

VANISHING HACHTROUDI CURVATURE AND LOCAL EQUIVALENCE TO THE HEISENBERG PSEUDOSPHERE

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ABSTRACT. To any completely integrable second-order system of real or complex partial differential equations:

$$y_{x^{k_1}x^{k_2}} = F_{k_1,k_2}(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})$$

with $1 \leq k_1, k_2 \leq n$ and with $F_{k_1,k_2} = F_{k_2,k_1}$ in $n \geq 2$ independent variables (x^1, \dots, x^n) and in one dependent variable y , Mohsen Hachtroudi associated in 1937 a normal projective (Cartan) connection, and he computed its curvature. By means of a natural transfer of jet polynomials to the associated submanifold of solutions, what the vanishing of the Hachtroudi curvature gives can be precisely translated in order to characterize when both families of Segre varieties and of conjugate Segre varieties associated to a Levi nondegenerate real analytic hypersurface M in \mathbb{C}^n ($n \geq 3$) can be straightened to be affine complex (conjugate) lines. In continuation to a previous paper devoted to the quite distinct \mathbb{C}^2 -case, this then characterizes in an effective way those hypersurfaces of \mathbb{C}^{n+1} in higher complex dimension $n + 1 \geq 3$ that are locally biholomorphic to a piece of the $(2n+1)$ -dimensional Heisenberg quadric, without any special assumption on their defining equations.

Table of contents

| | |
|--|------------|
| 1. Introduction | 1. |
| 2. Segre varieties and differential equations | 5. |
| 3. Geometry of associated submanifolds of solutions | 8. |
| 4. Effective differential characterization of pseudosphericality in \mathbb{C}^{n+1} | 11. |

§1. INTRODUCTION

The explicit characterization of pseudosphericality of an arbitrary real analytic local hypersurface sitting in the complex Euclidean space has been (re)studied recently by Isaev in [11], who employed the famous Chern(-Moser) tensorial approach [5, 4] to the concerned equivalence problem. But in the growing literature devoted to Lie-group symmetries of Cauchy-Riemann manifolds, only a very few articles underline that, already in his 1937 Ph.D. thesis [10] under the direction of his Élie Cartan — who was around the same period also the master of Chern —, the Iranian mathematician Mohsen Hachtroudi (cited briefly only in [4]) constructed directly an *explicit* normal projective Cartan connection canonically associated to any completely integrable system of real or complex partial differential equations:

$$(1) \quad y_{x^{k_1}x^{k_2}}(x) = F_{k_1,k_2}(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})$$

in $n \geq 2$ independent variables x^1, \dots, x^n and in one dependent variable y , by endeavouring in a successful way to generalize the celebrated paper [3]. Chern's clever observation in 1974 that Hachtroudi's 37 years-old approach was intrinsically related to the nascent higher-dimensional CR geometry was followed, in his

two papers in question, by his technical contribution of redoing (only) *parts* of Hachtroudi's effective computations, following the alternative (heavier, though essentially equivalent) strategy of constructing *a posteriori* the projective connection, after having reinterpreted at the beginning the problem in terms of the wide and powerful *Cartan Method of Equivalence*. Thus, one should be aware, historically speaking, that in the original reference [10], much more complete geometric and computational aspects were published long before, though they were expressed in a purely analytic and somewhat elliptic language which, unfortunately for us at present times, does not transmit in words and with figures all the underlying geometric meanings which were clear then to Élie Cartan.

Because Hachtroudi was able to write down explicitly his curvature tensors, he deduced the second-order system (5) — below — of partial differential equations that the functions F_{k_1, k_2} should satisfy in order that the system (1) be equivalent, through a point transformation $(x, y) \mapsto (x', y') = (x'(x, y), y'(x, y))$ to the simplest system: $y'_{x'^{k_1} x'^{k_2}}(x') = 0$, with all right-hand sides being zero. In the present article, a companion and a follower of a preceding one [22] devoted to the quite different \mathbb{C}^2 -case, we will apply, to the higher-dimensional characterization of pseudosphericality, this effective necessary and sufficient condition (5) due to Hachtroudi which, however and *inexplicably*, is totally inextant in the two contributions of Chern. We hope in this way to complete the explicit characterization of pseudosphericality for rigid or even tube hypersurfaces that was obtained recently by Isaev in [11], because apparently, the general (nonrigid) case was still open in the specialized field.

We now start the exposition. Let M be a local real analytic in \mathbb{C}^{n+1} . Though the basic definitions, lemmas and propositions of the theory are valid in any complex dimension $n + 1 \geq 2$, there is a strong computational difference between the two characterizations of sphericity for $n = 1$ (compare [21]) and of pseudosphericality for $n \geq 2$ (presently), so that, in order to fix the ideas, it will be assumed throughout the paper — and recalled when necessary — that the CR dimension n is always ≥ 2 .

Locally in a neighborhood of one of its points p , the hypersurface M may be represented, in any system of local holomorphic coordinates:

$$t = (w, z) \in \mathbb{C}^n \times \mathbb{C}$$

vanishing at p for which the w -axis is not complex-tangent to M at p , by a so-called *complex* defining equation — Section 2 provides further informations — of the form:

$$(2) \quad w = \Theta(z, \bar{z}, \bar{w}) = \Theta(z, \bar{t}),$$

or equivalently in a more expanded form which exhibits all the indices:

$$w = \Theta(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n, \bar{w}) = \Theta(z_1, \dots, z_n, \bar{t}_1, \dots, \bar{t}_n, \bar{t}_{n+1}).$$

Then M localized at p is called *pseudospherical* (at p) if it is biholomorphic to a piece of one Heisenberg pseudosphere:

$$(3) \quad \operatorname{Im} w' = |z'_1|^2 + \dots + |z'_q|^2 - |z'_{q+1}|^2 - \dots - |z'_n|^2,$$

for some q with $0 \leq q \leq n$, the number of positive eigenvalues of the nondegenerate Levi form. Next, let us introduce the following Jacobian-like determinant:

$$\Delta := \begin{vmatrix} \Theta_{\bar{z}_1} & \cdots & \Theta_{\bar{z}_n} & \Theta_{\bar{w}} \\ \Theta_{z_1\bar{z}_1} & \cdots & \Theta_{z_1\bar{z}_n} & \Theta_{z_1\bar{w}} \\ \cdots & \cdots & \cdots & \cdots \\ \Theta_{z_n\bar{z}_1} & \cdots & \Theta_{z_n\bar{z}_n} & \Theta_{z_n\bar{w}} \end{vmatrix} = \begin{vmatrix} \Theta_{\bar{t}_1} & \cdots & \Theta_{\bar{t}_n} & \Theta_{\bar{t}_{n+1}} \\ \Theta_{z_1\bar{t}_1} & \cdots & \Theta_{z_1\bar{t}_n} & \Theta_{z_1\bar{t}_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \Theta_{z_n\bar{t}_1} & \cdots & \Theta_{z_n\bar{t}_n} & \Theta_{z_n\bar{t}_{n+1}} \end{vmatrix}.$$

