NONALGEBRAIZABLE REAL ANALYTIC TUBES IN $\mathbb{C}^n$

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Abstract. We give necessary conditions for certain real analytic tube generic submanifolds in $\mathbb{C}^n$ to be locally algebraizable. As an application, we exhibit families of real analytic non locally algebraizable tube generic submanifolds in $\mathbb{C}^n$. During the proof, we show that the local CR automorphism group of a minimal, finitely nondegenerate real algebraic generic submanifold is a real algebraic local Lie group. We may state one of the main results as follows. Let M be a real analytic hypersurface tube in $\mathbb{C}^n$ passing through the origin, having a defining equation of the form $v = \varphi(y)$, where $(z, w) = (x + iy, u + iv) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Assume that M is Levi nondegenerate at the origin and that the real Lie algebra of local infinitesimal CR automorphisms of M is of minimal possible dimension n, *i.e.* generated by the real parts of the holomorphic vector fields $\partial_{z_1}, \dots, \partial_{z_{n-1}}, \partial_w$. Then M is locally algebraizable only if every second derivative $\partial^2_{y_k y_l}\varphi$ is an algebraic function of the collection of first derivatives $\partial_{y_1}\varphi, ..., \partial_{y_m}\varphi$.

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§1. Introduction

A real analytic submanifold M in $\mathbb{C}^n$ is called *algebraic* if it can be represented locally by the vanishing of a collection of Nash algebraic real analytic functions. We say that M is *locally algebraizable* at one of its points p if there exist some local holomorphic coordinates centered at p in which M is algebraic. For instance, every totally real, real analytic submanifold in $\mathbb{C}^n$ of dimension $k \leq n$ is locally biholomorphic to a k-dimensional linear real plane, hence locally algebraizable. Also, every complex manifold is locally algebraizable. Although every real analytic submanifold M is clearly locally equivalent to its tangent plane by a *real* analytic (in general not holomorphic) equivalence, the question whether M is biholomorphically equivalent to a real algebraic submanifold is subtle. In this article, we study the question whether every real analytic CR submanifold is locally algebraizable. One of the interests of algebraizability lies in the reflection principle, which is better understood in the algebraic category. Indeed, in the fundamental works of Pinchuk [Pi1975], [Pi1978] and of Webster [We1977], [We1978] and in the recent works of Sharipov-Sukhov [SS1996], Huang-Ji [HJ1998], Verma [Ve1999], Coupet-Pinchuk-Sukhov [CPS2000], and Shafikov [Sha2000], [Sha2002], the extendability of germs of CR mappings with target in a real algebraic hypersurface is achieved. On the contrary, even if some results previously shown under an algebraization hypothesis were proved recently under general assumptions (see the strong result obtained by Diederich-Pinchuk

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[DP2003]), most of the results cited above are still open in the case of a real analytic target hypersurface.

1.1. Brief history of the question. By the work of Moser and Webster [MW1983, Thm. 1], it is known that every real analytic two-dimensional surface $S \subset \mathbb{C}^2$ at an isolated elliptic (in the sense of Bishop) complex tangency $p \in S$ is biholomorphic to one of the surfaces $S_{\gamma,\delta,s} := \{(z_1, z_2) \in \mathbb{C}^2 : y_2 = 0, x_2 = z_1\bar{z}_1 + (\gamma + \delta(x_2)^s)(z_1^2 + \bar{z}_1^2)\}$, where p corresponds to the origin, where $0 < \gamma < 1/2$ is Bishop's invariant and where $\delta = \pm 1$ and $s \in \mathbb{N}$ or $\delta = 0$. The quantities γ, δ, s form a complete system of biholomorphic invariants for the surface S near p. In particular, every elliptic surface $S \subset \mathbb{C}^2$ is locally algebraizable. To the authors' knowledge, it is unknown whether there exist nonalgebraizable hyperbolic surfaces in $\mathbb{C}^2$. In fact, very few examples of nonalgebraizable submanifolds are known. In [Eb1996], the author constructed a nonminimal (and non Levi-flat) real analytic hypersurface M through the origin in $\mathbb{C}^2$ which is not locally algebraizable (*cf.* [BER2000, p. 330]). In a recent article [HJY] the authors prove that the strongly pseudoconvex real analytic hypersurface $\operatorname{Im} w = e^{|z|^2} - 1$ passing through the origin in $\mathbb{C}^2$ is not locally algebraizable at any of its points. Using an associated projective structure bundle $\mathcal{Y}$ introduced by Chern, they show that for every rigid algebraic hypersurface in $\mathbb{C}^n$, there exists an algebraic dependence relation between seven explicit Cartan-type holomorphic invariant functions on $\mathcal{Y}$. However a computational approach shows that when M is of the specific form $\operatorname{Im} w = e^{|z|^2} - 1$, no algebraic relation can be satisfied by these seven invariants.

1.2. Presentation of the main results. Our aim is to present a geometrical approach of the problem, valid in arbitrary dimension and in arbitrary codimension, and to exhibit a large class of nonalgebraizable real analytic generic submanifolds. We consider the class $\mathcal{T}_n^d$ of generic real analytic submanifolds in $\mathbb{C}^n$ passing through the origin, of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$, whose local CR automorphism group is n-dimensional, generated by the real parts of n holomorphic vector fields having holomorphic coefficients $X_1, \dots, X_n$ which are linearly independent at the origin and which commute: $[X_{i_1}, X_{i_2}] = 0$. We shall call $\mathcal{T}_n^d$ the class of *strong tubes* of codimension d. Indeed, since there exists a straightened system of coordinates $t = (t_1, \dots, t_n)$ over $\mathbb{C}^n$ in which $X_i = \partial_{t_i}$, we observe that every submanifold $M \in \mathcal{T}_n^d$ is tubifiable at the origin. By this, we mean that there exist holomorphic coordinates $t = (z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at the origin in which M is represented by d equations of the form $v_j = \varphi_j(y)$. Hence M is a tube, *i.e.* a product of the submanifold $\{v_j = \varphi_j(y),\ j = 1, \dots, d\} \subset \mathbb{R}^n_{y,v}$ by the n-dimensional real space $\mathbb{R}^n_{x,u}$. Since $M \in \mathcal{T}_n^d$, the only infinitesimal CR automorphisms of M are the real parts of the vector fields $\partial_{z_1}, \dots, \partial_{z_m}, \partial_{w_1}, \dots, \partial_{w_d}$, explaining the terminology. Notice that not every tube belongs to the class $\mathcal{T}_n^d$. For instance in codimension $d = 1$, the Heisenberg sphere $v = \sum_{k=1}^{n-1} y_k^2$ and more generally the Levi nondegenerate quadrics $v = \sum_{k=1}^{n-1} \varepsilon_k\, y_k^2$, where $\varepsilon_k = \pm 1$, have a CR automorphism group of dimension $(n+1)^2 - 1 > n$ and so do not belong to $\mathcal{T}_n^1$. We assume that $M \in \mathcal{T}_n^d$ is minimal at the origin, namely the local CR orbit of 0 in M contains a neighborhood of 0 in M. Furthermore, we assume that $M \in \mathcal{T}_n^d$ is finitely nondegenerate at 0, namely that there exists an integer $\ell \geq 1$ such that $\operatorname{Span}\{\overline{L}^\beta\, \nabla_t(r_j)(0,0) : \beta \in \mathbb{N}^m, |\beta| \leq \ell, j = 1, \dots, d\} = \mathbb{C}^n$, where $r_j(t, \bar{t}) = 0$, $j = 1, \dots, d$ are arbitrary real analytic defining functions for M near 0 satisfying $\partial r_1 \wedge \cdots \wedge \partial r_d \neq 0$ on M, where $\nabla_t(r_j)(t, \bar{t})$ is the holomorphic gradient with respect to t of r_j and where $\overline{L}^\beta$ denotes $(\overline{L}_1)^{\beta_1} \cdots (\overline{L}_m)^{\beta_m}$ for an arbitrary basis $\overline{L}_1, \dots, \overline{L}_m$ of $(0,1)$-vector fields tangent to M in a neighborhood of 0. In particular Levi nondegenerate hypersurfaces are finitely nondegenerate. Finally, assuming only that $\varphi_j(0) = 0$, $j = 1, \dots, d$, we shall observe in Lemma 3.2 below that a tube $v_j = \varphi_j(y)$ of codimension d is finitely nondegenerate at the origin if and only if there exist multi-indices

$\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \geq 1$ and integers $1 \leq j_*^1, \dots, j_*^m \leq d$ such that the real mapping

$$\psi(y) := \left(\frac{\partial^{|\beta_*^1|} \varphi_{j_*^1}(y)}{\partial y^{\beta_*^1}}, \dots, \frac{\partial^{|\beta_*^m|} \varphi_{j_*^m}(y)}{\partial y^{\beta_*^m}} \right) =: y' \in \mathbb{R}^m \tag{1.1}$$

is of rank m at the origin in $\mathbb{R}^m$. Our main theorem provides a necessary condition for the local algebraizability of strong tubes :

Theorem 1.1. *Let M be a real analytic generic tube of codimension d in $\mathbb{C}^n$ given in coordinates $(z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ by the equations $v_j = \varphi_j(y)$, where $\varphi_j(0) = 0$, $j = 1, \dots, d$. Assume that M is minimal and finitely nondegenerate at the origin, so the real mapping $\psi(y) = y'$ defined by (1.1) is of rank m at the origin in $\mathbb{R}_y^m$ and let $y = \psi'(y')$ denote the local inverse in $\psi(y)$. Assume that $M \in \mathcal{T}_n^d$, namely M is a strong tube of codimension d. If M is locally algebraizable at the origin, then all the derivative functions $\partial_{y'_k} \psi'_l(y')$, where $1 \leq k, l \leq m$, are real algebraic functions of y'. Equivalently, every second derivative $\partial^2_{y_k y_l} \varphi_j(y)$ is an algebraic function of the collection of first derivatives $\partial_{y_1} \varphi_j, \dots, \partial_{y_m} \varphi_j$.*

By contraposition, every real analytic strong tube $M \in \mathcal{T}_n^d$ for which one of the derivative functions $\partial_{y'_k} \psi'_l$ is not real algebraic is not locally algebraizable. We will argue in §8 that this is generically the case in the sense of Baire. It is however natural to look for explicit examples of nonalgebraizable real analytic submanifolds in $\mathbb{C}^n$. Since the real parts of the vector fields $\partial_{z_1}, \dots, \partial_{z_m}, \partial_{w_1}, \dots, \partial_{w_d}$ are infinitesimal CR automorphisms of every tube $v = \varphi(y)$, we must provide some sufficient conditions insuring that the dimension of the Lie algebra of such a tube is exactly n. We shall establish in §§7-8 below:

Corollary 1.2. *The tube hypersurface $M_{\chi_1, \dots, \chi_{n-1}}$ in $\mathbb{C}^n$ of equation $v = \sum_{k=1}^{n-1} [\varepsilon_k y_k^2 + y_k^6 + y_k^9 y_1 \cdots y_{k-1} + y_k^{n+8} \chi_k(y_1, \dots, y_{n-1})]$, where $\chi_1, \dots, \chi_{n-1}$ are arbitrary real analytic functions, belongs to the class $\mathcal{T}_n^1$ of strong tubes. Two such tubes $M_{\chi_1, \dots, \chi_{n-1}}$ and $M_{\widehat{\chi}_1, \dots, \widehat{\chi}_{n-1}}$ are biholomorphically equivalent if and only if $\chi_j = \widehat{\chi}_j$ for every j. Furthermore, for a generic choice in $\chi_1, \dots, \chi_{n-1}$ in the sense of Baire* (to be precised in §8), *$M_{\chi_1, \dots, \chi_{n-1}}$ is not locally algebraizable at the origin.*

Here we annihilate some Taylor coefficients in φ and keep some others to be nonzero to insure that M_χ is a strong tube. Furthermore, the terms $y_k^9 y_1 \cdots y_{k-1}$ insure that the M_χ are pairwise not biholomorphically equivalent. Using a classical direct algorithm (*cf.* [Bs1991], [St1991]), or the Lie theory of symmetries of differential equations, combined with Theorem 1.1 we may provide some other explicit strong tubes which are not locally algebraizable (see §§7-8 for the proof):

Corollary 1.3. *The following five explicit tubes belong to $\mathcal{T}_2^1$ and are not locally algebraizable at the origin: $v = \sin(y^2)$, $v = \tan(y^2)$, $v = e^{e^y - 1} - 1$, $v = \sinh(y^2)$ and $v = \tanh(y^2)$.*

In these five examples, the algebraic independence in $\partial_y \varphi$ and in $\partial^2_{yy} \varphi$ is clear; however, checking that each hypersurface is indeed a strong tube requires some formal computations, *see* §7. One may also check by a direct computation that in a neighborhood of every point $p = (z_p, w_p)$ with $z_p \neq 0$, the hypersurface $M_{\rm HJY}$ of global equation $\operatorname{Im} w = e^{|z|^2} - 1$ is a strong tube (*see* §7.5). Since it can be represented in a neighborhood of p under the tube form $v' = e^{|z_p|^2 (e^{y'} - 1)} - 1$ by means of the local change of coordinates $z' = 2i \ln(z/z_p)$, $w' = (w - w_p)\, e^{-|z_p|^2}$, applying Theorem 1.1 and inspecting the function $e^{|z_p|^2 (e^{y'} - 1)} - 1$, we may check that it is *not* algebraizable at such points p with $z_p \neq 0$ (*see* §7.5). It follows trivially that the hypersurface $M_{\rm HJY}$ is also not locally algebraizable at all the points p with $z_p = 0$, giving the result of [HJY, Theorem 1.1].

Using the same strategy as for Theorem 1.1, we obtain more generally the following criterion:

Theorem 1.4. *Let $M_\varphi : v = \varphi(z\bar{z})$ be a Levi nondegenerate real analytic hypersurface in $\mathbb{C}^2$ passing through the origin whose Lie algebra of local infinitesimal CR automorphisms is generated by ∂_w and $iz\,\partial_z$. If M_φ is locally algebraizable at the origin, then the first derivative $\partial_r\varphi$ in φ ($r \in \mathbb{R}$) is algebraic. For instance, the following seven explicit examples are not locally algebraizable at the origin: $v = e^{z\bar{z}} - 1$, $v = \sin(z\bar{z})$, $v = \tan(z\bar{z})$, $v = \sinh(z\bar{z})$, $v = \tanh(z\bar{z})$, $v = \sin(\sin(z\bar{z}))$ and $v = e^{e^{z\bar{z}}-1} - 1$.*

Finally, using the same recipe as for Theorems 1.1 and 1.4, we shall provide a very simple criterion for the local nonalgebraizability of some hypersurfaces having a local Lie CR automorphism group of dimension equal to one exactly. We consider the class $\mathcal{R}_n$ of Levi nondegenerate real analytic hypersurfaces passing through the origin in $\mathbb{C}^n$ ($n \geq 2$) such that the Lie algebra of infinitesimal CR automorphisms of M is generated by exactly one holomorphic vector field X_1 with holomorphic coefficients not all vanishing at the origin. We call $\mathcal{R}_n$ the class of *strongly rigid* hypersurfaces, in order to distinguish them from the so-called *rigid* ones whose local CR automorphism group may be of dimension larger than 1. By straightening X_1, we may assume that $X_1 = \partial_w$ and that M is given by a real analytic equation of the form $v = \varphi(z, \bar{z}) = \varphi(z_1, \dots, z_{n-1}, \bar{z}_1, \dots, \bar{z}_{n-1})$. By making some elementary changes of coordinates (*cf.* §3.3), we can furthermore assume without loss of generality that $\varphi(z, \bar{z}) = \sum_{k=1}^{n-1} \varepsilon_k\,|z_k|^2 + \chi(z, \bar{z})$, where $\varepsilon_k = \pm 1$ and $\chi(0, \bar{z}) \equiv \partial_{z_k}\chi(0, \bar{z}) \equiv 0$.

Theorem 1.5. *Let $M : v = \varphi(z, \bar{z}) = \sum_{k=1}^{n-1} \varepsilon_k\,|z_k|^2 + \chi(z, \bar{z})$ be a strongly rigid hypersurface in $\mathbb{C}^n$ with $\chi(0, \bar{z}) \equiv \partial_{z_k}\chi(0, \bar{z}) \equiv 0$. If M is locally algebraizable at the origin, then all the first derivatives $\partial_{z_k}\varphi$ are algebraic functions of $(z, \bar{z})$.*

This criterion enables us to exhibit a whole family of non locally algebraizable hypersurfaces in $\mathbb{C}^n$:

Corollary 1.6. *The rigid hypersurfaces $M_{\chi_1, \dots, \chi_{n-1}}$ in $\mathbb{C}^n$ of equation $v = \sum_{k=1}^{n-1} [\varepsilon_k\,|z_k|^2 + |z_k|^{10} + |z_k|^{14} + |z_k|^{16}(z_k + \bar{z}_k) + |z_k|^{18}|z_1|^2 \cdots |z_{k-1}|^2 + |z_k|^{2n+16}\,\chi_k(z, \bar{z})]$, where the χ_k are arbitrary real analytic functions, belong to the class $\mathcal{R}_n$ of strongly rigid hypersurfaces. Two such tubes $M_{\chi_1, \dots, \chi_{n-1}}$ and $M_{\widehat{\chi}_1, \dots, \widehat{\chi}_{n-1}}$ are biholomorphically equivalent if and only if $\chi_k = \widehat{\chi}_k$ for $k = 1, \dots, n-1$. Furthermore, for a generic choice of a $(n-1)$-tuple of real analytic functions $(\chi_1, \dots, \chi_{n-1})$ in the sense of Baire* (to be precised in §8)*, $M_{\chi_1, \dots, \chi_{n-1}}$ is not locally algebraizable at the origin.*

Finally, by computing generators of the Lie algebra of local infinitesimal CR automorphisms of some explicit examples, we obtain:

Corollary 1.7. *The following seven explicit examples of hypersurfaces in $\mathbb{C}^2$ are* strongly rigid *and are not locally algebraizable at the origin: $v = z\bar{z} + z^2\bar{z}^2 \sin(z + \bar{z})$, $v = z\bar{z} + z^2\bar{z}^2 \exp(z + \bar{z})$, $v = z\bar{z} + z^2\bar{z}^2 \cos(z + \bar{z})$, $v = z\bar{z} + z^2\bar{z}^2 \tan(z + \bar{z})$, $v = z\bar{z} + z^2\bar{z}^2 \sinh(z + \bar{z})$, $v = z\bar{z} + z^2\bar{z}^2 \cosh(z + \bar{z})$ and $v = z\bar{z} + z^2\bar{z}^2 \tanh(z + \bar{z})$.*

1.3. Content of the paper. To prove Theorem 1.1 we consider an algebraic equivalent M' of M. The main technical part of the proof consists in showing that an arbitrary real algebraic element M' of $\mathcal{T}_n^d$ can be straightened in some local complex *algebraic* coordinates $t' \in \mathbb{C}^n$ in order that its infinitesimal CR automorphisms are the real parts of n holomorphic vector fields of the form $X_i' = c_i'(t_i')\,\partial_{t_i'}$, $i = 1, \dots, n$, where the variables are separated and the functions $c_i'(t_i')$ are algebraic. For this, we need to show that the automorphism group of a minimal finitely nondegenerate real algebraic generic submanifold in $\mathbb{C}^n$ is a local real algebraic Lie group, a notion defined in §2.3. A large part of this article (§§4, 5, 6) is devoted to provide an explicit representation formula for the local biholomorphic self-transformations of a minimal finitely nondegenerate generic submanifold, *see* especially Theorem 2.1 and Theorem 4.1. Finally, using the specific simplified form of the vector fields X_i' and assuming that there exists a biholomorphic equivalence $\Phi : M \to M'$ satisfying $\Phi_*(\partial_{t_i}) = c_i'(t_i')\,\partial_{t_i'}$, we show by elementary computations that all the first order derivatives of the mapping $\psi'(y')$ must be algebraic. We follow a similar strategy for the proofs of Theorems 1.4 and 1.5. Finally, in §§7-8, we provide the proofs of Corollaries 1.2, 1.3, 1.6 and 1.7.

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§2. Preliminaries

We recall in this section the basic properties of the objects we will deal with.

2.1. Nash algebraic functions and manifolds. In this subsection, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(x_1, \ldots, x_n)$ denote coordinates over $\mathbb{K}^n$. Throughout the article, we shall use the norm $|x| := \max(|x_1|, \ldots, |x_n|)$ for $x \in \mathbb{K}^n$. Let $\mathcal{K}$ be an open polydisc centered at the origin in $\mathbb{K}^n$, namely $\mathcal{K} = \{x \in \mathbb{K}^n : |x| < \rho\}$ for some $\rho > 0$. Let $f : \mathcal{K} \to \mathbb{K}$ be a $\mathbb{K}$-analytic function, defined by a power series converging normally in $\mathcal{K}$. We say that f is (Nash) $\mathbb{K}$-*algebraic* if there exists a nonzero polynomial $P(X_1, \ldots, X_n, F) \in \mathbb{K}[X_1, \ldots, X_n, F]$ in $(n+1)$ variables such that the relation $P(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0$ holds for all $(x_1, \ldots, x_n) \in \mathcal{K}$. If $\mathbb{K} = \mathbb{R}$, we say that f is *real algebraic*. If $\mathbb{K} = \mathbb{C}$, we say that f is *complex algebraic*. The category of $\mathbb{K}$-algebraic functions is stable under elementary algebraic operations, under differentiation and under composition. Furthermore, implicit solutions of $\mathbb{K}$-algebraic equations (for which the real analytic implicit function theorem applies) are again $\mathbb{K}$-algebraic mappings. The theory of $\mathbb{K}$-algebraic manifolds is then defined by the usual axioms of manifolds, for which the authorized changes of chart are $\mathbb{K}$-algebraic mappings only (*cf.* [Za1995]). In this paper, we shall very often use the stability of algebraicity under differentiation.

2.2. Infinitesimal CR automorphisms. Let $M \subset \mathbb{C}^n$ be a generic submanifold of codimension $d \geq 1$ and CR dimension $m = n - d \geq 1$. Let $p \in M$, let $t = (t_1, \ldots, t_n)$ be some holomorphic coordinates vanishing at p and for some $\rho > 0$, let $\Delta_n(\rho) := \{t \in \mathbb{C}^n : |t| < \rho\}$ be an open polydisc centered at p. We consider the Lie algebra $\mathfrak{Hol}(\Delta_n(\rho))$ of holomorphic vector fields of the form $X = \sum_{j=1}^n a_j(t)\,\partial/\partial t_j$, where the a_j are holomorphic functions in $\Delta_n(\rho)$. Here, $\mathfrak{Hol}(\Delta_n(\rho))$ is equipped with the usual Jacobi-Lie bracket operation. We may consider the complex flow $\exp(\sigma X)(q)$ of a vector field $X \in \mathfrak{Hol}(\Delta_n(\rho))$. It is a holomorphic map of the variables (σ, q) which is well defined in some connected open neighborhood of $\{0\} \times \Delta_n(\rho)$ in $\mathbb{C} \times \Delta_n(\rho)$.

Let K denote the real vector field $K := X + \overline{X}$, considered as a *real* vector field over $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Again, the real flow of K is defined in some connected open neighborhood of $\{0\} \times \Delta_n(\rho)^{\mathbb{R}}$ in $\mathbb{R} \times \Delta_n(\rho)^{\mathbb{R}}$. We remind the following elementary relation between the flow of K and the flow of X. For a real time parameter $\sigma := s \in \mathbb{R}$, the flow $\exp(sX)(q)$ coincides with the real flow of $X + \overline{X}$, namely $\exp(sX)(q) = \exp(s(X + \overline{X}))(q^{\mathbb{R}})$, where for $q \in \mathbb{C}^n$, we denote $q^{\mathbb{R}}$ the corresponding real point in $\mathbb{R}^{2n}$. In the sequel, we shall always identify $\Delta_n(\rho)$ and its real counterpart $\Delta_n(\rho)^{\mathbb{R}}$.

Let now $\mathfrak{Hol}(M, \Delta_n(\rho))$ denote the real subalgebra of the vector fields $X \in \mathfrak{Hol}(\Delta_n(\rho))$ such that $X + \overline{X}$ is tangent to $M \cap \Delta_n(\rho)$. We also denote by $\mathfrak{Aut}_{CR}(M, \Delta_n(\rho))$ the Lie algebra of vector fields of the form $X + \overline{X}$, where X belongs to $\mathfrak{Hol}(M, \Delta_n(\rho))$, so $\mathfrak{Aut}_{CR}(M, \Delta_n(\rho)) = 2\,\mathrm{Re}\,\mathfrak{Hol}(M, \Delta_n(\rho))$. By the above considerations, the local flow $\exp(sX)(q)$ of X with $s \in \mathbb{R}$ real makes a one-parameter family of local biholomorphic transformations of M. In the sequel, we shall always identify $\mathfrak{Hol}(M, \Delta_n(\rho))$ and $\mathfrak{Aut}_{CR}(M, \Delta_n(\rho))$, namely we shall identify X and $X + \overline{X}$ and say by some abuse of language that X itself is an infinitesimal CR automorphism.

