LIE ALGEBRAS OF INFINITESIMAL CR-AUTOMORPHISMS OF FINITE TYPE, HOLOMORPHICALLY NONDEGENERATE, WEIGHTED HOMOGENEOUS CR-GENERIC SUBMANIFOLDS OF $\mathbb{C}^N$

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ABSTRACT. We consider the significant class of holomorphically nondegenerate CR manifolds of finite type that are represented by some weighted homogeneous polynomials and we derive some useful features which enable us to set up a fast effective algorithm to compute their Lie algebras of infinitesimal CR-automorphisms. This algorithm mainly relies upon a natural gradation of the sought Lie algebras, and it also consists in treating *separately* the related graded components. While some other methods are based on constructing and solving an associated PDE systems which become time consuming as soon as the number of variables increases, the new method presented here is based on plain techniques of linear algebra. Furthermore, it benefits from a *divide-and-conquer* strategy to break down the computations into some simpler sub-computations. Moreover, we consider the new and effective concept of comprehensive Gröbner systems which provides us some powerful tools to treat the computations in the parametric cases. The designed algorithm is also implemented in the MAPLE software.

1. INTRODUCTION

Let $M \subset \mathbb{C}^{n+k}$ be a Cauchy-Riemann (CR for short) submanifold of CR dimension $n \geqslant 1$ and of codimension $k \geqslant 1$ (*see* §2 for all pertinent definitions used in this introduction) represented in coordinates z_j and $w_l := u_l + iv_l$ for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. As is standard in CR-geometry ([4, 14, 7, 8]), one may often assign *weight* $[z_j] := 1$ to all the complex variables z_j and weights some certain integers $[w_l] \in \mathbb{N}$ with $1 < [w_1] \leqslant [w_2] \leqslant \cdots [w_k]$ to the variables $w_1, \ldots, w_k$. Accordingly, the weight of the conjugation of each complex variable and of its real and imaginary parts as well are all equal to that of the variable and, moreover, the assigned weight of any constant number $a \in \mathbb{C}$ and of coordinate vector fields are:

$$[a] := 0, \quad [\partial_{z_j}] := -[z_j], \quad [\partial_{w_l}] := -[w_l], \qquad {\scriptstyle (j=1,\ldots,n, \quad l=1,\ldots,k)}.$$

Furthermore, the weight of a monomial F in the z_j, w_k, $\overline{z}_j$, $\overline{w}_k$ is the sum taken over the weights of all variables of F with regards to their powers and also, the weight of each coordinate vector field of the form $F\,\partial_x$ with $x = z_i, w_l$ is defined as $[F] - [x]$. For instance, we have $[(a\,z\,w_1^2)\,\partial_{w_2}] = [z] + 2[w_1] - [w_2]$. A polynomial or a vector field is called *weighted homogeneous of weight* a certain integer d whenever each of its terms is of homogeneity d.

As is well-known (*see* [4] Theorem 4.3.2 and the Remark after it or [27] Theorem 2.12), every real analytic generic CR manifold M of CR dimension n and codimension k can be represented locally in a neighborhood of the origin as the graph of k defining equations of the form:

$$\left\{\begin{aligned} v_1 &:= \Phi_1(z, \overline{z}, u) + \mathrm{o}([w_1]), \\ &\;\;\vdots \\ v_k &:= \Phi_k(z, \overline{z}, u) + \mathrm{o}([w_k]), \end{aligned}\right. \tag{1}$$

where the weight of all variables z_j is 1 and the weights of the variables w_l are the Hörmander numbers of M at the origin. Moreover, each function Φ_l is a weighted homogeneous polynomial of weight $[w_l]$ and $\mathrm{o}(t)$ denotes remainder terms having weights $> t$.

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For a CR manifold M passing through the origin, the Lie group $\mathsf{Aut}_{CR}(M)$ is the holomorphic symmetry group of M, that is the local Lie group of local biholomorphisms mapping M to itself. The Lie algebra $\mathfrak{aut}_{CR}(M)$, associated to this group is called the *Lie algebra of infinitesimal CR-automorphisms* of M and it consists of all holomorphic vector fields — $(1,0)$ fields with holomorphic coefficients — whose real parts are tangent to M. Due to the fact that many geometric features of CR manifolds can be investigated by means of their associated Lie algebras of infinitesimal CR-automorphisms and because of central applications in Cartan geometry and in Tanaka theory exist, studying such algebras have gained an increasing interest during the recent decades (*cf.* [7, 11, 33, 28, 38]). As is known, the Lie algebra $\mathfrak{aut}_{CR}(M)$ is finite dimensional if and only if M is holomorphically nondegenerate and of finite type ([4, 38, 18, 19, 27]).

Consider the complex space $\mathbb{C}^{n+k}$ equipped with the coordinates $z_1, \ldots, z_n, w_1, \ldots, w_k$, where $w_j := u_j + iv_j$, assume again that certain weights have been assigned, and consider holomorphically nondegenerate weighted homogeneous CR manifolds $M \subset \mathbb{C}^{n+k}$ represented as graphs of k certain polynomials:

$$M := \left\{ (z,w) : \left[\begin{array}{l} \Xi_1(v_1, z, \overline{z}, u) := v_1 - \Phi_1(z, \overline{z}, u) \equiv 0, \\ \Xi_2(v_2, z, \overline{z}, u) := v_2 - \Phi_2(z, \overline{z}, u) \equiv 0, \\ \qquad \vdots \\ \Xi_j(v_j, z, \overline{z}, u) := v_j - \Phi_j(z, \overline{z}, u) \equiv 0, \\ \qquad \vdots \\ \Xi_k(v_k, z, \overline{z}, u) := v_k - \Phi_k(z, \overline{z}, u) \equiv 0, \end{array} \right. \right\}, \tag{2}$$

with the right-hand sides Φ_j being weighted homogeneous polynomials of weight equal to that $[v_j] = [w_j]$ of the left-hand sides. We will also assume that such M are *homogenous* in Lie theory's sense, namely that $\mathrm{Aut}_{CR}(M)$ is locally transitive near the origin. Since it may arise some confusion with the 'homogeneous spaces' terminology, let us stress that we will always use *weighted homogeneous* about functions, and plainly *homogeneous* about $\mathrm{Aut}_{CR}(M)$.

The so introduced class of CR-generic manifolds is already an extremely wide class in CR-geometry which includes of course well known quadric CR-models such as those of Poincaré [32] or of Chern-Moser [15], and also, there is nowadays an extensive literature dealing with constructing a great number of such weighted homogeneous CR manifolds (*see* for example [3, 4, 13, 24, 36] and [6]–[9]), their Lie groups being far from being completely understood.

Indeed, in a series of recent papers (for instance [6]–[10]), Valerii Beloshapka studied extensively the subject of model surfaces and found some considerable results in this respect. Specifically in [7], he introduced and established the structure of some nondegenerate models associated (uniquely) to totally nondegenerate germs having arbitrary CR dimensions and real codimensions. He also developed systematic tools for their construction. All of these models are again included in our already mentioned class of CR manifolds. Each of Beloshapka's model $M \subset \mathbb{C}^{n+k}$ of certain CR dimension and codimension n and k enjoys some *nice* properties ([7], page 484, Theorem 14) which have been encouraging enough to merit further investigation. In particular, computing their Lie algebras of infinitesimal CR-automorphisms and studying their structures may reveal some interesting features of these CR-models, and also of all totally nondegenerate CR manifolds corresponding to them (*cf.* [12]).

It is worth noting that, traditionally the subject of computing Lie algebras of infinitesimal CR-automorphisms is concerned with expensive computations and the cost of calculations increases as much as the number of the variables — namely the dimension or codimension of the CR manifolds — increases. Indeed, solving the PDE systems arising during these computations (*see* subsection 2.1) forms the most complicated part of the procedure. That may be the reason why, in contrast to the importance of the subject, the number of the relevant computational works is still limited (one finds some of them in [28, 10, 36, 24]).

Very recently, the authors provided in [34] a new general algorithm to compute the desired algebras by means of the effective techniques of differential algebra (*see* §2 below for a brief description of the algorithm). It enables one to use conveniently the ability of computer algebra for managing the associated computations of the concerning PDE systems. Although this (general) algorithm is able to decrease a lot the complexity of the computations and in particular to utilize systematically the ability of computer algebra, but *because of dealing with the* PDE *systems* — which are complicated in their spirit — the computations are still expensive in essence.

In the present paper we aim to study, by means of a *weight analysis approach and with an algorithmic treatment*, the intrinsic properties of the under consideration CR manifolds in order to provide an effective algorithm — entirely different and more powerful than that of [34] — to compute the associated Lie algebras of infinitesimal CR-automorphisms. The results enable us to bypass constructing and solving the arising systems of PDE (which is the classical method of [10, 28, 33, 24, 36, 34]) and to reach the sought algebras by employing just simple techniques of linear algebra. This decreases considerably the cost of computations, hence simultaneously increases the performance of the algorithm.

We show that for a homogeneous CR manifold M represented as (2), the sought algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ takes the graded form (in the sense of Tanaka):

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\varrho}}_{\mathfrak{g}_+} \qquad \rho,\ \varrho \in \mathbb{N}. \tag{3}$$

where each component $\mathfrak{g}_t$ is the Lie subalgebra of all weighted homogeneous vector fields having weight t. Summarizing, the results provide us with the ways of:

- using a *divide-and-conquer* method to break down the computations of the sought (graded) algebra $\mathfrak{g}$ into some simpler sub-computations of its (homogeneous) components $\mathfrak{g}_t$;
- employing some simple techniques of linear algebra for computing these components $\mathfrak{g}_t$ of infinitesimal CR-automorphisms *without* relying on solving the PDE systems;

We also find a simple criterion for how long it is necessary to compute the homogeneous components $\mathfrak{g}_t$ of $\mathfrak{g}$. In other words, it supplies us to recognize — in an algorithmic point of view — where the *maximum homogeneity* ϱ in (3) is, while the minimum homogeneity ρ is easy to determine. This criterion fails when $\mathrm{Aut}_{CR}(M)$, or equivalently $\mathfrak{aut}_{CR}(M)$, is not locally transitive. However without local transitivity, one still is able to compute the homogeneous components $\mathfrak{g}_t$, separately.

One of the main — somehow hidden — obstacles appearing among the computations arises when the set of defining equations includes some certain *parametric* polynomials. This case is quite usual as one observes in [7, 10, 24, 36]. To treat such cases, we suggest the modern and effective concept of *comprehensive Gröbner systems* [41, 20, 21, 30, 31] which enables us to consider and solve (linear) parametric systems appearing among the computations.

This weight analysis computational approach to the determination of infinitesimal CR-automorphisms can provide useful tools to study some general problems in CR-geometry, as for instance the models of Kolar ([22]), and also, it can offer numerical evidence to support a recent conjecture of Beloshapka [6] which asserts that:

> **Beloshapka's Maximum conjecture:** For a homogeneous CR-model as those of [7] with the graded Lie algebra of infinitesimal CR-automorphisms like (3), if $3 \leqslant \rho < \infty$ then the corresponding universal CR manifold has '*rigidity*' *i.e.* $\varrho = 0$.

This paper is organized as follows. Section 2 presents a brief description of required very basic definitions, notations and terminology and also a presentation of our recently prepared algorithm [34]. In Section 3, we inspect some of the Lie algebras computed in [10, 28, 36, 24] and we observe a striking *similarity* in the coefficients of the appearing holomorphic vector fields. This leads us to discover the key entrance to the desired algorithm. In Section 4, we employ the results of the previous section to provide the strategy of computing separately the homogeneous components of the sought algebras. We also provide the necessary criterion for terminating such computations. Section 5 presents the desired algorithm of computing the Lie algebras of infinitesimal CR-automorphisms associated to the universal CR manifolds by using the results obtained in the earlier sections. Finally, in section 6 we introduce briefly the modern concept of comprehensive Gröbner systems and show how it provides some effective tools to consider and solve appearing (linear) parametric systems.

The algorithm designed in this paper is implemented in MAPLE 15 as the library CRAUT, accessible online as [35]. To do this, at first we needed to implement the algorithm PGB introduced in the recently published paper [21] which enables us to consider the parametric defining equations in CRAUT.

2. BASIC PRELIMINARIES AND DEFINITIONS

On an arbitrary even-dimensional real vector space V, a *complex structure map* $J : V \to V$ is an $\mathbb{R}$-linear map satisfying $J \circ J = -\mathrm{Id}_V$. For example, in the simple case $V := T_p\mathbb{R}^{2N} = T_p\mathbb{C}^N,\ N \in \mathbb{N}$ with the local coordinates $(z_1 := x_1 + i\,y_1, \ldots, z_N := x_N + i\,y_N)$ and with $p \in \mathbb{C}^N$, one defines the (standard) complex structure by:

$$J\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j}, \qquad J\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}, \qquad j = 1, \ldots, N.$$

One should notice that in general, an arbitrary subspace H_p of $T_p\mathbb{C}^N$ is not invariant under the complex structure map J. Thus, one may give special designation to the largest J-invariant subspace of H_p as:

$$H_p \cap J(H_p) =: H_p^c,$$

which is called the *complex tangent subspace of* H_p. Due to the equality $J \circ J = -\mathrm{Id}$, this space is even-dimensional, too.

Similarly, one also introduces the smallest J-invariant real subspace of $T_p\mathbb{C}^N$ which *contains* H_p:

$$H_p^{i_c} := H_p + J(H_p)$$

and one calls its the *intrinsic complexification* of H_p.

As an application, consider the linear subspace $H_p := T_pM$ of $\mathbb{C}^N$, for some arbitrary connected differentiable submanifold M of $\mathbb{C}^N$. In general, it is not at all true that the complex-tangent planes:

$$T_p^cM = T_pM \cap J(T_pM)$$

have constant dimensions as p varies in M.

Definition 2.1. Let M be a real analytic submanifold of $\mathbb{C}^N$. Then M is called *Cauchy-Riemann* (CR for short), if the complex dimensions of T_p^cM are *constant* as p varies on M. Furthermore, M is called *generic* whenever:

$$T_p^{i_c}M := T_pM + J(T_pM) = T_p\mathbb{C}^N$$

for each $p \in M$, which implies that M is CR thanks to elementary linear algebra.

One knows ([14, 4, 27]) that any CR real analytic $M \subset \mathbb{C}^N$ is contained in a thin strip-like complex submanifold $\mathcal{C}' \cong \mathbb{C}^{N'}$ with $N' \leqslant N$ in which $M \subset \mathbb{C}^{N'}$ is CR-generic, hence there is no restriction to assume that M is CR-generic, as we will always do here, since we are interested in local CR geometry.

For such a CR-generic manifold M, we call the complex dimension of T_p^cM by the *CR-dimension* of M. Moreover, the subtraction of the real dimension of T_p^cM from that of T_pM, which is in fact the real dimension of the so-called *totally real part* T_pM/T_p^cM of T_pM, coincides then with the real *codimension* of $M \subset \mathbb{C}^N$.