For any index $\mu \in \{1, \dots, n, n+1\}$ and for any index $\ell \in \{1, \dots, n\}$, let also $\Delta_{[0_{1+\ell}]}^\mu$ denote the same determinant, but with its μ -th column replaced by the transpose of the line $(0 \cdots 1 \cdots 0)$ with 1 at the $(1 + \ell)$ -th place, and 0 elsewhere, its other columns being untouched. One easily convinces oneself (but see also Section 2) that M is Levi-nondegenerate at p — which is the origin of our system of coordinates — if and only if Δ does not vanish at the origin, whence Δ is nowhere zero in some sufficiently small neighborhood of the origin. Similarly, for any indices $\mu, \nu, \tau \in \{1, \dots, n, n+1\}$, denote by $\Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau$ the same determinant as Δ , but with only its τ -th column replaced by the transpose of the line:

$$(\Theta_{\bar{t}^\mu \bar{t}^\nu} \ \Theta_{z_1 \bar{t}^\mu \bar{t}^\nu} \ \cdots \ \Theta_{z_n \bar{t}^\mu \bar{t}^\nu}),$$

other columns being again untouched. All these determinants Δ , $\Delta_{[0_{1+\ell}]}^\mu$, $\Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau$ are visibly universal differential expressions depending upon the second-order jet $J_{z, \bar{z}, \bar{w}}^2 \Theta$ and upon the third-order jet $J_{z, \bar{z}, \bar{w}}^3 \Theta$.

Main Theorem. *An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with $n \geq 2$ which is Levi nondegenerate at one of its points p and has a complex defining equation of the form (2) in some system of local holomorphic coordinates $t = (z, w) \in \mathbb{C}^n \times \mathbb{C}$ vanishing at p , is pseudospherical at p if and only if its complex graphing function Θ satisfies the following explicit nonlinear fourth-order system of partial differential equations:*

$$\begin{aligned} 0 \equiv & \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \left[\Delta_{[0_{1+\ell_1}]}^\mu \cdot \Delta_{[0_{1+\ell_2}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}^\tau} \right\} - \right. \\ & - \frac{\delta_{k_1, \ell_1}}{n+2} \sum_{\ell'=1}^n \Delta_{[0_{1+\ell'}]}^\mu \cdot \Delta_{[0_{1+\ell_2}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}^\tau} \right\} - \\ & - \frac{\delta_{k_1, \ell_2}}{n+2} \sum_{\ell'=1}^n \Delta_{[0_{1+\ell_1}]}^\mu \cdot \Delta_{[0_{1+\ell'}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}^\tau} \right\} - \\ & - \frac{\delta_{k_2, \ell_1}}{n+2} \sum_{\ell'=1}^n \Delta_{[0_{1+\ell'}]}^\mu \cdot \Delta_{[0_{1+\ell_2}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}^\tau} \right\} - \\ & - \frac{\delta_{k_2, \ell_2}}{n+2} \sum_{\ell'=1}^n \Delta_{[0_{1+\ell_1}]}^\mu \cdot \Delta_{[0_{1+\ell'}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar{t}^\tau} \right\} + \\ & + \frac{1}{(n+1)(n+2)} \cdot [\delta_{k_1, \ell_1} \delta_{k_2, \ell_2} + \delta_{k_2, \ell_1} \delta_{k_1, \ell_2}] \cdot \\ & \cdot \sum_{\ell'=1}^n \sum_{\ell''=1}^n \Delta_{[0_{1+\ell'}]}^\mu \cdot \Delta_{[0_{1+\ell''}]}^\nu \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]}^\tau \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar{t}^\tau} \right\}, \end{aligned}$$

for all pairs of indices (k_1, k_2) with $1 \leq k_1, k_2 \leq n$, and for all pairs of indices (ℓ_1, ℓ_2) with $1 \leq \ell_1, \ell_2 \leq n$.

The written system is effective: no implicit formal expression is involved and pseudosphericality is characterized directly and only in terms of Θ .

Now, here is a summarized description of our arguments of proof. A bit similarly as for the \mathbb{C}^2 -case — but with major differences afterwards — which was already studied in [21], we may associate to any such Levi nondegenerate real analytic local hypersurface $M \subset \mathbb{C}^{n+1}$ of equation $w = \Theta(z, \bar{z}, \bar{w})$ a uniquely defined system of second-order partial differential equations:

$$(4) \quad w_{z_{k_1} z_{k_2}}(z) = \Phi_{k_1, k_2}(z, w(z), w_z(w)) \quad (1 \leq k_1, k_2 \leq n)$$

with $\Phi_{k_1, k_2} = \Phi_{k_2, k_1}$, simply by eliminating the two variables \bar{z} and \bar{w} , viewed as parameters, from the set of $n + 1$ equations¹:

$$w(z) = \Theta(z, \bar{z}, \bar{w}), \quad w_{z_1}(z) = \frac{\partial \Theta}{\partial z_1}(z, \bar{z}, \bar{w}), \dots, \quad w_{z_n}(z) = \frac{\partial \Theta}{\partial z_n}(z, \bar{z}, \bar{w}),$$

— the assumption that the Jacobian determinant Δ is nonvanishing at the origin being precisely the one which guarantees, technically speaking, that the classical (holomorphic) implicit function theorem applies — and then by replacing the so obtained values for \bar{z} and \bar{w} in all second order derivatives $\frac{\partial^2 \Theta}{\partial z_{k_1} \partial z_{k_2}}(z, \bar{z}, \bar{w})$, see (8) below. Trivially, this system is completely integrable, for we just derived it from its general solution $w(z) := \Theta(z, \bar{z}, \bar{w})$, where (\bar{z}, \bar{w}) are understood as parameters.

As we said, Hachtroudi showed that the curvature of the projective normal (Cartan) connection he associated with the system (1) vanishes if and only if the right-hand side functions F_{k_1, k_2} satisfy the following explicit differential system, which is *linear* in terms of their second-order derivatives (all of which, notably, appear only with respect to the y_{x^ℓ}):

$$(5) \quad 0 \equiv \frac{\partial^2 F_{k_1, k_2}}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}} - \frac{1}{n+2} \sum_{\ell'=1}^n \left(\delta_{k_1, \ell_1} \frac{\partial^2 F_{\ell', k_2}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell_2}}} + \delta_{k_1, \ell_2} \frac{\partial^2 F_{\ell', k_2}}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell'}}} + \delta_{k_2, \ell_1} \frac{\partial^2 F_{k_1, \ell'}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell_2}}} + \delta_{k_2, \ell_2} \frac{\partial^2 F_{k_1, \ell'}}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell'}}} \right) + \frac{1}{(n+1)(n+2)} [\delta_{k_1, \ell_1} \delta_{k_2, \ell_2} + \delta_{k_2, \ell_1} \delta_{k_1, \ell_2}] \sum_{\ell'=1}^n \sum_{\ell''=1}^n \frac{\partial^2 F_{\ell', \ell''}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell''}}} \quad \begin{array}{l} (1 \leq k_1, k_2 \leq n) \\ (1 \leq \ell_1, \ell_2 \leq n) \end{array}.$$