In the algebraic category, the main drawback of infinitesimal CR automorphism is that they do not have algebraic flow. For instance, the complex dilatation vector field $X = iz\,\partial_z$ has transcendent flow, even if it is an infinitesimal CR automorphism of every algebraic hypersurface in $\mathbb{C}^2$ whose equation is of the form $v = \varphi(z\bar{z})$, even if the coefficient of X is algebraic. Thus instead of infinitesimal CR automorphisms which generate one-parameter groups of biholomorphic transformations of M, we shall study algebraically dependent one-parameter families of biholomorphic transformations (not necessarily making a one parameter group). To begin with, we need to introduce some precise definitions about local algebraic Lie transformation groups.

2.3. Local Lie group actions in the $\mathbb{K}$-algebraic category. Often in real or in complex analytic geometry, the interest cannot be focalized on global Lie transformation groups, but only on local transformations which are close to the identity. For instance, the transformation group of a small piece of a real analytic CR manifold in $\mathbb{C}^n$ which is not contained in a global, large or compact CR manifold is almost never a true, global transformation group. Consequently the usual axioms of Lie transformation groups must be localized. Philosophically speaking, the local point of view is often the most adequate and the richest one, because a given analytico-geometric object often possesses much more local invariant than global invariants, if any. Historically speaking, the local Lie transformation groups were first studied, before the introduction of the now classical notion of global Lie group. Especially, in his first masterpiece work [Lie1880] on the subject, Sophus Lie essentially dealt with local "Lie" groups: he classified all continuous local transformation groups acting on an open subset of $\mathbb{C}^2$. This general classification provided afterwards in the years 1880–1890 many applications to the local study of differential equations: local normal forms, local solvability, *etc.*

In this paragraph we define precisely local actions of local Lie groups and we focus especially on the $\mathbb{K}$-algebraic category.

Let $c \in \mathbb{N}_*$, let $g = (g_1, \dots, g_c) \in \mathbb{K}^c$ and let two positive numbers satisfy $0 < \delta_2 < \delta_1$. We formulate the desired definition by means of the two precise polydiscs $\Delta_c(\delta_2) \subset \Delta_c(\delta_1) \subset \mathbb{K}^c$. A *local $\mathbb{K}$-algebraic Lie group of dimension c* consists of the following data:

- **(1)** A $\mathbb{K}$-algebraic *multiplication mapping* $\mu : \Delta_c(\delta_2) \times \Delta_c(\delta_2) \to \Delta_c(\delta_1)$ which is locally associative $(\mu(g, \mu(g', g'')) = \mu(\mu(g, g'), g''))$, whenever $\mu(g', g'') \in \Delta_c(\delta_2)$, $\mu(g, g') \in \Delta_c(\delta_2)$ and which satisfies $\mu(0, g) = \mu(g, 0) = g$, where the origin $0 \in \mathbb{K}^c$ corresponds to the identity element in the group structure.
- **(2)** A $\mathbb{K}$-algebraic *inversion mapping* $\iota : \Delta_c(\delta_2) \to \Delta_c(\delta_1)$ satisfying $\mu(g, \iota(g)) = \mu(\iota(g), g) = 0$ and $\iota(0) = 0$ whenever $\iota(g) \in \Delta_c(\delta_2)$.

Here, the integer $c \in \mathbb{N}_*$ is the *dimension* of G. We shall say that composition and inversion are defined locally in a neighborhood of the identity element. In the $\mathbb{K}$-analytic category, the corresponding definition is similar.

Now, we can define the notion of local $\mathbb{K}$-algebraic Lie group action. Let $n \in \mathbb{N}_*$, let $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ and let two positive numbers satisfy $0 < \rho_2 < \rho_1$. Let G be a local $\mathbb{K}$-algebraic Lie group as defined just above. We shall formulate the desired definition by means of the two precise polydiscs $\Delta_n(\rho_2) \subset \Delta_n(\rho_1)$. This pair of polydiscs represents a *local $\mathbb{K}$-algebraic manifold* up to changes of $\mathbb{K}$-algebraic coordinates. A *local $\mathbb{K}$-algebraic Lie group action on a local $\mathbb{K}$-algebraic manifold* consists of a *$\mathbb{K}$-algebraic* action mapping $x' = \Phi(x; g)$ defined over $\Delta_n(\rho_2) \times \Delta_c(\delta_2)$ with values in $\Delta_n(\rho_1)$ which satisfies:

- **(1)** $\Phi(\Phi(x; g); g')) = \Phi(x; \mu(g, g'))$ whenever $\Phi(x; g) \in \Delta_n(\rho_2)$ and $\mu(g, g') \in \Delta_c(\delta_2)$, where the local group multiplication $\mu(g, g')$ is $\mathbb{K}$-algebraic as above;
- **(2)** $\Phi(x; e) = x$ and $\Phi(\Phi(x; g); \iota(g)) = x$ whenever $\Phi(x; g) \in \Delta_n(\rho_2)$ and $\iota(g) \in \Delta_c(\rho_2)$, where the inverse group mapping $g \mapsto \iota(g)$ is $\mathbb{K}$-algebraic as above.

In this definition, it is allowed to suppose that $x \in \mathbb{C}^n$ and $g \in \mathbb{R}^c$, which is the case to be considered in the sequel. By differentiation, every local $\mathbb{K}$-algebraic action gives rise to vector fields defined over $\Delta_n(\rho_2)$ which are infinitesimal generators of the action. Indeed, let us consider the algebraically dependent one-parameter families of complex algebraic biholomorphic transformations $\Phi(x; 0, \dots, 0, g_i, 0, \dots, 0) =: \Phi_i(x; g_i) \equiv (\Phi_{i,1}(x; g_i), \dots, \Phi_{i,n}(x; g_i)) \in \mathbb{K}^n$, which we shall also denote by $\Phi_{i,g_i}(x)$. In general, such a family does not make a one-parameter group of transformations, but we can nevertheless introduce the vector fields $X_i(\Phi_{i,g_i}(x); g_i) := \partial_{g_i} \Phi_i(x; g_i) = \sum_{l=1}^n \partial_{g_i} \Phi_{i,l}(x; g_i)\, \partial/\partial x_l$. We notice that the coefficients of these vector fields do in general depend on the group parameter $g_i \in G$.

In fact, in the algebraic category, there is no hope to modify the coordinates on the group in order that the infinitesimal generators of the action are independent of the parameter coordinates g_j. For instance, the trivial one-dimensional action (complex dilatation) defined by $(z, w) \mapsto ((1+g)z, w) =: \Phi(z, w; g)$, where $(z, w) \in \mathbb{C}^2$ and $g \in \mathbb{C}$ is clearly an algebraic action. Here, the infinitesimal generator $X(x; g) = (1+g)^{-1} z \partial_z$ depends on the parameter g. The only way to avoid the dependence upon g of the coefficient of X is to change coordinates on the group by setting $1+g := e^\sigma$, $\sigma \in \mathbb{C}$, whence the action is represented by $(z, w) \mapsto (e^\sigma z, w) =: \Phi(z, w; \sigma)$. Indeed, from the group property $\Phi(\Phi(z, w; \sigma); \sigma') \equiv \Phi(z, w; \sigma + \sigma')$, it is classical and immediate to deduce that if we define the parameter independent vector field $X^0(z, w) := \partial_\sigma \Phi(z, w; \sigma)|_{\sigma=0} = z\, \partial_z$, then it holds that $\partial_\sigma \Phi(z, w; \sigma) = e^\sigma z\, \partial_z = X^0(\Phi(z, w; \sigma))$. So the infinitesimal generator of the action is independent of the parameter g. However, the main trouble here is that the algebraicity of the action is necessarily lost since the flow of X^0 is not algebraic (the reader may check that each right (or left) invariant vector field on an algebraic local Lie group defines in general a nonalgebraic one-parameter subgroup, *e.g.* for $\mathrm{SO}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$).

Consequently we may allow the infinitesimal generators of an algebraic local Lie group action $x' = \Phi(x; g)$, defined by $X_i(x; g_i) := [\partial_{g_i} \Phi_i](\Phi_{i,g_i}^{-1}(x); g_i)$ to depend on the group parameter g_i, even if the families $(\Phi_{i,g_i}(x))_{g_i \in \mathbb{K}}$ do not constitute one-dimensional subgroups of transformations.

2.4. Algebraicity of complex flow foliations. Suppose now that M is a real algebraic generic submanifold in $\mathbb{C}^n$, for instance a hypersurface which is Levi nondegenerate at a "center" point $p \in M$ corresponding to the origin in the coordinates $t = (t_1, \dots, t_n)$. Let $X \in \mathfrak{Hol}(M)$ be an infinitesimal CR automorphism. Even if, for fixed real s, the biholomorphic mapping $t \mapsto \exp(sX)(t)$ is complex algebraic, *i.e.* the n components of this biholomorphism are complex algebraic functions by Webster's theorem [We1977], we know by considering the infinitesimal CR automorphism $X_1 := i(z+1)\partial_z$ of the strong tube $\mathrm{Im}\, w = |z+1|^2 + |z+1|^6 - 2$ in $\mathbb{C}^2$ passing through the origin, that the flow of X is not necessarily algebraic with respect to all variables (s, t).

Nevertheless, we shall show that the local CR automorphism group of M is a local algebraic Lie group whose general transformations are of the form $t' = H(t; e_1, \dots, e_c)$, where $t \in \mathbb{C}^n$ and $(e_1, \dots, e_c) \in \mathbb{R}^c$ and where H is algebraic with respect to all its variables. Thus the "time" dependent vector fields defined by $X_i(t; e_i) := [\partial_{e_i} H_i](H_{i,e_i}^{-1}(t); e_i)$, where $H_{i,e_i}(t) := H_i(t; e_i) := H(t; 0, \dots, 0, e_i, 0, \dots, 0)$, have an algebraic flow, simply given by $(t, e_i) \mapsto H_i(t; e_i)$. It follows that each foliation defined by the complex integral curves of the time dependent complex vector fields X_i, $i = 1, \dots, c$, is a complex algebraic foliation, *see* §3 below. Now, we can state the main technical theorem of this paper, whose proof is postponed to §4, §5 and §6.

Theorem 2.1. *Let $M \subset \mathbb{C}^n$ be a* real algebraic *connected geometrically smooth generic submanifold of codimension $d \geq 1$ and CR dimension $m = n - d \geq 1$. Let $p \in M$ and assume that M is finitely nondegenerate and minimal at p. Then for every sufficiently small nonempty open polydisc Δ_1 centered at p, the following three properties hold*:

(1) *The complex Lie algebra* $\mathfrak{Hol}(M, \Delta_1)$ *is of finite dimension* $c \in \mathbb{N}$ *which depends only on the local geometry of* M *in a neighborhood of* p.

(2) *There exists a nonempty open polydisc* $\Delta_2 \subset \Delta_1$ *also centered at* p *and a* $\mathbb{C}^n$*-valued mapping* $H(t; e) = H(t; e_1, \dots, e_c)$ *with* $H(t; 0) \equiv t$ *which is defined in a neighborhood of the origin in* $\mathbb{C}^n \times \mathbb{R}^c$ *and which is algebraic with respect to both its variables* $t \in \mathbb{C}^n$ *and* $e \in \mathbb{R}^c$ *such that for every holomorphic map* $h : \Delta_2 \to \Delta_1$ *with* $h(\Delta_2 \cap M) \subset \Delta_1 \cap M$ *which is sufficiently close to the identity map, there exists a unique* $e \in \mathbb{R}^c$ *such that* $h(t) = H(t; e)$.

(3) *The mapping* $(t, e) \mapsto H(t; e)$ *constitutes a* $\mathbb{K}$*-algebraic local Lie transformation group action. More precisely, there exist a local multiplication mapping* $(e, e') \mapsto \mu(e, e')$ *and a local inversion mapping* $e \mapsto \iota(e)$ *such that* H, μ *and* ι *satisfy the axioms of local algebraic Lie group action as defined in §2.3.*

(4) *The* c *"time dependent" holomorphic vector fields*

$$X_i(t; e_i) := [\partial_{e_i} H_i](H_{i,e_i}^{-1}(t); e_i), \tag{2.1}$$

where $H_{i,e_i}(t) := H_i(t; e_i) := H(t; 0, \dots, 0, e_i, 0, \dots, 0)$*, have algebraic coefficients and have an algebraic flow, given by* $(t, e_i) \mapsto H_i(t; e_i)$.

In the case where M *is real analytic, the same theorem holds true with the word "algebraic" everywhere replaced by the word "analytic".*

A special case of Theorem 2.1 was proved in [BER1999b] where, apparently, the authors do not deal with the notion of local Lie groups and consider the isotropy group of the point p, namely the group of holomorphic self-maps of M fixing p. The consideration of the complete local Lie group of biholomorphic self-maps of a piece of M in a neighborhood of p (not only the isotropy group of p) is crucial for our purpose, since we shall have to deal with strong tubes $M \in \mathcal{T}_n^d$ for which the isotropy group of $p \in M$ is trivial. Sections §4, §5 and §6 are devoted to the proof of Theorem 4.1, a precise statement of Theorem 2.1. We mention that our method of proof of Theorem 2.1 gives a non optimal bound for the dimension of $\mathfrak{Hol}(M, \Delta_1)$. To our knowledge, the upper bound $c \leq (n+1)^2 - 1$ is optimal only in codimension $d = 1$ and in the Levi nondegenerate case.

§3. Proof of Theorem 1.1

We take in this section Theorem 2.1 for granted. As explained in §1.3 above, we shall conduct the proof of Theorem 1.1 in two essential steps (§§3.1 and 3.2). The strategy for the proof of Theorems 1.4 and 1.5 is similar and we prove them in §§3.3 and 3.4. Let $M \in \mathcal{T}_n^d$ be a strong tube of codimension d passing through the origin in $\mathbb{C}^n$ given by the equations $v_j = \varphi_j(y)$, $j = 1, \dots, d$. Assume that M is biholomorphically equivalent to a real algebraic generic submanifold M'.

First step. We show that an arbitrary real algebraic element $M' \in \mathcal{T}_n^d$ can be straightened in some local complex algebraic coordinates $t' = (t'_1, ..., t'_n) \in \mathbb{C}^n$ in order that its infinitesimal CR automorphisms are the n holomorphic vector fields of the specific form $X'_i = c'_i(t'_i)\, \partial_{t'_i}$, $i = 1, \dots, n$, where the functions $c'_i(t'_i)$ are algebraic.

Second step. Assuming that there exists a biholomorphic equivalence $\Phi : M \to M'$ satisfying $\Phi_*(\partial_{t_i}) = c'_i(t')\, \partial_{t'_i}$, we prove by direct computation that all the first order derivatives of the mapping $\psi'(y')$ must be algebraic.

3.1. Proof of the first step. Let $t' = \Phi(t)$ be such an equivalence, with $\Phi(0) = 0$ and $M' := \Phi(M)$ real algebraic. Let $X_i := \partial_{t_i}$, $i = 1, \dots, n$, be the n infinitesimal CR automorphisms of M and set $X'_i := \Phi_*(X_i)$. Of course, we have $[X'_{i_1}, X'_{i_2}] = \Phi_*([X_{i_1}, X_{i_2}]) = 0$, so the CR automorphism group of M' is also n-dimensional and commutative. Let us choose complex algebraic coordinates t' in a neighborhood of $0 \in M'$ such that $X'_i|_0 = \partial_{t'_i}|_0$. Let us apply Theorem 2.1 to the real algebraic submanifold M', noting all the datas with dashes. There exists an algebraic mapping $H'(t';e) = H'(t';e_1,\dots,e_n)$ such that every local biholomorphic self-map of M' writes uniquely $t' \mapsto H'(t';e)$, for some $e \in \mathbb{R}^n$. In particular, for every $i = 1,\dots,n$ and every small $s \in \mathbb{R}$, there exists $e_s \in \mathbb{R}^n$ depending on s such that $\exp(sX_i')(t') \equiv H'(t';e_s)$. From the commutativity of the flows of the X_i', i.e. from $\exp(s_1X_{i_1}'(\exp(s_2X_{i_2}'(t')))) \equiv \exp(s_2X_{i_2}'(\exp(s_1X_{i_1}'(t'))))$, we get

$$H'(H'(t';e_2);e_1) \equiv H'(H'(t';e_1);e_2). \tag{3.1}$$

This shows that the biholomorphisms $t' \mapsto H_e'(t') := H'(t';e)$ commute pairwise. In particular, if we define

$$G_i'(t';e_i) := H'(t';0,\dots,0,e_i,0,\dots,0), \tag{3.2}$$

we have $G_{i_1}'(G_{i_2}'(t';e_2);e_1) \equiv G_{i_2}'(G_{i_1}'(t';e_1);e_2)$.

Next, after making a linear change of coordinates in the e-space, we can insure that $\partial_{e_i}G_i'(0;e_i)|_{e_i=0} = \partial_{t_i'}|_0 = X_i'|_0$ for $i = 1,\dots,n$. Finally, complexifying the real variable e_i in a complex variable ϵ_i, we get mappings $G_i'(t_i';\epsilon_i)$ which are complex algebraic with respect to both variables $t' \in \mathbb{C}^n$ and $\epsilon_i \in \mathbb{C}$ and which commute pairwise. We can now state and prove the following crucial proposition (where we have dropped the dashes) according to which we can straighten commonly the n one-parameter families of biholomorphisms $t' \mapsto G_i'(t';\epsilon_i)$.

Proposition 3.1. *Let $t \mapsto G_i(t;\epsilon_i)$, $i = 1,\dots,n$, be n one complex parameter families of complex algebraic biholomorphic maps from a neighborhood of 0 in $\mathbb{C}^n$ onto a neighborhood of 0 in $\mathbb{C}^n$ satisfying $G_i(t;0) \equiv t$, $\partial_{\epsilon_i}G_i(0;\epsilon_i)|_{\epsilon_i=0} = \partial_{t_i}|_0$ and pairwise commuting: $G_{i_1}(G_{i_2}(t;\epsilon_2);\epsilon_1) \equiv G_{i_2}(G_{i_1}(t;\epsilon_1);\epsilon_2)$. Then there exists a complex algebraic biholomorphism of the form $t' \mapsto \Phi'(t') =: t$ of $\mathbb{C}^n$ fixing the origin with $d\Phi'(0) = \mathrm{Id}$ such that if we set $G_i'(t';\epsilon_i) := {\Phi'}^{-1}(G_i(\Phi'(t');\epsilon_i))$, where $t' = \Phi(t)$ denote the inverse of $t = \Phi'(t')$, then we have*

$$G_i'(t';\epsilon_i) \equiv (t_1',\dots,t_{i-1}',G_{i,i}'(t_i';\epsilon_i),t_{i+1}',\dots,t_n'), \tag{3.3}$$

where the functions $G_{i,i}'$ are complex algebraic, depend only on t_i' *(and on ϵ_i) and satisfy $G_{i,i}'(t_i';0) \equiv t_i'$ and $\partial_{\epsilon_i}G_{i,i}'(0;\epsilon_i)|_{\epsilon_i=0} = 1$.*

Proof. First of all, we define the complex algebraic biholomorphism

$$\Phi_1' : \quad (t_1',t_2',\dots,t_n') \longmapsto G_1(0,t_2',\dots,t_n';t_1') =: t. \tag{3.4}$$

We have $d\Phi_1'(0) = \mathrm{Id}$, because $G_1(t;0) \equiv t$ and $\partial_{\epsilon_1}G_1(0;\epsilon_1)|_{\epsilon_1=0} = \partial_{t_1}|_0$. Furthermore, since $\partial_{\epsilon_1}G_1(0;\epsilon_1)|_{\epsilon_1=0}$ is transversal to $\{(0,t_2,\dots,t_n)\}$, it also follows that a small neighborhood of the origin in $\mathbb{C}_t^n$ is algebraically foliated by the $(n-1)$-parameter family of complex curves $\mathcal{C}_{t_2',\dots,t_n'}' := \{G_1(0,t_2',\dots,t_n';t_1') : |t_1'| < \delta\}$ where $\delta > 0$ is small and $t_2',\dots,t_n'$ are fixed. The existence of this foliation shows that the relation

$$t^* \sim t \quad \text{iff} \quad \text{there exists } \epsilon_1 \text{ such that } t^* = G_1(t;\epsilon_1) \tag{3.5}$$

is a local equivalence relation, whose equivalence classes are the leaves $\mathcal{C}_{t_2',\dots,t_n'}'$ (*see* FIGURE 1).

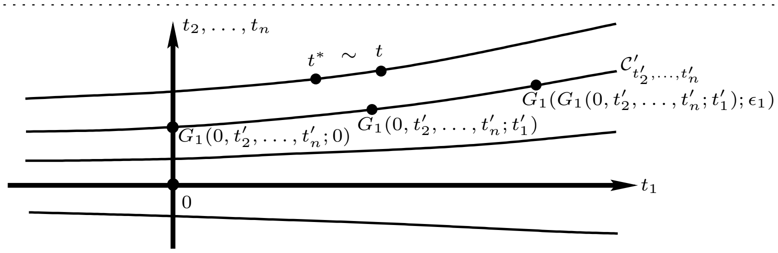


FIGURE 1: LOCAL ALGEBRAIC STRAIGHTENING OF THE ORBITS OF $G_1(t;\epsilon_1)$

Consequently, as we clearly have

$$(0, t_2', \dots, t_n') \sim G_1(0, t_2', \dots, t_n'; t_1') \sim G_1(G_1(0, t_2', \dots, t_n'; t_1'); \epsilon_1), \tag{3.6}$$

using the transitivity of the relation $\sim$, it follows that there exists a complex number $\varepsilon_{t',\epsilon_1}$ depending on t' and on ϵ_1 such that

$$G_1(G_1(0, t_2', \dots, t_n'; t_1'); \epsilon_1) = G_1(0, t_2', \dots, t_n'; \varepsilon_{t',\epsilon_1}). \tag{3.7}$$

By the very definition (3.4) of Φ_1', this is equivalent to

$$\Phi_1(G_1(\Phi_1'(t'); \epsilon_1)) = (\varepsilon_{t',\epsilon_1}, t_2', \dots, t_n'), \tag{3.8}$$

where $t' = \Phi_1(t)$ denotes the inverse of $t = \Phi_1'(t')$. Finally, since the left hand side of (3.8) is clearly a complex algebraic mapping of $(t'; \epsilon_1)$, it follows that there exists a complex algebraic function $G_{1,1}'(t'; \epsilon_1)$ such that we can write

$$\Phi_1(G_1(\Phi_1'(t'); \epsilon_1)) \equiv (G_{1,1}'(t'; \epsilon_1), t_2', \dots, t_n'). \tag{3.9}$$

So we have straightened the first family by means of Φ_1'.

Next, we drop the dashes and we restart with $G_1(t; \epsilon_1) = (G_{1,1}(t; \epsilon_1), t_2, \dots, t_n)$. Then, similarly as above, by introducing the complex algebraic biholomorphism

$$\Phi_2' : \quad (t_1', t_2', t_3', \dots, t_n') \longmapsto G_2(t_1', 0, t_3', \dots, t_n'; t_2'), \tag{3.10}$$

which satisfies $d\Phi_2'(0) = \mathrm{Id}$, and by denoting by $t' = \Phi_2(t)$ the inverse of $t = \Phi_2'(t')$, we get again that if we set $G_2'(t'; \epsilon_2) := \Phi_2(G_2(\Phi_2'(t'); \epsilon_2))$, then

$$G_2'(t'; \epsilon_2) \equiv (t_1', G_{2,2}'(t'; \epsilon_2), t_3', \dots, t_n'), \tag{3.11}$$

where the complex algebraic function $G_{2,2}'(t'; \epsilon_2)$ satisfies $\partial_{\epsilon_2} G_{2,2}'(0; \epsilon_2)|_{\epsilon_2=0} = 1$ and $G_{2,2}'(t'; 0) \equiv t_2'$.