A tangent vector field:

$$\mathsf{X} := \sum_{j=1}^{N} \bigg(a_j\,\frac{\partial}{\partial Z_j} + b_j\,\frac{\partial}{\partial \overline{Z}_j}\bigg)$$

of the complexified space $\mathbb{C}T_p\mathbb{C}^N := \mathbb{C} \otimes T_p\mathbb{C}^N$ is called of type $(1,0)$ whenever all $b_j \equiv 0$ and is called of type $(0,1)$ whenever all $a_j \equiv 0$. One denotes by $T_p^{1,0}\mathbb{C}^N$ and $T_p^{0,1}\mathbb{C}^N$ the corresponding subspaces of $\mathbb{C}T_p\mathbb{C}^N$. Accordingly, for a CR manifold M and an arbitrary point p of it, we denote:

$$T_p^{1,0}M := T_p^{1,0}\mathbb{C}^N \cap \mathbb{C}T_pM, \quad \text{and} \quad T_p^{1,0}M := T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM.$$

One easily verifies the equality $T_p^{0,1}M = \overline{T_p^{1,0}M}$. It is proved that (*see* [3], Proposition 1.2.8) the complex tangent space T_p^cM is the real part of $T_p^{1,0}M$, *i.e.* $T_p^cM = \{\mathsf{X} + \overline{\mathsf{X}} : \mathsf{X} \in T_p^{1,0}M\}$. Moreover, the complexified space $\mathbb{C}T_p^cM$ is equal to the direct sum $T_p^{1,0}M \oplus T_p^{0,1}M$.

If n and k are the CR dimension and the real codimension of a real analytic CR-generic submanifold $M \subset \mathbb{C}^N$, then of course $N = n + k$, and also M can be represented (locally) by k real analytic graphed equations:

$$\mathrm{Im}\, w_j := \Phi_j(z, \overline{z}, \mathrm{Re}\, w), \qquad {\scriptstyle (j=1,\dots,k)} \tag{4}$$

with some real-valued defining functions $\Phi_\bullet$ enjoying the *no-pluriharmonic term* condition:

$$0 \equiv \Phi_\bullet(z, 0, \mathrm{Re}\, w) = \Phi_\bullet(0, \overline{z}, \mathrm{Re}\, w).$$

Solving the above real-valued defining functions in w or in $\overline{w}$, one also reformulates the defining functions of M as complex defining equations of the kind:

$$w_j + \overline{w}_j = \overline{\Xi}_j(z, \overline{z}, w), \qquad {\scriptstyle (j=1,\dots,k)}. \tag{5}$$

The real analyticity of the functions Ξ_j enables us to employ their Taylor series for computing the associated algebras of infinitesimal CR-automorphisms (*see* the next subsection).

For a real analytic CR-generic manifold M represented by the above defining functions as (5) and for each $p \in M$, it is well-known ([33]) that the space $T_p^{(0,1)}M$ is generated by the following holomorphic vector fields, tangent to M:

$$\mathcal{L}_j := \frac{\partial}{\partial \overline{z}_j} + \sum_{l=1}^{k} \frac{\partial \overline{\Xi}_l}{\partial \overline{z}_j}(z, \overline{z}, w)\, \frac{\partial}{\partial \overline{w}_l} \qquad {\scriptstyle (j=1,\dots,n)}.$$

Definition 2.2. A CR-generic submanifold $M \subset \mathbb{C}^{n+k}$ of CR dimension n and of codimension k is called *of finite type* at a point $p \in M$ whenever the above generators $\mathcal{L}_1, \dots, \mathcal{L}_n, \overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_n$ together with all of their Lie brackets of any length span the complexified tangent space $\mathbb{C}T_pM$ at the point p.

Of course, finite-typeness at a point is an open condition. In [13], Bloom and Graham introduced an effective method to construct homogeneous CR manifolds that we now explain briefly. Consider the complex space $\mathbb{C}^{n+k}$ equipped with the variables $z_1, \dots, z_n, w_1 := u_1 + iv_1, \dots, w_k := u_k + iv_k$, assign weight 1 to the variables z_i and assign some arbitrary weights ℓ_j to each w_j for $j = 1, \dots, k$. Then a CR manifold $M \subset \mathbb{C}^{n+k}$ is called *represented in Bloom-Graham normal form* [13, 14] whenever it is defined as the graph of some real-valued functions[1]:

$$v_j = \Phi_j(z, \overline{z}, u_1, \dots, u_{j-1}) + \mathrm{o}(\ell_j) \qquad {\scriptstyle (j=1,\dots,k)}, \tag{6}$$

where each Φ_j is a weighted homogeneous polynomial of the weight ℓ_j enjoying the following two statements:

(i) there are no *pure* terms $z^\alpha u^\beta$ or $\overline{z}^\alpha u^\beta$ among the polynomials $\Phi_\bullet$ for some integers α and β;

(ii) for each $1 \leqslant j < i$ and for any nonnegative integers $\alpha_1, \dots, \alpha_j$, the polynomial Φ_i does not include any term of the form $u_1^{\alpha_1} \cdots u_j^{\alpha_j} \Phi_j$.

Every CR manifold represented in this form is of finite type (*see* [14] page 181). Bloom and Graham also showed that every CR manifold represented by the above expressions (6) can be transformed to such a normal form by means of some algebraic changes of coordinates (*see* [13], Theorem 6.2).

Definition 2.3. A real analytic CR-generic manifold $M \subset \mathbb{C}^N$ with coordinates $(Z_1, \dots, Z_N)$ is called *holomorphically nondegenerate* at $p \in M$ if there is no local *nonzero* vector field of type $(1, 0)$:

$$\mathsf{X} := \sum_{j=1}^{N} f_j(Z_1, \dots, Z_N) \frac{\partial}{\partial Z_j}$$

having coefficients f_j holomorphic in a neighborhood of p such that $\mathsf{X}|_M$ is tangent to M near p.

Every *connected* real analytic generic CR manifold is either holomorphically nondegenerate at every point or at no point ([27]).

[1] There is also a more general definition of Bloom-Graham normal form which one can find it for example in [5]

Definition 2.4. ([1, 3, 10, 28]) A *(local) infinitesimal CR-automorphism of M*, when understood extrinsically, is a local holomorphic vector field:

$$\mathsf{X} = \sum_{i=1}^{n} Z^i(z,w)\frac{\partial}{\partial z_i} + \sum_{j=1}^{k} W^j(z,w)\frac{\partial}{\partial w_j} \tag{7}$$

whose real part $\mathrm{Re}\, X = \frac{1}{2}(\mathsf{X} + \overline{\mathsf{X}})$ is tangent to M.

The collection of all infinitesimal CR-automorphisms of M constitutes a Lie algebra which is called the *Lie algebra of infinitesimal CR-automorphisms* of M and is denoted by $\mathfrak{aut}_{CR}(M)$.

The notion of holomorphically nondegeneracy was raised by Nancy Stanton in [38] where she proved that for a *hypersurface* $M \subset \mathbb{C}^{n+1}$ (always generic), $\mathfrak{aut}_{CR}(M)$ is finite-dimensional if and only of M is holomorphically nondegenerate. Amazingly enough, one realizes that the concept of tangent vector fields completely independent of $\overline{Z}_1, \ldots, \overline{Z}_N$ which points out a strong degeneracy can in fact be traced back at least to Sophus Lie's works (*cf.* pp. 13–14 of [29]). In general codimension $k \geqslant 1$, the Lie algebra $\mathfrak{aut}_{CR}(M)$ of infinitesimal CR-automorphisms of a CR-generic real analytic $M \subset \mathbb{C}^{n+k}$ is finite-dimensional if and only if M is holomorphically nondegenerate *and* of finite type ([18]).

Determining such Lie algebras $\mathfrak{aut}_{CR}(M)$ is the same as knowing the *CR-symmetries* of M, a question which lies at the heart of the (open) problem of classifying all local analytic CR manifolds up to biholomorphisms. In the groundbreaking works of Sophus Lie and his followers (Friedrich Engel, Georg Scheffers, Gerhard Kowalewski, Ugo Amaldi and others), the most fundamental question in concern was to draw up lists of possible Lie algebras $\mathfrak{aut}_{CR}(M)$ which would classify all possible M's according to their CR symmetries.

2.1. **The main idea.** Now, let us explain briefly the main strategy used in [34] for computing the Lie algebra $\mathfrak{aut}_{CR}(M)$ associated to an arbitrary real analytic generic CR manifold $M \subset \mathbb{C}^{n+k}$, represented as the graph of the k complex defining equations as (5). For this, we shall proceed to do successively the following steps:

- According to the definition, a holomorphic vector field:

$$\mathsf{X} = \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

belongs to $\mathfrak{aut}_{CR}(M)$ whenever it enjoys the tangency equations:

$$\begin{aligned}
0 \equiv (\overline{\mathsf{X}} + \mathsf{X})\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] &= \\
&= \overline{\mathsf{X}}\,\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] + \mathsf{X}\,\big[\overline{w}_j + w_j - \overline{\Xi}_j(\overline{z}, z, w)\big] \\
&= \overline{W}^j(\overline{z}, \overline{w}) - \sum_{i=1}^{n} \overline{Z}^i(\overline{z}, \overline{w})\,\frac{\partial \overline{\Xi}_j}{\partial \overline{z}_i}(\overline{z}, z, w) + \\
&\quad + W^j(z,w) - \sum_{i=1}^{n} Z^i(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial z_i}(\overline{z}, z, w) - \sum_{l=1}^{k} W^l(z,w)\,\frac{\partial \overline{\Xi}_j}{\partial w_l}(\overline{z}, z, w) \\
&\qquad (j = 1 \cdots k).
\end{aligned} \tag{8}$$

For each $j = 1, \ldots, k$, let us refer to the above equality as the *j-th tangency equation* of M. Now, the Taylor series formulaes:

$$Z^i(z,w) = \sum_{\alpha \in \mathbb{N}^n} z^\alpha\, Z^i_\alpha(w) \quad \text{and} \quad W^l(z,w) = \sum_{\alpha \in \mathbb{N}^n} z^\alpha\, W^l_\alpha(w), \tag{9}$$

bring the tangency equation into the form:

$$
\begin{aligned}
0 \equiv & \sum_{\alpha\in\mathbb{N}^n} \overline{z}^\alpha\, \overline{W}^j_\alpha\big(-w+\Xi\big) - \sum_{k=1}^n \sum_{\alpha\in\mathbb{N}^n} \overline{z}^\alpha\, \overline{Z}^k_\alpha\big(-w+\Xi\big)\, \frac{\partial\overline{\Xi}_j}{\partial\overline{z}_k}(\overline{z},z,w)+ \\
& + \sum_{\beta\in\mathbb{N}^n} z^\beta\, W^j_\beta(w) - \sum_{k=1}^n \sum_{\beta\in\mathbb{N}^n} z^\beta\, Z^k_\beta(w)\, \frac{\partial\overline{\Xi}_j}{\partial z_k}(\overline{z},z,w) - \sum_{l=1}^k \sum_{\beta\in\mathbb{N}^n} z^\beta\, W^l_\beta(w)\, \frac{\partial\overline{\Xi}_j}{\partial w_l}(\overline{z},z,w) \\
& \qquad\qquad {\scriptstyle (j=1\cdots k)}.
\end{aligned}
\tag{10}
$$

- After this, one replaces the appearing functions $\overline{W}^\bullet_\bullet(-w+\Xi)$ and $\overline{Z}^\bullet_\bullet(-w+\Xi)$ according to the following slightly artificial expansions:

$$
\begin{aligned}
\overline{A}\big(-w+\Xi\big) &= \overline{A}\big(w+(-2w+\Xi)\big) \\
&= \sum_{\gamma\in\mathbb{N}^d} \frac{\partial^{|\gamma|}\overline{A}}{\partial w_\gamma}(w)\,\frac{1}{\gamma!}\,\big(-2w+\overline{\Xi}(\overline{z},z,w)\big)^\gamma,
\end{aligned}
$$

  and one next substitutes each $-2w_j+\overline{\Xi}_j$ with:

$$
-2w_j+\overline{\Xi}_j(\overline{z},z,w) = \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n} \overline{z}^\alpha\, z^\beta\, \overline{\Xi}_{j,\alpha,\beta}(w) \qquad {\scriptstyle (j=1\cdots k)}.
$$

- Modifying the equations (10) by the two already presented formulaes, we reach k homogenous equations so that their right hand sides are some certain combinations of the (yet unknown) functions $Z^i_\alpha(w)$ and $W^l_\beta(w)$, their derivations and also the variables $w_1,\ldots,w_k$ with the coefficients of the form $z^\mu\overline{z}^\nu$ for $\mu=(\mu_1,\ldots,\mu_n)$ and $\nu=(\nu_1,\ldots,\nu_n)$ (here we set $z_\mu := z_1^{\mu_1}\ldots z_n^{\mu_n}$). To satisfy these equations, one should extract the coefficients of the various monomials $z^\mu\overline{z}^\nu$ and equate them to zero. Then, one attains a (usually lengthy and complicated) homogeneous linear system of complex partial differential equations with the unknowns $Z^i_\alpha(z,w)$ and $W^l_\alpha(z,w)$ — namely a linear system of differential polynomials of the differential algebra $R := \mathbb{C}(w,\overline{w})[Z^i_\alpha, W^l_\beta]$. The solution of this system yields the desired coefficients $Z^i(z,w)$ and $W^l(z,w)$ in (9).
- To solve this already constructed PDE system, we employ the effective tools of *differential algebra*, equipped with some additional operators such as *bar-reduction*. For more details we refer the reader to [34].

Let us conclude this section by a short discussion of the abilities and advantages of the algorithm introduced in [34] and also the difficulties and weaknesses inherent in it. This algorithm manages to compute *systematically* the desired algebras associated to *arbitrary* CR manifolds $M\subset\mathbb{C}^{n+k}$. Moreover, it employs the powerful techniques of differential algebra and the ability of computer algebra to provide a more effective method. In fact, it handles more appropriately the most complicated part of the computations, namely solving the associated PDE systems (*cf.* [10, 24, 33, 28, 36]). Specifically, for the significant class of *rigid* CR manifolds — those whose defining equations Ξ are independent of the variables w — the computations are considerabl eased up. *However*, the main obstacle we encounter is that, because of the complexity of the PDE systems, as much as the number of variables z_i and w_l increases, then the cost of the associated computation grows extensively and the implementation of the algorithm rapidly reveals limits concerning the capacity of computer systems.

3. RIGIDITY OF HOMOGENEOUS CR MANIFOLDS AND SIMILARITY IN THE EXPRESSIONS OF THE CORRESPONDING SOUGHT FUNCTIONS

Consider the complex space $\mathbb{C}^{n+k}$ equipped with n complex variables $z_1,\ldots,z_n$ of weight one, identically, and k complex variables $w_1 := u_1+iv_1,\ldots,w_k := u_k+iv_k$ of certain weights $1<[w_1]\leqslant$

$[w_2] \leqslant \ldots \leqslant [w_k]$ and define the homogeneous manifold M of CR dimension n and codimension k as:

$$M := \left\{ (z,w) : \begin{bmatrix} \Xi_1(v_1, z, \overline{z}, u) := v_1 - \Phi_1(z, \overline{z}, u) \equiv 0, \\ \Xi_2(v_2, z, \overline{z}, u) := v_2 - \Phi_2(z, \overline{z}, u) \equiv 0, \\ \vdots \\ \Xi_j(v_j, z, \overline{z}, u) := v_j - \Phi_j(z, \overline{z}, u) \equiv 0, \\ \vdots \\ \Xi_k(v_k, z, \overline{z}, u) := v_k - \Phi_k(z, \overline{z}, u) \equiv 0, \end{bmatrix} \right\}, \tag{11}$$

with the weighted homogeneous polynomials Φ_j of the certain weight $[w_j]$ for $j = 1, \ldots, k$. *Throughout this paper* we assume that these manifolds are holomorphically nondegenerate and of finite type, which guarantees the finite dimensionality of their associated algebras of infinitesimal CR-automorphisms.

