EXPLICIT EXPRESSION OF CARTAN'S CONNECTION FOR LEVI-NONDEGENERATE 3-MANIFOLDS IN COMPLEX SURFACES, AND IDENTIFICATION OF THE HEISENBERG SPHERE

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ABSTRACT. We study effectively the Cartan geometry of Levi-nondegenerate $\mathscr{C}^6$-smooth hypersurfaces M^3 in $\mathbb{C}^2$. Notably, we present the so-called curvature function of a related Tanaka-type normal connection *explicitly in terms of a graphing function for* M, which is the initial, single available datum. Vanishing of this curvature function then characterizes explicitly the local biholomorphic equivalence of such $M^3 \subset \mathbb{C}^2$ to the Heisenberg sphere $\mathbb{H}^3$, such M's being necessarily real analytic.

1. INTRODUCTION

The concept of *Cartan geometry* appeared at the beginning of the twentieth century, when Élie Cartan was working on the so-called *equivalence problem*, the aim of which is to determine whether two given geometric structures can be mapped bijectively onto each other by some diffeomorphism. This problem can be considered in many different contexts, such as equivalences of submanifolds, of differential equations, of frames, of coframes and of other geometric structures. In the specific case of local real analytic hypersurfaces in $\mathbb{C}^2$, Poincaré (1907) initiated the study of the *Cauchy-Riemann* (CR for short) equivalence problem under biholomorphic transformations. Later, in 1932, this problem was solved in an essentially complete way by Cartan [10].

In general, Cartan also developed appropriate concepts and showed that one can reformulate several — somewhat hard — initial equivalence questions (*see* [23]) in terms of equivalences of coframes. Granted this, he devised an algorithm to decide whether two given manifolds M_1 and M_2 equipped with certain specific geometric structures encoded by means of coframes are equivalent. The main thrust is to construct two principal bundles $\mathscr{G}_1$ and $\mathscr{G}_2$ over M_1 and M_2 having the same structure group together with two coframes $\omega^1 := \{\omega_1^1, \ldots, \omega_1^n\}$ on $\mathscr{G}_1$ and $\omega^2 := \{\omega_2^1, \ldots, \omega_2^n\}$ on $\mathscr{G}_2$, such that M_1 and M_2 are equivalent if and only if there exists a diffeomorphism $\Phi : \mathscr{G}_1 \to \mathscr{G}_2$ commuting with projections, which

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sends ω^1 to ω^2, i.e.:

$$\Phi^*(\omega_2^i) = \omega_1^i, \qquad (i=1,\dots,n).$$

This also motivated Cartan to introduce new elegant geometries, that he called *espaces généralisés* and that are nowadays defined as follows.

Definition 1.1. Let G be a Lie group with a closed subgroup H, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. A *Cartan geometry of type* (G,H) on a manifold M is a principal H-bundle:

$$\pi : \mathscr{G} \longrightarrow M$$

together with a $\mathfrak{g}$-valued 1-form ω, called the corresponding *Cartan connection*, on $\mathscr{G}$ subjected to the following three conditions:

- **(i)** $\omega_p : T_p\mathscr{G} \longrightarrow \mathfrak{g}$ is an isomorphism at every point $p \in \mathscr{G}$;
- **(ii)** if $R_h(p) := ph$ is the right translation on $\mathscr{G}$ by $h \in H$, then for any such h:
$$R_h^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega;$$
- **(iii)** $\omega(H^\dagger) = \mathsf{h}$ for every $\mathsf{h} \in \mathfrak{h}$, where:
$$H^\dagger|_p := \tfrac{d}{dt}\big|_0\big((R_{\exp(t\mathsf{h})}(p)\big)$$
is the left-invariant vector field on $\mathscr{G}$ corresponding to h.

Underlying a Cartan geometry, there always is a homogeneous space, namely G/H. In fact, among the Cartan geometries of type (G,H), the most symmetric one, called *Klein geometry of type* (G,H), arises when $M = G/H$, when $\pi\colon G \to G/H$ is the projection onto left-cosets, and when $\omega = \omega_{MC}\colon TG \to \mathfrak{g}$ is the *Maurer-Cartan form* on G (*see* [24]). Generally, Cartan geometries are a generalization of Klein geometries and also, are a generalization of *Riemannian geometries*. While the geometries of Klein present perfect homogeneity and while the ones of Riemann can be regarded as inhomogeneous types of Euclidean geometry, Cartan devised a broad synthesis between these two seemingly incompatible types of geometry.