We also have to consider the modification of the first family of biholomorphisms $G_1'(t'; \epsilon_1) := \Phi_2(G_1(\Phi_2'(t'); \epsilon_1))$. Using in an essential way the commutativity, we may compute

$$\begin{cases} \Phi_2'(G_1'(t'; \epsilon_1)) & = G_1(\Phi_2'(t'); \epsilon_1) \\ & = G_1(G_2(t_1', 0, t_3', \dots, t_n'; t_2'); \epsilon_1) \\ & = G_2(G_1(t_1', 0, t_3', \dots, t_n'; \epsilon_1); t_2') \\ & = G_2(G_{1,1}(t_1', 0, t_3', \dots, t_n'; \epsilon_1), 0, t_3', \dots, t_n'; t_2') \\ & = \Phi_2'(G_{1,1}(t_1', 0, t_3', \dots, t_n'; \epsilon_1), t_2', t_3', \dots, t_n'). \end{cases} \tag{3.12}$$

It follows that

$$G_1'(t'; \epsilon_1) \equiv (G_{1,1}(t_1', 0, t_3', \dots, t_n'; \epsilon_1), t_2', t_3', \dots, t_n') \tag{3.13}$$

whence

$$G_{1,1}'(t'; \epsilon_1) := G_{1,1}(t_1', 0, t_3', \dots, t_n'; \epsilon_1) \tag{3.14}$$

does not depend on t_2'. Finally, inserting (3.11) and (3.13) in the commutativity relation $G_1'(G_2'(t'; \epsilon_2); \epsilon_1) \equiv G_2'(G_1'(t'; \epsilon_1); \epsilon_2)$, we find

$$\begin{cases} G_{1,1}'(t_1', G_{2,2}'(t'; \epsilon_2), t_3', \dots, t_n'; \epsilon_1) \equiv G_{1,1}'(t'; \epsilon_1), \\ G_{2,2}'(G_{1,1}'(t'; \epsilon_1), t_2', t_3', \dots, t_n'; \epsilon_2) \equiv G_{2,2}'(t'; \epsilon_2). \end{cases} \tag{3.15}$$

The first relation gives nothing, since we already know that $G_{1,1}'$ is independent of t_2'. By differentiating the second relation with respect to ϵ_1 at $\epsilon_1 = 0$, we find that $G_{2,2}'$ is independent of t_1'.

In summary, after the change of coordinates $\Phi_2' \circ \Phi_1'(t') = t$ which is tangent to the identity map at $t' = 0$, we obtained that

$$\begin{cases} G_1'(t'; \epsilon_1) & = (G_{1,1}'(t_1', t_3', \dots, t_n'), t_2', t_3', \dots, t_n'), \\ G_2'(t'; \epsilon_2) & = (t_1', G_{2,2}'(t_2', t_3', \dots, t_n'), t_3', \dots, t_n'). \end{cases} \tag{3.16}$$

Using these arguments, the proof of Proposition 3.1 clearly follows by induction. □

Now, we come back to our CR manifold M' having the one-parameter families of algebraic biholomorphisms $G_i'(t'; e_i)$ given by (3.2) and pairwise commuting. Applying Proposition 3.1, after a change of complex algebraic coordinates of the form $t' = \Psi''(t'')$, we may assume that the $G_i''(t''; \epsilon_i)$ are algebraic and can be written in the specific form

$$G_i''(t''; \epsilon_i) \equiv (t_1'', \ldots, t_{i-1}'', G_{i,i}''(t_i''; \epsilon_i), t_{i+1}'', \ldots, t_n''), \tag{3.17}$$

with $\partial_{\epsilon_i} G_{i,i}''(0; \epsilon_i)|_{\epsilon_i=0} = 1$. Let $t'' = \Psi'(t')$ denote the inverse of $t' = \Psi''(t'')$. We thus have $t'' = \Psi'(t') = \Psi'(\Phi(t))$, where we remind that $t' = \Phi(t)$ provides the equivalence between the strong tube M and the algebraic CR generic M'.

Since Ψ' is algebraic, the image $M'' := \Psi'(M')$ is also algebraic. Let $r_j'(t', \bar{t}') = 0$, $j = 1, \ldots, d$, be defining equations for M'. Then $r_j''(t'', \bar{t}'') := r_j'(\Psi''(t''), \overline{\Psi''(t'')}) = 0$ are defining equations for M''. By assumption, for $\epsilon_i := e_i \in \mathbb{R}$ real, the family of algebraic biholomorphisms $G_i'(t'; \epsilon_i)$ maps a small piece of M' through the origin into M'. It follows trivially that $G_i''(t''; \epsilon_i) \equiv \Psi'(G_i'(\Psi''(t''); \epsilon_i))$ maps a small piece of M'' through the origin into M''. Furthermore, since $d\Psi''(0) = \mathrm{Id}$, it follows that if we denote $X_i'' := \Psi'_*(X_i')$, then $X_i''|_0 = \partial_{t_i''}|_0$.

Next, thanks to the specific form (3.17), by differentiating $\partial_{\epsilon_i} G_i''(t''; \epsilon_i)|_{\epsilon_i=0}$, we get n vector fields of the form $Z_i'' = c_i''(t_i'')\, \partial_{t_i''}$. By construction, the functions $c_i''(t_i'')$ are algebraic and satisfy $c_i''(0) = 1$. Differentiating with respect to e_i the identity $r_j''(G_i''(t''; e_i), \overline{G_i''(t''; e_i)}) = 0$ for $r_j''(t'', \bar{t}'') = 0$, *i.e.* for $t'' \in M''$, we see that Z_i'' is tangent to M'', *i.e.* we see that Z_i'' is an infinitesimal CR automorphism of M''. Consequently, there exist real constants $\lambda_{i,l}$ such that $Z_i'' = \sum_{l=1}^n \lambda_{i,l}\, X_l''$. Since $Z_i''|_0 = X_i''|_0 = \partial_{t_i''}|_0$, we have in fact $\lambda_{i,l} = 1$ for $i = l$ and $\lambda_{i,l} = 0$ for $i \neq l$. So $Z_i'' = X_i''$ and we have shown that

$$(\Psi' \circ \Phi)_*(X_i) = X_i'' = Z_i'' = c_i''(t_i'')\, \partial_{t_i''}. \tag{3.18}$$

We shall call a CR generic manifold M'' having infinitesimal CR automorphisms of the form $X_i'' = c_i''(t_i'')\, \partial_{t_i''}$ with $c_i''(0) \neq 0$ a *pseudotube*. Such a pseudotube is not in general a product by $\mathbb{R}^n$. In fact, there is no hope to tubify all algebraic peudotubes in algebraic coordinates, as shows the elementary example $\operatorname{Im} w = |z+1|^2 + |z+1|^6 - 2$ having infinitesimal CR automorphisms ∂_w and $i(z+1)\partial_z$, since the only change of coordinates for which $\Phi_*(\partial_w) = \partial_{w'}$ and $\Phi_*(i(z+1)\partial_z) = \partial_{z'}$ is $z+1 = e^{iz'}$, $w = w'$, which transforms M into M' of nonalgebraic defining equation $\operatorname{Im} w' = e^{-2y'} + e^{-6y'}$.

The constructions of this paragraph may be represented by the following symbolic picture.

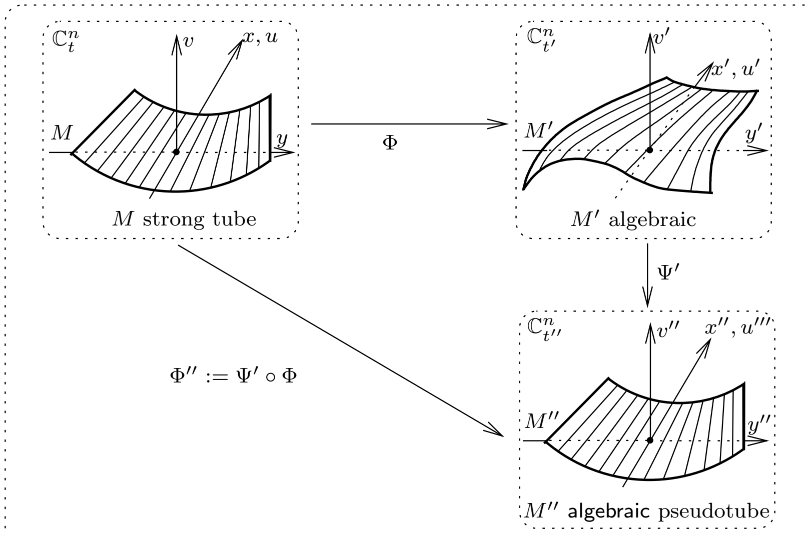


FIGURE 2: ALGEBRAIC STRAIGHTENING Ψ' OF M'

Summary and conclusion of the first step. To conclude, let us denote for simplicity M'' again by M', the coordinates t'' again by t' and $t'' = \Psi' \circ \Phi(t)$ by $t' = \Phi(t)$. After the above straightenings, we have shown that the infinitesimal CR automorphisms $X_i' := \Phi_*(X_i)$ of the algebraic generic manifold M' are of the sympathetic form $X_i' = c_i'(t_i')\, \partial_{t_i'}$, $i = 1, \ldots, n$, with algebraic coefficients $c_i'(t_i')$ satisfying $c_i'(0) = 1$.

3.2. Proof of the second step. We characterize first finite nondegeneracy for tubes of codimension d in $\mathbb{C}^n$.

Lemma 3.2. *Let M be a tube of codimension d in $\mathbb{C}^n$ equipped with coordinates $(z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ given by the equations $v_j = \varphi_j(y)$, $j = 1, \ldots, d$, where $\varphi_j(0) = 0$. Then M is finitely nondegenerate at the origin if and only if there exist m multi-indices $\beta_*^1, \ldots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \geq 1$ and integers $j_*^1, \ldots, j_*^m$ with $1 \leq j_*^k \leq d$ such that the real mapping*

$$\psi(y) := \left(\frac{\partial^{|\beta_*^1|} \varphi_{j_*^1}(y)}{\partial y^{\beta_*^1}}, \ldots, \frac{\partial^{|\beta_*^m|} \varphi_{j_*^m}(y)}{\partial y^{\beta_*^m}} \right) =: y' \in \mathbb{R}^m \tag{3.19}$$

is of rank m at the origin in $\mathbb{R}^m$.

Proof. We follow the definition of finite nondegeneracy given in §1.2. Let $r_j(t, \bar{t}) := v_j - \varphi_j(y) = 0$ be the defining equations of M. Let $\overline{L}_k := \partial_{\bar{z}_k} + \sum_{j=1}^d \varphi_{j, \bar{z}_k}\, \partial_{\bar{w}_j}$, $k = 1, \ldots, m$, be a basis of $(1,0)$-vector fields tangent to M. We write the first order terms in the Taylor series of $\varphi_j(y)$ as $\varphi_j(y) = \sum_{l=1}^n \lambda_{j,l}\, y_l + \mathrm{O}(|y|^2)$. Then the holomorphic gradient of r_j is given by

$$\begin{cases} \nabla_t(r_j) & = (\partial_{z_1} r_j, \ldots, \partial_{z_m} r_j, \partial_{w_1} r_j, \ldots, \partial_{w_d} r_j) \\ & = i2^{-1}(\partial_{y_1} \varphi_j, \ldots, \partial_{y_m} \varphi_j, 0, \ldots, 0, -1, 0, \ldots, 0) \\ & = i2^{-1}(\lambda_{j,1}, \cdots, \lambda_{j,m}, 0, \ldots, 0, -1, 0, \ldots, 0), \quad \text{at the origin.} \end{cases} \tag{3.20}$$

On the other hand, since for $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$ with $|\beta| \geq 1$ the order $|\beta|$ derivation $\overline{L}^\beta := \overline{L}_1^{\beta_1} \cdots \overline{L}_m^{\beta_m}$ acts on functions of y as the operator $(2i)^{-|\beta|}\, \partial_y^\beta$, we can compute

$$\begin{cases} \overline{L}^\beta(\nabla_t(r_j)) & = (\overline{L}^\beta \partial_{z_1} \varphi_j, \ldots, \overline{L}^\beta \partial_{z_m} \varphi_j, 0, \ldots, 0, \ldots, 0) \\ & = i^{-|\beta|+1} 2^{-|\beta|-1}\, (\partial_y^\beta \partial_{y_1} \varphi_j, \ldots, \partial_y^\beta \partial_{y_m} \varphi_j, 0, \ldots, 0, \ldots, 0). \end{cases} \tag{3.21}$$

By inspecting the expressions (3.20) and (3.21), we see that $\text{Span}\{(\overline{L}^\beta(\nabla_t(r_j)))(0) : \beta \in \mathbb{N}^m, j = 1, \ldots, d\} = \mathbb{C}^n$ if and only if $\text{Span}\{(\partial_y^\beta \partial_{y_1} \varphi_j(0), \ldots, \partial_y^\beta \partial_{y_m} \varphi_j(0)) : \beta \in \mathbb{N}^m, |\beta| \geq 1, j = 1, \ldots, d\} = \mathbb{R}^m$. This last condition is clearly equivalent to the one stated in Lemma 3.2. □

We can prove now that the inverse mapping $\psi'(y')$ of the mapping $\psi(y)$ defined by (1.1) (or (3.19)) has algebraic first order derivatives. By Step 1, there exists a biholomorphic transformation Φ mapping the strong tube M onto the algebraic pseudotube M' with the property that $\Phi_*(\partial_{t_i}) = c_i'(t_i')\, \partial_{t_i'}$. Writing $\Phi(t) = (h_1(t), \ldots, h_n(t))$, we have $\Phi_*(\partial_{t_i}) = \sum_{l=1}^n h_{l,t_i}(t)\, \partial_{t_l'} = c_i'(t_i')\, \partial_{t_i'}$, so $h_i(t)$ depends only on t_i which yields $\Phi(t) = (h_1(t_1), \ldots, h_n(t_n))$. We shall use the convenient notation $t_i' = h_i(t_i)$ and $t_i = h_i'(t_i')$ for the inverse $h_i' := h_i^{-1}$, $i = 1, \ldots, n$. If accordingly, $\Phi'(t') = t$ denotes the inverse of $\Phi(t) = t'$, we have $\Phi_*'(c_i'(t_i')\, \partial_{t_i'}) = c_i'(t_i')\, h_{i,t_i'}'(t_i')\, \partial_{t_i} = \partial_{t_i}$, which shows that $c_i'(t_i')\, h_{i,t_i'}'(t_i') \equiv 1$. Since $c_i'(0) = 1$, we see that $h_i'(t_i') = \int_0^{t_i'} 1/[c_i'(\sigma)]\, d\sigma$ is the complex primitive of an algebraic function. This observation will be important.

After a permutation of the coordinates, we may assume that M' is given in the coordinates $t' = (z', w') \in \mathbb{C}^m \times \mathbb{C}^d$ by the real defining equations $\text{Im}\, w_j' = \varphi_j'(z', \bar{z}', \text{Re}\, w')$, $j = 1, \ldots, d$, where the functions φ_j' are algebraic and vanish at the origin. Solving in terms of w' by means of the algebraic implicit function theorem, we can represent M' by the algebraic complex defining equations

$$w_j' = \overline{\Theta}_j'(z', \bar{z}', \bar{w}'), \qquad j = 1, \ldots, d, \tag{3.22}$$

where $\overline{\Theta}'$ satisfies the vectorial functional equation $w' \equiv \overline{\Theta}'(z', \bar{z}', \Theta'(\bar{z}', z', w'))$ (which we shall not use). According to the splitting (z', w') of coordinates, it is convenient to modify our previous notation by writing $z_k = f_k'(z_k')$, $k = 1, \ldots, m$ and $w_j = g_j'(w_j')$, $j = 1, \ldots, d$ instead of $t_i = h_i'(t_i')$, $i = 1, \ldots, n$, and also

$$\begin{cases} X_k' = a_k'(z_k')\, \partial_{z_k'}, & k = 1, \ldots, m, \qquad a_k'(0) = 1, \\ Y_j' = b_j'(w_j')\, \partial_{w_j'}, & j = 1, \ldots, d, \qquad b_j'(0) = 1, \end{cases} \tag{3.23}$$

instead of $X_i' = c_i'(t_i')\, \partial_{t_i'}$. The relation $c_i'(t_i')\, h_{i,t_i'}'(t_i') \equiv 1$ rewrites down in the form

$$\begin{cases} a_k'(z_k')\, f_{k,z_k'}'(z_k') \equiv 1, \\ b_j'(w_j')\, g_{j,w_j'}'(w_j') \equiv 1. \end{cases} \tag{3.24}$$

We remind that the derivatives of the f_k' and of the g_j' are algebraic. Let now $t' = (z', w') \in M'$, thus satisfying (3.22). Then $h'(t') = (f'(z'), g'(w'))$ belongs to M, namely we have for $j = 1, \ldots, d$:

$$\frac{g_j'(w_j') - \bar{g}_j'(\bar{w}_j')}{2i} = \varphi_j \left(\frac{f_1'(z_1') - \bar{f}_1'(\bar{z}_1')}{2i}, \ldots, \frac{f_m'(z_m') - \bar{f}_m'(\bar{z}_m')}{2i} \right), \tag{3.25}$$

where $i = \sqrt{-1}$ here. Replacing w_j' by $\overline{\Theta}_j'(z', \bar{z}', \bar{w}')$ in the left hand side, we get the following identity between converging power series of the $2m + d$ complex variables $(z', \bar{z}', \bar{w}')$:

$$\frac{g_j'(\overline{\Theta}_j'(z', \bar{z}', \bar{w}')) - \bar{g}_j'(\bar{w}_j')}{2i} \equiv \varphi_j \left(\frac{f_1'(z_1') - \bar{f}_1'(\bar{z}_1')}{2i}, \ldots, \frac{f_m'(z_m') - \bar{f}_m'(\bar{z}_m')}{2i} \right). \tag{3.26}$$

Let us differentiate this identity with respect to z_k', for $k = 1, \ldots, m$. Taking into account the relations (3.24), we obtain

$$\frac{a_k'(z_k')\, \overline{\Theta}_{j,z_k'}'(z', \bar{z}', \bar{w}')}{b_j'(\overline{\Theta}_j'(z', \bar{z}', \bar{w}'))} \equiv \frac{\partial \varphi_j}{\partial y_k} \left(\frac{f_1'(z_1') - \bar{f}_1'(\bar{z}_1')}{2i}, \ldots, \frac{f_m'(z_m') - \bar{f}_m'(\bar{z}_m')}{2i} \right). \tag{3.27}$$

Clearly, the left hand side is an algebraic function $\mathcal{A}'_{j,k}(z', \bar{z}', \bar{w}')$. Then differentiating again with respect to the variables z'_k the relations (3.27), we see that for every multi-index $\beta \in \mathbb{N}^m$ with $|\beta| \geq 1$, and every $j = 1, \dots, d$, there exists an algebraic function $\mathcal{A}'_{j,\beta}(z', \bar{z}', \bar{w}')$ such that the following identity holds:

$$\mathcal{A}'_{j,\beta}(z', \bar{z}', \bar{w}') \equiv \frac{\partial^{\beta_1+\cdots+\beta_m}\varphi_j}{\partial y_1^{\beta_1}\cdots\partial_{y_m}^{\beta_m}}\left(\frac{f'_1(z'_1)-\bar{f}'_1(\bar{z}'_1)}{2i}, \dots, \frac{f'_m(z'_m)-\bar{f}'_m(\bar{z}'_m)}{2i}\right). \tag{3.28}$$

Differentiating (3.28) with respect to $\bar{w}'$, we see immediately that $\mathcal{A}'_{j,\beta}$ is in fact independent of $\bar{w}'$. Furthermore, we see that $\mathcal{A}'_{j,\beta}$ is real, namely $\mathcal{A}'_{j,\beta}(z', \bar{z}') \equiv \overline{\mathcal{A}}'_{j,\beta}(\bar{z}', z')$. Now we extract from (3.28) the m identities written for $\beta := \beta_*^k$, $j := j_*^k$, $k = 1, \dots, m$ and we use the invertibility of the mapping ψ defined in (3.19) (recall that $\psi'(y') = y$ denotes the inverse of $y' = \psi(y)$), which yields

$$\frac{f'_k(z'_k)-\bar{f}'_k(\bar{z}'_k)}{2i} \equiv \psi'_k(\mathcal{A}'_{j_*^1,\beta_*^1}(z', \bar{z}'), \dots, \mathcal{A}'_{j_*^m,\beta_*^m}(z', \bar{z}')), \tag{3.29}$$

for $k = 1, \dots, m$. For simplicity, we shall write $\mathcal{A}'_k(z', \bar{z}')$ instead of $\mathcal{A}'_{j_*^k,\beta_*^k}(z', \bar{z}')$. Finally, we differentiate (3.29) with respect to z'_k, which yields, taking into account (3.24):

$$\begin{cases} \dfrac{1}{2i\, a'_k(z'_k)} \equiv \displaystyle\sum_{l=1}^m \frac{\partial \psi'_k}{\partial y'_l}(\mathcal{A}'_1(z', \bar{z}'), \dots, \mathcal{A}'_m(z', \bar{z}'))\, \frac{\partial \mathcal{A}'_l}{\partial z'_k}(z', \bar{z}'), \\ \qquad\qquad 0 \equiv \displaystyle\sum_{l=1}^m \frac{\partial \psi'_k}{\partial y'_l}(\mathcal{A}'_1(z', \bar{z}'), \dots, \mathcal{A}'_m(z', \bar{z}'))\, \frac{\partial \mathcal{A}'_l}{\partial z'_{\widetilde{k}}}(z', \bar{z}'), \qquad \widetilde{k} \neq k. \end{cases} \tag{3.30}$$

It follows from these relations (3.30) viewed in matrix form that the constant matrix $(\frac{\partial \mathcal{A}'_l}{\partial z'_k}(0,0))_{1\leq l,k\leq m}$ is invertible, because the diagonal matrix $(\delta_k^{\widetilde{k}}\,[2ia'_k(z'_k)]^{-1})_{1\leq k,\widetilde{k}\leq m}$ is evidently invertible at $z'_k = 0$ (recall $a'_k(0) = 1$). Consequently, there exist algebraic functions $\mathcal{B}'_{k,l}(z', \bar{z}')$ so that

$$\frac{\partial \psi'_k}{\partial y'_l}(\mathcal{A}'_1(z', \bar{z}'), \dots, \mathcal{A}'_m(z', \bar{z}')) \equiv \mathcal{B}'_{k,l}(z', \bar{z}'). \tag{3.31}$$

Next, setting $\widetilde{y}'_k = \mathcal{A}'_k(iy', -iy')$, $k = 1, \dots, m$ we see, from the invertibility of the matrix $(\frac{\partial \mathcal{A}'_k}{\partial z'_l}(0,0))_{1\leq k,l\leq m}$ and from the reality of $\mathcal{A}'_k(z', \bar{z}')$, that the Jacobian determinant at the origin of the mapping $y' \mapsto \mathcal{A}'(iy', -iy') = \widetilde{y}'_k$ is nonzero. Thus there are real algebraic functions $\mathcal{C}'_k$ so that we can express y' in terms of $\widetilde{y}'$ as $y'_k = \mathcal{C}'_k(\widetilde{y}')$. Finally, we get

$$\frac{\partial \psi'_k}{\partial y'_l}(\widetilde{y}'_1, \dots, \widetilde{y}'_m) = \mathcal{B}'_{k,l}(i\mathcal{C}'(\widetilde{y}'), -i\mathcal{C}'(\widetilde{y}')), \tag{3.32}$$

where the right hand sides are algebraic; this shows that the partial derivatives $\partial_{y'l}\psi'_k$ are algebraic functions of $\widetilde{y}'$.

To obtain the equivalent formulation of Theorem 1.1, we observe the following.

Lemma 3.3. *For every $k, l = 1, \dots, m$, the functions $\partial_{y'_k}\psi'_l(y')$ are algebraic functions of y' if and only if for every $k_1, k_2 = 1, \dots, m$, the second derivative $\partial^2_{y_{k_1}y_{k_2}}(y)$ is an algebraic function of $\psi(y) = (\partial_{y_1}\varphi(y), \dots, \partial_{y_m}\varphi(y))$.*

Proof. Differentiating the identities $y_k \equiv \psi'_k(\psi(y))$, $k = 1, \dots, m$, with respect to y_l, we get

$$\delta_k^l \equiv \sum_{j=1}^m \partial_{y'_j}\psi'_k(\psi(y))\,\partial_{y_j}\psi_j(y) \equiv \sum_{j=1}^m \partial_{y'_j}\psi'_k(y')\,\partial^2_{y_m y_l}\varphi(y). \tag{3.33}$$

Applying Cramer's rule, we see that there exist universal rational functions $R_{k,l}$ such that

$$(3.34)\qquad \begin{cases} \partial^2_{y_k y_l}\varphi(y) \equiv R_{k,l}(\{\partial_{y'_{k_2}}\psi'_{k_1}(y')\}_{1\le k_1,k_2\le m}) \\ \qquad\qquad \equiv R_{k,l}(\{\partial_{y'_{k_2}}\psi'_{k_1}(\partial_{y_1}\varphi(y),\dots,\partial_{y_m}\varphi(y))\}_{1\le k_1,k_2\le m}). \end{cases}$$

This implies the equivalence of Lemma 3.3. □

In conclusion, taking Theorem 2.1 for granted, the proof of Theorem 1.1 is now complete. □

3.3. Proof of Theorem 1.5. Let $M : v = \varphi(z,\bar z)$ be a rigid Levi nondegenerate hypersurface in $\mathbb{C}^n$ passing through the origin. We may assume that $v = \sum_{k=1}^{n-1}\varepsilon_k\,|z_k|^2 + \varphi^3(z,\bar z)$, where $\varepsilon_k = \pm 1$ and we may write $\varphi^3(z,\bar z) = \sum_{k=1}^{n-1}[\bar z_k\,\varphi_k^3(z) + z_k\bar\varphi_k^3(\bar z)] + \varphi^4(z,\bar z)$, with $\varphi^4(0,\bar z) \equiv \varphi^4_{z_k}(0,\bar z) \equiv 0$ and $\varphi_k^3 = \mathrm{O}(2)$. After making the change of coordinates $z'_k := z_k + \varepsilon_k\,\varphi_k^3(z)$, $w' := w$, we come to the simple equation $v' = \sum_{k=1}^{n-1}\varepsilon_k\,|z'_k|^2 + \chi'(z',\bar z')$, where $\chi'(0,\bar z') \equiv \chi_{z'_k}(0,\bar z') \equiv 0$, considered in Theorem 1.5.

