NONRIGID SPHERICAL REAL ANALYTIC HYPERSURFACES IN $\mathbb{C}^2$

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ABSTRACT. A Levi nondegenerate real analytic hypersurface M of $\mathbb{C}^2$ represented in local coordinates $(z, w) \in \mathbb{C}^2$ by a complex defining equation of the form $w = \Theta(z, \overline{z}, \overline{w})$ which satisfies an appropriate reality condition, is spherical if and only if its complex graphing function Θ satisfies an explicitly written sixth-order polynomial complex partial differential equation. In the rigid case (known before), this system simplifies considerably, but in the general nonrigid case, its combinatorial complexity shows well why the two fundamental curvature tensors constructed by Élie Cartan in 1932 in his classification of hypersurfaces have, since then, never been reached in parametric representation.

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§1. INTRODUCTION

A real analytic hypersurface M in $\mathbb{C}^2$ is called *spherical* at one of its points p if there exists a nonempty open neighborhood U_p of p in $\mathbb{C}^2$ such that $M \cap U_p$ is biholomorphic to a piece of the unit sphere $S^3 = \{(z, w) \colon |z|^2 + |w|^2 = 1\}$. When M is connected, sphericality at one point is known to propagate all over M, for it is equivalent to the vanishing of two certain *real analytic* curvature tensors that were constructed by Élie Cartan in [3]. However, the intrinsic computational complexity, in the Cauchy-Riemann (CR for short) context, of Élie Cartan's algorithm to derive an absolute parallelism on some suitable eight-dimensional principal bundle $\mathscr{P} \to M$ prevents from controlling explicitly all the appearing differential forms. As a matter of fact, the effective computation, in terms of a defining equation for M, of the two fundamental differential invariants the vanishing of which characterizes sphericality, appears nowhere in the literature (*see e.g.* [23, 5, 9] and the references therein as well), except notably when one makes the assumption that, in some suitable local holomorphic coordinates $(z, w) = (x + iy,\, u + iv)$ vanishing at the point p, the defining equation is of the so-called *rigid* form $u = \varphi(x, y)$ with the variable v missing, or even of the so-called (simpler) *tube* form $u = \varphi(x)$, with the two variables y and v missing, *see* [9] which showed recently a renewed interest, in CR geometry, for explicit characterizations of sphericality. But in general, a real analytic hypersurface $M \subset \mathbb{C}^2$ is represented at p by a *real* equation

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$u = \varphi(x, y, v)$ whose graphing function φ depends entirely arbitrarily upon v *also*, and then apparently, the characterization of sphericality is still unknown.

On the other hand, in the studies [13, 14, 15, 16, 17] devoted to the CR reflection principle, it was emphasized that all the adequate invariants of CR mappings between CR manifolds: *Pair of Segre foliations*, *Segre chains*, *Complexified CR orbits*, *Jets of complexified Segre varietes*, *Rigidity of formal CR mappings*, *Nondegeneracy conditions*, *CR-reflection function*[1], can be viewed correctly only when M is represented by a so-called *complex defining equation* of the form:

$$w = \Theta\big(z, \overline{z}, \overline{w}\big),$$

where the function $\Theta \in \mathbb{C}\big\{z, \overline{z}, \overline{w}\big\}$, vanishing at the origin, is the unique function obtained by solving with respect to w the equation: $\frac{w+\overline{w}}{2} = \varphi\big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}, \frac{w-\overline{w}}{2i}\big)$; then the fact that φ was *real* is reflected, in terms of this new function $\Theta(z, \overline{z}, \overline{w})$, by the constraint that, together with its complex conjugate $\overline{\Theta}\big(z, \overline{z}, \overline{w}\big)$, it satisfies the functional equation[2]:

$$w \equiv \Theta\big(z,\, \overline{z},\, \overline{\Theta}(\overline{z}, z, w)\big).$$

Accordingly, the author suspected since a few years — *cf.* the Open Question 2.35 in [20] — that sphericality of M at p should and could be expressed adequately in terms of Θ. The classical assumption that M be *Levi nondegenerate* at the point p (*see e.g.* [9]) — which is the origin of our present system of coordinates (z, w) — may then be expressed here (*cf.* [16, 17]) by requiring that $\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}$ does not vanish at the origin. In particular, this guarantees that the following explicit rational expression whose numerator is a polynomial in the fourth-order jet $J^4_{z,\overline{z},\overline{w}}\Theta$, is well defined and analytic in some sufficiently small neighborhood of the origin:

$$\begin{aligned}
\mathsf{AJ}^4(\Theta) := \frac{1}{[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}]^3} \bigg\{ & \Theta_{zz\overline{z}\overline{z}} \bigg(\Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) - \\
& - 2\Theta_{zz\overline{z}\overline{w}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) + \Theta_{zz\overline{w}\overline{w}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) + \\
& + \Theta_{zz\overline{z}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{w}\overline{w}} \end{vmatrix} - 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{w}} \end{vmatrix} + \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{z}} \end{vmatrix} \bigg) + \\
& + \Theta_{zz\overline{w}} \bigg(- \Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}\overline{w}} \end{vmatrix} + 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{w}} \end{vmatrix} - \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{z}} \end{vmatrix} \bigg) \bigg\}.
\end{aligned}$$

We hope, then, that the following precise statement will fill a gap in our understanding of the vanishing of CR curvature tensors.

Main (and unique) theorem. *An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^2$ which is Levi nondegenerate at one of its points p and has a complex defining equation of the form:*

$$w = \Theta\big(z,\, \overline{z},\, \overline{w}\big)$$

[1] For a presentation of these concepts, the reader is referred to the extensive introductions of [15, 17] and also to [20] for more about why dealing only with complex defining equations is natural and unavoidable when one wants to insert CR geometry in the wider universe of completely integrable systems of real or complex analytic partial differential equations.

[2] More will be said shortly in Section 2 below.

in some system of local holomorphic coordinates $(z,w) \in \mathbb{C}^2$ *centered at* p, *is spherical at* p if and only if *its graphing complex function* Θ *satisfies the following explicit sixth-order algebraic partial differential equation:*

$$0 \equiv \left(\frac{-\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}}\frac{\partial}{\partial \overline{z}} + \frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}}\frac{\partial}{\partial \overline{w}}\right)^2 \big[\mathsf{AJ}^4(\Theta)\big]$$

identically in $\mathbb{C}\{z, \overline{z}, \overline{w}\}$.

Here, it is understood that the first-order derivation in parentheses is applied twice to the fourth-order rational differential expression $\mathsf{AJ}^4(\Theta)$. The factor $\frac{1}{[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}]^7}$ then appears, and after clearing out this denominator, one obtains a universal *polynomial* differential expression $\mathsf{AJ}^6(\Theta)$ depending upon the sixth-order jet $J^6_{z,\overline{z},\overline{w}}\Theta$ and having integer coefficients. A partial expansion is provided in Section 5, and the already formidable incompressible length of this expansion perhaps explains the reason why no reference in the literature provides the explicit expressions, in terms of some defining function for M, of Élie Cartan's two fundamental differential invariants[3] which can (in principle) be used to classify real analytic hypersurfaces of $\mathbb{C}^2$ up to biholomorphisms, and to at least characterize sphericality.

Suppose in particular for instance that M is rigid, given by a complex equation of the form $w = -\overline{w} + \Xi(z,\overline{z})$, that is to say with $\Theta(z,\overline{z},\overline{w})$ of the form $-\overline{w} + \Xi(z,\overline{z})$, so that the reality condition simply reads here: $\Xi(z,\overline{z}) \equiv \overline{\Xi}(\overline{z},z)$. Then as a corollary-exercise, sphericality is explicitly characterized by a much simpler partial differential equation that we can write down in expanded form:

$$\begin{aligned}0 \equiv\; & \frac{\Xi_{z^2\overline{z}^4}}{(\Xi_{z\overline{z}})^4} - 6\,\frac{\Xi_{z^2\overline{z}^3}\,\Xi_{z\overline{z}^2}}{(\Xi_{z\overline{z}})^5} - 4\,\frac{\Xi_{z^2\overline{z}^2}\,\Xi_{z\overline{z}^3}}{(\Xi_{z\overline{z}})^5} - \frac{\Xi_{z^2\overline{z}}\,\Xi_{z\overline{z}^4}}{(\Xi_{z\overline{z}})^5} + \\ & + 15\,\frac{\Xi_{z^2\overline{z}^2}\,(\Xi_{z\overline{z}^2})^2}{(\Xi_{z\overline{z}})^6} + 10\,\frac{\Xi_{z\overline{z}^3}\,\Xi_{z^2\overline{z}}\,\Xi_{z\overline{z}^2}}{(\Xi_{z\overline{z}})^6} - 15\,\frac{\Xi_{z^2\overline{z}}\,(\Xi_{z\overline{z}^2})^3}{(\Xi_{z\overline{z}})^7},\end{aligned}$$

and this equation should of course hold identically in $\mathbb{C}\{z,\overline{z}\}$.

Now, here is a summarized description of our arguments of proof. Beniamino Segre ([24]) in 1931 and in fact much earlier Sophus Lie himself in the 1880's (*see e.g.* Chapter 10 of Volume I of the *Theorie der Transformationsgruppen* [6]) showed how to elementarily associate a unique second-order ordinary differential equation:

$$w_{zz}(z) = \Phi\big(z,\, w(z),\, w_z(z)\big)$$

to the Levi nondegenerate equation $w = \Theta(z,\overline{z},\overline{w})$ by eliminating the two variables $\overline{z}$ and $\overline{w}$, viewed as parameters, from the two equations $w = \Theta$ and $w_z = \Theta_z$. We check in great details the semi-known result that M is spherical at the origin if and only if its associated differential equation is equivalent, under some appropriate local holomorphic point transformation $(z,w) \longmapsto (z',w') = \big(z'(z,w), w'(z,w)\big)$ fixing the origin, to the simplest possible equation $w'_{z'z'}(z') = 0$ having null right-hand side, whose obvious solutions are just the affine complex lines. But since the doctoral dissertation of Arthur Tresse (defended in 1895 under the direction of

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[3] *See* [3] and also [23], where the tight analogy with second-order ordinary differential equations is well explained.

Lie in Leipzig), it is known that, attached to any such differential equation are two explicit differential invariants:

$$\begin{aligned} \mathsf{I}^1 &:= \Phi_{w_z w_z w_z w_z} \qquad \text{and:} \\ \mathsf{I}^2 &:= \mathsf{D}\mathsf{D}\big(\Phi_{w_z w_z}\big) - \Phi_{w_z}\,\mathsf{D}\big(\Phi_{w_z w_z}\big) - 4\,\mathsf{D}\big(\Phi_{w w_z}\big) + \\ &\qquad + 6\,\Phi_{ww} - 3\,\Phi_w\,\Phi_{w_z w_z} + 4\,\Phi_{w_z}\,\Phi_{w w_z}, \\ \text{where}\quad \mathsf{D} &:= \partial_z + w_z\,\partial_w + \Phi(z,w,w_z)\,\partial_{w_z}, \end{aligned}$$

depending both upon the fourth-order jet of Φ, which, together with all their covariant differentiations, enable one (in principle[4]) to completely determine when two arbitrarily given differential equations are equivalent one to another[5]. A very well-known application is: the vanishing of both I^1 and I^2 characterizes equivalence to $w'_{z'z'}(z') = 0$. So in order to characterize sphericality, one only has to reexpress the vanishing of I^1 and of I^2 in terms of the complex defining function $\Theta(z, \overline{z}, \overline{w})$. For this, we apply the techniques of computational differential algebra developed in [20] which enable us here to explicitly execute the two-ways transfer between algebraic expressions in the jet of Φ and algebraic expressions in the jet of Θ. It then turns out that the two equations which one obtains by transferring to Θ the vanishing of I^1 and of I^2 are *conjugate one to another*, so that a single equation suffices, and it is precisely the one enunciated in the theorem. In fact, this coincidence is caused by the famous projective duality, explained *e.g.* by Lie and Scheffers in Chapter 10 of [12] and restituted in modern language in [1, 5]. It is indeed well known that to any second-order ordinary differential equation ($\mathscr{E}$): $y_{xx}(x) = F\big(x, y(x), y_x(x)\big)$ is canonically associated a certain *dual* second-order ordinary differential equation, call it ($\mathscr{E}^*$): $b_{aa}(a) = F^*\big(a, b_a(a), b_{aa}(a)\big)$, which has the crucial property that:

$$\begin{aligned} &\mathsf{I}^1_{(\mathscr{E})} \quad \text{is a nonzero multiple of} \quad \mathsf{I}^2_{(\mathscr{E}^*)} \\ \text{and symmetrically also:}\quad &\mathsf{I}^2_{(\mathscr{E})} \quad \text{is a nonzero multiple of} \quad \mathsf{I}^1_{(\mathscr{E}^*)}. \end{aligned}$$

The doctoral dissertation [10] of Koppisch (Leipzig 1905) cited only *passim* by Élie Cartan in [2] contains the analytical details of this correspondence, which was well reconstituted recently in [5] within the context of projective Cartan connections. But the differential equation which is dual to the one $w_{zz}(z) = \Phi\big(z,\, w(z),\, w_z(z)\big)$ associated to $w = \Theta(z, \overline{z}, \overline{w})$ is easily seen to be just its *complex conjugate* ($\overline{\mathscr{E}}$): $\overline{w}_{\overline{z}\overline{z}}(\overline{z}) = \overline{\Phi}\big(\overline{z},\, \overline{w}(\overline{z}),\, \overline{w}_{\overline{z}}\big)$, and then as a consequence, $\mathsf{I}^1_{(\overline{\mathscr{E}})} = \overline{\mathsf{I}^1_{(\mathscr{E})}}$ is the conjugate of $\mathsf{I}^1_{(\mathscr{E})}$, and similarly also $\mathsf{I}^2_{(\overline{\mathscr{E}})} = \overline{\mathsf{I}^2_{(\mathscr{E})}}$ is the conjugate of $\mathsf{I}^2_{(\mathscr{E})}$. So it is no mystery that, as said, the sphericality of M at the origin:

$$0 \equiv \mathsf{I}^1_{(\mathscr{E})} \qquad \text{and} \qquad 0 \equiv \mathsf{I}^2_{(\mathscr{E})} = \text{nonzero} \cdot \mathsf{I}^1_{(\overline{\mathscr{E}})} = \text{nonzero} \cdot \overline{\mathsf{I}^1_{(\mathscr{E})}},$$

[4] To our knowledge, the only existing reference where this strategy is seriously endeavoured in order to classify second-order ordinary differential equations $y_{xx}(x) = F\big(x, y(x), y_x(x)\big)$ is [8], but only for certain point transformations — called there "*fiber-preserwing*" — of the special form $(x,y) \mapsto (x', y') = \big(x'(x),\, y'(x,y)\big)$, the first component of which is independent of y.

