tangent to M. We denote $\|\widehat{p}_\lambda\| =: \widehat{r}_\lambda$ with $\widehat{r}_1 < \cdots < \widehat{r}_\kappa$. In Figure 1, we have $\kappa = 6$. For an arbitrary fixed radius r with $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$, and some fixed λ with $1 \leqslant \lambda \leqslant \kappa - 1$, we consider all connected components $M^c_{>r}$, $1 \leqslant c \leqslant c_\lambda$, of the cut out hypersurface $M \cap \big\{\|z\| > r\big\}$. Their number c_λ is the same for all $r \in \big(\widehat{r}_\lambda, \widehat{r}_{\lambda+1}\big)$. In Figure 1, when $\widehat{r}_3 < r < \widehat{r}_4$, we see three such components.

By descending discrete induction $r \mapsto r-\eta$, we show that each such connected hypersurface $M^c_{>r} \subset \big\{\|z\| > r\big\}$ bounds a certain domain $\widetilde{\Omega}^c_{>r} \subset \big\{\|z\| > r\big\}$ which is relatively compact in $\mathbb{C}^n$ and that holomorphic functions in $\mathcal{V}_\delta(M)$ do extend holomorphically and uniquely to $\widetilde{\Omega}^c_{>r}$. While approaching a lower Morse point, three different topological processes will occur[2]: *creating* a new component $\widetilde{\Omega}^{c'}_{>r-\eta}$ to be filled in further; *merging* two components $\widetilde{\Omega}^{c'_1}_{>r-\eta}$ and $\widetilde{\Omega}^{c'_2}_{>r-\eta}$ which meet; and *suppressing* one superfluous component $\widetilde{\Omega}^{c'_1}_{>r-\eta}$.

The unavoidable multivaluation phenomenon will be tamed by the idea of *separating ab initio* the components $M^c_{>r}$, $1 \leqslant c \leqslant c_\lambda$. Indeed, an advantageous topological property will be shown to be inherited through the induction $r \mapsto r - \eta$, hence always true, namely that two different domains $\widetilde{\Omega}^{c_1}_{>r}$ and $\widetilde{\Omega}^{c_1}_{>r}$ are either disjoint or one is contained in the other. Consequently, the multivaluation aspect will only happen in the sense that the two *uniquely defined and univalent* holomorphic extensions $f^{c_1}_r$ to $\widetilde{\Omega}^{c_1}_{>r}$ and $f^{c_2}_r$ to $\widetilde{\Omega}^{c_2}_{>r}$ can be different on $\widetilde{\Omega}^{c_1}_{>r}$, in case $\widetilde{\Omega}^{c_1}_{>r} \subset \widetilde{\Omega}^{c_2}_{>r}$, or *vice versa*. In this way, we *avoid completely* to deal with Riemann domains spread over $\mathbb{C}^n$.

Some of the elements of our approach should be viewed in a broader context. In their celebrated paper [1] (*see* also [17]), Andreotti and Grauert observed that convenient exhaustion functions can be used to prove very general extension and finiteness results on q-concave complex varieties. Their arguments implicitly contained a geometrical proof of the Hartogs extension theorem in the case

[2]A certain number of other simpler cases will also happen, where the components $\widetilde{\Omega}^c_{>r}$ do grow regularly with respect to holomorphic extension, possibly changing topology.

where the domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex (whence Fornæss' counter-example *must* be nonpseudoconvex). However, in contrast to our finer method, the existence of an internal strongly pseudoconvex exhaustion function ρ on a complex manifold X excludes *ab initio* multisheetedness: indeed, in such a circumstance, extension holds stepwise from shells of the form $\{z \in X : a < \rho(z) < b\}$ just to deeper shells $\{a' < \rho < b\}$ with $a' < a$ (details are provided in [30]), namely the topology is controlled in advance by ρ and multiple domains as $\widetilde{\Omega}^c_{>r}$ above cannot at all appear.

There is a nice alternative approach to the (singular) Hartogs extension theorem via a global continuity principle, realized in [23] by Jöricke and the second author, with the purpose of understanding removable singularities by means of (geometric) envelopes of holomorphy. The idea is to perform holomorphic extensions along one-parameter families of holomorphic curves (not suppose to be discs). A basic extension theorem on some appropriate Levi flat 3-manifolds, called *Hartogs manifolds* in [23], is shown via stepwise extension in the direction of an increasing real parameter. The geometrical scheme of this construction has a common topological element with our method: the simultaneous holomorphic extension to collections of domains that are pairwise either disjoint or one is contained in the other.

On the other hand, our technique only rely upon the existence of appropriate exhaustion functions, without requiring neither the existence of Levi-flat 3-manifolds nor the existence of global holomorphic functions in the ambient complex manifold. In addition, inspired by a definition formulated by Fornæss in [10], we establish that only a *finite* number of Levi-Hartogs figures is needed in the filling process. Finally, we would like to mention that a straightforward adaptation of the proof developed here would yield a geometrical proof of the Hartogs-type extension theorem of Andreotti and Hill ([2]), which is valid for arbitrary domains in $(n-1)$-complete manifolds (in the sense of Andreotti-Grauert [13]).

Twenty-two colored illustrations appear, each one being inserted at the appropriate place in the text. Abstract geometrical thought being intrinsically pictural, we hope to address to a broad audience of complex analysts and geometers.

§2. Preparation of the boundary and unique extension

2.1. Preparation of a good $\mathcal{C}^\infty$ boundary. Denote by $\|z\| := \big(|z_1|^2 + \cdots + |z_n|^2\big)^{1/2}$ the Euclidean norm of $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and by $\mathbb{B}^n(p, \delta) := \big\{\|z-p\| < \delta\big\}$ the open ball of radius $\delta > 0$ centered at a point p. If $E \subset \mathbb{C}^n$ is any set,

$$\mathcal{V}_\delta(E) := \cup_{p \in E} \, \mathbb{B}^n(p, \delta)$$

is a concrete open neighborhood of E.

As in the Hartogs theorem, assume that the domain $\Omega \Subset \mathbb{C}^n$ has connected boundary $\partial\Omega$ and let $\mathcal{V}(\partial\Omega)$ be an open neighborhood of $\partial\Omega$, also connected. Clearly, there exists δ_1 with $0 < \delta_1 \ll 1$ such that $\partial\Omega \subset \mathcal{V}_{\delta_1}(\partial\Omega) \subset \mathcal{V}(\partial\Omega)$; of course, $\mathcal{V}_{\delta_1}(\partial\Omega)$ is then also connected. Choose a point $p_0 \in \mathbb{C}^n$ with

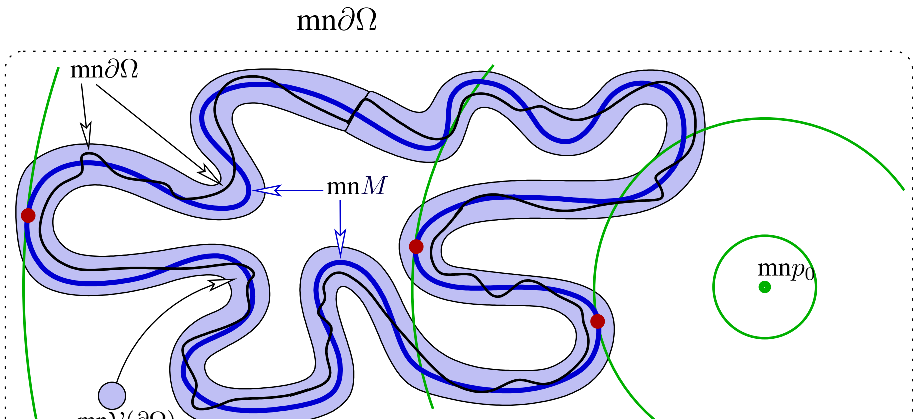
$\mathrm{dist}\,(p_0, \overline{\Omega}) = 3$, center the coordinates $(z_1, \dots, z_n)$ at p_0 and consider the distance function

$$r(z) := \|z - p_0\| = \|z\|. \tag{2.2}$$

It is crucial to prepare as follows the boundary, replacing $(\Omega, \partial\Omega)$ by (Ω_M, M), thanks to some transversality arguments that are standard in Morse theory ([31] and [19], Ch. 6).

Lemma 2.3. *There exists a $\mathcal{C}^\infty$ connected closed and oriented hypersurface $M \subset \mathcal{V}_{\delta_1/2}(\partial\Omega)$ such that*:

- **(i)** *M bounds a unique bounded domain Ω_M with $\Omega \subset \Omega_M \cup \mathcal{V}(\partial\Omega)$;*
- **(ii)** *the restriction $r_M(z) := r(z)\big|_M$ of the distance function $r(z) = \|z\|$ to M has only a finite number κ of critical points $\widehat{p}_\lambda \in M$, $1 \leqslant \lambda \leqslant \kappa$, located on different sphere levels, namely*
$$2 \leqslant r(\widehat{p}_1) < \cdots < r(\widehat{p}_\kappa) \leqslant 5 + \mathrm{diam}(\overline{\Omega});$$
- **(iii)** *all the $(2n-1) \times (2n-1)$ Hessian matrices $\mathsf{H}[r_M](\widehat{p}_1), \dots, \mathsf{H}[r_M](\widehat{p}_\kappa)$ have a nonzero determinant.*



mn**Fig. 2: Preparing the boundary**

Sometimes, r_M satisfying **(ii)** and **(iii)** is called a *good Morse function* on M. We will shortly say that M is a *good boundary*.

If k_λ is the number of positive eigenvalues of the (symmetric) Hessian matrix $\mathsf{H}[r_M](\widehat{p}_\lambda)$, the extrinsic Morse lemma ([31, 19]) shows that there exist $2n$ real coordinates $\big(v, x_1, \dots, x_{k_\lambda}, y_1, \dots, y_{2n-k_\lambda-1}\big)$ in a neighborhood of $\widehat{p}_\lambda$ in $\mathbb{C}^n$ such that

- the sets $\{v(z) = \mathrm{cst}\}$ simply correspond[3] to the spheres $\{r(z) = \mathrm{cst}\}$ near $\widehat{p}_\lambda$;
- $\big(x_1, \dots, x_{k_\lambda}, y_1, \dots, y_{2n-k_\lambda-1}\big)$ provide $(2n-1)$ local coordinates on the hypersurface M, whose graphed equation is normalized to be the simple hyperquadric
$$v = \sum_{1 \leqslant j \leqslant k_\lambda} x_j^2 - \sum_{1 \leqslant j \leqslant 2n-k_\lambda-1} y_j^2.$$

[3]In fact, one can just take the translated radius $r(z) - r(\widehat{p}_\lambda)$ as the coordinate $v = v(z)$.

Classically, the number $(2n - k_\lambda - 1)$ of *negatives* is called the *Morse index* of $r(z)\big|_M$ at $\widehat{p}_\lambda$; we will call k_λ its *Morse coindex*.

For rather general differential-geometric objects, Morse theory enables to control a significant part of homotopy groups and of (co)homologies, *e.g.* via Morse inequalities. In our case, we shall be able to control somehow the global topology of the cut-out domains $\Omega_M \cap \{\|z\| > r\}$ that re external to closed balls of radius r, filling them progressively by means of analytic discs contained in small (Levi-)Hartogs figures (Section 3). We start by checking rigorously that the Hartogs theorem can be reduced to some good boundary.

2.4. Unique holomorphic extension. If $\mathcal{U} \subset \mathbb{C}^n$ is open, $\mathcal{O}(\mathcal{U})$ denotes the ring of holomorphic functions in $\mathcal{U}$.

Definition 2.5. Given two connected open sets $\mathcal{U}_1 \subset \mathbb{C}^n$ and $\mathcal{U}_2 \subset \mathbb{C}^n$ with $\mathcal{U}_1 \cap \mathcal{U}_2$ nonempty, we will say[4] that $\mathcal{O}(\mathcal{U}_1)$ *extends holomorphically to* $\mathcal{U}_1 \cup \mathcal{U}_2$ if :

- the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is connected;
- there exists an open nonempty set $\mathcal{V} \subset \mathcal{U}_1 \cap \mathcal{U}_2$ such that for every $f_1 \in \mathcal{O}(\mathcal{U}_1)$, there exist $f_2 \in \mathcal{O}(\mathcal{U}_2)$ with $f_2|_{\mathcal{V}} = f_1|_{\mathcal{V}}$.

It then follows from the principle of analytic continuation that $f_1|_{\mathcal{U}_1\cap\mathcal{U}_2} = f_2|_{\mathcal{U}_1\cap\mathcal{U}_2}$, so that the joint function F, equal to f_j on $\mathcal{U}_j$ for $j = 1, 2$, is well defined, is holomorphic in $\mathcal{U}_1 \cup \mathcal{U}_2$ and extends f_1, namely $F|_{\mathcal{U}_1} = f_1$.

In concrete extensional situations, the coincidence of f_1 with f_2 is controlled only in some small $\mathcal{V} \subset \mathcal{U}_1 \cap \mathcal{U}_2$, so the connectedness of $\mathcal{U}_1 \cap \mathcal{U}_2$ appears to be useful to insure monodromy. Sometimes also, we shall briefly write $\mathcal{O}(\mathcal{U}_1) = \mathcal{O}(\mathcal{U}_1 \cup \mathcal{U}_2)\big|_{\mathcal{U}_1}$, instead of spelling rigorously:

$$\forall\, f_1 \in \mathcal{O}\big(\mathcal{U}_1\big) \quad \exists\, F \in \mathcal{O}\big(\mathcal{U}_1 \cup \mathcal{U}_2\big) \quad \text{such that} \quad F\big|_{\mathcal{U}_1} = f_1.$$

Lemma 2.6. *Suppose that for some* δ *with* $0 < \delta \leqslant \delta_1/2$ *so small that* $\mathcal{V}_\delta(M) \simeq M \times (-\delta, \delta)$ *is a thin tubular neighborhood of the good boundary* $M \subset \mathcal{V}_{\delta_1/2}(\partial\Omega) \subset \mathcal{V}(\partial\Omega)$*, the Hartogs theorem holds for the pair* $(\Omega_M, \mathcal{V}_\delta(M))$:

$$\mathcal{O}\big(\mathcal{V}_\delta(M)\big) = \mathcal{O}\big(\Omega_M \cup \mathcal{V}_\delta(M)\big)\big|_{\mathcal{V}_\delta(M)}.$$

Then the general Hartogs extension property holds:

$$\mathcal{O}\big(\mathcal{V}(\partial\Omega)\big) = \mathcal{O}\big(\Omega \cup \mathcal{V}(\partial\Omega)\big)\big|_{\mathcal{V}(\partial\Omega)}.$$

Proof. Let $f \in \mathcal{O}\big(\mathcal{V}(\partial\Omega)\big)$. By assumption, its restriction to $\mathcal{V}_\delta(M) \subset \mathcal{V}(\partial\Omega)$ enjoys an extension $F_\delta \in \mathcal{O}\big(\Omega_M \cup \mathcal{V}_\delta(M)\big)$. To ascertain that f and F_δ coincide in $\Omega_M \cap \mathcal{V}(\partial\Omega)$, connectedness of $\Omega_M \cap \mathcal{V}(\partial\Omega)$ is welcome.

[4]Because in the sequel, the union $\mathcal{U}_1 \cup \mathcal{U}_2$ would sometimes be a rather long, complicated expression (*see* e.g. (3.9)), hence uneasy to read, we will also say that $\mathcal{O}(\mathcal{U}_1)$ extends holomorphically *and uniquely* to $\mathcal{U}_2$.

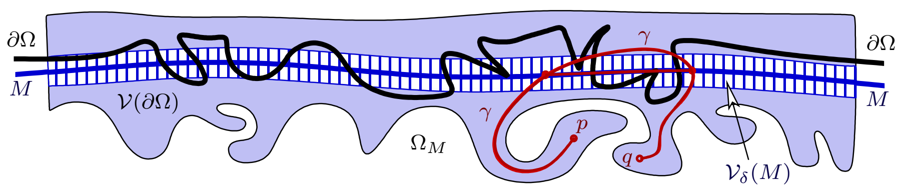


Fig. 3: Checking connectedness of $\Omega_M \cap \mathcal{V}(\partial\Omega)$

Letting $p, q \in \Omega_M \cap \mathcal{V}(\partial\Omega)$, there exists a $\mathcal{C}^\infty$ curve $\gamma : [0,1] \to \mathcal{V}(\partial\Omega)$ connecting p to q. If γ meets M, let p' be the first point on $\gamma \cap M$ and let q' be the last one. We then modify γ, joining p' to q' by means of a curve μ entirely contained in the connected hypersurface M. It suffices to push μ slightly inside Ω_M to get an appropriate curve running from p to q inside $\Omega_M \cap \mathcal{V}(\partial\Omega)$. Thus, $\Omega_M \cap \mathcal{V}(\partial\Omega)$ is connected. It follows, moreover, that the open set

$$\big[\Omega_M \cup \mathcal{V}_\delta(M)\big] \cap \mathcal{V}(\partial\Omega) = \big[\Omega_M \cap \mathcal{V}(\partial\Omega)\big] \bigcup \mathcal{V}_\delta(M)$$

is also connected, so the coincidence $f = F_\delta$, valid in $\mathcal{V}_\delta(M)$, propagates to $\big[\Omega_M \cap \mathcal{V}(\partial\Omega)\big] \cup \mathcal{V}_\delta(M)$. Finally, the function

$$F := \begin{cases} F_\delta & \text{in } \Omega_M \cup \mathcal{V}_\delta(M), \\ f & \text{in } \mathcal{V}(\partial\Omega)\backslash\overline{\Omega}_M, \end{cases}$$

is well defined $\big($since $F_\delta = f$ in $\mathcal{V}_\delta(M)\backslash\overline{\Omega}_M \simeq M \times (0,\delta)\big)$, is holomorphic in

$$\Omega_M \cup \mathcal{V}(\partial\Omega) = \Omega \cup \mathcal{V}(\partial\Omega)$$

and coincides with f in $\mathcal{V}(\partial\Omega)$. □

Thus, we are reduced to establish global holomorphic extension with some good, geometrically controlled data.