Assume that M is strongly rigid, locally algebraizable and let M' be an algebraic equivalent of M. Let $t' = h(t)$ be such an equivalence, or in our previous notation $z' = f(z,w)$ and $w' = g(z,w)$. We note $z = f'(z',w')$ and $w = g'(z',w')$ the inverse equivalence. Since M is strongly rigid, namely $\mathfrak{Hol}(M)$ is generated by the single vector field $X_1 := \partial_w$, it follows that $\mathfrak{Hol}(M')$ is also one-dimensional, generated by the single vector field $X'_1 := h_*(X_1)$. Taking again Theorem 2.1 for granted and proceeding as in the first step of the proof of Proposition 3.1, we may algebraically straighten the complex foliation induced by X'_1 to the "vertical" foliation by w'-lines. Equivalently, we may assume that $X'_1 = b'(z',w')\,\partial_{w'}$ with b' algebraic and $b'(0) = 1$. The assumption $h_*(\partial_w) = b'(z',w')\,\partial_{w'}$ yields that $f'(z',w')$ is independant of w' and that $b'(z',w')\,g'_{w'}(z',w') \equiv 1$, so that as in (3.24) above, the derivative $g'_{w'}$ is algebraic. Let $w' = \overline{\Theta}'(z',\bar z',\bar w')$ be the complex defining equation of M' in these coordinates. The assumption $h'(M') = M$ yields the following power series identity

$$(3.35)\qquad g'(z',\overline{\Theta}'(z',\bar z',\bar w')) - \bar g'(\bar z',\bar w') \equiv 2i\,\varphi(f'(z'),\bar f'(\bar z')).$$

By differentiating this identity with respect to z'_k, we get

$$(3.36)\quad \partial_{z'_k} g'(z',\overline{\Theta}'(z',\bar z',\bar w')) + \frac{\partial_{z'_k}\overline{\Theta}'(z',\bar z',\bar w')}{b'(\overline{\Theta}'(z',\bar z',\bar w'))} \equiv 2i\sum_{l=1}^{n-1}\partial_{z_l}\varphi(f'(z'),\bar f'(\bar z'))\,\partial_{z'_k}f'_l(z').$$

We notice that the second term in the left hand side of (3.36) is algebraic. By differentiating in turn (3.36) with respect to $\bar z'_k$ and using the algebraicity of $\partial^2_{z'_k w'}g'(z',\bar\Theta'(z',\bar z',\bar w'))$, we obtain that there exist algebraic functions $\mathcal{A}'_{k_1,k_2}(z',\bar z')$ such that

$$(3.37)\qquad \mathcal{A}'_{k_1,k_2}(z',\bar z') \equiv \sum_{l_1,l_2=1}^{n-1}\partial^2_{z_{l_1}\bar z_{l_2}}\varphi(f'(z'),\bar f'(\bar z'))\,\partial_{z'_{k_1}}f'_{l_1}(z')\,\partial_{\bar z'_{k_2}}\bar f'_{l_2}(\bar z').$$

Without loss of generality, we may assume that h' is tangent to the identity map at $t' = 0$. Then setting $\bar z' := 0$ in (3.37) and using the fact that $\partial^2_{z_{l_1}\bar z_{l_2}}\varphi(z,0) = \delta_{l_1}^{l_2}\varepsilon_{l_1} + \partial^2_{z_{l_1}\bar z_{l_2}}\chi(z,0) \equiv \delta_{l_1}^{l_2}\varepsilon_{l_1}$ by the properties of χ in Theorem 1.5 we get, since $\partial_{\bar z'_{k_2}}\bar f'_{l_2}(0) = \delta_{l_2}^{k_2}$:

$$(3.38)\qquad \mathcal{A}'_{k_1,k_2}(z',0) \equiv \varepsilon_{k_2}\,\partial_{z'_{k_1}}f'_{k_2}(z'),$$

which shows that all the first order derivatives $\partial_{z'_k}f'_l(z')$ are algebraic.

Next, since the canonical transformation to normalizing coordinates is algebraic and preserves the "horizontal" coordinates z' (*cf.* [CM1974]), hence does not perturb the complex foliation induced by X'_1, we may also assume that M' is given in normal coordinates, namely that the function Θ' satisfies $\Theta'(0,\bar z',\bar w') \equiv \Theta'(z',0,\bar w') \equiv \bar w'$. Since the coordinates are normal for both M and M', it follows by setting $\bar{z}' := 0$ and $\bar{w}' := 0$ in (3.35) that $g'(z',0) \equiv 0$. Consequently, $\partial_{z'_k} g'(z',0) \equiv 0$. Finally, by setting $z' := 0$ and $\bar{w}' := 0$ in (3.36), we see that the first term in the left hand side vanishes and that the second term is algebraic with respect to $\bar{z}'$, so we obtain that there exist algebraic functions $\overline{\mathcal{B}}'_k(\bar{z}')$ such that

$$\overline{\mathcal{B}}'_k(\bar{z}') \equiv \sum_{l=1}^{n-1} \partial_{z_l}\varphi(0, \bar{f}'(\bar{z}'))\, \partial_{z'_k} f'_l(0) \equiv \varepsilon_k\, \bar{f}'_k(\bar{z}'). \tag{3.39}$$

We have proved that the components $f'_k(z')$ are all algebraic.

Finally, coming back to the relation (3.36), we want to prove that the derivatives $\partial_{z_l}\varphi(f'(z'), \bar{f}'(\bar{z}'))$ are all algebraic. However, the first term of (3.36) is not algebraic in general. Fortunately, using the fact that $\overline{\Theta}' = \bar{w}' + \mathrm{O}(2)$, we see that there exists a unique algebraic solution $\bar{w}' = \overline{\Lambda}'(z', \bar{z}')$ of the implicit equation $\overline{\Theta}'(z', \bar{z}', \bar{w}') = 0$, namely satisfying $\overline{\Theta}'(z', \bar{z}', \overline{\Lambda}'(z', \bar{z}')) \equiv 0$. Then by replacing $\bar{w}'$ by $\overline{\Lambda}'$ in (3.36), we get that there exist algebraic functions $\mathcal{C}'_k(z', \bar{z}')$ such that

$$\mathcal{C}'_k(z', \bar{z}') \equiv \sum_{l=1}^{n-1} \partial_{z_l}\varphi(f'(z'), \bar{f}'(\bar{z}'))\, \partial_{z'_k} f'_l(z'). \tag{3.40}$$

Since f' is tangent to the identity map, we can solve by Cramer's rule this linear system for the derivatives $\partial_{z_l}\varphi$, which yields that the $\partial_{z_l}\varphi(f'(z'), \bar{f}'(\bar{z}'))$ are all algebraic. Since $f'(z')$ is also algebraic, we obtain in sum that the derivatives $\partial_{z_l}\varphi(z, \bar{z})$ are all algebraic. In conclusion, taking Theorem 2.1 for granted, the proof of Theorem 1.5 is complete. □

3.4. Proof of Theorem 1.4. Let $M : v = \varphi(z\bar{z})$ in $\mathbb{C}^2$ with $\mathfrak{Hol}(M)$ generated by ∂_w and $iz\partial_z$. Without loss of generality, we can assume that $\varphi(r) = r + \mathrm{O}(r^2)$. Let M' be an algebraic equivalent of M. Let $t = h'(t')$, or $z = f'(z', w')$, $w = g'(z', w')$ be a local holomorphic equivalence satisfying $h'(M') = M$. Let $t' = h(t)$ be its inverse. Then $\mathfrak{Hol}(M')$ is two-dimensional and generated by $h_*(\partial_w)$ and $h_*(iz\partial_z)$. First of all, using the algebraicity of the CR automorphism group of M' and proceeding as in the proof of Proposition 3.1, we can prove that there exist two generators of $\mathfrak{Hol}(M')$ of the form $X'_1 = b'(w')\, \partial_{w'}$ and $X'_2 = a'(z')\, \partial_{z'}$ where b' and a' are algebraic and satisfy $b'(0) = 1$ and $a'(z') = iz' + \mathrm{O}(z'^2)$. Furthermore, we may assume that h' is tangent to the identity map and that $h'_*(b'(w')\, \partial_{w'}) = \partial_w$ and $h'_*(a'(z')\, \partial_{z'}) = iz\partial_z$. As in (3.24), it follows that $b'(w')\, g'_{w'}(w') \equiv 1$ and $a'(z')\, f'_{z'}(z') \equiv i f'(z')$. Let $w' = \overline{\Theta}'(z', \bar{z}', \bar{w}')$ be the complex algebraic equation of M'. Then we get the following power series identity:

$$g'(\overline{\Theta}'(z', \bar{z}', \bar{w}')) - \bar{g}'(\bar{w}') \equiv 2i\, \varphi(f'(z')\bar{f}'(\bar{z}')), \tag{3.41}$$

which yields after differentiating with respect to z':

$$\begin{cases} \overline{\Theta}'_{z'}(z', \bar{z}', \bar{w}')/[b'(\overline{\Theta}'(z', \bar{z}', \bar{w}'))] \equiv 2i\, \partial_{z'} f(z')\, \bar{f}'(\bar{z}')\, \partial_r\varphi(f'(z')\bar{f}'(\bar{z}')) \\ \qquad\qquad\qquad \equiv\; -2\, f'(z')\bar{f}'(\bar{z}')\, \partial_r\varphi(f'(z')\bar{f}'(\bar{z}'))/[a'(z')]. \end{cases} \tag{3.42}$$

Here, we consider the function φ as a function $\varphi(r)$ of the real variable $r \in \mathbb{R}$. Since the left hand side is an algebraic function and $a'(z')$ is also algebraic, there exists an algebraic function $\mathcal{A}'(z', \bar{z}')$ such that we can write

$$\mathcal{A}'(z', \bar{z}') \equiv f'(z')\bar{f}'(\bar{z}')\, \partial_r\varphi(f'(z')\bar{f}'(\bar{z}')). \tag{3.43}$$

Next, using the property $\varphi(r) = r + \mathrm{O}(r^2)$, differentiating (3.43) with respect to $\bar{z}'$ at $\bar{z}' = 0$, we obtain that $f'(z')$ is algebraic. Coming back to (3.43), this yields that $\partial_r\varphi(f'(z')\bar{f}'(\bar{z}'))$ is algebraic. Since $f'(z')$ is also algebraic, we finally obtain that $\partial_r\varphi(r)$ is algebraic. Excepting the examples which will be treated in §7.5, the proof of Theorem 1.4 is complete. □

The next three sections are devoted to the statement of Theorem 4.1, which implies directly Theorem 2.1 (§4), and to its proof (§§5-6).

§4. Local Lie group structure for the CR automorphism group

4.1. Local representation of a real algebraic generic submanifold. We consider a connected real algebraic (or more generally, real analytic) generic submanifold M in $\mathbb{C}^n$ of codimension $d \geq 1$ and CR dimension $m = n - d \geq 1$. Pick a point $p \in M$ and consider some holomorphic coordinates $t = (t_1, \ldots, t_n) = (z_1, \ldots, z_m, w_1, \ldots, w_d) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at p in which $T_0M = \{\operatorname{Im} w = 0\}$. If we denote $w = u + iv$, it follows that there exists (Nash) real algebraic power series $\varphi_j(z, \bar{z}, u)$ with $\varphi_j(0) = 0$ and $d\varphi_j(0) = 0$ such that the defining equations of M are of the form $v_j = \varphi_j(z, \bar{z}, u)$, $j = 1, \ldots, d$ in a neighborhood of the origin. By means of the algebraic implicit function theorem, we can solve with respect to $\bar{w}$ the equations $w_j - \bar{w}_j = 2i\, \varphi_j(z, \bar{z}, (w+\bar{w})/2)$, $j = 1, \ldots, d$, which yields $\bar{w}_j = \Theta_j(\bar{z}, z, w)$ for some power series Θ_j which are complex algebraic with respect to their $2m + d$ variables. Here, we have $\Theta_j = w_j + \mathrm{O}(2)$, since $T_0M = \{\operatorname{Im} w = 0\}$. Without loss of generality, we shall assume that the coordinates are normal, namely the functions $\Theta_j(\bar{z}, z, w)$ satisfy $\Theta_j(0, z, w) \equiv w_j$ and $\Theta_j(\bar{z}, 0, w) \equiv w_j$. It may be shown that the power series $\Theta_j = w_j + \mathrm{O}(2)$ satisfy the vectorial functional equation $\Theta(\bar{z}, z, \overline{\Theta}(z, \bar{z}, \bar{w})) \equiv \bar{w}$ in $\mathbb{C}\{z, \bar{z}, \bar{w}\}^d$ and conversely that to every such power series mapping satisfying this vectorial functional equation, there corresponds a unique real algebraic generic manifold M (*cf.* for instance the manuscript [GM2001c] for the details). So we can equivalently take $\bar{w}_j = \Theta_j(\bar{z}, z, w)$ or $w_j = \overline{\Theta}_j(z, \bar{z}, \bar{w})$ as complex defining equations for M.

For arbitrary $\rho > 0$, we shall often consider the open polydisc $\Delta_n(\rho) := \{t \in \mathbb{C}^n : |t| < \rho\}$ where we denote by $|t| := \max_{1 \leq i \leq n} |t_i|$ the usual polydisc norm. Without loss of generality, we may assume that the power series Θ_j converge normally in the polydic $\Delta_{2m+d}(2\rho_1)$, where $\rho_1 > 0$. In fact, we shall successively introduce some other positive constants (radii) $0 < \rho_5 < \rho_4 < \rho_3 < \rho_2 < \rho_1$ afterwards. Finally, we define M as:

$$M = \{(z, w) \in \Delta_n(\rho_1) : \bar{w}_j = \Theta_j(\bar{z}, z, w),\ j = 1, \ldots, d\}. \tag{4.1}$$

Next, let ρ_2 arbitrary with $0 < \rho_2 < \rho_1$. For $h', h \in \mathcal{O}(\Delta_n(\rho_1), \mathbb{C}^n)$, we define

$$||h' - h||_{\rho_2} := \sup\{|h'(t) - h(t)| : t \in \Delta_n(\rho_2)\}. \tag{4.2}$$

For $k \in \mathbb{N}$, we shall also consider the $\mathcal{C}^k$ norms

$$||J^k h' - J^k h||_{\rho_2} := \sup\{|\partial_t^\alpha h'(t) - \partial_t^\alpha h(t)| : t \in \Delta_n(\rho_2),\ \alpha \in \mathbb{N}^n,\ |\alpha| \leq k\}. \tag{4.3}$$

For $k \in \mathbb{N}$ and $t \in \Delta_n(\rho_1)$, we denote by $J^k h(t)$ the collection of partial derivatives $(\partial_t^\alpha h_i(t))_{1 \leq i \leq n,\, |\alpha| \leq k}$ of length $\leq k$ of the components $h_1, \ldots, h_n$, so $J^k h(t) \in \mathbb{C}^{N_{n,k}}$, where $N_{n,k} := n \frac{(n+k)!}{n!\ k!}$. In particular, the expression $J^k h(0) = (\partial_t^\alpha h_i(0))_{1 \leq i \leq n,\, |\alpha| \leq k}$ denotes the k-jet of h at 0. So, the space of k-jets at the origin of holomorphic mappings $h \in \mathcal{O}(\Delta_n(\rho_1), \mathbb{C}^n)$ may be identified with the complex linear space $\mathbb{C}^{N_{n,k}}$. We denote the natural coordinates on $\mathbb{C}^{N_{n,k}}$ by $(J_i^\alpha)_{1 \leq i \leq n,\, |\alpha| \leq k}$. Sometimes, we abbreviate this collection of coordinates by $J^k \equiv (J_i^\alpha)_{1 \leq i \leq n,\, |\alpha| \leq k}$. Finally, we denote by J_{Id}^k the k-jet at the origin of the identity mapping. We introduce the important set of holomorphic self-mappings of M defined by

$$\begin{cases} \mathcal{H}_{M,k,\varepsilon}^{\rho_2,\rho_1} := \{h \in \mathcal{O}(\Delta_n(\rho_1), \mathbb{C}^n) : & ||J^k h - J_{\mathrm{Id}}^k||_{\rho_2} < \varepsilon, \\ & h(M \cap \Delta_n(\rho_2)) \subset M \cap \Delta_n(\rho_1)\}. \end{cases} \tag{4.4}$$

Here, $k \in \mathbb{N}$ and $\varepsilon > 0$ is a small positive number that we shall shrink many times in the sequel.

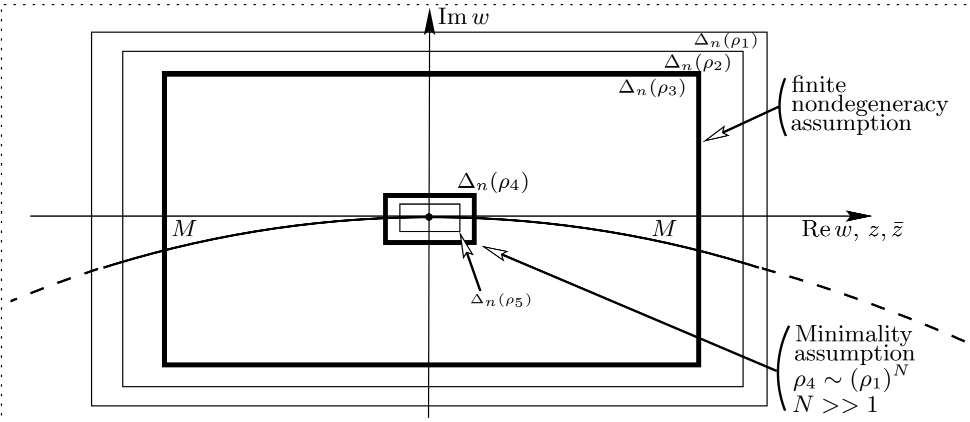


FIGURE 3: NEST OF POLYDISCS CENTERED AT $0 \in M$

We may now state the main theorem of §4, §5 and §6, namely Theorem 4.1, which provides a complete parametrized description of the set $\mathcal{H}_{M,k,\varepsilon}^{\rho_2,\rho_1}$ of local biholomorphic self-mappings of M, with k equal to an integer κ_0 depending on M. During the course of the (rather long) proof, for technical reasons, we shall have to introduce first a third positive radius ρ_3 with $0 < \rho_3 < \rho_2 < \rho_1$ which is related to the finite nondegeneracy of M, and then afterwards a fourth positive radius ρ_4 with $0 < \rho_4 < \rho_3 < \rho_2 < \rho_1$, which is related to the minimality of M. This is why the radius notation "ρ_4" appears after "ρ_2" and "ρ_1" without mention of "ρ_3" (*cf.* FIGURE 3).

Theorem 4.1. *Assume that the real algebraic generic submanifold M defined by* (4.1) *is minimal and finitely nondegenerate at the origin. As above, fix two radii ρ_1 and ρ_2 with $0 < \rho_2 < \rho_1$. Then there exists an* even *integer $\kappa_0 \in \mathbb{N}_*$ which depends only on the local geometry of M near the origin, there exists $\varepsilon > 0$, there exists $\rho_4 > 0$ with $\rho_4 < \rho_2$, there exists a complex algebraic $\mathbb{C}^n$-valued mapping $H(t, J^{\kappa_0})$ which is defined for $t \in \mathbb{C}^n$ with $|t| < \rho_4$ and for $J^{\kappa_0} \in \mathbb{C}^{N_{n,\kappa_0}}$ (where $N_{n,\kappa_0} = n\frac{(n+\kappa_0)!}{n!\ \kappa_0!}$) with $|J^{\kappa_0} - J^{\kappa_0}_{\mathrm{Id}}| < \varepsilon$ and which depends only on M and there exists a geometrically smooth real algebraic totally real submanifold E of $\mathbb{C}^{N_{n,\kappa_0}}$ passing through the identity jet $J^{\kappa_0}_{\mathrm{Id}}$ which depends only on M, which is defined by*

$$E = \{J^{\kappa_0} : |J^{\kappa_0} - J^{\kappa_0}_{\mathrm{Id}}| < \varepsilon,\ C_l(J^{\kappa_0}, \overline{J^{\kappa_0}}) = 0,\ l = 1, \ldots, \upsilon\}, \tag{4.5}$$

where the $C_l(J^{\kappa_0}, \overline{J^{\kappa_0}})$, $l = 1, \ldots, \upsilon$, are real algebraic functions defined on the polydisc $\{|J^{\kappa_0} - J^{\kappa_0}_{\mathrm{Id}}| < \varepsilon\}$, and which can be constructed algorithmically by means only of the defining equations of M, such that the following six statements hold:

(1) *Every local biholomorphic self-mapping $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$ of M (which is defined on the large polydisc $\Delta_n(\rho_1)$) is represented by*

$$h(t) = H(t, J^{\kappa_0} h(0)), \tag{4.6}$$

on the smallest polydisc $\Delta_n(\rho_4)$. In particular, each $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$ is a complex algebraic biholomorphic mapping. Furthermore, the κ_0-jet of h at the origin belongs to the real algebraic submanifold E, namely we have $C_l(J^{\kappa_0} h(0), J^{\kappa_0} \bar{h}(0)) = 0$, $l = 1, \ldots, \upsilon$.

(2) *Conversely, shrinking ε if necessary, given an arbitrary jet J^{κ_0} in E there exists a smaller positive radius $\rho_5 < \rho_4$ such that the mapping defined by $h(t) := H(t, J^{\kappa_0})$ for $|t| < \rho_5$ sends $M \cap \Delta_n(\rho_5)$ CR-diffeomorphically onto its image which is contained in $M \cap \Delta_n(\rho_4)$. We may therefore say that the set $\mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$ of local biholomorphic self-mappings of M is parametrized by the real algebraic submanifold E.*

(3) *For every choice of two smaller positive radii $\widetilde{\rho}_1 \leq \rho_1$ and $\widetilde{\rho}_2 \leq \rho_2$ with $\widetilde{\rho}_2 < \widetilde{\rho}_1$, there exists a positive radius $\widetilde{\rho}_4 \leq \rho_4$ with $\widetilde{\rho}_4 < \widetilde{\rho}_2$, and a positive $\widetilde{\varepsilon} \leq \varepsilon$*

such that the same complex algebraic mapping $H(t, J^{\kappa_0})$ as in statement **(1)** *above represents all local biholomorphic self-mappings $\widetilde{h} \in \mathcal{H}^{\widetilde{\rho}_2,\widetilde{\rho}_1}_{M,\kappa_0,\widetilde{\varepsilon}}$ of M, namely we have $\widetilde{h}(t) = H(t, J^{\kappa_0}\widetilde{h}(0))$ for all $|t| < \widetilde{\rho}_4$ as in (4.6). Furthermore, the corresponding real algebraic totally real submanifold $\widetilde{E}$ coincides with E in the polydisc $\{|J^{\kappa_0} - J^{\kappa_0}_{\rm Id}| < \widetilde{\varepsilon}\}$ and it is defined by the same real algebraic equations $C_l(J^{\kappa_0}, \overline{J^{\kappa_0}}) = 0$, $l = 1, \dots, \upsilon$, as in equation (4.5). In fact, the algebraic mapping $H(t, J^{\kappa_0})$ and the real algebraic totally real submanifold E depend only on the local geometry of M in a neighborhood of the origin, namely on the germ of M at 0.*

(4) *The set $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$, equipped with the law of composition of holomorphic mappings, is a real algebraic local Lie group. More precisely, let the positive integer c_0 denote the real dimension of E, which is independent of ρ_1, ρ_2 and consider a parametrization*

$$\mathbb{R}^{c_0} \ni e = (e_1, \dots, e_{c_0}) \mapsto j_{\kappa_0}(e) \in E \subset \mathbb{C}^{N_{n,\kappa_0}} \tag{4.7}$$

of the real algebraic totally real submanifold E. Then there exist a real algebraic associative local multiplication mapping $(e, e') \mapsto \mu(e, e')$ and a real algebraic local inversion mapping $e \mapsto \iota(e)$ such that if we define $H(t; e) := H(t, j_{\kappa_0}(e))$, then $H(H(t; e); e') \equiv H(t; \mu(e, e'))$ and $H(t; e)^{-1} \equiv H(t; \iota(e))$, with the local Lie transformation group axioms, as defined in §2.3, being satisfied by H, μ and ι.

