

APPLICATIONS OF DIFFERENTIAL ALGEBRA FOR COMPUTING LIE ALGEBRAS OF INFINITESIMAL CR-AUTOMORPHISMS

MASOUD SABZEVARI, AMIR HASHEMI, BENYAMIN M.-ALIZADEH,
AND JOËL MERKER

ABSTRACT. We perform *detailed* computations of Lie algebras of infinitesimal CR-automorphisms associated to three specific model real analytic CR-generic submanifolds in \mathbb{C}^9 by employing differential algebra computer tools — mostly within the MAPLE package `DifferentialAlgebra` — in order to automate the handling of the arising highly complex linear systems of PDE's. Before treating these new examples which prolong previous works of Beloshapka, of Shananina and of Mamai, we provide general formulas for the explicitation of the concerned PDE systems that are valid in arbitrary codimension $k \geq 1$ and in any CR dimension $n \geq 1$. Also, we show how Ritt's reduction algorithm can be adapted to the case under interest, where the concerned PDE systems admit so-called *complex conjugations*.

1. INTRODUCTION

The Lie algebras $\text{aut}_{CR}(M_{\text{model}})$ of infinitesimal CR-automorphisms of various *model* — in the sense of Beloshapka — Cauchy-Riemann (CR) generic submanifolds $M_{\text{model}} \subset \mathbb{C}^{n+k}$ of codimension $k \geq 1$ and of CR dimension $n \geq 1$ are key algebraic features which open the door to a potentially infinite number of new Cartan geometries. Indeed, the knowledge of $\text{aut}_{CR}(M_{\text{model}})$ and of its isotropy subalgebras $\text{aut}_{CR}(M_{\text{model}}, p)$ at points $p \in M_{\text{model}}$ strongly intervenes when one endeavours to build Cartan connections associated to *all geometry-preserving* real analytic deformations $M \subset \mathbb{C}^{n+k}$ of a chosen model. It is well known that procedures due to Cartan and to Tanaka exist to perform such constructions (*see* [11, 6, 13, 1, 19, 26, 25] in a CR context), although the practical outcome appears most of the times to be delicate and unpredictable.

In addition, there has recently been an increasing interest towards complete classification of CR-generic submanifolds according to their algebras of infinitesimal CR-automorphisms. Notably, Beloshapka and Kossovskiy [7] classified homogeneous CR-generic submanifolds $M^4 \subset \mathbb{C}^3$ of CR dimension 1, while a bit before, Fels and Kaup [14] classified the Levi-degenerate homogeneous 2-nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ of dimension five.

Far beyond for what concerns the appearing (co)dimensions, Beloshapka and his school in the last decade devised general procedures to cook up *model CR-generic submanifolds* that, most often, have rigid polynomial defining equations. But when some concrete equation of a CR-generic manifold is given, one unpleasant obstacle happens to be the *complexity* and the *length* of the computations that are required

Date: 2013-2-22.

2000 Mathematics Subject Classification. 68U05, 32M05, 32V40, 12H05.

to attain the full Lie algebras of infinitesimal CR-automorphisms (see [1, 5, 25, 26, 30]), an obstacle which motivates the present work.

By $n \geq 1$ and $k \geq 1$ throughout this paper, we shall mean the CR dimension and the codimension of a real analytic local CR-generic submanifold $M^{2n+k} \subset \mathbb{C}^{n+k}$ passing through a reference point, say through the origin. Beloshapka in [4], introduced a significant class of CR-generic manifolds with several *nice* properties (see e.g. Theorem 14 in [4]), that he denoted by $Q(n, k) \subset \mathbb{C}^{n+k}$ and called *universal models*. He also computed the algebras of infinitesimal CR-automorphisms associated to the simplest model $Q(1, 2)$ in [5], and he derived some interesting stability results. Subsequently, Shananina computed such algebras for the universal models $Q(1, k)$ with $3 \leq k \leq 7$ in [30] and derived expected consequences too (see Theorem 1, Propositions 1 and 2 and Corollary 1 of [30]). Finally, in [21], Mamai studied Lie algebras of infinitesimal CR-automorphisms associated to some universal models $Q(1, k)$ for $8 \leq k \leq 12$, though without presenting details. As far as the authors are aware of, in CR dimension $n = 1$, no higher codimensions have been explored. Understandably, as much as the dimension or codimension of a CR-generic submanifold $M^{2n+k} \subset \mathbb{C}^{n+k}$ increases, the size and the complexity of the corresponding computations of $\text{aut}_{CR}(M)$ grows rapidly, hence an automation would be welcome, even a partial one.

Serendipitously, an important, recently renewed, much related subject has been extensively studied: *Differential (Computer) Algebra*, cf. [9, 10, 15, 20, 28]. There, one employs algebraic tools — like the ones of Gröbner bases theory — in order to solve systems of partial differential equations, or in order to find the Lie symmetries of differential equations, a vast area too. Over the past few years, several relative packages have been developed within various computer algebra systems. For instance, two MAPLE packages were designed in this direction, firstly `DifferentialAlgebra` by Boulier, Lazard, Ollivier, Petitot ([9]), and secondly `diffgrob2` by Mansfield ([22]).

In this paper, we aim to provide an effective algorithm in order to compute the Lie algebras $\text{aut}_{CR}(M_{\text{model}})$ associated to model real analytic CR-generic submanifolds $M_{\text{model}}^{2n+k} \subset \mathbb{C}^{n+k}$, by the valuable means of differential algebra, supplemented by some new operations. For this purpose, we shall denote by `LinCons` the PDE systems that have constant (complex) coefficients, precisely as the ones we shall encounter several times. Since these systems admit complex-valued equations, we equip at first the fundamental *Ritt's reduction theorem* with an operator, which we call the *bar-reduction*, and we extend it to gain the following conclusion, more appropriate to treat the arising complex-valued `LinCons` PDE systems (see Theorem 3.2):

Theorem 1.1. (EXTENDED RITT'S REDUCTION THEOREM) *Consider a differential polynomial ring $R = \mathbb{C}\{u_1, \dots, u_n\}$ over the field of complex numbers, let Θ be the set of its derivation operators, and let ' $>$ ' be a ranking over ΘU . Furthermore, assume that $p \in R$ is a `LinCons` differential polynomial and let Q be a finite set of `LinCons` differential polynomials. Then, there exists $r \in R$, and for each $q \in Q$, there exists $\theta_q, \theta_{\bar{q}} \in \Theta$ and $c_q, c_{\bar{q}} \in K$ satisfying the following conditions:*

- $p = (\sum_{q \in Q} c_q \theta_q q) + (\sum_{q \in Q} c_{\bar{q}} \theta_{\bar{q}} \bar{q}) + r$,
- for each q appearing in this summation we have:

$$\text{rank}(r) < \min_{>} \{ \text{rank}(\theta_q q), \text{rank}(\theta_{\bar{q}} \bar{q}) \},$$

- *no term of r is a derivation of either $\text{rank}(q)$ or $\text{rank}(\bar{q})$ for each $q \in Q$.*

Using this result, we modify the *Rosenfeld-Gröbner* algorithm to an algorithm we call the *LRG* algorithm (Algorithm 3 below) in such a way that it works appropriately in the case of our LinCons complex-valued PDE systems. It is noteworthy that, for the sake of simplicity, we study only *rigid* CR-generic submanifolds $M^{10} \subset \mathbb{C}^{1+8}$ (see Subsection 4.4 for more details), because the arising PDE systems are LinCons; of course, a more general algorithm could be described for non-rigid $M_{\text{model}}^{2n+k} \subset \mathbb{C}^{n+k}$, the PDE system still being linear, though with non-necessarily constant coefficients.

Specifically, we employ our *LRG* algorithm for computing *explicitly* the Lie algebras of infinitesimal CR-automorphisms associated to the following three real analytic rigid CR-generic submanifolds $M^{10} \subset \mathbb{C}^9$, represented in coordinates $(z, w) = (z, w^1, \dots, w^8)$ as the graphs of the following *shared* six defining equations:

$$(1) \quad \begin{aligned} w^1 - \bar{w}^1 &= 2i z \bar{z}, & w^2 - \bar{w}^2 &= 2i (z^2 \bar{z} + z \bar{z}^2), \\ w^3 - \bar{w}^3 &= 2 (z^2 \bar{z} - z \bar{z}^2), & w^4 - \bar{w}^4 &= 2i (z^3 \bar{z} + z \bar{z}^3), \\ w^5 - \bar{w}^5 &= 2 (z^3 \bar{z} - z \bar{z}^3), & w^6 - \bar{w}^6 &= 2i z^2 \bar{z}^2, \end{aligned}$$

together with:

$$\text{First model } \mathbb{M}^1 : \begin{cases} w^7 - \bar{w}^7 = 2i (z^4 \bar{z} + z \bar{z}^4), \\ w^8 - \bar{w}^8 = 2 (z^4 \bar{z} - z \bar{z}^4), \end{cases}$$

$$\text{Second model } \mathbb{M}^2 : \begin{cases} w^7 - \bar{w}^7 = 2i (z^3 \bar{z}^2 + z^2 \bar{z}^3), \\ w^8 - \bar{w}^8 = 2 (z^3 \bar{z}^2 - z^2 \bar{z}^3), \end{cases}$$

$$\text{Third model } \mathbb{M}^3 : \begin{cases} w^7 - \bar{w}^7 = 2i (z^3 \bar{z}^2 + z^2 \bar{z}^3), \\ w^8 - \bar{w}^8 = 2i (z^4 \bar{z} + z \bar{z}^4). \end{cases}$$

For this, we use the MAPLE package `DifferentialAlgebra` to carry out the necessary computations. Nevertheless, these computations involve complex integers and functions and hence it is not possible to perform directly this package in this respect. That is why, we need also to utilize a new reduction called *bar-reduction* that uses complex conjugation to obtain a full remainder (see Section 3). Executing long computations, we achieve the following results. Of course, the main interest of our algorithmic partly automatize approach is to open the door to a wealth of other examples taking inspiration from Beloshapka's universal models.

Theorem 1.2. *The Lie algebras of infinitesimal CR-automorphisms $\text{aut}_{CR}(\mathbb{M}^1)$, $\text{aut}_{CR}(\mathbb{M}^2)$ and $\text{aut}_{CR}(\mathbb{M}^3)$ of the three real analytic generic CR-generic submanifolds $\mathbb{M}^1, \mathbb{M}^2$ and \mathbb{M}^3 of \mathbb{C}^{1+8} are of dimensions 12, 12 and 11, respectively and are generated by the \mathbb{R} -linearly independent real parts of the following collections*

of holomorphic vector fields:

$$(\mathbb{M}^1) \left\{ \begin{array}{l} X_i := \partial_{w_i}, \quad i = 1, \dots, 8, \\ X_9 := z\partial_z + 2w^1\partial_{w^1} + 3w^2\partial_{w^2} + 3w^3\partial_{w^3} + 4w^4\partial_{w^4} + \\ \quad + 4w^5\partial_{w^5} + 4w^6\partial_{w^6} + 5w^7\partial_{w^7} + 5w^8\partial_{w^8}, \\ X_{10} := iz\partial_z - w^3\partial_{w^2} + w^2\partial_{w^3} - 2w^5\partial_{w^4} + 2w^4\partial_{w^5} - 3w^8\partial_{w^7} + 3w^7\partial_{w^8}, \\ X_{11} := \partial_z + 2iz\partial_{w^1} + (4w^1 + 2iz^2)\partial_{w^2} + 2z^2\partial_{w^3} + (3w^2 + 2iz^3)\partial_{w^4} + \\ \quad + (3w^3 + 2z^3)\partial_{w^5} + 2w^2\partial_{w^6} + (4w^4 + 2iz^4)\partial_{w^7} + (4w^5 + 2z^4)\partial_{w^8}, \\ X_{12} := i\partial_z + 2z\partial_{w^1} + 2z^2\partial_{w^2} + (4w^1 - 2iz^2)\partial_{w^3} + (-3w^3 + 2z^3)\partial_{w^4} + \\ \quad + (3w^2 - 2iz^3)\partial_{w^5} + 2w^3\partial_{w^6} + (-4w^5 + 2z^4)\partial_{w^7} + (4w^4 - 2iz^4)\partial_{w^8}; \end{array} \right.$$

$$(\mathbb{M}^2) \left\{ \begin{array}{l} X_i := \partial_{w_i}, \quad i = 1, \dots, 8, \\ X_9 := z\partial_z + 2w^1\partial_{w^1} + 3w^2\partial_{w^2} + 3w^3\partial_{w^3} + 4w^4\partial_{w^4} + \\ \quad + 4w^5\partial_{w^5} + 4w^6\partial_{w^6} + 5w^7\partial_{w^7} + 5w^8\partial_{w^8}, \\ X_{10} := iz\partial_z - w^3\partial_{w^2} + w^2\partial_{w^3} - 2w^5\partial_{w^4} + 2w^4\partial_{w^5} - w^8\partial_{w^7} + w^7\partial_{w^8}, \\ X_{11} := \partial_z + 2iz\partial_{w^1} + (4w^1 + 2iz^2)\partial_{w^2} + 2z^2\partial_{w^3} + (3w^2 + 2iz^3)\partial_{w^4} + \\ \quad + (3w^3 + 2z^3)\partial_{w^5} + 2w^2\partial_{w^6} + (2w^4 + 6w^6)\partial_{w^7} + 2w^5\partial_{w^8}, \\ X_{12} := i\partial_z + 2z\partial_{w^1} + 2z^2\partial_{w^2} + (4w^1 - 2iz^2)\partial_{w^3} + (-3w^3 + 2z^3)\partial_{w^4} + \\ \quad + (3w^2 - 2iz^3)\partial_{w^5} + 2w^3\partial_{w^6} + 2w^5\partial_{w^7} + (-2w^4 + 6w^6)\partial_{w^8}; \end{array} \right.$$

$$(\mathbb{M}^3) \left\{ \begin{array}{l} X_i := \partial_{w_i}, \quad i = 1, \dots, 8, \\ X_9 := z\partial_z + 2w^1\partial_{w^1} + 3w^2\partial_{w^2} + 3w^3\partial_{w^3} + 4w^4\partial_{w^4} + \\ \quad + 4w^5\partial_{w^5} + 4w^6\partial_{w^6} + 5w^7\partial_{w^7} + 5w^8\partial_{w^8}, \\ X_{10} := \partial_z + 2iz\partial_{w^1} + (4w^1 + 2iz^2)\partial_{w^2} + 2z^2\partial_{w^3} + (3w^2 + 2iz^3)\partial_{w^4} + \\ \quad + (3w^3 + 2z^3)\partial_{w^5} + 2w^2\partial_{w^6} + (2w^4 + 6w^6)\partial_{w^7} + (4w^4 + 2iz^4)\partial_{w^8}, \\ X_{11} := i\partial_z + 2z\partial_{w^1} + 2z^2\partial_{w^2} + (4w^1 - 2iz^2)\partial_{w^3} + (-3w^3 + 2z^3)\partial_{w^4} + \\ \quad + (3w^2 - 2iz^3)\partial_{w^5} + 2w^3\partial_{w^6} + 2w^5\partial_{w^7} + (-4w^5 + 2z^4)\partial_{w^8}. \end{array} \right.$$

The paper is organized as follows. Section 2 contains an overview of the necessary background concerning the theory of differential algebras. In Section 3, we present the extended Ritt's reduction algorithm and we show how to utilize the differential algebraic tools to resolve a system of partial differential equations. In Section 4, we present a general method/strategy to compute the Lie algebra of infinitesimal CR-automorphisms of *arbitrary* generic real analytic CR-generic submanifolds $M \subset \mathbb{C}^{n+k}$. Section 5 is devoted to compute in detail the Lie algebra of infinitesimal CR-automorphisms $\text{aut}_{CR}(\mathbb{M}^1)$, associated to the first model \mathbb{M}^1 . However, we do not present the intermediate computations of $\text{aut}_{CR}(\mathbb{M}^2)$ and $\text{aut}_{CR}(\mathbb{M}^3)$ since they are similar to those of \mathbb{M}^1 and offer no new aspect. Tables of Lie brackets appear at the end.