In general, with a Cartan connection ω as above, if we associate the vector field $\widehat{X} := \omega^{-1}(\mathsf{x})$ on $\mathscr{G}$ to an arbitrary element x of $\mathfrak{g}$, then the infinitesimal version of condition **(ii)** reads as:

$$[\widehat{X}, \widehat{Y}] = \widehat{[\mathsf{x},\mathsf{y}]_{\mathfrak{g}}},$$

whenever y belongs to $\mathfrak{h}$. But in the special case of Klein geometries, this equality holds moreover for any arbitrary element y of $\mathfrak{g}$. This difference motivates one to define the *curvature function*:

$$\kappa\colon \quad \mathscr{G} \longrightarrow \mathrm{Hom}\big(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}\big)$$

associated to the Cartan connection ω by:

$$\kappa_p(\mathsf{x},\mathsf{y}) := \omega_p\big([\widehat{X}, \widehat{Y}]\big) - [\mathsf{x},\mathsf{y}]_{\mathfrak{g}} \qquad (p \in \mathscr{G},\ \mathsf{x},\mathsf{y} \in \mathfrak{g}/\mathfrak{h}).$$

In a way, the curvature function measures how far a Cartan geometry is from its corresponding Klein geometry. In particular, a Cartan geometry is locally equivalent to its corresponding Klein geometry if and only if its curvature function vanishes identically (*see* [24]).

In this paper, we aim to effectively build the Cartan geometry of real hypersurfaces in $\mathbb{C}^2$. After Cartan himself in 1932, several other mathematicians reconstructed and developed this geometry, especially, Chern-Moser [11] and Tanaka [26], who presented some alternative methods which enable one to construct the Cartan geometries in higher dimension. The powerful methods of Tanaka have been used widely in the important class of so-called *parabolic geometries*, which are a specific, rich type of Cartan geometries; *see* the recent extensive monograph [8, 9] by Čap and Slovák.

Recently, Ezhov, McLaughlin and Schmalz published the article [13] in the *Notices of the American Mathematical Society*, the purpose of which is to reconstruct Cartan's eight-dimensional coframe within Tanaka's framework. In this excellent expository paper, which in fact inspired us to prepare the current work, they computed again the Cartan curvatures of the mentioned real hypersurfaces in $\mathbb{C}^2$, taking account of the corresponding Lie algebra second cohomology space. Contrary to what is sometimes believed, neither Cartan's computations ([10]), nor Chern's computations ([11, 16, 15]) are really effective, though, *potentially*, they should be so after some (hard) work. Ezhov, McLaughlin and Schmalz ([13]) made a normalization of an initial frame for TM which requires an application of the Cauchy-Kowalewski theorem, hence requires real analyticity of M (*cf.* [16, 20] for some PDE aspects of CR geometry). By performing an alternative choice $\{H_1, H_2, T\}$ of an initial frame for TM which is explicit in terms of a local graphing function $\varphi(x, y, u)$ for M, we deviate from the normalization made in [13] (with a more geometric-minded approach), our computational objective being to provide a Cartan-Tanaka connection all elements of which are completely effective in terms of $\varphi(x, y, u)$ — assuming only $\mathscr{C}^6$-smoothness of M. One important obstacle on the way to performing completely explicit computations is that one has to divide by a complicated Levi-form factor Υ (*see* below) and then to execute several further differentiations of algebraic expressions involving Υ, *see* the functions A_i, A_{i,k_1}, A_{i,k_1,k_2}, A_{i,k_1,k_2,k_3} below whose full expansion costs hundreds of lines to Maple.