3.1. **Gradation and polynomiality.** At first, let us show two significant intrinsic features of the already mentioned homogeneous CR manifolds, namely gradation (in the sense of Tanaka) and polynomiality of their associated algebras of infinitesimal CR-automorphisms.

For a CR manifold M as above, consider the holomorphic vector field:

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\partial_{w_l}$$

as an element of $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$. Since the above coefficients Z^j and W^l are all holomorphic, then one can expand them as their Taylor series and thus decompose X into its weighted homogeneous components as follows:

$$\mathsf{X} := \mathsf{X}_{-\rho} + \cdots + \mathsf{X}_{-1} + \mathsf{X}_0 + \mathsf{X}_1 + \cdots + \mathsf{X}_t + \cdots, \qquad {\scriptstyle (\rho,\, t \in \mathbb{N})}. \tag{12}$$

We need the following facts:

Lemma 3.1. *The minimum homogeneous component* $\mathsf{X}_{-\rho}$ *in the above decomposition of* X *is of the weight* $-\rho = -[w_k]$*, where* w_k *has the maximum homogeneity among the complex variables appearing in* (11).

Proof. It is just enough to observe that the tangent space of holomorphic vector fields can be generated by the standard fields $\partial_{z_j}, \partial_{w_l}$ of the certain weights $-[z_j]$ and $-[w_l]$ for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. Among these standard fields, the minimum homogeneity belongs to $\frac{\partial}{\partial w_k}$. This completes the proof. □

Lemma 3.2. *Each of the above weighted homogeneous components* X_t*,* $t \geqslant -\rho$ *is again an infinitesimal CR-automorphism, namely belongs to* $\mathfrak{g}$.

Proof. Since X is an infinitesimal CR-automorphism then we have:

$$0 \equiv \big(\mathsf{X} + \overline{\mathsf{X}}\big)|_{\Xi_j}, \qquad {\scriptstyle (j=1,\ldots,k)}.$$

Now, for each component X_t of homogeneity t and since each defining function Ξ_j is homogeneous of weight $\ell_j := [w_j]$ then one verifies that the polynomial $(\mathsf{X}_t + \overline{\mathsf{X}}_t)|_{\Xi_j}$ is either zero or a homogeneous polynomial of weight $t + \ell_j$. Hence we have:

$$0 \equiv \big(\mathsf{X} + \overline{\mathsf{X}}\big)|_{\Xi_j} = \underbrace{\big(\mathsf{X}_{-\rho} + \overline{\mathsf{X}}_{-\rho}\big)|_{\Xi_j}}_{P_{\rho,j}} + \ldots + \underbrace{\big(\mathsf{X}_t + \overline{\mathsf{X}}_t\big)|_{\Xi_j}}_{P_{t,j}} + \cdots \qquad {\scriptstyle (j=1,\ldots,k)}, \tag{13}$$

in which each polynomial function $P_{t,j}$ is either zero or a homogeneous polynomial of the weight $t + \ell_j$. Hence, we have some certain weighted homogeneous polynomials $P_\bullet$ with distinct homogeneities and consequently with linear independency. Hence, one obtains from (13) that:

$$0 \equiv P_{t,j}(z,w) = \big(\mathsf{X}_t + \overline{\mathsf{X}}_t\big)|_{\Xi_j}, \quad {\scriptstyle t \geqslant -\rho, \quad (j=1,\ldots,k)}.$$

This equivalently means that each component X_t of X is an infinitesimal CR-automorphism. □

Now, we can prove the polynomiality of the sought algebras:

Corollary 3.3. *If:*

$$\mathsf{X}_t := \sum_{j=1}^{n} Z_t^j(z,w)\partial_{z_j} + \sum_{l=1}^{k} W_t^l(z,w)\partial_{w_l}$$

is a weighted homogeneous CR-automorphism of weight $t \geqslant -\rho$ then, its coefficients $Z_t^j(z,w)$ and $W_t^l(z,w)$ are weighted homogeneous polynomials of the weights $0,\ldots,t-1$.

Proof. It is a straightforward consequence of the decomposition (12). □

Now, let us consider the gradation of $\mathfrak{g}$. First we need the following definition:

Definition 3.4. The Lie algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ of an arbitrary CR manifold is called *graded* in the sense of Tanaka, whenever it can be expressed in the form:

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \mathfrak{g}_{-\rho+1} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho, \qquad \rho, \varrho \in \mathbb{N}$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Furthermore, we say that M has *rigidity* if the positive part:

$$\mathfrak{g}_+ := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\varrho$$

of $\mathfrak{aut}_{CR}(M)$ is zero.

Proposition 3.5. *For a finite type holomorphically nondegenerate CR manifold $M \subset \mathbb{C}^{n+k}$ represented by the above defining function* (11)*, the associated Lie algebra $\mathfrak{g}$ of infinitesimal CR-automorphisms is a graded Lie algebra, in the sense of Tanaka, of the form:*

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\varrho, \quad \rho, \varrho \in \mathbb{N}.$$

Proof. According to the above two lemmas, if $\mathfrak{g}_t$ is the collection of infinitesimal CR-automorphisms of weight t, then $\mathfrak{g}$ admits a gradation like:

$$\mathfrak{g} := \mathfrak{g}_{-\alpha} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t \oplus \cdots.$$

Furthermore, holomorphically nondegeneracy and finite typness of M guarantees that $\mathfrak{g}$ is finite dimensional and hence there exists an integer ϱ in which $g_\beta \equiv 0$ for each $\beta > \varrho$. Now, it remains to show that this gradation is in the sense of Tanaka. Namely, we shall prove that for each two homogeneous infinitesimal CR-automorphisms X_1 and X_2 of certain homogeneities t_1 and t_2, the Lie bracket $[\mathsf{X}_1, \mathsf{X}_2]$ belongs to $\mathfrak{g}_{t_1+t_2}$. For this, it is enough to show this statement for vector fields of the forms $\mathsf{X}_i := F_i(z,w)\frac{\partial}{\partial x_i}, i = 1,2$ where x_i is a complex variable $z_\bullet$ or $w_\bullet$. According to the homogeneities of these vector fields, F_1 and F_2 are two polynomials of the weights $t_1 + [x_1]$ and $t_2 + [x_2]$, respectively. Now, we have:

$$[\mathsf{X}_1, \mathsf{X}_2] = F_1 \frac{\partial F_2}{\partial x_1}\frac{\partial}{\partial x_2} - F_2 \frac{\partial F_1}{\partial x_2}\frac{\partial}{\partial x_1}.$$

In this expression, the derivations $\frac{\partial F_2}{\partial x_1}$ and $\frac{\partial F_1}{\partial x_2}$ are either zero or homogeneous polynomials of the weights $t_2 + [x_2] - [x_1]$ and $t_2 + [x_1] - [x_2]$, respectively. Now, simple simplifications yields that the above Lie bracket is a homogeneous vector field of the weight $t_1 + t_2$, as desired. □

3.2. **A comparison between the so far achieved results.** For a CR manifold $M \subset \mathbb{C}^{n+k}$, of *fixed* CR-dimension n and codimension k represented in coordinates $z_1,\ldots,z_n,w_1,\ldots,w_k$ as (11), consider the infinitesimal CR-automorphism:

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\partial_{w_l}.$$

In this subsection, we have a general look at the results through literatures. In [36], Shananina computed the Lie algebras of infinitesimal CR-automorphisms associated to the Beloshapka's homogeneous models — included in the class of under consideration CR manifolds — with $n = 1$ and $k = 3,\ldots,7$ and presented the expressions of the associated holomorphic coefficients $Z(z,w)$ and $W^l(z,w)$, $l = 1,\ldots,k$ in Theorem 1 page 386 of his paper (he used the notations $f(z,w)$ and $g^l(z,w)$, instead). Upon a close scrutiny, one observes a nice *similarity* in the expressions of those computed functions. More precisely, all the expressions of $Z(z,w)$ and $W^l(z,w)$, presented in the various types $k=3$ to $k=7$ are nearly the same. For instance, in the two cases $k=3$ and $k=4$ represented by the following defining equations:

$$\left(\begin{smallmatrix} n=1 \\ k=3 \end{smallmatrix}\right) : \begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2, \\ v_3 = -i(z^2\overline{z} - z\overline{z}^2), \end{cases} \qquad \left(\begin{smallmatrix} n=1 \\ k=4 \end{smallmatrix}\right) : \begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2, \\ v_3 = -i(z^2\overline{z} - z\overline{z}^2), \\ v_4 = z^3\overline{z} + z\overline{z}^3 + \lambda\, z^2\overline{z}^2, \quad {\scriptstyle \lambda\in\mathbb{C}} \end{cases}$$

one obtains the expressions:

$$\begin{aligned}
&\boxed{n=1,\, k=3}: \\
&\qquad Z(z,w) = \mathsf{a} + i\,\mathsf{b} + (\mathsf{d} + i\,\mathsf{e})\,z, \quad W^1(z,w) = \mathsf{c}_1 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z + 2\,\mathsf{d}w_1, \\
&\qquad W^2(z,w) = \mathsf{c}_2 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}w_1 + 3\,\mathsf{d}w_2 - \mathsf{e}w_3, \\
&\qquad W^3(z,w) = \mathsf{c}_3 + 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}w_1 + \mathsf{e}w_2 + 3\,\mathsf{d}w_3, \\
&\boxed{n=1,\, k=4}: \\
&\qquad Z(z,w) = \mathsf{a} + i\,\mathsf{b} + \mathsf{d}z, \quad W^1(z,w) = \mathsf{c}_1 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z + 2\,\mathsf{d}w_1, \\
&\qquad W^2(z,w) = \mathsf{c}_2 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}w_1 + 3\,\mathsf{d}w_2, \\
&\qquad W^3(z,w) = \mathsf{c}_3 + 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}w_1 + 3\,\mathsf{d}w_3, \\
&\qquad W^4(z,w) = \mathsf{c}_4 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z^3 + (2\lambda + 3)\,\mathsf{a}w_2 + 2\,(\lambda - 3)\,\mathsf{b}w_3 + 4\,\mathsf{d}w_4, \\
&\qquad\qquad {\scriptstyle (\mathsf{a},\mathsf{b},\mathsf{c}_i,\mathsf{d},\mathsf{e}\in\mathbb{R})}.
\end{aligned} \tag{14}$$

A plain comparison between the above two sets of the functions Z, W^1, W^2, W^3 can demonstrate the mentioned similarity. For example, even if a function is not equal to its correspondence, the degrees of each term in both of them and the variables which appear inside are same. This type of similarity and harmony is visible among all the expressions obtained in [36] for $n=1$ and $k=3,\ldots,7$. Moreover, a glance on the Beloshapka's results for his model with $n=1$, $k=2$ in [10], page 402 (*see* the expressions of f, g and h which are Z, W^1 and W^2 in our notations) indicates such similarity, too. In contrast, the computations of the infinitesimal CR-automorphisms associated to the CR manifold with $n, k = 1$, presented in [28], says that the expressions of the two functions $Z(z,w)$ and $W^1(z,w)$ are fairly different. Namely we have:

$$\begin{aligned}
Z(z,w_1) &= \mathsf{a} + i\mathsf{b} + (\mathsf{c} + i\mathsf{d})\,w_1 + (\tfrac{1}{2}\,\mathsf{f} + i\,\mathsf{e} + \mathsf{h}\,w_1)\,z + (2\,\mathsf{d} + 2\,i\,\mathsf{c})\,z^2, \\
W^1(z,w_1) &= \mathsf{f} + \mathsf{g}\,w_1 + \mathsf{h}\,w_1^2 + \big(2\,i\,(\mathsf{a} - i\mathsf{b}) + 2\,i\,(\mathsf{c} - i\mathsf{d})\,w_1\big)\,z,
\end{aligned}$$

for some eight real integers $\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots, \mathsf{h}$. In more details, in the above expression of $Z(z,w_1)$ we see the variable w_1 while the functions $Z(z,w)$ in the next types are independent of the variables w_i. Moreover, the degree of the variable z in $Z(z,w)$ in all already presented cases $n=1$ and $k=2,\ldots,7$ is 1 while in this case it is of degree 2 (compare the above expression of $Z(z,w_1)$ with (14)).

On the other hand — and *parallel* to the similarity of the obtained holomorphic functions — we know that all the mentioned CR manifolds with $n=1$ and $k=2,\ldots,7$ have *rigidity* while for $n, k = 1$ it does not retain this property [24].

We *claim* that the similarity in the expressions of the functions $Z^i(z,w)$ and $W^l(z,w)$ associated to the CR manifolds of CR-dimension and codimensions $n=1$ and $k=2,\ldots,7$ has a close relationship to their rigidity property. Before considering this assertion, let us illustrate by the following example that how one can extract the infinitesimal CR-automorphisms X from the obtained expressions of $Z(z,w)$ and $W^l(z,w)$.