2. DIFFERENTIAL ALGEBRA PRELIMINARIES

In this section, we present a brief overview of basic definitions, notation and results in differential algebra. Two extensive surveys of this subject are: [20, 28].

Definition 2.1. An operator $\delta : R \rightarrow R$ over the algebraic ring R is called a *derivation operator*, if for each $a, b \in R$ we have:

$$\begin{aligned}\delta(a + b) &= \delta(a) + \delta(b) & \text{and} \\ \delta(ab) &= \delta(a)b + a\delta(b).\end{aligned}$$

A *differential ring* is a pair (R, Δ) where R is a ring equipped with a collection $\Delta = \{\delta_1, \dots, \delta_m\}$ of commuting derivations operators over it, satisfying:

$$\delta_i \delta_j a = \delta_j \delta_i a, \quad (i, j = 1 \dots m; a \in R).$$

For simplicity, we suppress the dependence on Δ in the notation and denote a differential ring just by R . If $m = 1$, then R is called an *ordinary* differential ring; otherwise it will be called *partially*. An algebraic ideal I of R is called a *differential ideal* when it is closed under the action of derivations of R , namely $\delta a \in I$ for each $\delta \in \Delta$ and $a \in I$.

Example 2.2. The ring of polynomials $\mathbb{C}[x_1, \dots, x_m]$ over the variables x_1, \dots, x_m with rational coefficients together with the set of operators $\partial/\partial x_1, \dots, \partial/\partial x_m$ is a differential ring.

Let R be a differential ring with $\Delta = \{\delta_1, \dots, \delta_m\}$. Here, we introduce a collection of notations in differential algebra via the following itemized list:

- We denote by Θ the free multiplicative commutative semigroup generated by the elements of Δ , namely

$$\Theta := \{\delta_1^{t_1} \delta_2^{t_2} \dots \delta_m^{t_m} \mid t_1, \dots, t_m \in \mathbb{N}\}$$

Each element $\theta = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$ of Θ is called a *derivation operator* of R and furthermore the sum $\text{ord}(\theta) := \sum_{i=1}^m t_i$ is called the *order* of θ . Then θa is said to be a derivative of $a \in R$ of order $\text{ord}(\theta)$.

- For an arbitrary subset S of R , set $\Theta S := \{\theta s \mid s \in S, \theta \in \Theta\}$. It is the smallest subset of R containing S which is stable under derivation.
- An algebraic ideal of R is called a *differential ideal*, if it is closed under the derivation operators. We denote by (S) and $[S]$ respectively, the smallest algebraic and differential ideals of R containing S . In fact, $[S] = (\Theta S)$. This fact provides an algebraic approach to differential ideals which enables one to employ algebraic means.
- For a field of characteristic zero K , a *differential polynomial ring*:

$$R := K\{u_1, \dots, u_n\} := K[\Theta U]$$

is the usual commutative polynomial ring generated by ΘU over K , where $U := \{u_1, \dots, u_n\}$ is the set of *differential indeterminate*.

- For two certain derivatives θu and ϕu of a same differential indeterminate u , we denote by $\text{lcd}(\theta u, \phi u)$ the *least common derivative* between θu and ϕu , easily seen to be:

$$\text{lcd}(\theta u, \phi u) = \text{lcm}(\theta, \phi)u.$$

In this paper we let K be a differential field of characteristic zero.

Definition 2.3. Let $R = K\{U\}$ be a differential polynomial ring with the set of indeterminates $U = \{u_1, \dots, u_n\}$. A *ranking* $>$ is an ordering over ΘU compatible with the derivation act over ΘU , in the sense that for each derivation $\delta \in \Theta$ and for each $v, w \in \Theta U$ we have:

- $\delta v > v$,
- $v > w \Rightarrow \delta v > \delta w$.

For each $\theta, \phi \in \Theta$ and $v, w \in U$, a ranking $>$ for which the statement $\text{ord}(\theta) > \text{ord}(\phi)$ implies that $\theta v > \phi w$ is called *orderly*. Simultaneously, if the assumption $v > w$ gives $\theta v > \phi w$, then $>$ is called *elimination*. Moreover, for a fixed ranking $>$ over ΘU and for a differential polynomial $p \in R = K\{u_1, \dots, u_n\}$, the *leader* $\text{ld}(p)$ of p is the highest derivative appearing in p with respect to $>$. If $\text{ld}(p) = u$ and d is the degree of u in the expression of p then, the *initial* $\text{in}(p) \in K$ is defined to be the coefficient of u^d in p . Finally, u^d is called the *rank* of p , denoted by $\text{rank}(p)$.

3. DIFFERENTIAL ALGEBRA AND PDE SYSTEMS

Each differential polynomial ring $R = K\{U\}$ can be considered as the conventional polynomial ring $K[\Theta U]$ whose indeterminates are derivations of R . This enables one to use the conventional algebraic tools and get useful information about the differential polynomial ring and its differential ideals. In this section, we employ the *Rosenfeld-Gröbner* algorithm to discuss a system of partial differential equations, using algebraic operations. For a PDE system $\Sigma \subset R$, the Rosenfeld-Gröbner algorithm presents the radical differential ideal generated by Σ as an intersection of a finite number of differential ideals which are called *regular differential ideals*. Those are some differential ideals I represented by a *canonical representative* C , *i.e.* a set of differential polynomials which depends only on I and the given ranking. A canonical representative of the differential ideal I helps to solve ideal membership problem, which is a key computational tool to analyze a PDE system. An implementation of the Rosenfeld-Gröbner algorithm is available in the MAPLE package `diffalg`, and it was recently renovated into the package `DifferentialAlgebra`.

One of the main contributions of this paper is the use of Rosenfeld-Gröbner algorithm — followed by performing some further algebraic manipulations — for considering our PDE systems. It is worth emphasizing that the PDE systems that we deal with in this paper are *linear and admit complex equations*. Then, we have to equip the Rosenfeld-Gröbner algorithm with a certain operator which enables to treat with such systems. Moreover, in the considerably significant class of *rigid CR-generic submanifolds* (*see* the end of section 4 for definition), the under consideration PDE systems are not only complex and linear but also *with constant coefficients* and it is therefore reasonable to consider such systems more seriously. For brevity, let us call this type of systems by *LinCons* systems and also similarly, let us call each linear differential polynomial with constant coefficients *LinCons* polynomial¹. Computation with the Rosenfeld-Gröbner algorithm are comparatively less expensive. Furthermore, in this case *there is no need longer to decompose the*

¹Not every linear differential polynomial is a *LinCons* one, in general. For example, as an element of $\mathbb{C}(x, y)[u, v]$, the polynomial $p := x^2y u_x + 2y v_{xy}$ is linear while its coefficients are not constant.

differential ideal generated by the system into regular differential ideals (a complicated aspect of the general algorithm).

3.1. Extended Ritt's reduction algorithm. Let us therefore recall the Ritt's reduction algorithm, restricted to LinCons differential polynomials. Therefore, the reduction algorithm described here shall be a *weak version of the Ritt's reduction algorithm* in comparison with the version of [28, 20]. Let us recall the definition of partial divisibility for differential polynomials.

Definition 3.1. Consider two LinCons differential polynomials p_1 and p_2 . We say that p_2 reduces p_1 due to the Ritt's reduction algorithm, whenever there exists a certain derivation θ with $\text{rank}(p_1) = \text{rank}(\theta p_2)$. In this case, the result of reduction is:

$$r := p_1 - \frac{\text{in}(p_1)}{\text{in}(p_2)} \theta p_2.$$

When θ is proper, we call r , the *partial remainder* of p_1 on division by p_2 .

One notices that in this definition, if $\text{rank}(p_1) = \text{rank}(p_2)$ then θ must be the identity element of Θ and the Ritt's reduction coincides with the conventional division algorithm for multivariate polynomial rings.

Theorem 3.1. (RITT'S REDUCTION THEOREM) Consider a differential polynomial ring $R = K\{u_1, \dots, u_n\}$ over a field K of characteristic zero, let Θ be the set of derivation operators and let ' $>$ ' be a ranking over ΘU . Furthermore, assume that $p \in R$ is a LinCons differential polynomial and let Q be a finite set of LinCons differential polynomials. Then, there exists $r \in R$, and for each $q \in Q$, there exists $\theta_q \in \Theta$ and $c_q \in K$ satisfying the following conditions:

- $p = (\sum_{q \in Q} c_q \theta_q q) + r$,
- $\text{rank}(r) < \text{rank}(\theta_q q)$, for each q appearing in the summation,
- no term of r is a derivation of $\text{rank}(q)$ for each $q \in Q$.

Here, the differential polynomial $r \in R$ is called the *remainder* of p on division by Q . In order to prove this Theorem, let us display first the Ritt's reduction algorithm ensuing its assertion. Then, the correctness and termination of this algorithm proves Theorem 3.1.

Algorithm 1 RittReduction

Require: $p \in R$, $Q \subset R$; a finite set and $>$; a ranking

Ensure: r ; a remainder of p on division by Q

$h := p$;

$r := 0$;

while $h \neq 0$ **do**

if there is some $q \in Q$ and $\theta \in \Theta$ with $\text{rank}(h) = \text{rank}(\theta q)$ **then**

$h := h - \frac{\text{in}(h)}{\text{in}(q)} \theta q$;

else

$r := r + \text{in}(h) \text{rank}(h)$;

$h := h - \text{in}(h) \text{rank}(h)$;

end if

end while

Return (r)

Proof. The termination of this algorithm follows from the well-ordering property of $>$ (cf. [20]). Namely as one observes, the rank of h decreases as one goes along the steps of the algorithm and hence, it terminates after a finitely many steps, when we will have $h = 0$.

Now let us consider the correctness of the algorithm. For this, we claim that the equality:

$$(5) \quad p = \left(\sum_{q \in Q} c_q \theta_q q \right) + h + r$$

holds at each step of the algorithm. If we prove this claim then, the final value $h = 0$ of h can concludes the assertion. To prove the claim, we consider two cases:

Case 1: If a division occurs by a polynomial, say q_i , then the right hand side of (5) is equal to:

$$\left(\sum_{q \in Q} c_q \theta_q q + c \theta_{q_i} q_i \right) + (h - c \theta_{q_i} q_i) + r$$

with $c = \text{in}(q_i)/\text{in}(h)$. Therefore, it visibly is still equal to p .

Case 2: If no division arises, then the right hand side of (5) is equal to:

$$\sum_{q \in Q} c_q \theta_q q + (h - \text{in}(h) \text{rank}(h)) + (r + \text{in}(h) \text{rank}(h)),$$

and this is also equal to p .

In both cases, the equation (5) is satisfied at each step of the algorithm. Moreover, one convinces oneself that the last two assertions of the Theorem hold according to the structure of this algorithm. \square

One should notice that, as long as the field K is the one of real numbers \mathbb{R} , then the above version of the Ritt's reduction algorithm works as well to compute the remainder of the division of a LinCons differential polynomial by a finite system of LinCons PDE's. However, the coefficients of the PDE systems that we consider in this paper belong to the field $\mathbb{C} = \mathbb{R}(i)$ with $i = \sqrt{-1}$. In this case, we need to perform also the *complex conjugation* to obtain a full remainder. For this, we need the following definition and theorem.

Definition 3.2. Let $R := \mathbb{C}\{u_1, \dots, u_n\}$ be a differential polynomial ring with $u_j = \text{Re}(u_j) + i \text{Im}(u_j)$ for $j = 1, \dots, n$ as the unknown functions. We define the *bar operation* $\bar{\cdot} : R \rightarrow R$ by:

- $\overline{\text{Re}(a) + i \text{Im}(a)} = \text{Re}(a) - i \text{Im}(a)$ for each $a \in \mathbb{C}$,
- $\overline{\text{Re}(u_j) + i \text{Im}(u_j)} = \text{Re}(u_j) - i \text{Im}(u_j)$ for each $j = 1, \dots, n$.

As a matter of fact, the bar operator is compatible with the derivations; namely, for each (real) $\theta \in \Theta$ and each $j = 1, \dots, n$, one has:

$$\overline{\theta u_j} = \theta \bar{u}_j.$$

This allows one to insert the bar reduction operator in the Ritt's reduction algorithm.