[5] Three decades earlier, Christoffel in his famous memoir [4] of 1869 devoted to the equivalence problem for Riemannian metrics discovered that the covariant differentiations of the curvature provide a full list of differential invariants for positive definite quadratic infinitesimal metrics.

can in a simpler way be characterized by the vanishing of the two *mutually conjugate* (complex) equations:

$$0 \equiv \mathsf{I}^1_{(\mathscr{E})} \qquad \text{and} \qquad 0 \equiv \overline{\mathsf{I}^1_{(\mathscr{E})}},$$

which of course amount to just *one* (complex) equation.

To conclude this introduction, we would like to mention firstly that none of our computations — especially those of Sections 4 and 5 — was performed with the help of any computer, and secondly that the effective characterization of sphericality in higher complex dimension $n \geqslant 3$ will appear soon [22].

§2. Segre varieties and differential equations

Real analytic hypersurfaces in $\mathbb{C}^2$. Let us consider an arbitrary real analytic hypersurface M in $\mathbb{C}^2$ and let us localize it around one of its points, say $p \in M$. Then there exist complex affine coordinates:

$$(z, w) = \big(x + iy,\, u + iv\big)$$

vanishing at p in which $T_pM = \{u = 0\}$, so that M is represented in a neighborhood of p by a graphed defining equation of the form:

$$u = \varphi(x, y, v),$$

where the real-valued function:

$$\varphi = \varphi(x, y, v) = \sum_{\substack{k,l,m\in\mathbb{N} \\ k+l+m\geqslant 2}} \varphi_{k,l,m}\, x^k y^l v^m \in \mathbb{R}\big\{x, y, u\big\},$$

which possesses entirely arbitrary real coefficients $\varphi_{k,l,m}$, vanishes at the origin: $\varphi(0) = 0$, together with all its first order derivatives: $0 = \partial_x\varphi(0) = \partial_y\varphi(0) = \partial_v\varphi(0)$. All studies in the analytic reflection principle[6] show without doubt that the adequate geometric concepts: *Pair of Segre foliations*, *Segre chains*, *Complexified CR orbits*, *Jets of complexified Segre varietes*, *Rigidity of formal CR mappings*, *Nondegeneracy conditions*, *CR-reflection function*, can be viewed correctly only when M is represented by a so-called *complex defining equation*. Such an equation may be constructed by simply rewriting the initial real equation of M as:

$$\tfrac{w+\overline{w}}{2} = \varphi\big(\tfrac{z+\overline{z}}{2},\, \tfrac{z-\overline{z}}{2i},\, \tfrac{w-\overline{w}}{2i}\big),$$

and then by solving[7] the so written equation with respect to w, which yields an equation of the shape[8]:

$$w = \Theta\big(z,\, \overline{z},\, \overline{w}\big) = \sum_{\substack{\alpha,\,\beta,\,\gamma\,\in\,\mathbb{N} \\ \alpha+\beta+\gamma\geqslant 1}} \Theta_{\alpha,\beta,\gamma}\, z^\alpha\, \overline{z}^\beta\, \overline{w}^\gamma \in \mathbb{C}\big\{\overline{z},\, z,\, w\big\},$$

whose right-hand side converges of course near the origin $(0, 0, 0) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}$ and has *complex* coefficients $\Theta_{\alpha,\beta,\gamma} \in \mathbb{C}$. The paradox that any such *complex* equation provides in fact *two* real defining equations for the *real* hypersurface M which is

[6] The reader might for instance consult the survey [19], pp. 5–44 or the memoirs [16, 17], and look also at some of the concerned references therein.

[7] Thanks to $d\varphi(0) = 0$, the holomorphic implicit function theorem readily applies.

[8] Notice that since $d\varphi(0) = 0$, one has $\Theta = -\overline{w} + \text{order 2 terms}$.

one-codimensional, and also in addition the fact that one could as well have chosen to solve the above equation with respect to $\overline{w}$, instead of w, these two apparent "contradictions" are corrected by means of a fundamental, elementary statement that transfers to Θ (in a natural way) the condition of reality:

$$\overline{\varphi(x,y,u)} = \sum_{k+l+m\geqslant 1} \overline{\varphi_{k,l,m}}\, \overline{x}^k \overline{y}^l \overline{v}^m = \sum_{k+l+m\geqslant 1} \varphi_{k,l,m}\, x^k y^l v^m = \varphi(x,y,v)$$

enjoyed by the initial defininining function φ.

Theorem. ([19], p. 19[9]) *The complex analytic function* $\Theta = \Theta(z,\overline{z},\overline{w})$ *with* $\Theta = -\overline{w} + \mathsf{O}(2)$ *together with its complex conjugate*[10]*:*

$$\overline{\Theta} = \overline{\Theta}\big(\overline{z}, z, w\big) = \sum_{\alpha,\,\beta,\,\gamma\in\mathbb{N}} \overline{\Theta}_{\alpha,\beta,\gamma}\, \overline{z}^\alpha\, z^\beta\, w^\gamma \in \mathbb{C}\big\{\overline{z},\, z,\, w\big\}$$

satisfy the two (*equivalent by conjugation*) *functional equations*:

$$\begin{aligned} \overline{w} &\equiv \overline{\Theta}\big(\overline{z}, z, \Theta(z,\overline{z},\overline{w})\big), \\ w &\equiv \Theta\big(z,\overline{z},\overline{\Theta}(\overline{z},z,w)\big). \end{aligned} \tag{1}$$

Conversely, given a local holomorphic function $\Theta(z,\overline{z},\overline{w}) \in \mathbb{C}\{z,\overline{z},\overline{w}\}$, $\Theta = -\,\overline{w} + \mathsf{O}(2)$ *which, in conjunction with its conjugate* $\overline{\Theta}(\overline{z},z,w)$*, satisfies this pair of equivalent identities, then the two zero-sets:*

$$\big\{0 = -\,w + \Theta\big(z,\,\overline{z},\,\overline{w}\big)\big\} \qquad \textit{and} \qquad \big\{0 = -\,\overline{w} + \overline{\Theta}\big(\overline{z},\,z,\,w\big)\big\}$$

coincide and define a local one-codimensional *real analytic hypersurface* M *passing through the origin in* $\mathbb{C}^2$.

As before, let M be an arbitrary real analytic hypersurface passing through the origin in $\mathbb{C}^2$ equipped with coordinates (z,w), and assume that $T_0M = \{u = 0\}$. Without loss of generality, we can and we shall assume that the coordinates are chosen in such a way that a certain standard convenient normalization condition holds.

Theorem. ([16], p. 12) *There exists a local complex analytic change of holomorphic coordinates* $h\colon (z,w) \longmapsto (z',w') = h(z,w)$ *fixing the origin and tangent to the identity at the origin of the specific form*:

$$z' = z, \qquad w' = g(z,w),$$

such that the image $M' := h(M)$ *has a new complex defining equation* $w' = \Theta'\big(z',\overline{z}',\overline{w}'\big)$ *satisfying:*

$$\Theta'\big(0,\overline{z}',\overline{w}'\big) \equiv \Theta'\big(z',0,\overline{w}'\big) \equiv -\,\overline{w}',$$

[9] Compared to [19], we denote here by Θ the function denoted there by $\overline{\Theta}$.

[10] According to a general, common convention, given a power series $\Phi(t) = \sum_{\gamma\in\mathbb{N}^n} \Phi_\gamma\, t^\gamma$, $t \in \mathbb{C}^n$, $\Phi_\gamma \in \mathbb{C}$, one defines the series $\overline{\Phi}(t) := \sum_{\gamma\in\mathbb{N}^n} \overline{\Phi}_\gamma\, t^\gamma$ by conjugating only its complex coefficients. Then the complex conjugation operator distributes oneself simultaneously on functions and on variables: $\overline{\Phi(t)} \equiv \overline{\Phi}(\overline{t})$, a trivial property which is nonetheless frequently used in the formal CR reflection principle ([16, 17]).

or equivalently, which has a power series expansion of the form:

$$\Theta'\big(z',\overline{z}',\overline{w}'\big) = -\,\overline{w}' + \sum_{\alpha\geqslant 1,\ \beta\geqslant 1} \Theta'_{\alpha,\beta,0}\, z'^{\alpha}\overline{z}'^{\beta} + \sum_{\gamma\geqslant 1} \overline{w}'^{\gamma} \sum_{\alpha\geqslant 1,\ \beta\geqslant 1} \Theta'_{\alpha,\beta,\gamma}\, z'^{\alpha}\overline{z}'^{\beta}.$$

Levi nondegenerate hypersurfaces. Leaving aside the real defining equation of M, let us now rename the complex defining equation of M in such normalized coordinates simply as before: $w = \Theta(z,\overline{z},\overline{w})$, dropping all the prime signs. Quite concretely, the real analytic hypersurface M is said to be *Levi nondegenerate* at the origin if the coefficient $\Theta_{1,1,0}$ of $z\overline{z}$, which may be checked to always be real because of the reality condition (1), is *nonzero*. In fact, it is well known that Levi nondegeneracy is a biholomorphically invariant property, *see* for instance [19], p. 158, but in more conceptual terms, the following general characterization, which may be taken as a definition here, holds true. One then readily checks that it is equivalent to $\Theta_{1,1,0} \neq 0$ in normalized coordinates.

Lemma. ([16, 17, 20]) *The real analytic hypersurface $M \subset \mathbb{C}^2$ with $0 \in M$ represented in coordinates (z,w) by a complex defining equation of the form $w = \Theta(z,\overline{z},\overline{w})$ is Levi nondegenerate at the origin if and only if the map:*

$$\big(\overline{z},\overline{w}\big) \longmapsto \big(\Theta(0,\overline{z},\overline{w}),\ \Theta_z(0,\overline{z},\overline{w})\big)$$

has nonvanishing 2×2 Jacobian determinant at $(\overline{z},\overline{w}) = (0,0)$.

After a possible real dilation of the z-coordinate, we can therefore assume that $\Theta_{1,1,0} = 1$, and then we are provided with the following normalization:

$$w = -\,\overline{w} + z\overline{z} + z\overline{z}\,\mathsf{O}\big(|z| + |\overline{w}|\big), \tag{2}$$

that will be useful shortly. Another, even more convincing argument for consigning to oblivion the real defining equation $u = \varphi(x,y,v)$ dates back to Beniamino Segre [24], who observed that to any real analytic M are associated two deeply linked objects.

1) The nowadays so-called *Segre varieties*[11] $S_{\overline{q}}$ associated to any point $q \in \mathbb{C}^2$ near the origin of coordinates (z_q, w_q) that are the complex curves defined by the equation:

$$S_{\overline{q}} := \big\{0 = -\,w + \Theta\big(z,\,\overline{z}_q,\,\overline{w}_q\big)\big\},$$

quite appropriately in terms of the fundamental complex defining function Θ; this equation is *holomorphic* just because its antiholomorphic terms are set fixed.

2) When M is Levi nondegenerate at the origin, a second-order *complex* ordinary differential equation[12] of the form:

$$w_{zz}(z) = \Phi\big(z,\,w(z),\,w_z(z)\big),$$

[11] A presentation of the general theory, valuable for generic CR manifolds of arbitrary codimension $d \geqslant 1$ and of arbitrary CR dimension $m \geqslant 1$ in $\mathbb{C}^{m+d}$ enjoying no specific nondegeneracy condition, may be found in [16, 17, 19].

[12] This idea, usually attributed by contemporary CR geometers to B. Segre, dates in fact back (at least) to Chapter 10 of Volume 1 of the 2 100 pages long *Theorie der Transformationsgruppen* written by Sophus Lie and Friedrich Engel between 1884 and 1893, where it is even presented in the uppermost general context.

whose solutions are exactly the Segre varieties of M, parametrized by the two initial conditions $w(0)$ and $w_z(0)$ which correspond bijectively to the antiholomorphic variables $\overline{z}_q$ and $\overline{w}_q$.

In fact, the recipe for deriving the second-order differential equation associated to a local Levi-nondegenerate $M \subset \mathbb{C}^2$ with $0 \in M$ represented by a normalized[13] equation of the form (2) is very simple. Considering that $w = w(z)$ is given in the equation:

$$w(z) = \Theta\big(z,\, \overline{z},\, \overline{w}\big)$$

as a function of z with two supplementary (antiholomorphic) parameters $\overline{z}$ and $\overline{w}$ that one would like to eliminate, we solve with respect to $\overline{z}$ and $\overline{w}$, just by means of the implicit function theorem[14], the pair of equations:

$$\left[\begin{aligned} w(z) &= \Theta\big(z,\, \overline{z},\, \overline{w}\big) = -\,\overline{w} + z\overline{z} + z\overline{z}\,\mathsf{O}\big(|z| + |\overline{w}|\big) \\ w_z(z) &= \Theta_z\big(z,\, \overline{z},\, \overline{w}\big) = \overline{z} + \overline{z}\,\mathsf{O}\big(|z| + |\overline{w}|\big) \end{aligned}\right.$$

the second one being obtained by differentiating the first one with respect to z, and this yields a representation:

$$\overline{z} = \zeta\big(z,\, w(z),\, w_z(z)\big) \qquad \text{and} \qquad \overline{w} = \xi\big(z,\, w(z),\, w_z(z)\big)$$

for certain two uniquely defined local complex analytic functions $\zeta(z, w, w_z)$ and $\xi(z, w, w_z)$ of three complex variables. By means of these functions, we may then replace $\overline{z}$ and $\overline{w}$ in the second derivative:

$$\begin{aligned} w_{zz}(z) &= \Theta_{zz}\big(z,\, \overline{z},\, \overline{w}\big) \\ &= \Theta_{zz}\big(z,\, \zeta\big(z,\, w(z),\, w_z(z)\big),\ \xi\big(z,\, w(z),\, w_z(z)\big)\big) \\ &=: \Phi\big(z,\, w(z),\, w_z(z)\big), \end{aligned}$$

and this defines without ambiguity the associated differential equation. More about differential equations will be said in §3 below.

Of course, any spherical real analytic $M \subset \mathbb{C}^2$ must be Levi nondegenerate at every point, for the unit 3-sphere $S^3 \subset \mathbb{C}^2$ is. It is well known that S^3 minus one of its points, for instance: $S^3 \setminus \{p_\infty\}$ with $p_\infty := (0, -1)$, is biholomorphic, through the so-called *Cayley transform*:

$$(z, w) \longmapsto \Big(\tfrac{i\,z}{1+w},\ \tfrac{1-w}{2+2\,w}\Big) =: (z', w') \quad \text{having inverse:} \quad (z', w') \longmapsto \Big(\tfrac{-2iz'}{1+2w'},\ \tfrac{1-2w'}{1+2\,w'}\Big)$$

to the so-called *Heisenberg sphere* of equation:

$$w' = -\,\overline{w}' + z'\overline{z}',$$

in the target coordinate-space (z', w'), and this model will be more convenient to deal with for our purposes.