Theorem 2.7. *Let $M \Subset \mathbb{C}^n$ $(n \geqslant 2)$ be a* connected *$\mathcal{C}^\infty$ hypersurface bounding a domain $\Omega_M \Subset \mathbb{C}^n$. Suppose to fix ideas that $2 \leqslant \operatorname{dist}\big(0, \overline{\Omega}_M\big) \leqslant 5$ and assume that the restriction $r_M := r|_M$ of the distance function $r(z) = \|z\|$ to M is a Morse function having only a finite number κ of critical points $\widehat{p}_\lambda \in M$, $1 \leqslant \lambda \leqslant \kappa$, located on different sphere levels*:

$$2 \leqslant \widehat{r}_1 := r(\widehat{p}_1) < \cdots < \widehat{r}_\kappa := r(\widehat{p}_\kappa) \leqslant 5 + \operatorname{diam}\big(\overline{\Omega}_M\big).$$

Then there exists $\delta_1 > 0$ such that for every δ with $0 < \delta \leqslant \delta_1$, the (*tubular*) *neighborhood $\mathcal{V}_\delta(M)$ enjoys the global Hartogs extension property into Ω_M*:

$$\mathcal{O}\big(\mathcal{V}_\delta(M)\big) = \mathcal{O}\big(\Omega_M \cup \mathcal{V}_\delta(M)\big)\big|_{\mathcal{V}_\delta(M)},$$

by "pushing" analytic discs inside a finite number of Levi-Hartogs figures (§3.3), *without using neither the Martinelli kernel, nor solutions of an auxiliary $\overline{\partial}$ problem.*

§3. QUANTITATIVE HARTOGS-LEVI EXTENSION BY PUSHING ANALYTIC DISCS

3.1. The classical Hartogs figure. Local Hartogs phenomena can now enter the scene. They involve translating ("pushing") analytic discs and they will provide small, elementary extensional steps to fill in Ω_M.

Given $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \ll 1$ and $a \in \mathbb{N}$ with $1 \leqslant a \leqslant n-1$, we split the coordinates $z \in \mathbb{C}^n$ as $(z_1, \dots, z_a)$ together with $(z_{a+1}, \dots, z_n)$, and we define the $(n-a)$-*concave Hartogs figure* by

$$\mathcal{H}_\varepsilon^{n-a} := \Big\{ \max_{1 \leqslant i \leqslant a} |z_i| < 1, \ \max_{a+1 \leqslant j \leqslant n} |z_j| < \varepsilon \Big\} \bigcup \Big\{ 1 - \varepsilon < \max_{1 \leqslant i \leqslant a} |z_i| < 1, \ \max_{a+1 \leqslant j \leqslant n} |z_j| < 1 \Big\}.$$

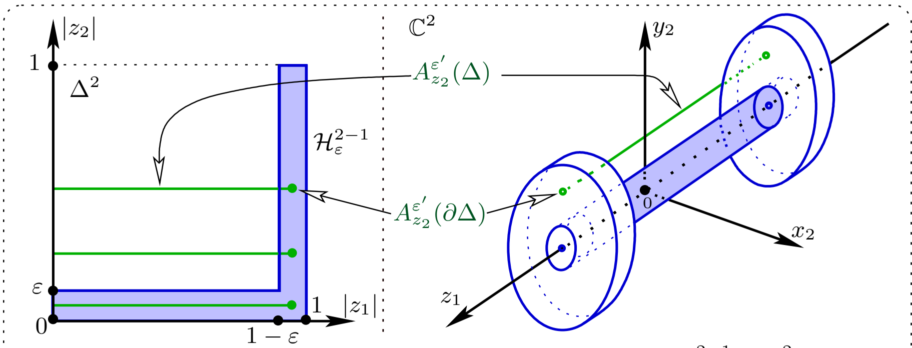


Fig. 4: Two views of the standard Hartogs figure $\mathcal{H}_\varepsilon^{2-1} \subset \mathbb{C}^2$

Lemma 3.2. $\mathcal{O}\big(\mathcal{H}_\varepsilon^{n-a}\big)$ *extends holomorphically to the unit polydisc*

$$\widehat{\mathcal{H}}_\varepsilon^{n-a} := \big\{ z \in \mathbb{C}^n : \max_{1 \leqslant i \leqslant n} |z_i| < 1 \big\} = \Delta^n.$$

Proof. As in the diagram, we consider only $n = 2$, $a = 1$, the general case being similar. Pick an arbitrary $f \in \mathcal{O}\big(\mathcal{H}_\varepsilon^{2-1}\big)$. Letting ε' with $0 < \varepsilon' < \varepsilon$, letting $z_2 \in \mathbb{C}$ with $|z_2| < 1$, the analytic disc

$$\zeta \longmapsto \big([1 - \varepsilon']\, \zeta, z_2\big) =: A_{z_2}^{\varepsilon'}(\zeta),$$

where ζ belongs to the closed unit disc $\overline{\Delta} = \{|\zeta| \leqslant 1\}$, has its boundary $A_{z_2}^{\varepsilon'}(\partial\Delta) = A_{z_2}^{\varepsilon'}\big(\{|\zeta| = 1\}\big)$ contained in $\mathcal{H}_\varepsilon^{2-1}$, the set where f is defined. Lowering dimensions by a unit, we draw discs as (green) segments and boundaries of discs as (green) bold points. Thus, we may compute the Cauchy integral

$$F(z_1, z_2) := \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f\big(A_{z_2}^{\varepsilon'}(\zeta)\big)}{\zeta - z_1}\, d\zeta.$$

Differentiating under the sum, the function F is seen to be holomorphic. In addition, for $|z_2| < \varepsilon$, it coincides with f, because the full closed disc $A_{z_2}^{\varepsilon'}\big(\overline{\Delta}\big)$ is contained in $\mathcal{H}_\varepsilon^{2-1}$ and thanks to Cauchy's formula. Clearly, the $A_{z_2}^{\varepsilon'}(\Delta)$ all together fill in the bidisc Δ^2. One may think that, as z_2 varies, discs are "pushed" gently by a virtual thumb. □

3.3. Levi extension and the Levi-Hartogs figure. Geometrically, the standard Hartogs figure is not best suited to perform holomorphic extension from a strongly (pseudo)concave boundary. For instance, in the proof of Theorem 2.7, we will encounter complements in $\mathbb{C}^n$ of some closed balls whose radius decreases step by step, and more generally spherical shells whose thickness increases interiorly. Thus, we delineate an appropriate set up.

For $r \in \mathbb{R}$ with $r > 1$ and for $\delta \in \mathbb{R}$ with $0 < \delta \ll 1$, the sphere $\mathsf{S}_r^{2n-1} = \{z \in \mathbb{C}^n : \|z\| = r\}$ of radius r is the interior (and strongly concave) boundary component of the spherical shell domain

$$\mathcal{S}_r^{r+\delta} := \{r < \|z\| < r+\delta\} = \bigcup_{p \in \mathsf{S}_r^{2n-1}} \mathbb{B}^n(p,\delta) \cap \{\|z\| > r\}.$$

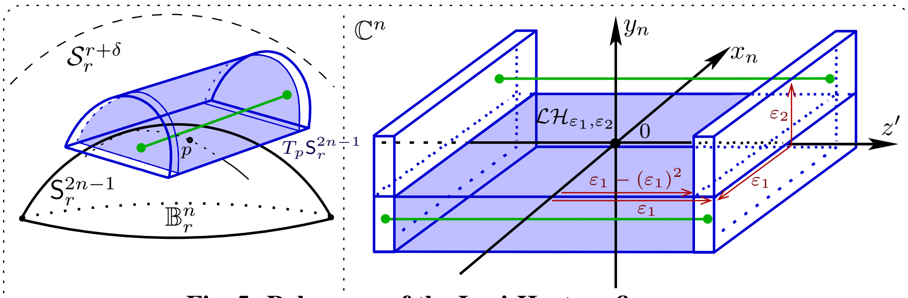


Fig. 5: Relevance of the Levi-Hartogs figure

Near a point $p \in \mathsf{S}_r^{2n-1}$ (left figure), all copies of $\mathbb{C}^{n-1}$ (in green) which are parallel to the complex tangent plane $T_p^c\mathsf{S}_r^{2n-1}$ and which lie above the real plane $T_p\mathsf{S}_r^{2n-1}$ are entirely contained in $\mathbb{C}^n \backslash \overline{\mathbb{B}}_r^n$. To remain inside the shell $\mathcal{S}_r^{r+\delta}$, we could (for instance) restraint our considerations to some half-cylinder of diameter $\approx \delta$, but it will be better to shape a convenient half parallelepiped. Accordingly, for two small $\varepsilon_j > 0$, $j = 1,2$, we introduce a geometrically relevant *Levi-Hartogs figure* (right illustration, reverse orientation):

$$\begin{aligned}\mathcal{LH}_{\varepsilon_1,\varepsilon_2} := &\Big\{ \max_{1 \leqslant i \leqslant n-1} |z_i| < \varepsilon_1, \quad |x_n| < \varepsilon_1, \quad -\varepsilon_2 < y_n < 0 \Big\} \\ &\bigcup \Big\{ \varepsilon_1 - (\varepsilon_1)^2 < \max_{1 \leqslant i \leqslant n-1} |z_i| < \varepsilon_1, \quad |x_n| < \varepsilon_1 \quad |y_n| < \varepsilon_2 \Big\}.\end{aligned}$$

To fill in this (bed-like) figure, we just compute the Cauchy integral on appropriate analytic discs (the (green) horizontal ones) whose boundaries remain in $\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$.

Lemma 3.4. *$\mathcal{O}(\mathcal{LH}_{\varepsilon_1,\varepsilon_2})$ extends holomorphically to the full parallelepiped*

$$\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}} := \Big\{ \max_{1 \leqslant i \leqslant n-1} |z_i| < \varepsilon_1, \quad |x_n| < \varepsilon_1, \quad |y_n| < \varepsilon_2 \Big\}.$$

Next, we must reorient and scale $\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$ in order to put it inside the shell. For every point $p \in \mathsf{S}_r^{2n-1}$, there exists some complex unitarian affine map

$$\Phi_p : \quad z \longmapsto p + Uz,$$

with $U \in \mathsf{SU}(n,\mathbb{C})$, sending the origin $0 \in \overline{\mathcal{LH}}_{\varepsilon_1,\varepsilon_2}$ to p and $T_0\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$ to $T_p\mathsf{S}_r^{2n-1}$, which in addition sends the half-parallelepiped (open) part outside $\overline{\mathbb{B}}_r^n$.

But we have to insure that $\Phi_p\big(\mathcal{LH}_{\varepsilon_1,\varepsilon_2}\big)$ *as a whole* (including the thin walls) lies outside $\overline{\mathbb{B}}_r^n$.

Lemma 3.5. *If $\varepsilon_1 = c\,\delta$ and $\varepsilon_2 = c\,\delta^2$ with some appropriate*[5] *positive constant $c < 1$, then $\Phi_p\big(\mathcal{LH}_{\varepsilon_1,\varepsilon_2}\big)$ is entirely contained in the shell $\mathcal{S}_r^{r+\delta}$. Furthermore, $\Phi_p\big(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}}\big)$ contains a rind of thickness $c\,\frac{\delta^2}{r}$ around some region $\mathsf{R}_p \subset \mathsf{S}_r^{2n-1}$ whose $(2n-1)$-dimensional area equals $\simeq c\,\delta^{2n-1}$.*

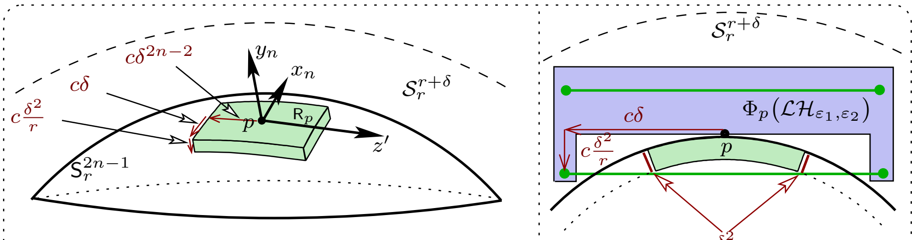


Fig. 6: Size of the piece of (green lemon) rind $c\,\frac{\delta^2}{r}$

By a (radial) *rind of thickness $\eta > 0$ around an open region* $\mathsf{R} \subset \mathsf{S}_r^{2n-1}$, we mean

$$\mathsf{Rind}\big(\mathsf{R},\eta\big) := \big\{(1+s)z : \ z \in \mathsf{R}, \ |s| < \eta/r\big\}.$$

We require that $|s| < \eta/r$ to insure that at every $z \in \mathsf{R}$, the half-line $(0z)^+$ emanating from the origin intersects $\mathsf{Rind}\big(\mathsf{R},\eta\big)$ along a symmetric segment of length $2\,\eta$ centered at z.

In the diagram above, we draw (in green) only the lower part of the small region R_p got in Lemma 3.5. Its shape, when projected onto $T_p\mathsf{S}_r^{2n-1}$, can either be (approximately) a parallelepiped $\big\{|z'| < c\,\delta, \ |x_n| < c\,\delta\big\}$, as in the figure, or say, a ball $\big\{\big(\|z'\|^2 + |x_n|^2\big)^{1/2} < c\,\delta\big\}$; only the scaling constant c changes.

The rigorous proof of the lemma (not developed here) involves elementary reasonings with geometric inequalities and a dry explicit control of the constants that does not matter for the sequel. The main argument uses the fact that S_r^{2n-1} detaches quadratically from $T_p\mathsf{S}_r^{2n-1}$, similarly as the parabola $\big\{y = -\frac{1}{r}\,x^2\big\}$ separates from the line $\{y = 0\}$ in $\mathbb{R}^2_{x,y}$.

Since the area of S_r^{2n-1} equals $\frac{2\,\pi^n}{(n-1)!}\,r^{2n-1} = C\,r^{2n-1}$, by covering S_r^{2n-1} with such adjusted $\mathsf{R}_p \subset \Phi_p\big(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}}\big)$ of area $c\,\delta^{2n-1}$ and by controlling monodromy (*see* rigorous arguments below) we deduce:

Corollary 3.6. *By means of a finite number $\leqslant C\,\big(\frac{r}{\delta}\big)^{2n-1}$ of Levi-Hartogs figures, $\mathcal{O}\big(\mathcal{S}_r^{r+\delta}\big)$ extends holomorphically to the slightly deeper spherical shell $\mathcal{S}^{r+\delta}_{r-c\,\frac{\delta^2}{r}}$.*

This application could seem superfluous, because large analytic discs with boundaries contained in $\mathcal{S}_r^{r+\delta}$ would yield holomorphic extension to the whole ball $\mathbb{B}^n_{r+\delta}$ in one single step. However, in our situation illustrated by Figure 1,

[5]We let the letter c (resp. C) denote a positive constant < 1 (resp. > 1), absolute or depending only on n, which is allowed to vary with the context.

when intersecting S_r^{2n-1} with the neighborhood $\mathcal{V}_\delta(M)$, we shall only get small subregions of S_r^{2n-1}. Hopefully, thanks to our local Levi-Hartogs figures, we may obtain a suitable semi-global extensional statement, valuable for proper subsets of the shell $\mathcal{S}_r^{r+\delta}$ whose shape is arbitrary. The next statement, not available by means of large discs, will be used a great number of times in the sequel.

Proposition 3.7. *Let* $\mathsf{R} \subset \mathsf{S}_r^{2n-1}$ (*with* $r > 1$ *and* $n \geqslant 2$) *be a relatively open set having* $\mathcal{C}^\infty$ *boundary* $\mathsf{N} := \partial\mathsf{R}$ *and let* $\delta > 0$ *with* $0 < \delta \ll 1$. *Then holomorphic functions in the open piece of shell* (*a one-sided neighborhood of* $\mathsf{R} \cup \mathsf{N}$):

$$\begin{aligned}\mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N}) &:= \big(\mathbb{C}^n\backslash\overline{\mathbb{B}}_r^n\big)\cap\mathcal{V}_\delta(\mathsf{R}\cup\mathsf{N})\\ &= \bigcup_{p\in\mathsf{R}\cup\mathsf{N}} \mathbb{B}^n(p,\delta)\cap\{\|z\|>r\}\end{aligned}$$

do extend holomorphically to a rind of thickness $c\,\frac{\delta^2}{r}$ *around* R *by means of a finite number* $\leqslant C\,\frac{\mathsf{area}(\mathsf{R})}{\delta^{2n-1}}$ *of Levi-Hartogs figures.*

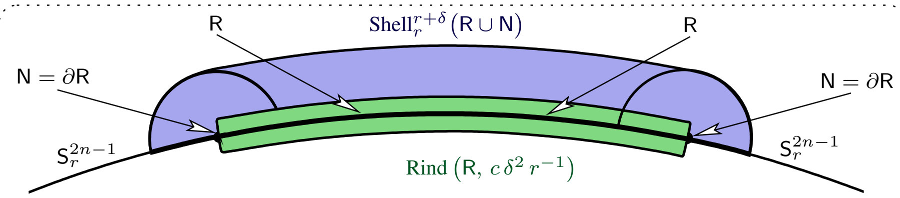


Fig. 7: Semi-global extension from a pseudoconcave piece of shell

Proof. We must control uniqueness of holomorphic extension (monodromy) into rinds covered by successively attached Levi-Hartogs figures. Noticing $c\,\delta^2\,r^{-1} < < \delta$, the considered rinds are much thinner than the piece of shell.