Theorem 3.2. (EXTENDED RITT'S REDUCTION THEOREM) *Consider a differential polynomial ring $R = \mathbb{C}\{u_1, \dots, u_n\}$ over the field of complex numbers, let Θ be the set of its derivation operators, and let ' $>$ ' be a ranking over ΘU . Furthermore, assume that $p \in R$ is a LinCons differential polynomial and let Q be a finite*

set of LinCons differential polynomials. Then, there exists $r \in R$, and for each $q \in Q$, there exists $\theta_q, \theta_{\bar{q}} \in \Theta$ and $c_q, c_{\bar{q}} \in K$ satisfying the following conditions:

- $p = \left(\sum_{q \in Q} c_q \theta_q q \right) + \left(\sum_{q \in Q} c_{\bar{q}} \theta_{\bar{q}} \bar{q} \right) + r$,
- for each q appearing in this summation we have:

$$\text{rank}(r) < \min_{>} \{ \text{rank}(\theta_q q), \text{rank}(\theta_{\bar{q}} \bar{q}) \},$$

- no term of r is a derivation of either $\text{rank}(q)$ or $\text{rank}(\bar{q})$ for each $q \in Q$.

Here, the differential polynomial r is called the *full-remainder* of p on division by Q . The proof of this Theorem is similar to that of Theorem 3.1. Let us display the following algorithm like Algorithm 1 extended by the bar reduction.

Algorithm 2 ExtendedRittReduction

Require: $p \in R, Q \subset R$; a finite set and $>$; a ranking

Ensure: r ; a remainder of p on division by Q

$h := p$;

$r := 0$;

while $h \neq 0$ **do**

if there is some $q \in Q$ and $\theta \in \Theta$ with $\text{rank}(h) = \text{rank}(\theta q)$ **then**

$h := h - \frac{\text{in}(h)}{\text{in}(q)} \theta q$;

else

if there is some $q \in Q$ and $\theta \in \Theta$ with $\text{rank}(h) = \text{rank}(\theta \bar{q})$ **then**

$h := h - \frac{\text{in}(h)}{\text{in}(q)} \theta \bar{q}$;

else

$r := r + \text{in}(h) \text{rank}(h)$;

$h := h - \text{in}(h) \text{rank}(h)$;

end if

end if

end while

Return (r)

3.2. Rosenfeld-Gröbner algorithm. As mentioned before, to solve a PDE system Σ , we use the Rosenfeld-Gröbner algorithm to decompose the radical of $[\Sigma]$, into some new PDE systems, presented by explicit generators. These generators have novel properties which leads to do a complete analysis of Σ . The main Rosenfeld-Gröbner algorithm as presented in [10, 9] requires some recursive loops to construct the mentioned decomposition. However, as we consider the Rosenfeld-Gröbner algorithm in the special case of the LinCons PDE systems, *there is no need longer to decompose the differential ideal generated by the system into regular differential ideals*. So, it is convenient to provide an adapted and computationally simpler version of the Rosenfeld-Gröbner algorithm to deal with just LinCons PDEs. Let us call this algorithm by *LRG* which stands for the **Lincons Rosenfeld-Gröbner** algorithm. First, we need the definition of Δ -polynomial — similar in spirit to that in Gröbner bases theory — which plays a crucial role.

Definition 3.3. Consider two LinCons differential polynomials p_1 and p_2 with $\text{ld}(p_i) = \theta_i u_i$, $i = 1, 2$. Then, the Δ -polynomial of p_1 and p_2 is defined as:

$$\Delta(p_1, p_2) = \begin{cases} \text{lc}(p_2) \frac{\theta_{1,2}}{\theta_1} p_1 - \text{lc}(p_1) \frac{\theta_{1,2}}{\theta_2} p_2 & u_1 = u_2, \\ 0 & u_1 \neq u_2, \end{cases}$$

where $\theta_{1,2} = \text{lcd}(\theta_1, \theta_2)$.

The aim of calculating the Δ -polynomial of two differential polynomials is in fact to remove their leading derivatives to obtain (probably) a new leading derivative.

Now, let us describe the LRG algorithm. If $\Sigma \subset R$ is a subset of a differential ring R (in fact, a PDE system) then, $[\Sigma]$ denotes the smallest differential ideal of R , containing Σ and *closed under complex conjugation*.

Algorithm 3 LRG algorithm

Require: Σ ; a finite set of LinCons differential polynomials, $>$; a ranking

Ensure: G ; a canonical representative for $[\Sigma]$

$G := \Sigma$;

$P := \{\{p_1, p_2\} \mid p_1, p_2 \in G\}$;

while $P \neq \{\}$ **do**

 Select and remove $\{p_1, p_2\} \in P$;

$h := \Delta(p_1, p_2)$;

$r := \text{ExtendedRittReduction}(h, G, >)$;

if $r \neq 0$ **then**

$P := P \cup \{\{r, g\} \mid g \in G\}$;

$G := G \cup \{r\}$;

end if

end while

Return (G)

The following theorem shows the termination and correctness of the algorithm.

Theorem 3.3. *The following statements hold:*

- (a) *LRG algorithm terminates in a finite number of steps.*
- (b) *If G is the canonical representative of $[\Sigma]$ then, any full-remainder of a LinCons differential polynomial $p \in R$ on division by G is zero if and only if $p \in [\Sigma]$.*

Proof. (a) The termination of the algorithm is guaranteed by the Ritt-Raudenbush basis Theorem (the analogue of the Hilbert basis theorem for polynomial rings). According to this theorem, since $K = \mathbb{C}$ is Noetherian with respect to the radical differential ideals, then R is too (cf. [20]).

Now, using *reductio ad absurdum*, let us assume that the algorithm does not terminate for a finite set Σ . Thus, we have an ascending chain of ideals $[\Sigma_1] \subset [\Sigma_2] \subset \dots$ which does not stabilize where Σ_i is the set of leading derivatives of the differential polynomials at the i -th step of the execution of the algorithm (by the i -th step we mean computing the i -th new polynomial and adding it to G). One can observe that Σ_i contains LinCons polynomials and therefore $[\Sigma_i]$ is a radical

ideal, namely a radical differential ideal². This contradicts the Noetherianity of R and so, proves the termination.

(b) To prove the correctness of the algorithm, it is sufficient to show that if $p \in [\Sigma]$ is a LinCons differential polynomial then its full-remainder on division by G is zero. Since $p \in [\Sigma]$, then one can conclude by Rosenfeld's Lemma (*see* [20], Chap. III, Sect. 8, Lemma 5), that a partial remainder of p , say p' , on division by G belongs to $(\Sigma \cup \overline{\Sigma})$. On the other hand, the set of Δ -polynomials contains also S-polynomials (*cf.* [3]) — note that the polynomials, under consideration in this paper, are LinCons. Moreover, in EXTENDED RITT REDUCTION algorithm, we consider the complex conjugation of any computed polynomial. These imply that

$$\text{EXTENDED RITT REDUCTION}(p', G, >) = 0,$$

according to the Buchberger's criterion (*see* [3], Theorem 5.48). Therefore, any full-remainder of p will be equal to zero. \square

At the end of this section, let us illustrate with the help of an example, how one can employ the MAPLE package DifferentialAlgebra³ to handle and solve a LinCons PDE system.

Example 3.4. Consider the following LinCons PDE system $\Sigma \subset \mathbb{Q}(x, y)[u, v]$:

$$\Sigma := \begin{cases} \frac{\partial^2}{\partial y^2} u(x, y) - \frac{\partial^2}{\partial x^2} v(x, y) - \frac{\partial^2}{\partial x \partial y} v(x, y) = 0, \\ \frac{\partial^2}{\partial x^2} v(x, y) - \frac{\partial^2}{\partial y^2} v(x, y) + v(x, y) = 0, \\ \frac{\partial^2}{\partial x^2} u(x, y) - \frac{\partial^2}{\partial x \partial y} u(x, y) = 0. \end{cases}$$

where $\mathbb{Q}(x, y)$ is the field of fractions of the polynomial ring $\mathbb{Q}[x, y]$. First, we must call the desired package to make it available:

```
[> with(DifferentialAlgebra);
```

and continue with the following three steps.

Step 1. In this step, we set:

$$\Sigma := \{p_1 := u_{y,y} - v_{x,x} - v_{x,y}, p_2 := v_{x,x} - v_{y,y} + v, p_3 := u_{x,x} - u_{x,y}\}.$$

Furthermore, we define the differential ring $\mathbb{Q}(x, y)[u, v]$ equipped with a ranking as follows.

```
[> R:=DifferentialRing(blocks=[[u, v]], derivations=[x, y]);
```

This ring has u, v as unknown functions of the variables x, y and also, it admits the set of derivations $\Delta = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Moreover, here the picked ranking is orderly, namely the following statement holds:

$$\theta u < \delta v \Leftrightarrow \text{ord } \theta < \text{ord } \delta.$$

for two arbitrary elements δ, θ of $\Theta = \{\frac{\partial^{i+j}}{\partial x^i \partial y^j}, i, j \in \mathbb{N}\}$.

Step 2. Next, we employ the Rosenfeld-Gröbner algorithm to compute a canonical representative for $[\Sigma]$:

```
[> CP := RosenfeldGroebner([p1, p2, p3], R);
```

Now to see the equations of, for example, the first component of CP , we enter the following command:

²One should notice that since the base field contains \mathbb{Q} , then the radical of a differential ideal is a radical differential ideal (*cf.* [29], Proposition 2.1).

³For more details on this package, we refer to <http://www.maplesoft.com/support/help>

```
[> Equations(CP[1]);
```

The result of this line is:

$$\{u_{x,x} - u_{x,y}, u_{y,y} - v_{x,y}, v_{x,x}, v_{y,y} - v\}.$$

Step 3. Finally, we find the formal power series solutions corresponding to the functions u, v around the point $(0, 0)$, up to the order 3:

```
PowerSeriesSolution(CP, order = 3);
```

and as the results it returns:

$$\begin{aligned} u(x, y) &= u(0, 0) + u_y(0, 0)y + u_x(0, 0)x + 1/2v_{x,y}(0, 0)y^2 + \\ &\quad + u_{x,y}(0, 0)xy + 1/2u_{x,y}(0, 0)x^2 + 1/6u_x(0, 0)y^3, \\ v(x, y) &= v(0, 0) + v_y(0, 0)y + v_x(0, 0)x + 1/2v(0, 0)y^2 + \\ &\quad + v_{x,y}(0, 0)xy + 1/6v_y(0, 0)y^3 + 1/2v_x(0, 0)xy^2, \end{aligned}$$

which is in fact the solution set of the above PDE system Σ .

Remark 3.5. It is worth noting that, if a PDE system contains some complex conjugates of unknown functions, then we can use only the MAPLE package `DifferentialAlgebra` and without implementing LRG algorithm to analyze it. For this purpose, we associate to each unknown function a *tag variable* as its complex conjugate, and we add it in the definition of the base differential ring. Furthermore, we add the complex conjugate of each input differential polynomial to Σ . Then, the result of this approach is the same as the output of LRG algorithm. However, the complexity of this computation is growing (and higher than LRG algorithm) when we add a new tag variable.

4. INFINITESIMAL LIE ALGEBRA OF REAL ANALYTIC CR-GENERIC SUBMANIFOLDS

4.1. Effective tangency equations. Hereafter, by $M \subset \mathbb{C}^{n+k}$ we mean a real analytic CR-generic submanifold of CR-dimension n and of codimension k ; recall ([2, 24]) that *CR-genericity* means that $TM + JTM = T\mathbb{C}^{n+k}|_M$, where J is the standard complex structure (multiplication by i). In order to compute $\text{aut}_{CR}(M)$ for an explicitly given such submanifold $M \subset \mathbb{C}^{n+k}$, it is most convenient to work with *complex defining equations* of the specific shape ([24, 1, 25]):

$$\bar{w}_j + w_j = \bar{\Xi}_j(\bar{z}, z, w) \quad (j = 1 \dots k),$$

where the coordinates $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_k)$ are centered at the origin $0 \in M$ and where $T_0M = \{\bar{w}_j + w_j = 0: j = 1, \dots, k\}$, so that all $\bar{\Xi}_j = O(2)$.

A general $(1, 0)$ vector field having holomorphic coefficients:

$$X = \sum_{i=1}^n Z^i(z, w) \frac{\partial}{\partial z_i} + \sum_{l=1}^k W^l(z, w) \frac{\partial}{\partial w_l}$$

is called an *infinitesimal CR-automorphism* of M , whenever it is *tangent* to it, namely whenever $(X + \bar{X})|_M \equiv 0$. Concretely and more precisely, the condition that a holomorphic vector field X belongs to $\text{aut}_{CR}(M)$ means that for $j = 1, \dots, k$,

the differentiated equation:

$$\begin{aligned}
0 &= (\bar{X} + X) [\bar{w}_j + w_j - \bar{\Xi}_j(\bar{z}, z, w)] = \\
&= \bar{X} [\bar{w}_j + w_j - \bar{\Xi}_j(\bar{z}, z, w)] + X [\bar{w}_j + w_j - \bar{\Xi}_j(\bar{z}, z, w)] \\
&= \bar{W}^j(\bar{z}, \bar{w}) - \sum_{i=1}^n \bar{Z}^i(\bar{z}, \bar{w}) \frac{\partial \bar{\Xi}_j}{\partial \bar{z}_i}(\bar{z}, z, w) + \\
&\quad + W^j(z, w) - \sum_{i=1}^n Z^i(z, w) \frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) - \sum_{l=1}^k W^l(z, w) \frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w)
\end{aligned}$$

should vanish for every $(z, w) \in M$. On the other hand, this condition holds if and only if, after extrinsic complexification and replacement of \bar{w} by $-w + \bar{\Xi}(\bar{z}, z, w)$, the k power series obtained in $\mathbb{C}\{\bar{z}, z, w\}$ vanish identically, and this yields the tangency equations:

$$\begin{aligned}
(6) \quad 0 &\equiv \bar{W}^j(\bar{z}, -w + \bar{\Xi}(\bar{z}, z, w)) - \sum_{i=1}^n \bar{Z}^i(\bar{z}, -w + \bar{\Xi}(\bar{z}, z, w)) \frac{\partial \bar{\Xi}_j}{\partial \bar{z}_i}(\bar{z}, z, w) + \\
&\quad + W^j(z, w) - \sum_{i=1}^n Z^i(z, w) \frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) - \sum_{l=1}^k W^l(z, w) \frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w) \\
&\quad (j=1, \dots, k).
\end{aligned}$$

Proposition 4.1. *The holomorphic vector field:*

$$X = \sum_{i=1}^n Z^i(z, w) \frac{\partial}{\partial z_i} + \sum_{l=1}^k W^l(z, w) \frac{\partial}{\partial w_l}$$

is an infinitesimal CR-automorphism of a generic real analytic CR-generic submanifold $M \subset \mathbb{C}^{n+k}$ represented in coordinates $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_k)$ as the graph of the k complex defining functions:

$$w_j + \bar{w}_j = \bar{\Xi}_j(\bar{z}, z, w) \quad (j=1, \dots, k)$$

if and only if its coefficients $Z^i(z, w)$ and $W^l(z, w)$ satisfy the k equations (6).