---

[13] In fact, such a normalization was made in advance just in order to make things concrete and clear, but thanks to what the Lemma on p. 7 expresses in a biholomorphically invariant way, everything which follows next holds in an arbitrary system of coordinates.

[14] Justification: by our preliminary normalization, the 2×2 Jacobian determinant $\frac{\partial(\Theta,\, \Theta_z)}{\partial(\overline{z},\, \overline{w})}$ computed at the origin equals $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$, hence is nonzero. Without the preliminary normalization, the condition of the Lemma on p. 7 also applies in any case.

Proposition. *A Levi nondegenerate local real analytic hypersurface M in $\mathbb{C}^2$ is locally biholomorphic to a piece of the Heisenberg sphere (hence spherical) if and only if its associated second-order ordinary complex differential equation is locally equivalent to the Newtonian free particle equation: $w'_{z'z'}(z') = 0$, with identically vanishing right-hand side.*

Proof. Indeed, any local equivalence of M to the Heisenberg sphere transforms its differential equation to the one associated with the Heisenberg sphere, and then trivially: $w'_{z'}(z') = \overline{z}'$, whence $w'_{z'z'}(z') = 0$.

Conversely, if the Segre varieties of M are mapped to the solutions of $w'_{z'z'}(z') = 0$, namely to the complex affine lines of $\mathbb{C}^2$, the complex defining equation of the transformed M' must necessarily be affine:

$$w' = \overline{\lambda}'\big(\overline{z}', \overline{w}'\big) + z'\, \overline{\mu}'\big(\overline{z}', \overline{w}'\big) =: \Theta'\big(z', \overline{z}', \overline{w}'\big), \tag{3}$$

with certain coefficients that are holomorphic with respect to $(\overline{z}', \overline{w}')$. Then $\overline{\lambda}'(0) = 0$ since the origin is fixed, and if $\overline{\mu}'(0)$ is nonzero, one performs the linear transformation $z' \mapsto z'$, $w' \mapsto w' - \overline{\mu}'(0)\, z'$, which stabilizes both $w'_{z'z'}(z') = 0$ and the form of (3), to insure then that $\overline{\mu}'(0) = 0$.

Next, the second reality condition (1) now reads:

$$w' \equiv \overline{\lambda}'\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big) + z'\, \overline{\mu}'\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big),$$

and by differentiating it with respect to $\overline{z}'$, we get, without writing the arguments for brevity:

$$\begin{aligned} 0 &\equiv \overline{\lambda}'_{\overline{z}'} + \overline{\Theta}'_{\overline{z}'}\, \overline{\lambda}'_{\overline{w}'} + z'\, \overline{\mu}'_{\overline{z}'} + z'\, \overline{\Theta}'_{\overline{z}'}\, \overline{\mu}'_{\overline{w}'} \\ &\equiv \overline{\lambda}'_{\overline{z}'} + \mu'\, \overline{\lambda}'_{\overline{w}'} + z'\, \overline{\mu}'_{\overline{z}'} + z'\, \mu'\, \overline{\mu}'_{\overline{w}'}, \end{aligned}$$

where we replace $\overline{\Theta}'_{\overline{z}'}$ in the second line by its value $\mu'(z', w')$. But with all the arguments, this identity reads in full length as the following identity holding in $\mathbb{C}\big\{\overline{z}', z', w'\big\}$:

$$\begin{aligned} &- \overline{\lambda}'_{\overline{z}'}\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big) - z'\, \overline{\mu}'_{\overline{z}'}\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big) \equiv \\ &\qquad \equiv \mu'(z', w')\, \overline{\lambda}'_{\overline{w}'}\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big) + z'\, \mu'(z', w')\, \overline{\mu}'_{\overline{w}'}\big(\overline{z}', \overline{\Theta}'\big(\overline{z}', z', w'\big)\big). \end{aligned}$$

For convenience, it is better to take $(z', \overline{z}', \overline{w}')$ as arguments of this identity instead of $(\overline{z}', z', w')$, so we simply replace w' in it by:

$$\Theta'\big(z', \overline{z}', \overline{w}'\big),$$

we apply the first reality condition (1) and we get what we wanted to pursue the reasonings:

$$\begin{aligned} -\overline{\lambda}'_{\overline{z}'}\big(\overline{z}', \overline{w}'\big) - z'\, \overline{\mu}'_{\overline{z}'}\big(\overline{z}', \overline{w}'\big) &\equiv \mu'\big(z', \overline{\lambda}'(\overline{z}', \overline{w}') + z'\, \overline{\mu}'(\overline{z}', \overline{w}')\big) \cdot \\ &\qquad \cdot \Big[\overline{\lambda}'_{\overline{w}'}\big(\overline{z}', \overline{w}'\big) + z'\, \overline{\mu}'_{\overline{w}'}\big(\overline{z}', \overline{w}'\big)\Big], \end{aligned} \tag{4}$$

i.e. an identity holding now in $\mathbb{C}\big\{z', \overline{z}', \overline{w}'\big\}$. The left-hand side being affine with respect to z', the same must be true of each one of the two factors of the right-hand side. In particular, the second order derivative of the first factor with respect to z' must vanish identically:

$$0 \equiv \partial_{z'}\partial_{z'}\big\{\mu'\big(z', \overline{\lambda}' + z'\overline{\mu'}\big)\big\}$$
$$\equiv \mu'_{z'z'} + 2\,\overline{\mu}'\,\mu'_{z'w'} + \overline{\mu}'\overline{\mu}'\mu'_{w'w'}.$$

Because M' is Levi nondegenerate at the origin, the lemma on p. 7 together with the affine form (3) of the defining equation entails that the map:

$$(5) \qquad \big(\overline{z}', \overline{w}'\big) \longmapsto \big(\overline{\lambda}'(\overline{z}', \overline{w}'),\ \overline{\mu}'(\overline{z}', \overline{w}')\big)$$

has nonvanishing Jacobian determinant at $(\overline{z}', \overline{w}') = (0, 0)$. Consequently, in the above identity (rewritten with some of the arguments):

$$0 \equiv \mu'_{z'z'}\big(z',\, \overline{\lambda}' + z'\,\overline{\mu}'\big) + 2\,\overline{\mu}'\,\mu'_{z'w'}\big(z',\, \overline{\lambda}' + z'\,\overline{\mu}'\big) + \overline{\mu}'\overline{\mu}'\,\mu'_{w'w'}\big(z',\, \overline{\lambda}' + z'\,\overline{\mu}'\big),$$

we can consider z', $\overline{\lambda}'$ and $\overline{\mu}'$ as being just three independent variables. Setting $\overline{\mu}' = 0$, we get $0 \equiv \mu'_{z'z'}\big(z',\, \overline{\lambda}'\big)$, that is to say: $\mu_{z'z'}(z', w') \equiv 0$ and then after division of $\overline{\mu}'$, we are left with only two terms:

$$0 \equiv 2\,\mu'_{z'w'}\big(z',\, \overline{\lambda}' + z'\,\overline{\mu}'\big) + \overline{\mu}'\,\mu'_{w'w'}\big(z',\, \overline{\lambda}' + z'\,\overline{\mu}'\big).$$

Then again $0 \equiv 2\,\mu_{z'w'}(z', w')$ and finally also $0 \equiv \mu_{w'w'}(z', w')$. This means that the function:

$$\mu'(z', w') = c_1' z' + c_2' w',$$

with some two constants $c_1', c_2' \in \mathbb{C}$, is *linear*.

Now, we claim that $c_2' = 0$ in fact. Indeed, setting $\overline{z}' = 0$ in (4), we get:

$$-\overline{\lambda}'_{\overline{z}'}\big(0, \overline{w}'\big) - z'\,\overline{c}_1' \equiv \big\{c_1'\,z' + c_2'\big(\overline{\lambda}'(0, \overline{w}') + z'\overline{c}_2'\overline{w}'\big)\big\} \cdot \big[\overline{\lambda}'_{\overline{w}'}(0, \overline{w}') + z'\overline{c}_2'\big].$$

The coefficient $c_2'\overline{c}_2'\overline{c}_2'$ of $(z')^2\overline{w}'$ in the right-hand side must vanish, so $c_2' = 0$. Since the rank at the origin of the map (5) equals 2, necessarily $\mu' \not\equiv 0$, so $c_1' \neq 0$, and then $c_1' = 1$ after a suitable dilation of the z'-axis. Next, rewriting the identity (4):

$$-\overline{\lambda}'_{\overline{z}'}\big(\overline{z}', \overline{w}'\big) - z' \equiv z'\big[\overline{\lambda}'_{\overline{w}'}\big(\overline{z}', \overline{w}'\big)\big],$$

we finally get $\overline{\lambda}'_{\overline{z}'} \equiv 0$ and $\overline{\lambda}'_{\overline{w}'} \equiv -1$, which means in conclusion that:

$$\lambda'(z', w') \equiv -\,w' \qquad \text{and} \qquad \mu'(z', w') \equiv z',$$

so that the equation of M' is the one: $w' = -\,\overline{w}' + z'\overline{z}'$ of the Heisenberg sphere in the target coordinates (z', w'). □

Thanks to this proposition, in order to characterize the sphericality of a local real analytic hypersurface $M \subset \mathbb{C}^2$ explicitly in terms of its complex defining function Θ, our strategy[15] will be to:

□ characterize the local equivalence to $w'_{z'z'}(z') = 0$ of the associated differential equation:

$$(6) \qquad w_{zz}(z) = \Theta_{zz}\big(z,\, \zeta\big(z,\, w(z),\, w_z(z)\big),\, \xi\big(z,\, w(z),\, w_z(z)\big)\big),$$

explicitly in terms of the three functions Θ_{zz}, ζ and ξ;

[15] — indicated already as the accessible Open Question 2.35 in [20] —

☐ eliminate any occurence of the two auxiliary functions ζ and ξ so as to re-express the obtained result only in terms of the sixth-order jet $J^6_{z,\overline{z},\overline{w}}\Theta$.

§3. GEOMETRY OF ASSOCIATED SUBMANIFOLDS OF SOLUTIONS

The characterization we will obtain holds in fact inside a broader context than just CR geometry, in terms of what we called in [20] the *submanifold of solutions* associated to any second-order ordinary differential equation, no matter whether it comes or not from a Levi nondegenerate $M \subset \mathbb{C}^2$. In fact, the elementary foundations towards a general theory embracing all systems of completely integrable partial differential equations was laid down [20], especially by producing explicit prolongation formulas for infinitesimal Lie symmetries, with many interesting problems that are still wide open as soon as the number of (independent or dependent) variables increases: construction of Cartan connections; production of differential invariants; full classification according to the Lie symmetry group.

Fortunately for our present purposes here, the geometry, the classification, and the Lie transformation group features of second order ordinary differential equations are essentially completely understood since the groundbreaking works of Lie [11], followed by a prized thesis by Tresse [25] and later by a celebrated memoir of Élie Cartan, *see* also [7] and the references therein.

Accordingly, letting $x \in \mathbb{K}$ and $y \in \mathbb{K}$ be two real or complex variables (with hence $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ throughout), consider any second-order ordinary differential equation:

$$y_{xx}(x) = F\big(x,\, y(x),\, y_x(x)\big)$$

having local $\mathbb{K}$-analytic right-hand side F, and denote it by $(\mathscr{E})$ for short. In the space of first-order jets of arbitrary graphing functions $y = y(x)$ that we equip with three independent coordinates denoted (x, y, y_x), let us introduce the vector field:

$$\mathsf{D} := \frac{\partial}{\partial x} + y_x\,\frac{\partial}{\partial y} + F(x,y,y_x)\,\frac{\partial}{\partial y_x},$$

whose integral curves inside the three-dimensional space (x, y, y_x) correspond, classically, to solving the equation $y_{xx}(x) = F(x, y(x), y_x(x))$ by transforming it into a system of two *first-order* differential equations with the two unknown functions $y(x)$ and $y_x(x)$.

Theorem. ([11, 25, 2, 7, 18]) *A second-order ordinary differential equation $y_{xx} = F(x, y, y_x)$ denoted $(\mathscr{E})$ with $\mathbb{K}$-analytic right-hand side possesses two fundamental differential invariants, namely:*

$$\begin{aligned}\mathsf{I}^1_{(\mathscr{E})} &:= F_{y_xy_xy_xy_x} \qquad \textit{and:}\\ \mathsf{I}^2_{(\mathscr{E})} &:= \mathsf{DD}\big(F_{y_xy_x}\big) - F_{y_x}\,\mathsf{D}\big(F_{y_xy_x}\big) - 4\,\mathsf{D}\big(F_{yy_x}\big) + \\ &\quad + 6\,F_{yy} - 3\,F_y\,F_{y_xy_x} + 4\,F_{y_x}\,F_{yy_x},\end{aligned}$$

while all other differential invariants are deduced from $\mathsf{I}^1_{(\mathscr{E})}$ and $\mathsf{I}^2_{(\mathscr{E})}$ by covariant (in the sense of Tresse) or coframe (in the sense of Cartan) diffentiations. Moreover, local equivalence to $y'_{x'x'}(x') = 0$ holds under some invertible local $\mathbb{K}$-analytic *point transformation:*

$$(x, y) \longmapsto (x', y') = \big(x'(x, y),\, y'(x, y)\big)$$

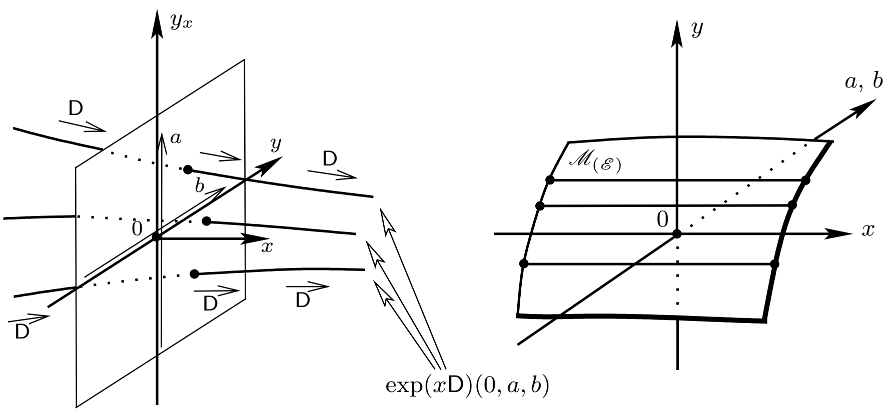
if and only if both invariants vanish:

$$0 = \mathsf{I}^1_{(\mathscr{E})} = \mathsf{I}^2_{(\mathscr{E})}.$$

In order to characterize sphericality of an $M \subset \mathbb{C}^2$, it is then natural and advisable to study what the vanishing of the above two differential invariants gives when applied to the second order ordinary differential equation (6) enjoyed by the defining function Θ. This goal will be pursued in §4 below.

