A NEW EXAMPLE OF UNIFORMLY LEVI DEGENERATE HYPERSURFACE IN $\mathbb{C}^3$

HERVÉ GAUSSIER AND JOËL MERKER

Abstract. We present a homogeneous real analytic hypersurface in $\mathbb{C}^3$, two-nondegenerate, Levi uniform of rank one, with a seven dimensional CR automorphism group such that the isotropy group of each point is two-dimensional and commutative. The classical tube over the two dimensional real cone in $\mathbb{R}^3$ is also homogeneous and has a seven-dimensional CR automorphism group. However, our example is *not* biholomorphic to the tube over the real cone, because the two-dimensional isotropy groups of $\Gamma_{\mathbb{C}}$ are, in contrast, noncommutative.

1. Introduction

In the paper *Uniformly Levi degenerate CR manifolds; the 5-dimensional case* [E2001], Peter Ebenfelt presented the tube $\Gamma_{\mathbb{C}} := C + i\mathbb{R}^3 \subset \mathbb{C}^3$ over the real two-dimensional cone $C := \{x_1^2 + x_2^2 - x_3^2 = 0\} \subset \mathbb{R}^3$ as a standard model among the class of uniformly Levi degenerate, 2-nondegenerate real analytic hypersurfaces M in $\mathbb{C}^3$. Following the Élie Cartan algorithm to solve the equivalence problem, $\Gamma_{\mathbb{C}}$ is characterized by the vanishing of some curvature in [E2001]. It is also proved that the isotropy subalgebra $\mathfrak{Aut}_{CR}(M, p)$ of a point $p \in M$, namely the Lie subalgebra of infinitesimal CR automorphisms $K \in \mathfrak{Aut}_{CR}(M)$ defined in a neighborhood of p in M and vanishing at p, satisfies $\dim_{\mathbb{R}} \mathfrak{Aut}_{CR}(M, p) \leq 2$ and consequently also $\dim_{\mathbb{R}} \mathfrak{Aut}_{CR}(M) \leq 7$. As it is known that $\Gamma_{\mathbb{C}}$ is homogeneous and that $\dim_{\mathbb{R}} \mathfrak{Aut}_{CR}(\Gamma_{\mathbb{C}}, p) = 2$ for every $p \in \Gamma_{\mathbb{C}}$ (*see* [E2001]), one might ask whether the standard model $\Gamma_{\mathbb{C}}$ is characterized by the maximal dimension of its isotropy CR automorphism subalgebras. However we present here a simple example of a homogeneous hypersurface M_0 in $\mathbb{C}^3$, having two-dimensional isotropy subalgebras, which is *not* locally biholomorphically equivalent to $\Gamma_{\mathbb{C}}$. In fact, the isotropy subalgebras $\mathfrak{Aut}_{CR}(M_0, p)$ are commutative whereas the corresponding $\mathfrak{Aut}_{CR}(\Gamma_{\mathbb{C}}, p)$ are noncommutative. Analogously, the sphere is *not* the unique flat standard model among Levi nondegenerate hypersurfaces in $\mathbb{C}^3$, since there is also the quadric $\operatorname{Im} w = |z_1|^2 - |z_2|^2$, *see* [CM1974], [M2001].

Acknowledgment. The authors thank Bernard Coupet for precious remarks concerning this work.

2. Construction of the example

2.1. Preliminaries. A real analytic hypersurface M of $\mathbb{C}^{n+1}$ is *uniformly Levi degenerate of rank r at p* if the Levi form is of constant rank equal to $r < n$ over M. To study the geometry of M, higher order nondegeneracy conditions

Date: 2011-6-1.

1991 *Mathematics Subject Classification.* Primary: 32V40. Secondary: 34C14.

are then necessary. Let $M = \{z \in \mathbb{C}^{n+1} : \rho(z, \bar{z}) = 0\}$ with ρ real analytic and $d\rho \neq 0$ on M. Let $p \in M$.

Definition 2.1. ([BER1999]) The hypersurface M is *finitely nondegenerate at* p if there exists a positive integer k such that

$$\operatorname{span}\{\overline{L}^{\alpha} \nabla \rho \, (p, \overline{p}) : \ \alpha \in \mathbb{N}^n, \ |\alpha| \leq k\} = \mathbb{C}^{n+1}, \tag{2.1}$$

where $\nabla \rho$ is the holomorphic gradient of ρ, where $(\overline{L}_1, \ldots, \overline{L}_n)$ is a basis of CR vector fields near p and where $\overline{L}^{\alpha} = \overline{L}_1^{\alpha_1} \ldots \overline{L}_n^{\alpha_n}$.