(5) *For $i = 1, \dots, c_0$, consider the one-parameter families of transformations defined by $H(t; 0, \dots, 0, e_i, 0, \dots, 0) =: H_i(t; e_i) =: H_{i,e_i}(t)$. Then for each $i = 1, \dots, c_0$, the vector field $X_i|_{(t;e_i)} := [\partial_{i,e_i} H_{e_i}(t')]_{t'=H_{e_i}^{-1}(t)}$, is defined for $t \in \Delta_n(\rho_5)$ and $|e_i| < \varepsilon$, has algebraic coefficients depending on the "time" parameter e_i, and has an algebraic flow, since this coincides with the algebraic mapping $(t, e_i) \mapsto H_{i,e_i}(t)$.*

(6) *Let ρ_5 be as in statement* **(2)**. *Then the dimension c_0 of the real Lie algebra $\mathfrak{Hol}(M, \Delta_n(\rho_5))$ is finite, bounded by the fixed integer $N_{n,\kappa_0} := n\frac{(n+\kappa_0)!}{n!\ \kappa_0!}$. Furthermore, each vector field $X \in \mathfrak{Hol}(M, \Delta_n(\rho_5))$ has complex algebraic coefficients.*

If M is real analytic, the same theorem holds with the word "algebraic" replaced everywhere by the word "analytic".

We shall explain below how the integer κ_0 is related to the minimality and to the finite nondegeneracy of M at the origin. The next §5 and §6 are devoted to the proof Theorem 4.1, namely the existence of the mapping $H(t, J^{\kappa_0})$, the existence of the real algebraic totally real submanifold E and the completion of the proof of properties **(1-6)**.

§5. Minimality and finite nondegeneracy

5.1. Local CR geometry of complexified real analytic generic submanifolds. Let $\zeta \in \mathbb{C}^m$ and $\xi \in \mathbb{C}^d$ denote some independent coordinates corresponging to the complexification of the variables $\bar{z}$ and $\bar{w}$, which we denote symbolically by $\zeta := (\bar{z})^c$ and $\xi := (\bar{w})^c$, where the letter "c" stands for the word "complexified". We also write $\tau := (\bar{t})^c$, so $\tau = (\zeta, \xi) \in \mathbb{C}^n$. The *extrinsic complexification* $\mathcal{M} := (M)^c$ of M is the complex submanifold of codimension d defined by

$$\mathcal{M} := \{(z, w, \zeta, \xi) \in \Delta_n(\rho_1) \times \Delta_n(\rho_1) : \xi = \Theta(\zeta, z, w)\}. \tag{5.1}$$

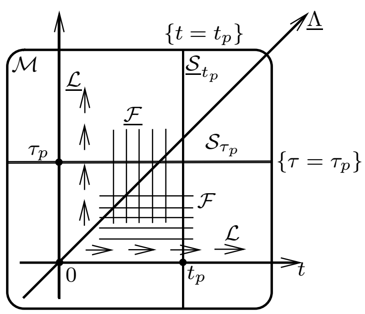
If M is (real, Nash) algebraic, so is $\mathcal{M}$. As remarked, we can choose the equivalent defining equation $w = \overline{\Theta}(z, \zeta, \xi)$ for $\mathcal{M}$. In the remainder of §5, we shall essentially deal with $\mathcal{M}$ instead of M. In fact, M clearly imbeds in $\mathcal{M}$ as the intersection of $\mathcal{M}$ with the antiholomorphic diagonal $\underline{\Lambda} := \{(t, \tau) \in \mathbb{C}^n \times \mathbb{C}^n : \tau = \bar{t}\}$.

Following [Me1998], [Me2001], we shall complexify a conjugate pair of generating families of CR vector fields tangent to M, namely $L_1, \dots, L_m$ of type $(1,0)$ and their conjugates $\overline{L}_1, \dots, \overline{L}_m$ which are of type $(0,1)$. Here, we can explicitely choose the generators $L_k = \partial/\partial z_k + \sum_{j=1}^d [\partial\overline{\Theta}_j/\partial z_k(z, \bar{z}, \bar{w})]\, \partial/\partial w_j$ for $k = 1, \dots, m$. Then their complexification yields a pair of collections of m vector fields defined over $\Delta_n(\rho_1)\times\Delta_n(\rho_1)$ by

$$(5.2)\qquad \begin{cases} \mathcal{L}_k := \dfrac{\partial}{\partial z_k} + \displaystyle\sum_{j=1}^{d} \dfrac{\partial \overline{\Theta}_j}{\partial z_k}(z,\zeta,\xi)\, \dfrac{\partial}{\partial w_j}, \qquad k=1,\dots,m, \\ \underline{\mathcal{L}}_k := \dfrac{\partial}{\partial \zeta_k} + \displaystyle\sum_{j=1}^{d} \dfrac{\partial \Theta_j}{\partial \zeta_k}(\zeta,z,w)\, \dfrac{\partial}{\partial \xi_j}, \qquad k=1,\dots,m. \end{cases}$$

The reader may check directly that $\mathcal{L}_k(w_j-\overline{\Theta}_j(z,\zeta,\xi))\equiv 0$, which shows that the vector fields $\mathcal{L}_k$ are tangent to $\mathcal{M}$. Similarly, $\underline{\mathcal{L}}_k(\xi_j-\Theta_j(\zeta,z,w))\equiv 0$, so the vector fields $\underline{\mathcal{L}}_k$ are also tangent to $\mathcal{M}$. Furthemore, we may check the commutation relations $[\mathcal{L}_k,\mathcal{L}_{k'}]=0$ and $[\underline{\mathcal{L}}_k,\underline{\mathcal{L}}_{k'}]=0$ for all $k,k'=1,\dots,m$. It follows from the Frobenius theorem that the two m-dimensional distributions spanned by each of these two collections of m vector fields has the integral manifold property. This is not surprising since the vector fields $\mathcal{L}_k$ are the vector fields tangent to the intersection of $\mathcal{M}$ with the sets $\{\tau=\tau_p=ct.\}$, which are clearly m-dimensional complex integral manifolds. Following [Me1998], [Me2001], we denote these manifolds by $\mathcal{S}_{\tau_p}:=\{(t,\tau_p):\, w=\overline{\Theta}(z,\zeta_p,\xi_p)\}$, where τ_p is a constant, and we call them *complexified Segre varieties*. Similarly, the integral manifolds of the vector fields $\underline{\mathcal{L}}_k$ are the *conjugate complexified Segre varieties* $\underline{\mathcal{S}}_{t_p}:=\{(t_p,\tau):\, \xi=\Theta(\zeta,z_p,w_p)\}$, where t_p is fixed. The union of the manifolds $\mathcal{S}_{\tau_p}$ induces a local complex algebraic foliation $\mathcal{F}$ of $\mathcal{M}$ by m-dimensional leaves. Similarly, there is a second foliation $\underline{\mathcal{F}}$ whose leaves are the $\underline{\mathcal{S}}_{t_p}$.

The following symbolic picture summarizes our constructions. However, we warn the reader that the codimension $d\geq 1$ of the union of the two foliations $\mathcal{F}$ and $\underline{\mathcal{F}}$ in $\mathcal{M}$ is not visible in this two-dimensional figure.



The complexification of a real analytic CR-generic manifold M carries two complex foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ directed by the complefixied CR-vector fields $\mathcal{L}$ and $\underline{\mathcal{L}}$ whose leaves coincide with the complexified Segre varieties $\mathcal{S}_{\tau_p}$ and $\underline{\mathcal{S}}_{t_p}$. These leaves also coincide with the intersection of $\mathcal{M}$ with the horizontal slices $\{\tau=\tau_p\}$ and with the vertical slices $\{t=t_p\}$.

FIGURE 4: GEOMETRY OF THE COMPLEXIFICATION $\mathcal{M}$

Now, we introduce the "multiple" flows of the two collections of conjugate vector fields $(\mathcal{L}_k)_{1\leq k\leq m}$ and $(\underline{\mathcal{L}}_k)_{1\leq k\leq m}$. For an arbitrary point $p=(w_p,z_p,\zeta_p,\xi_p)\in\mathcal{M}$ and for an arbitrary complex "multitime" parameter $z_1=(z_{1,1},\dots,z_{1,m})\in\mathbb{C}^m$, we define

$$(5.3)\qquad \begin{cases} \mathcal{L}_{z_1}(z_p,w_p,\zeta_p,\xi_p) := \exp(z_1\mathcal{L})(p) := \exp(z_{1,1}\mathcal{L}_1(\cdots(\exp(z_{1,m}\mathcal{L}_m(p)))\cdots)) := \\ \qquad := (z_p+z_1,\overline{\Theta}(z_p+z_1,\zeta_p,\xi_p),\zeta_p,\xi_p). \end{cases}$$

With this formal definition, there exists a maximal connected open subset Ω of $\mathcal{M}\times\mathbb{C}^m$ containing $\mathcal{M}\times\{0\}$ such that $\mathcal{L}_{z_1}(p)\in\mathcal{M}$ for all $(z_1,p)\in\Omega$. Analogously, for (ζ_1,p) running in a similar open subset $\underline{\Omega}$, we may also define the map

$$(5.4)\qquad \underline{\mathcal{L}}_{\zeta_1}(z_p,w_p,\zeta_p,\xi_p) := (z_p,w_p,\zeta_p+\zeta_1,\Theta(\zeta_p+\zeta_1,z_p,w_p)).$$

We notice that the two maps given by (5.3) and (5.4) are holomorphic in their variables. Since M is real algebraic, they are moreover complex algebraic.

5.2. Segre chains. Let us start from the point p being the origin and let us move alternately in the direction of $\mathcal{S}$ or of $\underline{\mathcal{S}}$, namely we consider the two maps $\Gamma_1(z_1) := \mathcal{L}_{z_1}(0)$ and $\underline{\Gamma}_1(z_1) := \underline{\mathcal{L}}_{z_1}(0)$. Next, we start from these endpoints and we move in the other direction, namely, we consider the two maps

$$\Gamma_2(z_1, z_2) := \underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0)), \qquad \underline{\Gamma}_2(z_1, z_2) := \mathcal{L}_{z_2}(\underline{\mathcal{L}}_{z_1}(0)), \tag{5.5}$$

where $z_1, z_2 \in \mathbb{C}^m$. Also, we define $\Gamma_3(z_1, z_2, z_3) := \mathcal{L}_{z_3}(\underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0)))$, *etc.* By induction, for every positive integer k, we obtain two maps $\Gamma_k(z_1, \dots, z_k)$ and $\underline{\Gamma}_k(z_1, \dots, z_k)$. In the sequel, we shall often use the notation $z_{(k)} := (z_1, \dots, z_k) \in \mathbb{C}^{mk}$. Since $\Gamma_k(0) = \underline{\Gamma}_k(0) = 0$, for every $k \in \mathbb{N}_*$, there exists a sufficiently small open polydisc $\Delta_{mk}(\delta_k)$ centered at the origin in $\mathbb{C}^{mk}$ with $\delta_k > 0$ such that $\Gamma_k(z_{(k)})$ and $\underline{\Gamma}_k(z_{(k)})$ belong to $\mathcal{M}$ for all $z_{(k)} \in \Delta_{mk}(\delta_k)$.

We also exhibit a simple link between the maps Γ_k and $\underline{\Gamma}_k$. Let σ be the antiholomorphic involution defined by $\sigma(t, \tau) := (\bar{\tau}, \bar{t})$. Since $w = \overline{\Theta}(z, \zeta, \xi)$ if and only if $\xi = \Theta(\zeta, z, w)$, this involution maps $\mathcal{M}$ onto $\mathcal{M}$ and it also fixes the antidiagonal $\underline{\Lambda}$ pointwise. Using the definitions (5.3) and (5.4), we see readily that $\sigma(\mathcal{L}_{z_1}(0)) = \underline{\mathcal{L}}_{\bar{z}_1}(0)$. It follows generally that $\sigma(\Gamma_k(z_{(k)})) = \underline{\Gamma}_k(\overline{z_{(k)}})$.

Next, we observe that $\Gamma_{k+1}(z_{(k)}, 0) = \Gamma_k(z_{(k)})$, since $\mathcal{L}_0$ and $\underline{\mathcal{L}}_0$ coincide with the identity map. So the ranks at the origin of the maps Γ_k increase with k.

Definition 5.1. The real analytic generic manifold M is said to be *minimal* at p if the maps Γ_k are of (maximal possible) rank equal to $2m + d = \dim_{\mathbb{C}} \mathcal{M}$ at the origin in $\Delta_{mk}(\delta_k)$ for all k large enough.

The following fundamental properties are established in [Me1998], [Me2001].

Theorem 5.2. *The minimality of M at 0 is a biholomorphically invariant property. It depends neither on the choice of a defining equation for M nor on the choice of a system of generating complexified CR vector fields $(\mathcal{L}_k)_{1\leq k\leq m}$ and $(\underline{\mathcal{L}}_k)_{1\leq k\leq m}$. Also, minimality is equivalent to the fact that the Lie algebra generated by the complexified CR vector fields $(\mathcal{L}_k)_{1\leq k\leq m}$ and $(\underline{\mathcal{L}}_k)_{1\leq k\leq m}$ spans $T\mathcal{M}$ in a neighborhood of 0. Furthermore, there exists an invariant integer ν_0, called the* Segre type *of M at 0 satisfying $\nu_0 \leq d + 1$ which is the smallest integer such that the mappings Γ_k and $\underline{\Gamma}_k$ are of generic rank equal to $2m + d$ over $\Delta_{mk}(\delta_k)$ for all $k \geq \nu_0 + 1$. Finally, with this integer ν_0, the odd integer $\mu_0 := 2\nu_0 + 1$, called the* Segre type *$\mathcal{M}$ at 0 is the smallest integer such that the mappings Γ_k and $\underline{\Gamma}_k$ are of rank equal to $2m + d$ at the origin in $\Delta_{mk}(\delta_k)$.*

Let $\mu_0 := 2\nu_0 + 1$ be the Segre type of $\mathcal{M}$ at 0 (notice that this is always odd). In the remainder of this section, we assume that M is minimal at 0 and we exploit the rank condition on Γ_k. More precisely we choose a positive η with $0 < \eta \leq \delta_{\mu_0}$ such that Γ_{μ_0} has rank $2m + d$ at every point of the polydisc $\Delta_{m\mu_0}(\eta)$. Without loss of generality, we can also assume that $\Gamma_{\mu_0}(\Delta_{m\mu_0}(\eta))$ contains $\mathcal{M} \cap (\Delta_n(\rho_4) \times \Delta_n(\rho_4))$. Simple examples in the hypersurface case show that $\rho_4 << \rho_1$ and in fact, one has necessarily an inequality of the form $\rho_4 \leq (\rho_1)^N$, where N is a certain integer depending on the vector fields $(\mathcal{L}_k)_{1\leq k\leq m}$ and $(\underline{\mathcal{L}}_k)_{1\leq k\leq m}$ (*cf.* [Be1996]).

5.3. Finite nondegeneracy. The last ingredient for Theorem 4.1 consists in developing the equations of M in powers of $\bar{z}$ as follows

$$\bar{w}_j = \sum_{\beta \in \mathbb{N}^m} (\bar{z})^\beta \, \Theta_{j,\beta}(t), \qquad j = 1, \dots, d, \tag{5.6}$$

where the functions $\Theta_{j,\beta}(t)$ are holomorphic in the polydisc $\Delta_n(2\rho_1)$. So we may introduce the holomorphic maps $\psi_k(t) := (\Theta_{j,\beta}(t))_{1\leq j\leq d,\, |\beta|\leq k}$ with values in $\mathbb{C}^{d\frac{(m+k)!}{m!\;k!}}$. Obviously, the ranks at the origin of the ψ_k increase with k.

Definition 5.3. The generic manifold M is said to be *finitely nondegenerate* at 0 if there exists a positive integer k such that the rank at the origin of the map ψ_k is equal to n.

It may be checked that this definition depends neither on the system of coordinates nor on the choice of a collection of d defining equations for M and that it coincides with the definition given in §1.2. If M is finitely nondegenerate at 0 we denote by ℓ_0 the smallest integer k given by definition 5.3 and we say that M is ℓ_0-nondegenerate at the origin.

Finite nondegeneracy is interesting for the following reason. In the sequel, we shall have to consider an infinite collection of equations of the form

$$\Theta_{j,\beta}(t) + \sum_{\gamma\in\mathbb{N}_*^m} (\zeta)^\gamma \, \frac{(\beta+\gamma)!}{\beta! \; \gamma!} \, \Theta_{j,\beta+\gamma}(t) = \omega_{j,\beta}, \tag{5.7}$$

where $\mathbb{N}_*^m := \mathbb{N}^m \backslash \{0\}$, where j runs from 1 to d, where β runs in $\mathbb{N}^m$ and where the right hand sides $\omega_{j,\beta}$ are independent complex variables. For $\beta = 0$, the equations (5.7) write simply $\Theta_j(\zeta,t) = \omega_{j,0}$. By definition, if M is ℓ_0-nondegenerate at 0, there exists n integers $j_*^1, \dots, j_*^n$ with $1 \leq j_*^i \leq d$ and n multi-indices $\beta_*^1, \dots, \beta_*^n \in \mathbb{N}^m$ with $|\beta_*^i| \leq \ell_0$ such that the local holomorphic self-mapping $t \mapsto (\Theta_{j_*^k, \beta_*^k}(t))_{1\leq k\leq n}$ of $\mathbb{C}^n$ is of rank n at the origin. Considering the equations (5.7) for $j = j_*^1, \dots, j_*^n$ and $\beta = \beta_*^1, \dots, \beta_*^n$ and applying the implicit function theorem, we observe that we can solve t in terms of $(\zeta, \omega_{j_*^1,\beta_*^1}, \dots, \omega_{j_*^n,\beta_*^n})$ by means of a holomorphic mapping, namely

$$t = \Psi(\tau, \omega_{j_*^1,\beta_*^1}, \dots, \omega_{j_*^n,\beta_*^n}). \tag{5.8}$$

Without loss of generality, we may assume that Ψ is holomorphic for $|\zeta| < \widetilde{\rho}_3$ and $|\omega_{j_*^i,\beta_*^i}| < \widetilde{\rho}_3$, where $0 < \widetilde{\rho}_3 < \rho_2 < \rho_1$.

§6. Algebraicity of local CR automorphism groups

6.1. Fundamental reflection identity for the mapping. So M is ℓ_0-nondegenerate at the origin. Recall that $\mu_0 = 2\nu_0 + 1$ is the Segre type of $\mathcal{M}$ and introduce the new integer $\kappa_0 := \ell_0(\mu_0 + 1)$. Notice that κ_0 is even. Let us take an arbitrary local holomorphic self map h of M close to the identity in the set $\mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$, *i.e.* with $k := \kappa_0$ in the definition (4.4). We denote the map h by $(h_1, \dots, h_n) = (f_1, \dots, f_m, g_1, \dots, g_d)$, according to the splitting $t = (z,w)$ of the coordinates. The complexification $h^c := (h, \bar{h})$ induces a local holomorphic self map of the complexification $\mathcal{M}$. More precisely, for all $(t,\tau) \in \mathcal{M}$ with $|t|$, $|\tau| \leq \rho_2$, we have $(h(t), \bar{h}(\tau)) \in \mathcal{M}$ and $|h(t)|$, $|h(\tau)| < \rho_1$, so we can write

$$\bar{g}_j(\tau) = \Theta_j(\bar{f}(\tau), h(t)), \tag{6.1}$$

for $j = 1, \dots, d$. Since h is a biholomorphism and $T_0^c M = \{w = 0\}$, it follows that the determinant

$$\det\left(\underline{\mathcal{L}}_k \bar{f}_l(\tau)\right)_{1\leq k,l\leq n}, \tag{6.2}$$

which is a $\mathbb{K}$-analytic function of $(t,\tau) \in \mathcal{M}$, does not vanish at the origin. Shrinking ε if necessary, we can assume that for every holomorphic map $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$, the determinant (6.2) does not vanish for all $|t|$, $|\tau| < \rho_2$. We now differentiate (6.2) by applying the vector fields $\underline{\mathcal{L}}_1, \dots, \underline{\mathcal{L}}_m$, which gives

$$\underline{\mathcal{L}}_k \bar{g}_j(\tau) = \sum_{l=1}^m \frac{\partial \Theta_j}{\partial \zeta_l}(\bar{f}(\tau), h(t)) \, \underline{\mathcal{L}}_k \bar{f}_l(\tau), \tag{6.3}$$

for $k = 1, \dots, m$ and $j = 1, \dots, d$. For fixed j, we consider the m equations (6.3) as an affine system satisfied by the partial derivatives $\partial\Theta_j/\partial\zeta_l$. By Cramer's rule, there exists universal polynomials $\Omega_{j,k}$ in their variables such that

$$\frac{\partial \Theta_j}{\partial \zeta_k}(\bar{f}(\tau), h(t)) = \frac{\Omega_{j,k}(\{\underline{\mathcal{L}}_l \, \bar{h}(\tau)\}_{1\leq l\leq m})}{\det\left(\underline{\mathcal{L}}_k \bar{f}_l(\tau)\right)_{1\leq k,l\leq n}} \tag{6.4}$$

for all $(t,\tau) \in \mathcal{M}$ with $|t|$, $|\tau| < \rho_2$ and for $k = 1, \dots, m$, $j = 1, \dots, d$.

Applying the derivations $\underline{\mathcal{L}}_k$ to (6.4) we see by induction that for every multi-index $\beta \in \mathbb{N}_*^m$ and for every $j = 1, \ldots, d$, there exists a universal polynomial $\Omega_{j,\beta}$ in its variables such that

$$\frac{1}{\beta!}\frac{\partial^{|\beta|}\Theta_j}{\partial\zeta^\beta}(\bar f(\tau), h(t)) = \frac{\Omega_{j,\beta}(\{\underline{\mathcal{L}}^\gamma\,\bar h(\tau)\}_{|\gamma|\le|\beta|})}{[\det(\underline{\mathcal{L}}_k \bar f_l(\tau))_{1\le k,l\le n}]^{2|\beta|-1}}, \tag{6.5}$$

for all $(t,\tau) \in \mathcal{M}$ with $|t|, |\tau| < \rho_2$. Here, for $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$, we denote by $\underline{\mathcal{L}}^\gamma$ the derivation $(\underline{\mathcal{L}}_1)^{\gamma_1} \ldots (\underline{\mathcal{L}}_m)^{\gamma_m}$. Next, denoting by $\omega_{j,\beta}(t,\tau)$ the right hand side of (6.5) and developing the left hand side in power series using (5.7), we may write

$$\Theta_{j,\beta}(h(t)) + \sum_{\gamma\in\mathbb{N}_*^m} (\bar f(\tau))^\gamma\,\Theta_{j,\beta+\gamma}(h(t)) = \omega_{j,\beta}(t,\tau). \tag{6.6}$$

Recall that M is ℓ_0-nondegenerate at 0. Using (5.8), we can solve $h(t)$ in terms of the derivatives of $\bar h(\tau)$, namely

$$\begin{cases} h(t) = \Psi\left(\bar f(\tau), \dfrac{\Omega_{j_*^1,\beta_*^1}(\{\underline{\mathcal{L}}^\gamma\,\bar h(\tau)\}_{|\gamma|\le|\beta_*^1|})}{[\det(\underline{\mathcal{L}}_k \bar f_l(\tau))_{1\le k,l\le n}]^{2|\beta_*^1|-1}}, \ldots \right. \\ \qquad\qquad \left. \ldots, \dfrac{\Omega_{j_*^n,\beta_*^n}(\{\underline{\mathcal{L}}^\gamma\,\bar h(\tau)\}_{|\gamma|\le|\beta_*^n|})}{[\det(\underline{\mathcal{L}}_k \bar f_l(\tau))_{1\le k,l\le n}]^{2|\beta_*^n|-1}}\right) = \\ \quad = \Psi(\bar f(\tau), \omega_{j_*^1,\beta_*^1}(t,\tau), \ldots, \omega_{j_*^n,\beta_*^n}(t,\tau)). \end{cases} \tag{6.7}$$

Here, the maximal length of the multi-indices $\beta_*^1, \ldots, \beta_*^n$ is equal to ℓ_0. According to (5.8), the representation (6.7) of $h(t)$ holds provided $|\bar g(\tau)| < \widetilde\rho_3$ and $|\omega_{j_*^i,\beta_*^i}| < \widetilde\rho_3$. Since the coordinates are normal, we have $\Theta_j(\bar z, 0, 0) \equiv 0$, or equivalently $\Theta_{j,\beta}(0) = 0$ for all $j = 1, \ldots, d$ and all $\beta \in \mathbb{N}^m$. It follows from (6.6) and from $h(0) = 0$ that $\omega_{j,\beta}(0) = 0$, for all $j = 1, \ldots, d$ and all $\beta \in \mathbb{N}^m$. Consequently, there exists a radius $\rho_3 \sim \widetilde\rho_3$ with $0 < \rho_3 < \rho_2 < \rho_1$ such that $|\omega_{j_*^i,\beta_*^i}(t,\tau)| < \widetilde\rho_3$, $i = 1, \ldots, n$ and such that $|\bar g(\tau)| < \widetilde\rho_3$ for all $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$ and for all $(t,\tau) \in \mathcal{M}$ with $|t|, |\tau| < \rho_3$.