Lemma 3.8. *If* $\mathsf{R}' \subset \mathsf{R}$ *is an arbitrary open subset and if* $\mathsf{R}_{p'} \subset \Phi_{p'}\big(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}}\big)$ *is a small Levi-Hartogs region centered at an arbitrary point* $p' \in \mathsf{R}$, *then the intersection*

$$\mathsf{Rind}\big(\mathsf{R}_{p'},\,c\,\delta^2\,r^{-1}\big)\bigcap\Big(\mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N})\bigcup\mathsf{Rind}\big(\mathsf{R}',\,c\,\delta^2\,r^{-1}\big)\Big) \tag{3.9}$$

is connected.

Admitting the lemma for a while, we pick a finite number $m \leqslant C\,\frac{\mathsf{area}(\mathsf{R})}{\delta^{2n-1}}$ of points $p_1,\dots,p_m \in \mathsf{R}\cup\mathsf{N}$ such that the associated local regions R_{p_k} contained in the filled Levi-Hartogs figures $\Phi_{p_k}\big(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}}\big)$ provided by Lemma 3.5 do cover $\mathsf{R}\cup\mathsf{N}$, namely $\mathsf{R}_{p_1}\cup\cdots\cup\mathsf{R}_{p_m}\supset\mathsf{R}\cup\mathsf{N}$.

Starting with $\mathsf{R}' := \emptyset$ and $p' := p_1$, unique holomorphic extension of $\mathcal{O}\big(\mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N})\big)$ to $\mathsf{Rind}\big(\mathsf{R}_{p'},c\,\delta^2\,r^{-1}\big)$ holds by means of Lemma 3.4, monodromy being assured thanks to the connectedness of the intersection (3.9). Reasoning by induction, fixing some k with $1 \leqslant k \leqslant m-1$, setting $\mathsf{R}' := \cup_{1\leqslant j\leqslant k}\mathsf{R}_{p_j}$, $p' := p_{k+1}$ and assuming that unique holomorphic extension is got from $\mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N})$ into

$$\mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N})\bigcup\mathsf{Rind}\big(\mathsf{R}',c\,\delta^2\,r^{-1}\big) = \mathsf{Shell}_r^{r+\delta}(\mathsf{R}\cup\mathsf{N})\bigcup_{1\leqslant j\leqslant k}\mathsf{Rind}\big(\mathsf{R}_{p_j},c\,\delta^2\,r^{-1}\big),$$

we add the Levi-Hartogs figure $\Phi_{p_{k+1}}\big(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}}\big)$ constructed in Lemma 3.5, and we get unique holomorphic extension to $\mathsf{Rind}\big(\mathsf{R}_{p_{k+1}}, c\,\delta^2\, r^{-1}\big)$, monodromy being assured again thanks to the connectedness of the intersection (3.9). Since $\mathsf{Rind}\big(\mathsf{R}, c\,\delta^2\, r^{-1}\big) \subset \bigcup_{1\leqslant k\leqslant m} \mathsf{Rind}\big(\mathsf{R}_{p_k}, c\,\delta^2\, r^{-1}\big)$, the proposition is proved. □

Proof of Lemma 3.8. To establish connectedness of the open set (3.9), picking two arbitrary points q_0, q_1 in it, we must produce a curve joining q_0 to q_1 inside (3.9). The two radial segments of length $2\,c\,\delta^2\, r^{-1}$ passing through q_0 and q_1 that are centered at two appropriate points of S_r^{2n-1} are by definition both entirely contained in $\mathsf{Rind}\big(\mathsf{R}_{p'}, c\,\delta^2\, r^{-1}\big)$ as well as in $\mathsf{Rind}\big(\mathsf{R}', c\,\delta^2\, r^{-1}\big)$. Thus, moving radially, we may join inside (3.9) q_0 to a new point q_0' and q_1 to a new point q_1', which both belong to the upper half-rind

$$\big\{(1+s)\,z:\ z\in \mathsf{R}_{p'},\ 0<s<c\,\delta^2\, r^{-1}/r\big\}.$$

Since this upper half-rind is connected and contained in $\mathrm{Shell}_r^{r+\delta}\big(\mathsf{R}\cup\mathsf{N}\big)$, we may finally join inside (3.9) the point q_0' to q_1'. □

In the sequel, in order to avoids several gaps and traps, we will put emphasis on rigourously checking univalence of holomorphic extensions.

§4. Filling domains outside balls of decreasing radius

4.1. Global Levi-Hartogs filling from the farthest point. We can now launch the proof of Theorem 2.7. The δ_1 is first chosen so small that $\mathcal{V}_\delta(M)$ is a true tubular neighborhood of M for every δ with $0<\delta\leqslant\delta_1$. Shrinking even more δ_1, in balls of radius δ_1 centered at its points, the hypersurface M is well approximated by its tangent planes.

The farthest point of $\overline{\Omega}_M$ from the origin is unique and it coincides with $\widehat{p}_\kappa$ since by assumption $\widehat{p}_\kappa$ is the single critical point of $r(z)\big|_M$ with $\|\widehat{p}_\kappa\| = \max_{1\leqslant\lambda\leqslant\kappa}\|\widehat{p}_\lambda\|$. By assumption also, the Hessian matrix of $r(z)\big|_M$ is nondegenerate at $\widehat{p}_\kappa$; this also follows automatically from the inclusion $\overline{\Omega}_M\subset\overline{\mathbb{B}}^n_{\widehat{r}_\kappa}$, which constrains strong convexity of M at $\widehat{p}_\kappa$. Consequently, according to the *Morse lemma* ([31], [19], Ch. 6), there exist local coordinates $(\theta_1,\dots,\theta_{2n-1})$ on M centered at $\widehat{p}_\kappa$ such that the intersection $M\cap\mathsf{S}_r^{2n-1}$ is given by the equation

$$-\theta_1^2-\cdots-\theta_{2n-1}^2 = r-\widehat{r}_\kappa,$$

for all r close to $\widehat{r}_\kappa$. Thus $M\cap\mathsf{S}_r^{2n-1}$ is empty for $r>\widehat{r}_\kappa$; it reduces to $\{\widehat{p}_\kappa\}$ for $r=\widehat{r}_\kappa$; and it is diffeomorphic to a $(2n-2)$-sphere for $r<\widehat{r}_\kappa$ close to $\widehat{r}_\kappa$.

Similarly, the nearest point of $\overline{\Omega}_M$ from the origin is unique and it coincides with $\widehat{p}_1$; notice that hence $\kappa\geqslant 2$. Also, the second farthest critical point $\widehat{p}_{\kappa-1}$ lies at a distance $\widehat{r}_{\kappa-1}<\widehat{r}_\kappa$ from 0. If necessary, we shrink δ_1 to insure

$$\delta_1 \ll \min_{1\leqslant\lambda\leqslant\kappa-1}\big\{\widehat{r}_{\lambda+1}-\widehat{r}_\lambda\big\}. \tag{4.2}$$

Next, for every radius r with $\widehat{r}_{\kappa-1}<r<\widehat{r}_\kappa$, we introduce the cut out domain

$$\Omega_{>r} := \Omega_M\cap\big\{\|z\|>r\big\}$$

together with the cut out hypersurface

$$M_{>r} := M \cap \{\|z\| > r\}.$$

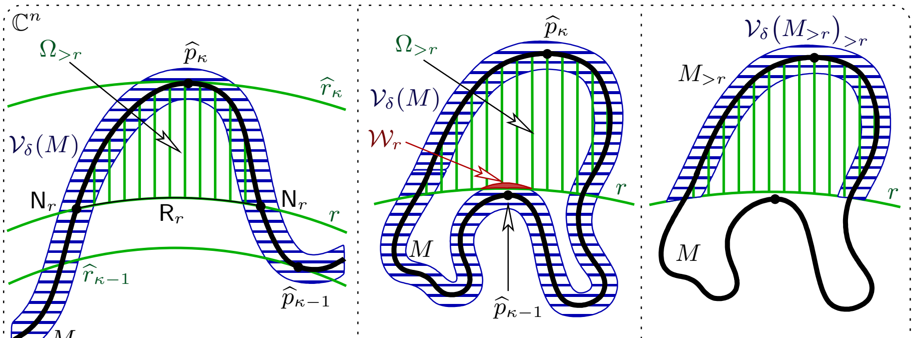


Fig. 8: Filling the domain from the farthest point

Since there are no critical points of $r(z)\big|_M$ in the interval $(\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$, Morse theory shows that $M_{>r}$ is a deformed spherical cap diffeomorphic to $\mathbb{R}^{2n-1}$ for every r with $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$. Also, $\Omega_{>r}$ is then a piece of deformed ball diffeomorphic to $\mathbb{R}^{2n}$.

The boundary in $\mathbb{C}^n$ of $\Omega_{>r}$

$$\partial\Omega_{>r} = M_{>r} \cup \mathsf{R}_r \cup \mathsf{N}_r$$

consists of $M_{>r}$ together with the open subregion $\mathsf{R}_r := \Omega_M \cap \{\|z\| = r\}$ of S_r^{2n-1} which is diffeomorphic to $\mathbb{R}^{2n-1}$ and has boundary $\mathsf{N}_r := M \cap \{\|z\| = r\}$ diffeomorphic to the unit $(2n-2)$-sphere. Thus, the global geometry of $\Omega_{>r}$ is understood.

We can also cut out $\mathcal{V}_\delta(M)$, getting $\mathcal{V}_\delta(M)_{>r}$. The central figure shows that when $r > \widehat{r}_{\kappa-1}$ is very close to $\widehat{r}_{\kappa-1}$, a parasitic connected component $\mathcal{W}_{>r}$ of $\mathcal{V}_\delta(M)_{>r}$ might appear near $\widehat{p}_{\kappa-1}$. After filling $\Omega_{>r}$ progressively by means of Levi-Hartogs figures (*see* below), because $\Omega_{>r} \cap \mathcal{V}_\delta(M)_{>r}$ is *not* connected in such a situation, *no* unique holomorphic extension can be assured, and in fact, multivalence might well occur.

A trick to erase such parasitic components $\mathcal{W}_{>r}$ is to consider instead the open set

$$\mathcal{V}_\delta\big(M_{>r}\big)_{>r} = \mathcal{V}_\delta\big(M_{>r}\big) \cap \{\|z\| > r\},$$

putting a double "$_{>r}$". It is drawn in the right figure and it is always diffeomorphic to $M_{>r} \times (-\delta, \delta)$.

From pieces of shells as in Proposition 3.7 which embrace spheres of varying radius r, holomorphic extension holds to (symmetric) rinds whose thickness $c\,\delta\,r^{-1}$ also varies. To simplify, we introduce the smallest appearing thickness

$$\eta := \min_{\widehat{r}_1 \leqslant r \leqslant \widehat{r}_\kappa} c\,\delta\,r^{-1} = c\,\delta\,\widehat{r}_\kappa^{\,-1}, \tag{4.3}$$

and we observe that it follows trivially from Proposition 3.7 (just by shrinking and by restricting) that holomorphic extension holds to some rind around R of *arbitrary* smaller thickness $\eta' > 0$ with $0 < \eta' \leqslant \eta$. In the sequel, our rinds shall most often have the uniform thickness η, and sometimes also, a smaller one η'. Shrinking the constant c of η in (4.3), we insure $\eta \ll \delta_1$.

Summarizing, we list and we compare the quantities introduced so far:

$$(4.4)\qquad \begin{cases} 0 < \delta \leqslant \delta_1 & \text{neighborhood } \mathcal{V}_\delta(M) \\ 2 \leqslant r(\widehat{p}_1) < \cdots < r(\widehat{p}_\kappa) \leqslant 5 + \operatorname{diam}(\overline{\Omega}_M) & \text{Morse radii} \\ \delta \leqslant \delta_1 \ll \min\limits_{1 \leqslant \lambda \leqslant \kappa - 1} \{\widehat{r}_{\lambda+1} - \widehat{r}_\lambda\} & \text{smallness of } \mathcal{V}_\delta(M) \\ \eta := c\,\delta^2\,\widehat{r}_\kappa^{\,-1} & \text{uniform useful rind thickness} \\ \eta \ll \delta & \text{thickness of extensional rinds is tiny} \end{cases}$$

Proposition 4.5. *For every cutting radius r with $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$ arbitrarily close to $\widehat{r}_{\kappa-1}$, holomorphic functions in the open set*

$$\mathcal{V}_\delta\big(M_{>r}\big)_{>r} = \mathcal{V}_\delta\big(M_{>r}\big) \cap \big\{\|z\| > r\big\}$$

do extend holomorphically and uniquely to $\Omega_{>r}$ by means of a finite number $\leqslant C\left(\frac{\widehat{r}_\kappa}{\delta}\right)^{2n-1}\left[\frac{\widehat{r}_\kappa - r}{\eta}\right]$ of Levi-Hartogs figures.

Proof. We fix such a radius r with $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$. Putting a single Levi-Hartogs figure at $\widehat{p}_\kappa$ as in Proposition 3.7, we get unique holomorphic extension to $\Omega_{>\widehat{r}_\kappa - \eta}$. Since $\eta \ll \delta$, we have $\widehat{r}_\kappa - \eta > \widehat{r}_{\kappa-1}$. If the radius $\widehat{r}_\kappa - \eta$ is already $< r$, we just shrink to $\eta' := \widehat{r}_\kappa - r < \eta$ the thickness of our single rind, getting unique holomorphic extension to $\Omega_{>\widehat{r}_\kappa - \eta'} = \Omega_{>r}$.

Performing induction on an auxiliary integer $k \geqslant 1$, we suppose that, by descending from $\widehat{r}_\kappa$ to a lower radius $r' := \widehat{r}_\kappa - k\eta$ assumed to be still $\geqslant r$, holomorphic functions in $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ extend holomorphically *and uniquely* (remind Definition 2.5) to $\Omega_{>r'}$.

Lemma 4.6. *For every radius r' with $\widehat{r}_{\kappa-1} < r < r' < \widehat{r}_\kappa$,*

$$(4.7)\qquad \mathrm{Shell}_{r'}^{r'+\delta}\big(\mathsf{R}_{r'} \cup \mathsf{N}_{r'}\big) \text{ is contained in } \Omega_{>r'} \bigcup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}.$$

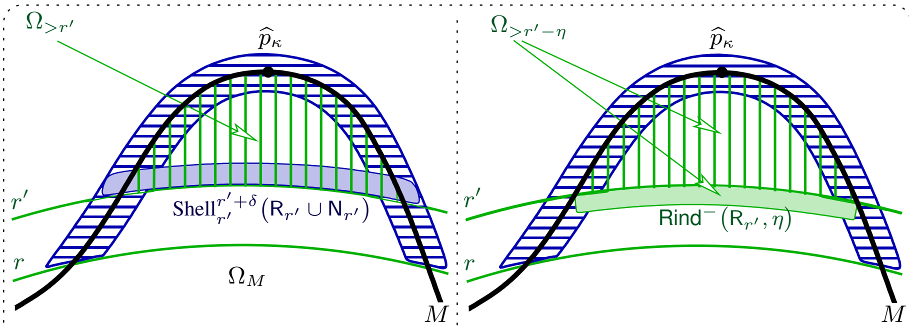


Fig. 9: A shell contained in the cap-shaped domain and the associated rind

Proof. Picking an arbitrary point $p \in \mathsf{R}_{r'} \cup \mathsf{N}_{r'}$, we must verify that

$$\mathbb{B}^n(p, \delta) \cap \{\|z\| > r'\}$$

is contained in the right hand side of (4.7).

If $p \in \mathsf{N}_{r'} \subset M$, whence $p \in M_{>r}$, we get simply what we want:

$$\begin{aligned}\mathbb{B}^n(p,\delta) \cap \{\|z\| > r'\} &\subset \mathcal{V}_\delta\big(M_{>r}\big) \cap \{\|z\| > r'\} \\ &\subset \mathcal{V}_\delta\big(M_{>r}\big) \cap \{\|z\| > r\} \\ &= \mathcal{V}_\delta\big(M_{>r}\big)_{>r}.\end{aligned}$$

If $p \in \mathsf{R}_{r'} \backslash \mathsf{N}_{r'}$, whence $p \in \Omega_M$, reasoning by contradiction, we assume that there exists a point $q \in \mathbb{B}^n(p,\delta) \cap \{\|z\| > r'\}$ in the cut out ball which does not belong to the right hand side of (4.7). Since $\Omega_{>r'} = \Omega_M \cap \{\|z\| > r'\}$, we have $q \notin \Omega_M$.

Reminding $\mathsf{R}_{r'} \subset \mathsf{S}_{r'}^{2n-1}$, the tangent plane $T_p\mathsf{S}_{r'}^{2n-1} = T_p\mathsf{R}_{r'}$ divides $\mathbb{C}^n$ in two *closed* half-spaces, $\overline{T}_p^+ \mathsf{S}_{r'}^{2n-1}$ exterior to $\mathbb{B}_{r'}^n$ and the opposite one $\overline{T}_p^- \mathsf{S}_{r'}^{2n-1}$. We distinguish two (nonexclusive) cases.