4.2. General formulae. According to Proposition 4.1, to find the infinitesimal CR-automorphisms X associated to a CR-generic submanifold M of \mathbb{C}^{n+k} , it is necessary and sufficient to determine the holomorphic functions $Z^i(z, w)$ and $W^l(z, w)$ satisfying the tangency equations (6). Then, the main question that immediately arises here is to ask: *How can one specify such functions?* To answer this question, we focus our attention on providing an effective algorithm. To this aim, first we introduce the expansions of the coefficients of such a sought X with respect to the powers of z :

$$(7) \quad Z^i(z, w) = \sum_{\alpha \in \mathbb{N}^n} z^\alpha Z^{i, \alpha}(w) \quad \text{and} \quad W^l(z, w) = \sum_{\alpha \in \mathbb{N}^n} z^\alpha W^{l, \alpha}(w),$$

where the $Z^{i, \alpha}(w)$ and the $W^{l, \alpha}(w)$ are local holomorphic functions. We will show that the identical vanishing of the k equations (6) in $\mathbb{C}\{\bar{z}, z, w\}$ is equivalent to a certain (in general complicated) linear system of partial differential equations involving the $\frac{\partial^{|\gamma|} Z^{k, \alpha}}{\partial w^\gamma}(w)$, the $\frac{\partial^{|\gamma'|} Z^{k', \alpha'}}{\partial w^{\gamma'}}(w)$, the $\frac{\partial^{|\gamma''|} W^{l, \alpha''}}{\partial w^{\gamma''}}(w)$ and the $\frac{\partial^{|\gamma'''}| W^{l', \alpha'''}}{\partial w^{\gamma'''}}(w)$.

Inserting these expansions with respect to the powers of z in the tangency equations (6), we get:

$$0 \equiv \sum_{\alpha \in \mathbb{N}^n} \bar{z}^\alpha \bar{W}^{j,\alpha}(-w + \bar{\Xi}) - \sum_{i=1}^n \sum_{\alpha \in \mathbb{N}^n} \bar{z}^\alpha \bar{Z}^{i,\alpha}(-w + \bar{\Xi}) \frac{\partial \bar{\Xi}_j}{\partial \bar{z}_i}(\bar{z}, z, w) + \\ + \sum_{\beta \in \mathbb{N}^n} z^\beta W^{j,\beta}(w) - \sum_{i=1}^n \sum_{\beta \in \mathbb{N}^n} z^\beta Z^{i,\beta}(w) \frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) - \sum_{l=1}^k \sum_{\beta \in \mathbb{N}^n} z^\beta W^{l,\beta}(w) \frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w) \\ (j = 1 \dots k).$$

Since in these equations, w is the argument both of all the $Z^{i,\beta}$ and of all the $W^{l,\beta}$ appearing in the second line, one should arrange that the same argument w takes place inside the functions $\bar{W}^{j,\alpha}$ and $\bar{Z}^{i,\alpha}$ appearing in the first line. Thus, one is led, for an arbitrary converging holomorphic power series $\bar{A} = \bar{A}(w) = \sum_{\gamma \in \mathbb{N}^k} \frac{\partial^{|\gamma|} \bar{A}}{\partial w^\gamma}(0) w^\gamma$, to apply the well known basic infinite Taylor series formulae under the following slightly artificial form:

$$(8) \quad \begin{aligned} \bar{A}(-w + \bar{\Xi}) &= \bar{A}(w + (-2w + \bar{\Xi})) \\ &= \sum_{\gamma \in \mathbb{N}^k} \frac{\partial^{|\gamma|} \bar{A}}{\partial w^\gamma}(w) \frac{1}{\gamma!} (-2w + \bar{\Xi}(\bar{z}, z, w))^\gamma. \end{aligned}$$

When one does this, one transforms the first lines of the previous k equations as follows:

$$(9) \quad \begin{aligned} 0 \equiv \sum_{\alpha \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^k} \frac{1}{\gamma!} \bar{z}^\alpha (-2w + \bar{\Xi}(\bar{z}, z, w))^\gamma \frac{\partial^{|\gamma|} \bar{W}^{j,\alpha}}{\partial w^\gamma}(w) - \\ - \sum_{i=1}^n \sum_{\alpha \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^k} \frac{1}{\gamma!} \bar{z}^\alpha (-2w + \bar{\Xi}(\bar{z}, z, w))^\gamma \frac{\partial^{|\gamma|} \bar{Z}^{i,\alpha}}{\partial w^\gamma}(w) + \\ + \sum_{\beta \in \mathbb{N}^n} z^\beta W^{j,\beta}(w) - \sum_{i=1}^n \sum_{\beta \in \mathbb{N}^n} z^\beta Z^{i,\beta}(w) \frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) - \sum_{l=1}^k \sum_{\beta \in \mathbb{N}^n} z^\beta W^{l,\beta}(w) \frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w) \\ (j = 1 \dots k). \end{aligned}$$

But still, we must expand and reorganize everything in terms of the powers $\bar{z}^\alpha z^\beta$ of (\bar{z}, z) . At first, we must do this for the multipowers:

$$(-2w + \bar{\Xi}(\bar{z}, z, w))^\gamma = \prod_{j=1}^k \left(-2w_j + \bar{\Xi}_j(\bar{z}, z, w) \right)^{\gamma_j}.$$

4.3. Expansion, reorganization and associated LinCons PDE system. To begin with, let us denote the (\bar{z}, z) -power series expansion of $-2w_j + \bar{\Xi}_j$ by:

$$(10) \quad -2w_j + \bar{\Xi}_j(\bar{z}, z, w) = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \bar{\Xi}_{j,\alpha,\beta}^\sim(w) \quad (j = 1 \dots k),$$

with the understanding that the coefficients of the expansion of $\bar{\Xi}_j$ would be denoted plainly $\bar{\Xi}_{j,\alpha,\beta}(w)$, without \sim sign. Reminding $\bar{\Xi}_j = O(2)$, we adopt the convention that in this right-hand side, the $\bar{\Xi}_{j,\alpha,\beta}^\sim(w)$ for $\alpha = \beta = 0$ comes not from $\bar{\Xi}_j$ itself, but just from the supplementary first-order term $-2w_j$.

Thus, denoting:

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in \mathbb{N}^k,$$

we may expand explicitly the exponentiated product under consideration, and the intermediate, detailed computations read as follows:

$$\begin{aligned}
 & \prod_{j=1}^k (-2w_j + \Xi_j(\bar{z}, z, w))^{\gamma_j} = \\
 & = \prod_{j=1}^k \left(\sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \Xi_{j,\alpha,\beta}^{\sim}(w) \right)^{\gamma_j} \\
 & = \prod_{j=1}^k \left[\sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left(\sum_{\substack{\alpha_1 + \dots + \alpha_{\gamma_j} = \alpha \\ \beta_1 + \dots + \beta_{\gamma_j} = \beta}} \Xi_{j,\alpha_1,\beta_1}^{\sim}(w) \cdots \Xi_{j,\alpha_{\gamma_j},\beta_{\gamma_j}}^{\sim}(w) \right) \right] \\
 & = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left[\sum_{\substack{\alpha^1 + \dots + \alpha^k = \alpha \\ \beta^1 + \dots + \beta^k = \beta}} \sum_{\substack{\alpha_1^1 + \dots + \alpha_1^k = \alpha^1 \\ \beta_1^1 + \dots + \beta_1^k = \beta^1}} \cdots \sum_{\substack{\alpha_1^k + \dots + \alpha_1^k = \alpha^k \\ \beta_1^k + \dots + \beta_1^k = \beta^k}} \right. \\
 & \quad \left. \Xi_{1,\alpha_1^1,\beta_1^1}^{\sim}(w) \cdots \Xi_{1,\alpha_1^1,\beta_1^1}^{\sim}(w) \cdots \cdots \Xi_{k,\alpha_1^k,\beta_1^k}^{\sim}(w) \cdots \Xi_{k,\alpha_1^k,\beta_1^k}^{\sim}(w) \right] \\
 & =: \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \mathcal{A}_{\alpha,\beta,\gamma} \left(\left\{ \Xi_{j,\hat{\alpha},\hat{\beta}}^{\sim}(w) \right\}_{j \in \mathbb{N}, \hat{\alpha} \in \mathbb{N}^n, \hat{\beta} \in \mathbb{N}^n} \right),
 \end{aligned}$$

where we introduce a collection of certain polynomial functions $\mathcal{A}_{\alpha,\beta,\gamma}$ of all the $\Xi_{j,\hat{\alpha},\hat{\beta}}^{\sim}(w)$ that appear naturally in the large brackets of the penultimate equality, namely where we set:

$$\begin{aligned}
 \mathcal{A}_{\alpha,\beta,\gamma} \left(\left\{ \Xi_{j,\hat{\alpha},\hat{\beta}}^{\sim}(w) \right\}_{j \in \mathbb{N}, \hat{\alpha} \in \mathbb{N}^n, \hat{\beta} \in \mathbb{N}^n} \right) := & \sum_{\substack{\alpha^1 + \dots + \alpha^k = \alpha \\ \beta^1 + \dots + \beta^k = \beta}} \sum_{\substack{\alpha_1^1 + \dots + \alpha_1^k = \alpha^1 \\ \beta_1^1 + \dots + \beta_1^k = \beta^1}} \cdots \sum_{\substack{\alpha_1^k + \dots + \alpha_1^k = \alpha^k \\ \beta_1^k + \dots + \beta_1^k = \beta^k}} \\
 & \Xi_{1,\alpha_1^1,\beta_1^1}^{\sim}(w) \cdots \Xi_{1,\alpha_1^1,\beta_1^1}^{\sim}(w) \cdots \cdots \Xi_{k,\alpha_1^k,\beta_1^k}^{\sim}(w) \cdots \Xi_{k,\alpha_1^k,\beta_1^k}^{\sim}(w).
 \end{aligned}$$

At present, coming back to the k equations (9) we left momentarily untouched, we see that in them, five sums are extant and we now want to expand and to reorganize properly each one of these sums as a (\bar{z}, z) -power series of the form:

$$\sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta (\text{coeff}_{j,\alpha,\beta}).$$

For the sum in (9), we therefore compute, changing in advance the index α to α' :

$$\begin{aligned}
 & \sum_{\alpha' \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^k} \frac{1}{\gamma!} \bar{z}^{\alpha'} \frac{\partial^{|\gamma|} \overline{W}^{j,\alpha'}}{\partial w^\gamma}(w) (-2w + \Xi(\bar{z}, z, w))^\gamma = \\
 (11) \quad & = \sum_{\alpha' \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \frac{1}{\gamma!} \bar{z}^{\alpha'} \frac{\partial^{|\gamma|} \overline{W}^{j,\alpha'}}{\partial w^\gamma}(w) \sum_{\alpha'' \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^{\alpha''} z^\beta \mathcal{A}_{\alpha'',\beta,\gamma} \left(\left\{ \Xi_{j,\hat{\alpha},\hat{\beta}}^{\sim}(w) \right\} \right) \\
 & = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left[\sum_{\gamma \in \mathbb{N}^k} \sum_{\alpha = \alpha' + \alpha''} \frac{1}{\gamma!} \mathcal{A}_{\alpha'',\beta,\gamma} \left(\left\{ \Xi_{j,\hat{\alpha},\hat{\beta}}^{\sim}(w) \right\} \right) \cdot \frac{\partial^{|\gamma|} \overline{W}^{j,\alpha'}}{\partial w^\gamma}(w) \right].
 \end{aligned}$$

The computations for the second sum in (9) are the same:

$$\begin{aligned}
(12) \quad & - \sum_{i=1}^n \sum_{\alpha' \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^k} \frac{1}{\gamma!} \bar{z}^{\alpha'} \frac{\partial^{|\gamma|} \bar{Z}^{i, \alpha'}}{\partial w^\gamma}(w) (-2w + \bar{\Xi}(\bar{z}, z, w))^\gamma \\
& = - \sum_{i=1}^n \sum_{\alpha' \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^k} \frac{1}{\gamma!} \bar{z}^{\alpha'} \frac{\partial^{|\gamma|} \bar{Z}^{i, \alpha'}}{\partial w^\gamma}(w) \sum_{\alpha'' \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^{\alpha''} z^\beta \mathcal{A}_{\alpha'', \beta, \gamma}(\{\bar{\Xi}_{\tilde{j}, \hat{\alpha}, \hat{\beta}}(w)\}) \\
& = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left[- \sum_{i=1}^n \sum_{\gamma \in \mathbb{N}^k} \sum_{\alpha' + \alpha'' = \alpha} \frac{1}{\gamma!} \mathcal{A}_{\alpha'', \beta, \gamma}(\{\bar{\Xi}_{\tilde{j}, \hat{\alpha}, \hat{\beta}}(w)\}) \cdot \frac{\partial^{|\gamma|} \bar{Z}^{i, \alpha'}}{\partial w^\gamma}(w) \right].
\end{aligned}$$

The third sum in (9) is already almost well written, for we indeed have, if we denote by $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^n$ the zero-multiindex:

$$(13) \quad \sum_{\beta \in \mathbb{N}^n} z^\beta W^{j, \beta}(w) = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta [\delta_\alpha^{\mathbf{0}} \cdot W^{j, \beta}(w)],$$

where $\delta_a^b = 0$ if $a \neq b$ and equals 1 if $a = b$. To transform the fourth sum in (9), we must at first compute, for each $i = 1, \dots, n$ (and for each $j = 1, \dots, k$), the first-order partial derivatives $\frac{\partial \bar{\Xi}_j}{\partial z_i}$ from (10), which gives, if we denote simply by $\mathbf{1}_i$ the multiindex $(0, \dots, 1, \dots, 0)$ of \mathbb{N}^n with 1 at the i -th place and zero elsewhere:

$$\begin{aligned}
\frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) & = \sum_{\alpha \in \mathbb{N}^n} \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta_i \geq 1}} \bar{z}^\alpha \beta_i z^{\beta - \mathbf{1}_i} \bar{\Xi}_{j, \alpha, \beta}^\sim(w) \\
& = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta (\beta_i + 1) \bar{\Xi}_{j, \alpha, \beta + \mathbf{1}_i}^\sim(w).
\end{aligned}$$