Thus, let $M^3 \subset \mathbb{C}^2$ be a local Levi-nondegenerate real 3-dimensional hypersurface passing through the origin, represented in coordinates $(z, w) = (x+iy, u+iv)$ as a graph:

$$v = \varphi(x, y, u) = x^2 + y^2 + \mathrm{O}(3),$$

for a certain real-valued $\mathscr{C}^6$-smooth graphing function φ defined in a neighborhood of the origin in $\mathbb{R}^3$. Throughout the paper, $\mathscr{C}^6$-smoothness will be regularly assumed, because all objects (curvatures, frames, coframes) will happen to depend upon partial derivatives of order $\leqslant 6$ of φ. In this paper, our intention is to reformulate Cartan's results in terms of the graphing function φ, which is the initial, single datum of this study. The goal is to build the Cartan geometry of such hypersurfaces M^3 and to characterize explicitly when they are locally biholomorphic to the distinguished *Heisenberg sphere* $\mathbb{H}^3$ defined by the simplest equation having no $\mathrm{O}(3)$ remainder:

$$v = x^2 + y^2 \qquad \text{or equivalently:} \qquad w - \overline{w} = 2i\, z\overline{z}.$$

To do this, we use Tanaka's powerful methods in several steps. At first, we compute the Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ of infinitesimal CR automorphisms of the Heisenberg sphere, namely the $\mathbb{R}$-linear space of all $(1,0)$-vector fields:

$$\mathsf{X} = Z(z,w)\frac{\partial}{\partial z} + W(z,w)\frac{\partial}{\partial w}$$

having holomorphic coefficients Z and W, whose real part is tangent to $\mathbb{H}^3$, *i.e.*:

$$\big(\mathsf{X} + \overline{\mathsf{X}}\big)|_{\mathbb{H}^3} \equiv 0.$$

Easy computations yield at first some standard, known generators (*see* section 2 for details):

Proposition 1.2. *The Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ of infinitesimal CR automorphisms of the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$ is of dimension 8 and is generated by the following eight $\mathbb{R}$-linearly independent fields:*

$$\mathsf{H}_1 := \partial_z + 2iz\,\partial_w \quad \mathsf{H}_2 := i\,\partial_z + 2z\,\partial_w, \quad \mathsf{I}_1 := (w + 2iz^2)\,\partial_z + 2izw\,\partial_w,$$
$$\mathsf{I}_2 := (iw + 2z^2)\,\partial_z + 2zw \quad \mathsf{T} := \partial_w, \quad \mathsf{D} := z\,\partial_z + 2w\,\partial_w,$$
$$\mathsf{J} := zw\,\partial_z + w^2\,\partial_w, \quad \mathsf{R} := iz\,\partial_z.$$

This Lie algebra is two-graded (*see* section 2 for definition) of the form:

$$\mathfrak{hol}(\mathbb{H}^3) = \underbrace{\mathfrak{l}_{-2} \oplus \mathfrak{l}_{-1}}_{\mathfrak{l}_-} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2,$$

where: $\mathfrak{l}_{-2} = \mathbb{R}\,\mathsf{T}$; $\mathfrak{l}_{-1} = \mathbb{R}\,\mathsf{H}_1 \oplus \mathbb{R}\,\mathsf{H}_2$; $\mathfrak{l}_0 = \mathbb{R}\,\mathsf{D} \oplus \mathbb{R}\,\mathsf{R}$; $\mathfrak{l}_1 = \mathbb{R}\,\mathsf{I}_2 \oplus \mathbb{R}\,\mathsf{I}_2$; $\mathfrak{l}_2 = \mathbb{R}\,\mathsf{J}$. It is known (*see* [5]) that $\mathfrak{l}_-$ is in fact the Levi-Tanaka symbol algebra of every Levi nondegenerate real hypersurface $M^3 \subset \mathbb{C}^2$, up to isomorphism.

As for the second step, we apply the Tanaka prolongation procedure (*see* section 3) to the nilpotent Lie algebra $\mathfrak{l}_-$ just above, which we rename $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, and we recover an eight-dimensional two-graded algebra:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

which is isomorphic to $\mathfrak{hol}(\mathbb{H}^3)$ *via* a trivial map:

$$\mathsf{T} \to \mathsf{t}, \quad \mathsf{H}_1 \to \mathsf{h}_1, \quad \mathsf{H}_2 \to \mathsf{h}_2, \quad \mathsf{D} \to \mathsf{d}, \quad \mathsf{R} \to \mathsf{r}, \quad \mathsf{I}_1 \to \mathsf{i}_1, \quad \mathsf{I}_2 \to \mathsf{i}_2, \quad \mathsf{J} \to \mathsf{j}.$$

where $\mathsf{t}, \mathsf{h}_1, \mathsf{h}_2, \mathsf{r}, \mathsf{d}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}$ are the eight generators of $\mathfrak{g}$ we construct. Knowing that, generally speaking, the Lie algebra of infinitesimal automorphisms of a nonholonomic homogeneous distribution is isomorphic to the Tanaka prolongation of its nilpotent $\mathfrak{h}_-$-part ([26, 28]), our computations verify this fact in the specific case of Levi-nondegenerate real hypersurfaces $M^3 \subset \mathbb{C}^2$ (*cf.* also [13]) .