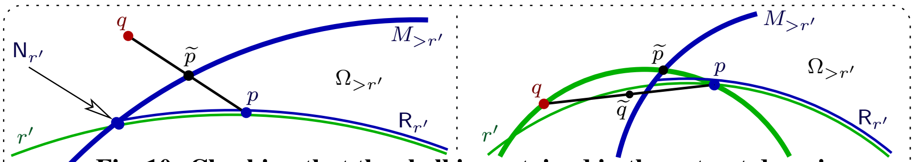


Fig. 10: Checking that the shell is contained in the cut out domain

Firstly, suppose that the half-line $(pq)^+$ is contained in $\overline{T}_p^+ \mathsf{S}_{r'}^{2n-1}$, as in the left figure. Since $p \in \Omega_M$ and $q \notin \Omega_M$, there exists at least one point $\widetilde{p}$ of the open segment (p,q) which belongs to M, hence $\widetilde{p} \in M_{>r}$. Then

$$\|q - \widetilde{p}\| < \|q - p\| < \delta,$$

whence $q \in \mathbb{B}^n(\widetilde{p},\delta) \cap \{\|z\| > r\}$ and we deduce that $q \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ belongs to the right hand side of (4.7), contradiction.

Secondly, suppose that the half-line $(pq)^+$ is contained in $\overline{T}_p^- \mathsf{S}_{r'}^{2n-1}$, as in the right figure. Let $\widetilde{q} \in (p,q)$ be the middle point. In the plane passing through 0, p and q, consider a circle passing through p and q and centered at some point close to 0 in the open segment $(0,\widetilde{q})$. It has radius $< r'$ close to r'. The open arc of circle between p and q is fully contained in $\{\|z\| > r'\}$.

Since $p \in \Omega_M$ and $q \notin \Omega_M$, there exists at least one point $\widetilde{p}$ of the open arc of circle between p and q which belongs to M, hence $\widetilde{p} \in M_{>r}$. But then (p,q) is the hypothenuse of the triangle $pq\widetilde{p}$ (remind $r' > 1$ and $\|q - p\| < \delta \ll 1$), whence $\|q - \widetilde{p}\| < \|q - p\| < \delta$, hence again as in the first case, we deduce that $q \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$, contradiction. □

If the slightly smaller radius

$$r'' := r' - \eta = \widehat{r}_\kappa - (k+1)\eta$$

is already $< r$, we will shrink to $\eta' := \widehat{r}_\kappa - r - k\eta < \eta$ the thickness of the final extensional rind. Otherwise, in the generic case, $\widehat{r}_\kappa - (k+1)\eta$ is still $> r$. The final (exceptional) case being formally similar, we continue the proof with $r' = \widehat{r}_\kappa - k\eta$ and $r'' = r' - \eta$, assuming that $r'' \geqslant r$.

Setting $r' := \widehat{r}_\kappa - k\eta$ in the auxiliary Lemma 4.6, functions holomorphic in $\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ restrict to $\mathsf{Shell}_{r'}^{r'+\delta}\big(\mathsf{R}_{r'} \cup \mathsf{N}_{r'}\big)$ and then, thanks to Proposition 3.7, these restricted functions extend holomorphically to $\mathsf{Rind}\big(\mathsf{R}_{r'}, \eta\big)$.

Lemma 4.8. *The following intersection of two open sets is connected*:

$$\mathsf{Rind}\big(\mathsf{R}_{r'},\, \eta\big) \bigcap \Big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\Big). \tag{4.9}$$

Furthermore, the union of the same two open sets contains

$$\Omega_{>r'-\eta} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}. \tag{4.10}$$

Thus we get unique holomorphic extension to (4.10) and finally, by induction on k and taking account of the final step where η should be shrunk appropriately, we get unique holomorphic extension to $\Omega_{>r} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$.

The number of utilized Levi-Hartogs figures is majorated by the product of the number of needed rinds $\sim \frac{\widehat{r}_\kappa - r}{\eta}$ times the maximal area of $\mathsf{R}_{r'}$, which we roughly majorate by the area $C\,(\widehat{r}_\kappa)^{2n-1}$ of the biggest sphere $\mathsf{S}_{\widehat{r}_\kappa}^{2n-1}$, everything being divided by the area $c\,\delta^{2n-1}$ covered by a small Levi-Hartogs figure. This yields the finite number claimed in Proposition 4.5, achieving its proof. □

Proof of Lemma 4.8. [May be skipped in a first reading] To establish connectedness, we decompose the rind as

$$\begin{aligned}
\mathsf{Rind}^+ &:= \big\{(1+s)z :\ z \in \mathsf{R}_{r'},\ 0 < s < \eta/r'\big\} \\
\mathsf{Rind}^0 &:= \mathsf{R}_{r'}, \\
\mathsf{Rind}^- &:= \big\{(1-s)z :\ z \in \mathsf{R}_{r'},\ 0 < s < \eta/r'\big\},
\end{aligned}$$

so that $\mathsf{Rind} = \mathsf{Rind}^- \cup \mathsf{Rind}^0 \cup \mathsf{Rind}^+$, without writing the common argument $\big(\mathsf{R}_{r'}, \eta\big)$.

Obviously, the upper Rind^+ is diffeomorphic to $\mathsf{R}_{r'} \times (0, \eta) \simeq \mathbb{R}^{2n-1} \times (0, \eta)$, hence is connected. We claim that, moreover, the full Rind^+ is contained in $\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$, whence

$$\mathsf{Rind}^+ = \mathsf{Rind}^+ \bigcap \Big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\Big). \tag{4.11}$$

Indeed, let $q' \in \mathsf{Rind}^+$, hence of the form $q' = (1+s)p'$ for some $p' \in \mathsf{Rind}^0 = \mathsf{R}_{r'}$ and some s with $0 < s < \eta/r'$. If the half-open-closed segment $(p', q']$ is contained in Ω_M, hence in $\Omega_{>r'} = \Omega_M \cap \big\{\|z\| > r'\big\}$, we get for free $q' \in \Omega_{>r'}$.

If on the contrary, $(p', q']$ is *not* contained in Ω_M, then there exists a point $\widetilde{q}' \in (p', q']$ with $\widetilde{q}' \in M = \partial\Omega_M$, whence $\widetilde{q}' \in M_{>r'} \subset M_{>r}$ (remind $r' - \eta \geqslant r$). The ball $\mathbb{B}^n(\widetilde{q}', \delta)$ then contains q', because $\|q' - \widetilde{q}'\| < \|q' - p'\| \leqslant \eta \ll \delta$. This shows $q' \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$, achieving the claim.

Thus, the (upper) subpart (4.11) of the intersection (4.9) is already connected.

To conclude the proof of connectedness, it suffices to show that every point p' of the remaining part

$$\Big(\mathsf{Rind}^0 \cup \mathsf{Rind}^-\Big) \bigcap \Big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\Big) \tag{4.12}$$

can be joined, by means of some appropriate continuous curve running inside the intersection (4.9), to some point q' of the connected upper subpart (4.11). Thus, let p' in (4.12) be arbitrary.

If $p' \in \mathsf{Rind}^0 \cap \big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$, it suffices to join radially p' to $q' = (1+s_\varepsilon)p'$, for some s_ε with $0 < s_\varepsilon \ll \eta$. Indeed, such a q' then belongs to $\mathsf{Rind}^+ \cap \big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$.

If $p' \in \mathsf{Rind}^- \cap \big(\Omega_{>r'} \cup \mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$, then necessarily $p' \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$, because by definition:

$$\mathsf{Rind}^-\big(\mathsf{R}_{r'}, \eta\big) \cap \Omega_{>r'} = \emptyset.$$

So there is a point $q \in M_{>r}$ with $p' \in \mathbb{B}^n(q, \delta)$.

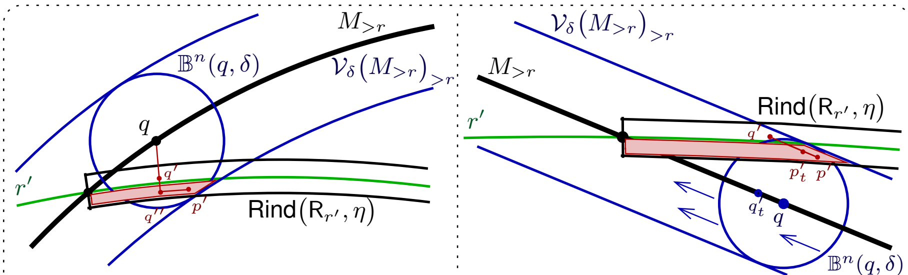


Fig. 11: Joining a point p' of the lower rind to the connected upper rind

We then distinguish two exclusive cases: either $r(q) \geqslant r'$ or $r(q) < r'$.

Firstly, assume $r(q) \geqslant r'$ (left diagram).

If 0, p' and q are aligned, we simply join p' to the point $q' := (1+s_\varepsilon)\,\frac{r'}{r(p')}\,p'$ which belongs to Rind^+. The segment $[p', q']$ is then entirely contained in $\mathsf{Rind} \cap \mathbb{B}^n(q,\delta)_{>r}$, hence in (4.9).

Otherwise, in the unique plane passing through 0, p' and q, consider the point $q'' := \frac{r(p')}{r(q)}\,q$, satisfying $r(q'') = r(p')$ and belonging to $(0, q)$. Since q'' is the orthogonal projection of q onto $\overline{\mathbb{B}^n(0, r(p'))}$, we get $\|q - q''\| < \|q - p'\| < \delta$, whence $q'' \in \mathbb{B}^n(q, \delta)$. The circle of radius $r(p')$ centered at 0 joins p' to q'' by means of a small arc which is entirely contained in $\mathbb{B}^n(q,\delta)$. Denote by $\gamma : [0,1] \to \mathbb{B}^n(q,\delta)$ a parametrization of this arc of circle, with $\gamma(0) = p'$ and $\gamma(1) = q''$.

If $\gamma[0,1]$ is entirely contained in Rind^-, we conclude by joining q'' radially to the point $q' := (1+s_\varepsilon)\,\frac{r'}{r(q'')}\,q''$.

If $\gamma[0,1]$ is not contained in Rind, let $t_1 \in (0,1)$ satisfying $\gamma[0,t_1) \subset \mathsf{Rind}^-$ but $\gamma(t_1) \not\in \mathsf{Rind}^-$. Then $\gamma(t_1)$ belongs to $\partial\mathsf{Rind}^-$ and since $r(\gamma(t_1)) = r(p')$ still satisfies $r' - \eta < r(p') < r'$, necessarily $\gamma(t_1)$ belongs "vertical part" of $\partial\mathsf{Rind}^-$, namely to the strip $\big\{(1-s)z : \ z \in \mathsf{N}_{r'},\ 0 \leqslant s \leqslant \eta/r'\big\}$. Hence the point $q''' := \frac{r'}{r(\gamma(t_1))}\,\gamma(t_1)$ belongs to $\mathsf{N}_{r'}$. We now modify γ by constructing a curve which remains entirely inside $\mathbb{B}^n(q''',\delta)_{>r} \subset \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ as follows: choose $t_2 < t_1$ very close to t_1, join p' to $\gamma(t_2) \in \mathsf{Rind}^-$ through γ and then $\gamma(t_2)$ radially to the point $q' := (1+s_\varepsilon)\,\frac{r'}{r(\gamma(t_2))}\,\gamma(t_2) \in \mathsf{Rind}^+$. The resulting curve is entirely contained in (4.9). In conclusion, we have joined p' to a suitable point q', as announced.

Secondly, assume that $r(q) < r'$. Consider the normalized gradient vector field $\frac{\nabla r_M}{\|\nabla r_M\|}$, defined and nowhere singular on $M \cap \big\{\widehat{r}_{\kappa-1} < \|z\| < \widehat{r}_\kappa\big\}$, hence on $M_{>r}\big\backslash\{\widehat{p}_\kappa\}$. For $t \in [0, 2\,\eta]$, denote by $t \mapsto q_t$ the integral curve of $\frac{\nabla r_M}{\|\nabla r_M\|}$ passing through q, satisfying $q_0 = q$, $q_t \in M$ and $r(q_t) = r(q) + t$. Together with its center q, the ball is translated as $\mathbb{B}^n(q_t, \delta)$. Accordingly, the point p' is moved, yielding a curve p'_t such that p'_t occupies a fixed position with respect to the moving ball. Explicitly: $p'_t = p' + q'_t - q$. Thanks to $r' > 1$ and $\delta \ll 1$, one may check[6] that $\frac{dr(p'_t)}{dt} \geqslant 1 - c_{r',\delta}$, for some small positive constant $c_{r',\delta}$ <1.

Thus, as t increases, the point p'_t moves away from 0 at speed almost equal to 1. Since $r' - \eta < r(p'_0) < r'$, we deduce that for $t = 2\,\eta$, we have $r(p'_{2\eta}) > r'$, namely $p'_{2\eta}$ has escaped from Rind^-. Consequently, there exists $t_1 \in (0, 2\eta)$ with $p'_t \in \mathsf{Rind}^-$ for $0 \leqslant t < t_1$ such that $p'_{t_1} \in \partial\mathsf{Rind}^-$.

The boundary of Rind^- has three parts: the top $\mathsf{R}_{r'}$, the bottom $\big\{(1-\eta/r')z : z \in \mathsf{R}_{r'}\big\}$ and the (closed) strip $\big\{(1-s)z :\, z \in \mathsf{N}_{r'},\, 0 \leqslant s \leqslant \eta/r'\big\}$. The limit point p'_{t_1} cannot belong to the bottom, since $r(p'_{t_1}) > r(p'_0) > r' - \eta$.

Since by construction $p'_t \in \mathbb{B}^n(q_t, \delta)$ with $q_t \in M_{>r}$, we observe that $p'_t \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ for every $t \in [0, 2\,\eta]$. Consequently:

$$p'_t \in \mathsf{Rind}^- \bigcap \mathcal{V}_\delta\big(M_{>r}\big)_{>r}, \qquad \forall\, t \in [0, t_1).$$

Assuming that $p'_{t_1} \in \partial\mathsf{Rind}^-$ belongs to the top $\mathsf{R}_{r'} = \mathsf{Rind}^0$, we may join p'_{t_1} radially to $q' := (1 + s_\varepsilon)p'_{t_1}$. In this way, p' is joined, by means of a continuous curve running in the intersection (4.9), to the point $q' = (1 + s_\varepsilon)p'_{t_1}$ belonging to the connected upper subpart (4.11).

Finally, assume that $p'_{t_1} \in \partial\mathsf{Rind}^-$ belongs to the strip $\big\{(1-s)z : z \in \mathsf{N}_{r'},\, 0 \leqslant s \leqslant \eta/r'\big\}$. The point $q'' := \frac{r'}{r(p'_{t_1})}\, p'_{t_1}$ belongs to $\mathsf{N}_{r'} \subset M_{>r}$, and we will construct a small curve running entirely inside $\mathbb{B}^n(q'', \delta)_{>r} \subset \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$. Choose $t_2 \in (0, t_1)$ very close to t_1, join p' to $p'_{t_2} \in \mathsf{Rind}^-$ as above (but do not go up to p'_{t_1}) and then join p'_{t_2} radially to the point $q' := (1 + s_\varepsilon)\frac{r'}{r(p'_{t_2})}\, p'_{t_2}$, which belongs to Rind^+. The small radial segment from p'_{t_2} to q' is entirely contained in $\mathbb{B}^n(q'', \delta)$ and in the full Rind. In conclusion, p' is joined, by means of a continuous curve running in the intersection (4.9), to this point $q' = (1 + s_\varepsilon)\frac{r'}{r(p'_{t_2})}\, p'_{t_2}$ which belongs to the connected upper subpart (4.11).

The proof of the connectedness of the intersection (4.9) is complete.

We now show that the union, instead of the intersection in (4.9), contains (4.10).

[6] If the spheres S^{2n-1}_r for r close to r' would be hyperplanes — they almost are in comparison to $\mathbb{B}^n(q_t, \delta)$ — we would have exactly $r(p'_t) = r(p') + t$, whence $\frac{dr(p'_t)}{dt} = 1$.

Let $p' \in \Omega_{>r'-\eta} \backslash \Omega_{>r'}$, whence $r' - \eta < \|p'\| \leqslant r'$. The radial half line $\big\{t\, p' \,:\, 0 < t < \infty\big\}$ emanating from the origin and passing through p' meets $\mathsf{S}_{r'}^{2n-1}$ at the point $q' = \frac{r'}{\|p'\|}\, p'$.

If the closed segment $[p', q']$ is contained in $\Omega_{>r'-\eta}$, then $q' \in \Omega_M$. Since $\|q'\| = r'$ and since $\mathsf{R}_{r'} = \Omega_M \cap \big\{\|z\| = r'\big\}$, we get $q' \in \mathsf{R}_{r'}$, whence $p' \in \mathsf{Rind}\big(\mathsf{R}_{r'}, \eta\big)$.

If on the contrary, the closed segment $[p', q']$ is *not* contained in $\Omega_{>r'-\eta}$, then there exists $\widetilde{q}' \in (p', q']$ with $\widetilde{q}' \in M = \partial\Omega_M$, whence $\widetilde{q}' \in M_{>r'-\eta} \subset M_{>r}$. Since $\eta \ll \delta$, we deduce $p' \in \mathbb{B}^n(\widetilde{q}', \delta)$ and finally $p' \in \mathcal{V}_\delta\big(M_{>r}\big)_{>r}$.

The proofs of Lemma 4.8 and hence also of Proposition 4.5 are complete. □

§5. CREATING DOMAINS, MERGING AND SUPPRESSING CONNECTED COMPONENTS

5.1. Topological stability and global extensional geometry between regular values of r_M. In the preceding Section 4, for r with $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$, we described the simple shape of the cut out domain $\Omega_{>r} = \Omega_M \cap \{\|z\| > r\}$, just diffeomorphic to a piece of ball. Decreasing the radius under $\widehat{r}_{\kappa-1}$, the topological picture becomes more complex. At least for radii comprised between two singular values of $r(z)\big|_M$, Morse theory assures geometrical control together with constancy properties.