Example 3.6. According to [33] (*see* also the case $K=3$, [36], Theorem 1 for another presentation), an infinitesimal CR-automorphism for the CR manifold $M \subset \mathbb{C}^{1+3}$, represented in coordinates

$\mathbb{C}\{z, w_1, w_2, w_3\}$ as the graph of the defining functions:

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z} \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}) \\ w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}) \end{cases}$$

is a holomorphic vector field:

$$\mathsf{X} := Z(z,w)\,\partial_z + \sum_{l=1}^{3} W^l(z,w)\,\partial_{w_l},$$

with the desired coefficients (*cf.* (14))

$$\begin{aligned} Z(z,w) &= \mathsf{a} + i\,\mathsf{b} + (\mathsf{d} + i\,\mathsf{e})\,z, \\ W^1(z,w) &= \mathsf{c}_1 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z + 2\,\mathsf{d}\,w_1, \\ W^2(z,w) &= \mathsf{c}_2 + 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}\,w_1 + 3\,\mathsf{d}\,w_2 - \mathsf{e}\,w_3, \\ W^3(z,w) &= \mathsf{c}_3 + 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}\,w_1 + \mathsf{e}\,w_2 + 3\,\mathsf{d}\,w_3, \end{aligned}$$

for some seven real integers $\mathsf{a}, \mathsf{b}, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3, \mathsf{d}, \mathsf{e}$. Throughout this paper, we call such integers by the *free parameters*. Then, the Lie algebra $\mathfrak{aut}_{CR}(M)$ is seven dimensional, with the basis elements extracted as the coefficients of the seven free parameters in the above general form of X. Namely, it is generated by the following seven holomorphic vector fields:

$$\begin{aligned} \boxed{\mathsf{a}}:&\quad \mathsf{X}_1 := \partial_z + 2\,i\,z\partial_{w_1} + (2\,i\,z^2 + 4\,w_1)\,\partial_{w_2} + 2\,z^2\,\partial_{w_3}, \\ \boxed{\mathsf{b}}:&\quad \mathsf{X}_2 := i\,\partial_z + 2\,z\,\partial_{w_1} + 2\,z^2\,\partial_{w_2} + (-2\,i\,z^2 + 4\,w_1)\,\partial_{w_3}, \\ \boxed{\mathsf{c}_1}:&\quad \mathsf{X}_3 := \partial_{w_1}, \\ \boxed{\mathsf{c}_2}:&\quad \mathsf{X}_4 := \partial_{w_2}, \\ \boxed{\mathsf{c}_3}:&\quad \mathsf{X}_5 := \partial_{w_3}, \\ \boxed{\mathsf{d}}:&\quad \mathsf{X}_6 := z\,\partial_z + 2\,w_1\,\partial_{w_1} + 3\,w_2\,\partial_{w_2} + 3\,w_3\,\partial_{w_3}, \\ \boxed{\mathsf{e}}:&\quad \mathsf{X}_7 := i\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}. \end{aligned} \tag{15}$$

The weights associated in [36] to the appearing complex variables are:

$$[z] = 1, \quad [w_1] = 2, \quad [w_2] = [w_3] = 3.$$

Hence, a close look at the obtained basis holomorphic vector fields $\mathsf{X}_1, \ldots, \mathsf{X}_7$ gives the following weighted homogeneities for them:

X	X_1	X_2	X_3	X_4	X_5	X_6	X_7
Hom	-1	-1	-2	-3	-3	0	0

Therefore, the Lie algebra $\mathfrak{aut}_{CR}(M)$ can be represented as:

$$\mathfrak{aut}_{CR}(M) := \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

with $\mathfrak{g}_{-3} := \langle \mathsf{X}_4, \mathsf{X}_5 \rangle$, with $\mathfrak{g}_{-2} := \langle \mathsf{X}_3 \rangle$, with $\mathfrak{g}_{-1} := \langle \mathsf{X}_1, \mathsf{X}_2 \rangle$ and with $\mathfrak{g}_0 := \langle \mathsf{X}_6, \mathsf{X}_7 \rangle$. Now, one easily verifies that M is rigid.

3.3. **Homogeneous components.** Now, let us inspect the structure of homogeneous components of the desired algebras of infinitesimal CR-automorphisms. At first, the following proposition demonstrates the influence of the rigidity of the under consideration homogeneous CR manifold M on the structure. It also reveals the close connection between the similarity property of the holomorphic vector fields Z^i and W^l of the CR manifolds discussed in the last subsection and their rigidity.

Proposition 3.7. *Let M be a holomorphically nondegenerate CR manifold of CR-dimension n and codimension k represented as* (11). *Then, M has rigidity if and only if for any weighted homogeneous infinitesimal CR-automorphism:*

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

of M, each weighted homogeneous polynomial $Z^j(z,w)$ (respectively $W^l(z,w)$) is of weight at most $[z_i]$ (respectively of weight at most $[w_l]$). In particular, $Z^j(z,w)$ (respectively $W^l(z,w)$) is independent of the variables of the weights $\gneqq [z_j]$ (respectively of the weights $\gneqq [w_l]$).

Proof. If X is of weight homogeneity d, then since the standard fields ∂_{z_j} and ∂_{w_l} have the constant weights $-[z_i]$ and $-[w_l]$, respectively, then the weighted homogeneous polynomials Z^j and W^l have the constant degrees $d+[z_i]$ and $d+[w_l]$, respectively. Now, assume that M has rigidity. The dimension of the Lie algebra $\mathfrak{aut}_{CR}(M)$ is equal to the number of free parameters involved in the expressions of the functions $Z^j(z,w), i = 1,\ldots,n$ and $W^l(z,w), l = 1,\ldots,k$ (*see* Example 3.6). Each generator is extracted from one of such free parameters as its coefficient in the general form $\mathsf{X} := \sum_{j=1}^n Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^k W^l(z,w)\,\partial_{w_l}$. In such expression, coefficients of the standard fields ∂_{z_j} come from the found functions $Z^j(z,w)$. Now, rigidity of M means to have no any (homogeneous) basis element belonging to $\mathfrak{aut}_{CR}(M)$ of the positive weighted homogeneity. Hence, when we have standard field ∂_{z_j} of the weight $-[z_j]$, then no term of weight bigger than $[z_j]$ appears in its coefficient. Consequently, $Z^j(z,w)$ — which provides the coefficients of ∂_{z_j} in the basis elements — is independent of the variables of the weights bigger than $[z_j]$. Similar fact holds when one considers the coefficients $W^l(z,w)$ of the standard fields ∂_{w_l}.
For the converse, if none of the (weighted homogeneous) holomorphic coefficients $Z^j(z,w), j = 1,\ldots,n$ admits the terms of weight larger than $[z_j]$, then the weight $[Z^j(z,w)]-[z_j]$ of each term $Z^j(z,w)\partial_{z_j}$ of X is non-positive. Similar fact holds for the terms $W^l(z,w)\partial_{w_l}, l = 1,\ldots,k$. Consequently, $\mathfrak{aut}_{CR}(M)$ does not contain any (weighted homogeneous) basis element X of the positive weight. In other words, M has rigidity. □

For a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho,$$

let us denote by $\mathfrak{g}^{(t)}$ the graded subspace:

$$\mathfrak{g}^{(t)} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_t, \qquad {\scriptstyle (t=-\rho,\ldots,\varrho)}.$$

According to definition of the graded algebras, one easily convinces oneself that for $t=-\rho,\ldots,0$ each subspace $\mathfrak{g}^{(t)}$ is in fact a Lie subalgebra of $\mathfrak{g}$. The idea behind the proof of Proposition 3.7 can lead one to obtain the following more general conclusion.

Proposition 3.8. *Let M be a homogeneous CR manifold of CR-dimension n and codimension k represented as* (11). *Let $\mathfrak{g} = \mathfrak{aut}_{CR}(M)$ be of the graded form:*

$$\mathfrak{g} = \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\varrho \tag{16}$$

and let the (weighted homogeneous) infinitesimal CR-automorphism:

$$\mathsf{X} = \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

belongs to $\mathfrak{g}$. Then,

- (i) X *belongs to $\mathfrak{g}_t$ for $t=-\rho,\ldots,\varrho$, if and only if each coefficient $Z^j(z,w)$ (respectively $W^l(z,w)$) is homogeneous of the precise weight $[z_j]+t$ (respectively $[w_l]+t$). In particular, each $Z^j(z,w)$ (respectively $W^l(z,w)$) is independent of the variables of weights $\gneqq [z_j]+t$ (respectively of weights $\gneqq [w_l]+t$).*

(ii) X *belongs to* $\mathfrak{g}^{(t)}$ *if and only if each coefficient* $Z^j(z,w)$ *(respectively* $W^l(z,w)$*) is homogeneous of weight at most* $[z_j]+t$ *(respectively of weights at most* $[w_l]+t$*). Specifically, each function* $Z^j(z,w)$ *(respectively,* $W^l(z,w)$*) is independent of the variables of weights* $\gneqq [z_j]+t$*) (respectively of weights* $\gneqq [w_l]+t$*)*

(iii) *In particular, the negative part* $\mathfrak{g}_- = \mathfrak{g}^{(-1)}$ *of the CR manifold* M *is generated by the elements* X *of* $\mathfrak{g}$ *with the coefficients* $Z^j(z,w)$ *(respectively* $W^l(z,w)$*) independent of the variables of weights* $\gneqq [z_j]-1$ *(respectively of weights* $\gneqq [w_l]-1$*).*

Proof. The proof is similar to that of Proposition 3.7. Here, we prove the first item (i). If a holomorphic vector field X belongs to $\mathfrak{g}_t$, then it is homogeneous of the weight t. In the expression of X, each standard field ∂_{z_j} is of the fixed weight $-[z_j]$ and hence having the field $Z^j(z,w)\,\partial_{z_j}$ of the precise homogeneity t, then the coefficient $Z^j(z,w)$ must be of the weigh $[z_j]+t$. Similar conclusion holds for the functions $W^l(z,w)$. The converse of the assertion can be concluded in a very similar way. In particular, since all the variables in $Z^j(z,w)$ have the positive weight then, each function $Z^j(z,w)$ — which must be of the weight $[z_j]+t$ — is independent of the variables of the weights bigger than $[z_j]+t$. □

Remark 3.9. This proposition suggests an appropriate way to compute each subspace $\mathfrak{g}^{(t)}$ (specifically, to compute $\mathfrak{g}^{(0)} = \mathfrak{g}_- \oplus \mathfrak{g}_0$ in the rigid case). Namely, it states that to find each $\mathfrak{g}^{(t)}$ it is not necessary to compute the surrounding algebra $\mathfrak{g} = \mathfrak{aut}_{CR}(M)$ of infinitesimal CR-automorphisms, but it is sufficient to compute just the (homogeneous) coefficients $Z^j(z,w)$ and $W^l(z,w)$ of homogeneities less than $[z_j]+t$ and $[w_l]+t$, respectively. This extremely reduces the size of computation. Similar process can be employed when we aim to compute only the t-th component $\mathfrak{g}_t$ of $\mathfrak{g}$. Moreover, one can divide the general computation of the Lie algebra $\mathfrak{g}$ of infinitesimal CR-automorphisms into some distinct sub-computations of its components $\mathfrak{g}_t$ for $t = -\rho,\ldots,\varrho$. In the next section, we will use the results to provide an algorithm for computing $\mathfrak{aut}_{CR}(M)$.

4. Computing the homogeneous components

As mentioned in Remark 3.9, to compute the holomorphic coefficients $Z^j(z,w)$ and $W^l(z,w)$ of the vector fields $\mathsf{X} \in \mathfrak{g}^{(t)}$, one can assume these functions independent of the variables with the associated weights larger than $[z_j]+t$ and $[w_l]+t$, respectively. In this section, we aim to develop this result for constructing a very convenient method of computing each subspace $\mathfrak{g}^{(t)}$ and each component $\mathfrak{g}_t$ associated to our homogeneous CR manifold.

4.1. **Computing each component $\mathfrak{g}_t$.** For each element X of the t-th component $\mathfrak{g}_t$, Proposition 3.8 enables one to attain an upper bound for the weight degree of each of its desired (polynomial) coefficients $Z^j(z,w)$ and $W^l(z,w)$. Hence, we can predict the expression of these polynomials as the elements of the polynomial ring $\mathbb{C}[z,w]$. Then finding these expressions, it is necessary and sufficient to seek their constant coefficients. One can pick the following convenient strategy for computing the t-th component $\mathfrak{g}_t$ of the desired algebra $\mathfrak{g}$ as (16), for a fixed integer $t = -\rho,\ldots,\varrho$.

(s1) First, we construct the tangency equations (8) of M corresponding to (in general form) holomorphic vector fields:

$$\mathsf{X} = \sum_{j=1}^{n} Z^j(z,w)\,\partial_{z_j} + \sum_{l=1}^{k} W^l(z,w)\,\partial_{w_l}$$

of $\mathfrak{g}_t \subset \mathfrak{g}$.

(s2) Now, to compute the coefficients $Z^j(z,w)$ and $W^l(z,w)$, it is no longer necessary to use the Taylor series (9) and construct and solve the arising PDE systems as is the classical method of [10, 24, 28, 34, 36]. Here there is another, entirely different and much simpler way to proceed the computation. Namely by Corollary 3.3, all desired functions $Z^j(z,w)$ and $W^l(z,w)$ are polynomials with bounded degrees. Then, according to Proposition 3.8(i), the desired coefficients

$Z^j(z,w)$ and $W^l(z,w)$ are weighted homogeneous polynomials of the precise weights $[z_j]+t$ and $[w_l]+t$, respectively. Hence, we can assume the following expressions for them:

$$\boxed{\begin{aligned} Z^j(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]=[z_j]+t}} \mathsf{c}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \\ W^l(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]=[w_l]+t}} \mathsf{d}_{\alpha,\beta}\,.\,z^\alpha\,w^\beta, \qquad {\scriptstyle (i=1,\ldots,n, \quad l=1,\ldots,k)}, \end{aligned}} \tag{17}$$

for some (unknown yet) complex *free parameters* $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$. Here, by z^α we mean $z_1^{\alpha_1}\,z_2^{\alpha_2}\ldots z_n^{\alpha_n}$ for $\alpha:=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$. Furthermore, we have:

$$[z^\alpha]=\sum_{j=1}^n[z_j^{\alpha_j}]=\sum_{j=1}^n\alpha_j\,[z_j].$$

Similar notations hold for w^β and $[w^\beta]$.

(s3) Determining the parameters $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$ is in fact equivalent to find the explicit expressions of the CR-automorphisms X of $\mathfrak{g}_t$. For this, we should just put the already assumed expressions (17) in the k tangency equations (8) and next solve the extracted (*not* PDE) homogeneous linear system of equations with the unknowns $\mathsf{c}_{\alpha,\beta}$ and $\mathsf{d}_{\alpha,\beta}$, and with the equations obtained as the coefficients of the various powers $z^\alpha w^\beta$ of the variables z and w.

Let us denote by $\mathsf{Sys}^{t,j}$ the system of equations, mentioned in the step (**s3**), associated to a CR manifold M of CR dimension and codimension n and k, extracted from the j-th tangency equation for $j=1,\ldots,k$ along the way of computing the t-th component $\mathfrak{g}_t$ of $\mathfrak{g}=\mathfrak{aut}_{CR}(M)$. Furthermore, we denote by Sys^t the general system of equations:

$$\mathsf{Sys}^t:=\bigcup_{j=1}^k\mathsf{Sys}^{t,j}.$$

Definition 4.1. A graded Lie algebra of the form:

$$\mathfrak{g}_-:=\mathfrak{g}_{-\rho}\oplus\cdots\oplus\mathfrak{g}_{-1},\qquad\rho\in\mathbb{N}$$

is called *fundamental* whenever it can be generated by $\mathfrak{g}_{-1}$.

In the case that the negative part $\mathfrak{g}_-$ of the desired algebra $\mathfrak{g}$ is fundamental, it is even not necessary to compute the homogeneous components $\mathfrak{g}_{-\rho},\ldots,\mathfrak{g}_{-2}$. In this case, one can easily compute the $(-t)$-th components $\mathfrak{g}_{-t}$ inductively as the length t iterated Lie brackets $\mathfrak{g}_{-t}=[\mathfrak{g}_{-1},\mathfrak{g}_{-t+1}]$. There are many situations where this occurs. For example, all Beloshapka's homogeneous models enjoy it.

Proposition 4.2. *(see* [7]*, Proposition 4). For a Beloshapka's homogeneous CR manifold M as those of* [7]*, the associated Levi-Tanaka algebra is the negative part $\mathfrak{g}_-$ of its Lie algebra of infinitesimal CR-automorphisms. Moreover, each component $\mathfrak{g}_{-m}$, $m\in\mathbb{N}$, is a linear combination of brackets of degree m of vector fields in $\mathfrak{g}_{-1}$.*

Then, in the case that $\mathfrak{g}_-$ is fundamental, one plainly can add the following item to the already presented strategy **(s1)-(s3)**.