For the time being, with the aim of extending such a kind of characterization to a broader scope, following §2 of [20], let us now recall how one may in a natural way construct a *sumanifold of solutions* $\mathscr{M}_{\mathscr{E}}$ associated to the differential equation $(\mathscr{E})$ which, when $(\mathscr{E})$ comes from a Levi nondegenerate local real analytic hypersurface $M \subset \mathbb{C}^2$, regives without any modification its complex defining equation $w = \Theta\big(z, \overline{z}, \overline{w}\big)$.

To begin with, in the first-order jet space (x, y, y_x) that we simply draw as a common three-dimensional space:



we *duplicate* the two dependent coordinates (y, y_x) by introducing a new subspace of coordinates $(a, b) \in \mathbb{K} \times \mathbb{K}$, and we draw a vertical plane containing the two new axes that are just parallel copies (for the moment, just look at the left-hand side). Then the leaves of the local foliation associated to the integral curves of the vector field D are uniquely determined by their intersection with this plane, because thanks to the presence of $\frac{\partial}{\partial x}$ in D, all these curves are approximately directed by the x-axis in a neighborhood of the origin: no tangent vector can be vertical. But we claim that all such intersection points of coordinates $(0, b, a) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K}$ correspond bijectively to the two initial conditions $y(0) \equiv b$ and $y_x(0) = a$ for solving uniquely the differential equation. In fact, the flow of D at time x starting from all such points $(0, b, a)$ of the duplicated vertical plane:

$$\exp(x\,\mathsf{D})(0, b, a) =: \big(x,\, Q(x, a, b),\, S(x, a, b)\big)$$

(*see* again the diagram) expresses itself in terms of two certain local $\mathbb{K}$-analytic functions Q and S that satisfy, by the very definition of the flow of our vector field

$\partial_x + y_x \, \partial_y + F \, \partial_{y_x}$, the following two differential equations:

$$\frac{d}{dx}Q(x,a,b) = S(x,a,b) \qquad \text{and:} \qquad \frac{d}{dx}S(x,a,b) = F\big(x, Q(x,a,b), S(x,a,b)\big)$$

together with the (obious) initial condition for $x = 0$:

$$(0,b,a) = \exp(0\,\mathsf{D})(0,b,a) = \big(0,\; Q(0,a,b),\; S(0,a,b)\big).$$

We notice *passim* that $S \equiv Q_x$ (no two symbols were in fact needed), and most importantly, we emphasize that in this way, we have viewed in a somewhat geometric-minded way of thinking that the *general solution*:

$$y = y(x) = Q\big(x,\; y_x(0),\; y(0)\big) = Q(x,a,b)$$

to the original differential equation arises naturally as the first (amongst two) graphing function for the integral curves of D in the first order jet space, these curves being parametrized by (a,b).

Definition. The *sumanifold of solutions*[16] $\mathscr{M}_{(\mathscr{E})}$ associated with the second-order ordinary differential equation $(\mathscr{E})$: $y_{xx}(x) = F\big(x,\, y(x),\, y_x(x)\big)$ is the local $\mathbb{K}$-analytic submanifold of the four-dimensional Euclidean space $\mathbb{K}_x \times \mathbb{K}_y \times \mathbb{K}_a \times \mathbb{K}_b$ represented as the zero-set:

$$0 = -y + Q(x,a,b),$$

where $Q(x,a,b)$ is the general local $\mathbb{K}$-analytic solution of $(\mathscr{E})$, satisfying therefore:

$$Q_{xx}\big(x,a,b\big) \equiv F\big(x,\; Q(x,a,b),\; Q_x(x,a,b)\big),$$

and $Q(0,a,b) = b$, $Q_x(0,a,b) = a$.

Conversely, let us assume we are given a submanifold $\mathscr{M}$ of $\mathbb{K}_x \times \mathbb{K}_y \times \mathbb{K}_a \times \mathbb{K}_b$ of the specific equation $y = Q(x,a,b)$, for a certain local $\mathbb{K}$-analytic function Q of the three variables (x,a,b). Call (x,y) the *variables*, (a,b) the parameters, and call $\mathscr{M}$ *solvable with respect to the parameters* (at the origin) if the map:

$$(a,b) \longmapsto \big(Q(0,a,b),\; Q_x(0,a,b)\big)$$

has rank two at the central point $(a,b) = (0,0)$. Of course, the submanifold of solutions associated to any second-order ordinary differential equation is solvable with respect to parameters, for in this case $Q(0,a,b) \equiv b$ and $Q_x(0,a,b) \equiv a$.

Similarly as what we did for deriving **2)** on p. 7, if an arbitrarily given submanifold $\mathscr{M}$ of $\mathbb{K}_x \times \mathbb{K}_y \times \mathbb{K}_a \times \mathbb{K}_b$ is assumed to be solvable with respect to parameters, then viewing y in $y = Q(x,a,b)$ as a parametrized function of x, the implicit function theorem enables one to solve (a,b) in the two equations:

$$\left[\begin{aligned} y(x) &= Q(x,a,b) \\ y_x(x) &= Q_x(x,a,b), \end{aligned}\right.$$

[16] At this point, the reader is referred to [20] for more about how one can develope the whole theory of Lie symmetries of partial differential equations intrinsically within submanifolds of solutions only; the theory of Cartan connections associated to certain exterior differential systems could (and should also) be transferred to submanifolds of solutions.

to yield both a representation for a and and a representation for b of the form:

$$\begin{cases} a = A\big(x, y(x), y_x(x)\big) \\ b = B\big(x, y(x), y_x(x)\big), \end{cases} \tag{7}$$

for certain two local $\mathbb{K}$-analytic functions A and B of three independent variables (x, y, y_x), that one may insert afterwards in the second order derivative:

$$\begin{aligned} y_{xx}(x) &= Q_{xx}\big(x, a, b\big) \\ &= Q_{xx}\big(x,\ A(x, y(x), y_x(x)),\ B(x, y(x), y_x(x))\big) \\ &=: F\big(x,\ y(x),\ y_x(x)\big), \end{aligned}$$

which yields the differential equation $(\mathscr{E}_{\mathscr{M}})$ associated to the submanifold $\mathscr{M}$ solvable with respect to the parameters. In summary:

Proposition. ([20]) *There is a one-to-one correspondence:*

$$(\mathscr{E}_{\mathscr{M}}) = (\mathscr{E}) \longleftrightarrow \mathscr{M} = \mathscr{M}_{(\mathscr{E})}$$

between second-order ordinary differential equations $(\mathscr{E})$ *of the general form:*

$$y_{xx}(x) = F\big(x,\ y(x),\ y_x(x)\big)$$

and submanifolds (*of solutions*) $\mathscr{M}$ *of equation:*

$$y = Q(x, a, b)$$

that are solvable with respect to the parameters, and this correspondence satisfies:

$$\big(\mathscr{E}_{\mathscr{M}_{(\mathscr{E})}}\big) = (\mathscr{E}) \qquad \textit{and} \qquad \mathscr{M}_{(\mathscr{E}_{\mathscr{M}})} = \mathscr{M}.$$

We now claim that solvability with respect to the parameters is an invariant condition, independently of the choice of coordinates. Indeed, let $y = Q(x, a, b)$ be any submanifold of solutions, call it $\mathscr{M}$, and let:

$$\big(x, y, a, b\big) \longmapsto \big(x'(x, y),\ y'(x, y),\ a,\ b\big)$$

be an arbitrary local $\mathbb{K}$-analytic diffeomorphism fixing the origin which leaves untouched the parameters. The vector of coordinates $\big(1,\ Q_x(x, a, b), 0, 0\big)$ based at the point $\big(x, Q(x, a, b),\ a, b\big)$ of $\mathscr{M}$ is sent, through such a diffeomorphism, to a vector whose x'-coordinate equals: $\frac{d}{dx}\big[x'(x, Q)\big] = x'_x + Q_x\, x'_y$. Therefore the implicit function theorem insures that, provided the expression:

$$x'_x(x, y) + Q_x(x, a, b)\, x'_y(x, y) \neq 0$$

does not vanish, the image $\mathscr{M}'$ of $\mathscr{M}$ through such a diffeomorphism can still be represented, locally in a neighborhood of the origin, as a graph of a similar form:

$$y' = Q'\big(x',\ a,\ b\big),$$

for a certain local $\mathbb{K}$-analytic new function $Q' = Q'(x', a, b)$. Since $\mathscr{M} : y = Q(x, a, b)$ is sent to $\mathscr{M}' : y' = Q'(x', a, b)$, it follows that $x'(x, y)$, $y'(x, y)$, $Q(x, a, b)$ and $Q'(x', a, b)$ are all linked by the following fundamental identity:

$$y'\big(x,\ Q(x, a, b)\big) \equiv Q'\big(x'(x,\ Q(x, a, b)),\ a, b\big), \tag{8}$$

which holds in $\mathbb{C}\big\{x, a, b\big\}$.

Claim. *If $\mathscr{M}$ is solvable with respect to the parameters (at the origin), then $\mathscr{M}'$ is also solvable with respect to the parameters (at the origin too), and conversely.*

Proof. The assumption that $\mathscr{M}$ is solvable with respect to the parameters is equivalent to the fact that its first order x-jet map:

$$\big(x,\,a,\,b\big)\longmapsto\big(x,\,Q(x,a,b),\,Q_x(x,a,b)\big)\big)$$

is (locally) of rank three. One should therefore look at the same first order jet map attached to $\mathscr{M}'$, represented in the right part of the following diagram:

$$\begin{array}{ccccc}
(x,a,b) & \longrightarrow & \big(x'(x,Q(x,a,b)),\,a,\,b\big) & & (x',a,b) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\big(x,\,Q(x,a,b),\,Q_x(x,a,b)\big) & \xrightarrow{\ \mathsf{X}\,?\ } & \big(x',\,Q'(x',a,b),\,Q'_{x'}(x',a,b)\big) & & \big(x',\,Q'(x',a,b),\,Q'_{x'}(x',a,b)\big)
\end{array},$$

and ask how these two x- and x'-jet maps can be related to each other, namely search for a map:

$$\mathsf{X}\,\boldsymbol{?}\colon\;\big(x,\,Q,\,Q_x\big)\longmapsto\big(x,\,Q',\,Q'_x\big)$$

which would close up the diagram and make it commutative.

The answer for the second component of the sought map is simply:

$$\mathsf{X}_2\colon\;\big(x,\,Q,\,Q_x\big)\longmapsto y'\big(x,\,Q\big),$$

since (9) indeed shows that composing the right vertical arrow with the upper horizontal one gives the same result, concerning a second component, as composing the bottom horizontal arrow with the left vertical one.

The answer for the third component of the sought map then proceeds by differentiating with respect to x the fundamental identity (9), which yields, without writing the arguments:

$$y'_x+Q_x\,y'_y\equiv\big[x'_x+Q_x\,x'_y\big]\,Q'_{x'},$$

and since $x'_x+Q_x\,x'_y\neq 0$ by assumption, it suffices to set:

$$\mathsf{X}_3\colon\;\big(x,\,Q,\,Q_x\big)\longmapsto\frac{y'_x(x,Q)+Q_x\,y'_y(x,Q)}{x'_x(x,Q)+Q_x\,x'_y(x,Q)},$$

in order to complete the commutativity of the diagram, namely to get:

$$Q'_{x'}\big(f(x,Q(x,a,b)),\,a,\,b\big)\equiv\frac{y'_x(x,Q(x,a,b))+Q_x(x,a,b)\,y'_y(x,Q(x,a,b))}{x'_x(x,Q(x,a,b))+Q_x(x,a,b)\,x'_y(x,Q(x,a,b))},$$

as was required. But now considering instead the inverse diffeomorphisme changes nothing to the reasonings, hence we have at the same time a right-inverse:

$$\begin{array}{ccccc}
(x,a,b) & \longrightarrow & \big(x'(x,Q(x,a,b)),\,a,\,b\big) & \longrightarrow & (x,a,b) \\
\big\downarrow{\scriptstyle x\text{-jet}} & & \big\downarrow{\scriptstyle x'\text{-jet}} & & \big\downarrow{\scriptstyle x\text{-jet}} \\
\big(x,\,Q(x,a,b),\,Q_x(x,a,b)\big) & \xrightarrow{\ \mathsf{X}\ } & \big(x',\,Q'(x',a,b),\,Q'_{x'}(x',a,b)\big) & \xrightarrow{\ \mathsf{X}^{-1}\ } & \big(x,\,Q(x,a,b),\,Q(x,a,b)\big)
\end{array}$$

of our commutative diagram, so that the x-jet map and the x'-jet map have coinciding ranks at pairs of points which correspond one to another. □

We are now in a position to generalize the characterization of sphericality derived earlier on p. 9.

Proposition. *A second-order ordinary differential equation $y_{xx}(x) = F(x, y(x), y_x(x))$ with $\mathbb{K}$-analytic right-hand side is equivalent, under some invertible local $\mathbb{K}$-analytic point transformation $(x, y) \mapsto (x', y')$, to the free particle Newtonian equation $y'_{x'x'}(x') = 0$ if and only if its associated submanifold of solutions $y = Q(x, a, b)$ is equivalent, under some local $\mathbb{K}$-analytic map in which variables are separated from parameters:*

$$(x, y, a, b) \longmapsto \big(x'(x, y),\, y'(x, y),\; a'(a, b),\, b'(a, b)\big)$$

to the affine submanifold of solutions of equation $y' = b' + x'a'$.

Before proceeding to the proof, let us observe that when one looks at a real analytic hypersurface $M \subset \mathbb{C}^2$, the corresponding transformation in the parameter space is constrained to be the *conjugate transformation* of the local biholomorphism:

$$\big(z, w,\, \overline{z}, \overline{w}\big) \longmapsto \big(z'(z, w),\, w'(z, w),\, \overline{z}'(\overline{z}, \overline{w}),\, \overline{w}'(\overline{z}, \overline{w})\big),$$

while one has more freedom for general differential equations, in the sense that transformations of variables and transformations of parameters are entirely *decoupled*.