Lemma 5.2. *Fix a radius r satisfying $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$ for some λ with $1 \leqslant \lambda \leqslant \kappa - 1$, hence noncritical for the distance function $r(z)|_M$. Then*:

- **(a)** *$T_zM + T_z\mathsf{S}_r^{2n-1} = T_z\mathbb{C}^n$ at every point $z \in M \cap \mathsf{S}_r^{2n-1}$*;
- **(b)** *the intersection $M \cap \mathsf{S}_r^{2n-1}$ is a $\mathcal{C}^\infty$ compact hypersurface $\mathsf{N}_r \subset \mathsf{S}_r^{2n-1}$ of codimension 2 in $\mathbb{C}^n$, without boundary and having finitely many connected components*;
- **(c)** *$\mathsf{N}_{r''}$ is diffeomorphic to $\mathsf{N}_{r'}$, whenever $\widehat{r}_\lambda < r'' < r' < \widehat{r}_{\lambda+1}$*;
- **(d)** *$M_{>r} = M \cap \{\|z\| > r\}$ has finitely many connected components $M_{>r}^c$, with $1 \leqslant c \leqslant c_\lambda$, for some $c_\lambda < \infty$ which is independent of r*;
- **(e)** *$M_{>r''}^c$ is diffeomorphic to $M_{>r'}^c$, whenever $\widehat{r}_\lambda < r'' < r' < \widehat{r}_{\lambda+1}$, for all c with $1 \leqslant c \leqslant c_\lambda$*;
- **(f)** *$M \cap \{r'' < \|z\| < r'\}$ is diffeomorphic to $\mathsf{N}_{r'} \times (r'', r')$, hence also to $\mathsf{N}_{r''} \times (r'', r')$, whenever $\widehat{r}_\lambda < r'' < r' < \widehat{r}_{\lambda+1}$*;

Proof. We summarize the known arguments of proof (*cf.* [31] and [19], Ch. 6). Equivalently, **(a)** says that $dr : T_zM \to T_{r(z)}\mathbb{R}$ is onto, and this holds true since by assumption $M \cap \big\{\widehat{r}_\lambda < \|z\| < \widehat{r}_{\lambda+1}\big\}$ contains no critical points of $r(z)|_M$. Then **(b)** follows from this transversality **(a)**.

Next, consider the Euclidean metric $(v, w) := \sum_{k=1}^{2n} v_k\, w_k$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, which induces a Riemannian metric $(\cdot,\cdot)_M$ on M, a nondegenerate positive bilinear form on TM. The gradient $\nabla(r|_M)$ of $r(z)|_M$ is the vector field on M defined by requiring that $\big(\nabla(r|_M), X\big)_M = d\big(r|_M\big)(X)$ for all $\mathcal{C}^\infty$ (locally defined) vector fields X on M. Let $\mathsf{D} := 2\,\mathrm{Re}\,\sum_{k=1}^n z_k\, \frac{\partial}{\partial z_k}$ be the radial vector field which is obviously orthogonal to spheres and consider the orthogonal projection X_{D} of $\mathsf{D}|_M$ on TM, a $\mathcal{C}^\infty$ vector field on M. We want to scale the gradient as $\mathsf{V}_{r,M} := \lambda \cdot \nabla(r|_M)$ so that its radial component is identically equal to one, namely, so that $\big(\mathsf{V}_{r,M}, \mathsf{D}\big) \equiv 1$, which gives the equation:

$$1 = \lambda\,\big(\nabla(r|_M),\, \mathsf{D}\big) = \lambda\,\big(\nabla(r|_M),\, X_{\mathsf{D}}\big) = \lambda\,\big(\nabla(r|_M),\, X_{\mathsf{D}}\big)_M = \lambda\, d\big(r|_M\big)(X_{\mathsf{D}}).$$

To simply set $\lambda := \frac{1}{d(r|_M)(X_{\mathsf{D}})}$, we must establish that X_{D} cannot belong to $\operatorname{Ker} d\big(r|_M\big)$ at any point $z \in M \cap \big\{\widehat{r}_\lambda < \|z\| < \widehat{r}_{\lambda+1}\big\}$ of a noncritical shell.

We check this. At such a point z, $\mathsf{D}(z)$ is not orthogonal to T_zM (otherwise T_zM would coincide with $T_z\mathsf{S}^{2n-1}_{\|z\|}$), whence its orthogonal projection $X_{\mathsf{D}}(z)$ is $\neq 0$. By definition, $(\mathsf{D} - X_{\mathsf{D}})(z)$ is orthogonal to $T_zM \ni X_{\mathsf{D}}(z)$, hence it is orthogonal to $X_{\mathsf{D}}(z)$ inside the 2-dimensional plane Π_z generated by $X_{\mathsf{D}}(z) \neq 0$ and by $\mathsf{D}(z) \neq 0$. If, contrary to what we want, $X_{\mathsf{D}}(z)$ would belong to $\operatorname{Ker} d\big(r|_M\big) = T_z\mathsf{S}^{2n-1}_{\|z\|}$, then it would be orthogonal to $\mathsf{D}(z)$, and in the plane Π_z, we would have both $\mathsf{D}(z)$ and the hypothenuse $\big(\mathsf{D} - X_{\mathsf{D}}\big)(z)$ orthogonal to $X_{\mathsf{D}}(z)$, which is impossible.

Thus, in spherical coordinates $(r, \vartheta_1, \dots, \vartheta_{2n-1})$ restricted to a noncritical shell, the r-component of the $\mathcal{C}^\infty$ scaled gradient vector field $\mathsf{V}_{r,M} := \frac{\nabla(r|_M)}{(\nabla(r|_M),\, \mathsf{D})}$ is $\equiv 1$. We deduce that the flow (wherever defined) $z_s := \exp\!\big(s\,\mathsf{V}_{r,M}\big)(z)$ simply increases the norm as $\|z_s\| = \|z\| + s$, whence $\exp\big((r' - r'')\mathsf{V}_{r,M}\big)(\cdot)$ induces a diffeomorphism from $\mathsf{N}_{r''}$ onto $\mathsf{N}_{r'}$: this yields **(c)**. Also, $(z'', s) \longmapsto \exp\big((r'' + s)\mathsf{V}_{r,M}\big)(z'')$ gives the diffeomorphism of $\mathsf{N}_{r''} \times (r' - r'')$ onto the strip $M \cap \big\{r'' < \|z\| < r'\big\}$, which is **(f)**.

Next, the compact manifold with boundary $M_{>r} \cup \mathsf{N}_r$ surely has finitely many connected components, whose number is constant for all $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$, because when r increases or decreases, the connected components of the slices N_r do slide smoothly in S^{2n-1}_r without encountering each other: this is **(d)**. Finally, **(e)** follows from **(f)** and the trivial fact that the two segments (r'', r^0) and (r', r^0) are diffeomorphic, whenever $\widehat{r}_\lambda < r'' < r' < r^0 < \widehat{r}_{\lambda+1}$. □

We can now state the very main technical proposition of this paper.

Proposition 5.3. *Fix a radius r satisfying $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$ for some λ with $1 \leqslant \lambda \leqslant \kappa - 1$ and let $M^c_{>r}$, $c = 1, \dots, c_\lambda$, denote the collection of connected components of $M \cap \{\|z\| > r\}$. Then*:

- **(i)** *each $M^c_{>r}$ bounds in $\{\|z\| > r\}$ a* unique *domain $\widetilde{\Omega}^c_{>r}$ which is relatively compact in $\mathbb{C}^n$;*
- **(ii)** *the boundary in $\mathbb{C}^n$ of each $\widetilde{\Omega}^c_{>r}$, namely*:

$$\partial\widetilde{\Omega}^c_{>r} = M^c_{>r} \cup \mathsf{N}^c_r \cup \widetilde{\mathsf{R}}^c_r$$

consists of $M^c_{>r}$ together with some appropriate union N^c_r of finitely many connected components of $\mathsf{N}_r = M \cap \{\|z\| = r\}$ and with an appropriate region $\widetilde{\mathsf{R}}^c_r \subset \mathsf{S}^{2n-1}_r$ delimited by N^c_r;

(iii) *two such domains $\widetilde{\Omega}_{>r}^{c_1}$ and $\widetilde{\Omega}_{>r}^{c_2}$, associated to two different connected components $M_{>r}^{c_1}$ and $M_{>r}^{c_2}$ of $M_{>r}$, are either disjoint or one is contained in the other*;

(iv) *for $c_1 \neq c_2$, the regions $\widetilde{\mathsf{R}}_r^{c_1}$ and $\widetilde{\mathsf{R}}_r^{c_2}$ are either disjoint or one is contained in the other, while their boundaries $\mathsf{N}_r^{c_1}$ and $\mathsf{N}_r^{c_2}$ are always disjoint*;

(v) *for each $c = 1, \dots, c_\lambda$, every function f holomorphic in $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ has a restriction to $\mathcal{V}_\delta\big(M_{>r}^c\big)_{>r}$ which extends holomorphically and uniquely to $\widetilde{\Omega}_{>r}^c$ by means of a finite number of Levi-Hartogs figures.*

We point out that in **(i)** and **(ii)**, neither $\widetilde{\Omega}_r^c$ nor $\widetilde{\mathsf{R}}_r^c$ need be contained in our original domain Ω_M (as it was the case in Section 4 for $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$): this is why we introduced a widetilde notation. We refer to the middle Figure 1 for an illustration. Similarly, neither $\widetilde{\Omega}_r^c$ nor $\widetilde{\mathsf{R}}_r^c$ need be contained in $\mathbb{C}^n \backslash \overline{\Omega}_M$: they both may intersect Ω_M *and* $\mathbb{C}^n \backslash \overline{\Omega}_M$. Also, the number of connected components of N_r^c is $\geqslant$ that of $\widetilde{\mathsf{R}}_r^c$ and may be $>$, as illustrated below.

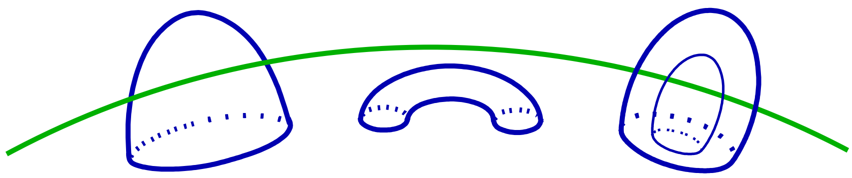


Fig. 12: Possible topologies of the cut out hypersurfaces $M_{>r}$

As a direct application, we may achieve the proof of our principal result.

Theorem 5.4. *Under the precise assumptions of Theorem 2.7, holomorphic functions in $\mathcal{V}_\delta(M)$ do extend holomorphically and uniquely to Ω_M by means of a finite number of Levi-Hartogs figures*:

$$\forall\, f \in \mathcal{O}\big(\mathcal{V}_\delta(M)\big) \quad \exists\, F \in \mathcal{O}\big(\Omega_M \cup \mathcal{V}_\delta(M)\big) \ \text{ s.t. } F\big|_{\mathcal{V}_\delta(M)} = f.$$

Proof. In the main Proposition 5.3, we choose $r = \widehat{r}_1 + \varepsilon$ (where $\varepsilon > 0$ satisfies $\varepsilon \ll \delta$) very close to the last, smallest singular radius. Then $M_{>r}$ has a single connected component, $M_{>r}$ itself, and it simply bounds $\big(\Omega_M\big)_{>r}$. The remainder part of M, namely $M \cap \big\{\|z\| \leqslant \widehat{r}_1 + \varepsilon\big\}$ is diffeomorphic to a very small closed $(2n-1)$-dimensional spherical cap and is entirely contained in $\mathcal{V}_\delta(M)$.

Fix an arbitrary function $f \in \mathcal{O}\big(\mathcal{V}_\delta(M)\big)$ and restrict it to $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$. Thanks to the proposition, f extend holomorphically and uniquely to $\big(\Omega_M\big)_{>r}$ by means of a finite number of Levi-Hartogs figures. Since

$$\mathcal{V}_\delta(M) \bigcap \Big(\mathcal{V}_\delta\big(M_{>r}\big)_{>r} \cup \big(\Omega_M\big)_{>r}\Big)$$

is easily seen to be connected, we get a globally defined extended function which is holomorphic in

$$\mathcal{V}_\delta(M) \bigcup \Big(\mathcal{V}_\delta\big(M_{>r}\big)_{>r} \cup \big(\Omega_M\big)_{>r}\Big) = \mathcal{V}_\delta(M) \cup \Omega_M.$$

This completes the proof. □

Proof of Proposition 5.3. In **(i)**, let us check the uniqueness of a relatively compact $\widetilde{\Omega}_{>r}^c$. Since $M_{>r}^c$ inherits an orientation from M, the complement $\big\{ \|z\| > r \big\} \backslash M_{>r}^c$ has at most 2 connected components. As $M \Subset \mathbb{C}^n$ is bounded, at least one component contains the points at infinity, hence there can remain at most one component of $\big\{ \|z\| > r \big\} \backslash M_{>r}^c$ that is relatively compact in $\mathbb{C}^n$.

If r satisfies $\widehat{r}_{\kappa-1} < r < \widehat{r}_\kappa$, Proposition 4.5 already completes the proof.

Assume therefore that r satisfies $\widehat{r}_\mu < r < \widehat{r}_{\mu+1}$, for some $\mu \in \mathbb{N}$ with $1 \leqslant \mu \leqslant \kappa - 1$. For every λ with $2 \leqslant \lambda \leqslant \kappa - 1$, it will be convenient to flank each singular radius $\widehat{r}_\lambda$ by the following two very close nonsingular radii

$$\boxed{\widehat{r_\lambda^-} := \widehat{r}_\lambda - \eta/2} \qquad \text{and} \qquad \boxed{\widehat{r_\lambda^+} := \widehat{r}_\lambda + \eta/2}, \tag{5.5}$$

with η being the same uniform thickness of extensional rinds as before. We fix once for all an arbitrary function f holomorphic in $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$. Letting λ be arbitrary with $\mu \leqslant \lambda \leqslant \kappa - 1$, the logic of the proof shows up two topologically distinct phenomena that we overview.

A: Filling domains through regular radii intervals. Assume that at the regular radius $\widehat{r_{\lambda+1}^-} = \widehat{r}_{\lambda+1} - \frac{\eta}{2}$, all domains $\widetilde{\Omega}^c_{>\widehat{r_{\lambda+1}^-}}$, $c = 1, \dots, c_\lambda$, as well as the corresponding holomorphic extensions, have been constructed. Then prolong the domains (without topological change) as $\widetilde{\Omega}^c_{>\widehat{r_\lambda^+}}$, $c = 1, \dots, c_\lambda$, up to $\widehat{r_\lambda^+} = \widehat{r}_\lambda + \frac{\eta}{2}$ and fill in the conquered territory by means of a finite number of Levi-Hartogs figures.

B: Jumping across singular radii and changing the domains. Restarting at $\widehat{r_\lambda^+}$ with the domains $\widetilde{\Omega}^c_{>\widehat{r_\lambda^+}}$, $c = 1, \dots, c_\lambda$, distinguish three cases as follows. Remind from §2.3 that M is represented by $v = \sum_{1 \leqslant j \leqslant k_\lambda} x_j^2 - \sum_{1 \leqslant j \leqslant 2n-k_\lambda-1} y_j^2$ in suitable coordinates (x, y, v) centered at $\widehat{p}_\lambda$, where k_λ is the *Morse coindex* of $r(z)|_M$ at $\widehat{p}_\lambda$.

- **(I)** Firstly, assume $k_\lambda = 0$, namely $z \mapsto r(z)|_M$ has a local maximum at $\widehat{p}_\lambda$, or inversely, assume $k_\lambda = 2n - 1$, namely $z \mapsto r(z)|_M$ has a local minimum at $\widehat{p}_\lambda$. This is the easiest case, the only one in which new domains can be born or die, locally.
- **(II)** Secondly, assume $k_\lambda = 1$. This is the most delicate case, because in a small neighborhood of $\widehat{p}_\lambda$, the cut out hypersurface $M_{>\widehat{r_\lambda^+}}$ has exactly 2 connected components, so that two different enclosed domains $\widetilde{\Omega}^{c_1}_{>\widehat{r_\lambda^+}}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r_\lambda^+}}$ can meet here; it may also occur that the two parts near $\widehat{p}_\lambda$ belong to the *same* domain, i.e. that $c_2 = c_1$. While descending down to $\widehat{r_\lambda^-}$, we must analyze the way how the two (maybe the single) component(s) merge. Three subcases will be distinguished, one of which showing a crucial trick of *subtracting* one growing component from a larger one which also grows (right Figure 1).

(III) Thirdly, assume that $2 \leqslant k_\lambda \leqslant 2n-2$. In all these cases, locally in a neighborhood of $\widehat{p}_\lambda$, the cut out hypersurface $M_{>\widehat{r}_\lambda^+}$ has exactly 1 connected component and the way how the corresponding single enclosed domain $\widetilde{\Omega}^c_{>\widehat{r}_\lambda^+}$ grows will be topologically constant.

Reasoning by induction on λ and applying the filling processes A and B, we then descend progressively inside deeper spherical shells, checking all properties of Proposition 5.3. When approaching the bottom radius r of Proposition 5.3, it will suffice to shortcut A or B appropriately in order to complete the proof.