In conclusion, the relation (6.7) holds for all $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$ and for all $(t,\tau) \in \mathcal{M}$ with $|t|, |\tau| < \rho_3$.

Next, using the explicit expressions of the vector fields $\underline{\mathcal{L}}_k$ given in (5.2), we may develop the higher order derivatives $\underline{\mathcal{L}}^\gamma \bar h(\tau)$ as polynomials in the $|\gamma|$-jet $(\partial_\tau^{\gamma'}\bar h(\tau))_{|\gamma'|\le|\gamma|}$ of $\bar h(\tau)$ with coefficients being certain holomorphic functions of (t,τ) obtained as certain polynomials with respect to the partial derivatives of the functions $\Theta_j(\zeta,t)$.

To be more explicit in this desired new representation of (6.7), we remind first our jet notation. For each $i = 1, \ldots, n$ and each $\alpha \in \mathbb{N}^n$, we introduced a new *independent* coordinate J_i^α corresponding to the partial derivative $\partial_\tau^\alpha \bar h_i(\tau)$ (or $\partial_t^\alpha h_i(t)$). The space of k-jets of holomorphic mappings $\bar h(\tau)$ is then the complex space $\mathbb{C}^{n\frac{(n+k)!}{n!\;k!}}$ with coordinates $(J_i^\alpha)_{1\le i\le n,\,|\alpha|\le k}$. It will be convenient to use the abbreviations $J^k := (J_i^\alpha)_{1\le i\le n,\,|\alpha|\le k}$ and $J^k\bar h(\tau) := (\partial_\tau^\alpha \bar h_i(\tau))_{1\le i\le n,\,|\alpha|\le k}$.

So pursuing with (6.7), we argue that for every $\gamma \in \mathbb{N}^m$, there exists a polynomial in the jet $J^{|\gamma|}\bar h(\tau)$ with holomorphic cooeficients depending only on Θ such that

$$\underline{\mathcal{L}}^\gamma\,\bar h(\tau) \equiv P_\gamma(t,\tau,J^{|\gamma|}\bar h(\tau)). \tag{6.8}$$

Putting all these expressions in (6.7), we obtain an important relation between h and the ℓ_0-jet of $\bar h$ which we may now summarize. At first, as $\kappa_0 = \ell_0(\mu_0+1) \ge \ell_0$, observe that for every $h \in \mathcal{H}_{M,\kappa_0,\varepsilon}^{\rho_2,\rho_1}$, we have $||J^{\ell_0}h - J_{\mathrm{Id}}^{\ell_0}||_{\rho_2} \le ||J^{\kappa_0}h - J_{\mathrm{Id}}^{\kappa_0}||_{\rho_2} \le \varepsilon$. Shrinking ε if necessary, we have proved the following lemma.

Lemma 6.1. *There exists a complex algebraic $\mathbb{C}^n$-valued mapping $\Pi(t,\tau,J^{\ell_0})$ defined for $|t|, |\tau| < \rho_3$ and for $|J^{\ell_0} - J_{\mathrm{Id}}^{\ell_0}| < \varepsilon$ which depends only on the defining functions*

$\xi_j - \Theta_j(\zeta, t)$ of $\mathcal{M}$, such that for every local holomorphic self-mapping $h \in \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$ of M (hence satisfying $||J^{\ell_0}h - J^{\ell_0}_{\mathrm{Id}}||_{\rho_2} < \varepsilon$), the relation

$$h(t) = \Pi(t, \tau, J^{\ell_0}\bar{h}(\tau)) \tag{6.9}$$

holds for all $(t, \tau) \in \mathcal{M}$ with $|t|$, $|\tau| < \rho_3$.

6.2. Reflection identity for arbitrary jets. Let now Υ_j and $\underline{\Upsilon}_j$ be the vector fields tangent to $\mathcal{M}$ defined by

$$\Upsilon_j := \frac{\partial}{\partial w_j} + \sum_{l=1}^{d} \Theta_{l,w_j}(\zeta, t)\,\frac{\partial}{\partial \xi_l}, \qquad \underline{\Upsilon}_j := \frac{\partial}{\partial \xi_j} + \sum_{l=1}^{d} \overline{\Theta}_{l,\xi_j}(z, \tau)\,\frac{\partial}{\partial w_l}, \tag{6.10}$$

for $j = 1, \dots, d$. We observe that the collection of $2m + d$ vector fields $\mathcal{L}_k$, $\underline{\mathcal{L}_k}$, Υ_j span $T\mathcal{M}$. The same holds for the collection $\mathcal{L}_k$, $\underline{\mathcal{L}_k}$, $\underline{\Upsilon}_j$. We also have the commutation relations $[\Upsilon_j, \underline{\mathcal{L}_k}] = 0$ and $[\underline{\Upsilon}_j, \mathcal{L}_k] = 0$. We observe that $\Upsilon^\gamma h(t) = \partial^\gamma_w h(t)$ for all $\gamma \in \mathbb{N}^d$. Let $\alpha = (\beta, \gamma) \in \mathbb{N}^m \times \mathbb{N}^d$. By expanding $\mathcal{L}^\beta\, \Upsilon^\gamma\, h(t)$ using the explicit expressions (5.2), we obtain a polynomial $Q_{\beta,\gamma}(t, \tau, (\partial^{\alpha'}_t h(t))_{|\alpha'| \leq |\alpha|})$, where $Q_{\beta,\gamma}$ is a polynomial in its last variables with coefficients depending on $\overline{\Theta}$ and its partial derivatives. Conversely, since $\mathcal{L}_k|_0 = \partial_{z_k}$ at the origin, we can invert these formulas, so there exist polynomials P_α in their last variables with coefficients depending only on $\overline{\Theta}$ such that

$$\partial^\alpha_t h(t) = P_\alpha(t, \tau, (\mathcal{L}^{\beta'}\, \Upsilon^{\gamma'}\, h(t))_{|\beta'| \leq |\beta|,\, |\gamma'| \leq |\gamma|}). \tag{6.11}$$

Lemma 6.2. *For every $\ell \in \mathbb{N}$, there exists a complex algebraic mapping Π_ℓ with values in $\mathbb{C}^{N_{n,\ell}}$ defined for $|t|$, $|\tau| < \rho_3$ and $|J^{\ell_0} - J^{\ell_0}_{\mathrm{Id}}| < \varepsilon$ which is relatively polynomial with respect to the higher order jets J^α_i with $|\alpha| \geq \ell_0 + 1$, $i = 1, \dots, n$, such that for every local holomorphic self-mapping $h \in \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$, the two conjugate relations*

$$\begin{cases} J^\ell h(t) = \Pi_\ell(t, \tau, J^{\ell_0+\ell}\bar{h}(\tau)), \\ J^\ell \bar{h}(\tau) = \overline{\Pi}_\ell(\tau, t, J^{\ell_0+\ell}h(t)). \end{cases} \tag{6.12}$$

hold for all $(t, \tau) \in \mathcal{M}$ with $|t|$, $|\tau| < \rho_3$.

Proof. Applying the derivations $\mathcal{L}^\beta\, \Upsilon^\gamma$ to (6.9), and using the chain rule, we obtain

$$\mathcal{L}^\beta\, \Upsilon^\gamma\, h(t) = \Pi_{\beta,\gamma}(t, \tau, J^{\ell_0+|\beta|+|\gamma|}\bar{h}(\tau)), \tag{6.13}$$

where the function $\Pi_{\beta,\gamma}$ (as the function Π) is holomorphic for $|t|, |\tau| < \rho_3$ and $|J^{\ell_0} - J^{\ell_0}_{\mathrm{Id}}| < \varepsilon$ and relatively polynomial with respect to the jets J^α_i with $|\alpha| \geq \ell_0 + 1$. Applying (6.11), we obtain the function Π_ℓ, which completes the proof. □

6.3. Substitutions of reflection identities. Let $\pi_t(t, \tau) := t$ and $\pi_\tau(t, \tau) := \tau$ denote the two canonical projections. We write $h^c(t, \tau) := (h(t), \bar{h}(\tau))$. We make the following slight abuse of notation: instead of rigorously writing $h(\pi_t(t, \tau))$, we write $h(t, \tau) = h(t)$ and $\bar{h}(t, \tau) = \bar{h}(\tau)$.

Let $x \in \mathbb{C}^\nu$ and let $\mathcal{Q}(x) = (\mathcal{Q}_1(x), \dots, \mathcal{Q}_{2n}(x)) \in \mathbb{C}\{x\}^{2n}$. As the multiple flow of $\underline{\mathcal{L}}$ given by (5.3) does not act on the (z, w) variables, we have the trivial but important property $h(\underline{\mathcal{L}}_{z_1}(\mathcal{Q}(x))) = h(\mathcal{Q}(x))$. More generally, for every multi-index $\alpha \in \mathbb{N}^n$, we have $\partial^\alpha_t h(\underline{\mathcal{L}}_{z_1}(\mathcal{Q}(x))) = \partial^\alpha_t h(\mathcal{Q}(x))$. Analogously, we have $\partial^\alpha_\tau \bar{h}(\mathcal{L}_{z_1}(\mathcal{Q}(x))) = \partial^\alpha_\tau \bar{h}(\mathcal{Q}(x))$. Since for k even, we have $\Gamma_k(z_{(k)}) = \underline{\mathcal{L}}_{z_k}(\Gamma_{k-1}(z_{(k-1)}))$, the following two properties hold:

$$\begin{cases} J^\ell h(\Gamma_k(z_{(k)})) = J^\ell h(\Gamma_{k-1}(z_{(k-1)})), & \text{if } k \text{ is even;} \\ J^\ell \bar{h}(\Gamma_k(z_{(k)})) = J^\ell \bar{h}(\Gamma_{k-1}(z_{(k-1)})), & \text{if } k \text{ is odd.} \end{cases} \tag{6.14}$$

Let now $\kappa_0 := \ell_0(\mu_0+1)$ be the product of the Levi type with the Segre type of $\mathcal{M}$ plus 1 and consider the open subset of the κ_0-order jet space $\mathbb{C}^{N_{n,\kappa_0}}$ defined by the inequality $|J^{\kappa_0} - J^{\kappa_0}_{\mathrm{Id}}| < \varepsilon$. Let $z_{(k)} \in \Delta_{mk}$ as in §5.6 above. Since the maps Γ_k are holomorphic and satisfy $\Gamma_k(0) = 0$, we may choose $\delta > 0$ sufficiently small in order that the following two conditions are satisfied for every $k \leq \mu_0$ and for and for every $|z_{(k)}| < \delta$:

$$|\Gamma_k(z_{(k)})| < \rho_3 \qquad \text{and} \qquad |J^{\kappa_0} h(\Gamma_k(z_{(k)})) - J^{\kappa_0}_{\mathrm{Id}}| < \varepsilon. \tag{6.15}$$

This choice of δ is convenient to make several susbtitutions by means of formulas (6.12). The formulas (6.16) that we will obtain below strongly differ from the previous formulas (6.12), because they depend on the jet of h at the origin only.

Lemma 6.3. *Shrinking ε if necessary, for every integer $k \leq \mu_0 + 1$ and for every integer $\ell \geq 0$, there exists a complex algebraic mapping $\Pi_{\ell,k}$ with values in $\mathbb{C}^{N_{n,\ell}}$ defined for $|t|$, $|\tau| < \rho_3$ and for $|J^{k\ell_0} - J^{k\ell_0}_{\mathrm{Id}}| < \varepsilon$, which is relatively polynomial with respect to the higher order jets J^α_i with $|\alpha| \geq k\ell_0 + 1$, $i = 1, \ldots, n$, and which depends only on the defining functions $\xi_j - \Theta_j(\zeta, t)$ of $\mathcal{M}$, such that the following two families of conjugate identities are satisfied*

$$\begin{cases} J^\ell h(\Gamma_k(z_{(k)})) = \Pi_{\ell,k}(\Gamma_k(z_{(k)}), J^{k\ell_0+\ell}\bar{h}(0)), & \text{if } k \text{ is odd;} \\ J^\ell \bar{h}(\Gamma_k(z_{(k)})) = \overline{\Pi_{\ell,k}}(\Gamma_k(z_{(k)}), J^{k\ell_0+\ell}\bar{h}(0)), & \text{if } k \text{ is even.} \end{cases} \tag{6.16}$$

Proof. For $k = 1$, replacing (t, τ) by $\Gamma_1(z_{(1)})$ in the first relation (6.12) and using the second property (6.14), we get

$$\begin{cases} J^\ell h(\Gamma_1(z_{(1)})) = \Pi_\ell(\Gamma_1(z_{(1)}), J^{\ell_0+\ell}\bar{h}(\Gamma_1(z_{(1)}))) = \\ \qquad\qquad = \Pi_\ell(\Gamma_1(z_{(1)}), J^{\ell_0+\ell}\bar{h}(0)), \end{cases} \tag{6.17}$$

so the lemma holds true for $k = 1$ if we simply choose $\Pi_{\ell,1} := \Pi_\ell$. By induction, suppose that the lemma holds true for $k \leq \mu_0$. To fix the ideas, let us assume that this k is even (the odd case is completely similar). Then replacing the arguments (t, τ) in the first relation (6.12) by $\Gamma_{k+1}(z_{(k+1)})$, using again the second property (6.14), and using the induction assumption, namely using the conjugate of the second relation (6.16) with ℓ replaced by $\ell_0 + \ell$, we get

$$\begin{cases} J^\ell h(\Gamma_{k+1}(z_{(k+1)})) &= \Pi_\ell(\Gamma_{k+1}(z_{(k+1)}), J^{\ell_0+\ell}\bar{h}(\Gamma_{k+1}(z_{(k+1)}))) = \\ &= \Pi_\ell(\Gamma_{k+1}(z_{(k+1)}), J^{\ell_0+\ell}\bar{h}(\Gamma_k(z_{(k)}))) = \\ &= \Pi_\ell(\Gamma_{k+1}(z_{(k+1)}), \overline{\Pi_{\ell_0+\ell,k}}(\Gamma_k(z_{(k)}), J^{k\ell_0+\ell_0+\ell}\bar{h}(0))) =: \\ &=: \Pi_{\ell,k+1}(\Gamma_{k+1}(z_{(k+1)}), J^{(k+1)\ell_0+\ell}\bar{h}(0)), \end{cases} \tag{6.18}$$

which yields the desired formula at level $k+1$. For the above formal composition formulas to be correct, we possibly have to shrink ε. Finally, a direct inspection of relative polynomialness shows that $\Pi_{\ell,k+1}$ is polynomial with respect to the jet variables J^α_i with $|\alpha| \geq (k+1)\ell_0 + 1$, $i = 1, \ldots, n$. The proof of Lemma 6.21 is complete. □

6.4. Algebraic parameterization of CR mappings by their jet at the origin. Finally, as in the paragraph after Theorem 5.2, we choose $\rho_4 > 0$ sufficiently small such that Γ_{μ_0} maps the polydisc $\Delta_{m\mu_0}(\eta)$ submersively onto an open neighborhood of the origin in $\mathcal{M}$ which contains the open subset $\mathcal{M} \cap (\Delta_n(\rho_4) \times \Delta_n(\rho_4))$. From the relation $\Gamma_{\mu_0+1}(z_{(\mu_0)}, 0) \equiv \Gamma_{\mu_0}(z_{(\mu_0)})$, it follows trivially that Γ_{μ_0+1} also induces a submersion from $\Delta_{m(\mu_0+1)}(\eta)$ onto $\mathcal{M} \cap (\Delta_n(\rho_4) \times \Delta_n(\rho_4))$. It follows that the composition $\pi_t \circ \Gamma_{\mu_0+1}$ also maps submersively the polydisc $\Delta_{m(\mu_0+1)}(\eta)$ onto an open neighborhood of the origin in $\mathbb{C}^n$ which contains $\Delta_n(\rho_4)$. Consequently, in the representation obtained in Lemma 6.21 with $\ell = 0$ and $k := \mu_0 + 1 = 2\nu_0 + 2$ (which is even), namely in the representation

$$\bar{h}(\Gamma_{\mu_0+1}(z_{(\mu_0+1)})) = \overline{\Pi_{0,\mu_0+1}}(\Gamma_{\mu_0+1}(z_{(\mu_0+1)}), J^{(\mu_0+1)\ell_0}\bar{h}(0)), \tag{6.19}$$

we can write an arbitrary $t \in \Delta_n(\rho_4)$ in the form $\Gamma_{\mu_0+1}(z_{(\mu_0+1)})$, and finally, conjugating (6.19), we obtain a complex algebraic mapping H with the property that $h(t) = H(t, J^{(\mu_0+1)\ell_0} h(0))$. We may now summarize what we have proved so far.

Theorem 6.4. *Let M be a real algebraic generic submanifold in $\mathbb{C}^n$ passing through the origin, of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$. Assume that M is ℓ_0-nondegenerate at 0. Assume that M is minimal at 0, let ν_0 be the Segre type of M at 0 and let $\mu_0 := 2\nu_0 + 1$ be the Segre type of $\mathcal{M}$ at 0. Let $\kappa_0 := (\mu_0 + 1)\ell_0$. Let $t = (z, w) \in \mathbb{C}^m \times \mathbb{C}^d$ be holomorphic coordinates vanishing at 0 with $T_0M = \{\operatorname{Im} w = 0\}$ and let $\rho_1 > 0$ be such that M is represented by the complex analytic defining equations $\xi_j = \Theta_j(\zeta, t)$, $j = 1, \dots, d$ in $\Delta_n(\rho_1)$. Then there exist $\varepsilon > 0$, $\rho_4 > 0$ and there exists a complex algebraic $\mathbb{C}^n$-valued mapping $H(t, J^{\kappa_0})$ defined for $|t| < \rho_4$ and for $|J^{\kappa_0} - J^{\kappa_0}_{\rm Id}| < \varepsilon$ which satisfies $H(t, J^{\kappa_0}_{\rm Id}) \equiv t$ and which depends only on the defining functions $\bar{w}_j - \Theta_j(\bar{z}, t)$ of $\mathcal{M}$, such that for every local holomorphic self-mapping h of M belonging to $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$, we have the representation formula*

$$h(t) = H(t, J^{\kappa_0} h(0)), \tag{6.20}$$

for all $t \in \mathbb{C}^n$ with $|t| < \rho_4$. Furthermore the mapping H depends neither on the choice of smaller radii $\widetilde{\rho}_1 \leq \rho_1$, $\widetilde{\rho}_2 \leq \rho_2$, $\widetilde{\rho}_3 \leq \rho_3$ and $\widetilde{\rho}_4 \leq \rho_4$ satisfying $0 < \widetilde{\rho}_4 < \widetilde{\rho}_3 < \widetilde{\rho}_2 < \widetilde{\rho}_1$ nor on the choice of a smaller constant $\widetilde{\varepsilon} < \varepsilon$, so that the first sentence of property **(3)** *in Theorem 4.1 holds true. Finally, if M is real analytic, the same statement holds with the word "algebraic" everywhere replaced by the word "analytic".*

It remains now to construct the submanifold E whose existence is stated in Theorem 4.1 and to establish that $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$ may be endowed with the structure of a local real algebraic Lie group.

6.5. Local real algebraic Lie group structure. In order to construct this submanifold E, we introduce the κ_0-th jet mapping $\mathcal{J}^{\kappa_0} : \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon} \to \mathbb{C}^{N_{n,\kappa_0}}$ defined by $\mathcal{J}^{\kappa_0}(h) := (\partial_t^\alpha h(0))_{|\alpha| \leq \kappa_0} = J^{\kappa_0} h(0)$. The following lemma is crucial.

Lemma 6.5. *Shrinking ε if necessary, the set*

$$E := \mathcal{J}^{\kappa_0}(\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}) = \{J^{\kappa_0} h(0) : h \in \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}\} \tag{6.21}$$

is a real algebraic totally real submanifold of the polydisc $\{J^{\kappa_0} \in \mathbb{C}^{N_{n,\kappa_0}} : |J^{\kappa_0} - J^{\kappa_0}_{\rm Id}| < \varepsilon\}$.

Proof. Let $h \in \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$. Substituting the representation formula $h(t) = H(t, J^{\kappa_0} h(0))$ given by Theorem 6.4 in the defining equations of M, we get

$$r_j(H(t, J^{\kappa_0} h(0)), \overline{H}(\tau, J^{\kappa_0} \bar{h}(0))) = 0, \tag{6.22}$$

for $j = 1, \dots, d$ and $(t, \tau) \in \mathcal{M}$ with $|t|$, $|\tau| < \rho_4$. As $(t, \tau) \in \mathcal{M}$, we replace ξ by $\Theta(\zeta, t)$ and we use the $2m + d$ coordinates (t, ζ) on $\mathcal{M}$. So, by expanding the functions (6.22) in power series with respect to (t, ζ), we can write

$$r_j(H(t, J^{\kappa_0}), \overline{H}(\zeta, \Theta(\zeta, t), \overline{J^{\kappa_0}})) = \sum_{\alpha \in \mathbb{N}^n,\, \beta \in \mathbb{N}^m} t^\alpha \, \zeta^\beta \, C_{j,\alpha,\beta}(J^{\kappa_0}, \overline{J^{\kappa_0}}). \tag{6.23}$$

Here, we obtain an infinite collection of complex-valued real algebraic functions $C_{j,\alpha,\beta}$ defined in $\{|J^{\kappa_0} - J^{\kappa_0}_{\rm Id}| < \varepsilon\}$ with the property that a mapping $H(t, J^{\kappa_0})$ sends $M \cap \Delta_n(\rho_4)$ into M if and only if

$$C_{j,\alpha,\beta}(J^{\kappa_0}, \overline{J^{\kappa_0}}) = 0, \qquad \forall\ j,\ \alpha,\ \beta. \tag{6.24}$$

Consequently, the set E defined by the vanishing of all the equations (6.24) is a real algebraic subset.

It follows from the representation formula (6.20) that the mapping $\mathcal{J}^{\kappa_0}$ is injective and from the Cauchy integral formula that $\mathcal{J}^{\kappa_0}$ is continuous on its domain of definition $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$ endowed with the topology of uniform convergence on compact sets.

On the reverse side, let $J^{\kappa_0} \in E$. Then the mapping $h(t) := H(t, J^{\kappa_0})$ defined for $|t| < \rho_4$ maps $M \cap \Delta_n(\rho_4)$ into M. Applying Theorem 6.4 to this mapping $h(t)$, with ρ_1 replaced by ρ_4, we deduce that there exists a radius $\rho_6 < \rho_4$ such that we can represent $h(t) = H(t, J^{\kappa_0} h(0))$ for $|t| < \rho_6$, with the same mapping H, as stated in the end of Theorem 6.4. By differentiating this representation with respect to t at $t = 0$, we deduce that $J^{\kappa_0}h(0) = ([\partial_t^\alpha H(t, J^{\kappa_0}h(0))]_{t=0})_{|\alpha|\leq\kappa_0}$. Consequently, since $h(t) = H(t, J^{\kappa_0})$ by definition, we get $J^{\kappa_0} = ([\partial_t^\alpha H(t, J^{\kappa_0})]_{t=0})_{|\alpha|\leq\kappa_0}$. In conclusion, we proved that $\mathcal{J}^{\kappa_0}(H(t, J^{\kappa_0})) = J^{\kappa_0}$ for every $J^{\kappa_0} \in E$, so $\mathcal{J}^{\kappa_0}$ has a continuous local inverse on E, formally defined by $H(t, J^{\kappa_0})$.

It follows from the above two paragraphs that the mapping $\mathcal{J}^{\kappa_0}$ is a local homeomorphism from a neighborhood of the identity in $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$ onto its image E.