Proof. One direction is clear: if $y = Q(x, a, b)$ is equivalent to:

$$y' = b' + x'a' = b'(a, b) + x'a'(a, b), \tag{9}$$

then its associated differential equation $y_{xx}(x) = F\big(x, y(x),\, y_x(x)\big)$ is equivalent, through the same diffeomorphism $(x, y) \mapsto (x', y')$ of the variables, to the differential equation associated with (9), which trivially is: $y'_{x'x'}(x') = 0$.

Conversely, if $y_{xx}(x) = F\big(x, y(x),\, y_x(x)\big)$ is equivalent, through a diffeomorphism $(x, y) \mapsto (x', y')$, to $y'_{x'x'}(x') = 0$, then its submanifold of solutions $y = Q(x, a, b)$ is transformed to $y' = Q'(x', a, b)$ and since $y'_{x'x'}(x') = 0$, the function Q' is necessarily of the form:

$$y' = b'(a, b) + x'\, a'(a, b).$$

Because the condition of solvability with respect to the parameters is invariant, the rank of $(a, b) \mapsto \big(a'(a, b),\, b'(a, b)\big)$ is again equal to 2, which concludes the proof. □

Coming now back to the wanted characterization of sphericality, our more general goal now amounts to characterize, *directly in terms of its fundamental solution function* $Q(x, a, b)$, the local equivalence to $y'_{x'x'}(x') = 0$ of a second-order ordinary differential equation $y_{xx}(x) = F\big(x,\, y(x),\, y_x(x)\big)$. Afterwards at the end, it will suffice to replace $Q(x, a, b)$ simply by $\Theta(z, \overline{z}, \overline{w})$ in the obtained equations.

But before going further, let us explain how a certain generalized projective duality will simplify our task, as already said in the Introduction. Thus, let $(\mathscr{E})$: $y_{xx}(x) = F\big(x, y(x), y_x(x)\big)$ be a differential equation as above having general solution $y = Q(x, a, b) = -b + xa + \mathrm{O}(x^2)$, with initial conditions $b = -y(0)$ and $a = y_x(0)$. The implicit function theorem enables us to solve b in the equation

$y = Q(x,a,b)$ of the associated submanifold of solutions $\mathscr{M}_{(\mathscr{E})}$ in terms of the other quantities, which yields an equation of the shape:

$$b = Q^*(a,x,y) = -y + ax + \mathsf{O}(x^2),$$

for some new local $\mathbb{K}$-analytic function $Q^* = Q^*(a,x,y)$. Then similarly as previously, we may eliminate x and y from the two equations:

$$\begin{aligned} b(a) &= Q^*(a,x,y) = -y + ax + \mathsf{O}(x^2) \\ b_a(a) &= Q^*_a(a,x,y) = x + \mathsf{O}(x^2), \end{aligned}$$

that is to say: $x = X\big(a, b(a), b_a(a)\big)$ and $y = Y\big(a, b(a), b_a(a)\big)$, and we then insert these two solutions in:

$$\begin{aligned} b_{aa}(a) &= Q^*_{aa}(a,x,y) \\ &= Q^*_{aa}\big(a,\, X(a,b(a),b_a(a)),\, Y(a,b(a),b_a(a))\big) \\ &=: F^*\big(a,\, b(a),\, b_a(a)\big). \end{aligned}$$

We shall call the so obtained second-order ordinary differential equation the *dual* of $y_{xx}(x) = F\big(x, y(x), y_x(x)\big)$.

In the case of a hypersurface $M \subset \mathbb{C}^2$, solving $\overline{w}$ in the equation $w = \Theta(z,\overline{z},\overline{w})$ gives nothing else but the *conjugate* equation $\overline{w} = \overline{\Theta}(\overline{z}, z, w)$, just by virtue of the reality identities (1). It also follows rather trivially that the dual differential equation:

$$\begin{aligned} \overline{w}_{\overline{z}\overline{z}}(\overline{z}) &= \overline{\Theta}_{\overline{z}\overline{z}}\big(\overline{z},\, \overline{\zeta}(\overline{z},\overline{w}(\overline{z}),\overline{w}_{\overline{z}}(\overline{z})),\, \overline{\xi}(\overline{z},\overline{w}(\overline{z}),\overline{w}_{\overline{z}}(\overline{z}))\big) \\ &= \overline{\Phi}\big(\overline{z},\overline{w}(\overline{z}),\overline{w}_{\overline{z}}(\overline{z})\big) \end{aligned}$$

is also just the *conjugate* differential equation.

To the differential equation $y_{xx} = F$ and to its dual $b_{aa} = F^*$ are associated two submanifolds of solutions:

$$\mathscr{M} = \mathscr{M}_{(\mathscr{E})} := \big\{\big(x,y,a,b\big) \in \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}\colon\ y = Q(x,a,b)\big\},$$

together with:

$$\mathscr{M}^* = \mathscr{M}_{(\mathscr{E}^*)} := \big\{\big(a,b,x,y\big) \in \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}\colon\ b = Q^*(a,x,y)\big\},$$

and as one obviously guesses, the duality, when viewed within submanifolds of solutions, just amounts to permute variables and parameters:

$$\mathscr{M} \ni (x,y,a,b) \longleftrightarrow (a,b,x,y) \in \mathscr{M}^*.$$

In the CR case, if we denote by $\widetilde{z}$ and $\widetilde{w}$ two independent complex variables which correspond to the complexifications of $\overline{z}$ and $\overline{w}$ (respectively of course), the duality takes place between the so-called *extrinsic complexification* ([14, 15, 16, 17, 19, 20]):

$$\mathscr{M} = M^c := \big\{\big(z,w,\widetilde{z},\widetilde{w}\big) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\colon\ w = \Theta\big(z,\widetilde{z},\widetilde{w}\big)\big\}$$

of M in one hand, and in the other hand, its own transformation[17]:

$$\mathscr{M}^* = *^c(M^c) := \big\{\big(\widetilde{z},\widetilde{w},z,w\big) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\colon\ \widetilde{w} = \overline{\Theta}\big(\widetilde{z},z,w\big)\big\}$$

[17] Be careful not to write $\big\{\big(z,w,\widetilde{z},\widetilde{w}\big)\colon\ \widetilde{w} = \overline{\Theta}\big(\widetilde{z},z,w\big)\big\}$, because this would regive the same subset $\mathscr{M}$ of $\mathbb{C}^2\times\mathbb{C}^2$, due to the reality identities (1).

under the involution:

$$*^c\big(z, w, \widetilde{z}, \widetilde{w}\big) := \big(\widetilde{z}, \widetilde{w}, z, w\big)$$

which clearly is the complexification of the natural antiholomorphic involution:

$$*\big(z, w, \overline{z}, \overline{w}\big) := \big(\overline{z}, \overline{w}, z, w\big)$$

that fixes M pointwise, as it fixes any other *real* analytic subset of $\mathbb{C}^2$. Here, one has $\mathscr{M}^* = *(\mathscr{M})$ — which is $\neq \mathscr{M}$ in general — and of course also $\big(\mathscr{M}^*\big)^* = \mathscr{M}$.

So in terms of the coordinates (x, a, b) on $\mathscr{M}$ and of the coordinates (a, x, y) on $\mathscr{M}^*$, the duality is the map:

$$(x, a, b) \longmapsto \big(a,\ x,\ Q(x, a, b)\big)$$

with inverse:

$$(a, x, y) \longmapsto \big(x, a, Q^*(a, x, y)\big).$$

But we may also express the duality from the first jet (x, y, y_x)-space to the first jet (a, b, b_a)-space by simply composing the following three maps, the central one being the duality $\mathscr{M} \to \mathscr{M}^*$:

$$\left(\begin{array}{c} (a, x, y) \\ \downarrow \\ \big(a, Q^*(a, x, y), Q_a^*(a, x, y)\big) \end{array}\right) \circ \Big((x, a, b) \to \big(a, x, Q(x, a, b)\big)\Big) \circ \left(\begin{array}{c} \big(x, A(x, y, y_x), B(x, y, y_x)\big) \\ \uparrow \\ (x, y, y_x) \end{array}\right),$$

which in sum gives us the map:

$$(x, y, y_x) \longmapsto \left(\begin{array}{cc} A(x, y, y_x), & Q^*\big(A(x, y, y_x), x, Q\big(x, A(x, y, y_x), B(x, y, y_x)\big)\big), \\ & Q_a^*\big(A(x, y, y_x), x, Q\big(x, A(x, y, y_x), B(x, y, y_x)\big)\big) \end{array}\right).$$

With the approximations, one checks that:

$$(x, y, y_x) \longmapsto \big(y_x + \cdots,\ -y + x y_x + \cdots,\ x + \cdots\big),$$

where the remainder terms "$+\cdots$" are all $\mathsf{O}(x^2)$. For the differential equation $y_{xx}(x) = 0$ of affine lines, these remainders disappear completely and we recover the classical projective duality written in inhomogeneous coordinates ([5], pp. 156–157). Furthermore, one shows (*see e.g.* [5]) that the above duality map within first order jet spaces is a *contact transformation*, namely through it, the pullback of the standard contact form $db - b_a da$ in the target space is a nonzero multiple of the standard contact form $dy - y_x dx$ in the source space.

But what matters more for us is the following. The two fundamental differential invariants of $b_{aa}(a) = F^*\big(a, b(a), b_a(a)\big)$ are functions exactly similar to the ones written on p. 11, namely:

$$\begin{aligned} \mathsf{I}^1_{(\mathscr{E}^*)} &:= F^*_{b_a b_a b_a b_a} \\ \mathsf{I}^2_{(\mathscr{E}^*)} &:= \mathsf{D}^*\mathsf{D}^*\big(F^*_{b_a b_a}\big) - F^*_{b_a}\,\mathsf{D}^*\big(F^*_{b_a b_a}\big) - 4\,\mathsf{D}^*\big(F^*_{b b_a}\big) + \\ &\qquad + 6\,F^*_{bb} - 3\,F^*_b\,F^*_{b_a b_a} + 4\,F^*_{b_a}\,F^*_{b b_a}, \end{aligned}$$

where $\mathsf{D}^* := \partial_a + b_a\,\partial_b + F^*(a, b, b_a)\,\partial_{b_a}$. Then according to Koppisch ([10]), through the duality map, $\mathsf{I}^1_{(\mathscr{E})}$ is transformed to a nonzero multiple of $\mathsf{I}^2_{(\mathscr{E}^*)}$, and simultaneously also, $\mathsf{I}^2_{(\mathscr{E})}$ is transformed to a nonzero multiple[18] of $\mathsf{I}^1_{(\mathscr{E}^*)}$, so that:

$$\begin{aligned} 0 = \mathsf{I}^1_{(\mathscr{E})} \quad &\Longleftrightarrow \quad \mathsf{I}^2_{(\mathscr{E}^*)} = 0 \\ 0 = \mathsf{I}^2_{(\mathscr{E})} \quad &\Longleftrightarrow \quad \mathsf{I}^1_{(\mathscr{E}^*)} = 0. \end{aligned}$$

Consequently, the differential equation ($\mathscr{E}$): $y_{xx}(x) = F\big(x, y(x), y_x(x)\big)$ is equivalent to $y'_{x'x'}(x') = 0$ if and only if:

$$\boxed{F_{y_x y_x y_x y_x} = 0 \qquad \text{and} \qquad F^*_{b_a b_a b_a b_a} = 0}\,.$$

This observation has essentially no practical interest, because the computation of F^* in terms of F relies upon the composition of three maps ... *except notably in the CR case*, since the duality in this case is complex conjugation: $\Phi^* = \overline{\Phi}$. In summary, we have established the following.

Proposition. *An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^2$ which is Levi nondegenerate at one of its points p and has a complex defining equation of the form:*

$$w = \Theta\big(z,\, \overline{z},\, \overline{w}\big)$$

in some system of local holomorphic coordinates $(z, w) \in \mathbb{C}^2$ centered at p, is spherical at p if and only if *the right-hand side Φ of its uniquely associated second-order ordinary complex differential equation:*

$$w_{zz}(z) = \Phi\big(z,\, w(z),\, w_z(w)\big)$$

satisfies the single *fourth-order partial differential equation:*

$$0 \equiv \Phi_{w_z w_z w_z w_z}\big(z, w, w_z\big).$$

It now only remains to re-express this fourth-order partial differential equation in terms of the complex graphing function $\Theta(z, \overline{z}, \overline{w})$ for M. We will achieve this more generally for $F_{y_x y_x y_x y_x}$.

§4. EFFECTIVE DIFFERENTIAL CHARACTERIZATION OF SPHERICALITY IN $\mathbb{C}^2$

Reminding the reasonings and notations introduced in a neighborhood of equation (7), the transformation:

$$\big(x, y, y_x\big) \longmapsto \big(x, a, b\big)$$

and its inverse are given by the two triples of functions:

$$\left[\begin{aligned} x &= x \\ a &= A(x, y, y_x) \\ b &= B(x, y, y_x) \end{aligned}\right. \qquad \text{and} \qquad \left[\begin{aligned} x &= x \\ y &= Q(x, a, b) \\ y_x &= Q_x(x, a, b). \end{aligned}\right.$$

Equivalently, one has the two pairs of identically satisfied equations:

$$\begin{aligned} a &\equiv A\big(x,\, Q(x,a,b),\, Q_x(x,a,b)\big) \\ b &\equiv B\big(x,\, Q(x,a,b),\, Q_x(x,a,b)\big) \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &\equiv Q\big(x,\, A(x,y,y_x),\, B(x,y,y_x)\big) \\ y_x &\equiv Q_x\big(x,\, A(x,y,y_x),\, B(x,y,y_x)\big). \end{aligned}$$

[18] To be precise, both factors of multiplicity ([5], p. 165) are nonvanishing in a neighborood of the origin, but for our purposes, it suffices just that they are not identically zero power series.

Differentiating the second column of equations with respect to x, to y and to y_x yields:

$$
\begin{aligned}
0 &= Q_x + Q_a\,A_x + Q_b\,B_x & \qquad 0 &= Q_{xx} + Q_{xa}\,A_x + Q_{xb}\,B_x \\
1 &= \qquad Q_a\,A_y + Q_b\,B_y & \qquad 0 &= \qquad Q_{xa}\,A_y + Q_{xb}\,B_y \\
0 &= \qquad Q_a\,A_{y_x} + Q_b\,B_{y_x} & \qquad 1 &= \qquad Q_{xa}\,A_{y_x} + Q_{xb}\,B_{y_x}.
\end{aligned}
$$

Then thanks to a straightforward application of the rule of Cramer for 2×2 linear systems, we derive six useful formulas.