5.6. Filling domains through regular radii intervals. Recall that $\widehat{r}_\mu < r < \widehat{r}_{\mu+1}$, let λ with $\mu \leqslant \lambda \leqslant \kappa - 1$ and consider the regular radius interval $\big[\widehat{r}_\lambda^+, \widehat{r}_{\lambda+1}^-\big]$. We suppose first that $r \leqslant \widehat{r}_\lambda^+$, so that we may descend inside the whole spherical shell $\big\{\widehat{r}_\lambda^+ < \|z\| \leqslant \widehat{r}_{\lambda+1}^-\big\}$. Afterwards, we explain how we stop in the case where $\lambda = \mu$ and $\widehat{r}_\mu^+ < r < \widehat{r}_{\mu+1}^-$.

By descending induction on λ through A and B, we may assume that at the superlevel set $(\cdot)_{>\widehat{r}_{\lambda+1}^-}$, the domains $\widetilde{\Omega}^c_{>\widehat{r}_{\lambda+1}^-}$ enclosed by $M^c_{>\widehat{r}_{\lambda+1}^-}$ for $1 \leqslant c \leqslant c_\lambda$ have been constructed and that each restriction $f^c_{\widehat{r}_{\lambda+1}^-}$ of $f \in \mathcal{O}\big(\mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$ to $\mathcal{V}_\delta\big(M^c_{>\widehat{r}_{\lambda+1}^-}\big)_{>\widehat{r}_{\lambda+1}^-}$ extends holomorphically and uniquely to the domain

$$\widetilde{\Omega}^c_{>\widehat{r}_{\lambda+1}^-} \bigcup \mathcal{V}_\delta\big(M^c_{>\widehat{r}_{\lambda+1}^-}\big)_{>\widehat{r}_{\lambda+1}^-}. \tag{5.7}$$

For every radius r' with $\widehat{r}_\lambda^+ \leqslant r' < \widehat{r}_{\lambda+1}^-$, the cut out hypersurface $M_{>r'} = \bigcup_{1 \leqslant c \leqslant c_\lambda} M^c_{>r'}$ has the same number of connected components, each $M^c_{>r'}$ is diffeomorphic to $M^c_{>\widehat{r}_{\lambda+1}^-}$ and the difference $M^c_{>r'} \backslash M^c_{>\widehat{r}_{\lambda+1}^-}$ is diffeomorphic to $N^c_{\widehat{r}_{\lambda+1}^-} \times \big(r', \widehat{r}_{\lambda+1}^-\big]$. Furthermore, each prolongation $\widetilde{\Omega}^c_{>r'}$ of $\widetilde{\Omega}^c_{>\widehat{r}_{\lambda+1}^-}$ is obviously defined just by adding the tube domain surrounded by $M^c_{>r'} \backslash M^c_{\widehat{r}_{\lambda+1}^-}$. Then each $N^c_{r'} = \partial \widetilde{\mathsf{R}}^c_{r'}$ has finitely many connected components $N^{c,j}_{r'}$, with $1 \leqslant j \leqslant j_{\lambda,c}$, where $j_{\lambda,c}$ is independent of r'.

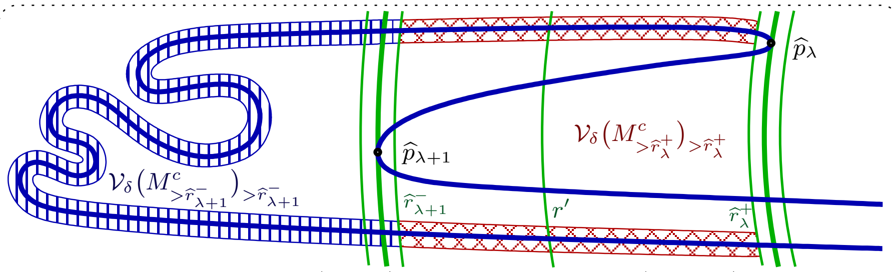


Fig. 13: Getting $\mathcal{V}_\delta\big(M^c_{>\widehat{r}_\lambda^+}\big)_{\widehat{r}_\lambda^+}$ by adding legs to $\mathcal{V}_\delta\big(M^c_{>\widehat{r}_{\lambda+1}^-}\big)_{\widehat{r}_{\lambda+1}^-}$

Since f was defined in $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$ and since $r \leqslant \widehat{r}_\lambda^+$, we claim that each restriction $f^c_{\widehat{r}_{\lambda+1}^-}$ may be extended holomorphically and uniquely to

$$\widetilde{\Omega}^c_{>\widehat{r}_{\lambda+1}^-} \bigcup \mathcal{V}_\delta\big(M^c_{>\widehat{r}_\lambda^+}\big)_{>\widehat{r}_\lambda^+}. \tag{5.8}$$

Indeed, to the original domain of definition (5.7) of $f^c_{\widehat{r}^-_{\lambda+1}}$ which was contained in $\big\{\|z\| > \widehat{r}^-_{\lambda+1}\big\}$, we add in the enlarged domain (5.8) a finite number $j_{\lambda,c}$ of tubular domains around the connected components of $M^c_{>r'}\big\backslash M^c_{>\widehat{r}^-_{\lambda+1}}$. Because δ was chosen so small that $\mathcal{V}_\delta(M)$ is a small tubular neighborhood of M, and because $f \in \mathcal{O}\big(\mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$ is uniquely defined, we get a unique extension, still denoted by $f^c_{\widehat{r}^-_{\lambda+1}}$, to (5.8).

We can now apply the same reasoning as in Proposition 4.5, which consists of progressive holomorphic extension by means of thin rinds. Reproducing the proof of Lemma 4.6 (with changes of notation only), we get for every radius r' with $\widehat{r}^+_\lambda < r' \leqslant \widehat{r}^-_{\lambda+1}$ that

$$\text{(5.9)} \qquad \mathsf{Shell}^{r'+\delta}_{r'}\big(\widetilde{\mathsf{R}}^c_{r'} \cup \mathsf{N}^c_{r'}\big) \text{ is contained in } \widetilde{\Omega}^c_{>r'} \bigcup \mathcal{V}_\delta\big(M^c_{>\widehat{r}^+_\lambda}\big)_{>\widehat{r}^+_\lambda}.$$

Similarly, reproducing the proof of Lemma 4.8 yields the connectedness of

$$\mathsf{Rind}\big(\mathsf{R}^c_{r'},\,\eta\big) \bigcap \Big(\widetilde{\Omega}^c_{>r'} \cup \mathcal{V}_\delta\big(M^c_{>\widehat{r}^+_\lambda}\big)_{>\widehat{r}^+_\lambda}\Big),$$

and furthermore, this yields that the union, instead of the intersection, contains

$$\widetilde{\Omega}^c_{>r'-\eta} \bigcup \mathcal{V}_\delta\big(M^c_{>\widehat{r}^+_\lambda}\big)_{>\widehat{r}^+_\lambda},$$

whenever $r' - \eta$ is still $\geqslant \widehat{r}^+_\lambda$ (otherwise, shrink conveniently the thickness of the last extensional rind, as in the proof of Proposition 4.5). Thus, by piling up $\frac{\widehat{r}^-_{\lambda+1} - \widehat{r}^+_\lambda}{\eta}$ rinds and by using a finite number $\leqslant C\left(\frac{\widehat{r}_\kappa}{\delta}\right)^{2n-1}\left[\frac{\widehat{r}^-_{\lambda+1} - \widehat{r}^+_\lambda}{\eta}\right]$ of Levi-Hartogs figures, we get unique holomorphic extension to

$$\text{(5.10)} \qquad \mathcal{V}_\delta\big(M^c_{>\widehat{r}^+_\lambda}\big)_{>\widehat{r}^+_\lambda} \bigcup \widetilde{\Omega}^c_{>\widehat{r}^+_\lambda}.$$

Finally, if r satisfies $\widehat{r}^+_\mu < r < \widehat{r}^-_{\mu+1}$, descending from $(\cdot)_{>\widehat{r}^-_{\mu+1}}$ with $\lambda = \mu$ as above, we just stop the construction of rinds to $(\cdot)_{>r}$ by shrinking appropriately the thickness of the last extensional rind.

The property **(iii)** that enclosed domains $\widetilde{\Omega}^c_{>r}$ are either disjoint or one is contained in the other remains stable as r decreases through the whole nonsingular interval $\big(\widehat{r}_\lambda, \widehat{r}_{\lambda+1}\big)$, because their (moving) boundaries always remain disjoint, so that property **(iv)** is also simultaneously transmitted to lower regular radii. This completes A.

5.11. Localizing (pseudo)cubes at Morse points. We now study B. Recall that $\widehat{r}_\mu < r < \widehat{r}_{\mu+1}$, let λ with $\mu \leqslant \lambda \leqslant \kappa - 1$ and suppose that $r \leqslant \widehat{r}^-_\lambda$, so that starting from $(\cdot)_{>\widehat{r}^+_\lambda}$, we may (and we must) continue the Hartogs-Levi filling inside the whole thin spherical shell $\big\{\widehat{r}^-_\lambda < \|z\| \leqslant \widehat{r}^+_\lambda\big\}$. Similarly as above, the way how we should stop the process in the case where $\lambda = \mu$ and $\widehat{r}_\mu < r < \widehat{r}^+_\mu$ is obvious.

By descending induction on λ through A and B, we may assume that at $\widehat{r}^+_\lambda$, the domains $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda}$ enclosed by $M^c_{>\widehat{r}^+_\lambda}$ for $1 \leqslant c \leqslant c_\lambda$ have been constructed and that each restriction $f^c_{\widehat{r}^+_\lambda}$ of $f \in \mathcal{O}\big(\mathcal{V}_\delta\big(M^c_{>r}\big)_{>r}\big)$ to $\mathcal{V}_\delta\big(M^c_{>\widehat{r}^+_\lambda}\big)_{>\widehat{r}^+_\lambda}$ extends holomorphically to the domain (5.10) of the previous paragraph.

By an elementary analysis of the Morse normalizing quadric, we will see that in some small (pseudo)cube centered at $\widehat{p}_\lambda$, there passes in most cases only one component $M^c_{>\widehat{r}^+_\lambda}$, while in a single exceptional case, there can pass two (at most) different connected components $M^{c_1}_{>\widehat{r}^+_\lambda}$ and $M^{c_2}_{>\widehat{r}^+_\lambda}$. We will consider only this single (or these two) component(s), because the other components do pass regularly and without topological change accross $\widehat{p}_\lambda$, hence are filled in by Levi-Hartogs figures exactly as in A.

Shrinking the δ_1 of Theorem 2.7 if necessary (remind $0 < \delta \leqslant \delta_1$), we may assume that the Morse normalizing coordinates $\big(v, x_1, \dots, x_{k_\lambda}, y_1, \dots, y_{2n-1-k_\lambda}\big)$ near $\widehat{p}_\lambda$ are defined in the ball $\mathbb{B}^n(\widehat{p}_\lambda, \delta_1)$ and that the map

$$z \longmapsto \big(v(z), x(z), y(z)\big), \qquad \mathbb{B}^n(\widehat{p}_\lambda, \delta_1) \longrightarrow \mathbb{R}^{2n}$$

is close in $\mathcal{C}^1$ norm to its differential at $\widehat{p}_\lambda$, so that it is almost not distorting. Then δ_1 shall not be shrunk anymore.

Because in the estimates of the (finite) number of Levi-Hartogs figures, η only appears as a denominator in a factor $\frac{r'-r''}{\eta}$ (*cf.* Proposition 4.5), it is allowed to work with extensional rinds of smaller universal positive thickness, at the cost of spending a number of pushed analytic discs that is greater, of course, but still finite. If necessary, we shrink $\eta > 0$ to insure that $\eta^{1/2} \ll \delta$. Then η will not be shrunk anymore.

Thanks to these preliminaries, we may define a convenient (pseudo)cube centered at $\widehat{p}_\lambda$ by

(5.12)
$$\mathsf{C}_\eta := \Big\{ z \in \mathbb{B}^n(\widehat{p}_\lambda, \delta_1) : \ |v(z)| < \eta, \ \ \|x(z)\| < 2\,\eta^{1/2}, \ \ \|y(z)\| < 2\,\eta^{1/2} \Big\}.$$

It then follows that C_η is properly contained in $\mathcal{V}_\delta(M)$ and is relatively small. Reminding that $v(z) = r(z) - r(\widehat{p}_\lambda)$, the radial thickness of C_η is equal to 2η, twice the difference $\widehat{r}^+_\lambda - \widehat{r}^-_\lambda = \eta$. We draw a diagram assuming $k_\lambda = 2n-1$ (see only the left one).

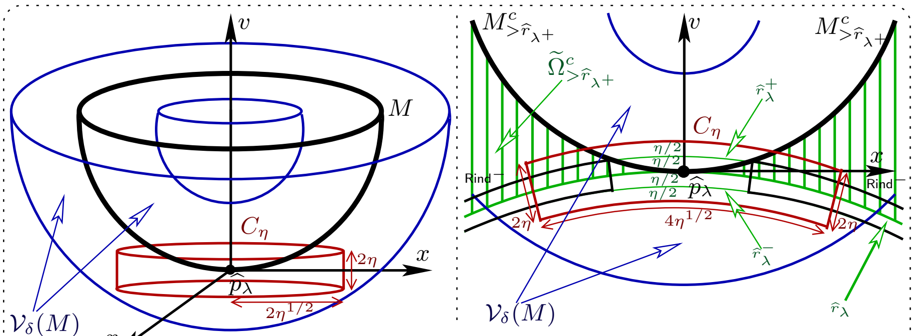


Fig. 14: The radial (pseudo)cube C_η centered at $\widehat{p}_\lambda$

5.13. Topology of horizontal super-level sets in the complement of quadrics. Simultaneously to the proof, we provide an auxiliary elementary study. Let $n \in \mathbb{N}$ with $n \geqslant 2$, let $k \in \mathbb{N}$ with $0 \leqslant k \leqslant 2n-1$, let $x = (x_1, \dots, x_k) \in \mathbb{R}^k$, let $y = (y_1, \dots, y_{2n-1-k}) \in \mathbb{R}^{2n-1-k}$, let $v \in \mathbb{R}$, and in $\mathbb{R}^{2n}$ equipped with the coordinates (x, y, v), consider the quadric of equation

$$v = \sum_{1 \leqslant j \leqslant k} x_j^2 - \sum_{1 \leqslant j \leqslant 2n-1-k} y_j^2, \tag{5.14}$$

which we will denote by Q_k. The coordinate v playing the rôle of $r(z) - r(\widehat{p}_\lambda)$ near a singular radius $\widehat{r}_\lambda$ having Morse coindex k_λ, we want to understand how the topology of the super-level sets

$$\{v > \varepsilon\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_k\big)$$

(which relate to the possible domains $\widetilde{\Omega}^c_{>r}$ for r close to $\widehat{r}_\lambda$) do change when the parameter ε descends from a small positive value to a small negative value.

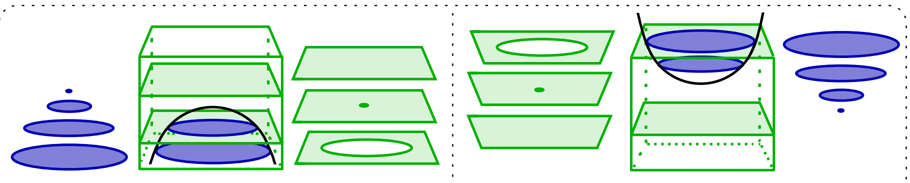


Fig. 15: Growing of superlevel domains near a local maximum or minimum

In the case $k = 0$ (left figure) the quadric looks like a spherical cap, its complement $\mathbb{R}^{2n} \backslash \mathsf{Q}_0$ having exactly two connected components. For positive values of ε, there is only one (green) super-level component $\{v > \varepsilon\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_0\big)$. As ε becomes negative, this component grows regularly, allowing a newly created hole to widen inside the slices $\{v = \varepsilon\}$. The (blue) holes then pile up to constitute a newly created, local component $M^c_{>\widehat{r}_\lambda^-}$.

The (reverse) case $k = 2n-1$ exhibits the local end of some component $M^c_{>\widehat{r}_\lambda^-}$. In a while, we will see that there is a salient topological difference between the two remaining (less obvious) cases $2 \leqslant k \leqslant 2n-2$ and $k = 1$, the exceptional one. Before pursuing, we conclude the proof of B in case $\widehat{p}_\lambda$ is a local maximum or minimum.

We assume $k_\lambda = 2n - 1$, the case $k_\lambda = 0$ being already considered (essentially completely) in Section 4. Observe that $M_{>\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ is diffeomorphic to $\mathsf{S}^{2n-2} \times (c/2, c)$, hence connected. Thus, let $M^c_{>\widehat{r}_\lambda^+}$ denote the single component entering C_η. By descending induction through A and B, $M^c_{>\widehat{r}_\lambda^+}$ bounds a relatively compact domain of holomorphic extension $\widetilde{\Omega}^c_{>\widehat{r}_\lambda^+}$, with $\partial\widetilde{\Omega}^c_{>\widehat{r}_\lambda^+} = M^c_{>\widehat{r}_\lambda^+} \cup \mathsf{N}^c_{\widehat{r}_\lambda^+} \cup \widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$, as in property **(ii)** of Proposition 5.3, all the other properties also holding true on $(\cdot)_{>\widehat{r}_\lambda^+}$. Denote by $\widetilde{\mathsf{R}}^{c,k}_{\widehat{r}_\lambda^+}$, $1 \leqslant k \leqslant k_{\lambda,c}$, the connected components of $\widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$ and by $\mathsf{N}^{c,j}_{\widehat{r}_\lambda^+}$, $1 \leqslant j \leqslant j_{\lambda,c}$, with $j_{\lambda,c} \geqslant k_{\lambda,c}$, the components of $\mathsf{N}^c_{\widehat{r}_\lambda^+}$.