Furthermore, we claim that the real algebraic subset E is in fact geometrically smooth at every point, namely it is a real algebraic submanifold. Indeed, let $J_1^{\kappa_0}$ be a regular point of E where E is of maximal geometrical dimension c_0, with $J_1^{\kappa_0}$ arbitrarily close to the identity jet $J_{\rm Id}^{\kappa_0}$. Let $h_1 \in \mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon}$ such that $J_1^{\kappa_0} = \mathcal{J}^{\kappa_0}(h_1)$. Let $\mathcal{U}_1$ be a small neighborhood of $J_1^{\kappa_0}$ in $\mathbb{C}^{N_{n,\kappa_0}}$ in which $E \cap \mathcal{U}_1$ is a regular c_0-dimensional real algebraic submanifold and consider the complex algebraic mapping defined over $\mathcal{U}_1$ by

$$\mathcal{F}_1(J^{\kappa_0}) := ([\partial_t^\alpha\, (h_1^{-1}(H(t, J^{\kappa_0})))]_{t=0})_{|\alpha|\leq\kappa_0} \in \mathbb{C}^{N_{n,\kappa_0}}. \tag{6.25}$$

We have $\mathcal{F}_1(J_1^{\kappa_0}) = J_{\rm Id}^{\kappa_0}$ and the restriction of $\mathcal{F}_1$ to $E \cap \mathcal{U}_1$ induces a homeomorphism onto its image, which is a neighborhood of $J_{\rm Id}^{\kappa_0}$ in E. We remind that the mapping $J^{\kappa_0} \to ([\partial_t^\alpha\, (H(t, J^{\kappa_0}))]_{t=0})_{|\alpha|\leq\kappa_0}$ restricted to $E \cap \mathcal{U}_1$ is the identity and consequently of constant rank equal to c_0. As h_1 is invertible, it follows from the chain rule by developing (6.25) that $\mathcal{F}_1|_{E\cap\mathcal{U}_1}$ is also of locally constant rank equal to c_0. This proves that E is a c_0-dimensional real algebraic submanifold in $\mathbb{C}^{N_{n,\kappa_0}}$ through $J_{\rm Id}^{\kappa_0}$. More generally, this reasoning shows that E is geometrically smooth at every point.

Finally, applying Lemma 6.3 with the odd integer $k = \mu_0 = 2\nu_0 + 1$ (instead of $k = \mu_0 + 1$), we get a new, different representation formula $h(t) = \widetilde{H}(t, J^{\ell_0\mu_0}\bar{h}(0))$ (notice $\bar{h}(0)$). Accordingly, we can define a real algebraic submanifold $\widetilde{E}$. It is clear that we can identify E and $\widetilde{E}$, since they both parametrize the local biholomorphic self-mappings of M, so they are algebraically equivalent by means of the natural projection from the $\ell_0(\mu_0+1)$-th jet space onto the $\ell_0\mu_0$-th jet space. Next, we see by differentiating $h(t) = \widetilde{H}(t, J^{\ell_0\mu_0}\bar{h}(0))$ with respect to t that

$$J^{\ell_0\mu_0}h(0) = ([\partial_t^\alpha \widetilde{H}(t, J^{\ell_0\mu_0}\bar{h}(0))]_{t=0})_{|\alpha|\leq\ell_0\mu_0}. \tag{6.26}$$

Consequently, if K is the holomorphic map defined by

$$K(J^{\ell_0\mu_0}) := ([\partial_t^\alpha \widetilde{H}(t, J^{\ell_0\mu_0}]_{t=0})_{|\alpha|\leq\ell_0\mu_0}), \tag{6.27}$$

we get the equality $J^{\ell_0\mu_0} = K(\overline{J^{\ell_0\mu_0}})$ for every $J^{\ell_0\mu_0} \in \widetilde{E}$, which proves that $\widetilde{E}$ is totally real. It follows that E is totally real, which completes the proof. □

Lemma 6.6. *The submanifold E is naturally equipped with a local real algebraic Lie group structure in a neighborhood of $J_{\rm Id}^{\kappa_0}$.*

Proof. Indeed, let us parametrize E by a real algebraic mapping

$$\mathbb{R}^{c_0} \ni (e_1, \ldots, e_{c_0}) \longmapsto j_{\kappa_0}(e) \in \mathbb{C}^{N_{n,\kappa_0}}, \tag{6.28}$$

where c_0 is the dimension of E. Here, to avoid excessive formal complexity, we shall avoid to mention all the polydiscs of variation of the variables. For $e \in E$, we shall use the notation

$$H(t; e) := H(t, j_{\kappa_0}(e)). \tag{6.29}$$

Let $e \in E$ and $e' \in E$, set $J^{\kappa_0} := j_{\kappa_0}(e)$ and $'J^{\kappa_0} := j_{\kappa_0}(e')$. Then we can define the Lie group multiplication μ_J by

$$\mu_J('J^{\kappa_0}, J^{\kappa_0}) := ([\partial_t^\alpha(H(H(t, J^{\kappa_0}), 'J^{\kappa_0}))]_{t=0})_{|\alpha|\leq\kappa_0}. \tag{6.30}$$

Accordingly, in terms of the coordinates $(e_1, \ldots, e_{c_0})$ on E, the Lie group multiplication μ is defined by

$$\mu(e, e') := (j_{\kappa_0})^{-1}(\mu_J(j_{\kappa_0}(e'), j_{\kappa_0}(e))) \in \mathbb{R}^{c_0} \tag{6.31}$$

It follows from the algebraicity of the mappings H and j_{κ_0} that the mappings μ_J and μ are algebraic.

We must check the associativity of μ, namely $\mu(\mu(e,e'),e'') = \mu(e,\mu(e',e''))$. So we set $h(t) := H(t, j_{\kappa_0}(e))$, $h'(t) := H(t, j_{\kappa_0}(e'))$ and $h''(t) := H(t, j_{\kappa_0}(e''))$. By the definition (6.30), we have $\mu_J(j_{\kappa_0}(e), j_{\kappa_0}(e')) = J^{\kappa_0}(h \circ h')(0)$. Applying then Theorem 6.4, we get $H(t, J^{\kappa_0}(h\circ h')(0)) \equiv (h \circ h')(t)$. Consequently, using again (6.30) and the associativity of the composition of mappings, we may compute

$$
(6.32) \qquad \left\{ \begin{aligned} \mu_J(\mu_J(j_{\kappa_0}(e), j_{\kappa_0}(e')), j_{\kappa_0}(e'')) &= \mu_J(J^{\kappa_0}((h\circ h')(0), j_{\kappa_0}(e'')) \\ &= J^{\kappa_0}((h\circ h')\circ h'')(0) \\ &= J^{\kappa_0}(h\circ (h'\circ h''))(0) \\ &= \mu_J(j_{\kappa_0}(e), J^{\kappa_0}(h'\circ h'')(0)) \\ &= \mu_J(j_{\kappa_0}(e), \mu_J(j_{\kappa_0}(e), j_{\kappa_0}(e'))), \end{aligned} \right.
$$

which proves the associativity.

Finally, we may define an algebraic inversion mapping ι as follows. First of all, for J^{κ_0} close to $J^{\kappa_0}_{\mathrm{Id}}$, the mapping $h(t) := H(t, J^{\kappa_0}) = t + \sum_{\alpha\in\mathbb{N}^n} t^\alpha \, H_\alpha(J^{\kappa_0})$ is an invertible algebraic biholomorphic mapping. Here, the coefficients $H_\alpha(J^{\kappa_0})$ are algebraic functions of J^{κ_0} which vanish at $J^{\kappa_0}_{\mathrm{Id}}$ (since $H(t, J^{\kappa_0}_{\mathrm{Id}}) \equiv t$ in Theorem 6.4). From the algebraic implicit function theorem, it follows that the local inverse $h^{-1}(t)$ writes uniquely in the form $h^{-1}(t) = t + \sum_{\alpha\in\mathbb{N}^n} t^\alpha \, \widetilde{H}_\alpha(J^{\kappa_0}) =: \widetilde{H}(t, J^{\kappa_0})$, where the $\widetilde{H}_\alpha(J^{\kappa_0})$ are algebraic functions of J^{κ_0} also satisfying $\widetilde{H}_\alpha(J^{\kappa_0}_{\mathrm{Id}}) = 0$. Consequently, choosing $e \in E$ such that $J^{\kappa_0} = j_{\kappa_0}(e)$, we can define

$$
(6.33) \qquad \iota_J(J^{\kappa_0}) := ([\partial_t^\alpha \widetilde{H}(t, J^{\kappa_0})]_{t=0})_{|\alpha|\leq\kappa_0}.
$$

Accordingly, in terms of the coordinates $(e_1, \dots, e_{c_0})$ on E, the Lie group inverse mapping is defined by

$$
(6.34) \qquad \iota(e) := (j^{\kappa_0})^{-1}(i_J(j_{\kappa_0}(e))).
$$

Of course, with this definition we have $\iota_J(J^{\kappa_0}_{\mathrm{Id}}) = J^{\kappa_0}_{\mathrm{Id}}$. Finally, we leave to the reader to verify that $\mu_J(j_{\kappa_0}(e), i_J(j_{\kappa_0}(e))) = J^{\kappa_0}_{\mathrm{Id}}$. This completes the proof of property **(4)** of Theorem 4.1. □

End of proof of Theorem 4.1. We notice that statement **(5)** does not need to be proved. Furthermore that the dimensional inequality $c_0 \leq \frac{(n+\kappa_0)!}{n!\ \kappa_0!}$ in **(6)** follows from the fact each local biholomorphic mapping in the local Lie group $\mathcal{H}^{\rho_2,\rho_1}_{M,\kappa_0,\varepsilon} \cong E$ writes uniquely as $h(t) = H(t, J^{\kappa_0} h(0))$, so the complex dimension of the local Lie group E is $\leq \frac{(n+\kappa_0)!}{n!\ \kappa_0!}$, the dimension of the κ_0-th jet space. As E is totally real, the real dimension of E is also $\leq \frac{(n+\kappa_0)!}{n!\ \kappa_0!}$. Finally, it follows that the real local Lie algebra of vector fields $\mathfrak{Hol}(M, \Delta_n(\rho_5))$ is of dimension $\leq \frac{(n+\kappa_0)!}{n!\ \kappa_0!}$. The proof of Theorem 4.1 is complete. □

§7. Description of explicit families of strong tubes in $\mathbb{C}^n$

7.1. Introduction. Theorems 1.1, 1.4 and 1.5 provide sufficient conditions for some real analytic real submanifold in $\mathbb{C}^n$ to be not locally algebraizable. For the sake of completeness, we exhibit explicit examples of such nonalgebraizable submanifolds which are effectively strong tubes and effectively nonalgebraizable, proving corollaries 1.2, 1.3, 1.6 and 1.7. Consequently we will deal with the two following families of nonalgebraizable real analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^n$ $(n \geq 2)$: the Levi nondegenerate strong tube hypersurfaces in $\mathbb{C}^n$ and the strongly rigid hypersurfaces in $\mathbb{C}^n$. For heuristic reasons, we shall sometimes start with the case $n = 2$ and treat the general case $n \geq 2$ afterwards. In fact, our goal will be to construct infinite families of pairwise non biholomorphically equivalent and non locally algebraizable hypersurfaces. Our computations for the construction of families of manifolds with a control on the structure of their automorphism group are all based on the Lie theory of symmetries of differential equations. For the convenience of the reader, we recall briefly the procedure (*see* [Su2001a,b], [GM2001a,b,c] for more details).

7.2. Hypersurfaces and differential equations. Let M be a real analytic hypersurface in $\mathbb{C}^n$. Assume that M is Levi nondegenerate at one of its points p. Then there exist some local holomorphic coordinates $(z,w) = (z, u+iv) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p such that M is given by the real analytic equation

$$v = \varphi(z,\bar{z},u) = \varepsilon_1 |z_1|^2 + \cdots + \varepsilon_{n-1}|z_{n-1}|^2 + \psi(z,\bar{z},u), \tag{7.1}$$

where $\varepsilon_k = \pm 1$, $k = 1, \dots, n-1$ and where $\psi = \mathrm{O}(3)$. Passing to the extrinsic complexification $\mathcal{M}$ of M, we may consider the variables $\bar{z}$ and $\bar{w}$ as independent complex parameters $\zeta \in \mathbb{C}^{n-1}$ and $\xi \in \mathbb{C}$. Then the associated complex defining equation is of the form

$$w = \overline{\Theta}(z,\zeta,\xi) = \xi + 2i(\varepsilon_1 z_1 \zeta_1 + \cdots + \varepsilon_{n-1} z_{n-1}\zeta_{n-1} + \overline{\Xi}(z,\zeta,\xi)), \tag{7.2}$$

where $\overline{\Xi} = \mathrm{O}(3)$. By [Me1998] (*cf.* §5.1 above), for $\tau_p = (\zeta_p, \xi_p)$ fixed, the family of complexified Segre varieties $\mathcal{S}_{\tau_p} := \{(t,\tau_p) : w = \overline{\Theta}(z,\tau_p)\}$ is invariantly and biholomorphically attached to M.

Following [Se1931] and [Su2001a,b], we may consider this family as a family of graphs of the solutions of a second order completely integrable system of partial differential equations as follows. By differentiating the left and the right hand sides of (7.2) with respect to z_k, we get

$$\partial_{z_k} w = \partial_{z_k} \overline{\Theta}(z,\tau) = 2i(\varepsilon_k \zeta_k + \partial_{z_k}\overline{\Xi}(z,\tau)), \tag{7.3}$$

for $k = 1, \dots, n-1$. Here, we consider w as a function of z. Using the analytic implicit function theorem to solve τ in the $1 + (n-1) = n$ equations (7.2) and (7.3), we may express τ in terms of w, of z and of the first order derivative w_{z_l}, which yields

$$\tau = \Pi(z, w, (\partial_{z_l} w)_{1 \le l \le n-1}), \tag{7.4}$$

where Π is holomorphic in its variables. If we take the second derivative $w_{z_{k_1} z_{k_2}}$ of w and replace the value of τ, we get the desired system of partial differential equations:

$$\begin{aligned} \partial^2_{z_{k_1} z_{k_2}} w &= \partial^2_{z_{k_1} z_{k_2}} \overline{\Theta}(z,\tau) = \partial^2_{z_{k_1} z_{k_2}} \overline{\Theta}(z, \Pi(z,w,(\partial_{z_l} w)_{1\le l \le n-1})) =: \\ &=: F_{k_1,k_2}(z,w,(\partial_{z_l} w)_{1 \le l \le n-1}). \end{aligned} \tag{7.5}$$

Here, $k_1, k_2 = 1, \dots, n-1$ and the $F_{k_1,k_2} \equiv F_{k_2,k_1}$ are holomorphic in their variables. We denote by $\mathcal{E}_M$ this system of partial differential equations (here, to construct $\mathcal{E}_M$, we have used the Levi nondegeneracy of M but we note that if M were finitely nondegenerate the same conclusion would be true, by considering some derivatives of w of larger order). Since the solutions of $\mathcal{E}_M$ are precisely the complexified Segre varieties $\mathcal{S}_\tau$, the system $\mathcal{E}_M$ is completely integrable.

To study the local geometry of M, we may consider on one hand the real Lie algebra of infinitesimal CR automorphisms of M (*cf.* §2.2), namely $\mathfrak{Aut}_{CR}(M) = 2\,\mathrm{Re}\,\mathfrak{Hol}(M)$. On the other hand, following the general ideas of Lie (*cf.* the modern restitution by Olver in [Ol1986, Ch 2]), we may consider the Lie algebra of infinitesimal generators of the local symmetry group of the system of partial differential equations $\mathcal{E}_M$, which we shall denote by $\mathfrak{Sym}(\mathcal{E}_M)$. By definition, $\mathfrak{Sym}(\mathcal{E}_M)$ consists of holomorphic vector fields in the (z,w)-space whose local flow transforms the graph of every solution of $\mathcal{E}_M$ (namely a complexified Segre variety) into the graph of another solution of $\mathcal{E}_M$ (namely into another complexified Segre variety). The link between $\mathfrak{Aut}_{CR}(M)$ and $\mathfrak{Sym}(\mathcal{E}_M)$ is as follows: by [Ca1932, p. 30–32], one can prove that $\mathfrak{Aut}_{CR}(M)$ is a maximally real subspace of $\mathfrak{Sym}(\mathcal{E}_M)$ (*see* also [Su2001a,b], [GM2001a,b,c]).

The computation of explicit generators of $\mathfrak{Sym}(\mathcal{E}_M)$ may be performed using the Lie theory of symmetries of differential equations. By inspecting some examples, it appears that dealing with $\mathfrak{Sym}(\mathcal{E}_M)$ generally shortens the complexity of the computation of $\mathfrak{Aut}_{CR}(M)$ by at least one half.

The Lie procedure to compute $\mathfrak{Sym}(\mathcal{E}_M)$ is as follows. Let $J^2_{n-1,1}(\mathbb{C})$ denote the space of second order jets of a function $w(z_1, \dots, z_{n-1})$ of $(n-1)$ complex variables, equipped with independent coordinates $(z, w, W^1_l, W^2_{k_1,k_2})$ corresponding to $(z, w, w_{z_l}, w_{z_{k_1} z_{k_2}})$, where $l = 1, \dots, n-1$, where $k_1, k_2 = 1, \dots, n-1$, and where we of course identify $W^2_{k_1,k_2}$ with $W^2_{k_2,k_1}$. To the system $\mathcal{E}_M$, we associate the complex submanifold of $J^2_{n-1,1}(\mathbb{C})$ defined by replacing the derivatives of w by the independent jet variables in the system $\mathcal{E}_M$, which yields (*cf.* (7.5)):

$$W^2_{k_1,k_2} = F_{k_1,k_2}(z, w, (W^1_l)_{1 \le l \le n-1}), \tag{7.6}$$

for $k_1, k_2 = 1, \dots, n-1$. Let Δ_M denote this submanifold. By Lie's theory, every vector field $X = \sum_{k=1}^{n-1} Q^k(z,w)\,\partial_{z_k} + R(z,w)\,\partial_w$ defined in a neighborhood of the origin in $\mathbb{C}^n$ can be uniquely lifted to a vector field $X^{(2)}$ in $J^2_{n-1,1}(\mathbb{C})$, which is called the *second prolongation* of X (by definition, the lift $X^{(2)}$ shows how the flow of X transforms second order jets of graphs of functions $w(z)$). The coefficients R^1_l and $R^2_{k_1,k_2}$ of the second prolongation

$$X^{(2)} = \sum_{k=1}^{n-1} Q^k \frac{\partial}{\partial z_k} + R \frac{\partial}{\partial w} + \sum_{l=1}^{n-1} R^1_l \frac{\partial}{\partial W^1_l} + \sum_{k_1,k_2=1}^{n-1} R^2_{k_1,k_2} \frac{\partial}{\partial W^2_{k_1,k_2}}, \tag{7.7}$$

are completely determined by the following universal formulas (*cf.* [Ol1986], [Su2001a,b], [GM2001a]):

(7.8)

$$\left\{ \begin{aligned} R^1_l &= \partial_{z_l} R + \sum_{m_1} [\delta^{m_1}_l\, \partial_w R - \partial_{z_l} Q^{m_1}]\, W^1_{m_1} + \sum_{m_1,m_2} [-\delta^{m_1}_l\, \partial_w Q^{m_2}]\; W^1_{m_1}\, W^1_{m_2}. \\ R^2_{k_1,k_2} &= \partial^2_{z_{k_1} z_{k_2}} R + \sum_{m_1} \left[\delta^{m_1}_{k_1}\, \partial^2_{z_{k_2} w} R + \delta^{m_1}_{k_2}\, \partial^2_{z_{k_1} w} R - \partial^2_{z_{k_1} z_{k_2}} Q^{m_1}\right] W^1_{m_1} + \\ &\quad + \sum_{m_1,m_2} \left[\delta^{m_1,m_2}_{k_1,\,k_2}\, \partial^2_{w^2} R - \delta^{m_1}_{k_1}\, \partial^2_{z_{k_2} w} Q^{m_2} - \delta^{m_1}_{k_2}\, \partial^2_{z_{k_1} w} Q^{m_2}\right] W^1_{m_1}\, W^1_{m_2} + \\ &\quad + \sum_{m_1,m_2,m_3} \left[-\delta^{m_1,m_2}_{k_1,\,k_2}\, \partial^2_{w^2} Q^{m_3}\right] W^1_{m_1}\, W^1_{m_2}\, W^1_{m_3} + \\ &\quad + \sum_{m_1,m_2} \left[\delta^{m_1,m_2}_{k_1,\,k_2}\, \partial_w R - \delta^{m_1}_{k_1}\, \partial_{z_{k_2}} Q^{m_2} - \delta^{m_1}_{k_2}\, \partial_{z_{k_1}} Q^{m_2}\right] W^2_{m_1,m_2} + \\ &\quad + \sum_{m_1,m_2,m_3} \left[-\delta^{m_1,m_2}_{k_1,\,k_2}\, \partial_w Q^{m_3} - \delta^{m_2,m_3}_{k_1,\,k_2}\, \partial_w Q^{m_1} - \delta^{m_3,m_1}_{k_1,\,k_2}\, \partial_w Q^{m_2}\right] W^1_{m_1}\, W^2_{m_2,m_3}. \end{aligned} \right.$$

In these formulas, by δ^m_l we denote the Kronecker symbol equal to 1 if $l = m$ and to 0 otherwise. The multiple Kronecker symbol $\delta^{m_1,m_2}_{l_1,\,l_2}$ is defined to be the product $\delta^{m_1}_{l_1} \cdot \delta^{m_2}_{l_2}$. Finally, in the sums $\sum_{m_1}$, $\sum_{m_1,m_2}$ and $\sum_{m_1,m_2,m_3}$, the integers m_1, m_2, m_3 run from 1 to $n-1$. We would like to mention that in [GM2001a], we also provide some explicit expression of the k-th prolongation $X^{(k)}$ for $k \ge 3$.

Then the *Lie criterion* states that *a holomorphic vector field X belongs to $\mathfrak{Sym}(\mathcal{E}_M)$ if and only if its second prolongation $X^{(2)}$ is tangent to* Δ_M ([Ol1986, Ch 2]). This gives the following equations:

$$R^2_{k_1,k_2} - \sum_{k=1}^{n-1} Q^k\, \partial_{z_k} F_{k_1,k_2} - R\, \partial_w F_{k_1,k_2} - \sum_{l=1}^{n-1} R^1_l\, \partial_{W^1_l} F_{k_1,k_2} \equiv 0, \tag{7.9}$$

where $1 \leq k_1, k_2 \leq n-1$ and where each occurence of $W^2_{l_1,l_2}$ is replaced by its value F_{l_1,l_2} on Δ_M. By developping (7.9) in power series with respect to the variables W^1_l, we get an expression of the form

$$\sum_{l_1,\dots,l_{n-1}\geq 0} W^1_{l_1}\cdots W^1_{l_{n-1}}\,\Phi_{l_1,\dots,l_{n-1}} \equiv 0, \tag{7.10}$$

where each term $\Phi_{l_1,\dots,l_{n-1}}$ is a certain linear partial differential expression involving the derivatives of $Q^1,\dots,Q^{n-1},R$ up to order two with coefficients being holomorphic functions of (z,w). The determination of a system of generators $X_1,\dots,X_c$ of $\mathfrak{Sym}(\mathcal{E}_M)$ is obtained by solving the infinite collection of these linear partial differential equations $\Phi_{l_1,\dots,l_{n-1}} = 0$ (*cf.* [Ol1986], [Su2001a,b], [GM2001a,b,c]). We shall apply this general procedure to provide different families of nonalgebraizable real analytic hypersurfaces in $\mathbb{C}^n$.

7.3. Hypersurfaces in $\mathbb{C}^2$ with control of their CR automorphism group. The goal of this paragraph is to construct some classes of strong tubes, namely tubes having the smallest possible CR automorphism group. We start with the case $n=2$ and study afterwards the case $n\geq 3$ in the next subparagraph. Let M_χ be the strong tube hypersurface in $\mathbb{C}^2$ defined by the equation

$$M_\chi: \qquad v = \varphi(y) := y^2 + y^6 + y^9 + y^{10}\,\chi(y). \tag{7.11}$$

where χ is a real analytic function defined in a neighborhood of the origin in $\mathbb{R}$.