Thanks to this, the fourth sum in (9) may be reorganized as wanted:

$$\begin{aligned}
(14) \quad & - \sum_{i=1}^n \sum_{\beta' \in \mathbb{N}^n} z^{\beta'} Z^{i, \beta'}(w) \frac{\partial \bar{\Xi}_j}{\partial z_i}(\bar{z}, z, w) = \\
& = - \sum_{i=1}^n \sum_{\beta' \in \mathbb{N}^n} z^{\beta'} Z^{i, \beta'}(w) \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta'' \in \mathbb{N}^n} \bar{z}^\alpha z^{\beta''} (1 + \beta''_i) \bar{\Xi}_{j, \alpha, \beta'' + \mathbf{1}_i}^\sim(w) \\
& = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left[- \sum_{i=1}^n \sum_{\beta' + \beta'' = \beta} (\beta''_i + 1) \bar{\Xi}_{j, \alpha, \beta'' + \mathbf{1}_i}^\sim(w) \cdot Z^{i, \beta'}(w) \right].
\end{aligned}$$

Lastly, in order to transform the fifth sum in (9), we must at first compute, for each $l = 1, \dots, k$ (and for each $j = 1, \dots, k$), the first-order partial derivatives $\frac{\partial \bar{\Xi}_j}{\partial w_l}$, and to this aim, we start by rewriting from (10):

$$\bar{\Xi}_j(\bar{z}, z, w) = 2w_j + \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \bar{\Xi}_{j, \alpha, \beta}^\sim(w),$$

whence it immediately follows:

$$\frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w) = 2\delta_j^l + \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \frac{\partial \bar{\Xi}_{j, \alpha, \beta}^\sim(w)}{\partial w_l}.$$

Thanks to this, the fifth sum in (9), too, may be reorganized appropriately:

$$\begin{aligned}
 & - \sum_{l=1}^k \sum_{\beta' \in \mathbb{N}^n} z^{\beta'} W^{l, \beta'}(w) \frac{\partial \bar{\Xi}_j}{\partial w_l}(\bar{z}, z, w) = \\
 (15) \quad & = - \sum_{l=1}^k \sum_{\beta' \in \mathbb{N}^n} z^{\beta'} W^{l, \beta'}(w) \left[2 \delta_j^l + \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta'' \in \mathbb{N}^n} \bar{z}^\alpha z^{\beta''} \frac{\partial \bar{\Xi}_{j, \alpha, \beta''}^{\sim}(w)}{\partial w_l} \right] \\
 & = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \left[-2 \delta_\alpha^0 \cdot W^{j, \beta}(w) - \sum_{l=1}^k \sum_{\beta' + \beta'' = \beta} \frac{\partial \bar{\Xi}_{j, \alpha, \beta''}^{\sim}(w)}{\partial w_l} W^{l, \beta'}(w) \right].
 \end{aligned}$$

Summing up these five reorganized sums appearing in (9) as a double sum $\sum_{\alpha} \sum_{\beta} \bar{z}^\alpha z^\beta (\text{coeff}_{j, \alpha, \beta})$, and equating to zero all the obtained coefficients (11), (12), (13), (14) and (15), we deduce the following fundamental statement.

Theorem 4.1. *Let M be a local generic real analytic CR-generic submanifold of \mathbb{C}^{n+k} having positive codimension $k \geq 1$ and positive CR dimension $n \geq 1$ which is represented, in local holomorphic coordinates $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_k)$ centered at the origin $0 \in M$ by k complex defining equations of the shape:*

$$\bar{w}_j + w_j = \bar{\Xi}_j(\bar{z}, z, w) \quad (j=1 \dots k),$$

with $\bar{\Xi}_j = O(2)$, and introduce the power series expansion with respect to the variables (\bar{z}, z) :

$$-2w_j + \bar{\Xi}_j(\bar{z}, z, w) =: \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \bar{z}^\alpha z^\beta \bar{\Xi}_{j, \alpha, \beta}^{\sim}(w) \quad (j=1 \dots k).$$

For every multiindex $\alpha \in \mathbb{N}^n$, every multiindex $\beta \in \mathbb{N}^n$ and every multiindex $\gamma \in \mathbb{N}^k$, introduce also the explicit universal polynomial:

$$\begin{aligned}
 \mathcal{A}_{\alpha, \beta, \gamma} \left(\{ \bar{\Xi}_{\hat{j}, \hat{\alpha}, \hat{\beta}}^{\sim}(w) \}_{\hat{j} \in \mathbb{N}, \hat{\alpha} \in \mathbb{N}^n, \hat{\beta} \in \mathbb{N}^n} \right) & := \sum_{\substack{\alpha^1 + \dots + \alpha^k = \alpha \\ \beta^1 + \dots + \beta^k = \beta}} \sum_{\substack{\alpha_1^1 + \dots + \alpha_{\gamma_1}^1 = \alpha^1 \\ \beta_1^1 + \dots + \beta_{\gamma_1}^1 = \beta^1}} \dots \sum_{\substack{\alpha_1^k + \dots + \alpha_{\gamma_k}^k = \alpha^k \\ \beta_1^k + \dots + \beta_{\gamma_k}^k = \beta^k}} \\
 & \bar{\Xi}_{1, \alpha_1^1, \beta_1^1}^{\sim}(w) \dots \bar{\Xi}_{1, \alpha_{\gamma_1}^1, \beta_{\gamma_1}^1}^{\sim}(w) \dots \bar{\Xi}_{k, \alpha_1^k, \beta_1^k}^{\sim}(w) \dots \bar{\Xi}_{k, \alpha_{\gamma_k}^k, \beta_{\gamma_k}^k}^{\sim}(w).
 \end{aligned}$$

Then a general holomorphic vector field:

$$X = \sum_{i=1}^n Z^i(z, w) \frac{\partial}{\partial z_i} + \sum_{l=1}^k W^l(z, w) \frac{\partial}{\partial w_l}$$

is an infinitesimal CR-automorphism of M belonging to $\text{aut}_{CR}(M)$, namely it has the property that $\bar{X} + X$ is tangent to M if and only if, for every $j = 1, \dots, k$, for every $\alpha \in \mathbb{N}^n$ and for every $\beta \in \mathbb{N}^n$, the following linear holomorphic partial

differential equation:

$$\begin{aligned}
(16) \quad 0 \equiv & \sum_{\gamma \in \mathbb{N}^k} \sum_{\alpha = \alpha' + \alpha''} \frac{1}{\gamma!} \mathcal{A}_{\alpha'', \beta, \gamma} \left(\{ \widetilde{\Xi}_{\hat{j}, \hat{\alpha}, \hat{\beta}}(w) \}_{\hat{j} \in \mathbb{N}, \hat{\alpha} \in \mathbb{N}^n, \hat{\beta} \in \mathbb{N}^n} \right) \cdot \frac{\partial^{|\gamma|} \overline{W}^{j, \alpha'}}{\partial w^\gamma} (w) - \\
& - \sum_{i=1}^n \sum_{\gamma \in \mathbb{N}^k} \sum_{\alpha' + \alpha'' = \alpha} \frac{1}{\gamma!} \mathcal{A}_{\alpha'', \beta, \gamma} \left(\{ \widetilde{\Xi}_{\hat{j}, \hat{\alpha}, \hat{\beta}}(w) \} \right) \cdot \frac{\partial^{|\gamma|} \overline{Z}^{i, \alpha'}}{\partial w^\gamma} (w) + \\
& + \delta_\alpha^0 \cdot W^{j, \beta} (w) - \\
& - \sum_{i=1}^n \sum_{\beta' + \beta'' = \beta} (\beta'' + 1) \widetilde{\Xi}_{j, \alpha, \beta'' + \mathbf{1}_i}(w) \cdot Z^{i, \beta'} (w) - \\
& - 2 \delta_\alpha^0 \cdot W^{j, \beta} (w) - \sum_{l=1}^k \sum_{\beta' + \beta'' = \beta} \frac{\partial \widetilde{\Xi}_{j, \alpha, \beta''}(w)}{\partial w_l} W^{l, \beta'} (w)
\end{aligned}$$

which is linear with respect to the partial derivatives:

$$\frac{\partial^{|\gamma|} Z^{i, \alpha}}{\partial w^\gamma} (w), \quad \frac{\partial^{|\gamma'|} \overline{Z}^{i', \alpha'}}{\partial w^{\gamma'}} (w), \quad \frac{\partial^{|\gamma''|} W^{l, \alpha''}}{\partial w^{\gamma''}} (w), \quad \frac{\partial^{|\gamma'''} \overline{W}^{l', \alpha'''}}{\partial w^{\gamma'''}} (w),$$

together with its conjugate, are satisfied identically in $\mathbb{C}\{w\}$ by the four families of functions:

$$Z^{i, \alpha}(w), \quad \overline{Z}^{i', \alpha'}(w), \quad W^{l, \alpha''}(w), \quad \overline{W}^{l', \alpha'''}(w).$$

depending only upon the k holomorphic variables (w_1, \dots, w_k) .

4.4. The main strategy. According to Theorem 4.1, finding the sought infinitesimal CR-automorphisms X is equivalent to solve the linear PDE system constructed by the equations (16) with the unknowns $Z^{i, \alpha}, \overline{Z}^{i', \alpha'}, W^{l, \alpha''}, \overline{W}^{l', \alpha'''}$ and afterwards, finding the expressions of the holomorphic coefficients $Z^i(z, w)$ and $W^l(z, w)$ of X for $i = 1, \dots, n$ and $l = 1, \dots, k$, according to the formulae (7) and thanks to the achieved solution. Then, we can choose the following strategy to compute the desired infinitesimal Lie algebra $\text{aut}_{CR}(M)$ associated to a specific real analytic CR-generic manifold $M \subset \mathbb{C}^{n+k}$:

- Constructing the k fundamental equations (6).
- Expanding these equations according to the formulae (7), (8) and (10).
- Extracting the coefficients of each $z^\alpha \overline{z}^\beta$ for $\alpha, \beta \in \mathbb{N}^n$ and constituting the linear homogeneous PDE system of the partial differential equations (16), introduced in Theorem 4.1.
- Solving the obtained system by the means of techniques in differential algebra theory.
- Substituting the solution of the PDE system into the formulae (7) to find the holomorphic functions $Z^{i, \alpha}(w)$ and $W^{l, \alpha''}(w)$ as the coefficients of the infinitesimal CR-automorphisms X .

It is worth noting that the already introduced linear PDE system defined underlying the differential ring:

$$R := \mathbb{C}(w) \left[Z^{i, \alpha}, \overline{Z}^{i', \alpha'}, W^{l, \alpha''}, \overline{W}^{l', \alpha'''} \right]$$

admits *complex* linear equations and hence at the fourth step of the above strategy, it is not possible to employ the Rosenfeld-Gröbner algorithm directly for solving it. But, it is feasible to equip this algorithm with the conjugate operator bar-reduction

for being able to treat the complex equations. Nevertheless, one should notice that such a linear PDE system is not necessarily a LinCons one, since its coefficients may contain some powers of the variables $w_l, l = 1, \dots, k$. However, a careful look at the equations (10) and (16) reveals that if the k defining functions $\bar{\Xi}_i$ are *independent* of the variables w_1, \dots, w_k , then the concerned linear PDE system has constant complex coefficients, namely it will be a LinCons PDE system. Such CR-generic submanifolds $M \subset \mathbb{C}^{n+k}$ which are defined as the graph of the k defining complex equations:

$$\bar{w}_j + w_j = \bar{\Xi}_j(\bar{z}, z) \quad (j=1 \dots k)$$

are usually called the *rigid submanifolds*. They constitute a wide and considerably significant class of CR-generic submanifolds (see [2, 24, 5, 8, 25, 26, 30, 31]). Hence, in the case of the rigid real analytic generic CR-generic submanifolds, one can employ the computationally much simpler algorithm *LRG* (Algorithm 3) to perform the fourth step of the above strategy. In the sequel, using this method, we compute the Lie algebras of infinitesimal CR-automorphisms associated to three rigid real analytic generic CR-generic submanifolds of \mathbb{C}^{1+8} .

5. LIE ALGEBRA OF INFINITESIMAL CR-AUTOMORPHISMS OF CR-GENERIC SUBMANIFOLDS $\mathbb{M}^1, \mathbb{M}^2$ AND \mathbb{M}^3 OF \mathbb{C}^{1+8}

The aim of the current section is to compute the Lie algebras of infinitesimal CR-automorphisms, associated to the three rigid real analytic CR-generic submanifolds $\mathbb{M}^1, \mathbb{M}^2$ and \mathbb{M}^3 of \mathbb{C}^{1+8} of CR dimension 1, represented in coordinates (z, w_1, \dots, w_8) by (1). We give in detail the computations of $\text{aut}_{CR}(\mathbb{M}^1)$ and since the remaining computations of $\text{aut}_{CR}(\mathbb{M}^2)$ and $\text{aut}_{CR}(\mathbb{M}^3)$ are fairly similar, then shall not report them in detail.