Hachtroudi also showed that this latter condition, better known nowadays amongst the *Several Complex Variables* community as *vanishing of Chern(-Moser) curvature* to which it indeed amounts, characterizes the local equivalence, through a point transformation $(x, y) \mapsto (x', y') = (x'(x, y), y'(x, y))$, to the simplest system: $y'_{x'^{k_1} x'^{k_2}}(x') = 0$. We then remind the semi-known fact that M is pseudo-spherical if and only if its associated second-order system (4) is equivalent, through a local biholomorphism $(z, w) \mapsto (z', w') = (z'(z, w), w'(z, w))$ fixing the origin, to the simplest system $w'_{z'^{k_1} z'^{k_2}}(z') = 0$. So we may apply to the functions Φ_{k_1, k_2} Hachtroudi's vanishing curvature equations (5), but still, the Φ_{k_1, k_2} are not expressed in terms of Θ , for they were constructed by employing some unpleasant implicit functions when solving above for \bar{z} and \bar{w} . Fortunately, here similarly as in [21], we may apply the techniques of computational differential algebra sketched

¹ This process appears for instance in the references [8, 10, 4, 24, 25, 1, 19].

in [19] in order to explicitly express any algebraic expressions in the second-order jet of the Φ_{k_1, k_2} in terms of the fourth-order jet of Θ , and the appropriate general equation which we shall need:

$$\frac{\partial^2 \Phi_{k_1, k_2}}{\partial w_{z_{\ell_1}} \partial w_{z_{\ell_2}}} = \frac{1}{\Delta^3} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \Delta_{[0_{1+\ell_1}]^\mu} \cdot \Delta_{[0_{1+\ell_2}]^\nu} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}^\mu \partial \bar{t}^\nu} - \sum_{\tau=1}^{n+1} \Delta_{[\bar{t}^\mu \bar{t}^\nu]^\tau} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}^\tau} \right\}$$

will be obtained in Section 4 below, after rather lengthy but elementary calculations, parts of which are inspired from [17]. It is now essentially clear how one obtains the (boxed) long fourth-order differential equations stated in the theorem, but in any case, some complete details will be provided at the very end of the paper.

To conclude this extensive introduction which was designed for readers wanting to quickly embrace the contents, we would like to draw the attention on the work [22], whose manual calculations were finalized in manuscript form already in 2003², and which will soon confirm the above theorem by following another route, *viz.* by calculating explicitly the so-called Chern(-Moser) tensor differential forms, which might interest some contemporary CR geometers better than the (essentially equivalent) original Cartan-Hachtroudi(-Tanaka) approach.

§2. SEGRE VARIETIES AND DIFFERENTIAL EQUATIONS

Real analytic hypersurfaces in \mathbb{C}^{n+1} . Let us therefore consider an arbitrary real analytic hypersurface M in \mathbb{C}^{n+1} with $n \geq 2$, and let us localize it around one of its points, say $p \in M$. Then there exist complex affine coordinates:

$$(z, w) = (z_1, \dots, z_n, w) = (x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n, u + iv) = (x + \sqrt{-1}y, u + \sqrt{-1}v)$$

vanishing at p in which $T_p M = \{u = 0\}$, so that M is represented in a neighborhood of p by a graphed defining equation of the form:

$$u = \varphi(x, y, v) = \varphi(x_1, \dots, x_n, y_1, \dots, y_n, v),$$

where the real-valued function:

$$\varphi = \varphi(x, y, v) = \sum_{\substack{k \in \mathbb{N}^n, l \in \mathbb{N}^n, m \in \mathbb{N} \\ |k| + |l| + m \geq 2}} \varphi_{k, l, m} x^k y^l v^m \in \mathbb{R}\{x, y, u\},$$

which possesses entirely arbitrary real coefficients $\varphi_{k, l, m}$, vanishes at the origin: $\varphi(0) = 0$, together with all its first order derivatives: $0 = \partial_{x^k} \varphi(0) = \partial_{y^l} \varphi(0) = \partial_v \varphi(0)$. By simply rewriting this initial real equation of M as:

$$\frac{w + \bar{w}}{2} = \varphi\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\sqrt{-1}}, \frac{w - \bar{w}}{2\sqrt{-1}}\right),$$

and then by solving the so written equation with respect to w , one obtains an equation of the shape:

$$w = \Theta(z, \bar{z}, \bar{w}) = \sum_{\substack{k \in \mathbb{N}^n, l \in \mathbb{N}^n, m \in \mathbb{N} \\ |k| + |l| + m \geq 1}} \Theta_{k, l, m} z^k \bar{z}^l \bar{w}^m \in \mathbb{C}\{\bar{z}, z, w\},$$

² At the conference *Cauchy-Riemann Analysis and Geometry* organized by Ingo Lieb and Gerd Schmalz at the *Max-Planck Institut* of Bonn, 22–27 September 2003, the author gave a talk the title of which was “*Explicit Chern-Moser tensors*”.

whose right-hand side converges of course near the origin $(0, 0, 0) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ and whose coefficients $\Theta_{k,l,m} \in \mathbb{C}$ are *complex*. Since $d\varphi(0) = 0$, one has $\Theta = -\bar{w} + \text{order } 2 \text{ terms}$.

The paradox that any such *complex* equation provides in fact *two* real defining equations for the *real* hypersurface M which is *one*-codimensional, and also in addition the fact that one could as well have chosen to solve the above equation with respect to \bar{w} , instead of w , these two apparent ‘‘contradictions’’ are corrected by means of a fundamental, elementary statement that transfers to Θ (in a natural way) the condition of reality:

$$\overline{\varphi(x, y, u)} = \sum_{|k|+|l|+m \geq 1} \overline{\varphi_{k,l,m}} \bar{x}^k \bar{y}^l \bar{v}^m = \sum_{|k|+|l|+m \geq 1} \varphi_{k,l,m} x^k y^l v^m = \varphi(x, y, v)$$

enjoyed by the initial defining function φ . In the sequel, we shall work exclusively with Θ ; the reader is referred to [21] for justifications and motivations.

Theorem. ([18], p. 19) *The complex analytic function $\Theta = \Theta(z, \bar{z}, \bar{w})$ with $\Theta = -\bar{w} + O(2)$ together with its complex conjugate:*

$$\bar{\Theta} = \bar{\Theta}(\bar{z}, z, w) = \sum_{k \in \mathbb{N}^n, l \in \mathbb{N}^n, m \in \mathbb{N}} \bar{\Theta}_{k,l,m} \bar{z}^k z^l w^m \in \mathbb{C}\{\bar{z}, z, w\}$$

satisfy the two (equivalent by conjugation) functional equations:

$$(6) \quad \begin{aligned} \bar{w} &\equiv \bar{\Theta}(\bar{z}, z, \Theta(z, \bar{z}, \bar{w})), \\ w &\equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)). \end{aligned}$$

Conversely, given a local holomorphic function $\Theta(z, \bar{z}, \bar{w}) \in \mathbb{C}\{z, \bar{z}, \bar{w}\}$, $\Theta = -\bar{w} + O(2)$ which, in conjunction with its conjugate $\bar{\Theta}(\bar{z}, z, w)$, satisfies this pair of equivalent identities, then the two zero-sets:

$$\{0 = -w + \Theta(z, \bar{z}, \bar{w})\} \quad \text{and} \quad \{0 = -\bar{w} + \bar{\Theta}(\bar{z}, z, w)\}$$

coincide and define a local one-codimensional real analytic hypersurface M passing through the origin in \mathbb{C}^{n+1} .