Afterward, we compute the second cohomology of the obtained pair of graded Tanaka-type Lie algebras $(\mathfrak{g}_-, \mathfrak{g})$ (*see* Section 4). This enables us to find in advance some significant algebraic properties of the desired curvature function κ before starting the main computations in order to construct the sought $\mathfrak{g}$-valued connection. For example, we can find the homogeneity of the first nonzero homogeneous component of this curvature function, and also, we can find in advance how many essential curvature components there are. To compute the cohomology space, we have used the implementation of an algorithm provided in [1] which is workable within the Maple software. Section 4 is devoted to this part of computations.

Next, we start the computation of an initial frame for any hypersurface M^3 in $\mathbb{C}^2$. At first we construct two basis elements H_1 and H_2 for the complex tangent bundle T^cM in terms of the defining function φ and we get:

Lemma 1.3. *For any local $\mathscr{C}^6$-smooth hypersurface M^3 of $\mathbb{C}^2$ which is represented as a graph:*

$$v = \varphi(x, y, u)$$

in coordinates $(z, w) = (x + iy,\, u + iv)$, the complex tangent bundle $T^c M = \operatorname{Re} T^{0,1} M$ is generated by the two explicit vector fields:

$$\begin{cases} H_1 := \dfrac{\partial}{\partial x} + \left(\dfrac{\varphi_y - \varphi_x\, \varphi_u}{1 + \varphi_u^2}\right) \dfrac{\partial}{\partial u}, \\ H_2 := \dfrac{\partial}{\partial y} + \left(\dfrac{-\varphi_x - \varphi_y\, \varphi_u}{1 + \varphi_u^2}\right) \dfrac{\partial}{\partial u}. \end{cases}$$

Next, assuming that M is furthermore Levi nondegenerate, we also compute the Lie bracket $T := \frac{1}{4}[H_1, H_2]$ in terms of the defining function. For each hypersurface M^3 defined as the graph of the function φ, the associated vector fields H_1, H_2 and T constitute a local frame on M, which will be what we call the *initial frame*. For later use, we also compute the two length-three brackets $[H_1, T]$ and $[H_2, T]$ and, fortunately, we see that both of them are certain multiples of T (*see* Section 5):

Lemma 1.4. *Allowing the two notational coincidences: $x_1 \equiv x$ and $x_2 \equiv y$, one has:*

$$[H_1, T] = \Phi_1\, T \qquad \text{and} \qquad [H_2, T] = \Phi_2\, T,$$

where the two rational functions Φ_1 and Φ_2 of the variables (x_1, x_2, u) are of the form:

$$\Phi_1 = \frac{A_1}{\Delta^2 \Upsilon} \qquad \text{and} \qquad \Phi_2 = \frac{A_2}{\Delta^2 \Upsilon},$$

in which the two functions Δ and Υ (the Levi-form factor, nonzero by assumption) have the explicit expressions:

$$\begin{aligned} \Delta &= 1 + \varphi_u^2, \\ \Upsilon &= -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + \\ &\quad + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy}. \end{aligned}$$

and in which the two numerators are given by:

$$A_i := \Delta^2\, \Upsilon_{x_i} + \Delta\big(-2\,\Delta_{x_i}\,\Upsilon + \Lambda_i\,\Upsilon_u - \Upsilon\,\Lambda_{i,u}\big) - \Lambda_i\,\Upsilon\,\Delta_u \qquad (i = 1, 2),$$

where we set:

$$\Lambda_1 := \varphi_y - \varphi_x\,\varphi_u, \qquad \Lambda_2 := -\,\varphi_x - \varphi_y\,\varphi_u.$$