The smallest integer $k =: \ell_p$ satisfying this condition is a local biholomorphic invariant called the *Levi type of* M *at* p (we put $\ell_p = \infty$ if no finite k satisfies (2.1)). We remind that M is called *holomorphically nondegenerate* if there is no holomorphic vector field tangent to an open subset of M. It is known (*see*, for instance, [BER1999]) that if M is *connected* and holomorphically nondegenerate, then there is a biholomorphic invariant ℓ_M called the *Levi type of* M with $1 \leq \ell_M \leq n-1$, and a *proper* real analytic subset Σ of M such that M is ℓ_M-nondegenerate at each point of $M \backslash \Sigma$.

Hence there are three different types of (connected) real analytic hypersurfaces in $\mathbb{C}^3$ *at a generic point*: **A:** the holomorphically degenerate ones, which are locally biholomorphic to a product $N \times \Delta$, where N is a real hypersurface in $\mathbb{C}^2$ and Δ is the unit disc in $\mathbb{C}$; **B:** the Levi nondegenerate ones, whose Levi type equals 1; and **C:** the 2-nondegenerate ones which are uniformly Levi-degenerate of rank one.

2.2. General form of a 2-nondegenerate hypersurface in $\mathbb{C}^3$. Let now M be a small piece of a *rigid* real analytic hypersurface passing through the origin in $\mathbb{C}^3$ given by $M =: \{w + \overline{w} = F(z, \overline{z})\}$, where $z = (z_1, z_2)$. Assume that M is uniformly Levi degenerate of rank 1, namely that the Levi form of M has exactly one nonzero eigenvalue at every point. So $M =: \{w + \overline{w} = z_1 \overline{z_1} + \sum_{k \geq 3} F^k(z, \overline{z})\}$, locally at the origin, where F^k is a homogeneous polynomial of degree k with respect to $z, \overline{z}$ and the Levi determinant of M must vanish identically:

$$\begin{vmatrix} 1 + \sum_{k \geq 3} F^k_{z_1 \overline{z_1}} & \sum_{k \geq 3} F^k_{z_2 \overline{z_1}} \\ \\ \sum_{k \geq 3} F^k_{z_1 \overline{z_2}} & \sum_{k \geq 3} F^k_{z_2 \overline{z_2}} \end{vmatrix} \equiv 0. \tag{2.2}$$

In particular $F^3_{z_2 \overline{z_2}} \equiv 0$ in a neighborhood of the origin, so there are four complex constants a_{2100}, a_{2001}, a_{1110} and a_{0120} such that : $F^3(z, \overline{z}) = a_{2100} z_1^2 \overline{z_1} + a_{2001} z_1^2 \overline{z_2} + a_{1110} z_1 z_2 \overline{z_1} + a_{0120} z_2^2 \overline{z_1} + \overline{a_{2100}} z_1 \overline{z_1}^2 + \overline{a_{2001}} z_2 \overline{z_1}^2 + \overline{a_{1110}} z_1 \overline{z_1} \overline{z_2} + \overline{a_{0120}} z_1 \overline{z_2}^2$.

The transformation $z \mapsto y = (y_1, y_2) := (z_1 + a_{2100} z_1^2 + a_{1110} z_1 z_2 + a_{0120} z_2^2, z_2)$ is a local biholomorphic map at the origin. In the (w, y_1, y_2) coordinates M is represented by $M = \{w + \overline{w} = y_1 \overline{y_1} + a_{2001} y_1^2 \overline{y_2} + \overline{a_{2001} y_1}^2 y_2 + \sum_{k \geq 4} G^k(y, \overline{y})\}$ where G^k is a homogeneous polynomial of degree k with respect to $(y, \overline{y})$. Assuming that M is 2-nondegenerate it follows that $a_{2001} \neq 0$ and so by a rescaling of the y_2 axis we may write :

$$M = \{w + \overline{w} = y_1 \overline{y_1} + y_1^2 \overline{y_2} + \overline{y_1}^2 y_2 + \sum_{k \geq 4} G^k(y, \overline{y})\}. \tag{2.3}$$