(ib) After computing the basis elements of $\mathfrak{g}_{-1}$, to achieve each component $\mathfrak{g}_{-m}$, $m=2,\ldots,\rho$ one should just compute all the length m iterated brackets like:

$$[\mathsf{x}_{i_1},[\mathsf{x}_{i_2},[\mathsf{x}_{i_3},\ldots,[\mathsf{x}_{i_m}]]]\ldots],\qquad {\scriptstyle i_1<i_2<\cdots<i_m}$$

of the basis elements $\mathsf{x}_\bullet$ of $\mathfrak{g}_{-1}$.

Example 4.3. (*compare with Example* 3.6). Consider the CR-submanifold $M \subset \mathbb{C}^{1+3}$ of CR-dimension $n = 1$ and codimension $k = 3$ as in Example 3.6. Here, we have $[z] = 1, [w_1] = 2$ and $[w_2] = [w_3] = 3$. According to (8), the three fundamental tangency equations are:

$$
\begin{aligned}
(18) \qquad & 0 \equiv \big[W^1 - \overline{W}^1 - 2i\overline{z}Z - 2iz\overline{Z}\big]_{w=\overline{w}+\Xi(z,\overline{z},\overline{w})}, \\
& 0 \equiv \big[W^2 - \overline{W}^2 - 4iz\overline{z}Z - 2i\overline{z}^2Z - 2iz^2\overline{Z} - 4iz\overline{z}\overline{Z}\big]_{w=\overline{w}+\Xi(z,\overline{z},\overline{w})}, \\
& 0 \equiv \big[W^3 - \overline{W}^3 - 4z\overline{z}Z - 2z^2\overline{Z} + 2\overline{z}^2Z + 4z\overline{z}\overline{Z}\big]_{w=\overline{w}+\Xi(z,\overline{z},\overline{w})}.
\end{aligned}
$$

First let us compute the subalgebra $\mathfrak{g}_{-1}$. For this aim, we may set the following expressions for the unknown functions $Z(z, w_1, w_2, w_3), W^l(z, w_1, w_2, w_3)$ for $l = 1, 2, 3$ with their homogeneities at their left hand sides (*cf.* (17)):

$$
\begin{aligned}
\boxed{1+(-1)} &: \quad Z(z,w) := \mathsf{p}, \\
\boxed{2+(-1)} &: \quad W^1(z,w) := \mathsf{q}\, z, \\
\boxed{3+(-1)} &: \quad W^2(z,w) := \mathsf{r}\, w_1 + \mathsf{s}\, z^2, \\
\boxed{3+(-1)} &: \quad W^3(z,w) := \mathsf{t}\, w_1 + \mathsf{u}\, z^2,
\end{aligned}
$$

for some six complex functions $\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{u}$. Putting these expressions into the tangency equations and equating to zero the coefficients of the appeared polynomials of $\mathbb{C}[z, w_1, w_2, w_3]$, we obtain the following three systems of linear homogeneous equations:

$$
\begin{aligned}
\mathsf{Sys}^{-1,1} &= \Big\{ -2\,i\,\overline{\mathsf{p}} + \mathsf{q} = 0, \quad -2\,i\,\mathsf{p} - \overline{\mathsf{q}} = 0 \Big\}, \\
\mathsf{Sys}^{-1,2} &= \Big\{ \mathsf{s} - 2\,i\,\overline{\mathsf{p}} = 0, \quad -2\,\mathsf{p} - 2\,\overline{\mathsf{p}} + \mathsf{r} = 0, \quad 2\,\mathsf{p} + 2\,\overline{\mathsf{p}} - \mathsf{r} = 0 \Big\}, \\
\mathsf{Sys}^{-1,3} &= \Big\{ \mathsf{u} - 2\,\overline{\mathsf{p}} = 0, \quad -2\,i\,\overline{\mathsf{p}} + 2\,i\,\mathsf{p} + \mathsf{t} = 0 \Big\}.
\end{aligned}
$$

Solving the homogeneous linear system Sys^{-1} of all the above equations, we can write:

$$
\mathsf{p} := \mathsf{a} + i\,\mathsf{b}, \quad \mathsf{q} = \mathsf{s} := 2(\mathsf{b} + i\,\mathsf{a}), \quad \mathsf{r} := 4\,\mathsf{a}, \quad \mathsf{t} := 4\,\mathsf{b}, \quad \mathsf{u} := 2(\mathsf{a} - i\,\mathsf{b}),
$$

for two real constants a and b which brings the following general expressions for the desired holomorphic coefficients of the elements $\mathsf{X} \in \mathfrak{g}_{-1}$ (*compare with* Example: 3.6)

$$
\begin{aligned}
Z(z,w) &= \mathsf{a} + i\,\mathsf{b}, \\
W^1(z,w) &= 2\,(\mathsf{b} + i\,\mathsf{a})\,z, \\
W^2(z,w) &= 2\,(\mathsf{b} + i\,\mathsf{a})\,z^2 + 4\,\mathsf{a}\,w_1, \\
W^3(z,w) &= 2\,(\mathsf{a} - i\,\mathsf{b})\,z^2 + 4\,\mathsf{b}\,w_1.
\end{aligned}
$$

Thanks to the two free parameters a, b appeared in the above expressions, we will have two infinitesimal CR-automorphisms as the generators of $\mathfrak{g}_{-1}$:

$$
\begin{aligned}
\mathsf{X}_1 &:= \partial_z + 2\,i\,z\partial_{w_1} + (2\,i\,z^2 + +4\,w_1)\,\partial_{w_2} + 2\,z^2\,\partial_{w_3}, \\
\mathsf{X}_2 &:= i\,\partial_z + 2\,z\,\partial_{w_1} + 2\,z^2\,\partial_{w_2} + (-2\,i\,z^2 + 4\,w_1)\,\partial_{w_3}.
\end{aligned}
$$

Now, we can follow the step (**ib**) to seek the elements of the homogeneous components $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-3}$ by computing the iterated Lie brackets of X_1 and X_2 (here notice that M is one of the Beloshapka's homogeneous models and hence $\mathfrak{g}$ is fundamental. Moreover, since the maximum homogeneity of the appearing variables is 3 then, the minimum homogeneity in $\mathfrak{g}$ is -3). For $\mathfrak{g}_{-2}$ we have only one generator:

$$
\mathsf{X}_3 := [\mathsf{X}_1, \mathsf{X}_2] = 4\,\partial_{w_1}.
$$

Then, $\mathfrak{g}_{-3}$ includes two basis elements

$$
\begin{aligned}
\mathsf{X}_4 &:= [\mathsf{X}_1, \mathsf{X}_3] = -4\,\partial_{w_2}, \\
\mathsf{X}_5 &:= [\mathsf{X}_2, \mathsf{X}_3] = -4\,\partial_{w_3}.
\end{aligned}
$$

At present we found $5 = 2 \times 1 + 3$ basis elements for the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Similarly, one achieves the zeroth component $\mathfrak{g}_0$ of $\mathfrak{g}$. For this, we may assume the following expressions for the functions $Z(z, w_1, w_2, w_3), W^l(z, w_1, w_2, w_3)$ for $l = 1, 2, 3$ with their homogeneities at their left hand sides:

$$\begin{aligned}
\boxed{1+0}: &\quad Z(z,w) := \mathsf{p}_1\, z, \\
\boxed{2+0}: &\quad W^1(z,w) := \mathsf{q}_1\, w_1 + \mathsf{q}_2\, z^2, \\
\boxed{3+0}: &\quad W^2(z,w) := \mathsf{r}_1\, w_2 + \mathsf{r}_2\, w_3 + \mathsf{r}_3\, z^3 + \mathsf{r}_4\, z w_1, \\
\boxed{3+0}: &\quad W^3(z,w) := \mathsf{s}_1\, w_2 + \mathsf{s}_2\, w_3 + \mathsf{s}_3\, z^3 + \mathsf{s}_4\, z w_1.
\end{aligned}$$

Again, substituting the recent expressions in the tangency equations (18) and equating to zero the coefficients of the appeared polynomials of $\mathbb{C}[z, w_1, w_2, w_3]$, we get the following systems of equations:

$$\begin{aligned}
\mathsf{Sys}^{0,1} &= \Big\{ \mathsf{q}_2 = 0, \quad -\mathsf{p}_1 - \overline{\mathsf{p}_1} + \mathsf{q}_1 = 0 \Big\}, \\
\mathsf{Sys}^{0,2} &= \Big\{ \mathsf{r}_3 = 0, \quad -\mathsf{p}_1 + \overline{\mathsf{p}_1} + \mathsf{r}_4 - \overline{\mathsf{r}_4} = 0, \quad \overline{\mathsf{r}_4} = 0, \quad \mathsf{r}_1 - \mathsf{p}_1 - 2\,\overline{\mathsf{p}_1} + \overline{\mathsf{r}_4} = 0, \quad \mathsf{r}_2 = 0 \Big\}, \\
\mathsf{Sys}^{0,3} &= \Big\{ \mathsf{s}_3 = 0, \quad \mathsf{s}_4 - \overline{\mathsf{s}_4} = 0, \quad \overline{\mathsf{s}_4} = 0, \quad \overline{\mathsf{s}_4} + \mathsf{s}_1 + \frac{i}{2}\,\mathsf{p}_1 - \frac{i}{2}\,\overline{\mathsf{p}_1} = 0, \quad -\overline{\mathsf{s}_4} - \overline{\mathsf{s}_1} - \frac{i}{2}\,\mathsf{p}_1 + \frac{i}{2}\,\overline{\mathsf{p}_1} = 0, \\
&\qquad -\frac{3}{2}\,\mathsf{p}_1 - \frac{3}{2}\,\overline{\mathsf{p}_1} + \mathsf{s}_2 = 0 \Big\}.
\end{aligned}$$

Solving the linear homogeneous system Sys^0 of all the above equations, we have the solutions:

$$\begin{aligned}
&\mathsf{p}_1 = \mathsf{d} + i\mathsf{e}, \quad \mathsf{q}_1 = 2\,\mathsf{d}, \quad \mathsf{r}_1 = \mathsf{s}_2 = 3\,\mathsf{d}, \quad \mathsf{r}_2 = -\mathsf{e}, \quad \mathsf{s}_1 = \mathsf{e}, \\
&\mathsf{q}_2 = \mathsf{r}_3 = \mathsf{r}_4 = \mathsf{s}_3 = 0,
\end{aligned}$$

which implies the following expressions for the desired functions:

$$\begin{aligned}
Z(z,w) &= (\mathsf{d} + i\,\mathsf{e})\, z, \\
W^1(z,w) &= 2\,\mathsf{d}\, w_1, \\
W^2(z,w) &= 3\,\mathsf{d}\, w_2 - \mathsf{e}\, w_3, \\
W^3(z,w) &= \mathsf{e}\, w_2 + 3\,\mathsf{d}\, w_3,
\end{aligned}$$

where d and e are two real constants. Extracting the coefficients of these two integers brings the following two tangent vector fields, belonging to $\mathfrak{g}_0$:

$$\begin{aligned}
\mathsf{X}_6 &:= z\,\partial_z + 2\,w_1\,\partial_{w_1} + 3\,w_2\,\partial_{w_2} + 3\,w_3\,\partial_{w_3}, \\
\mathsf{X}_7 &:= i\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned}$$

Remark 4.4. It is worth noting that for computing each subspace $\mathfrak{g}^{(t)}$, although one can achieve its basis elements by computing the corresponding components $\mathfrak{g}_s$, $t = -\rho, \ldots, t$, it is also possible to adopt the above strategy **(s1)-(s3)** by modifying the assumed expressions of the functions $Z^j(z,w)$ and $W^l(z,w)$ as follows (*cf.* Proposition 3.8(ii)):

$$\begin{aligned}
Z^j(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]\leqslant[z_j]+t}} \mathsf{c}_{\alpha,\beta}\,.\, z^\alpha\, w^\beta, \\
W^l(z,w) &:= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\in\mathbb{N}^k \\ [z^\alpha]+[w^\beta]\leqslant[w_l]+t}} \mathsf{d}_{\alpha,\beta}\,.\, z^\alpha\, w^\beta, \qquad {\scriptstyle (i=1,\ldots,n, \quad l=1,\ldots,k)}.
\end{aligned} \tag{19}$$

In particular when M has rigidity, we can obtain the sought algebra $\mathfrak{g}$ by setting $t = 0$.

4.2. **Finding the maximum homogeneity ϱ.** For an arbitrary homogeneous CR manifold, represented as (2), so far we have provided an effective way to compute homogeneous components $\mathfrak{g}_t$ of the graded desired algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ of the form:

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\varrho.$$

We also find that the value of ρ in this gradation is equal to the maximum weight $[w_k]$ appearing among the complex variables. The only not-yet-fixed problem is to answer how much we have to compute the homogeneous components $\mathfrak{g}_t$ to arrive at the last one $\mathfrak{g}_\varrho$. Here, we do not aim to find the precise value of ϱ but — in an algorithmic point of view — it suffices to find a criterion to stop the computations. For this aim, we employ the transitivity of the Lie algebra $\mathfrak{g}$. For every homogeneous CR manifold M, the associated Lie algebra of its infinitesimal CR-automorphisms is transitive:

Definition 4.5. A graded Lie algebra $\mathfrak{g}$ as above is called *transitive* whenever for each element $\mathsf{x} \in \mathfrak{g}_{\mathsf{t}}$ with $t \geqslant 0$, the equality $[\mathsf{x}, \mathfrak{g}_-] = 0$ implies that $\mathsf{x} = 0$. In the case that $\mathfrak{g}_-$ is fundamental then, the transitivity means that for any x as above, the equality $[\mathsf{x}, \mathfrak{g}_{-1}] = 0$ implies that $\mathsf{x} = 0$.