Levi nondegeneracy. Within the hierarchy of nondegeneracy conditions for real hypersurfaces initiated by Diederich and Webster ([7], see also [15, 16] for generalizations and a unification), Levi nondegeneracy is the most studied. The classical definition may be found in [2] and in the survey of Chirka [6], but the following basic equivalent characterization can also be understood as a definition in the present paper. One may show ([15, 16, 18]) that it is biholomorphically invariant.

Lemma. ([18], p. 28) *The real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with $0 \in M$ represented in coordinates (z_1, \dots, z_n, w) by a complex defining equation of the form $w = \Theta(z, \bar{z}, \bar{w})$ is Levi nondegenerate at the origin if and only if the map:*

$$(\bar{z}_1, \dots, \bar{z}_n, \bar{w}) \longmapsto \left(\Theta(0, \bar{z}, \bar{w}), \frac{\partial \Theta}{\partial z_1}(0, \bar{z}, \bar{w}), \dots, \frac{\partial \Theta}{\partial z_n}(0, \bar{z}, \bar{w}) \right)$$

has nonvanishing $(n+1) \times (n+1)$ Jacobian determinant at $(\bar{z}, \bar{w}) = (0, 0)$.

It follows then that this Jacobian determinant, not restricted to the origin:

$$(7) \quad \Delta = \Delta(z, \bar{z}, \bar{w}) := \begin{vmatrix} \Theta_{\bar{z}_1} & \cdots & \Theta_{\bar{z}_n} & \Theta_{\bar{w}} \\ \Theta_{z_1 \bar{z}_1} & \cdots & \Theta_{z_1 \bar{z}_n} & \Theta_{z_1 \bar{w}} \\ \cdots & \cdots & \cdots & \cdots \\ \Theta_{z_n \bar{z}_1} & \cdots & \Theta_{z_n \bar{z}_n} & \Theta_{z_n \bar{w}} \end{vmatrix}$$

does not vanish in some small neighborhood of the origin in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. Levi nondegeneracy at the central point, *i.e.* $\Delta \neq 0$ locally, will be assumed throughout the present paper.

Associated system of partial differential equations. At least since the publication in 1888 by Lie and Engel in Leipzig of the *Theorie der Transformationsgruppen*, it is known in a very general context — see Chapter 10 of [8] and also [23, 10, 4, 9, 19, 1, 21] — that, to the whole family of Segre varieties:

$$S_{\bar{z}, \bar{w}} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : w = \Theta(z, \bar{z}, \bar{w})\}$$

parametrized by the $n + 1$ antiholomorphic variables $(\bar{z}_1, \dots, \bar{z}_n, \bar{w})$, one may canonically associate a completely integrable second-order system of partial differential equations whose general solution is precisely the function $\Theta(z, \bar{z}, \bar{w})$. Indeed, considering w as a function $w = w(z)$ of (z_1, \dots, z_n) in the defining equation of M , one differentiates it once with respect to each variable z_1, \dots, z_n so that one gets the $n + 1$ equations:

$$w(z) = \Theta(z, \bar{z}, \bar{w}), \quad w_{z_1}(z) = \frac{\partial \Theta}{\partial z_1}(z, \bar{z}, \bar{w}), \quad \dots, \quad w_{z_n}(z) = \frac{\partial \Theta}{\partial z_n}(z, \bar{z}, \bar{w}).$$

Then by means of the implicit function theorem — which applies precisely thanks to the nonvanishing of Δ —, one may clearly solve for the $n + 1$ antiholomorphic “parameters” (\bar{z}, \bar{w}) , and this procedure provides a representation:

$$\bar{z}_1 = \zeta_1(z, w(z), w_z(z)), \quad \dots, \quad \bar{z}_n = \zeta_n(z, w(z), w_z(z)), \quad \bar{w} = \xi(z, w(z), w_z(z))$$

with certain $n + 1$ uniquely defined local complex analytic functions $\zeta_i(z, w, w_z)$ and $\xi(z, w, w_z)$ of $2n + 1$ complex variables. Utilizing these functions, one is then pushed to replace \bar{z} and \bar{w} in all possible second-order derivative:

$$(8) \quad \begin{aligned} w_{z_{k_1} z_{k_2}}(z) &= \frac{\partial^2 \Theta}{\partial z_{k_1} \partial z_{k_2}}(z, \bar{z}, \bar{w}) \\ &= \frac{\partial^2 \Theta}{\partial z_{k_1} \partial z_{k_2}}(z, \zeta(z, w(z), w_z(z)), \xi(z, w(z), w_z(z))) \\ &=: \Phi_{k_1, k_2}(z, w(z), w_z(z)) \quad (k_1, k_2 = 1 \dots n), \end{aligned}$$

and this defines without ambiguity the associated system of partial differential equations. It is of second order. It is *complete*: all second-order derivatives are functions of derivatives of lower order ≤ 1 . In a sense to be precised right now, it is also *completely integrable* because by construction, its general solution is $\Theta(z, \bar{z}, \bar{w})$.

Geometric characterization of pseudosphericality. It is well known that the unit sphere:

$$S^{2n+1} = \{(z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C} : |z_1|^2 + \cdots + |z_n|^2 + |w|^2 = 1\}$$

in \mathbb{C}^n minus one of its points, for instance: $S^{2n+1} \setminus \{p_\infty\}$ with $p_\infty := (0, \dots, 0, -1)$, is biholomorphic, through the so-called *Cayley transform*:

$$(z_1, \dots, z_n, w) \longmapsto \left(\frac{iz_1}{1+w}, \dots, \frac{iz_n}{1+w}, \frac{1-w}{2+2w} \right) =: (z'_1, \dots, z'_n, w')$$

having inverse:

$$(z'_1, \dots, z'_n, w') \longmapsto \left(\frac{-2iz'_1}{1+2w'}, \dots, \frac{-2iz'_n}{1+2w'}, \frac{1-2w'}{1+2w'} \right) = (z_1, \dots, z_n, w)$$

to the so-called *standard Heisenberg sphere* of equation:

$$w' = -\overline{w'} + z'_1 \overline{z'_1} + \dots + z'_n \overline{z'_n}$$

which sits in the target space (z', w') . Hence in the particular case when the Levi form of M has only positive eigenvalues, namely when $q = n$ in (3), it follows clearly that M is *spherical* in the sense given in the Introduction if and only if there exists a nonempty open neighborhood U_0 of 0 in \mathbb{C}^{n+1} such that $M \cap U_0$ is biholomorphic to a piece of the unit sphere. In general, there are $n - q$ negative eigenvalues in the Levi form, and this justifies adding a “*pseudo*”.