2.3. Construction of the example. Coming back to the previous notation z instead of y in (2.3), the vanishing of the Levi determinant (2.2) is now equivalent to the following equations:

$$\begin{cases} (\star)_4 \quad G^4_{z_2\overline{z_2}} \equiv 4z_1\overline{z_1} \\ (\star)_5 \quad G^5_{z_2\overline{z_2}} \equiv 2(z_1 G^5_{z_1\overline{z_2}} + \overline{z_1} G^5_{z_2\overline{z_1}}) \\ (\star)_k \quad G^k_{z_2\overline{z_2}} \equiv 2(z_1 G^{k-1}_{z_1\overline{z_2}} + \overline{z_1} G^{k-1}_{z_2\overline{z_1}}) + \sum_{j=4}^{k-2} G^j_{z_1\overline{z_2}} G^{k+2-j}_{z_2\overline{z_1}}, \\ \qquad\qquad \text{for every } k \geq 6. \end{cases} \tag{2.4}$$

The integration of $(\star)_4$ gives $G^4(z,\overline{z}) = 4z_1\overline{z_1}z_2\overline{z_2} + \tilde{G}^4(z,\overline{z})$ where $\tilde{G}^4_{z_2\overline{z_2}} \equiv 0$. We set $\tilde{G}^4 \equiv 0$ for the construction of a particular hypersurface denoted by M_0 in the sequel.

The integration of $(\star)_5$ gives $G^5(z,\overline{z}) = 4(z_1^2 z_2\overline{z_2}^2 + \overline{z_1}^2 z_2^2\overline{z_2}) + \tilde{G}^5(z,\overline{z})$ where $\tilde{G}^5_{z_2\overline{z_2}} \equiv 0$. We set $\tilde{G}^5 \equiv 0$ and subsequently $\tilde{G}^k \equiv 0$ for every $k \geq 6$, so we obtain the following expansion for the defining equation of M_0:

$$M_0 = \{w + \overline{w} = (z_1\overline{z_1} + z_1^2\overline{z_2} + \overline{z_1}^2 z_2) \sum_{k\geq 0} 4^k (z_2\overline{z_2})^k\}, \tag{2.5}$$

and M_0 is uniformly degenerate. By the change of variables $z \mapsto (z_1, 2z_2)$ we obtain the final form of M_0. More precisely we denote by M_0 the connected piece passing through the origin with $|z_2| < 1$:

$$M_0 = \{w + \overline{w} = \frac{2z_1\overline{z_1} + z_1^2\overline{z_2} + \overline{z_1}^2 z_2}{1 - z_2\overline{z_2}} : \ |z_2| < 1\}. \tag{2.6}$$

2.4. Geometry of M_0. The $(1,0)$ vector fields tangent to M_0 are generated by

$$\begin{cases} L_1 := \dfrac{\partial}{\partial z_1} + \left[\dfrac{2\overline{z_1} + 2z_1\overline{z_2}}{1 - z_2\overline{z_2}}\right] \dfrac{\partial}{\partial w}, \\ L_2 := \dfrac{\partial}{\partial z_2} + \left[\dfrac{(\overline{z_1} + z_1\overline{z_2})^2}{(1 - z_2\overline{z_2})^2}\right] \dfrac{\partial}{\partial w}. \end{cases} \tag{2.7}$$

The kernel of the Levi form is generated by the vector field

$$T := -\left[\frac{\overline{z_1} + z_1\overline{z_2}}{1 - z_2\overline{z_2}}\right] \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} - \left[\frac{(\overline{z_1} + z_1\overline{z_2})^2}{(1 - z_2\overline{z_2})^2}\right] \frac{\partial}{\partial w}. \tag{2.8}$$

Indeed, we compute:

$$[L_1, \overline{T}] = -\left(\frac{1}{1 - z_2\overline{z_2}}\right) \overline{L_1}, \qquad [L_2, \overline{T}] = -\left(\frac{\overline{z_1} + z_1\overline{z_2}}{(1 - z_2\overline{z_2})^2}\right) \overline{L_1}. \tag{2.9}$$

Finally, according to a theorem of Freeman [F1976], M_0 is necessarily foliated by complex curves. In fact, M_0 is foliated by the complex *lines* $z_1 := z_0 - \overline{z_0}\zeta$, $z_2 := \zeta$, $w := z_0\overline{z_0} + i\lambda - \zeta\overline{z_0}^2$, where $z_0 \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and where $\zeta \in \mathbb{C}$ satisfies $|\zeta| < 1$.