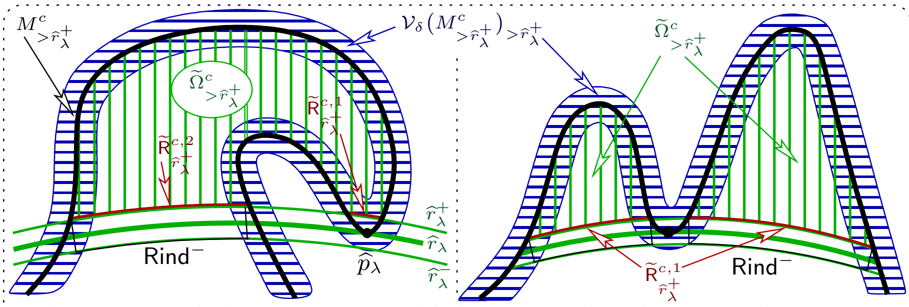


Fig. 16: Two distinct Hartogs-Levi fillings at a point of Morse coindex $2n-1$

We do the numbering so that C_η encloses the first (small) $\mathsf{N}^{c,1}_{\widehat{r}^+_\lambda}$, which is diffeomorphic to a small $(2n-2)$-dimensional sphere. Also, we number so that the boundary of $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}^+_\lambda}$ in $\mathsf{S}^{2n-1}_{\widehat{r}^+_\lambda}$ contains $\mathsf{N}^{c,1}_{\widehat{r}^+_\lambda}$, whence $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}^+_\lambda}$ meets C_η. We do not draw C_η.

Observe that, by means of extensional rinds that are symmetric around the other components $\widetilde{\mathsf{R}}^{c,2}_{\widehat{r}^+_\lambda}, \dots, \widetilde{\mathsf{R}}^{c,k_{\lambda,c}}_{\widehat{r}^+_\lambda}$, we may achieve the Hartogs-Levi filling exactly as in A, because $r(z)|_M$ is regular in $\mathcal{V}_\delta\big(\mathsf{N}^{c,j}_{\widehat{r}^+_\lambda}\big)$, for every j such that $\mathsf{N}^{c,j}_{\widehat{r}^+_\lambda}$ is contained in the boundary of each of these other components. Hence it remains only to discuss what is happening in a neighborhood of the single component $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}^+_\lambda}$, and especially near $\widehat{p}_\lambda$.

For the disposition of $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$, or equivalently of $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$, two cases occur. Let $\big(v, x_1, \dots, x_{2n-1}\big)$ be the Morse coordinates centered at $\widehat{p}_\lambda$.

- (a) As illustrated by the left figure above, $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$ consists of the space[7] lying above $\big\{v = \eta/2\big\}$ and above $\big\{v = x_1^2 + \cdots + x_{2n-1}^2\big\}$, a cap-shaped space which is clearly connected; the region $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}^+_\lambda}$ is then diffeomorphic to a small $(2n-1)$-dimensional ball.
- (b) As illustrated by the right figure above, $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$ consists of the space lying above $\big\{v = \eta/2\big\}$ but below $\big\{v = x_1^2 + \cdots + x_{2n-1}^2\big\}$; the dimension of $\mathsf{S}^{2n-1}_{\widehat{r}^+_\lambda}$ being $\geqslant 3$, the region $\widetilde{R}^{c,1}_{\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$ is connected, a fact that a one-dimensional diagram cannot show adequately; then $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda} \cap \mathsf{C}_\eta$ is also connected.

In case (a), near $\widehat{p}_\lambda$, a piece of $\widetilde{\Omega}^c_{>\widehat{r}^+_\lambda}$ ends up while descending to the lower super-level set $(\cdot)_{>\widehat{r}^-_\lambda}$. We do not use any extensional rind there, we just observe that unique holomorphic extension is got for free in

$$\Big[\mathcal{V}_\delta\big(M^c_{>\widehat{r}^-_\lambda}\big)_{>\widehat{r}^-_\lambda}\Big] \cap \mathsf{C}_\eta,$$

since this domain is fully contained in $\mathcal{V}_\delta\big(M_{>r}\big)_{>r}$.

[7]Sets written "$\{\cdot\}$" here are understood to be subsets of C_η.

In case (b), we apply Hartogs Levi extension to $\mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}, \eta\big)$ and we get unique holomorphic extension from (5.10) to

$$\Big[\mathcal{V}_\delta\big(M^c_{>\widehat{r}_\lambda^-}\big)_{>\widehat{r}_\lambda^-}\Big] \bigcup \mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}, \eta\big).$$

The union of this open set together with (5.10) contains a unique well defined domain $\widetilde{\Omega}^c_{>\widehat{r}_\lambda^-}$ with the property that the passage from $\widetilde{\mathsf{R}}^{c,1}_{>\widehat{r}_\lambda^+}$ to $\widetilde{\mathsf{R}}^{c,1}_{>\widehat{r}_\lambda^-}$ fills a hole, as illustrated by the right diagram above, whence $\mathsf{N}^c_{>\widehat{r}_\lambda^-}$ has one less connected component, because the $(2n-2)$-sphere $\mathsf{N}^{c,1}_{>\widehat{r}_\lambda+\varepsilon}$ drops when $\varepsilon < 0$.

The properties that two different domains $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+}$ are either disjoint or one is contained in the other is easily seen to be inherited by $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^-}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^-}$: it suffices to distinguish two cases: $c_2 \neq c$ and $c_1 \neq c$, or $c_2 \neq c$ and $c_1 = c$; to look at (a) or (b) and then to conclude.

The proof of B in case $k_\lambda = 2n-1$ is complete. The case $k_\lambda = 0$ is similar: two subcases (a') — reverse (a) — and (b') — reverse (b) — then appear; subcase (a') exhibits the birth of a new component (blue left Figure 15), as already fully studied in Section 4 while subcase (b') (green left Figure 15) shows that an external component descends regularly as do clouds around a hill.

5.15. The regular cases $2 \leqslant k_\lambda \leqslant 2n-2$. Let k with $2 \leqslant k \leqslant 2n-2$ and consider the quadric Q_k of (5.14). We claim that $\mathsf{Q}_k \cap \big\{v > \varepsilon\big\}$ has exactly one connected component for every $\varepsilon > 0$. Indeed, $\mathsf{Q}_k \cap \big\{v > \varepsilon\big\}$ can be represented as

$$\bigcup_{y_1,\dots,y_{2n-k-1}} \bigcup_{\varepsilon' > \varepsilon} \big\{x_1^2 + \cdots + x_k^2 = \varepsilon' + y_1^2 + \cdots + y_{2n-1-k}^2\big\}.$$

Since ε' is always positive, we hence have a smoothly parameterized family of $(k-1)$-dimensional spheres that are all connected. Consequently, the union is also connected, as claimed.

To view the topology more adequately, in the case $n = 2$, we draw a short movie consisting of the 3-dimensional slices $\big\{v = \varepsilon'\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_k\big)$, where $\varepsilon' = \frac{2}{3}\,\eta,\ \frac{1}{2}\,\eta,\ 0,\ -\frac{1}{2}\,\eta$. To conceptualize (in case $n = 2$) the super-level sets

$$\big\{v > \varepsilon\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_k\big) = \bigcup_{\varepsilon' > \varepsilon} \big\{v = \varepsilon'\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_k\big),$$

it suffices to pile up intuitively the images of the corresponding movie.

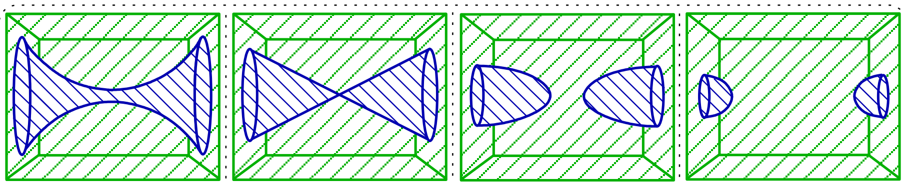


Fig. 17: Sliced view of the growing of the two possible domains in case $2 \leqslant k_\lambda \leqslant 2n-2$

So let $M^c_{>\widehat{r}_\lambda^+}$ be the single connected component of $M \cap \big\{\|z\| > \widehat{r}_\lambda^+\big\}$ that enters C_η. The corresponding domain $\widetilde{\Omega}^c_{>\widehat{r}_\lambda^+}$ can be located from one or the other side.

Its prolongation up to the deeper sublevel set $(\cdot)_{>\widehat{r}_\lambda^-}$ (viewed only inside C_η) consists of piling up the (blue) small symmetric regions or the (green) surrounding regions drawn above.

We do the numbering so that $\mathsf{N}^{c,1}_{\widehat{r}_\lambda^+}$ enters C_η, being a (connected) hyperboloid as drawn in the first picture of Figure 17 and so that $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}$ enters C_η as one (connected, blue or green) side of this hyperboloid. As previously in the two cases $k_\lambda = 0$ and $k_\lambda = 2n-1$, the Hartogs-Levi filling goes through exactly as in the regular case **A** for all other $\widetilde{\mathsf{R}}^{c,2}_{\widehat{r}_\lambda^+}, \dots, \widetilde{\mathsf{R}}^{c,k_\lambda,c}_{\widehat{r}_\lambda^+}$. Next, by putting finitely many Levi-Hartogs figures in $\mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}, \eta\big)$ we get holomorphic extension from the domain (5.10) to

$$\Big[\mathcal{V}_\delta\big(M^c_{>\widehat{r}_\lambda^-}\big)_{>\widehat{r}_\lambda^-}\Big] \bigcup \mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}, \eta\big).$$

The intersection of (5.10) with this open set is connected because $\widetilde{\mathsf{R}}^{c,1}_{\widehat{r}_\lambda^+}$ is connected, and the union of both contains a well defined domain $\widetilde{\Omega}^c_{>\widehat{r}_\lambda^-}$ obtained by adding the (blue or green) slices of Figure 17.

§6. THE EXCEPTIONAL CASE $k_\lambda = 1$

6.1. Illustration. To begin with the most delicate case, we draw a 3-dimensional diagram showing a saddle-like M localized in a (pseudo)cube C_η centered at $\widehat{p}_\lambda$.

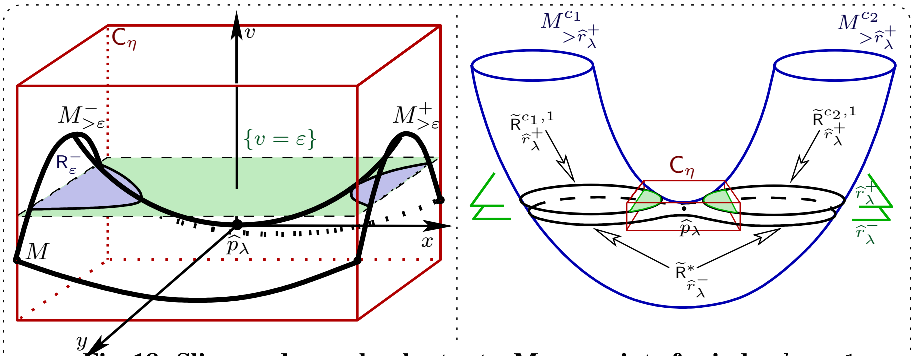


Fig. 18: Slices and superlevel sets at a Morse point of coindex $k_\lambda = 1$

For every ε satisfying $0 < \varepsilon < \eta$, there are two connected components $M^-_{>\varepsilon}$ and $M^+_{>\varepsilon}$ of $M_{>\widehat{r}_\lambda+\varepsilon} \cap \mathsf{C}_\eta$, namely the two upper tips of the saddle, defined in equations by

$$M^\pm_{>\varepsilon} := \big\{v = x^2 - y_1^2 - \cdots - y_{2n-2}^2\big\} \cap \big\{\pm x > 0\big\} \cap \big\{v > \varepsilon\big\}.$$

With $\varepsilon = \frac{1}{2}\eta$, we are simply looking at $M_{>\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$. By descending induction through A and B, we are given two domains of holomorphic extension $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+}$ whose boundary contains $M^-_{>\eta/2}$ and $M^+_{>\eta/2}$, respectively.

Firstly, we assume that $c_2 \neq c_1$. Since each one of the two pieces of hypersurfaces $M^-_{>\eta/2}$ and $M^+_{>\eta/2}$ has two sides, there are $2 \times 2 = 4$ subcases to be considered for the relative disposition of $\Omega_{>\eta/2}^{-} := \widetilde{\Omega}_{>\widehat{r}_\lambda^+}^{c_1} \cap \mathsf{C}_\eta$ and of $\Omega_{>\eta/2}^{+} := \widetilde{\Omega}_{>\widehat{r}_\lambda^+}^{c_2} \cap \mathsf{C}_\eta$, with $c_2 \neq c_1$.

(a) $\Omega_{>\eta/2}^{-}$ $\big($resp. $\Omega_{>\eta/2}^{+}\big)$ consists of the space lying above the hyperplane $\{v = \eta/2\}$ and below the left (resp. right) tip of the saddle, namely in equations:

$$\Omega_{>\eta/2}^{\pm} = \big\{v > \eta/2\big\} \bigcap \big\{ \pm x > 0\big\} \bigcap \big\{v < x^2 - y_1^2 - \cdots - y_{2n-2}^2\big\}.$$

(b) $\Omega_{>\eta/2}^{-}$ is the small nose as in (a) but $\Omega_{>\eta/2}^{+}$ consists of the other side, *i.e.* of the (rather bigger) space lying inside $\big\{v > \eta/2\big\}$ left to $M_{>\eta/2}^{+}$, namely in equations:

$$\Omega_{>\eta/2}^{+} = \big\{v > \eta/2\big\} \Big\backslash \Big(\big\{x > 0\big\} \bigcap \big\{v \leqslant x^2 - y_1^2 - \cdots - y_{2n-2}^2\big\}\Big).$$

(c) Symetrically to (b), $\Omega_{>\eta/2}^{+}$ is the small nose as in (a) but

$$\Omega_{>\eta/2}^{-} = \big\{v > \eta/2\big\} \Big\backslash \Big(\big\{x < 0\big\} \bigcap \big\{v \leqslant x^2 - y_1^2 - \cdots - y_{2n-2}^2\big\}\Big).$$

(d) Finally, $\Omega_{>\eta/2}^{-}$ is as in (c) and $\Omega_{>\eta/2}^{+}$ is as in (b).

The last subcase (d) cannot occur, because it is ruled out by property **(iii)** of Proposition 5.3, which holds on the super-level set $(\cdot)_{>\widehat{r}_\lambda^+}$ by the inductive assumption.

Secondly, we assume that $c_2 = c_1$. Then there can occur a subcase (a') very similar to (a), in which $c_2 = c_1$, so that $\Omega_{>\eta/2}^{-}$ and $\Omega_{>\eta/2}^{+}$ belong to the *same* enclosed relatively compact domain. But with $c_2 = c_1$, no subcase similar to (b) — or to (c) — can occur, because $M_{>\eta/2}^{-} \subset \partial\Omega_{>\eta/2}^{+}$ — or $M_{\eta/2}^{+} \subset \partial\Omega_{>\eta/2}^{-}$ — would then bound the *same* relatively compact domain from its both sides, but we already know from the beginning of the proof, that one side at least must always contain the points at infinity.

Finally, with $c_2 = c_1 = c$, there remains the following last subcase (unseen previously).

(e) $\Omega_{>\eta/2} := \widetilde{\Omega}_{>\widehat{r}_\lambda^+}^{c} \cap \mathsf{C}_\eta$ consists of the space lying above $\big\{v = \eta/2\big\}$ and above the saddle, namely

$$\Omega_{>\eta/2} = \big\{v > \eta/2\big\} \bigcap \big\{v > x^2 - y_1^2 - \cdots - y_{2n-2}^2\big\}.$$

As $M = \partial\Omega_M$ lies in $\mathbb{C}^n$ with $n \geqslant 2$, whence $2n - 2 \geqslant 2$, there is at least one dimension of $y \in \mathbb{R}^{2n-2}$ which is missing in the left figure above. To view the topology more adequately, coming back to the abstract quadric Q_1 and assuming $n = 2$, we plan to draw a short movie consisting of the 3-dimensional slices $\big\{v = \varepsilon'\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_1\big)$, where $\varepsilon' = \frac{2}{3}\,\eta,\ \frac{1}{2}\,\eta,\ 0,\ -\frac{1}{2}\,\eta$.

Recall that we are interested in the connected components of the super-level sets

$$\big\{v > \varepsilon\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_1\big) = \bigcup_{\varepsilon' > \varepsilon} \big\{v = \varepsilon'\big\} \cap \big(\mathbb{R}^{2n} \backslash \mathsf{Q}_1\big).$$

As suggested by this sliced union, to conceptualize these 4-dimensional (in case $n = 2$) super-level sets, it suffices to pile up intuitively the images of the corresponding movie.

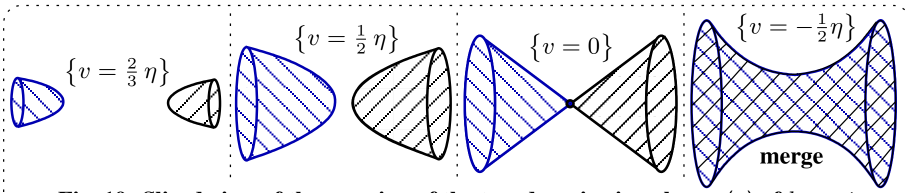


Fig. 19: Sliced view of the merging of the two domains in subcase (a) **of** $k_\lambda = 1$

Here, the second picture shows $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ (in blue, to the left) together with $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ (in black, to the right). Then the third picture shows how the two components do touch and the fourth one shows how they should be merged as $\varepsilon' = -\frac{1}{2}\,\eta$ becomes negative. The complete discussion follows in a while.