Proposition. *A Levi nondegenerate local real analytic hypersurface M in \mathbb{C}^{n+1} is locally biholomorphic to a piece of the Heisenberg pseudosphere (hence pseudo-spherical) if and only if its associated second-order ordinary complex differential equation is locally equivalent to the second-order system:*

$$w'_{z'_{k_1} z'_{k_2}}(z') = 0 \quad (1 \leq k_1, k_2 \leq n),$$

with identically vanishing right-hand side.

Proof. The $n = 1$ case, treated in great details by a previous reference [21], generalizes here with rather evident adaptations, hence will be skipped. As $n \geq 2$ throughout the present paper, one may also argue by slicing \mathbb{C}^{n+1} by all possible copies of \mathbb{C}^2 which pass through the origin and which contain the w -axis, so as to be able to apply the already detailed $n = 1$ case. \square

Geometrically, the local equivalence of M to the Heisenberg pseudosphere means that, through some suitable local biholomorphism $(z, w) \mapsto (z', w')$ fixing the origin, both its Segre varieties and its conjugate Segre varieties ([15, 16, 18]):

$$S_{\overline{z}, \overline{w}} := \{(z, w) : w = \Theta(z, \overline{z}, \overline{w})\} \quad \text{and} \quad \overline{S}_{z, w} := \{(\overline{z}, \overline{w}) : \overline{w} = \overline{\Theta}(\overline{z}, z, w)\}$$

are mapped to the Segre and conjugate Segre varieties of the Heisenberg pseudosphere:

$$S'_{\overline{z}', \overline{w}'} = \{w' = -\overline{w}' + z' \overline{z}'\} \quad \text{and} \quad \{\overline{w}' = -w' + \overline{z}' z'\}$$

which, visibly, are plain complex affine lines.

§3. GEOMETRY OF ASSOCIATED SUBMANIFOLDS OF SOLUTIONS

Completely integrable systems of partial differential equations. The characterization of pseudosphericality we are dealing with holds in a context more general than just CR geometry³. Accordingly, let \mathbb{K} denote either the field \mathbb{C} of complex

³ We will be very brief here, the reader being referred to [19, 21] for the general theoretical considerations.

numbers or the field \mathbb{R} of real numbers, let $x = (x^1, \dots, x^n) \in \mathbb{K}^n$ with again $n \geq 2$ — since the case $n = 1$ was already studied in [21] —, let $y \in \mathbb{K}$, and consider a system of the form (1). We will assume that it is *completely integrable* in the sense that the natural commutativity of partial derivatives enjoyed trivially by the left-hand sides:

$$\partial^2 y_{x^{k_1} x^{k_2}} / \partial y_{x^{k_3}} = \partial^2 y_{x^{k_1} x^{k_3}} / \partial y_{x^{k_2}} \\ (1 \leq k_1, k_2, k_3 \leq n)$$

imposes immediately to the right-hand side functions F_{k_1, k_2} that they satisfy the so-called *compatibility conditions*:

$$D_{k_3}(F_{k_1, k_2}) = D_{k_2}(F_{k_1, k_3}),$$

where we have introduced the following n *total differentiation* operators:

$$D_k := \frac{\partial}{\partial x^k} + y_{x^k} \frac{\partial}{\partial y} + \sum_{\ell=1}^n F_{k, \ell} \frac{\partial}{\partial y_{x^\ell}} \\ (1 \leq k \leq n)$$

living on the first-order jet space $(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})$. One verifies that these compatibility conditions amount to the fact that the n -dimensional tangential distribution spanned by D_1, \dots, D_n in the $(2n + 1)$ -dimensional first-order jet space satisfies the classical Frobenius integrability condition $[D_{k'}, D_{k''}] = 0$, and then the Clebsch-Frobenius theorem tells us that this distribution comes from a local foliation by n -dimensional manifolds graphed over the x -space that are naturally parametrized by $n + 1$ auxiliary constants (transversal directions) — call them $a^1, \dots, a^n, b \in \mathbb{K}$ —, namely the leaves of this local foliation may be explicitly represented as sets of the shape:

$$\left\{ (x^1, \dots, x^n, Q(x^1, \dots, x^n, a^1, \dots, a^n, b), \right. \\ \left. S^1(x^1, \dots, x^n, a^1, \dots, a^n, b), \dots, S^n(x^1, \dots, x^n, a^1, \dots, a^n, b)) \right\},$$

where x^1, \dots, x^n vary freely and where Q, S^1, \dots, S^n are certain graphing functions. In fact, the functions S^k are the first-order derivatives:

$$S^1 = Q_{x^1}, \dots, S^n = Q_{x^n}$$

of the function Q , because by definition the integral curves of every vector field D_k must be contained in such leaves, so that one has:

$$\frac{\partial Q}{\partial x^k} = y_{x^k} \Big|_{\text{any leaf}} = S^k$$

and furthermore also:

$$\frac{\partial S^l}{\partial x^k} = F^{k, l} \Big|_{\text{any leaf}},$$

whence we see that the *fundamental graphing function* $Q = Q(x, a, b)$ happens to be the *general solution* to the initially given system of partial differential equations:

$$Q_{x^{k_1} x^{k_2}}(x, a, b) \equiv F_{k_1, k_2}(x, Q(x, a, b), Q_{x^1}(x, a, b), \dots, Q_{x^n}(x, a, b)) \\ (1 \leq k_1, k_2 \leq n).$$

In the CR case, the fundamental function which is the general solution to the associated system of partial differential equations (4) is obviously the complex defining function $\Theta(z, \bar{z}, \bar{w})$, where the $n + 1$ quantities (\bar{z}, \bar{w}) , viewed as independent variables, play the role of the constants (a, b) .

As in the $n = 1$ case, the constants (a^1, \dots, a^n, b) are best interpreted as a set of $n + 1$ *initial conditions* $(y_{x^1}(0), \dots, y_{x^n}(0), -y(0))$ or *integration constants*, so that we can assume without loss of generality that the first-order terms in the fundamental function Q are⁴:

$$Q(x, a, b) = -b + x^1 a^1 + \dots + x^n a^n + \mathcal{O}(|x|^2).$$

It is then clear that the map:

$$(9) \quad \begin{aligned} (a^1, \dots, a^n, b) &\longmapsto (Q(0, a, b), Q_{x^1}(0, a, b), \dots, Q_{x^n}(0, a, b)) \\ &= (-b, a^1, \dots, a^n) \end{aligned}$$

is of rank $n + 1$ at the origin, and this property remains also true whatever one chooses as a fundamental function $Q(x, a, b)$, that is to say, without necessarily assuming it to be normalized as above, which amounts to saying that⁵, in the parameter (a, b) -space, everything holds invariantly up to any local \mathbb{K} -analytic transformation $(a, b) \mapsto (a', b')$ which does *not* involve the variables (x, y) .