3. Biholomorphic inequivalence of $\Gamma_{\mathbb{C}}$ and M_0

The inequivalence of $\Gamma_{\mathbb{C}}$ and M_0 is based on the comparison of the isotropy subalgebras of two reference points for these hypersurfaces. In the two forecoming subsections, we determine the Lie algebra of the infinitesimal CR automorphisms of $\Gamma_{\mathbb{C}}$ and M_0.

3.1. Geometry of $\Gamma_{\mathbb{C}}$. The tube $\Gamma_{\mathbb{C}}$ is invariant under translations in the tube directions, dilatations and automorphisms of the real quadratic form $x_1^2+x_2^2-x_3^2$. The infinitesimal generators $K_1,\dots,K_7$ of these seven independent transformations form a Lie algebra and are the real parts of the following holomorphic vector fields :

$$
\begin{cases}
X^1 = i\dfrac{\partial}{\partial z_1}, \qquad X^2 = i\dfrac{\partial}{\partial z_2}, \qquad X^3 = i\dfrac{\partial}{\partial z_3},\\[2mm]
X^4 = z_1\dfrac{\partial}{\partial z_1} + z_2\dfrac{\partial}{\partial z_2} + z_3\dfrac{\partial}{\partial z_3}, \qquad X^5 = z_2\dfrac{\partial}{\partial z_1} - z_1\dfrac{\partial}{\partial z_2},\\[2mm]
X^6 = z_3\dfrac{\partial}{\partial z_1} + z_1\dfrac{\partial}{\partial z_3}, \qquad X^7 = z_3\dfrac{\partial}{\partial z_2} + z_2\dfrac{\partial}{\partial z_3}.
\end{cases}
\tag{3.1}
$$

These transformations $K_1,\dots,K_7$ exhaust the infinitesimal CR automorphisms of $\Gamma_{\mathbb{C}}$. Indeed, let $p \in \Gamma_{\mathbb{C}}$ outside the singular locus. One can prove that any biholomorphic local self-map of $\Gamma_{\mathbb{C}}$ defined in a neighborhood of p must be affine of the form $\mathbb{C}^3 \ni Z \mapsto \Phi(Z) + ib \in \mathbb{C}^3$, where $b \in \mathbb{R}^3$ and Φ is a *linear* mapping with real coefficients which stabilizes C near $\operatorname{Re} p$. Then Φ is necessarily a dilatation or an automorphism of $x_1^2+x_2^2-x_3^2$ (*see* [Pi1966] for the study of global biholomorphisms of tube domains or [E2001] for the local study).

Since $\Gamma_{\mathbb{C}}$ is homogeneous, we can study its local geometry in a neighborhood of the point $p_0 := (1,0,1)$. The $(1,0)$ vector fields tangent to $\Gamma_{\mathbb{C}}$ are then generated near p_0 by

$$
\begin{cases}
L_1 := \dfrac{\partial}{\partial z_1} + \dfrac{x_1}{x_3}\dfrac{\partial}{\partial z_3},\\[2mm]
L_2 := \dfrac{\partial}{\partial z_2} + \dfrac{x_2}{x_3}\dfrac{\partial}{\partial z_3}.
\end{cases}
\tag{3.2}
$$

Furthermore, the vector field $T := x_1\frac{\partial}{\partial z_1} + x_2\frac{\partial}{\partial z_2} + x_3\frac{\partial}{\partial z_3}$ spans the kernel of the Levi-form, because $[L_1,\overline{T}] = \frac{1}{2}\overline{L_1}$ and $[L_2,\overline{T}] = \frac{1}{2}\overline{L_2}$. Also, the regular locus of $\Gamma_{\mathbb{C}}$ (*cf.* [F1976]) is globally foliated by complex *lines* as follows: $z_1 := (r+is)\cos\theta + i\lambda$, $z_2 := (r+is)\sin\theta + i\mu$, $z_3 := r+is$. Also, computing (2.1), it is easy to check that $\Gamma_{\mathbb{C}}$ is 2-nondegenerate at every point.