Lemma. ([20], p. 9) *All the six first order derivatives A_x, A_y, A_{y_x}, B_x, B_y, B_{y_x} of the two functions A and B with respect to their three arguments (x, y, y_x) may be expressed as follows in terms of the second jet $J^2(Q)$ of the defining function Q:*

$$
\begin{aligned}
A_x &= \frac{Q_b\,Q_{xx} - Q_x\,Q_{xb}}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}, & \qquad B_x &= \frac{Q_x\,Q_{xa} - Q_a\,Q_{xx}}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}, \\
A_y &= \frac{Q_{xb}}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}, & \qquad B_y &= \frac{-Q_{xa}}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}, \\
A_{y_x} &= \frac{-Q_b}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}, & \qquad B_{y_x} &= \frac{Q_a}{Q_a\,Q_{xb} - Q_b\,Q_{xa}}.
\end{aligned}
$$

For future abbreviation, we shall denote the single appearing denominator — which evidently is the common determinant of all the three 2×2 linear systems involved above — simply by a square symbol:

$$\Delta := Q_a Q_{xb} - Q_b Q_{xa}.$$

The two-ways transfer between functions G defined in the (x, y, y_x)-space and functions T defined in the (x, a, b)-space, namely the one-to-one correspondence:

$$G(x, y, y_x) \longleftrightarrow T(x, a, b)$$

may be read very concretely as the following two equivalent identities:

$$
\begin{aligned}
G(x, y, y_x) &\equiv T\big(x,\ A(x, y, y_x),\ B(x, y, y_x)\big) \\
G\big(x,\ Q(x, a, b),\ Q_x(x, a, b)\big) &\equiv T(x, a, b),
\end{aligned}
$$

holding in $\mathbb{K}\{x, y, y_x\}$ and in $\mathbb{K}\{x, a, b\}$ respectively. By differentiating the first identity, the chain rule shows how the three first-order derivation operators (basic vector fields) ∂_x, ∂_y and ∂_{y_x} living in the (x, y, y_x)-space are transformed into the (x, a, b)-space:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x} + \left(\frac{Q_b\,Q_{xx} - Q_x\,Q_{xb}}{\Delta}\right)\frac{\partial}{\partial a} + \left(\frac{Q_x\,Q_{xa} - Q_a\,Q_{xx}}{\Delta}\right)\frac{\partial}{\partial b} \\
\frac{\partial}{\partial y} &= \left(\frac{Q_{xb}}{\Delta}\right)\frac{\partial}{\partial a} + \left(\frac{-Q_{xa}}{\Delta}\right)\frac{\partial}{\partial b} \\
\frac{\partial}{\partial y_x} &= \left(\frac{-Q_b}{\Delta}\right)\frac{\partial}{\partial a} + \left(\frac{Q_a}{\Delta}\right)\frac{\partial}{\partial b}.
\end{aligned}
$$

Lemma. *The total differentiation operator $\mathsf{D} = \partial_x + y_x\,\partial_y + F\,\partial_{y_x}$ associated to $y_{xx} = F(x, y, y_x)$ simply transfers to the basic derivation operator along the x-direction:*

$$\mathsf{D} \longleftrightarrow \partial_x.$$

Proof. Reading the three formulas just preceding, by adding the first one to the second one multiplied by $y_x = Q_x$ together with the third one multiplied by $F = Q_{xx}$, one visibly sees that the coefficients of both $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ do vanish in the obtained sum, as announced. $\square$

Keeping in mind — so as to avoid any confusion — that the same letter x is used to denote simultaneously the independent variable of the differential equation $y_{xx} = F(x, y, y_x)$ and the non-parameter variable of the associated submanifold of solutions $y = Q(x, a, b)$, we may now write this two-ways transfer $\mathsf{D} \longleftrightarrow \partial_x$ exactly as we did in the above three equations, namely simply as an equality between two derivations living in the (x, y, y_x)-space and in the (x, a, b)-space:

$$\mathsf{D} = \partial_x.$$

Lemma. *With $G = G(x, y, y_x)$ being any local $\mathbb{K}$-analytic function in the (x, y, y_x)-space, the three second-order derivatives $G_{y_x y_x}$, $G_{y y_x}$ and G_{yy} express as follows in terms of the second-order jet $J^2_{x,a,b}(T)$ of the defining function T:*

$$\begin{aligned}
G_{y_x y_x} = & \frac{Q_b\,Q_b}{\Delta^2}\,T_{aa} - \frac{2\,Q_a\,Q_b}{\Delta^2}\,T_{ab} + \frac{Q_a\,Q_a}{\Delta^2}\,T_{bb} + \\
& + \frac{T_a}{\Delta^3}\bigg(Q_a\,Q_a \begin{vmatrix} Q_b & Q_{bb} \\ Q_{xb} & Q_{xbb} \end{vmatrix} - 2\,Q_a\,Q_b \begin{vmatrix} Q_b & Q_{ab} \\ Q_{xb} & Q_{xab} \end{vmatrix} + Q_b\,Q_b \begin{vmatrix} Q_b & Q_{aa} \\ Q_{xb} & Q_{xaa} \end{vmatrix}\bigg) + \\
& + \frac{T_b}{\Delta^3}\bigg(-\,Q_a\,Q_a \begin{vmatrix} Q_a & Q_{bb} \\ Q_{xa} & Q_{xbb} \end{vmatrix} + 2\,Q_a\,Q_b \begin{vmatrix} Q_a & Q_{ab} \\ Q_{xa} & Q_{xab} \end{vmatrix} - Q_b\,Q_b \begin{vmatrix} Q_a & Q_{aa} \\ Q_{xa} & Q_{xaa} \end{vmatrix}\bigg)
\end{aligned}$$

$$\begin{aligned}
G_{y y_x} = & -\frac{Q_b\,Q_{xb}}{\Delta^2}\,T_{aa} + \frac{Q_a\,Q_{xb} + Q_b\,Q_{xa}}{\Delta^2}\,T_{ab} - \frac{Q_a\,Q_{xa}}{\Delta^2}\,T_{bb} + \\
& + \frac{T_a}{\Delta^3}\bigg(-\,Q_a\,Q_{xa} \begin{vmatrix} Q_b & Q_{bb} \\ Q_{xb} & Q_{xbb} \end{vmatrix} + \big(Q_a\,Q_{xb} + Q_b\,Q_{xa}\big) \begin{vmatrix} Q_b & Q_{ab} \\ Q_{xb} & Q_{xab} \end{vmatrix} - Q_b\,Q_{xb} \begin{vmatrix} Q_b & Q_{aa} \\ Q_{xb} & Q_{xaa} \end{vmatrix}\bigg) + \\
& + \frac{T_b}{\Delta^3}\bigg(Q_a\,Q_{xa} \begin{vmatrix} Q_a & Q_{bb} \\ Q_{xa} & Q_{xbb} \end{vmatrix} - \big(Q_a\,Q_{xb} + Q_b\,Q_{xa}\big) \begin{vmatrix} Q_a & Q_{ab} \\ Q_{xa} & Q_{xab} \end{vmatrix} + Q_b\,Q_{xb} \begin{vmatrix} Q_a & Q_{aa} \\ Q_{xa} & Q_{xaa} \end{vmatrix}\bigg)
\end{aligned}$$

$$\begin{aligned}
G_{yy} = & \frac{Q_{xb}\,Q_{xb}}{\Delta^2}\,T_{aa} - \frac{2\,Q_{xa}\,Q_{xb}}{\Delta^2}\,T_{ab} + \frac{Q_{xa}\,Q_{xa}}{\Delta^2}\,T_{bb} + \\
& + \frac{T_a}{\Delta^3}\bigg(Q_{xa}\,Q_{xa} \begin{vmatrix} Q_b & Q_{bb} \\ Q_{xb} & Q_{xbb} \end{vmatrix} - 2\,Q_{xa}\,Q_{xb} \begin{vmatrix} Q_b & Q_{ab} \\ Q_{xb} & Q_{xab} \end{vmatrix} + Q_{xb}\,Q_{xb} \begin{vmatrix} Q_b & Q_{aa} \\ Q_{xb} & Q_{xaa} \end{vmatrix}\bigg) + \\
& + \frac{T_b}{\Delta^3}\bigg(-\,Q_{xa}\,Q_{xa} \begin{vmatrix} Q_a & Q_{bb} \\ Q_{xa} & Q_{xbb} \end{vmatrix} + 2\,Q_{xa}\,Q_{xb} \begin{vmatrix} Q_a & Q_{ab} \\ Q_{xa} & Q_{xab} \end{vmatrix} - Q_{xb}\,Q_{xb} \begin{vmatrix} Q_a & Q_{aa} \\ Q_{xa} & Q_{xaa} \end{vmatrix}\bigg).
\end{aligned}$$

Proof. We apply the operator $\frac{\partial}{\partial y_x}$, wiewed in the (x, a, b)-space, to the first order derivative G_{y_x}, namely we consider:

$$\partial_{y_x}\big(G_{y_x}\big) = \frac{\partial}{\partial y_x}\bigg[-\,\frac{Q_b}{\Delta}\,T_a + \frac{Q_a}{\Delta}\,T_b\bigg],$$

and we then expand carefully the result by collecting somewhat in advance the obtained terms with respect to the derivatives of T:

$$\begin{aligned} G_{y_x y_x} &= \Big(-\frac{Q_b}{\Delta}\frac{\partial}{\partial a}+\frac{Q_a}{\Delta}\frac{\partial}{\partial b}\Big)\Big[-\frac{Q_b}{\Delta}T_a+\frac{Q_a}{\Delta}T_b\Big] \\ &= \Big(\frac{Q_b}{\Delta}\frac{Q_{ab}}{\Delta}-\frac{Q_b}{\Delta}\frac{Q_b\,\Delta_a}{\Delta^2}\Big)T_a+\frac{Q_b}{\Delta}\frac{Q_b}{\Delta}T_{aa}+ \\ &\quad+\Big(-\frac{Q_b}{\Delta}\frac{Q_{aa}}{\Delta}+\frac{Q_b}{\Delta}\frac{Q_a\,\Delta_a}{\Delta^2}\Big)T_b-\frac{Q_b}{\Delta}\frac{Q_a}{\Delta}T_{ab}+ \\ &\quad+\Big(-\frac{Q_a}{\Delta}\frac{Q_{bb}}{\Delta}+\frac{Q_a}{\Delta}\frac{Q_b\,\Delta_b}{\Delta^2}\Big)T_a-\frac{Q_a}{\Delta}\frac{Q_b}{\Delta}T_{ab}+ \\ &\quad+\Big(\frac{Q_a}{\Delta}\frac{Q_{ab}}{\Delta}-\frac{Q_a}{\Delta}\frac{Q_a\,\Delta_b}{\Delta^2}\Big)T_b+\frac{Q_a}{\Delta}\frac{Q_a}{\Delta}T_{bb}. \end{aligned}$$

The terms involving T_{aa}, T_{ab}, T_{bb} are exactly the ones exhibited by the lemma for the expression of $G_{y_x y_x}$. In the four large parentheses which are coefficients of T_a, T_b, T_a, T_b, we replace the occurences of Δ_a, Δ_a, Δ_b, Δ_b simply by:

$$\begin{aligned} \Delta_a &= Q_{xb}\,Q_{aa}+Q_a\,Q_{xab}-Q_{xa}\,Q_{ab}-Q_b\,Q_{xaa} \\ \Delta_b &= Q_{xb}\,Q_{ab}+Q_a\,Q_{xbb}-Q_{xa}\,Q_{bb}-Q_b\,Q_{xab}, \end{aligned}$$

and the total sum of terms coefficiented by T_a in our expression now becomes:

$$\begin{aligned} &\frac{T_a}{\Delta^3}\Big(Q_b\,Q_{ab}\big[Q_a\,Q_{xb}-Q_b\,Q_{xa}\big]-Q_b\,Q_b\big[Q_{xb}\,Q_{aa}+Q_a\,Q_{xab}-Q_{xa}\,Q_{ab}-Q_b\,Q_{xaa}\big]- \\ &\qquad -Q_a\,Q_{bb}\big[Q_a\,Q_{xb}-Q_b\,Q_{xa}\big]+Q_a\,Q_b\big[Q_{xb}\,Q_{ab}+Q_a\,Q_{xbb}-Q_{xa}\,Q_{bb}-Q_b\,Q_{xab}\big]\Big)= \\ &=\frac{T_a}{\Delta^3}\Big(Q_a\,Q_b\,Q_{xb}\,Q_{ab}-\underline{Q_b\,Q_b\,Q_{xa}\,Q_{ab}}_{①}-Q_b\,Q_b\,Q_{xb}\,Q_{aa}-Q_a\,Q_b\,Q_b\,Q_{xab}+ \\ &\qquad\qquad +\underline{Q_b\,Q_b\,Q_{xa}\,Q_{ab}}_{①}+Q_b\,Q_b\,Q_b\,Q_{xaa}- \\ &\qquad -Q_a\,Q_a\,Q_{xb}\,Q_{bb}+\underline{Q_a\,Q_b\,Q_{xa}\,Q_{bb}}_{②}+Q_a\,Q_b\,Q_{xb}\,Q_{ab}+Q_a\,Q_a\,Q_b\,Q_{xbb}- \\ &\qquad\qquad -\underline{Q_a\,Q_b\,Q_{xa}\,Q_{bb}}_{②}-Q_a\,Q_b\,Q_b\,Q_{xab}\Big)= \\ &=\frac{T_a}{\Delta^3}\Big(Q_a\,Q_a\big[Q_b\,Q_{xbb}-Q_{xb}\,Q_{bb}\big]-2\,Q_a\,Q_b\big[Q_b\,Q_{xab}-Q_{xb}\,Q_{ab}\big]+ \\ &\qquad\qquad +Q_b\,Q_b\big[Q_b\,Q_{xaa}-Q_{xb}\,Q_{aa}\big]\Big), \end{aligned}$$

so that we now have effectively reconstituted the three 2×2 determinants appearing in the second line of the expression claimed by the lemma for the transfer of $G_{y_x y_x}$ to the (x,a,b)-space. The treatment of the coefficient of $\frac{T_b}{\Delta^3}$ makes only a few differences, hence will be skipped here (but not in the manuscript). Finally, the two remaining expressions for $G_{y y_x}$ and for G_{yy} are obtained by performing entirely analogous algebrico-differential computations. □