The way how one recovers the system of partial differential equations is very similar to what we did in the CR case (4). Suppose indeed a bit more generally that we are given any local \mathbb{K} -analytic function $Q = Q(x, a, b)$ having the property that its first-order x -jet map (9) is of rank $n + 1$ at $(a, b) = (0, 0)$. Then in the $n + 1$ equations:

$$y(x) = Q(x, a, b), \quad y_{x^1} = Q_{x^1}(x, a, b), \dots, \quad y_{x^n} = Q_{x^n}(x, a, b),$$

we can solve, by means of the implicit function theorem, for the $n + 1$ constants (a^1, \dots, a^n, b) , and this yields a representation:

$$\begin{aligned} a^k &= A^k(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \\ b &= B(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \end{aligned}$$

for certain functions A^1, \dots, A^n, B of $(2n + 1)$ variables. Then by replacing these obtained values for the a^k and for b in all the possible second-order derivatives:

$$\begin{aligned} y_{x^{k_1} x^{k_2}} &= Q_{x^{k_1} x^{k_2}}(x, a, b) \\ &= Q_{x^{k_1} x^{k_2}}(x, A(x, y, y_x), B(x, y, y_x)) \\ &= F_{k_1, k_2}(x, y, y_x) \end{aligned}$$

it is rigorously clear that one may only recover the functions F_{k_1, k_2} we started with.

⁴ We put a minus sign in front of $y(0)$ so as to match up with our choice of complex defining equation $w = -\bar{w} + \mathcal{O}(2)$.

⁵ Much more theoretical information is provided in [19].

§4. EFFECTIVE DIFFERENTIAL CHARACTERIZATION
OF PSEUDOSPHERICALITY IN \mathbb{C}^{n+1}

The $2n + 1$ coordinates of the transformation considered at the moment:

$$(10) \quad (x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \longmapsto (x^1, \dots, x^n, a^1, \dots, a^n, b)$$

and those of its inverse are given by the collection of functions:

$$\begin{cases} x^j = x^j \\ a^k = A^k(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \\ b = B(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \end{cases} \quad \text{and} \quad \begin{cases} x^j = x^j \\ y = Q(x^1, \dots, x^n, a^1, \dots, a^n, b) \\ y_{x^k} = Q_{x^k}(x^1, \dots, x^n, a^1, \dots, a^n, b). \end{cases}$$

For uniformity and harmony, we shall admit by convention the equivalences of notation:

$$b \equiv a^{n+1} \quad \text{and} \quad B \equiv A^{n+1}.$$

Then by differentiating with respect to y_{x^ℓ} each one of the following $n+1$ identically satisfied equations:

$$\begin{aligned} y &\equiv Q(x^1, \dots, x^n, A^1(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), \dots, \\ &\quad A^n(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), A^{n+1}(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})) \\ y_{x^k} &\equiv Q_{x^k}(x^1, \dots, x^n, A^1(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), \dots, \\ &\quad A^n(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), A^{n+1}(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})), \end{aligned}$$

we get the following $n + n^2$ equations:

$$\begin{aligned} 0 &\equiv Q_{a^1} \frac{\partial A^1}{\partial y_{x^\ell}} + \dots + Q_{a^n} \frac{\partial A^n}{\partial y_{x^\ell}} + Q_{a^{n+1}} \frac{\partial A^{n+1}}{\partial y_{x^\ell}} \\ \delta_{k,\ell} &= Q_{x^k a^1} \frac{\partial A^1}{\partial y_{x^\ell}} + \dots + Q_{x^k a^n} \frac{\partial A^n}{\partial y_{x^\ell}} + Q_{x^k a^{n+1}} \frac{\partial A^{n+1}}{\partial y_{x^\ell}} \\ &\quad (k, \ell = 1 \dots n). \end{aligned}$$

Fixing any $\ell \in \{1, \dots, n\}$, thanks to the assumption (Levi nondegeneracy) that the Jacobian determinant:

$$\square = \square(a^1 | \dots | a^n | a^{n+1}) := \begin{vmatrix} Q_{a^1} & \dots & Q_{a^n} & Q_{a^{n+1}} \\ Q_{x^1 a^1} & \dots & Q_{x^1 a^n} & Q_{x^1 a^{n+1}} \\ \vdots & \ddots & \vdots & \vdots \\ Q_{x^n a^1} & \dots & Q_{x^n a^n} & Q_{x^n a^{n+1}} \end{vmatrix},$$

does not vanish, we may solve — just by means of Cramer's rule — for the $n + 1$ unknowns $\frac{\partial A^\mu}{\partial y_{x^\ell}}$, the above system of $n + 1$ equations, and this gives us:

$$(11) \quad \frac{\partial A^\mu}{\partial y_{x^\ell}} = \frac{\square_{[0_{1+\ell}^\mu]}}{\square} := \frac{\square(a^1 | \dots | a^{\mu-1} | 0_{1+\ell} | a^{\mu+1} | \dots | a^{n+1})}{\square(a^1 | \dots | a^{\mu-1} | a^\mu | a^{\mu+1} | \dots | a^{n+1})},$$

where $0_{1+\ell}$ is a specific notation to denote the column consisting of $n + 1$ zeros piled up, except at the $(1 + \ell)$ -th level from its top, where instead of 0, one reads 1, and where, as our notation with vertical bars helps to guess:

$$\square_{[0_{1+\ell}^\mu]}^\mu = \square(a^1 | \dots | a^{\mu-1} |^\mu 0_{1+\ell} | a^{\mu+1} | \dots | a^{n+1}) :=$$

$$:= \left(\begin{array}{ccccccc} Q_{a^1} & \cdots & Q_{a^{\mu-1}} & 0 & Q_{a^{\mu+1}} & \cdots & Q_{a^{n+1}} \\ Q_{x^1 a^1} & \cdots & Q_{x^1 a^{\mu-1}} & 0 & Q_{x^1 a^{\mu+1}} & \cdots & Q_{x^1 a^{n+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Q_{x^k a^1} & \cdots & Q_{x^k a^{\mu-1}} & 1 & Q_{x^k a^{\mu+1}} & \cdots & Q_{x^k a^{n+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Q_{x^n a^1} & \cdots & Q_{x^n a^{\mu-1}} & 0 & Q_{x^n a^{\mu+1}} & \cdots & Q_{x^n a^{n+1}} \end{array} \right).$$

To avoid any ambiguity, we shall sometimes put the integer μ in the upper index position of the vertical bar to indicate precisely which column is concerned. As is clear, this notation allows one to view and to remember what are the involved partial derivatives of the fundamental function Q that appear inside each column. In summary, $\square_{[0_{1+\ell}]}^\mu$ comes from \square by changing just its μ -th column, as Cramer's rule classically says.