We next offer the movie of (b), the movie of (c) being obtained from it just by a reflection across the hyperplane $\{x = 0\}$.

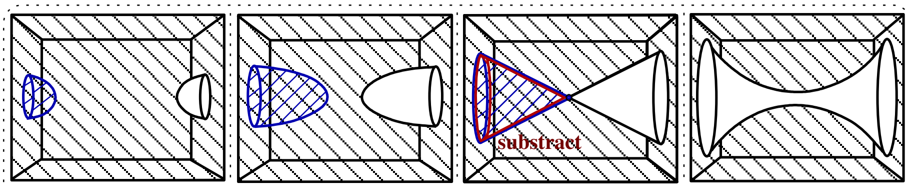


Fig. 20: Sliced view of the substraction of the left domain in subcase (b) **of** $k_\lambda = 1$

Here again, the second picture shows $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ (in blue, to the left) together with $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ (the large (black) region, containing the small (blue) one). Then the third picture, namely the slice $\varepsilon' = 0$, shows a not allowed situation: the left cone does bound *two* regions from its *two* sides, contrary to the *a priori* unique relatively compact domain $\widetilde{\Omega}^{c_1}_{\widehat{r}_\lambda} \subset \{\|z\| > \widehat{r}_\lambda\}$ we are seeking to construct, when starting from $\widetilde{\Omega}^{c_1}_{\widehat{r}_\lambda^+}$. The trick is then to *suppress* the (blue) small slice, or equivalently to subtract it from the (black) large slice which contains it. Then the black winning slice continues to grow up to $\{v = -\eta/2\}$ (fourth picture). The complete discussion follows in a while.

Finally, here is the (simpler) movie of **(e)**.

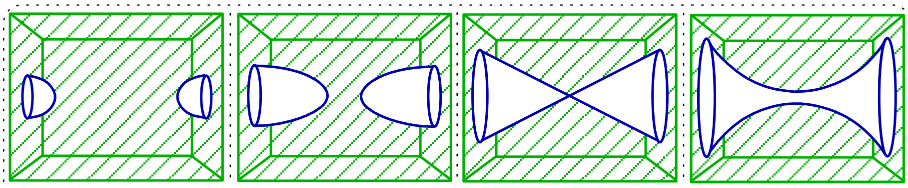


Fig. 21: Sliced view of the growing of the external domain in subcase (e) **of** $k_\lambda = 1$

6.2. Jumping across the singular radius: merging process. Assuming $k_\lambda = 1$, we can now complete B in subcase (a), postponing subcase (a'). We look at Figures 17 and 18.

Let $M^{c_1}_{>\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ and $M^{c_2}_{>\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ be the two "nose" components of $M_{>\widehat{r}_\lambda^+}$ entering C_η. Here, $c_2 \neq c_1$. By descending induction through A and B, $M^{c_1}_{>\widehat{r}_\lambda^+}$ and $M^{c_2}_{>\widehat{r}_\lambda^+}$ bound some two relatively compact domains of holomorphic extension $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+}$ with $\partial\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+} = M^{c_1}_{>\widehat{r}_\lambda^+} \cup \mathsf{N}^{c_1}_{\widehat{r}_\lambda^+} \cup \widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}$ and $\partial\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+} = M^{c_2}_{>\widehat{r}_\lambda^+} \cup \mathsf{N}^{c_2}_{\widehat{r}_\lambda^+} \cup \widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}$ as in property **(ii)** of Proposition 5.3, all the other properties also holding true on $(\cdot)_{>\widehat{r}_\lambda^+}$.

We remind that the other domains $\widetilde{\Omega}^{c}_{>\widehat{r}_\lambda^+}$ for $c \neq c_1$ and $c \neq c_2$ with $1 \leqslant c \leqslant c_\lambda$ do pass regularly through $\widehat{r}_\lambda$ up to $(\cdot)_{>\widehat{r}_\lambda^-}$, thanks to A.

For $i = 1, 2$, denote by $\widetilde{\mathsf{R}}^{c_i,k}_{\widehat{r}_\lambda^+}$, $1 \leqslant k \leqslant k_{\lambda,c_i}$, the connected components of $\widetilde{\mathsf{R}}^{c_i}_{\widehat{r}_\lambda^+}$ and by $\mathsf{N}^{c_i,j}_{\widehat{r}_\lambda^+}$, $1 \leqslant j \leqslant j_{\lambda,c_i}$, with $j_{\lambda,c_i} \geqslant k_{\lambda,c_i}$, the components of $\mathsf{N}^{c_i}_{\widehat{r}_\lambda^+}$. We do the numbering so that $\widetilde{\mathsf{R}}^{c_1,1}_{\widehat{r}_\lambda^+}$ (resp. $\widetilde{\mathsf{R}}^{c_2,1}_{\widehat{r}_\lambda^+}$) enters C_η to the left (resp. right), together with $\mathsf{N}^{c_1,1}_{\widehat{r}_\lambda^+}$ (resp. $\mathsf{N}^{c_2,1}_{\widehat{r}_\lambda^+}$), as illustrated by Figure 17.

As in the case $k_\lambda = 2n-1$, for $i = 1, 2$, by means of extensional rinds that are symmetric around the other components $\widetilde{\mathsf{R}}^{c_i,2}_{\widehat{r}_\lambda^+}, \dots, \widetilde{\mathsf{R}}^{c_i,k_{\lambda,c_i}}_{\widehat{r}_\lambda^+}$, we may achieve the Hartogs-Levi filling exactly as in A, because $r(z)|_M$ is regular in $\mathcal{V}_\delta\big(\mathsf{N}^{c_i,j}_{\widehat{r}_\lambda^+}\big)$, for every j such that $\mathsf{N}^{c_i,j}_{\widehat{r}_\lambda^+}$ is contained in the boundary of each of these other components. Hence it remains only to discuss what is happening in a neighborhood of the two components $\widetilde{\mathsf{R}}^{c_i,1}_{\widehat{r}_\lambda^+}$, $i = 1, 2$, and especially near the saddle point $\widehat{p}_\lambda$.

While descending from $\widehat{r}_\lambda^+$ to $\widehat{r}_\lambda^-$, the two regions $\widetilde{\mathsf{R}}^{c_1,1}_{\widehat{r}_\lambda^+} \subset \mathsf{S}^{2n-1}_{\widehat{r}_\lambda^+}$ and $\widetilde{\mathsf{R}}^{c_2,1}_{\widehat{r}_\lambda^+} \subset \mathsf{S}^{2n-1}_{\widehat{r}_\lambda^+}$ do merge as a single connected region contained in $\mathsf{S}^{2n-1}_{\widehat{r}_\lambda^-}$ that we will denote by $\widetilde{\mathsf{R}}^{*}_{\widehat{r}_\lambda^-}$, *see* the right Figure 17. In Morse theory ([31, 19]), one speaks of *attaching a one-cell*, since in the merging process, the two regions are essentially joined by means of a (thickened) segment directed along the x-axis. It follows that the two hypersurfaces $M^{c_1}_{>\widehat{r}_\lambda^+}$ and $M^{c_2}_{>\widehat{r}_\lambda^+}$ do merge as a connected hypersurface $M^{*}_{>\widehat{r}_\lambda^-}$ containing them, and furthermore, that the two domains $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+}$ and $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+}$ do prolong uniquely up to the slightly deeper super-level set $(\cdot)_{>\widehat{r}_\lambda^-}$, merging as a uniquely defined domain $\widetilde{\Omega}^{*}_{>\widehat{r}_\lambda^-}$ which is relatively compact in $\mathbb{C}^n$ and which contains $\widetilde{\mathsf{R}}^{*}_{\widehat{r}_\lambda^+}$ in its boundary $\partial\widetilde{\Omega}^{*}_{>\widehat{r}_\lambda^+}$.

As $c_2 \neq c_1$, the new number of domains in the interval $(\widehat{r}_{\lambda-1}, \widehat{r}_\lambda)$ is lowered by a unit, *i.e.* $c_{\lambda-1} = c_\lambda - 1$ (if $c_2 = c_1$ as in (a'), the number would not change, *i.e.* $c_{\lambda-1} = c_\lambda$).

For $i = 1, 2$, let $f^{c_i}_{\widehat{r}_\lambda^+}$ denote the restriction of $f \in \mathcal{O}\big(\mathcal{V}_\delta\big(M_{>r}\big)_{>r}\big)$ to $\mathcal{V}_\delta\big(M^{c_i}_{>\widehat{r}_\lambda^+}\big)_{>\widehat{r}_\lambda^+}$. By descending induction through A and B, $f^{c_i}_{\widehat{r}_\lambda^+}$ extends holomorphically and uniquely to $\widetilde{\Omega}^{c_i}_{>\widehat{r}_\lambda^+}$. Then both functions do extend holomorphically and uniquely to

$$\mathcal{V}_\delta\big(M^*_{>\widehat{r}_\lambda^-}\big)_{>\widehat{r}_\lambda^-},$$

since they coincide with f near $\widehat{p}_\lambda$. We then introduce the two extensional rinds $\mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c_i}_{\widehat{r}_\lambda^+}, \eta\big)$, drawn in the right Figure 17. Two applications of Proposition 3.7 together with a geometrically clear connectedness property yield unique holomorphic extension to

$$\mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}, \eta\big) \bigcup \mathsf{Rind}\big(\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}, \eta\big) \bigcup \mathcal{V}_\delta\big(M^*_{>\widehat{r}_\lambda^-}\big)_{\widehat{r}_\lambda^-} \bigcup \widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda^+} \bigcup \widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda^+}.$$

In sum, we have got unique holomorphic extension to

$$\mathcal{V}_\delta\big(M^*_{>\widehat{r}_\lambda^-}\big)_{\widehat{r}_\lambda^-} \bigcup \widetilde{\Omega}^*_{>\widehat{r}_\lambda^-}.$$

To establish **(iv)** of Proposition 5.3 at $(\cdot)_{>\widehat{r}_\lambda^-}$, it suffices to show **(iii)**, which is checked to be equivalent. We observe that, for logical reasons only, a given region $\widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$ for $c \neq c_1$ and $c \neq c_2$ can:

- be disjoint from $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}$ and also disjoint from $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}$;
- be contained in $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}$ or (exclusive "or") in $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}$;
- contain $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}$ or (inclusive "or") $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}$.

But we claim that in the latter case, $\widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$ necessarily contains both regions $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+}$ and $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+}$. Indeed, otherwise the boundary $\mathsf{N}^c_{\widehat{r}_\lambda^+}$ of $\widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$ should separate $\widetilde{\mathsf{R}}^{c_1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ from $\widetilde{\mathsf{R}}^{c_2}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ in the level set $\big\{v = \frac{\eta}{2}\big\} \cap \mathsf{C}_\eta$, which is impossible since $\mathsf{N}^c_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta$ is exactly equal to $\big(\mathsf{N}^{c_1,1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta\big) \bigcup \big(\mathsf{N}^{c_2,1}_{\widehat{r}_\lambda^+} \cap \mathsf{C}_\eta\big)$, not more.

It follows in all cases that $\mathsf{N}^c_{\widehat{r}_\lambda^+} = \partial \widetilde{\mathsf{R}}^c_{\widehat{r}_\lambda^+}$ is disjoint from C_η, hence it lies in $\big\{\widehat{r}_\lambda^- \leqslant \|z\| \leqslant \widehat{r}_\lambda^+\big\} \backslash \mathsf{C}_\eta$. Consequently, the regular flow of $\frac{\nabla r_M}{\|\nabla r_M\|}$ on

$$\big[M \cap \big\{\widehat{r}_\lambda^- \leqslant \|z\| \leqslant \widehat{r}_\lambda^+\big\}\big] \backslash \mathsf{C}_\eta$$

pushes down regularly $\mathsf{N}^c_{\widehat{r}_\lambda^+}$, as a uniquely defined compact 2-codimensional $\mathsf{N}^c_{\widehat{r}_\lambda^-} \subset \mathsf{S}^{2n-1}_{\widehat{r}_\lambda^+}$, disjointly from the newly created merged boundary $\mathsf{N}^*_{\widehat{r}_\lambda^-} = \partial \widetilde{\Omega}^*_{>\widehat{r}_\lambda^-} \subset \mathsf{S}^{2n-1}_{\widehat{r}_\lambda^-}$. This information suffices now to check that **(iii)** and **(iv)** of Proposition 5.3 are transmitted to $(\cdot)_{>\widehat{r}_\lambda^-}$, just for logical reasons.

The proof of B in case $k_\lambda = 1$, subcase (a) is complete. Subcase (a') involves only minor differences.

6.3. Subtracting process. We now summarize the discussion of subcase (b), focusing only on topological aspects and dropping the formal considerations about holomorphic extensions. For an adequate three-dimensional illustration, think of a smoothly cut cylindrical piece of modelling clay in which a thin finger drills a hole.

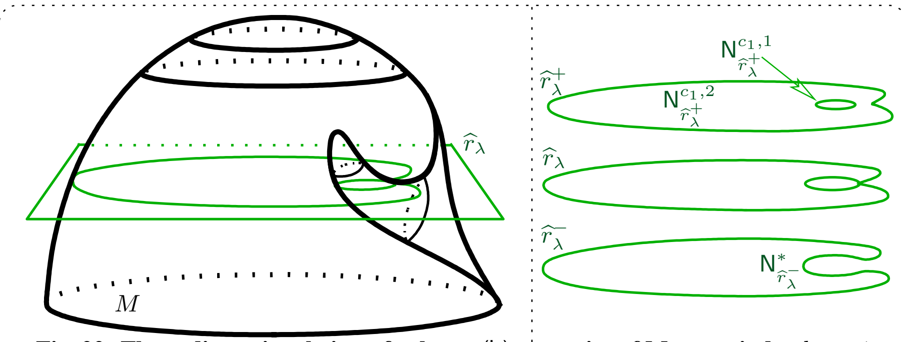


Fig. 22: Three-dimensional view of subcase (b) **at a point of Morse coindex** $k_\lambda = 1$

As in §5.6, in C_η, there enter exactly two domains $\widetilde{\Omega}^{c_i}_{>\widehat{r}^+_\lambda}$, $i = 1, 2$, with $\widetilde{\Omega}^{c_1}_{>\widehat{r}^+_\lambda} \subset \widetilde{\Omega}^{c_2}_{>\widehat{r}^+_\lambda}$ by the induction assumption. Also, there enter two connected regions $\widetilde{\mathsf{R}}^{c_i,1}_{\widehat{r}^+_\lambda} \subset \mathsf{S}^{2n-1}_{\widehat{r}^+_\lambda}$, $i = 1, 2$, with $\widetilde{\mathsf{R}}^{c_1,1}_{\widehat{r}^+_\lambda} \subset \widetilde{\mathsf{R}}^{c_2,1}_{\widehat{r}^+_\lambda}$. Their boundaries contain two connected hypersurfaces $\mathsf{N}^{c_i,1}_{\widehat{r}^+_\lambda}$ of $\mathsf{S}^{2n-1}_{\widehat{r}^+_\lambda}$, $i = 1, 2$, which enter C_η as the two caps of the third pic of Figure 19.

By descending the interval $(\widehat{r}_\lambda, \widehat{r}^+_\lambda)$ up to $(\cdot)_{>\widehat{r}_\lambda}$, we get two regions $\widetilde{\mathsf{R}}^{c_i,1}_{\widehat{r}_\lambda}$, $i = 1, 2$, that touch at $\widehat{p}_\lambda$, namely the left cone and the exterior of the right cone in the second pic of Figure 19.

While descending further to $(\cdot)_{>\widehat{r}_\lambda - \varepsilon}$, with $\varepsilon > 0$ very small, the left cone does merge with the right (white) cone. Observe that the points of this (white) cone may be joined continuously to points of the (white) right cap of the first pic, which by hypothesis lies outside $\widetilde{\Omega}^{c_2}_{>\widehat{r}^+_\lambda}$, hence in the same connected component as the points at infinity. Consequently, we cannot prolong the left domain $\widetilde{\Omega}^{c_1}_{>\widehat{r}^+_\lambda}$ so that its prolongation contains the left cone in the slice $\{v = 0\}$ (third pic), because no admissible prolongation would enjoy the relative compactness **(i)** of Proposition 5.3. Hence we have no other choice except to suppress $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda}$ when attaining $(\cdot)_{>\widehat{r}_\lambda}$. We then get a new domain $\widetilde{\Omega}^*_{>\widehat{r}_\lambda}$ defined as $\widetilde{\Omega}^{c_2}_{>\widehat{r}_\lambda}$ minus the closure of $\widetilde{\Omega}^{c_1}_{>\widehat{r}_\lambda}$ (subtraction process), which is checked to be relatively compact in $\mathbb{C}^n$. This domain then descends as a uniquely defined domain $\widetilde{\Omega}^*_{>\widehat{r}^-_\lambda}$ at $(\cdot)_{>\widehat{r}^-_\lambda}$. We also get a corresponding connected region $\widetilde{\mathsf{R}}^*_{\widehat{r}^-_\lambda}$ approximately equal to $\widetilde{\mathsf{R}}^{c_2,1}_{\widehat{r}_\lambda}$ minus the closure of $\widetilde{\mathsf{R}}^{c_1,1}_{\widehat{r}_\lambda}$ whose boundary contains a connectedd $\mathsf{N}^*_{\widehat{r}^-_\lambda}$ (bottom right Figure 21), obtained by merging $\mathsf{N}^{c_1,1}_{\widehat{r}_\lambda}$ with $\mathsf{N}^{c_2,1}_{\widehat{r}_\lambda}$.

The last subcase (e) above is topologically similar to what happens in §5.15, hence the proof of Proposition 5.3 is complete. □

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