End of the proof of the Main Theorem. Applying the above formula for $G_{y_x y_x}$ with $x := z$, with $a := \overline{z}$, with $b := \overline{w}$, with $\Delta := \Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}$, with $G := \Phi$ and with $T := \Theta_{zz}$, we exactly get the expression $\mathsf{AJ}^4(\Theta)$ of the Introduction, and then its further derivative $\partial_{y_x}\partial_{y_x}\big[G_{y_xy_x}\big] = G_{y_xy_xy_xy_x}$ is exactly:

$$0 \equiv \bigg(\frac{-\,\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}}\,\frac{\partial}{\partial\overline{z}} + \frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}}\,\frac{\partial}{\partial\overline{w}}\bigg)^2 \big[\mathsf{AJ}^4(\Theta)\big]$$
$$=: \frac{\mathsf{AJ}^6(\Theta)}{[\Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}]^7}.$$

As we have said, the vanishing of the second invariant of $w_{zz}(z) = \Phi\big(z, w(z), w_z(z)\big)$ amounts to the complex conjugation of the above equation, which is then obviously redundant. Thus, the proof of the Main Theorem is now complete, but we will nevertheless discuss in a specific final section what $\mathsf{AJ}^6(\Theta)$ would look like in purely expanded form. □

§5. SOME COMPLETE EXPANSIONS: EXAMPLES OF EXPRESSION SWELLINGS

Coming back to the non-CR context with the submanifold of solutions $\mathscr{M}_{(\mathscr{E})} = \big\{y = Q(x,a,b)\big\}$, let us therefore figure out how to expand the expression differentiated twice:

$$G_{y_xy_xy_xy_x} = \bigg(-\,\frac{Q_b}{\Delta}\,\frac{\partial}{\partial a} + \frac{Q_a}{\Delta}\,\frac{\partial}{\partial b}\bigg)^2 \bigg\{\frac{Q_b\,Q_b}{\Delta^2}\,T_{aa} - \frac{2\,Q_a\,Q_b}{\Delta^2}\,T_{ab} + \frac{Q_a\,Q_a}{\Delta^2}\,T_{bb} +$$
$$+\frac{T_a}{\Delta^3}\bigg(Q_a\,Q_a\begin{vmatrix} Q_b & Q_{bb} \\ Q_{xb} & Q_{xbb}\end{vmatrix} - 2\,Q_a\,Q_b\begin{vmatrix} Q_b & Q_{ab} \\ Q_{xb} & Q_{xab}\end{vmatrix} + Q_b\,Q_b\begin{vmatrix} Q_b & Q_{aa} \\ Q_{xb} & Q_{xaa}\end{vmatrix}\bigg) +$$
$$+\frac{T_b}{\Delta^3}\bigg(-\,Q_a\,Q_a\begin{vmatrix} Q_a & Q_{bb} \\ Q_{xa} & Q_{xbb}\end{vmatrix} + 2\,Q_a\,Q_b\begin{vmatrix} Q_a & Q_{ab} \\ Q_{xa} & Q_{xab}\end{vmatrix} - Q_b\,Q_b\begin{vmatrix} Q_a & Q_{aa} \\ Q_{xa} & Q_{xaa}\end{vmatrix}\bigg)\bigg\},$$

which would make the Main Theorem a bit more precise and explicit.

First of all, we notice that, in the formulas for $G_{y_xy_x}$, for G_{yy_x}, for G_{yy}, all the appearing 2×2 determinants happen to be modifications of the basic Jacobian-like Δ-determinant:

$$\Delta\big(a|b\big) := \Delta = \begin{vmatrix} Q_a & Q_b \\ Q_{xa} & Q_{xb}\end{vmatrix},$$

and we will denote them accordingly by employing the following (formally and intuitively clear) notations:

$$\Delta\big(b|bb\big) := \begin{vmatrix} Q_b & Q_{bb} \\ Q_{xb} & Q_{xbb}\end{vmatrix} \qquad \Delta\big(b|ab\big) := \begin{vmatrix} Q_b & Q_{ab} \\ Q_{xb} & Q_{xab}\end{vmatrix} \qquad \Delta\big(b|aa\big) := \begin{vmatrix} Q_b & Q_{aa} \\ Q_{xb} & Q_{xaa}\end{vmatrix}$$
$$\Delta\big(a|bb\big) := \begin{vmatrix} Q_a & Q_{bb} \\ Q_{xa} & Q_{xbb}\end{vmatrix} \qquad \Delta\big(a|ab\big) := \begin{vmatrix} Q_a & Q_{ab} \\ Q_{xa} & Q_{xab}\end{vmatrix} \qquad \Delta\big(a|aa\big) := \begin{vmatrix} Q_a & Q_{aa} \\ Q_{xa} & Q_{xaa}\end{vmatrix},$$

the bottom line always coinciding with the differentiation with respect to x of the top line. These abbreviations will be very appropriate for the next explicit computation, so let us rewrite the formula for $G_{y_xy_x}$ using this newly introduced formalism:

$$G_{y_xy_x} = \frac{1}{\Delta(a|b)^3}\bigg\{T_{aa}\big[Q_b\,Q_b\,\Delta(a|b)\big] + T_{ab}\big[-\,2\,Q_a\,Q_b\,\Delta(a|b)\big] + T_{bb}\big[Q_a\,Q_a\,\Delta(a|b)\big] +$$
$$+\,T_a\big[Q_a\,Q_a\,\Delta(b|bb) - 2\,Q_a\,Q_b\,\Delta(b|ab) + Q_b\,Q_b\,\Delta(b|aa)\big] +$$
$$+\,T_b\big[-\,Q_a\,Q_a\,\Delta(a|bb) + 2\,Q_a\,Q_b\,\Delta(a|ab) - Q_b\,Q_b\,\Delta(a|aa)\big]\bigg\}.$$

Then the twelve partial derivatives with respect to a and with respect to b of all the six determinants $\Delta\big(*|*\big)$ appearing in the the second line are easy to write down:

$$\begin{aligned}
&\tfrac{\partial}{\partial b}\big[\Delta\big(b|bb\big)\big] = \underline{\Delta\big(bb|bb\big)}_0 + \Delta\big(b|bbb\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(b|bb\big)\big] = \Delta\big(ab|bb\big) + \Delta\big(b|abb\big)\\
&\tfrac{\partial}{\partial b}\big[\Delta\big(b|ab\big)\big] = \Delta\big(bb|ab\big) + \Delta\big(b|abb\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(b|ab\big)\big] = \underline{\Delta\big(ab|ab\big)}_0 + \Delta\big(b|aab\big)\\
&\tfrac{\partial}{\partial b}\big[\Delta\big(b|aa\big)\big] = \Delta\big(bb|aa\big) + \Delta\big(b|aab\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(b|aa\big)\big] = \Delta\big(ab|aa\big) + \Delta\big(b|aaa\big)\\
&\tfrac{\partial}{\partial b}\big[\Delta\big(a|bb\big)\big] = \Delta\big(ab|bb\big) + \Delta\big(a|bbb\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(a|bb\big)\big] = \Delta\big(aa|bb\big) + \Delta\big(a|abb\big)\\
&\tfrac{\partial}{\partial b}\big[\Delta\big(a|ab\big)\big] = \underline{\Delta\big(ab|ab\big)}_0 + \Delta\big(a|abb\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(a|ab\big)\big] = \Delta\big(aa|ab\big) + \Delta\big(a|aab\big)\\
&\tfrac{\partial}{\partial b}\big[\Delta\big(a|aa\big)\big] = \Delta\big(ab|aa\big) + \Delta\big(a|aab\big) && \tfrac{\partial}{\partial a}\big[\Delta\big(a|aa\big)\big] = \underline{\Delta\big(aa|aa\big)}_0 + \Delta\big(a|aaa\big),
\end{aligned}$$

and the underlined terms vanish for the trivial reason that any 2×2 determinant, two columns of which coincide, vanishes. Consequently, we may now endeavour the computation of the third order derivative:

$$G_{y_xy_xy_x} = \bigg(-\frac{Q_b}{\Delta}\frac{\partial}{\partial a} + \frac{Q_a}{\Delta}\frac{\partial}{\partial b}\bigg)\big[G_{y_xy_x}\big].$$

When applying the two derivations in parentheses to:

$$G_{y_xy_x} = \tfrac{1}{\Delta^3}\big\{\mathsf{expression}\big\}$$

we start out by differentiating $\frac{1}{\Delta^3}$ multiplied by $\mathsf{expression}$, and then we differentiate $\mathsf{expression}$. Before any contraction, the full expansion of:

$$\Delta^5\, G_{y_xy_xy_x} =$$

(we indeed clear out the denominator Δ^5) is then:

$$\begin{aligned}
&= T_{aa}\big[3Q_bQ_bQ_b\Delta(a|b)\Delta(aa|b) + 3Q_bQ_bQ_b\Delta(a|b)\Delta(a|ab) - 3Q_aQ_bQ_b\Delta(a|b)\Delta(ab|b) - 3Q_aQ_bQ_b\Delta(a|b)\Delta(a|bb)\big]+\\
&+ T_{ab}\big[-6Q_aQ_bQ_b\Delta(a|b)\Delta(aa|b) - 6Q_aQ_bQ_b\Delta(a|b)\Delta(a|ab) + 6Q_aQ_aQ_b\Delta(a|b)\Delta(ab|a) + 6Q_aQ_aQ_b\Delta(a|b)\Delta(a|bb)\big]+\\
&+ T_{bb}\big[3Q_aQ_aQ_b\Delta(a|b)\Delta(aa|b) + 3Q_aQ_aQ_b\Delta(a|b)\Delta(a|ab) - 3Q_aQ_aQ_a\Delta(a|b)\Delta(ab|b) - 3Q_aQ_aQ_a\Delta(a|b)\Delta(a|bb)\big]+\\
&+T_a\Big[3Q_aQ_aQ_b\Delta(b|bb)\Delta(aa|b) + 3Q_aQ_aQ_b\Delta(b|bb)\Delta(a|ab) - 3Q_aQ_aQ_a\Delta(b|bb)\Delta(ab|b) - 3Q_aQ_aQ_a\Delta(b|bb)\Delta(a|bb)-\\
&\qquad - 6Q_aQ_bQ_b\Delta(b|ab)\Delta(aa|b) - 6Q_aQ_bQ_b\Delta(b|ab)\Delta(a|ab) + 6Q_aQ_aQ_b\Delta(b|ab)\Delta(ab|b) + 6Q_aQ_aQ_b\Delta(b|ab)\Delta(a|bb)+\\
&\qquad + 3Q_bQ_bQ_b\Delta(b|aa)\Delta(aa|b) + 3Q_bQ_bQ_b\Delta(b|aa)\Delta(a|ab) - 3Q_aQ_bQ_b\Delta(b|aa)\Delta(ab|b) - 3Q_aQ_bQ_b\Delta(b|aa)\Delta(a|bb)\Big]+\\
&+T_b\Big[3Q_aQ_aQ_b\Delta(a|bb)\Delta(aa|b) + 3Q_aQ_aQ_b\Delta(a|bb)\Delta(a|ab) - 3Q_aQ_aQ_a\Delta(a|bb)\Delta(ab|b) - 3Q_aQ_aQ_a\Delta(a|bb)\Delta(a|bb)-\\
&\qquad - 6Q_aQ_bQ_b\Delta(a|ab)\Delta(aa|b) - 6Q_aQ_bQ_b\Delta(a|ab)\Delta(a|ab) + 6Q_aQ_aQ_b\Delta(a|ab)\Delta(ab|b) + 6Q_aQ_aQ_b\Delta(a|ab)\Delta(a|bb)+\\
&\qquad + 3Q_bQ_bQ_b\Delta(a|aa)\Delta(aa|b) + 3Q_bQ_bQ_b\Delta(a|aa)\Delta(a|ab) - 3Q_aQ_bQ_b\Delta(a|aa)\Delta(ab|b) - 3Q_aQ_bQ_b\Delta(a|aa)\Delta(a|bb)\Big]+\\
&+\Delta(a|b)T_{aaa}\big[-Q_bQ_bQ_b\Delta(a|b)\big] + T_{aab}\big[3Q_aQ_bQ_b\Delta(a|b)\big] + T_{abb}\big[-3Q_aQ_aQ_b\Delta(a|b)\big] + T_{bbb}\big[Q_aQ_aQ_a\Delta(a|b)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{aa}\big[-2Q_bQ_bQ_{ab}\Delta(a|b) - Q_bQ_bQ_b\Delta(aa|b) - Q_bQ_bQ_b\Delta(a|ab)+\\
&\qquad\qquad\qquad + 2Q_aQ_bQ_b\Delta(a|b) + Q_aQ_bQ_bQ_b\Delta(ab|b) + Q_aQ_bQ_b\Delta(a|bb)\big]+\\
&\qquad +\Delta(a|b)T_{ab}\big[2Q_bQ_bQ_{aa}\Delta(a|b) + 2Q_aQ_bQ_{ab}\Delta(a|b) + 2Q_aQ_bQ_b\Delta(aa|b) + 2Q_aQ_bQ_b\Delta(a|ab)-\\
&\qquad\qquad\qquad - 2Q_aQ_bQ_{ab}\Delta(a|b) - 2Q_aQ_aQ_{bb}\Delta(a|b) - 2Q_aQ_aQ_b\Delta(ab|b) - 2Q_aQ_aQ_b\Delta(a|bb)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{bb}\big[-2Q_aQ_bQ_{aa}\Delta(a|b) - Q_aQ_aQ_b\Delta(aa|b) - Q_aQ_aQ_b\Delta(a|ab)+\\
&\qquad\qquad\qquad + 2Q_aQ_aQ_{ab}\Delta(a|b) + Q_aQ_aQ_a\Delta(ab|b) + Q_aQ_aQ_a\Delta(a|bb)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{aa}\big[-Q_aQ_aQ_b\Delta(b|bb) + 2Q_aQ_bQ_b\Delta(b|ab) - Q_bQ_bQ_b\Delta(b|aa)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{ab}\big[Q_aQ_aQ_a\Delta(b|bb) - 2Q_aQ_aQ_b\Delta(b|ab) + Q_aQ_bQ_b\Delta(b|aa)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{ba}\big[Q_aQ_aQ_b\Delta(a|bb) - 2Q_aQ_bQ_b\Delta(a|ab) + Q_bQ_bQ_b\Delta(a|aa)\big]+\\
&\qquad\qquad +\Delta(a|b)T_{bb}\big[-Q_aQ_aQ_a\Delta(a|bb) + 2Q_aQ_aQ_b\Delta(a|ab) - Q_aQ_bQ_b\Delta(a|aa)\big]+
\end{aligned}$$