Next, the two-ways transfer between local functions G defined in the (x, y, y_x) -space and local functions T defined in the (x, a, b) -space, namely the one-to-one correspondence:

$$G(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \longleftrightarrow T(x^1, \dots, x^n, a^1, \dots, a^n, b)$$

through the diffeomorphism (10), may be viewed concretely, in the direction we are interested in, as the following identity:

$$\begin{aligned} & G(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}) \equiv \\ \equiv & T(x^1, \dots, x^n, \\ & A^1(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), \dots, A^n(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}), \\ & A^{n+1}(x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n})) \end{aligned}$$

holding of course in $\mathbb{C}\{x^1, \dots, x^n, y, y_{x^1}, \dots, y_{x^n}\}$. We therefore readily deduce how the derivation $\frac{\partial}{\partial y_{x^\ell}}$ is transferred to the (x, a, b) -space:

$$\frac{\partial G}{\partial y_{x^\ell}} = \frac{\partial A^1}{\partial y_{x^\ell}} \cdot \frac{\partial T}{\partial a^1} + \cdots + \frac{\partial A^n}{\partial y_{x^\ell}} \cdot \frac{\partial T}{\partial a^n} + \frac{\partial A^{n+1}}{\partial y_{x^\ell}} \cdot \frac{\partial T}{\partial a^{n+1}}.$$

By applying twice any two such derivations $\partial/\partial y_{x^{\ell_1}}$ and $\partial/\partial y_{x^{\ell_2}}$ to an arbitrary function G , we may see, after a few computations, what such a composed differentiation corresponds to, in terms of the function T defined in the (x, a, b) -space:

$$\begin{aligned} \frac{\partial^2 G}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}} &= \left(\sum_{\mu=1}^{n+1} \frac{\partial A^\mu}{\partial y_{x^{\ell_1}}} \frac{\partial}{\partial a^\mu} \right) \left[\sum_{\nu=1}^{n+1} \frac{\partial A^\nu}{\partial y_{x^{\ell_2}}} \frac{\partial T}{\partial a^\nu} \right] \\ &= \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\partial A^\mu}{\partial y_{x^{\ell_1}}} \frac{\partial A^\nu}{\partial y_{x^{\ell_2}}} \frac{\partial^2 T}{\partial a^\mu \partial a^\nu} + \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\partial A^\mu}{\partial y_{x^{\ell_1}}} \frac{\partial}{\partial a^\mu} \left[\frac{\partial A^\nu}{\partial y_{x^{\ell_2}}} \right] \frac{\partial T}{\partial a^\nu}. \end{aligned}$$

Here, by a helpful formal convention, the three Greek letters μ, ν and τ will be used as summation indices in the total set $\{1, \dots, n, n+1\}$, while the four Latin letters i, j, k, ℓ will always run in the restricted set $\{1, \dots, n\}$. Replacing then the partial

derivatives of the A^μ by their values (11) obtained previously, we thus get:

$$\begin{aligned} \frac{\partial^2 G}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}} &= \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\square_{[0_{1+\ell_1}]}^\mu}{\square} \frac{\square_{[0_{1+\ell_2}]}^\nu}{\square} \frac{\partial^2 T}{\partial a^\mu \partial a^\nu} + \\ &+ \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \left\{ \frac{\square_{[0_{1+\ell_1}]}^\mu}{\square} \cdot \frac{\square \cdot \frac{\partial}{\partial a^\mu} (\square_{[0_{1+\ell_2}]}^\nu) - \square_{[0_{1+\ell_2}]}^\nu \cdot \frac{\partial}{\partial a^\mu} (\square)}{\square \cdot \square} \right\} \frac{\partial T}{\partial a^\nu} \end{aligned}$$

Here, the coefficients of the $\frac{\partial^2 T}{\partial a^\mu \partial a^\nu}$ will not be touched anymore, but the coefficients of the $\frac{\partial T}{\partial a^\nu}$ must be subjected to further transformations towards formal harmony, especially the numerator involving a subtraction.

First of all, let us rewrite in length the concerned partial derivative of the appearing modified Jacobian determinant⁶:

$$\begin{aligned} \frac{\partial}{\partial a^\mu} (\square_{[0_{1+\ell_2}]}^\nu) &= \frac{\partial}{\partial a^\mu} \left[\square (a^1 | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1}) \right] \\ &= \square (a^1 a^\mu | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1}) + \dots + \\ &+ \square (a^1 | \dots | a^{\nu-1} a^\mu | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1}) \\ &+ 0 + \\ &+ \square (a^1 | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} a^\mu | \dots | a^{n+1}) + \dots + \\ &+ \square (a^1 | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1} a^\mu), \end{aligned}$$

and also at the same time the partial derivative of the plain Jacobian determinant:

$$\begin{aligned} \frac{\partial}{\partial a^\mu} (\square) &= \frac{\partial}{\partial a^\mu} \left[\square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) \right] \\ &= \square (a^1 a^\mu | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) + \dots + \\ &+ \square (a^1 | \dots | a^{\nu-1} a^\mu | a^\nu | a^{\nu+1} | \dots | a^{n+1}) + \\ &+ \square (a^1 | \dots | a^{\nu-1} | a^\nu a^\mu | a^{\nu+1} | \dots | a^{n+1}) + \\ &+ \square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} a^\mu | \dots | a^{n+1}) + \dots + \\ &+ \square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1} a^\mu). \end{aligned}$$

Consequently, the numerator with a subtraction that we want to simplify may be rewritten in length as follows:

$$\begin{aligned} (12) \quad & \square \cdot \frac{\partial}{\partial a^\mu} (\square_{[0_{1+\ell_2}]}^\nu) - \square_{[0_{1+\ell_2}]}^\nu \cdot \frac{\partial}{\partial a^\mu} (\square) = \\ &= \frac{\square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) \cdot \square (a^1 a^\mu | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1})}{\textcircled{a}} + \dots + \\ &+ \frac{\square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) \cdot \square (a^1 | \dots | a^{\nu-1} a^\mu | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1})}{\textcircled{b}} + \\ &+ 0 + \\ &+ \frac{\square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) \cdot \square (a^1 | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} a^\mu | \dots | a^{n+1})}{\textcircled{c}} + \dots + \\ &+ \frac{\square (a^1 | \dots | a^{\nu-1} | a^\nu | a^{\nu+1} | \dots | a^{n+1}) \cdot \square (a^1 | \dots | a^{\nu-1} | 0_{[1+\ell_2]} | a^{\nu+1} | \dots | a^{n+1} a^\mu)}{\textcircled{d}} - \end{aligned}$$

⁶ Remind that, in order to differentiate a determinant, one should differentiate separately each column once and then sum all the obtained terms.