For the first model \mathbb{M}^1 , represented by the eight real analytic equations:

$$(17) \quad \begin{aligned} w^1 - \bar{w}^1 &= \Xi_1(z, \bar{z}) := 2i z \bar{z}, & w^2 - \bar{w}^2 &= \Xi_2(z, \bar{z}) := 2i (z^2 \bar{z} + z \bar{z}^2), \\ w^3 - \bar{w}^3 &= \Xi_3(z, \bar{z}) := 2 (z^2 \bar{z} - z \bar{z}^2), & w^4 - \bar{w}^4 &= \Xi_4(z, \bar{z}) := 2i (z^3 \bar{z} + z \bar{z}^3), \\ w^5 - \bar{w}^5 &= \Xi_5(z, \bar{z}) := 2 (z^3 \bar{z} - z \bar{z}^3), & w^6 - \bar{w}^6 &= \Xi_6(z, \bar{z}) := 2i z^2 \bar{z}^2, \\ w^7 - \bar{w}^7 &= \Xi_7(z, \bar{z}) := 2i (z^4 \bar{z} + z \bar{z}^4), & w^8 - \bar{w}^8 &= \Xi_8(z, \bar{z}) := 2 (z^4 \bar{z} - z \bar{z}^4), \end{aligned}$$

a holomorphic vector field $X := Z(z, w) \partial_z + \sum_{l=1}^8 W^l(z, w) \partial_{w_l}$ is tangent to \mathbb{M}^1 if and only if the restriction of the real vector field $X = X + \bar{X}$ to each of the above eight defining equations vanishes identically. In other words, if and only if the holomorphic functions Z and $W^l, l = 1, \dots, 8$, and their conjugates enjoy the

following eight equations (cf. (6)):

$$\begin{aligned}
(18) \quad & 0 \equiv [W^1 - \bar{W}^1 - 2i\bar{z}Z - 2iz\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^2 - \bar{W}^2 - 4iz\bar{z}Z - 2i\bar{z}^2Z - 2iz^2\bar{Z} - 4iz\bar{z}\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^3 - \bar{W}^3 - 4z\bar{z}Z + 2\bar{z}^2Z - 2z^2\bar{Z} + 4z\bar{z}\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^4 - \bar{W}^4 - 6iz^2\bar{z}Z - 2i\bar{z}^3Z - 2iz^3\bar{Z} - 6iz\bar{z}^2\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^5 - \bar{W}^5 - 6z^2\bar{z}Z + 2\bar{z}^3Z - 2z^3\bar{Z} + 6z\bar{z}^2\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^6 - \bar{W}^6 - 4iz\bar{z}^2Z - 4iz^2\bar{z}\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^7 - \bar{W}^7 - 8iz^3\bar{z}Z - 2i\bar{z}^4Z - 2iz^4\bar{Z} - 8iz\bar{z}^3\bar{Z}]_{w=\bar{w}+\Xi}, \\
& 0 \equiv [W^8 - \bar{W}^8 - 8z^3\bar{z}Z + 2\bar{z}^4Z - 2z^4\bar{Z} + 8z\bar{z}^3\bar{Z}]_{w=\bar{w}+\Xi}.
\end{aligned}$$

These functions are real analytic and hence one may expand them with respect to the powers of z (cf. (7)):

$$Z(z, w) = \sum_{k \in \mathbb{N}} z^k Z_k(w) \quad \text{and} \quad W^l(z, w) = \sum_{k \in \mathbb{N}} z^k W_k^l(w).$$

Our current aim is to find a closed form expression for the holomorphic functions $Z(z, w)$ and $W^l(z, w)$ by using their corresponding Taylor series. One sees simplified expressions of these functions in the following lemma:

Lemma 5.1. *The holomorphic functions $Z(z, w)$ and $W^l(z, w)$, $l = 1, \dots, 8$ are all polynomial with respect to z :*

$$\left[\begin{array}{l}
Z(z, w) = Z_0(w) + zZ_1(w) + z^2Z_2(w) + z^3Z_3(w) + z^4Z_4(w) + z^5Z_5(w), \\
W^1(z, w) = W_0^1(w) + zW_1^1(w), \\
W^2(z, w) = W_0^2(w) + zW_1^2(w) + z^2W_2^2(w), \\
W^3(z, w) = W_0^3(w) + zW_1^3(w) + z^2W_2^3(w), \\
W^4(z, w) = W_0^4(w) + zW_1^4(w) + z^2W_2^4(w) + z^3W_3^4(w), \\
W^5(z, w) = W_0^5(w) + zW_1^5(w) + z^2W_2^5(w) + z^3W_3^5(w), \\
W^6(z, w) = W_0^6(w), \\
W^7(z, w) = W_0^7(w) + zW_1^7(w) + z^2W_2^7(w) + z^3W_3^7(w) + z^4W_4^7(w), \\
W^8(z, w) = W_0^8(w) + zW_1^8(w) + z^2W_2^8(w) + z^3W_3^8(w) + z^4W_4^8(w).
\end{array} \right.$$

Proof. After expansion, the first equation (18) reads:

$$\begin{aligned}
(19) \quad & 0 \equiv \sum_{k \in \mathbb{N}} z^k [W_k^1(\bar{w}_1 + \Xi_1, \bar{w}_2 + \Xi_2, \dots, \bar{w}_8 + \Xi_8) - \\
& \quad - 2i\bar{z}Z_k(\bar{w}_1 + \Xi_1, \bar{w}_2 + \Xi_2, \dots, \bar{w}_8 + \Xi_8)] + \\
& \quad + \sum_{k \in \mathbb{N}} \bar{z}^k [-\bar{W}_k^1(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_8) - 2iz\bar{Z}_k(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_8)].
\end{aligned}$$

Then, we expand further each holomorphic function Z^k and W_k^1 according to the formulae (cf. (8)):

$$\begin{aligned}
 A(\bar{w}_1 + \Xi_1, \bar{w}_2 + \Xi_2, \dots, \bar{w}_8 + \Xi_8) = \\
 \sum_{l_1, \dots, l_8 \in \mathbb{N}} A_{w_1^{l_1} w_2^{l_2} \dots w_8^{l_8}}(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_8) \frac{(2i z \bar{z})^{l_1}}{l_1!} \cdot \frac{(2i z^2 \bar{z} + 2i z \bar{z}^2)^{l_2}}{l_2!} \cdot \frac{(2z^2 \bar{z} - 2z \bar{z}^2)^{l_3}}{l_3!} \cdot \\
 \frac{(2i z^3 \bar{z} + 2i z \bar{z}^3)^{l_4}}{l_4!} \cdot \frac{(2z^3 \bar{z} + 2z \bar{z}^3)^{l_5}}{l_5!} \cdot \frac{(2i z^2 \bar{z}^2)^{l_6}}{l_6!} \cdot \\
 \frac{(2i z^4 \bar{z} + 2i z \bar{z}^4)^{l_7}}{l_7!} \cdot \frac{(2z^4 \bar{z} + 2z \bar{z}^4)^{l_8}}{l_8!}.
 \end{aligned}$$

Chasing the coefficient of \bar{z}^k for every $k \geq 2$ after further expansion, we therefore see that the first two lines give absolutely no contribution, and that from the third line, it only comes: $0 \equiv \bar{W}_k^1(\bar{w})$, which is what was claimed about the W_k^1 .

Next, chasing the coefficient of $z \bar{z}^{k'}$ for every $k' \geq 6$, we get $0 \equiv \bar{Z}_{k'}(\bar{w})$. The seven remaining families of vanishing equations $0 \equiv \bar{W}_{k_l}^l(\bar{w})$ for $l = 2, \dots, 8$ with $k_2, k_3 \geq 3$, with $k_4, k_5 \geq 4$, $k_6 \geq 1$ and with $k_7, k_8 \geq 5$ are obtained in a completely similar way by looking at the second to eighth equations of (18). \square

After this extensive simplification, we try to determine expressions of the remaining 33 holomorphic functions Z_0, Z_1, \dots, W_4^8 in the above lemma. Substituting in (18) the already obtained expressions for the nine functions $Z, W^l, l = 1, \dots, 8$, the fundamental equations (18) change into the form:

$$(20) \quad 0 \equiv \sum_{t=0}^1 (z^t W_t^1(\bar{w} + \Xi)) - \sum_{t=0}^1 (\bar{z}^t \bar{W}_t^1(\bar{w})) - 2i \sum_{t=0}^5 (z^t \bar{z} Z(\bar{w} + \Xi)) - 2i \sum_{t=0}^5 (z \bar{z}^t \bar{Z}_t(\bar{w})),$$

$$(21) \quad 0 \equiv \sum_{t=0}^2 (z^t W_t^2(\bar{w} + \Xi)) - \sum_{t=0}^2 (\bar{z}^t \bar{W}_t^2(\bar{w})) - 4i \sum_{t=0}^5 (z^{t+1} \bar{z} Z(\bar{w} + \Xi)) - \\
 - 2i \sum_{t=0}^5 (z^t \bar{z}^2 Z_t(\bar{w} + \Xi)) - 2i \sum_{t=0}^5 (z^2 \bar{z}^t \bar{Z}_t(\bar{w})) - 4i \sum_{t=0}^5 (z \bar{z}^{t+1} \bar{Z}_t(\bar{w})),$$

$$(22) \quad 0 \equiv \sum_{t=0}^2 (z^t W_t^3(\bar{w} + \Xi)) - \sum_{t=0}^2 (\bar{z}^t \bar{W}_t^3(\bar{w})) - 4 \sum_{t=0}^5 (z^{t+1} \bar{z} Z(\bar{w} + \Xi)) + \\
 + 2 \sum_{t=0}^5 (z^t \bar{z}^2 Z_t(\bar{w} + \Xi)) - 2 \sum_{t=0}^5 (z^2 \bar{z}^t \bar{Z}_t(\bar{w})) + 4 \sum_{t=0}^5 (z \bar{z}^{t+1} \bar{Z}_t(\bar{w})),$$

$$(23) \quad 0 \equiv \sum_{t=0}^3 (z^t W_t^4(\bar{w} + \Xi)) - \sum_{t=0}^3 (\bar{z}^t \bar{W}_t^4(\bar{w})) - 6i \sum_{t=0}^5 (z^{t+2} \bar{z} Z(\bar{w} + \Xi)) - \\
 - 2i \sum_{t=0}^5 (z^t \bar{z}^3 Z_t(\bar{w} + \Xi)) - 2i \sum_{t=0}^5 (z^3 \bar{z}^t \bar{Z}_t(\bar{w})) - 6i \sum_{t=0}^5 (z \bar{z}^{t+2} \bar{Z}_t(\bar{w})),$$

$$(24) \quad 0 \equiv \sum_{t=0}^3 (z^t W_t^5(\bar{w} + \Xi)) - \sum_{t=0}^3 (\bar{z}^t \bar{W}_t^5(\bar{w})) - 6 \sum_{t=0}^5 (z^{t+2} \bar{z} Z(\bar{w} + \Xi)) + \\
 + 2 \sum_{t=0}^5 (z^t \bar{z}^3 Z_t(\bar{w} + \Xi)) - 2 \sum_{t=0}^5 (z^3 \bar{z}^t \bar{Z}_t(\bar{w})) + 6 \sum_{t=0}^5 (z \bar{z}^{t+2} \bar{Z}_t(\bar{w})),$$

$$(25) \quad 0 \equiv W_0^6(\bar{w} + \Xi) - \bar{W}_t^6(\bar{w}) - 4i \sum_{t=0}^5 (z^{t+1} \bar{z}^2 Z(\bar{w} + \Xi)) - 4i \sum_{t=0}^5 (z^2 \bar{z}^{t+1} \bar{Z}_t(\bar{w})),$$

$$(26) \quad 0 \equiv \sum_{t=0}^4 (z^t W_t^7(\bar{w} + \Xi)) - \sum_{t=0}^4 (\bar{z}^t \bar{W}_t^7(\bar{w})) - 8i \sum_{t=0}^5 (z^{t+3} \bar{z} Z(\bar{w} + \Xi)) - \\ - 2i \sum_{t=0}^5 (z^t \bar{z}^4 Z_t(\bar{w} + \Xi)) - 2i \sum_{t=0}^5 (z^4 \bar{z}^t \bar{Z}_t(\bar{w})) - 8i \sum_{t=0}^5 (z \bar{z}^{t+3} \bar{Z}_t(\bar{w})),$$

$$(27) \quad 0 \equiv \sum_{t=0}^4 (z^t W_t^8(\bar{w} + \Xi)) - \sum_{t=0}^4 (\bar{z}^t \bar{W}_t^8(\bar{w})) - 8 \sum_{t=0}^5 (z^{t+3} \bar{z} Z(\bar{w} + \Xi)) + \\ + 2 \sum_{t=0}^5 (z^t \bar{z}^4 Z_t(\bar{w} + \Xi)) - 2 \sum_{t=0}^5 (z^4 \bar{z}^t \bar{Z}_t(\bar{w})) + 8 \sum_{t=0}^5 (z \bar{z}^{t+3} \bar{Z}_t(\bar{w})).$$

Now we arrive at the point where we should expand the functions $Z_\bullet(\bar{w} + \Xi)$ and $W_\bullet(\bar{w} + \Xi)$ appearing in the above eight equations (20)–(27) using the Taylor series (8). Afterwards, the coefficients of $z^\mu \bar{z}^\nu$ in these equations should be extracted and set equal to zero, identically. This is equivalent to solve the LinCons PDE system constructed by these coefficients underlying the differential ring $R := \mathbb{C}(w)[Z_\bullet, W_\bullet^1, \dots, W_\bullet^8]$, equipped with the functional conjugation operator.