$$
\begin{aligned}
+\Delta(a|b)T_a\Big[& -2Q_aQ_bQ_{aa}\Delta(b|bb) - Q_aQ_aQ_b\Delta(ab|bb) - Q_aQ_aQ_b\Delta(b|abb) + \\
& + 2Q_bQ_bQ_{aa}\Delta(b|ab) + 2Q_aQ_bQ_{ab}\Delta(b|ab) + \underline{2Q_aQ_bQ_b\Delta(ab|ab)}_0 + 2Q_aQ_bQ_b\Delta(b|aab) - \\
& - 2Q_bQ_bQ_{ab}\Delta(b|aa) - Q_bQ_bQ_b\Delta(ab|aa) - Q_bQ_bQ_b\Delta(b|aaa) + \\
& + 2Q_aQ_aQ_{ab}\Delta(b|bb) + \underline{Q_aQ_aQ_a\Delta(bb|bb)}_0 + Q_aQ_aQ_a\Delta(b|bbb) - \\
& - 2Q_aQ_bQ_{ab}\Delta(b|ab) - 2Q_aQ_aQ_{bb}\Delta(b|ab) - 2Q_aQ_aQ_b\Delta(bb|ab) - 2Q_aQ_aQ_b\Delta(b|abb) + \\
& + 2Q_aQ_bQ_{bb}\Delta(b|aa) + Q_aQ_bQ_b\Delta(bb|aa) + Q_aQ_bQ_b\Delta(b|aab)\Big] +
\end{aligned}
$$

$$
\begin{aligned}
+\Delta(a|b)T_b\Big[& 2Q_aQ_bQ_{aa}\Delta(a|bb) + Q_aQ_aQ_b\Delta(aa|bb) + Q_aQ_aQ_b\Delta(a|abb) - \\
& - 2Q_bQ_bQ_{aa}\Delta(a|ab) - 2Q_aQ_bQ_{ab}\Delta(a|ab) - \underline{2Q_aQ_bQ_b\Delta(aa|ab)}_0 - 2Q_aQ_bQ_b\Delta(a|aab) + \\
& + 2Q_bQ_bQ_{ab}\Delta(a|aa) + Q_bQ_bQ_b\Delta(aa|aa) + Q_bQ_bQ_b\Delta(a|aaa) - \\
& - 2Q_aQ_aQ_{ab}\Delta(a|bb) - \underline{Q_aQ_aQ_a\Delta(ab|bb)}_0 - Q_aQ_aQ_a\Delta(a|bbb) + \\
& + 2Q_aQ_bQ_{ab}\Delta(a|ab) + 2Q_aQ_aQ_{bb}\Delta(a|ab) + 2Q_aQ_aQ_b\Delta(ab|ab) + 2Q_aQ_aQ_b\Delta(a|abb) - \\
& - 2Q_aQ_bQ_{bb}\Delta(a|aa) - Q_aQ_bQ_b\Delta(ab|aa) - Q_aQ_bQ_b\Delta(a|aab)\Big].
\end{aligned}
$$

The simplification (collecting all terms) gives:

$$
G_{y_xy_xy_x} = \frac{1}{[\Delta(a|b)]^5}\left\{
\begin{aligned}
& T_{aaa}\big[-Q_b^3\Delta(a|b)^2\big] + T_{aab}\big[3Q_aQ_b^2\Delta(a|b)^2\big] + \\
& + T_{abb}\big[-3Q_a^2Q_b\Delta(a|b)^2\big] + T_{bbb}\big[Q_a^3\Delta(a|b)^2\big] +
\end{aligned}\right.
$$

$$
\begin{aligned}
+T_{aa}\Big[& -2Q_b^2Q_{ab}\Delta(a|b)^2 + 2Q_aQ_bQ_{bb}\Delta(a|b)^2 + 3Q_b^3\Delta(a|b)\Delta(aa|b) + 2Q_b^3\Delta(a|b)\Delta(a|ab) - \\
& - 4Q_aQ_b^2\Delta(a|b)\Delta(ab|b) - 2Q_aQ_b^2\Delta(a|b)\Delta(a|bb) - Q_a^2Q_b\Delta(a|b)\Delta(b|bb)\Big] +
\end{aligned}
$$

$$
\begin{aligned}
+T_{ab}\Big[& -2Q_a^2Q_{bb}\Delta(a|b)^2 + 2Q_bQ_bQ_{aa}\Delta(a|b)^2 + Q_a^3\Delta(a|b)\Delta(b|bb) + 6Q_a^2Q_b\Delta(a|b)\Delta(ab|b) + \\
& + Q_b^3\Delta(a|b)\Delta(a|aa) - 6Q_aQ_b^2\Delta(a|b)\Delta(a|ab) + 5Q_a^2Q_b\Delta(a|b)\Delta(a|bb) - 5Q_aQ_b^2\Delta(a|b)\Delta(aa|b)\Big] +
\end{aligned}
$$

$$
\begin{aligned}
+T_{bb}\Big[& -2Q_aQ_bQ_{aa}\Delta(a|b)^2 + 2Q_a^2Q_{ab}\Delta(a|b)^2 - 3Q_a^3\Delta(a|b)\Delta(a|bb) - 2Q_a^3\Delta(a|b)\Delta(ab|b) + \\
& + 4Q_a^2Q_b\Delta(a|b)\Delta(a|ab) + 2Q_a^2Q_b\Delta(a|b)\Delta(aa|b) - Q_aQ_b^2\Delta(a|b)\Delta(a|aa)\Big] +
\end{aligned}
$$

$$
\begin{aligned}
+T_a\Big[& 3Q_a^2Q_b\Delta(aa|b)\Delta(b|bb) + 3Q_a^2Q_b\Delta(a|ab)\Delta(b|bb) - 3Q_a^3\Delta(ab|b)\Delta(b|bb) - 3Q_a^3\Delta(a|bb)\Delta(b|bb) - \\
& - 6Q_aQ_b^2\Delta(aa|b)\Delta(b|ab) - 6Q_aQ_b^2\Delta(a|ab)\Delta(b|ab) + 6Q_a^2Q_b\Delta(ab|b)\Delta(b|ab) + 6Q_a^2Q_b\Delta(a|bb)\Delta(b|ab) - \\
& - 3Q_b^3\Delta(aa|b)\Delta(b|aa) - 3Q_b^3\Delta(a|ab)\Delta(b|aa) + 3Q_aQ_b^2\Delta(ab|b)\Delta(b|aa) + 3Q_aQ_b^2\Delta(a|bb)\Delta(b|aa) - \\
& - 2Q_aQ_bQ_{aa}\Delta(a|b)\Delta(b|bb) + 2Q_b^2Q_{aa}\Delta(a|b)\Delta(b|ab) + 2Q_aQ_bQ_{ab}\Delta(a|b)\Delta(b|ab) - 2Q_b^2Q_{ab}\Delta(a|b)\Delta(b|aa) - \\
& - Q_a^2Q_b\Delta(a|b)\Delta(ab|bb) - Q_a^2Q_b\Delta(a|b)\Delta(b|abb) + 2Q_aQ_b^2\Delta(a|b)\Delta(b|aab) - Q_b^3\Delta(a|b)\Delta(ab|aa) - Q_b^3\Delta(a|b)\Delta(b|aaa) + \\
& + 2Q_a^2Q_{ab}\Delta(a|b)\Delta(b|bb) - 2Q_aQ_bQ_{ab}\Delta(a|b)\Delta(b|ab) - 2Q_a^2Q_{bb}\Delta(a|b)\Delta(b|ab) + 2Q_aQ_bQ_{bb}\Delta(a|b)\Delta(b|aa) + \\
& + Q_a^3\Delta(a|b)\Delta(b|bbb) - 2Q_a^2Q_b\Delta(a|b)\Delta(bb|ab) - 2Q_a^2Q_b\Delta(a|b)\Delta(b|abb) + Q_aQ_b^2\Delta(a|b)\Delta(bb|aa) + Q_aQ_b^2\Delta(a|b)\Delta(b|aab)\Big] +
\end{aligned}
$$

$$
\begin{aligned}
+T_b\Big[& 3Q_a^2Q_b\Delta(a|bb)\Delta(aa|b) + 3Q_a^2Q_b\Delta(a|bb)\Delta(a|ab) - 3Q_a^3\Delta(a|bb)\Delta(ab|b) - 3Q_a^3\Delta(a|bb)\Delta(a|bb) - \\
& - 6Q_aQ_b^2\Delta(a|ab)\Delta(aa|b) - 6Q_aQ_b^2\Delta(a|ab)\Delta(a|ab) + 6Q_a^2Q_b\Delta(a|ab)\Delta(ab|b) + 6Q_aQ_aQ_b\Delta(a|ab)\Delta(a|bb) + \\
& + 3Q_b^2\Delta(a|aa)\Delta(aa|b) + 3Q_b^2\Delta(a|aa)\Delta(a|ab) - 3Q_aQ_b^2\Delta(a|aa)\Delta(ab|b) - 3Q_aQ_b^2\Delta(a|aa)\Delta(a|bb) + \\
& + 2Q_aQ_bQ_{aa}\Delta(a|b)\Delta(a|bb) - 2Q_b^2Q_{aa}\Delta(a|b)\Delta(a|ab) - 2Q_aQ_bQ_{ab}\Delta(a|b)\Delta(a|ab) + 2Q_b^2Q_{ab}\Delta(a|b)\Delta(a|aa) + \\
& + Q_a^2Q_b\Delta(a|b)\Delta(aa|bb) + Q_a^2Q_b\Delta(a|b)\Delta(a|abb) - 2Q_aQ_b^2\Delta(a|b)\Delta(a|aab) + Q_b^3\Delta(a|b)\Delta(aa|aa) + Q_b^3\Delta(a|b)\Delta(a|aaa) - \\
& - 2Q_a^2Q_{ab}\Delta(a|b)\Delta(a|bb) + 2Q_aQ_bQ_{ab}\Delta(a|b)\Delta(a|ab) + 2Q_a^2Q_{bb}\Delta(a|b)\Delta(a|ab) - 2Q_aQ_bQ_{bb}\Delta(a|b)\Delta(a|aa) - \\
& - Q_a^3\Delta(a|b)\Delta(a|bbb) + 2Q_a^2Q_b\Delta(a|b)\Delta(ab|ab) + 2Q_a^2Q_b\Delta(a|b)\Delta(a|abb) - Q_aQ_b^2\Delta(a|b)\Delta(ab|aa) - Q_aQ_b^2\Delta(a|b)\Delta(a|aab)\Big].
\end{aligned}
$$

The full expansion of $G_{y_xy_xy_xy_x}$ will not be presented here.

REFERENCES

[1] Bryant, R.L.: *Élie Cartan and geometric duality*, Journées Élie Cartan 1998 & 1999, Inst. É. Cartan **16** (2000), 5–20.

[2] Cartan, É.: *Sur les variétés à connexion projective*, Bull. Soc. Math. France **52** (1924), 205–241.

[3] Cartan, É.: *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, II*, Ann. Scuola Norm. Sup. Pisa **1** (1932), 333–354.

[4] Christoffel, E.B.: *Über die Transformation der homogenen Differentialausdrücke zweiten Grades*, J. reine angew. Math. **70** (1869), 46–70.

[5] Crampin, M.; Saunders, D.J.: *Cartan's concept of duality for second-order ordinary differential equations*, J. Geom. Phys. **54** (2005), 146–172.

[6] Engel, F.; Lie, S.: *Theorie der transformationsgruppen. Erster Abschnitt*. Unter Mitwirkung von Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1888. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).

[7] Grissom, C.; Thompson, G.; Wilkens, G.: *Linearization of second-order ordinary differential equations via Cartan's equivalence method*, J. Diff. Eq. **77** (1989), no. 1, 1–15.

[8] Hsu, L.; Kamran, N.: *Classification of second order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries*, Proc. London Math. Soc. **58** (1989), no. 3, 387–416.

[9] Isaev, A.V.: *Zero CR-curvature equations for rigid and tube hypersurfaces*, Complex Variables and Elliptic Equations, **54** (2009), no. 3-4, 317–344.

[10] Koppisch, M.A.: *Zur Invariantentheorie der gewöhlichen Differentialgleichungen zweiter Ordnung*, Inaugural Dissertation, Leipzig, B.-G. Teubner, 1905.

[11] Lie, S.: *Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestaten I-IV*. In: Gesammelte Abhandlungen, Vol. 5, B.G. Teubner, Leipzig, 1924, pp. 240–310; 362–427, 432–448.

[12] Lie, S.; Scheffers, G.: *Vorlesungen über continuirlichen gruppen, mit geometrischen und anderen anwendungen*, B.-G. Teubner, Leipzig, 1893. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1971).

[13] Merker, J.: *Convergence of formal biholomorphisms between minimal holomorphically nondegenerate real analytic hypersurfaces*, Int. J. Math. Math. Sci. **26** (2001), no. 5, 281–302.

[14] Merker, J.: *On the partial algebraicity of holomorphic mappings between real algebraic sets*, Bull. Soc. Math. France **129** (2001), no. 3, 547–591.

[15] Merker, J.: *On envelopes of holomorphy of domains covered by Levi-flat hats and the reflection principle*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 5, 1443–1523.

[16] Merker, J.: *On the local geometry of generic submanifolds of $\mathbb{C}^n$ and the analytic reflection principle*, Journal of Mathematical Sciences (N. Y.) **125** (2005), no. 6, 751–824.

[17] Merker, J.: *Étude de la régularité analytique de l'application de réflexion CR formelle*, Annales Fac. Sci. Toulouse, **XIV** (2005), no. 2, 215–330.

[18] Merker, J.: *Explicit differential characterization of the Newtonian free particle system in $m \geqslant 2$ dependent variables*, Acta Mathematicæ Applicandæ, **92** (2006), no. 2, 125–207.

[19] Merker, J.; Porten, E.: *Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities*, International Mathematics Research Surveys, Volume **2006**, Article ID 28295, 287 pages.

[20] Merker, J.: *Lie symmetries of partial differential equations and CR geometry*, Journal of Mathematical Sciences (N.Y.), to appear (2009), arxiv.org/abs/math/0703130

[21] Merker, J.: *Sophus Lie, Friedrich Engel et le problème de Riemann-Helmholtz*, Hermann Éditeurs, Paris, 2010, à paraître, 307 pp., arxiv.org/abs/0910.0801

[22] Merker, J.: *Vanishing Hachtroudi curvature and spherical real analytic hypersurfaces*, arxiv.org, to appear.

[23] Nurowski, P.; Sparling, G.A.J.: *3-dimensional Cauchy-Riemann structures and $2^{\underline{nd}}$ order ordinary differential equations*, Class. Quant. Gravity, **20** (2003), 4995–5016.

[24] Segre, B.: *Intorno al problema di Poincaré della rappresentazione pseudoconforme*, Rend. Acc. Lincei, VI, Ser. **13** (1931), 676–683.

[25] Tresse, A.: *Détermination des invariants ponctuels de l'équation différentielle du second ordre $y'' = \omega(x, y, y')$*, Hirzel, Leipzig, 1896.

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