$$\begin{aligned}
& - \underbrace{\square(a^1 \cdots |a^{\nu-1}|_{0_{[1+\ell_2]}} |a^{\nu+1}| \cdots |a^{n+1}|) \cdot \square(a^1 a^\mu \cdots |a^{\nu-1}| a^\nu |a^{\nu+1}| \cdots |a^{n+1}|)}_{\text{a}} - \cdots - \\
& - \underbrace{\square(a^1 \cdots |a^{\nu-1}|_{0_{[1+\ell_2]}} |a^{\nu+1}| \cdots |a^{n+1}|) \cdot \square(a^1 \cdots |a^{\nu-1}| a^\mu |a^\nu |a^{\nu+1}| \cdots |a^{n+1}|)}_{\text{b}} - \\
& - \underbrace{\square(a^1 \cdots |a^{\nu-1}|_{0_{[1+\ell_2]}} |a^{\nu+1}| \cdots |a^{n+1}|) \cdot \square(a^1 \cdots |a^{\nu-1}| a^\nu a^\mu |a^{\nu+1}| \cdots |a^{n+1}|)}_{\text{OK}} - \\
& - \underbrace{\square(a^1 \cdots |a^{\nu-1}|_{0_{[1+\ell_2]}} |a^{\nu+1}| \cdots |a^{n+1}|) \cdot \square(a^1 \cdots |a^{\nu-1}| a^\nu |a^{\nu+1}| a^\mu \cdots |a^{n+1}|)}_{\text{c}} - \cdots - \\
& - \underbrace{\square(a^1 \cdots |a^{\nu-1}|_{0_{[1+\ell_2]}} |a^{\nu+1}| \cdots |a^{n+1}|) \cdot \square(a^1 \cdots |a^{\nu-1}| a^\nu |a^{\nu+1}| \cdots |a^{n+1}| a^\mu)}_{\text{d}}.
\end{aligned}$$

The ante-penultimate underlined term “OK” will be kept untouched. To the pairs of (subtracted) \square -binomials that are underlined with a, b, c, d appended (including of course all terms present in the four “...”), we need an elementary instance of the Plücker identities.

To state it generally, let $m \geq 2$, let $C_1, C_2, \dots, C_m, D, E$ be $(m+2)$ column vectors in \mathbb{K}^m and introduce the following notation for the $m \times (m+2)$ matrix consisting of these vectors:

$$[C_1|C_2| \cdots |C_m|D|E].$$

Extracting columns from this matrix, we shall construct $m \times m$ determinants that are modification of the following “ground” determinant:

$$\|C_1| \cdots |C_m\| \equiv \|C_1| \cdots |^{j_1} C_{j_1}| \cdots |^{j_2} C_{j_2}| \cdots |C_m\|.$$

We use a double vertical line in the beginning and in the end to denote a determinant. Also, we emphasize two distinct columns, the j_1 -th and the j_2 -th, where $j_2 > j_1$, since we will modify them. For instance in this matrix, let us replace these two columns by the column D and by the column E , which yields the determinant:

$$\|C_1| \cdots |^{j_1} D| \cdots |^{j_2} E| \cdots |C_m\|.$$

In this notation, one should understand that *only* the j_1 -th and the j_2 -th columns are distinct from the columns of the fundamental $m \times m$ “ground” determinant.

Lemma. ([17], p. 155) *The following quadratic identity between determinants holds true:*

$$\begin{aligned}
& \|C_1| \cdots |^{j_1} D| \cdots |^{j_2} E| \cdots |C_n\| \cdot \|C_1| \cdots |^{j_1} C_{j_1}| \cdots |^{j_2} C_{j_2}| \cdots |C_n\| = \\
& = \|C_1| \cdots |^{j_1} D| \cdots |^{j_2} C_{j_2}| \cdots |C_n\| \cdot \|C_1| \cdots |^{j_1} C_{j_1}| \cdots |^{j_2} E| \cdots |C_n\| - \\
& - \|C_1| \cdots |^{j_1} E| \cdots |^{j_2} C_{j_2}| \cdots |C_n\| \cdot \|C_1| \cdots |^{j_1} C_{j_1}| \cdots |^{j_2} D| \cdots |C_n\|.
\end{aligned}$$

Admitting this elementary statement without redoing its proof and applying it to all the above underlined pairs of (subtracted) monomials, after checking that all final signs are “−”, we obtain the following neat expression for (12):

$$\begin{aligned}
& \square \cdot \frac{\partial}{\partial a^\mu} (\square_{[0_{1+\ell_2}]}^\nu) - \square_{[0_{1+\ell_2}]}^\nu \cdot \frac{\partial}{\partial a^\mu} (\square) = \\
& = -\square(0_{[1+\ell_2]}| \cdots |^\nu a^\nu| \cdots |a^{n+1}|) \cdot \square(a^1| \cdots |^\nu a^1 a^\mu| \cdots |a^{n+1}|) - \cdots - \\
& - \square(a^1| \cdots |^\nu 0_{[1+\ell_2]}| \cdots |a^{n+1}|) \cdot \square(a^1| \cdots |^\nu a^\nu a^\mu| \cdots |a^{n+1}|) - \cdots - \\
& - \square(a^1| \cdots |^\nu a^\nu| \cdots |0_{[1+\ell_2]}|) \cdot \square(a^1| \cdots |^\nu a^{n+1} a^\mu| \cdots |a^{n+1}|),
\end{aligned}$$

or equivalently, in contracted form:

$$\square \cdot \frac{\partial}{\partial a^\mu} (\square_{[0_1+\ell_2]}^\nu) - \square_{[0_1+\ell_2]}^\nu \cdot \frac{\partial}{\partial a^\mu} (\square) = - \sum_{\tau=1}^{n+1} \square_{[0_1+\ell_2]}^\tau \cdot \square_{[a^\tau a^\mu]}^\nu.$$

Thanks to this sidework, coming back to the expression for $\frac{\partial^2 G}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}}$ we left pending above, we obtain:

$$\begin{aligned} \frac{\partial^2 G}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}} &= \frac{1}{\square^2} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \left\{ \square_{[0_1+\ell_1]}^\mu \cdot \square_{[0_1+\ell_2]}^\nu \right\} \frac{\partial^2 T}{\partial a^\mu \partial a^\nu} - \\ &- \frac{1}{\square^3} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \sum_{\tau=1}^{n+1} \left\{ \square_{[0_1+\ell_1]}^\mu \cdot \square_{[0_1+\ell_2]}^\tau \cdot \square_{[a^\mu a^\tau]}^\nu \right\} \frac{\partial T}{\partial a^\nu}. \end{aligned}$$

To really finalize this expression, we factor everything by $\frac{1}{\square^3}$ and we exchange the two summation indices ν and τ in the second line:

$$\boxed{\frac{\partial^2 G}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell_2}}} = \frac{1}{\square^3} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \square_{[0_1+\ell_1]}^\mu \cdot \square_{[0_1+\ell_2]}^\nu \left\{ \square \cdot \frac{\partial^2 T}{\partial a^\mu \partial a^\nu} - \sum_{\tau=1}^{n+1} \square_{[a^\mu a^\tau]}^\tau \cdot \frac{\partial T}{\partial a^\tau} \right\}}.$$

End of proof of the Main Theorem. As already explained in the Introduction, one applies to the system (8) Hachtroudi's characterization (5) of equivalence to the system $w'_{z'_{k_1} z'_{k_2}}(z') = 0$ with $x := z$, with $y := w$, with $a := \bar{z}$, with $b := \bar{w}$, with $(a, b) := \bar{t}$, with $Q := \Theta$, with $\square := \Delta$, with $G := \Phi_{k_1, k_2}$ and with $T := \frac{\partial^2 \Theta}{\partial z_{k_1} \partial z_{k_2}}$.

The denominator $\frac{1}{\Delta^3}$ can be cleared out, and we simply get the explicit fourth-order partial differential equation satisfied by Θ . This completes the proof of our Main Theorem and the paper may end up now. \square

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