Finally, the isotropy Lie algebra $\mathfrak{Aut}_{CR}(\Gamma_{\mathbb{C}}, p_0)$ is generated by the two vector fields $K_8 = K_4 - K_6$ and $K_9 = K_5 + K_7$. We observe that $[2K_8, 2K_9] = 2K_9$. Since the regular part of $\Gamma_{\mathbb{C}}$ is homogeneous, it follows that the isotropy algebras $\mathfrak{Aut}_{CR}(\Gamma_{\mathbb{C}}, p)$ are all noncommutative.

3.2. Computation of the Lie algebra of M_0. The computation of the Lie algebra of $\mathfrak{Aut}_{CR}(M_0)$ at the origin is based on the Lie theory of the prolongation of vector fields (*see* [Ol1986], [Ol1995]) and on the fundamental observation by Alexandre Sukhov in [Su2000] that the Segre varieties are solutions of nonlinear systems of PDE's. We would like to mention that the direct computation of this Lie algebra, using only the tangency condition for an infinitesimal CR automorphism of M, involves the resolution of a huge system of approximatively sixty partial linear differential equations. Using the Lie theory we restrict the study to a much simpler system of only nine partial linear differential equations.

Let us write $(x_1, x_2, u, \overline{x_1}, \overline{x_2}, \overline{u})$ instead of $(z_1, z_2, w, \overline{z_1}, \overline{z_2}, \overline{w})$ and let us consider $\overline{u}$, $\overline{x_1}$, $\overline{x_2}$ as complex parameters, x_1, x_2 as two independent variables and u as a dependent variable, *i.e.* as a function of x_1, x_2 in the defining equation (2.6) of M. Then the two differential terms:

$$u_{x_1} := \frac{\partial u}{\partial x_1} = \frac{2(\overline{x_1} + x_1\overline{x_2})}{1 - x_2\overline{x_2}}, \qquad u_{x_1^2} := \frac{\partial^2 u}{\partial x_1^2} = \frac{2\overline{x_2}}{1 - x_2\overline{x_2}} \tag{3.4}$$

are sufficient to express any partial derivative $u_{x_1^k x_2^l} = \frac{\partial^{k+l} u}{\partial x_1^k \partial x_2^l}$:

$$\begin{cases} u_{x_2} = \frac{1}{4}(u_{x_1})^2, & u_{x_1^3} = 0, \\ u_{x_1 x_2} = \frac{1}{2}\, u_{x_1}\, u_{x_1^2}, & u_{x_1^2 x_2} = \frac{1}{2}\,(u_{x_1^2})^2, \\ u_{x_2^2} = \frac{1}{4}\,(u_{x_1})^2\, u_{x_1^2}, & u_{x_1 x_2^2} = \frac{1}{2}\, u_{x_1}\,(u_{x_1^2})^2, \\ & u_{x_2^3} = \frac{3}{8}\,(u_{x_1})^2\,(u_{x_1^2})^2. \end{cases} \tag{3.5}$$

Let us denote $J^3_{2,1}(\mathbb{C})$ the jet space of partial derivatives up to order three of one function u depending on two complex variables x_1, x_2, equipped with coordinates

$$(x_1, x_2, u, U^1_1, U^2_1, U^2_{1,1}, U^2_{1,2}, U^2_{2,2}, U^3_{1,1,1}, U^3_{1,1,2}, U^3_{1,2,2}, U^3_{2,2,2}). \tag{3.6}$$

To the system (3.5) corresponds the following complex submanifold of $J^3_{2,1}(\mathbb{C})$:

$$\begin{cases} U^1_2 = \frac{1}{4}\,(U^1_1)^2, & U^3_{1,1,1} = 0, \\ U^2_{1,2} = \frac{1}{2}\, U^1_1\, U^2_{1,1}, & U^3_{1,1,2} = \frac{1}{2}\,(U^2_{1,1})^2, \\ U^2_{2,2} = \frac{1}{4}\,(U^1_1)^2\, U^2_{1,1}, & U^3_{1,2,2} = \frac{1}{2}\, U^1_1\,(U^2_{1,1})^2, \\ & U^3_{2,2,2} = \frac{3}{8}\,(U^1_1)^2\,(U^2_{1,1})^2. \end{cases} \tag{3.7}$$

The computation of the Lie algebra $\mathfrak{Aut}_{CR}(M_0)$ is based on the following three observations:

1. The solutions of the system (3.5) are exactly the Segre varieties of M_0.
2. Every local biholomorphic map stabilizing M_0 maps every Segre variety onto another one and hence is a symmetry of the system (3.5).
3. Since M_0 is holomorphically nondegenerate, it follows from [Ca1932, pp. 30–32] that the group of CR automorphisms of M_0 is a maximally real subspace of the symmetry group of the differential system (3.5).