Proposition 4.6. *Consider a transitive graded algebra $\mathfrak{g}$ as above. For each integer $t \geqslant 0$, if $\mathfrak{g}_{\mathsf{t}} = \mathfrak{g}_{t+1} = \cdots = \mathfrak{g}_{t+\rho-1} \equiv 0$ then we have $\mathfrak{g}_{t+\rho} = 0$. Moreover, if $\mathfrak{g}$ is also fundamental then the equality $\mathfrak{g}_t = 0$ implies independently that $\mathfrak{g}_{t+1} = 0$.*

Proof. Assume the following gradation for the transitive algebra $\mathfrak{g}$:

$$\mathfrak{g} := \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{t-1} \oplus \underbrace{0}_{\mathfrak{g}_t} \oplus \underbrace{0}_{\mathfrak{g}_{t+1}} \oplus \ldots \oplus \underbrace{0}_{\mathfrak{g}_{t+\rho-1}} \oplus \mathfrak{g}_{t+\rho} \oplus \ldots$$

and let $\mathsf{x} \in \mathfrak{g}_{t+\rho}$. According to the inequality $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, we have:

$$\begin{aligned}
[\mathsf{x}, \mathfrak{g}_{-1}] &\subset \mathfrak{g}_{t+\rho-1} = 0, \\
[\mathsf{x}, \mathfrak{g}_{-2}] &\subset \mathfrak{g}_{t+\rho-2} = 0, \\
&\vdots \\
[\mathsf{x}, \mathfrak{g}_{-\rho}] &\subset \mathfrak{g}_{t+\rho-\rho} = 0,
\end{aligned}$$

which implies that $[\mathsf{x}, \mathfrak{g}_-] = 0$. Now, the transitivity of $\mathfrak{g}$ immediately implies that $\mathsf{x} = 0$. For the second part of the assertion, similarly assume the following gradation for $\mathfrak{g}$:

$$\mathfrak{g} := \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{t-1} \oplus \underbrace{0}_{\mathfrak{g}_t} \oplus \mathfrak{g}_{t+1} \oplus \ldots,$$

and let $\mathsf{x} \in \mathfrak{g}_{\mathsf{t}+1}$. Consequently we have:

$$[\mathsf{x}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{t+1-1} = \mathfrak{g}_t = 0.$$

Again, the definition of fundamental transitive algebras immediately implies that $\mathsf{x} = 0$. This completes the proof. □

Accordingly, for a homogeneous CR manifold M and to realize how much we have to compute the homogeneous components of $\mathfrak{g}$ to arrive at the last weighted homogeneous component $\mathfrak{g}_\varrho$ we can apply the following plain strategy:

- **When $\mathfrak{g}_-$ is fundamental.** Compute the homogeneous components $\mathfrak{g}_t$ of $\mathfrak{g}$ as much as it appears the first trivial component.
- **When $\mathfrak{g}_-$ is not fundamental.** Compute the homogeneous components $\mathfrak{g}_t$ of $\mathfrak{g}$ as much as they appear ρ successive trivial components.

Example 4.7. In Example 4.3, we computed the negative part $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and also zeroth component $\mathfrak{g}_0$ of the graded algebra $\mathfrak{g} := \mathfrak{aut}_{CR}(M)$ associated to the presented homogeneous CR manifold $M \subset \mathbb{C}^{1+3}$. Here, let us finalize computation of the desired algebra. Proceeding further in this direction, now let us compute the next component $\mathfrak{g}_1$. In this case, we can set the following expressions for four desired coefficients $Z(z,w)$ and $W^l(z,w)\, l=1,2,3$, with their weighted homogeneities at their left hand sides (*cf.* (17))

$$\begin{aligned}
\boxed{1+1}:&\quad Z(z,w) := \mathsf{a}_1\, w_1 + \mathsf{a}_2\, z^2,\\
\boxed{2+1}:&\quad W^1(z,w) := \mathsf{a}_3\, w_2 + \mathsf{a}_4\, w_3 + \mathsf{a}_5\, z^3 + \mathsf{a}_6\, zw_1,\\
\boxed{3+1}:&\quad W^2(z,w) := \mathsf{a}_7\, w_1^2 + \mathsf{a}_8\, z^2 w_1 + \mathsf{a}_9\, z^4 + \mathsf{a}_{10}\, zw_2 + \mathsf{a}_{11}\, zw_3,\\
\boxed{3+1}:&\quad W^3(z,w) := \mathsf{a}_{12}\, w_1^2 + \mathsf{a}_{13}\, z^2 w_1 + \mathsf{a}_{14}\, z^4 + \mathsf{a}_{15}\, zw_2 + \mathsf{a}_{16}\, zw_3.
\end{aligned}$$

Now, we should check these predefined expressions in the tangency equations (18). This gives us the total system $\mathsf{Sys}^1 = \bigcup_{j=1}^3 \mathsf{Sys}^{1,j}$ as follows:

$$\mathsf{Sys}^1 =$$

$$\left\{\begin{array}{l}
\mathsf{a}_5 = 0, \quad i\left(\mathsf{a}_6 + \mathsf{a}_3 - \mathsf{a}_2 = 0\right) + \mathsf{a}_4 = 0, \quad i\left(\mathsf{a}_3 - \bar{\mathsf{a}}_2\right) - \mathsf{a}_4 + 2\,\mathsf{a}_1 = 0, \quad -2i\,\bar{\mathsf{a}}_1 + \mathsf{a}_6 = 0,\\
\bar{\mathsf{a}}_5 = 0, \quad -2i\,\mathsf{a}_1 - \bar{\mathsf{a}}_6 = 0, \quad \mathsf{a}_3 - \bar{\mathsf{a}}_3 = 0, \quad \mathsf{a}_4 - \bar{\mathsf{a}}_4 = 0, \quad \mathsf{a}_9 = 0, \quad i\left(\mathsf{a}_{10} - 2\,\mathsf{a}_2 + \mathsf{a}_8\right) + \mathsf{a}_{11} = 0,\\
i\left(\mathsf{a}_{10} - \mathsf{a}_2 - \bar{\mathsf{a}}_2\right) + 8\,\mathsf{a}_1 - 2\,\mathsf{a}_{11} - 4\,\mathsf{a}_7 = 0, \quad -2i\,\bar{\mathsf{a}}_1 + \mathsf{a}_8 = 0, \quad \mathsf{a}_1 - i\,\bar{\mathsf{a}}_2 = 0, \quad \mathsf{a}_{10} = 0,\\
\mathsf{a}_{11} = 0, \quad i\left(\mathsf{a}_7 - \mathsf{a}_1 - \bar{\mathsf{a}}_1 = 0\right) = 0, \quad \bar{\mathsf{a}}_9 = 0, \quad -\bar{\mathsf{a}}_8 - 2i\,\mathsf{a}_1 = 0, \quad \bar{\mathsf{a}}_{10} = 0, \quad \bar{\mathsf{a}}_{11} = 0, \quad \mathsf{a}_{14} = 0,\\
\mathsf{a}_7 - \bar{\mathsf{a}}_7 = 0, \quad i\left(\mathsf{a}_{13} + \mathsf{a}_{15}\right) + \mathsf{a}_{16} - 2\,\mathsf{a}_2 = 0, \quad -2\,\mathsf{a}_{12} + \mathsf{a}_2 - \mathsf{a}_{16} - \bar{\mathsf{a}}_2 + i\left(\mathsf{a}_{15} - 4\,\mathsf{a}_1\right) = 0,\\
\mathsf{a}_{13} - 2\,\bar{\mathsf{a}}_1 = 0, \quad \bar{\mathsf{a}}_2 + i\,\mathsf{a}_1 = 0, \quad i\,\mathsf{a}_{12} - \mathsf{a}_1 + \bar{\mathsf{a}}_1 = 0, \quad \mathsf{a}_{15} = 0, \quad \mathsf{a}_{16} = 0, \quad \bar{\mathsf{a}}_{14} = 0,\\
-\bar{\mathsf{a}}_{13} + 2\,\mathsf{a}_1 = 0, \quad -\bar{\mathsf{a}}_{15} = 0, \quad -\bar{\mathsf{a}}_{16} = 0, \quad \mathsf{a}_{12} - \bar{\mathsf{a}}_{12} = 0
\end{array}\right\}.$$

This system has only the trivial solution $\mathsf{a}_1 = \cdots = \mathsf{a}_{16} = 0$ which means that we have $\mathfrak{g}_1 = 0$. Furthermore as we know, the desired algebra $\mathfrak{g}$ is fundamental which guarantees that the next components are trivial, too. Summing up the results of this example with those of Example 4.3, one finds the sought 7-dimensional Lie algebra of infinitesimal CR-automorphisms associated to M as the gradation:

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

with $\mathfrak{g}_{-3} = \langle \mathsf{X}_4, \mathsf{X}_5 \rangle$, with $\mathfrak{g}_{-2} = \langle \mathsf{X}_3 \rangle$, with $\mathfrak{g}_{-1} = \langle \mathsf{X}_1, \mathsf{X}_2 \rangle$ and with $\mathfrak{g}_0 = \langle \mathsf{X}_6, \mathsf{X}_7 \rangle$ with the Lie commutators displayed in the following table:

	X_5	X_4	X_3	X_2	X_1	X_6	X_7
X_5	0	0	0	0	0	$3\mathsf{X}_5$	$-\mathsf{X}_4$
X_4	$*$	0	0	0	0	$3\mathsf{X}_4$	X_5
X_3	$*$	$*$	0	$4\mathsf{X}_5$	$4\mathsf{X}_4$	$2\mathsf{X}_3$	0
X_2	$*$	$*$	$*$	0	$-4\mathsf{X}_3$	X_2	$-\mathsf{X}_1$
X_1	$*$	$*$	$*$	$*$	0	X_1	X_2
X_6	$*$	$*$	$*$	$*$	$*$	0	0
X_7	$*$	$*$	$*$	$*$	$*$	$*$	0

One observes that the achieved algebra is exactly that of Example 3.6.

One can finds another computation of the Lie algebra achieved in the above example via the classical method of solving the arisen PDE system in [33]. Comparing the above process with that of this paper clarify the effectiveness of the prepared algorithm.

5. Summing up the results

Here, let us gather the results obtained so far to provide an algorithm for computing the sought Lie algebras of infinitesimal CR-automorphisms associated to the holomorphically nondegenerate homogeneous CR manifolds, represented as (11). The strategy introduced in subsection 4.1 enabled one to compute separately the homogeneous components $\mathfrak{g}_t$, $t = -\rho, \ldots, \varrho$ of the graded algebra $\mathfrak{g}$ of infinitesimal CR-automorphisms of such CR manifolds as:

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\varrho. \tag{20}$$

One may follow the following two points for computing the desired algebras associated to the under consideration homogeneous CR manifolds:

Point 1 Executing three steps $(\mathbf{s1})-(\mathbf{s2})-(\mathbf{s3})$ introduced in subsection 4.1 and finding homogeneous components $\mathfrak{g}_t$ successively. In particular if $\mathfrak{g}_-$ is fundamental then one can execute the step **(ib)**.

Point 2 In the above gradation, the minimum homogeneity $-\rho$ is equal to $[w_k]$ where w_k has the maximum homogeneity among the complex variables appearing in (11). Moreover, it suffices to compute successively the homogeneous components $\mathfrak{g}_0, \mathfrak{g}_1, \ldots$ as much as we find ρ successive trivial algebras. In particular if $\mathfrak{g}_-$ is fundamental, we can terminate the computations as much as we find first trivial component.

Remark 5.1. In [7], Beloshapka called his introduced CR manifolds by *nice* universal CR-models. These manifolds have the ability of enjoying all properties, required for launching the presented algorithm. They are of finite type, holomorphically nondegenerate, generic and real analytic with the graded Lie algebras of infinitesimal CR-automorphisms their negative parts are fundamental. Having such properties, we shall confirm that these models are really deserved to be called by the phrase *nice*.

Let us conclude this paper by computing Lie algebras of infinitesimal CR-automorphisms associated to the following CR-manifold:

$$\mathcal{M} := \begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}\,\big(z+\overline{z}\big), \\ w_3 - \overline{w}_3 = 2i\, z\overline{z}\,\big(z^2 + \frac{3}{2}\, z\overline{z} + \overline{z}^2\big). \end{cases} \tag{21}$$

It is worth noting that this CR-manifold — which admits some interesting features [26, 33] — does not belong to the class of CR-models introduced by Beloshapka [7]. Before computations, let us show that $\mathcal{M}$ is homogeneous.

Proposition 5.2. *The already introduced CR-manifold $\mathcal{M} \subset \mathbb{C}^{1+3}$ is homogeneous.*

Proof. To prove the assertion, we show that for each arbitrary point (p, q_1, q_2, q_3) of $\mathcal{M}$, there is a CR-automorphism mapping the origin to it. For this aim, it suffices to prove that after applying the four replacements $z \mapsto z + p$ and $w_j \mapsto w_j + q_j$, $j = 1, 2, 3$, one can transform the model into its initial form by means of some certain changes of coordinates. At first, let us consider the effect of these transformations on the third equation. In fact, it gives:

(22)

$$\begin{aligned} w_3 - \overline{w}_3 + q_3 - \overline{q}_3 = {} & 2i\, z\overline{z}\,\big(z^2 - 3\, z\overline{z} + \overline{z}^2\big) + \\ & + 3i\, p^2\overline{p}^2 + 2i\, p^3\overline{p} + 2i\, p\overline{p}^3 + (6i\,\overline{p}^2 p + 2i\,\overline{p}^3 + 6i\,\overline{p}p^2)z + (6i\,\overline{p}p^2 + 6i\,\overline{p}^2 p + 2i\, p^3)\overline{z} + \\ & + (6i\,\overline{p}^2 + 6i\, p^2 + 12i\, p\overline{p})z\overline{z} + (3i\,\overline{p}^2 + 6i\, p\overline{p})z^2 + (6i\, p\overline{p} + 3i\, p^2)\overline{z}^2 + \\ & + (6i\, p + 6i\,\overline{p})z^2\overline{z} + (6i\,\overline{p} + 6i\, p)z\overline{z}^2 + 2i\,\overline{p}z^3 + 2i\, p\overline{z}^3. \end{aligned}$$

It is possible to kill all pluriharmonic terms by the change of variables $w_3 \mapsto w_3 + P_3(z)$ with:

$$P_3(z) = -q_3 + \tfrac{3}{2} i\, p^2\overline{p}^2 + 2i\, p^3\overline{p} + \big(6i\, p\overline{p}^2 + 2i\,\overline{p}^3 + 6i\, p^2\overline{p}\big)\, z + \big(3i\,\overline{p}^2 + 6i\, p\overline{p}\big)\, z^2 + 2i\overline{p}\, z^3.$$

After such change of variables we will have:

$$\begin{aligned} w_3 - \overline{w}_3 = {} & 2i\, z\overline{z}\,\big(z^2 - 3\, z\overline{z} + \overline{z}^2\big) \\ & + (6i\overline{p}^2 + 6ip^2 + 12ip\overline{p})z\overline{z} + \\ & + (6ip + 6i\overline{p})z^2\overline{z} + (6i\overline{p} + 6ip)z\overline{z}^2. \end{aligned}$$

Replacing the term $z\overline{z}$ by the equal expression $\frac{1}{2i}\,(w_1 - \overline{w}_1)$ at the second line, one plainly verifies that it is also possible to eliminate this line by the change of coordinates $w_3 \mapsto w_3 + \big(3\,\overline{p}^2 + 3\, p^2 + 6\, p\overline{p}\big)\, w_1$ and obtain:

$$\begin{aligned} w_3 - \overline{w}_3 = {} & 2i\, z\overline{z}\,\big(z^2 - 3\, z\overline{z} + \overline{z}^2\big) \\ & + (6ip + 6i\overline{p})z^2\overline{z} + (6i\overline{p} + 6ip)z\overline{z}^2. \end{aligned}$$

Now it remains to eliminate the second line of the above expression again by some holomorphic change of coordinates. First, one should notice that according to the defining equations of the model, the second line can be represented into the form:

$$(6ip+6i\overline{p})z^2\overline{z}+(6i\overline{p}+6ip)z\overline{z}^2=(3\overline{p}+3p)\big(w_2-\overline{w}_2\big).$$

Hence, to eliminate this line from the last expression, it suffices to use the holomorphic change of coordinates $w_3\mapsto w_3+(3\overline{p}+3p)\,w_2$. This convert the third expressions into the initial form:

$$w_3-\overline{w}_3=2i\,z\overline{z}\,\big(z^2-3\,z\overline{z}+\overline{z}^2\big),$$

as desired. Simpler procedure works in the cases of two first defining equations. □