Doing so, we have extracted these coefficients as much as it was needed⁴ and obtained the following system of 63 equations with 33 unknowns Z_0, Z_1, \dots, W_4^8 , which are in fact the indeterminates of the differential ring R . Here by $(\mu, \nu : \ell)$ we mean the coefficient of $z^\mu \bar{z}^\nu$ in the fundamental equation (ℓ) for $\ell = 20, \dots, 27$:

$$(28) \quad \begin{aligned} (0, 1 : 20) : -2i Z_0 - \bar{W}_1^1 &\equiv 0, & (0, 1 : 21) : \bar{W}_1^2 &\equiv 0, & (0, 1 : 22) : \bar{W}_1^3 &\equiv 0, & (0, 1 : 23) : \bar{W}_1^4 &\equiv 0, \\ (0, 1 : 24) : \bar{W}_1^5 &\equiv 0, & (0, 1 : 25) : \bar{W}_1^6 &\equiv 0, & (0, 1 : 26) : \bar{W}_1^7 &\equiv 0, & (0, 1 : 27) : \bar{W}_1^8 &\equiv 0, \\ (1, 1 : 20) : -2i Z_1 - 2i \bar{Z}_1 + 2i \bar{W}_{0,w^1}^1 &\equiv 0, & (1, 1 : 21) : -4i Z_0 - 4i \bar{Z}_0 + 2i \bar{W}_{0,w^1}^2 &\equiv 0, \\ (1, 1 : 22) : -4i Z_0 + 4i \bar{Z}_0 + 2i \bar{W}_{0,w^1}^3 &\equiv 0, & (1, 1 : 23) : W_{0,w^1}^4 &\equiv 0, & (1, 1 : 24) : W_{0,w^1}^5 &\equiv 0, \\ (1, 1 : 25) : W_{0,w^1}^6 &\equiv 0, & (1, 1 : 26) : W_{0,w^1}^7 &\equiv 0, & (1, 1 : 27) : W_{0,w^1}^8 &\equiv 0, \\ (0, 2 : 21) : 2i Z_0 + \bar{W}_2^2 &\equiv 0, & (0, 2 : 22) : 2i Z_0 - \bar{W}_2^3 &\equiv 0, & (0, 2 : 23) : \bar{W}_2^4 &\equiv 0, \\ (0, 2 : 24) : \bar{W}_2^5 &\equiv 0, & (0, 2 : 26) : \bar{W}_2^7 &\equiv 0, & (0, 2 : 27) : \bar{W}_2^8 &\equiv 0, \\ (1, 2 : 20) : -2i \bar{Z}_2 + 2i W_{0,w^2}^1 - 2i W_{0,w^3}^1 + 4i Z_{0,w^1} &\equiv 0, & (1, 2 : 21) : -2i Z_1 - 4i \bar{Z}_1 - 2i W_{0,w^3}^2 + 2i W_{0,w^2}^2 &\equiv 0, \end{aligned}$$

⁴ The number of the equations that we found during this step was more than 63 but many of them could be obtained from the remaining ones.

$$\begin{aligned}
 (1, 2 : 22) : 2 Z_1 + 4 \bar{Z}_1 - 2 W_{0,w^3}^3 + 2 i W_{0,w^2}^3 &\equiv 0, & (1, 2 : 23) : -6 i \bar{Z}_0 - 2 W_{0,w^3}^4 + 2 i W_{0,w^2}^4 &\equiv 0, \\
 (1, 2 : 24) : 6 \bar{Z}_0 - 2 W_{0,w^3}^5 + 2 i W_{0,w^2}^5 &\equiv 0, & (1, 2 : 25) : -4 i Z_0 - 2 W_{0,w^3}^6 + 2 i W_{0,w^2}^6 &\equiv 0, \\
 (1, 2 : 26) : -2 W_{0,w^3}^7 + 2 i W_{0,w^2}^7 &\equiv 0, & (1, 2 : 27) : -2 W_{0,w^3}^8 + 2 i W_{0,w^2}^8 &\equiv 0, \\
 (0, 3 : 23) : 2 i Z_0 + \bar{W}_3^4 &\equiv 0, & (0, 3 : 24) : 2 Z_0 - \bar{W}_3^5 &\equiv 0, & (0, 3 : 26) : \bar{W}_3^7 &\equiv 0, & (0, 3 : 27) : \bar{W}_3^8 &\equiv 0, \\
 (1, 3 : 20) : -\bar{W}_{0,w^5}^1 - i \bar{Z}_3 + i W_{0,w^4}^1 &\equiv 0, & (1, 3 : 21) : -2 i \bar{Z}_2 + i W_{0,w^4}^2 - \bar{W}_{0,w^5}^2 &\equiv 0, \\
 (1, 3 : 22) : 2 \bar{Z}_2 - W_{0,w^5}^3 + i \bar{W}_{0,w^4}^3 &\equiv 0, & (1, 3 : 23) : -2 i Z_1 - 6 i \bar{Z}_1 - 2 W_{0,w^5}^4 + 2 i \bar{W}_{0,w^4}^4 &\equiv 0, \\
 (1, 3 : 24) : 2 Z_1 + 6 \bar{Z}_1 - 2 W_{0,w^5}^5 + 2 i \bar{W}_{0,w^4}^5 &\equiv 0, & (1, 3 : 25) : -2 W_{0,w^5}^6 + 2 i \bar{W}_{0,w^4}^6 &\equiv 0, \\
 (1, 3 : 26) : -8 i \bar{Z}_0 - 2 W_{0,w^5}^7 + 2 i \bar{W}_{0,w^4}^7 &\equiv 0, & (1, 3 : 27) : 8 \bar{Z}_0 - 2 W_{0,w^5}^8 + 2 i \bar{W}_{0,w^4}^8 &\equiv 0, \\
 (2, 2 : 20) : i W_{0,w^6}^1 - Z_{1,w^1} &\equiv 0, & (2, 2 : 21) : i W_{0,w^6}^2 - i Z_2 - i \bar{Z}_2 &\equiv 0, \\
 (2, 2 : 22) : -W_{0,w^6}^3 + i Z_2 - i \bar{Z}_2 &\equiv 0, & (2, 2 : 23) : i W_{0,w^6}^4 &\equiv 0, & (2, 2 : 24) : i W_{0,w^6}^5 &\equiv 0, \\
 (2, 2 : 25) : i W_{0,w^6}^6 - 2 i Z_1 - 2 i \bar{Z}_1 &\equiv 0, & (2, 2 : 26) : i W_{0,w^6}^7 &\equiv 0, & (2, 2 : 27) : i W_{0,w^6}^8 &\equiv 0, \\
 (0, 4 : 26) : 2 i Z_0 + \bar{W}_4^7 &\equiv 0, & (0, 4 : 27) : 2 Z_0 - \bar{W}_4^8 &\equiv 0, \\
 (1, 4 : 20) : -i \bar{Z}_4 + i \bar{W}_{0,w^7}^1 - W_{0,w^8}^1 &\equiv 0, & (1, 4 : 21) : -2 i \bar{Z}_3 + i \bar{W}_{0,w^7}^2 - W_{0,w^8}^2 &\equiv 0, \\
 (1, 4 : 22) : W_{0,w^8}^3 - i W_{0,w^7}^3 &\equiv 0, & (1, 4 : 23) : -3 i \bar{Z}_2 + i \bar{W}_{0,w^7}^4 - W_{0,w^8}^4 &\equiv 0, \\
 (1, 4 : 24) : 3 \bar{Z}_2 - \bar{W}_{0,w^8}^5 + i W_{0,w^7}^5 &\equiv 0, & (1, 4 : 25) : 2 i \bar{W}_{0,w^7}^6 - 2 i W_{0,w^8}^6 &\equiv 0, \\
 (1, 4 : 26) : -2 i Z_1 - 8 i \bar{Z}_1 - 2 \bar{W}_{0,w^8}^7 + 2 i W_{0,w^7}^7 &\equiv 0, & (1, 4 : 27) : 2 i Z_1 + 8 \bar{Z}_1 - 2 \bar{W}_{0,w^8}^8 + 2 i W_{0,w^7}^8 &\equiv 0, \\
 (1, 4 : 21) : -i \bar{Z}_3 &\equiv 0, & (1, 4 : 23) : -i Z_2 &\equiv 0, \\
 (1, 5 : 20) : -i \bar{Z}_5 &\equiv 0, & (1, 5 : 21) : -i \bar{Z}_4 &\equiv 0.
 \end{aligned}$$

We employ the MAPLE package `DifferentialAlgebra`, performing the approach described in Section 3 based on the Rosenfeld-Gröbner algorithm to solve such a LinCons PDE system, and we obtain the following general solution:

$$\begin{aligned}
 Z(z, w) &:= \underbrace{c + i d}_{Z_0(w)} + \underbrace{(a + i b) z}_{Z_1(w)}, \\
 W^1(z, w) &:= \underbrace{c_1 + 2 a w^1}_{W_0^1(w)} + 2 \underbrace{(d + i c) z}_{W_1^1(w)}, \\
 W^2(z, w) &:= \underbrace{c_2 + 4 c w^1 + 3 a w^2 - b w^3}_{W_0^2(w)} + 2 \underbrace{(d + i c) z^2}_{W_2^2(w)}, \\
 W^3(z, w) &:= \underbrace{c_3 + 4 d w^1 + b w^2 + 3 a w^3}_{W_0^3(w)} + 2 \underbrace{(c - i d) z^2}_{W_2^3(w)}, \\
 W^4(z, w) &:= \underbrace{c_4 + 3 c w^2 - 3 d w^3 + 4 a w^4 - 2 b w^5}_{W_0^4(w)} + 2 \underbrace{(d + i c) z^3}_{W_3^4(w)}, \\
 W^5(z, w) &:= \underbrace{c_5 + 3 d w^2 + 3 c w^3 + 2 b w^4 + 4 a w^5}_{W_0^5(w)} + 2 \underbrace{(c - i d) z^3}_{W_3^5(w)}, \\
 W^6(z, w) &:= \underbrace{c_6 + 2 c w^2 + 2 d w^3 + 4 a w^6}_{W_0^6(w)}, \\
 W^7(z, w) &:= \underbrace{c_7 + 4 c w^4 - 4 d w^5 + 5 a w^7 - 3 b w^8}_{W_0^7(w)} + 2 \underbrace{(d + i c) z^4}_{W_4^7(w)}, \\
 W^8(z, w) &:= \underbrace{c_8 + 4 d w^4 + 4 c w^5 + 3 b w^7 + 5 a w^8}_{W_0^8(w)} + 2 \underbrace{(c - i d) z^4}_{W_4^8(w)},
 \end{aligned}$$

5.1. Two remaining sought Lie algebras $\text{aut}_{CR}(\mathbb{M}^2)$ and $\text{aut}_{CR}(\mathbb{M}^3)$. One can perform long computations similar to what we did for $\text{aut}_{CR}(\mathbb{M}^1)$ and obtain the structure of the two remaining Lie algebras $\text{aut}_{CR}(\mathbb{M}^2)$ and $\text{aut}_{CR}(\mathbb{M}^3)$. Here, we omit the corresponding intermediate computations — since they are similar to those of \mathbb{M}^1 and offer no new aspect.

Theorem 5.2. *The Lie algebra of infinitesimal CR-automorphisms $\text{aut}_{CR}(\mathbb{M}^2)$ of the rigid real analytic generic CR-generic submanifold $\mathbb{M}^2 \subset \mathbb{C}^{1+8}$, represented as the graph of the eight defining equations:*

$$\begin{aligned} w^j - \bar{w}^j &= \Xi_j(z, \bar{z}), & (j=1 \dots 6), \\ w^7 - \bar{w}^7 &= \Xi'_7(z, \bar{z}) := 2i(z^3 \bar{z}^2 + z^2 \bar{z}^3), \\ w^8 - \bar{w}^8 &= \Xi'_8(z, \bar{z}) := 2(z^3 \bar{z}^2 - z^2 \bar{z}^3), \end{aligned}$$

is 12-dimensional with the holomorphic coefficients:

$$\begin{aligned} Z(z, w) &:= \underbrace{c + i d}_{Z_0(w)} + \underbrace{(a + i b)}_{Z_1(w)} z, \\ W^1(z, w) &:= \underbrace{c_1 + 2 a w^1}_{W_0^1(w)} + 2 \underbrace{(d + i c)}_{W_1^1(w)} z, \\ W^2(z, w) &:= \underbrace{c_2 + 4 c w^1 + 3 a w^2 - b w^3}_{W_0^2(w)} + \underbrace{2(d + i c)}_{W_2^2(w)} z^2, \\ W^3(z, w) &:= \underbrace{c_3 + 4 d w^1 + b w^2 + 3 a w^3}_{W_0^3(w)} + \underbrace{2(c - i d)}_{W_2^3(w)} z^2, \\ W^4(z, w) &:= \underbrace{c_4 + 3 c w^2 - 3 d w^3 + 4 a w^4 - 2 b w^5}_{W_0^4(w)} + \underbrace{2(d + i c)}_{W_3^4(w)} z^3, \\ W^5(z, w) &:= \underbrace{c_5 + 3 d w^2 + 3 c w^3 + 2 b w^4 + 4 a w^5}_{W_0^5(w)} + \underbrace{2(c - i d)}_{W_3^5(w)} z^3, \\ W^6(z, w) &:= \underbrace{c_6 + 2 c w^2 + 2 d w^3 + 4 a w^6}_{W_0^6(w)}, \\ W^7(z, w) &:= \underbrace{c_7 + 2 c w^4 + 2 d w^5 + 6 c w^6 + 5 a w^7 - b w^8}_{W_0^7(w)}, \\ W^8(z, w) &:= \underbrace{c_8 - 2 d w^4 + 2 c w^5 + 6 d w^6 + b w^7 + 5 a w^8}_{W_0^8(w)}, \end{aligned}$$

and is generated by the twelve \mathbb{R} -linearly independent holomorphic vector fields:

$$\left\{ \begin{array}{l} X_i := \partial_{w_i}, \quad i = 1, \dots, 8, \\ X_9 := z \partial_z + 2 w^1 \partial_{w^1} + 3 w^2 \partial_{w^2} + 3 w^3 \partial_{w^3} + 4 w^4 \partial_{w^4} + \\ \quad + 4 w^5 \partial_{w^5} + 4 w^6 \partial_{w^6} + 5 w^7 \partial_{w^7} + 5 w^8 \partial_{w^8}, \\ X_{10} := i z \partial_z - w^3 \partial_{w^2} + w^2 \partial_{w^3} - 2 w^5 \partial_{w^4} + 2 w^4 \partial_{w^5} - w^8 \partial_{w^7} + w^7 \partial_{w^8}, \\ X_{11} := \partial_z + 2 i z \partial_{w^1} + (4 w^1 + 2 i z^2) \partial_{w^2} + 2 z^2 \partial_{w^3} + (3 w^2 + 2 i z^3) \partial_{w^4} + \\ \quad + (3 w^3 + 2 z^3) \partial_{w^5} + 2 w^2 \partial_{w^6} + (2 w^4 + 6 w^6) \partial_{w^7} + 2 w^5 \partial_{w^8}, \\ X_{12} := i \partial_z + 2 z \partial_{w^1} + 2 z^2 \partial_{w^2} + (4 w^1 - 2 i z^2) \partial_{w^3} + (-3 w^3 + 2 z^3) \partial_{w^4} + \\ \quad + (3 w^2 - 2 i z^3) \partial_{w^5} + 2 w^3 \partial_{w^6} + 2 w^5 \partial_{w^7} + (-2 w^4 + 6 w^6) \partial_{w^8}. \end{array} \right.$$