It remains to determine the symmetry group of (3.7), using the

Lie criterion. ([Ol1986]) *A vector field* $Y = Q^1 \frac{\partial}{\partial x_1} + Q^2 \frac{\partial}{\partial x_2} + R \frac{\partial}{\partial u}$ *is an infinitesimal symmetry of the system* (3.5) if and only if *its* third prolongation Y^3 *is tangent to the complex manifold defined by equations* (3.7) *in* $J^3_{2,1}(\mathbb{C})$.

The third prolongation of Y can be written (*cf.* [Su2000], [GM2001]):

$$(3.8)\qquad \begin{cases} Y^3 = Y + \sum_{1\leq j_1\leq 2} R^1_{j_1}\,\frac{\partial}{\partial U^1_{j_1}} + \sum_{1\leq j_1,j_2\leq 2} R^2_{j_1,j_2}\,\frac{\partial}{\partial U^2_{j_1,j_2}} + \\ \qquad + \sum_{1\leq j_1,j_2,j_3\leq 2} R^3_{j_1,j_2,j_3}\,\frac{\partial}{\partial U^3_{j_1,j_2,j_3}}, \end{cases}$$

where the terms $R^1_{j_1}$, $R^2_{j_1,j_2}$, $R^3_{j_1,j_2,j_3}$, $j_i = 1,2$, are computed by induction:

$$(3.9)\qquad \begin{cases} R^1_{j_1} & = & D_{j_1}(R) - \sum_{k=1}^2 D_{j_1}(Q^k)\,U^1_k, \\ R^2_{j_1,j_2} & = & D_{j_2}(R^1_{j_1}) - \sum_{k=1}^2 D_{j_2}(Q^k)\,U^2_{j_1,k}, \\ R^3_{j_1,j_2,j_3} & = & D_{j_3}(R^2_{j_1,j_2}) - \sum_{k=1}^2 D_{j_3}(Q^k)\,U^3_{j_1,j_2,k}, \end{cases}$$

using the operators of *total derivative* for $j = 1,2,3$:

$$(3.10)\qquad D_j = \frac{\partial}{\partial x_j} + U^1_j\,\frac{\partial}{\partial u} + \sum_{1\leq j_1\leq 2} U^2_{j,j_1}\,\frac{\partial}{\partial U^1_{j_1}} + \sum_{1\leq j_1,j_2\leq 2} U^3_{j,j_1,j_2}\,\frac{\partial}{\partial U^2_{j_1,j_2}}.$$

For instance, the expression of $R^3_{1,1,1}$ is as follows:

(3.11)

$$\begin{cases} R^3_{1,1,1} = R_{x_1^3} + [3R_{ux_1^2} - Q^1_{x_1^3}]\,U^1_1 + [-Q^2_{x_1^3}]\,U^1_2 + [3R_{u^2x_1} - 3Q^1_{ux_1^2}]\,(U^1_1)^2 + \\ \qquad + [-3Q^2_{ux_1^2}]\,U^1_1\,U^1_2 + [R_{u^3} - 3Q^1_{u^2x_1}]\,(U^1_1)^3 + [-3Q^2_{u^2x_1}]\,(U^1_1)^2\,U^1_2 + \\ \qquad + [-Q^1_{u^3}]\,(U^1_1)^4 + [-Q^2_{u^3}]\,(U^1_1)^3\,U^1_2 + [3R_{ux_1} - 3Q^1_{x_1^2}]\,U^2_{1,1} + \\ \qquad + [-2Q_{x_1^2}]\,U^2_{1,2} + [3R_{u^2} - 9Q^1_{ux_1}]\,U^1_1\,U^2_{1,1} + [-3Q^2_{ux_1}]\,U^1_2\,U^2_{1,1} + \\ \qquad + [-6Q^2_{ux_1}]\,U^1_1\,U^2_{1,2} + [-6Q^1_{u^2}]\,(U^1_1)^2\,U^2_{1,1} + [-3Q^2_{u^2}]\,(U^1_1)^2\,U^2_{1,2} + \\ \qquad + [-3Q^2_{u^2}]\,U^1_1\,U^1_2\,U^2_{1,1} + [-3Q^1_u]\,(U^2_{1,1})^2 + [-3Q^2_u]\,U^2_{1,1}\,U^2_{1,2} + \\ \qquad + [R_u - 3Q^1_{x_1}]\,U^3_{1,1,1} + [-5Q^2_{x_1}]\,U^3_{1,1,2} + [-4Q^1_u]\,U^1_1\,U^3_{1,1,1} + \\ \qquad + [-5Q^2_u]\,U^1_1\,U^3_{1,1,2} + +[-Q^2_u]\,U^1_2\,U^3_{1,1,1}. \end{cases}$$