Example 5.3. Now, let us compute the Lie algebra $\mathfrak{g}:=\mathfrak{aut}_{CR}(\mathcal{M})$ associated to the CR-manifold $\mathcal{M}$ defined as (21). First, one notices that we have the following weights of the appearing complex variables:

$$[z]=1,\quad [w_1]=2,\quad [w_2]=3,\quad [w_3]=4.$$

Hence the minimum homogeneity of the homogeneous components will be $-\rho=-4$. Here, an infinitesimal CR-automorphism is of the form:

$$\mathsf{X}:=Z(z,w)\,\partial_z+W^1(z,w)\,\partial_{w_1}+W^2(z,w)\,\partial_{w_2}+W^3(z,w)\,\partial_{w_3},$$

enjoying the following three fundamental tangency equations:

$$\begin{aligned}
&0\equiv\big[W^1-\overline{W}^1-2i\,\overline{z}Z-2i\,z\overline{Z}\big]_{\mathcal{M}},\\
&0\equiv\big[W^2-\overline{W}^2-4i\,z\overline{z}Z-2i\,\overline{z}^2Z-2i\,z^2\overline{Z}-4i\,z\overline{z}\overline{Z}\big]_{\mathcal{M}},\\
&0\equiv\big[W^3-\overline{W}^3-6i\,z^2\overline{z}Z-6i\,z\overline{z}^2Z-2i\,\overline{z}^3Z-2i\,z^3\overline{Z}-6i\,z^2\overline{z}\overline{Z}-6i\,z\overline{z}^2\overline{Z}\big]_{\mathcal{M}}.
\end{aligned}\tag{23}$$

Let us start by computing the negative part $\mathfrak{g}_-$. For the (-1)-th component $\mathfrak{g}_{-1}$, the sought coefficients are of the forms:

$$\left[\begin{array}{l}
Z(z,w):=\mathsf{a}_1,\\
W^1(z,w):=\mathsf{a}_2\,z,\\
W^3(z,w):=\mathsf{a}_3\,z^2+\mathsf{a}_4\,w_1,\\
W^4(z,w):=\mathsf{a}_5\,z^3+\mathsf{a}_6\,zw_1+\mathsf{a}_7\,w_2.
\end{array}\right.$$

Checking these predefined polynomials into the tangency equations (23) gives the following system:

$$\mathsf{Sys}^{-1}:=\left\{\begin{array}{l}
\mathsf{a}_2-2i\,\overline{\mathsf{a}}_1=0\quad -4i\,\mathsf{a}_1-4i\,\overline{\mathsf{a}}_1+2i\,\overline{\mathsf{a}}_4=0\quad \mathsf{a}_3-2i\,\overline{\mathsf{a}}_1=0\\
\mathsf{a}_4-\overline{\mathsf{a}}_4=0\quad \mathsf{a}_5-2i\,\overline{\mathsf{a}}_1=0\quad \mathsf{a}_6=0\quad \mathsf{a}_7-\overline{\mathsf{a}}_7=0
\end{array}\right\}$$

which has the solution:

$$\begin{aligned}
&\mathsf{a}_1:=\mathsf{a}+i\,\mathsf{b},\quad \mathsf{a}_2=2\,\mathsf{b}+2i\,\mathsf{a},\quad \mathsf{a}_3=2\,\mathsf{b}+2i\,\mathsf{a},\quad \mathsf{a}_4=4\,\mathsf{a},\\
&\mathsf{a}_5=2\,\mathsf{b}+2i\,\mathsf{a},\quad \mathsf{a}_6=0,\quad \mathsf{a}_7=6\,\mathsf{a},\qquad {\scriptstyle(\mathsf{a},\mathsf{b}\in\mathbb{R})}.
\end{aligned}$$

Consequently, the sought homogeneous component $\mathfrak{g}_{-1}$ is 2-dimensional with the generators:

$$\begin{aligned}
&\mathsf{X}_1=\partial_z+2iz\,\partial_{w_1}+2iz^2\,\partial_{w_2}+4w_1\,\partial_{w_2}+2iz^3\,\partial_{w_3}+6w_2\,\partial_{w_3},\\
&\mathsf{X}_2=i\,\partial_z+2z\,\partial_{w_1}+2z^2\,\partial_{w_2}+2z^3\,\partial_{w_3}.
\end{aligned}$$

Similar (and even simpler) computations give three vector fields:

$$\begin{aligned}
\mathsf{X}_3&:=\partial_{w_1},\\
\mathsf{X}_4&:=\partial_{w_2},\\
\mathsf{X}_5&:=\partial_{w_3},
\end{aligned}$$

of homogeneities $-2,-3,-4$, respectively. So far, we have computed the negative component:

$$\mathfrak{g}_-:=\mathfrak{g}_{-4}\oplus\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$$

with $\mathfrak{g}_{-1}:=\langle\mathsf{X}_1,\mathsf{X}_2\rangle$, with $\mathfrak{g}_{-2}:=\langle\mathsf{X}_3\rangle$, with $\mathfrak{g}_{-3}:=\langle\mathsf{X}_4\rangle$ and with $\mathfrak{g}_{-4}:=\langle\mathsf{X}_5\rangle$. One can see all the possible Lie commutators of these generators in the table presented below. From this table, one easily verifies that the negative part $\mathfrak{g}_-$ of $\mathfrak{g}$ is in fact *fundamental*. Hence, we have to compute nonnegative components as much as we encounter first trivial one. Now, we have to continue with computing $\mathfrak{g}_0$. In this case, the sought coefficients are of the form:

$$\left[\begin{array}{l} Z := \mathsf{a}_1\, z \\ W_1 := \mathsf{a}_2\, w_1 + \mathsf{a}_3\, z^2 \\ W_2 := \mathsf{a}_4\, w_2 + \mathsf{a}_5\, z^3 + \mathsf{a}_6\, zw_1 \\ W_3 := \mathsf{a}_7\, w_3 + \mathsf{a}_8\, w_1^2 + \mathsf{a}_9\, z^2 w_1 + \mathsf{a}_{10}\, z^4 + \mathsf{a}_{11}\, zw_2. \end{array}\right.$$

Checking these predefined functions in the tangency equations (18) gives the following complex system:

$$\mathsf{Sys}^0 =$$

$$\left\{\begin{array}{l} \mathsf{a}_3 = 0, \quad -2i\,\mathsf{a}_1 - 2i\,\bar{\mathsf{a}}_1 + 2i\,\mathsf{a}_2 = 0, \quad -\bar{\mathsf{a}}_3 = 0, \quad -\bar{\mathsf{a}}_2 + \mathsf{a}_2 = 0, \quad \mathsf{a}_5 = 0, \\ -4i\,\mathsf{a}_1 - 2i\,\bar{\mathsf{a}}_1 + 2i\,\mathsf{a}_4 + 2i\,\mathsf{a}_6 = 0, \quad -4i\,\bar{\mathsf{a}}_1 - 2i\,\mathsf{a}_1 + 2i\,\mathsf{a}_4 = 0, \quad \mathsf{a}_6 - \bar{\mathsf{a}}_5 = 0, \quad -\bar{\mathsf{a}}_6 = 0, \\ \mathsf{a}_{10} - 2i\,\bar{\mathsf{a}}_1 + 2i\,\mathsf{a}_9 + 2i\,\mathsf{a}_7 + 2i\,\mathsf{a}_{11} - 6i\,\mathsf{a}_1 = 0, \quad 3i\,\mathsf{a}_7 + 2i\,\mathsf{a}_{11} - 4\mathsf{a}_8 - 6i\,\mathsf{a}_1 - 6i\,\bar{\mathsf{a}}_1 = 0, \\ \mathsf{a}_4 - \bar{\mathsf{a}}_4 = 0, \quad \mathsf{a}_9 = 0, \quad 2i\,\mathsf{a}_7 - 6i\,\bar{\mathsf{a}}_1 - 2i\,\mathsf{a}_1 = 0, \quad 4i\,\mathsf{a}_8 = 0, \quad \mathsf{a}_{11} = 0, \quad -\bar{\mathsf{a}}_{10} = 0, \\ -\bar{\mathsf{a}}_9 = 0, \quad -\bar{\mathsf{a}}_{11} = 0, \quad \mathsf{a}_8 - \bar{\mathsf{a}}_8 = 0, \quad \mathsf{a}_7 - \bar{\mathsf{a}}_7 = 0 \end{array}\right\}.$$

This system has the solution:

$$\mathsf{a}_1 = \mathsf{a}, \quad \mathsf{a}_2 = 2\,\mathsf{a}, \quad \mathsf{a}_4 = 3\,\mathsf{a}, \quad \mathsf{a}_7 = 4\,\mathsf{a}, \quad \mathsf{a}_3 = \mathsf{a}_5 = \mathsf{a}_6 = \mathsf{a}_8 = \mathsf{a}_9 = \mathsf{a}_{10} = \mathsf{a}_{11} \equiv 0$$

for some real number a. Therefore, $\mathfrak{g}_0$ is 1-dimensional with the generator:

$$\mathsf{X}_0 = z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 4w_3\,\partial_{w_3}.$$

This nonnegative component was not trivial; hence we have to proceed by computing the next component $\mathfrak{g}_1$. Similar computations that we do not present them for saving space shows that this component *is trivial*. Then, according to the fundamentality of $\mathfrak{g}_-$ we can terminate the computations. Consequently, the sought graded algebra $\mathfrak{g}$ is of the form:

$$\mathfrak{g} := \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with the negative components as above, with $\mathfrak{g}_0 = \langle \mathsf{X}_0 \rangle$ and with the table of commutators displayed as follows:

	X_0	X_1	X_2	X_3	X_4	X_5
X_0	0	$-\mathsf{X}_1$	$-\mathsf{X}_2$	$-2\mathsf{X}_3$	$-3\mathsf{X}_4$	$-4\mathsf{X}_5$
X_1	$*$	0	$-4\mathsf{X}_3$	$-4\mathsf{X}_4$	$6\mathsf{X}_5$	0
X_2	$*$	$*$	0	0	0	0
X_3	$*$	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	$*$	0

6. Parametric defining equations and Gröbner systems

Actually, one of the main — somehow hidden — obstacles appearing among the computations arises when the set of defining equations includes some certain *parametric* polynomials. This case is quite usual as one observes in [7, 10, 24, 36]. To treat such cases, we suggest the modern and effective concept of *comprehensive Gröbner systems* [41, 20, 21, 30, 31] which enables us to consider and solve (linear) parametric systems appearing among the computations.

To begin, let $\mathbb{K}$ be a field and let $\mathbf{a} := a_1, \ldots, a_t$ and $\mathbf{x} := x_1, \ldots, x_n$ be two certain sequences of parameters and variables, respectively. Naturally, we call the ring:

$$\mathbb{K}[\mathbf{a}][\mathbf{x}] := \left\{ \sum_{i=1}^{m} p_{\alpha_i} x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \,|\, p_{\alpha_i} \in \mathbb{K}[\mathbf{a}],\ \alpha_{ij} \in \mathbb{N} \cup \{0\} \right\}$$

the *parametric polynomial ring* over $\mathbb{K}$ with parameters $\mathbf{a}$ and variables $\mathbf{x}$. Let P be a set of parametric polynomials which generates the parametric ideal $\mathcal{I}$. Obviously, the solutions of the *parametric system*

$P = 0$ depend on the extant parameters. The main idea behind the modern concept of *comprehensive Gröbner systems* is to treat such solutions by discussing the values of the parameters of the system defined by $\mathcal{I}$. This concept provides some effective and powerful tools in which enables one to divide the space of parameters into a finite number of partitions for which the *general form* of solutions arising from each partition is unique.

Definition 6.1. Let $\mathcal{I} \subset \mathbb{K}[\mathbf{a}][\mathbf{x}]$ be a parametric ideal, $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and $\prec$ be a monomial ordering on $\mathbf{x}$. Then the set:

$$\mathbf{G}(\mathcal{I}) = \{(E_i, N_i, G_i) \mid i = 1, \ldots, \ell\} \subset \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}][\mathbf{x}]$$

is called a comprehensive Gröbner system for $\mathcal{I}$ if for each homomorphism $\sigma_{(\lambda_1,\ldots,\lambda_t)} : \mathbb{K}[\mathbf{a}][\mathbf{x}] \longrightarrow \overline{\mathbb{K}}[\mathbf{x}]$, associated to a t-tuple $(\lambda_1, \ldots, \lambda_t) \in \overline{\mathbb{K}}^t$ and defined by:

$$\textstyle\sum_{i=1}^m p_{\alpha_i}(a_1, \cdots, a_t)\, x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \quad \mapsto \quad \sum_{i=1}^m p_{\alpha_i}(\lambda_1, \ldots, \lambda_t)\, x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}},$$

there exists a pair (E_i, N_i) with $(\lambda_1, \ldots, \lambda_t) \in V(E_i) \setminus V(N_i)$ such that $\sigma(G_i)$ is a Gröbner basis for $\sigma(I)$ with respect to $\prec$. Here by $V(E_i)$ and $V(N_i)$ we mean the algebraic varieties associated to the polynomial sets E_i and N_i. In this case, E_i and N_i are called *null* and *non-null conditions*, respectively.

Remark that, by [41, Theorem 2.7], every parametric ideal possesses a (*finite*) comprehensive Gröbner system, however, by Definition 6.1, we can observe that such a system may be not unique. The concept of Gröbner systems was introduced first by Weispfenning in 1992 [41]. Later on, Montes [30] proposed DISPGB algorithm for computing Gröbner systems. In 2006, Sato and Suzuki [37] provided an important improvement for computing Gröbner systems by doing only computation of the reduced Gröbner bases in polynomial rings over ground fields. Furthermore, Montes and Wibmer in [31] presented the GRÖBNERCOVER algorithm which computes a finite partition of the parameter space into locally closed subsets together with polynomial data from which the *reduced* Gröbner basis for a given values of parameters can immediately be determined. Kapur, Sun and Wang [20, 21] in 2010 and 2013 suggested two new algorithms for computing Gröbner systems by combining Weispfenning's algorithm with Suzuki and Sato's.

It is worth noting that if $V(E_i) \setminus V(N_i) = \emptyset$, for some i, then the triple (E_i, N_i, G_i) is useless and it must be omitted from the comprehensive Gröbner system. In this case, the pair (E_i, N_i) is called *inconsistent*. It is known that inconsistency occurs if and only if $N_i \subset \sqrt{\langle E_i \rangle}$ and thus we need to an efficient radical membership test to determine it.

In the recently published paper [21], Kapur, Sun and Wang introduced an effective algorithm to compute comprehensive Gröbner system of a parametric polynomial ideal. This algorithm which is called by PGB uses a new and efficient radical membership criterion based on linear algebra methods. To the best of our knowledge, it is the most powerful algorithm of computing comprehensive Gröbner systems introduced so far and it is for this reason that we prefer to employ this algorithm in our computations.

Besides the deep theory encompassing this subject, the concept of comprehensive Gröbner bases provides some effective tools to consider and to solve parametric systems by decomposing the space of the extant parameters. To illustrate this ability let us borrow the following example from [21].