Lemma 7.1. *The hypersurfaces M_χ are pairwise not biholomorphically equivalent strong tubes.*

Proof. To check that M_χ is a strong tube, it suffices to show that every hypersurface of the form $v = y^2 + y^6 + \mathrm{O}(y^9)$ is a strong tube (the term y^9 will be used afterwards). Writing $v = (w-\bar w)/2i$ and $y = (z-\bar z)/2i$, considering w as a function of z and $\bar w$, $\bar z$ as constants, the differentiation of w with respect to z in (7.11) yields:

$$\partial_z w = 2y + 6y^5 + \mathrm{O}(y^8). \tag{7.12}$$

The implicit function theorem yields:

$$y = (1/2)\partial_z w - (3/2^5)(\partial_z w)^5 + \mathrm{O}((\partial_z w)^8). \tag{7.13}$$

One further differentiation of equation (7.12) with respect to z gives:

$$\partial^2_{zz} w = -i - (15i)\,y^4 + \mathrm{O}(y^7). \tag{7.14}$$

Replacing y in this equation by its value obtained in (7.13), we obtain the following second order ordinary equation $\mathcal{E}_M$ satisfied by $\partial_z w$ and $\partial^2{}_z zw$:

$$\partial^2_{zz} w = -i - (15i/2^4)(\partial_z w)^4 + \mathrm{O}((\partial_z w)^7). \tag{7.15}$$

In the four dimensional jet space $J^2_{1,1}(\mathbb{C})$ equipped with the coordinates (z,w,W^1,W^2) the equation of the corresponding complex hypersurface Δ_M is of course:

$$W^2 = -i - (15i/2^4)(W^1)^4 + \mathrm{O}((W^1)^7). \tag{7.16}$$

Then the Lie criterion states that a holomorphic vector field $X = Q\,\partial_z + R\,\partial_w$ belongs to $\mathfrak{Sym}(\mathcal{E}_M)$ if and only if its second prolongation $X^{(2)} = Q\,\partial_z + R\,\partial_w + R^1\,\partial_{W^1} + R^2\,\partial_{W^2}$ is tangent to Δ_M, where the coefficients R^1 and R^2 are given by the formulas (7.8) specified for $n=2$, namely:

$$\begin{cases} R^1 = \partial_z R + [\partial_w R - \partial_z Q]\,W^1 - \partial_w Q\,(W^1)^2. \\ R^2 = \partial^2_{zz} R + [2\partial^2_{zw} R - \partial^2_{zz} Q]\,W^1 + [\partial^2_{ww} R - 2\partial^2_{zw} Q]\,(W^1)^2 - \partial^2_{ww} Q\,(W^1)^3 + \\ \qquad\qquad + [\partial_w R - 2\partial_z Q]\,W^2 - 3\partial_w Q\,W^1 W^2. \end{cases} \tag{7.17}$$

The tangency condition yields the following equation which is satisfied on Δ_M, *i.e.* after replacing W^2 by its value given by (7.16):

$$R^2 + (15i/2^2)R^1(W^1)^3 + \mathrm{O}((W^1)^6) = 0. \tag{7.18}$$

By expanding equation (7.18) in powers of W^1 up to order five, we obtain the following system of six linear partial differential equations which must be satisfied by the derivatives of Q and R up to order two:

$$\begin{cases} (e_0): & \partial^2_{zz}R - i(\partial_w R - 2\partial_z Q) \equiv 0. \\ (e_1): & 2\partial^2_{zw}R - \partial^2_{zz}Q \equiv 0. \\ (e_2): & \partial^2_{ww}R - 2\partial^2_{zw}Q \equiv 0. \\ (e_3): & -\partial^2_{ww}Q + \frac{15i}{2^2}\partial_z R \equiv 0. \\ (e_4): & -\frac{15i}{2^4}(\partial_w R - 2\partial_z Q) + \frac{15i}{2^2}(\partial_w R - \partial_z Q) \equiv 0. \\ (e_5): & -\frac{15i}{2^4}(-3\partial_w Q) - \frac{15i}{2^2}(\partial_w Q) \equiv 0. \end{cases} \tag{7.19}$$

It follows from the equation (e_5) that $\partial_w Q \equiv 0$ which implies $\partial^2_{ww}Q \equiv 0$. Then by equation (e_3) we obtain $\partial_z R \equiv 0$, implying $\partial^2_{zz}R \equiv 0$. From equation (e_0) we get $\partial_w R \equiv 2\partial_z Q$ and, from equation (e_4), we get $\partial_w R \equiv \partial_z Q$. Consequently $\partial_z R \equiv \partial_w R \equiv \partial_z Q \equiv \partial_w Q \equiv 0$. Since the two vector fields ∂_z and ∂_w evidently belong to $\mathfrak{Sym}(\mathcal{E}_M)$, it follows that $\dim_{\mathbb{C}} \mathfrak{Sym}(\mathcal{E}_M) = 2$. Finally, this implies that $\dim_{\mathbb{R}} \mathfrak{Aut}_{CR}(M) = 2$ and that $\mathfrak{Aut}_{CR}(M)$ is generated by $\partial_w + \partial_{\bar{w}}$ and $\partial_z + \partial_{\bar{z}}$.

Next, let $\chi(y)$ and $\chi'(y')$ be two real analytic functions, and assume that M_χ and $M'_{\chi'}$ are biholomorphically equivalent. Let $t' = h(t)$ be such an equivalence. Reasoning as in §4 and taking into account that both are strong tubes, we see that $h_*(\partial_z)$ and $h_*(\partial_w)$ must be linear combinations of $\partial_{z'}$ and $\partial_{w'}$ with real coefficients. It follows that h must be linear, of the form $z' = az + bw$, $w' = cz + dw$, where a, b, c and d are real. Since $T_0 M_\chi = \{v = 0\}$ and $T_0 M'_{\chi'} = \{v' = 0\}$, we have $c = 0$. Next, in the equation

$$\begin{cases} d(y^2 + y^6 + y^9 + y^{10}\chi(y)) \equiv [ay + b(y^2 + y^6 + y^9 + y^{10}\chi(y))]^2 + \\ + [ay + b(y^2 + y^6 + y^9 + y^{10}\chi(y))]^6 + [ay + b(y^2 + y^6 + y^9 + y^{10}\chi(y))]^9 + \\ + [ay + b(y^2 + y^6 + y^9 + y^{10}\chi(y))]^{10}\chi'(ay + b(y^2 + y^6 + y^9 + y^{10}\chi(y))), \end{cases} \tag{7.20}$$

we firstly see that $b = 0$, and then from

$$d(y^2 + y^6 + y^9 + y^{10}\chi(y)) \equiv a^2y^2 + a^6y^6 + a^9y^9 + a^{10}y^{10}\chi'(ay), \tag{7.21}$$

we see that $a = d = 1$. In other words, $h = \mathrm{Id}$, whence $y' = y$ and $\chi'(y') \equiv \chi(y)$. This proves Lemma 7.1. □

In the remainder of §7, we shall exhibit other classes of hypersurfaces with a control on their CR automorphism group. Since the computations are generally similar, we shall summarize them.

7.4. Some classes of strong tube hypersurfaces in $\mathbb{C}^n$. Generalizing Lemma 7.1, we may state:

Lemma 7.2. *The real analytic hypersurfaces $M_{\chi_1,\dots,\chi_{n-1}} \subset \mathbb{C}^n$ of equation*

$$v = \sum_{k=1}^{n-1} [\varepsilon_k\, y_k^2 + y_k^6 + y_k^9 y_1 \cdots y_{k-1} + y_k^{n+8}\chi_k(y_1, \dots, y_{n-1})], \tag{7.22}$$

where $\varepsilon_k = \pm 1$, are pairwise not biholomorphically equivalent strong tubes.

Proof. The associated system of partial differential equations is of the form

$$\begin{cases} \partial^2_{z_k z_k} w = \, -i\varepsilon_k - (15i/2^4)\,(\partial_{z_k} w)^4 + \mathrm{O}((\partial_{z_1} w)^7) + \cdots + \mathrm{O}((\partial_{z_{n-1}} w)^7), \\ \partial^2_{z_{k_1} z_{k_2}} w = 0, \qquad\qquad \text{for} \;\; k_1 \neq k_2. \end{cases} \tag{7.23}$$

Using the formulas (7.8) and inspecting the coefficients of the monomials in the W_l^1 up to order five in the $(n-1)$ equations extracted from the set of Lie equations

$$\begin{cases} R^2_{k,k} + (15i/2^2)(W_k^2)\, R_k^1 + \mathrm{O}((W_1^1)^6) + \cdots + \mathrm{O}((W_{n-1}^1)^6) = 0, \\ R^2_{k_1,k_2} = 0, \qquad\qquad \text{for} \;\; k_1 \neq k_2, \end{cases} \tag{7.24}$$

we get $\partial_{z_l} R \equiv \partial_w R \equiv \partial_{z_l} Q^k \equiv \partial_w Q^k \equiv 0$ for $l, k = 1, \dots, n-1$. Thus, $M_{\chi_1,\dots,\chi_{n-1}}$ is a strong tube.

Next, reasoning as in the end of the proof of Lemma 7.1, we see first that an equivalence between $M_{\chi_1,\dots,\chi_{n-1}}$ and $M'_{\chi'_1,\dots,\chi'_{n-1}}$ must be of the form $z'_k = \sum_{l=1}^{n-1} \lambda_k^l \, z_l$, $w' = \mu\, w$, where λ_k^l, $1 \leq l, k \leq n-1$ and μ are real. Inspecting the terms of degree $9, 10, \dots, n+7$, we get $\lambda_k^l = 0$ if $k \neq l$, *i.e.* $y'_k = \lambda_k^k \, y_k$ and $w' = \mu\, w$. Finally, $\lambda_k^k = 1$ and $\mu = 1$, which completes the proof. □

7.5. Families of strongly rigid hypersurfaces. Alongside the same recipe, we can study some classes of hypersurfaces of the form $v = \varphi(z\bar{z})$.

Lemma 7.3. *The Lie algebra $\mathfrak{Hol}(M_\chi)$ of the rigid real analytic hypersurfaces M_χ in $\mathbb{C}^2$ of equation $v = \varphi(z\bar{z}) = z\bar{z} + z^5\bar{z}^5 + z^7\bar{z}^7 + z^8\bar{z}^8\chi(z\bar{z})$ is two-dimensional and generated by ∂_w and $iz\partial_z$. Furthermore, M_χ is biholomorphically equivalent to $M'_{\chi'}$ if and only if $\chi = \chi'$.*

Proof. The associated differential equation is of the form

$$\partial^2_{zz} w = [5z^3/4](\partial_z w)^5 - [21z^5/32](\partial_z w)^7 + \mathrm{O}((\partial_z w)^9). \tag{7.25}$$

Extracting from the associated Lie equations (7.10) the coefficients of the monomials $(W^1)^4$, $(W^1)^5$, $(W^1)^6$ and $(W^1)^7$, we obtain four equations which are solved by $z\partial_z Q - Q \equiv 0$, $\partial_w Q \equiv 0$, $\partial_z R \equiv 0$ and $\partial_w R \equiv 0$. Next, if M_χ and $M'_{\chi'}$ are biholomorphically equivalent, reasoning as in §4, taking into account that $h_*(iz\partial_z)$ and $h_*(\partial_w)$ are linear combinations of $iz'\partial_{z'}$ and $\partial_{w'}$ with real coefficients, we see first that $z' = \lambda\, z\, e^{\gamma w/2i}$ and $w' = \mu\, w$ for some three real constants γ, $\lambda \neq 0$ and $\mu \neq 0$. Replacing z' and w' in the equation of $M'_{\chi'}$, we get $\gamma = 0$, $\mu = 1$ and $\lambda \pm 1$. In other words, $z' = \pm z$ and $w' = w$, which entails $\chi'(z'\bar{z}') \equiv \chi(z\bar{z})$, as claimed. □

Perturbing this family we may exhibit other strongly rigid hypersurfaces :

Lemma 7.4. *The Lie algebra $\mathfrak{Hol}(M_\chi)$ of the real analytic hypersurfaces M_χ in $\mathbb{C}^2$ of equation $v = \varphi(z, \bar{z}) = z\bar{z} + z^5\bar{z}^5 + z^7\bar{z}^7 + z^8\bar{z}^8(z + \bar{z}) + z^{10}\bar{z}^{10}\chi(z, \bar{z})$ is one-dimensional and generated by ∂_w. Furthermore, M_χ is biholomorphically equivalent to $M'_{\chi'}$ if and only if $\chi = \chi'$.*

Proof. We already know that $z\partial_z Q - Q \equiv \partial_w Q \equiv \partial_z R \equiv \partial_w R \equiv 0$. Extracting from the associated Lie equations (7.10) the coefficient of the monomials $(W^1)^8$, we also get $Q \equiv 0$. Next, let M_χ and $M'_{\chi'}$ be biholomorphically equivalent. Let $t' = h(t)$ be such an equivalence. Using $h_*(\partial_w) = \mu\, \partial_{w'}$, where $\mu \in \mathbb{R}$ is nonzero, we get $z' = f(z)$ and $w' = \mu w + g(z)$. Next from the equation

$$\begin{cases} \mu(z\bar{z} + z^5\bar{z}^5 + z^7\bar{z}^7 + z^8\bar{z}^8(z + \bar{z}) + \mathrm{O}(z^9\bar{z}^9)) + [g(z) - \bar{g}(\bar{z})]/2i \equiv \\ \quad \equiv f(z)\bar{f}(\bar{z}) + f(z)^5\bar{f}(\bar{z})^5 + f(z)^7\bar{f}(\bar{z})^7 + f(z)^8\bar{f}(\bar{z})^8(f(z) + \bar{f}(\bar{z})) + \mathrm{O}(z^9\bar{z}^9), \end{cases} \tag{7.26}$$

we get firstly $f(z) = \sqrt{|\mu|}e^{i\theta}z$ by differentiating with respect to $\bar{z}$ at $\bar{z} = 0$ and secondly $\mu = e^{i\theta} = 1$, which completes the proof. □

We provide a second family of strongly rigid hypersurfaces in $\mathbb{C}^2$ with a one-dimensional Lie algebra:

Lemma 7.5. *The Lie algebra $\mathfrak{Hol}(M_\chi)$ of the real analytic hypersurfaces $M_\chi \subset \mathbb{C}^2$ of equation $v = z\bar{z} + z^5\bar{z}^5(z+\bar{z}) + z^{10}\bar{z}^{10}\chi(z,\bar{z})$ is one-dimensional and generated by ∂_w. Furthermore M_χ is biholomorphically equivalent to $M'_{\chi'}$ if and only if $\chi = \chi'$.*

Proof. The derivatives $\partial_z w$ and $\partial^2_{z^2} w$ of w with respect to z are given by:

$$\begin{cases} \partial_z w = 2i\bar{z} + 12iz^5\bar{z}^5 + 10iz^4\bar{z}^6 + \mathrm{O}(\bar{z}^{10})), \\ \partial^2_{zz} w = 60iz^4\bar{z}^5 + 40iz^3\bar{z}^6 + \mathrm{O}(\bar{z}^{10}). \end{cases} \tag{7.27}$$

Replacing $\bar{z}$ in the second equation by its expression given by the first equation we obtain the following second order differential equation, interpreted in the jet space:

$$W_2 = [15z^4/8]\,(W^1)^5 - [5iz^3/8]\,(W^1)^6 - [225z^9/64]\,(W^1)^9 + \mathrm{O}((W^1)^{10}). \tag{7.28}$$

Solving the partial differential equations involving Q, $\partial_z Q$, $\partial_w Q$, $\partial_z R$ and $\partial_w R$ given in the coefficients of $(W^1)^4$, $(W^1)^5$, $(W^1)^6$, $(W^1)^7$ and $(W^1)^9$ we obtain $Q \equiv \partial_z Q \equiv \partial_w Q \equiv \partial_z R \equiv \partial_w R \equiv 0$ which is the desired information. Finally, proceeding exactly as in the end of the proof of Lemma 7.4, we see that the M_χ are pairwise biholomorphically not equivalent. □

The dimension of $\mathfrak{Hol}(M)$ for the five examples of Corollary 1.3, for the seven examples of Theorem 1.4, for the seven examples of Corollary 1.7 and for the hypersurface $v = e^{z\bar{z}} - 1$ at a point p with $z_p \neq 0$ was computed with the package `diffalg` of Maple Release 6. Since at a point p with $z_p \neq 0$ the hypersurface $v = e^{z\bar{z}} - 1$ is biholomorphically equivalent to the hypersurface M_a of equation $v = \varphi^a(y) := e^{a(e^y - 1)} - 1$ with $a = |z_p|^2$, this defines a strong tube. Applying Theorem 1.1 and Lemma 3.3, we see that M_a is not locally algebraizable at the origin, because $\varphi^a_{yy}(y) = ae^y e^{a(e^y-1)} + a^2 e^{2y} e^{a(e^y-1)}$ and $\varphi^a_y(y) = ae^y e^{a(e^y-1)}$ are algebraically independent. Finally all the examples of Corollary 1.3, Theorem 1.4 and Corollary 1.6 are not locally algebraic since they satisfy the required transcendence conditions.

§8. Analyticity versus algebraicity

Intuitively there seems to be much more analytic mappings, manifolds and varieties than algebraic ones. Our goal is to elaborate a precise statement about this. By complexification, every local real analytic object yields a local complex analytic object, so we shall only work in the holomorphic category. Let Δ_n be the complex polydisc of radius one in $\mathbb{C}^n$ and $\overline{\Delta}_n$ its closure. Let $k \in \mathbb{N}$. We consider the space $\mathcal{O}^k(\overline{\Delta}_n) := \mathcal{O}(\Delta_n) \cap \mathcal{C}^k(\overline{\Delta}_n)$ of holomorphic functions extending up to the boundary as a function of class $\mathcal{C}^k$. This is a Banach space for the $\mathcal{C}^k$ norm $||\varphi||_k := \sum_{l=0}^k \sup_{z \in \overline{\Delta}_n} |\varphi_{z^l}(z)|$. The last statements of Corollaries 1.2 and 1.6 are a direct consequence of the following lemma.

Lemma 8.1. *The set of holomorphic functions $\varphi \in \mathcal{O}^k(\overline{\Delta}_n)$ such that there exists a polynomial P such that*

$$P(z, j^k\varphi(z)) \equiv 0, \tag{8.1}$$

is of first category, namely it can be represented as the countable union of nowhere dense closed subsets. Conversely, the set of functions $\varphi \in \mathcal{O}^k(\overline{\Delta}_n)$ such that there is no algebraic dependence relation like (8.1) *is generic in the sense of Baire, namely it can be represented as the countable intersection of everywhere dense open subsets.*

Proof. Let $N \in \mathbb{N}$. Consider the set F_N of functions φ such that there exists a polynomial of degree N satisfying (8.1). It suffices to show that F_N is closed and that its complement is everywhere dense. Suppose that a sequence $(\varphi^{(m)})_{m \in \mathbb{N}}$ converges to $\varphi \in \mathcal{O}^k(\overline{\Delta}_n)$. Let the zero-set of a degree N polynomial $P_N^{(m)}(z, J_k)$ contain the graph of the k-jet of $\varphi^{(m)}$.

The coefficients of $P_N^{(m)}$ belong to a certain complex projective space $P_A(\mathbb{C})$, where the integer $A = A(n,k)$ is independent of m. By compactness of $P_A(\mathbb{C})$, passing to a subsequence if necessary, the $P_N^{(m)}$ converge to a nonzero polynomial P_N. By continuity, $P_N(z, j^k\varphi(z)) = 0$ for all $z \in \mathcal{O}(\overline{\Delta}_n)$. We claim that the complement of the union of the F_N is dense in $\mathcal{O}^k(\overline{\Delta}_n)$. Indeed, let $\varphi(z)$ be such that there exists a degree N polynomial P satisfying (8.1). Fix $z_0 \in \Delta_n$ having rational real and imaginary parts. Then the complex numbers z_0, $\partial_z^\alpha \varphi(z_0)$, $|\alpha| \leq k$, are algebraically dependent. By a Cantorian argument, there exists complex numbers χ_0^α arbitrarily close to $\partial_z^\alpha \varphi(z_0)$ such that z_0, χ_0^α are algebraically independent. Let $\chi(z)$ be a polynomial with $\partial_z^\alpha(z_0) = \chi_0^\alpha - \partial_t^\alpha \varphi(z_0)$. We can choose χ to be arbitrarily close to zero in the $\mathcal{C}^k(\overline{\Delta}_n)$ norm. Then the function $\varphi(z) + \chi(z)$ is not Nash algebraic. □

References

[Ar1974] Arnold, V.I.: *Équations différentielles ordinaires. Champs de vecteurs, groupes à un paramètre, difféomorphismes, flots, systèmes linéaires, stabilité des positions d'équilibre, théorie des oscillations, équations différentielles sur les variétés.* Traduit du russe par Djilali Embarek, Éditions Mir, Moscou, 1974. 267pp.

[BER1999] Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: *Rational dependence of smooth and analytic CR mappings on their jets.* Math. Ann. **315** (1999), 205–249.

[BER2000] Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. **37** (2000), no.3, 309–336.

[Be1996] Bellaïche, A.: *SubRiemannian Geometry*, Progress in Mathematics **144**, Birkhäuser Verlag, Basel/Switzerland, 1996, 1–78.

[Bs1991] Beloshapka, V.K.: *On holomorphic transformations of a quadric*, Mat. Sb. **182** (1991), no.2, 203–219; English transl. in Math. USSR Sb. **72** (1992), no.1, 189–205.

[Ca1932] Cartan, É.: *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I*, Ann. Math. Pura Appl. **11** (1932), 17–90.

[CM1974] Chern, S.S.; Moser, J.K.: *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), no.2, 219–271.

[CPS2000] Coupet, B.; Pinchuk, S.; Sukhov, A.: *On partial analyticity of CR mappings*, Math. Z. **235** (2000), 541–557.

[DP2003] Diederich, D.; Pinchuk, S.: *Regularity of continuous CR-maps in arbitrary dimension*, to appear in Michigan Math. J. (2003).

[Eb1996] Ebenfelt, P.: *On the unique continuation problem for CR mappings into nonminimal hypersurfaces*, J. Geom. Anal. **6** (1996), no.3, 385–405.

[GM2001a] Gaussier, H.; Merker, J.: *Estimates on the dimension of the symmetry group of a system of k-order partial differential equations*, Université de Provence, Prépublication LATP, **12**, 2001, 27 pp.

[GM2001b] Gaussier, H.; Merker, J.: *A new example of uniformly Levi degenerate hypersurface in* $\mathbb{C}^3$, Ark. Mat. (to appear).

[GM2001c] Gaussier, H.; Merker, J.: *Symmetries of differential equations and infinitesimal CR automorphisms of real analytic CR submanifolds of* $\mathbb{C}^n$, manuscript, 34 pp.

[HJ1998] Huang, X.; Ji, S.: *Global holomorphic extension of a local map and a Riemann mapping theorem for algebraic domains*, Math. Res. Lett. **5** (1998), no.1-2, 247–260.

[HJY2001] Huang, X.; Ji, S.; Yau, S.T.: *An example of a real analytic strongly pseudoconvex hypersurface which is not holomorphically equivalent to any algebraic hypersurface*, Ark. Mat. **39** (2001), no.1, 75–93.

[Lie1880] Lie, S.: *Theorie der Transformationsgruppen*, Math. Ann. **16** (1880), 441–528.

[Me1998] Merker, J.: *Vector field construction of Segre sets*, Preprint 1998, augmented in 2000. Downloadable at `arXiv.org/abs/math.CV/9901010`.

[Me2001] Merker, J.: *On the partial algebraicity of holomorphic mappings between two real algebraic sets*, Bull. Soc. Math. France **129** (2001), no.3, 547–591.

[MW1983] Moser, J.K.; Webster S.M.: *Normal forms for real surfaces in* $\mathbb{C}^2$ *near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), no.3-4, 255–296.

[Ol1986] Olver, P.J.: *Applications of Lie groups to differential equations.* Springer Verlag, Heidelberg, 1986.

[Pi1975] Pinchuk, S.: *On the analytic continuation of holomorphic mappings* (Russian), Mat. Sb. (N.S.) **98(140)** (1975) no.3(11), 375–392, 416–435, 495–496.

[Pi1978] Pinchuk, S.: *Holomorphic mappings of real-analytic hypersurfaces* (Russian), Mat. Sb. (N.S.) **105(147)** (1978), no. 4, 574–593, 640.

[Se1931] SEGRE, B.: *Intorno al problema di Poincaré della rappresentazione pseudoconforme*, Rend. Acc. Lincei, VI, Ser. **13** (1931), 676–683.

[Sha2000] SHAFIKOV, R.: *Analytic continuation of germs of holomorphic mappings between real hypesurfaces in* $\mathbb{C}^n$, Michigan Math. J. **47** (2000), no.1, 133–149.

[Sha2002] SHAFIKOV, R.: *Analytic continuation of holomorphic correspondences and equivalence of domains in* $\mathbb{C}^n$, to appear in Inventiones Math.

[SS1996] SHARIPOV, R.; SUKHOV, A.: *On CR mappings between algebraic Cauchy-Riemann manifolds and separate algebraicity for holomorphic functions*, Trans. Amer. Math. Soc. **348** (1996), no.2, 767–780.

[St1991] STANTON, N.: *Infinitesimal CR automorphisms of rigid hypersurfaces in* $\mathbb{C}^2$, J. Geom. Anal. **1** (1991), no.3, 231–267.

[Sto2000] STORMARK, O.: *Lie's structural approach to PDE systems*, Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, xv+572 pp.s

[Su2001a] SUKHOV, A.: *Segre varieties and Lie symmetries*, Math. Z. **238** (2001), no.3, 483–492.

[Su2001b] SUKHOV, A.: *On transformations of analytic CR structures*, Pub. Irma, Lille 2001, Vol. **56**, no. II.

[Ve1999] VERMA, K: *Boundary regularity of correspondences in* $\mathbb{C}^2$, Math. Z. **231** (1999), no.2, 253–299.

[We1977] WEBSTER, S.M.: *On the mapping problem for algebraic real hypersurfaces*, Invent. Math. **43** (1977), no.1, 53–68.

[We1978] WEBSTER, S.M.: *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), no.1, 26–28.

[Za1995] ZAITSEV, D.: *On the automorphism groups of algebraic bounded domains*, Math. Ann. **302** (1995), no.1, 105–129.

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