The Lie criterion is equivalent to saying that the following equalities are satisfied on the variety defined by (3.7):

(3.12)

$$\begin{cases} R^1_2 = \frac{1}{2}\,R^1_1U^1_1, & R^3_{1,1,1} = 0, \\ R^2_{1,2} = \frac{1}{2}\,(R^2_{1,1}U^1_1 + R^1_1U^2_{1,1}), & R^3_{1,1,2} = R^2_{1,1}U^2_{1,1}, \\ R^2_{2,2} = \frac{1}{2}R^1_1U^1_1U^2_{1,1} + \frac{1}{4}\,R^2_{1,1}(U^1_1)^2, & R^3_{1,2,2} = \frac{1}{2}\,R^1_1(U^2_{1,1})^2 + R^2_{1,1}U^1_1, \\ & R^3_{2,2,2} = \frac{3}{4}\,R^1_1U^1_1(U^2_{1,1})^2 + R^2_{1,1}(U^1_1)^2U^2_{1,1}. \end{cases}$$

Substituting the explicit expressions of the R terms, we get a system of linear partial differential equations. We extract the following ones, which are sufficient to determine Y completely:

(3.13)
$$
\begin{cases}
\text{Constant term in } R^1_2 = \frac{1}{2}R^1_1U^1_1: & R_{x_2} = 0,\\
U^1_1 \text{ term in } R^1_2 = \frac{1}{2}R^1_1U^1_1: & Q^1_{x_2} = -\frac{1}{2}R_{x_1},\\
(U^1_1)^2 \text{ term in } R^1_2 = \frac{1}{2}R^1_1U^1_1: & \frac{1}{4}Q^2_{x_2} = \frac{1}{2}Q^1_{x_1} - \frac{1}{4}R_u,\\
(U^1_1)^3 \text{ term in } R^1_2 = \frac{1}{2}R^1_1U^1_1: & \frac{1}{4}Q^1_u = -\frac{1}{8}Q^2_{x_1},\\
(U^1_1)^4 \text{ term in } R^1_2 = \frac{1}{2}R^1_1U^1_1: & \frac{1}{16}Q^2_u = 0,\\
(U^1_1)^2 \text{ term in } R^2_{1,2} = \frac{1}{2}(R^2_{1,1}U^1_1 + R^1_1U^2_{1,1}): & -Q^1_{ux_2} + \frac{1}{4}(R_{ux_1} - Q^2_{x_1x_2}) =\\
& = \frac{1}{2}(2R_{ux_1} - Q^1_{x_1^2}),\\
(U^1_1)^3 \text{ term in } R^2_{1,2} = \frac{1}{2}(R^2_{1,1}U^1_1 + R^1_1U^2_{1,1}): & \frac{1}{4}(R_{u^2} - Q^1_{ux_1} - Q^2_{ux_2}) =\\
& = \frac{1}{2}(R_{u^2} - 2Q^1_{ux_1} - \frac{1}{4}Q^2_{x_1^2}),\\
\text{Constant term in } R^3_{1,1,1} = 0: & R_{x_1^3} = 0,\\
(U^2_{1,1})^2 \text{ term in } R^3_{1,1,1} = 0: & -\frac{5}{2}Q^2_{x_1} - 3Q^1_u = 0.
\end{cases}
$$