Example 6.2. Consider the following parametric polynomial system in $\mathbb{C}[a, b, c][x, y]$:

$$\Sigma : \begin{cases} ax - b &= 0 \\ by - a &= 0 \\ cx^2 - y &= 0 \\ cy^2 - x &= 0. \end{cases}$$

Choosing the graded reverse lexicographical ordering $y \prec x$ and computing the sought comprehensive Gröbner system using the algorithm PGB give the results displayed in the following table:

E_i	N_i	G_i
$\{a,b,c\}$	$\{\,\}$	$\{x,y\}$
$\{a,b\}$	$\{c\}$	$\{cx^2-y, cy^2-x\}$
$\{a^6-b^6, a^3c-b^3, b^3c-a^3, ac^2-a, bc^2-b\}$	$\{b\}$	$\{bx-acy, by-a\}$
$\{\,\}$	$\{a^6-b^6, a^3c-b^3, b^3c-a^3, ac^2-a, bc^2-b\}$	$\{1\}$

Accordingly, the algorithm divides the solution set of the system Σ into four partitions, each of them corresponds to one of the above rows. Let us explain what each of these rows means. For the first row, we have the null conditions $E_1 = \{a, b, c\}$ and there is no any non-null condition. This means that if the elements of E_1 are null, namely if $a = b = c = 0$ then the system Σ reduces to the system $G_1 = \{x = 0, y = 0\}$ which obviously has the single solution $(0, 0)$. For the second row, we have null conditions $E_2 = \{a, b\}$ and non-null condition $N_2 = \{c\}$. This means that if $a = b = 0$ and $c \neq 0$ then the system Σ reduces to $G_2 = \{cx^2 - y = 0, cy^2 - x = 0\}$ which has the solution set $\{(\frac{1}{c}, \frac{1}{c}), c \in \mathbb{C}\} \cup \{(0,0), (\frac{-1+\sqrt{3}i}{2c}, \frac{-1-\sqrt{3}i}{2c}), (\frac{-1-\sqrt{3}i}{2c}, \frac{-1+\sqrt{3}i}{2c})\}$. Similar interpretation holds for the third row. Finally, the last row means that if none of the previous null conditions holds, namely if $E_1, E_2, E_3 \neq 0$, then the under consideration system Σ reduces to $G_4 = \{1 = 0\}$ which of course its solution set is empty.

6.1. **Implementation.** In [35], we have implemented the designed algorithm in MAPLE 15 which is available online as a library entitled CRAUT (together with a sample file relevant to the following example). To do it, at first we implemented the recent algorithm PGB of Kapur, Sun and Wang [21]. By means of this auxiliary algorithm, we employ techniques of comprehensive Gröbner systems to consider and solve the appearing parametric *linear* systems Sys in the parametric case. Our implementation CRAUT enables one to compute the desired algebras of infinitesimal CR-automorphisms associated to homogeneous and weighted homogeneous CR-manifolds in two non-parametric and parametric cases and also to evaluate the effectiveness of the algorithm designed in this paper.

Example 6.3. (*reverification of the Beloshapka's conjecture in the lengths* $\rho = 1, \ldots, 5$, *cf.* [6, 24, 36]). Consider the following weighted homogeneous rigid defining equations:

(24)
$$\begin{aligned}
w_1 - \overline{w}_1 &= 2i\, z\overline{z},\\
w_2 - \overline{w}_2 &= 2i\left(z^2\overline{z} + \overline{z}^2 z\right), \quad w_3 - \overline{w}_3 = 2\left(z^2\overline{z} - \overline{z}^2 z\right),\\
w_4 - \overline{w}_4 &= 2i\left(z^3\overline{z} + \overline{z}^3 z\right) + 2i\mathbf{a}\, z^2\overline{z}^2, \quad w_5 - \overline{w}_5 = 2\left(z^3\overline{z} - \overline{z}^3 z\right) + 2i\mathbf{b}\, z^2\overline{z}^2, \quad w_6 - \overline{w}_6 = 2i\, z^2\overline{z}^2, \qquad a \in \mathbb{R},\\
w_7 - \overline{w}_7 &= 2i\left(z^4\overline{z} + z\overline{z}^4\right) + \mathbf{c}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \quad w_8 - \overline{w}_8 = 2\left(z^4\overline{z} - z\overline{z}^4\right) + \mathbf{d}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right),\\
w_9 - \overline{w}_9 &= 2i\left(z^3\overline{z}^2 + z^2\overline{z}^3\right) + \mathbf{e}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right),\\
w_{10} - \overline{w}_{10} &= 2\left(z^3\overline{z}^2 - z^2\overline{z}^3\right) + \mathbf{f}i\left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right),\\
w_{11} - \overline{w}_{11} &= \left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} - z\overline{z}^2\right), \quad w_{12} - \overline{w}_{12} = \left(w_1 + \overline{w}_1\right)\left(z^2\overline{z} + z\overline{z}^2\right), \qquad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathbb{R},
\end{aligned}$$

and let M_{k} be the Beloshapka's CR-model of CR-dimension 1 and codimension $\mathsf{k} = 1, \ldots, 12$, represented in coordinates $(z, w_1, \ldots, w_{\mathsf{k}})$ in $\mathbb{C}^{\mathsf{k}+1}$. For $\mathsf{k} = 1, \ldots, 5$, M_{k} is represented as the graph of the above first k equations. For $\mathsf{k} = 6, \ldots, 11$, it is represented again by the first k equations but with the assumption $\mathbf{a}, \mathbf{b} = 0$. Finally, M_{12} is represented as the graph of the above 12 equations with the assumption that all the appearing six parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ are vanished — namely a non-parametric model as M_1, M_2, M_3 and M_6 are. These twelve CR-manifolds encompass all the Beloshapka's models up to the length five and are constructed by Shananina and Mamai [24, 36]. They also computed the associated Lie algebras of infinitesimal CR-automorphisms, thought Mamai did not present the outputs, perhaps because of the length of them. It is also known that these models are all homogeneous ([7]). By means of our implementation, we have computed the associated Lie algebras of infinitesimal CR-automorphisms. The following table displays some properties of the obtained results, where the timings were conducted on a personal laptop with Intel(R) Core(TM) *i*7 CPU@2.80 GHz and 6.00 GB of RAM:

<table>
<tr><td>Model</td><td>M_1</td><td>M_2</td><td>M_3</td><td>M_4</td><td colspan="2">M_5</td><td>M_6</td><td colspan="2">M_7</td><td colspan="2">M_8</td><td colspan="2">M_9</td><td colspan="2">M_{10}</td><td>M_{11}</td><td>M_{12}</td></tr>
<tr><td>Time (sec.)</td><td>0.5</td><td>0.5</td><td>1</td><td>3</td><td colspan="2">6.2</td><td>5.8</td><td colspan="2">17.6</td><td colspan="2">52.2</td><td colspan="2">131.5</td><td colspan="2">340</td><td>198</td><td>22</td></tr>
<tr><td>ρ</td><td>2</td><td>3</td><td>3</td><td>4</td><td colspan="2">4</td><td>4</td><td colspan="2">5</td><td colspan="2">5</td><td colspan="2">5</td><td colspan="2">5</td><td>5</td><td>5</td></tr>
<tr><td>ϱ</td><td>2</td><td>0</td><td>0</td><td>0</td><td colspan="2">0</td><td>0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td>-1</td><td>-1</td></tr>
<tr><td>dim.</td><td>8</td><td>5</td><td>7</td><td>7</td><td>9</td><td>8</td><td>10</td><td>10</td><td>9</td><td>12</td><td>10</td><td>12</td><td>11</td><td>14</td><td>12</td><td>13</td><td>14</td></tr>
</table>

The last row of the above table needs some explanation. In fact for some models, the dimension of the associated algebra is not unique and depends on the values of the extant parameters. Being more precise, in this table we observe two different values for the dimensions associated to some models. In such cases, the left number is the dimension of the desired Lie algebra associated to the model whereas all the appearing parameters vanish identically; otherwise the dimension is equal to the number at the right hand side. For example for M_8, we have $\dim\big(\mathfrak{aut}_{CR}(M_8)\big) = 12$ if $\mathbf{c}, \mathbf{d} = 0$ and $= 10$, otherwise. More precisely, in the case that $\mathbf{c}, \mathbf{d} \neq 0$, we have the basis elements of the components of $\mathfrak{aut}_{CR}(M_8)$ as:
(25)

$$
\begin{aligned}
&\mathfrak{g}_{-5} = \langle \partial_{w_7}, \partial_{w_8} \rangle, \qquad \mathfrak{g}_{-4} = \langle \partial_{w_6}, \partial_{w_5}, \partial_{w_4} \rangle, \qquad \mathfrak{g}_{-3} = \langle \partial_{w_3}, \partial_{w_2} \rangle, \\
&\mathfrak{g}_{-2} = \langle \partial_{w_1} + \mathbf{a} w_2 \partial_{w_7} + \mathbf{b} w_2 \partial_{w_8} \rangle, \\
&\mathfrak{g}_{-1} = \Big\langle -2w_5\partial_{w_7} + 2w_4\,\partial_{w_8} + z\partial_{w_1} + w_3\partial_{w_6} + z^3\partial_{w_4} + 2w_1\partial_{w_3} + z^2\partial_{w_2} + z^4\partial_{w_7} - \frac{3}{2}w_3\partial_{w_4} + \frac{3}{2}w_2\partial_{w_5} + \\
&\quad + \mathbf{c}z^2w_1\partial_{w_7} + \mathbf{c}w_6\partial_{w_7} + \mathbf{d}z^2w_1\partial_{w_8} + \mathbf{d}w_6\partial_{w_8} + \frac{i}{2}\big(\partial_z - 2z^2\partial_{w_3} - 2z^3\partial_{w_5} - 2z^4\partial_{w_8}\big), \\
&\quad \mathbf{c}z^2w_1\partial_{w_7} + \mathbf{d}z^3\partial_{w_4} + \mathbf{d}z^2w_1\partial_{w_8} + \mathbf{d}z\partial_{w_1} + \mathbf{d}z^4\partial_{w_7} + \mathbf{d}z^2\partial_{w_2} - \mathbf{cb}w_1^2\partial_{w_7} - \frac{3}{2}\mathbf{d}w_3\partial_{w_5} - \frac{3}{2}\mathbf{d}w_2\partial_{w_4} \\
&\quad - \mathbf{d}w_2\partial_{w_6} - \mathbf{d}z^4\partial_{w_8} - \mathbf{d}z^3\partial_{w_5} - \mathbf{d}z^2\partial_{w_3} - \frac{1}{2}\mathbf{d}\partial_z - 2\mathbf{d}w_4\partial_{w_7} - 2\mathbf{d}w_1\partial_{w_2} - 2\mathbf{d}w_5\partial_{w_8} - \mathbf{d}w_1^2\partial_{w_8}\Big\rangle, \\
&\mathfrak{g}_0 = \langle\,\rangle.
\end{aligned}
$$

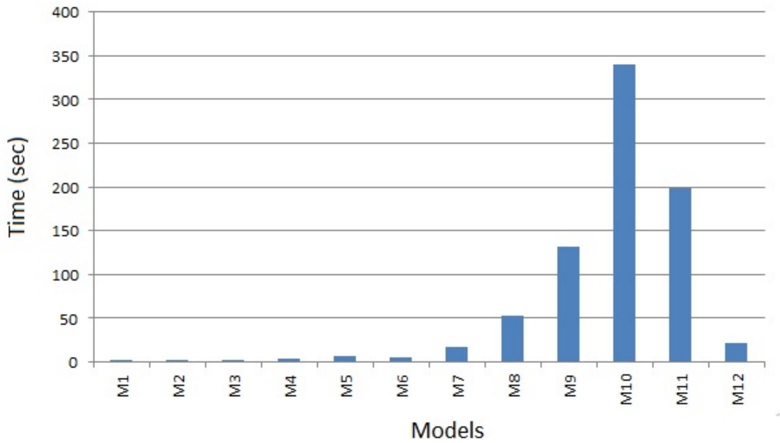
In the case that $\mathbf{c}, \mathbf{d} = 0$, the basis elements of $\mathfrak{g}_i, i = -1, \ldots, -5$ are as above with of course $\mathbf{c}, \mathbf{d} = 0$ while in this case $\mathfrak{g}_0$ has two basis elements as follows:

$$
\begin{aligned}
\mathfrak{g}_0 = \Big\langle & -w_8\partial_{w_7} + \frac{1}{3}w_2\partial_{w_3} + w_7\partial_{w_8} - \frac{1}{3}w_3\partial_{w_2} + \frac{2}{3}w_4\partial_{w_5} - \frac{2}{3}w_5\partial_{w_4} + \frac{1}{3}iz\partial_z, \\
& \frac{1}{5}z\partial_z + \frac{3}{5}w_3\partial_{w_3} + \frac{3}{5}w_2\partial_{w_2} + \frac{4}{5}w_5\partial_{w_5} + \frac{4}{5}w_6\partial_{w_6} + \frac{4}{5}w_4\partial_{w_4} + w_7\partial_{w_7} + w_8\partial_{w_8} + \frac{2}{5}w_1\partial_{w_1}\Big\rangle.
\end{aligned}
$$

One easily concludes from the results of the above table, all the above twelve models have rigidity, as is the main result of Mamai in [24]. Together with the library CRAUT, we also have put a sample file concerning the computations of this example.

According to the above table, it took just 13 minutes from the implementation to verify the Beloshapka's conjecture in the lengths $\rho = 1, \ldots, 5$. This shows the effectiveness of the algorithm, designed in this paper.

Remark 6.4. A glance on the above timings shows how it increases mostly the complexity of computations as one passes from each model to the next by adding just one variable and one defining equation to the previous ones. The following diagram may be helpful to compare the appearing timings. Moreover, actually the computations in the case of parametric defining equations are more complicated in comparison to those of the non-parametric case. For example, compare the timings corresponding to the models M_{11} and M_{12} — notice that the defining equations of M_{12} are non-parametric.



Example 6.5. Add the following rigid defining equations to the list (24):
(26)

$$w_{13} - \overline{w}_{13} = 2i\left(z^5\overline{z} + \overline{z}^5 z\right), \quad w_{14} - \overline{w}_{14} = 2\left(z^5\overline{z} - \overline{z}^4 z\right), \quad w_{15} - \overline{w}_{15} = 2i\left(z^4\overline{z}^2 + \overline{z}^4 z^2\right),$$
$$w_{16} - \overline{w}_{16} = 2\left(z^4\overline{z}^2 - \overline{z}^4 z^2\right), \quad w_{17} - \overline{w}_{17} = 2i\, z^3\overline{z}^3,$$

and for $\mathsf{k} = 13, \ldots, 17$ let M_{k} to be the CR-manifold of CR-dimension one and codimension k represented as the graph of the equations of (24) together the first $\mathsf{k} - 12$ equations of the above list. These are the next five rigid Beloshapka's models which are the very first models of the length six. Here, we also compute — for the first time — the desired Lie algebras associated to these models and the results are displayed in the following table:

Model	M_{13}	M_{14}	M_{15}	M_{16}	M_{17}
Time (sec.)	83.5	152	286	545	1157
ρ	6	6	6	6	6
ϱ	-1	-1	-1	-1	-1
dim	15	16	17	18	19

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