Furthermore, it is graded of the form:

$$\text{aut}_{CR}(\mathbb{M}^1) := \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with $\mathfrak{g}_{-5} := \langle X_7, X_8 \rangle$, with $\mathfrak{g}_{-4} := \langle X_4, X_5, X_6 \rangle$, with $\mathfrak{g}_{-3} := \langle X_2, X_3 \rangle$, with $\mathfrak{g}_{-2} := \langle X_1 \rangle$, with $\mathfrak{g}_{-1} := \langle X_{11}, X_{12} \rangle$, and with $\mathfrak{g}_0 := \langle X_9, X_{10} \rangle$ and together with the following table of commutators:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
X_1	0	0	0	0	0	0	0	0	$2X_1$	0	$4X_2$	$4X_3$
X_2	*	0	0	0	0	0	0	0	$3X_2$	X_3	$3X_4+2X_6$	$3X_5$
X_3	*	*	0	0	0	0	0	0	$3X_3$	$-X_2$	$3X_5$	$-3X_4+2X_6$
X_4	*	*	*	0	0	0	0	0	$4X_4$	$2X_5$	$2X_7$	$-2X_8$
X_5	*	*	*	*	0	0	0	0	$4X_5$	$-2X_4$	$2X_8$	$2X_7$
X_6	*	*	*	*	*	0	0	0	$4X_6$	0	$6X_7$	0
X_7	*	*	*	*	*	*	0	0	$5X_7$	X_8	0	0
X_8	*	*	*	*	*	*	*	0	$5X_8$	$-X_7$	0	0
X_9	*	*	*	*	*	*	*	*	0	0	$-X_{11}$	$-X_{12}$
X_{10}	*	*	*	*	*	*	*	*	*	0	$-X_{12}$	X_{11}
X_{11}	*	*	*	*	*	*	*	*	*	*	0	$4X_1$
X_{12}	*	*	*	*	*	*	*	*	*	*	*	0

Theorem 5.3. *The Lie algebra of infinitesimal CR-automorphisms $\text{aut}_{CR}(\mathbb{M}^3)$ of the rigid real analytic CR-generic submanifold $\mathbb{M}^3 \subset \mathbb{M}^{1+8}$, represented as the graph of the eight defining equations:*

$$\begin{aligned} w^j - \bar{w}^j &= \Xi_j(z, \bar{z}), & (j=1 \dots 6), \\ w^7 - \bar{w}^7 &= \Xi'_7(z, \bar{z}), \\ w^8 - \bar{w}^8 &= \Xi''_8(z, \bar{z}) := 2i(z^4 \bar{z} + z \bar{z}^4), \end{aligned}$$

is 11-dimensional with the coefficients:

$$\begin{aligned} Z(z, w) &:= \underbrace{c + id}_{Z_0(w)} + \underbrace{a}_{Z_1(w)} z, \\ W^1(z, w) &:= \underbrace{c_1 + 2aw^1}_{W_0^1(w)} + 2 \underbrace{(d + ic)}_{W_1^1(w)} z, \\ W^2(z, w) &:= \underbrace{c_2 + 4cw^1 + 3aw^2}_{W_0^2(w)} + 2 \underbrace{(d + ic)}_{W_2^2(w)} z^2, \\ W^3(z, w) &:= \underbrace{c_3 + 4dw^1 + 3aw^3}_{W_0^3(w)} + 2 \underbrace{(c - id)}_{W_2^3(w)} z^2, \\ W^4(z, w) &:= \underbrace{c_4 + 3cw^2 - 3dw^3 + 4aw^4}_{W_0^4(w)} + 2 \underbrace{(d + ic)}_{W_3^4(w)} z^3, \\ W^5(z, w) &:= \underbrace{c_5 + 3dw^2 + 3cw^3 + 4aw^5}_{W_0^5(w)} + 2 \underbrace{(c - id)}_{W_3^5(w)} z^3, \\ W^6(z, w) &:= \underbrace{c_6 + 2cw^2 + 2dw^3 + 4aw^6}_{W_0^6(w)}, \\ W^7(z, w) &:= \underbrace{c_7 + 2cw^4 + 2dw^5 + 6cw^6 + 5aw^7}_{W_0^7(w)}, \\ W^8(z, w) &:= \underbrace{c_8 + 4cw^4 - 4dw^5 + 5aw^8}_{W_0^8(w)} + 2 \underbrace{(d + ic)}_{W_4^8(w)} z^4, \end{aligned}$$

and is generated by the eleven \mathbb{R} -linearly independent holomorphic vector fields:

$$\begin{cases} X_i := \partial_{w_i}, & i = 1, \dots, 8, \\ X_9 := z\partial_z + 2w^1\partial_{w^1} + 3w^2\partial_{w^2} + 3w^3\partial_{w^3} + 4w^4\partial_{w^4} + \\ & + 4w^5\partial_{w^5} + 4w^6\partial_{w^6} + 5w^7\partial_{w^7} + 5w^8\partial_{w^8}, \\ X_{10} := \partial_z + 2iz\partial_{w^1} + (4w^1 + 2iz^2)\partial_{w^2} + 2z^2\partial_{w^3} + (3w^2 + 2iz^3)\partial_{w^4} + \\ & + (3w^3 + 2z^3)\partial_{w^5} + 2w^2\partial_{w^6} + (2w^4 + 6w^6)\partial_{w^7} + (4w^4 + 2iz^4)\partial_{w^8}, \\ X_{11} := i\partial_z + 2z\partial_{w^1} + 2z^2\partial_{w^2} + (4w^1 - 2iz^2)\partial_{w^3} + (-3w^3 + 2z^3)\partial_{w^4} + \\ & + (3w^2 - 2iz^3)\partial_{w^5} + 2w^3\partial_{w^6} + 2w^5\partial_{w^7} + (-4w^5 + 2z^4)\partial_{w^8}. \end{cases}$$

Furthermore, it is graded of the form:

$$\text{aut}_{CR}(\mathbb{M}^1) := \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with $\mathfrak{g}_{-5} := \langle X_7, X_8 \rangle$, with $\mathfrak{g}_{-4} := \langle X_4, X_5, X_6 \rangle$, with $\mathfrak{g}_{-3} := \langle X_2, X_3 \rangle$, with $\mathfrak{g}_{-2} := \langle X_1 \rangle$, with $\mathfrak{g}_{-1} := \langle X_{10}, X_{11} \rangle$, and with $\mathfrak{g}_0 := \langle X_9 \rangle$ and together with the following table of commutators:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
X_1	0	0	0	0	0	0	0	0	$2X_1$	$4X_2$	$4X_3$
X_2	*	0	0	0	0	0	0	0	$3X_2$	$3X_4 + 2X_6$	$3X_5$
X_3	*	*	0	0	0	0	0	0	$3X_3$	$3X_5$	$-3X_4 + 2X_6$
X_4	*	*	*	0	0	0	0	0	$4X_4$	$2X_7 + 4X_8$	0
X_5	*	*	*	*	0	0	0	0	$4X_5$	0	$2X_7 - 4X_8$
X_6	*	*	*	*	*	0	0	0	$4X_6$	$6X_7$	0
X_7	*	*	*	*	*	*	0	0	$5X_7$	0	0
X_8	*	*	*	*	*	*	*	0	$5X_8$	0	0
X_9	*	*	*	*	*	*	*	*	0	$-X_{10}$	$-X_{11}$
X_{10}	*	*	*	*	*	*	*	*	*	0	$4X_1$
X_{11}	*	*	*	*	*	*	*	*	*	*	0

REFERENCES

- [1] M.Aghasi, J. Merker, M. Sabzevari, *Effective Cartan-Tanaka connections on C^6 -smooth strongly pseudoconvex hypersurfaces $M^3 \subset \mathbb{C}^2$* , C. R. Acad. Sci. Paris, Sér. I **349** (2011), 845–848; arxiv:1104.1509v1, 2011, 113 pages.
- [2] S. Baouendi, P. Ebenfelt, L. P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Math. Series, 1999.
- [3] T. Becker, V. Weispfenning, *Gröbner bases, a computational approach to commutative Algebra*, Springer, 1993.
- [4] V. K. Beloshapka, *Universal models for real submanifolds*, Mathematical Notes, **75**(4), pages 475–488, 2004.
- [5] V. K. Beloshapka, *CR-Varieties of the type (1, 2) as varieties of super-high codimension*, Russian J. Mathematical Physics, **5**(2), pages 399–404, 1997.
- [6] V. K. Beloshapka, V. Ezhov, G. Schmalz, *Canonical Cartan connection and holomorphic invariants on Engel CR manifolds*, Russian J. Mathematical Physics **14** (2007), no. 2, 121–133.
- [7] V. K. Beloshapka, I. G. Kossovskiy, *Classification of homogeneous CR-manifolds in dimension 4*, J. Mathematical Analysis and Applications, **374**, pages 655–672, 2011.
- [8] A. Boggess, *CR-manifolds and the tangential Cauchy-Riemann complex.*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991.
- [9] F. Boulier, D. Lazard, F. Ollivier, M. Petitot, *Computing representations for radicals of finitely generated differential ideals.*, AAECC, **20**, pages 273–121, 2009.

- [10] F. Boulier, D. Lazard, F. Ollivier, M. Petitot, *Representation for the radical of a finitely generated differential ideal.*, In ISSAC'95, ACM Press, pages 158–166, 1995.
- [11] S. S. Chern, J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1975), 219–271 .
- [12] S. C. Chou, *Mechanical Geometry theorem proving.*, D. Reidel, Dordrecht, 1988.
- [13] V. Ezhov, B. McLaughlin, G. Schmalz, *From Cartan to Tanaka: getting real in the complex world*, Notices of the AMS **58** (2011), no. 1, 20–27
- [14] G. Fels, W. Kaup, *Classification of Levi degenerate homogeneous CR-manifolds in dimension 5*, Acta Math., **201**, pages 1–82, 2008.
- [15] G. Gallo, *Complexity issues in computational Algebra.*, Ph.D. thesis, Courant Institute of Mathematical Sciences, New York University, 1992.
- [16] G. Gallo, B. Mishra, *Efficient algorithms and bounds for Wu-Ritt characteristic sets.*, **94** Progress in Mathematics, *Effective Methods in Algebraic Geometry*, pages 119–142. Birkhäuser, Boston, 1991.
- [17] G. Gallo, B. Mishra, *Wu-Ritt characteristic sets and their complexity.*, **6** Discrete and Computational Geometry: Papers from the DIMACS (Discrete Mathematics and Computer Science) Special Year, pages 111–136, 1991.
- [18] H. Gaussier, J. Merker, *Nonalgebraizable real analytic tubes in \mathbb{C}^n* , Math. Z. **247**(2), pages 337–383, 2004.
- [19] A. Isaev, *Spherical tube hypersurfaces*, Lecture Notes in Mathematics, 2011, to appear.
- [20] E. R. Kolchin, *Differential Algebra and algebraic groups.*, Academic Press, New York, 1973.
- [21] I. B. Mamai, *Model CR-manifolds with one-dimensional complex tangent*, Russian J. Mathematical Physics, **16**(1), pages 97–102, 2009.
- [22] E. N. Mansfield, P. A. Clarkson, *Applications of the differential algebra package diffgrob2 to classical symmetries of differential equations*, J. Symb. Comp., **23**, pages 517–533, 1997.
- [23] J. Merker, *Lie symmetries and CR Geometry*, J. Mathematical Sciences, **154**(6), pages 817–922, 2008.
- [24] J. Merker, E. Porten, *Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities*, International Mathematics Research Surveys, Volume **2006**, Article ID 28295, 287 pages.
- [25] J. Merker, M. Sabzevari, *Cartan equivalence problem for 5-dimensional CR-manifolds in \mathbb{C}^4* , In progress, 2012.
- [26] J. Merker, M. Sabzevari, *Explicit expression of Cartan's connections for Levi-nondegenerate 3-manifolds in complex surfaces, and identification of the Heisenberg sphere*, Cent. Eur. J. Math., **10**(5), pages 1801–1835, 2012.
- [27] F. Richman, *Constructive aspects of Noetherian rings.*, Proceedings of Amer. Math. Soc., **44**, pages 436–441, 1974.
- [28] J. F. Ritt, *Differential Algebra.*, American Mathematical Society, New York, 1948.
- [29] S. Stenström, *Differential Gröbner bases.*, Master thesis, Lulea University of Technology, 2002.
- [30] E. N. Shananina, *Models for CR-manifolds of type $(1, K)$ for $3 \leq K \leq 7$ and their automorphisms*, Mathematical Notes, **67**(3), pages 382–388, 2000.
- [31] A. E. Tumanov, *Finite-dimensionality of the group of CR automorphisms of a standard CR-manifold, and proper holomorphic mappings of Siegel domains*, Math. USSR Izvestiya, **32**(3), pages 655–662, 1989.
- [32] W-T. Wu, *Basic principles of mechanical theorem proving in geometries.*, Journal of Sys. Sci. and Math. Sci., **4**(3), pages 207–235, 1984. (see also Journal of Automated Reasoning, **2**(4), pages 221–252, 1986).
- [33] W-T. Wu, *On the decision problem and the mechanization of theorem proving in elementary Geometry.*, Scientia Sinica, **21**, pages 157–179, 1978.
- [34] W-T. Wu, *Some recent advances in mechanical theorem proving of geometries.*, **29** Automated Theorem Proving: After 25 Years, Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, pages 235–242, 1984.

Applications of Differential Algebra to the computation of Lie algebras of infinitesimal CR-automorphisms²⁹

DEPARTMENT OF MATHEMATICS, SHAHREKORD UNIVERSITY, 115 SHAHREKORD, IRAN
E-mail address: sabzevari@math.iut.ac.ir

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN, IRAN
E-mail address: a.hashemi@cc.iut.ac.ir

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN, IRAN AND YOUNG RESEARCHERS CLUB, SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY, TEHRAN, 461/15655, IRAN.
E-mail address: benyamin.m.alizadeh@gmail.com

DÉPARTEMENT DE MATHÉMATIQUES D'ORSAY, BÂTIMENT 425, FACULTÉ DES SCIENCES, UNIVERSITÉ PARIS XI - ORSAY, F-91405 ORSAY CEDEX, FRANCE
E-mail address: merker@dma.ens.fr