The resolution of these nine linear PDE's gives the following general form for a generator of the symmetry group of the system (3.5):

$$
\begin{cases}
Y = (\gamma + \delta x_1 + \alpha x_2 + \beta x_1 x_2)\frac{\partial}{\partial x_1} + (\mu + (2\delta + \varepsilon)x_2 + \beta x_2^2)\frac{\partial}{\partial x_2} +\\
\qquad + (\lambda - 2\alpha x_1 - \varepsilon u - \beta x_1^2)\frac{\partial}{\partial u},
\end{cases}
\tag{3.14}
$$

where α, β, γ, δ, ε, λ, μ are complex constants.
A basis of the infinitesimal CR automorphisms of M_0, expressed in the (z_1, z_2, w) coordinates, is given by the real parts of the following linear complex combinations of the seven generators given by (3.14) [Ca1932, pp. 30–32]):

$$
\begin{cases}
X^1 = i\frac{\partial}{\partial w},\\
X^2 = z_1\frac{\partial}{\partial z_1} + 2w\frac{\partial}{\partial w},\\
X^3 = i(z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2}),\\
X^4 = (z_2 - 1)\frac{\partial}{\partial z_1} - 2z_1\frac{\partial}{\partial w},\\
X^5 = i((1 + z_2)\frac{\partial}{\partial z_1} - 2z_1\frac{\partial}{\partial w}),\\
X^6 = z_1z_2\frac{\partial}{\partial z_1} + (z_2^2 - 1)\frac{\partial}{\partial z_2} - z_1^2\frac{\partial}{\partial w},\\
X^7 = i(z_1z_2\frac{\partial}{\partial z_1} + (z_2^2 + 1)\frac{\partial}{\partial z_2} - z_1^2\frac{\partial}{\partial w}).
\end{cases}
\tag{3.15}
$$

3.3. Local biholomorphic inequivalence of $\Gamma_{\mathbb{C}}$ and M_0. The isotropy algebra $\mathfrak{Aut}_{CR}(M_0, 0)$ is generated over $\mathbb{R}$ by the real parts of X^2 and X^3 and hence is commutative since $[X^2, X^3] = 0$. On the contrary the isotropy algebra $\mathfrak{Aut}_{CR}(\Gamma_{\mathbb{C}}, p_0)$ is generated by the vector fields K_8 and K_9 (*see* subsection 3.1), satisfying the condition $[2K_8, 2K_9] = 2K_9$. This implies that $\Gamma_{\mathbb{C}}$ and M_0 are locally biholomorphically inequivalent.

Open question. It would be of great interest to provide a complete classification of real analytic homogeneous hypersurfaces in $\mathbb{C}^3$, in the spirit of Élie Cartan's list in dimension two [Ca 1932].

References

[BER1999] Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, **47**, Princeton University Press, Princeton, NJ, 1999, xii+404 pp.

[Ca1932] Cartan, É.: *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I*, Ann. Math. Pura Appl. **11** (1932), 17–90, (or Oeuvres II, 1231–1304); *II*, Ann. Scuola Norm. Sup. Pisa **1** (1932), 333–354, (or Oeuvres III, 1217–1238).

[CM1974] Chern, S.-S.; Moser, J.K.: *Real analytic hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[E1998] Ebenfelt, P.: *Normal forms and the biholomorphic equivalence problem for real hypersurfaces in* $\mathbb{C}^3$, Indiana Univ. Math. J. **47** (1998), no.2, 311–366.

[E2001] Ebenfelt, P.: *Uniformly Levi degenerate CR manifolds; the 5-dimensional case*, Duke Math. J. **110** (2001), no. 1, 37–80.

[F1976] Freeman, M.: *The Levi form and local complex foliations*, Proc. Amer. Math. Soc. **57** (1976), no.2, 369–370.

[GM2001] Gaussier, H.; Merker, J.: *Estimates on the dimension of the symmetry group of a system of k-th order partial differential equations*, Preprint Université de Provence, **12** (2001), 27pp.

[M2001] Merker, J.: *Lie symmetries of integrable systems of holomorphic second order partial differential equations*, In preparation.

[Ol1986] Olver, P.J.: *Applications of Lie groups to differential equations.* Springer Verlag, Heidelberg, 1986.

[Ol1995] Olver, P.J.: *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.

[Pi1966] Piatetsky-Chapiro, I.I.: *Géométrie des domaines classiques et théorie des fonctions automorphes*, Travaux et recherches mathématiques 12, Dunod, 1966.

[Su2000] Sukhov, A.: *Segre varieties and Lie symmetries*, Math. Z. (to appear); e-print: `arXiv/abs/math.CV/0002197`.

(Gaussier) Université de Provence, CMI, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France

E-mail address: `Herve.Gaussier@cmi.univ-mrs.fr`

(Merker) Université de Provence, CMI, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France

E-mail address: `Joel.Merker@cmi.univ-mrs.fr`