From now on, we are using a different normalization than Ezhov, McLaughlin and Schmalz ([13]), so that the computations begin to be substantially distinct. One should notice here that the two functions Φ_1 and Φ_2 which encode the Lie structure of the initial frame $\{H_1, H_2, T\}$ already necessitate a division by the complicated function Υ, which coincides, in the real coordinates (x, y, u), with the Levi determinant of M, of course of size 1×1, because $\operatorname{CRdim}(M) = 1$. Then the A_i require differentiations of Υ, and furthermore, higher order invariants of the normal Tanaka connection we will construct — which corresponds to a known, basic parabolic geometry — will require further partial differentiations of Υ up to order 4. This will make computations really explode when expressing back everything in terms of partial derivatives of the graphing function $\varphi(x, y, u)$ of order $\leqslant 6$. In particular, we shall have to introduce furthermore the H_k-iterated derivatives of the functions Φ_i up to order 3, where $i, k_1, k_2, k_3 = 1, 2$:

$$H_{k_1}(\Phi_i) = \frac{A_{i,k_1}}{\Delta^4\,\Upsilon^2}, \quad H_{k_2}(H_{k_1}(\Phi_i)) = \frac{A_{i,k_1,k_2}}{\Delta^6\,\Upsilon^3}, \quad H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) = \frac{A_{i,k_1,k_2,k_3}}{\Delta^8\,\Upsilon^4}.$$

Proposition 1.5. (*see* [2]) *All the numerators appearing above are explicitly given by:*

$$\begin{aligned}
A_i &:= \Delta^2\,\Upsilon_{x_i} + \Delta\big(-2\,\Delta_{x_i}\,\Upsilon + \Lambda_i\,\Upsilon_u - \Upsilon\,\Lambda_{i,u}\big) - \Lambda_i\,\Upsilon\,\Delta_u,\\
A_{i,k_1} &:= \Delta^2\big(\Upsilon\,A_{i,x_{k_1}} - \Upsilon_{x_{k_1}}\,A_i\big) + \Delta\big(-2\,\Delta_{x_{k_1}}\,\Upsilon\,A_i + \Upsilon\,\Lambda_{k_1}\,A_{i,u} - \Upsilon_u\,\Lambda_{k_1}\,A_i\big) - 2\,\Delta_u\,\Upsilon\,\Lambda_{k_1}\,A_i,\\
A_{i,k_1,k_2} &:= \Delta^2\big(\Upsilon\,A_{i,k_1,x_{k_2}} - 2\,\Upsilon_{x_{k_2}}\,A_{i,k_1}\big) + \Delta\big(-4\,\Delta_{x_{k_2}}\,\Upsilon\,A_{i,k_1} + \Upsilon\,\Lambda_{k_2}\,A_{i,k_1,u} - 2\,\Upsilon_u\,\Lambda_{k_2}\,A_{i,k_1}\big) -\\
&\quad - 4\,\Delta_u\,\Upsilon\,\Lambda_{k_2}\,A_{i,k_1},\\
A_{i,k_1,k_2,k_3} &= \Delta^2\big(\Upsilon\,A_{i,k_1,k_2,x_{k_3}} - 3\,\Upsilon_{x_{k_3}}\,A_{i,k_1,k_2}\big) + \Delta\big(-6\,\Delta_{x_{k_3}}\,\Upsilon\,A_{i,k_1,k_2} + \Upsilon\,\Lambda_{k_3}\,A_{i,k_1,k_2,u} - 3\,\Upsilon_u\,\Lambda_{k_3}\,A_{i,k_1,k_2}\big) -\\
&\quad - 6\,\Delta_u\,\Upsilon\,\Lambda_{k_3}\,A_{i,k_1,k_2}.
\end{aligned}$$

Furthermore, these iterated derivatives identically satisfy:

$$H_2(\Phi_1) \equiv H_1(\Phi_2)$$

and also:

$$\begin{aligned}
0 &\equiv -H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)),\\
0 &\equiv -H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)),\\
0 &\equiv -H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)),\\
0 &\equiv -H_2(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_2(\Phi_2))) - H_1(H_2(H_2(\Phi_2))) + \Phi_2\,H_2(H_1(\Phi_2)) - \Phi_2\,H_1(H_2(\Phi_2)).
\end{aligned}$$

The latter statement corresponds to an observation which seems to be new in the subject, seemingly absent in the papers [10, 11, 16, 22, 17, 13, 15].

Subsequently, we will be able to start the main computations of the curvature function κ. We compute in fact *curvature coefficients* $\kappa_{q_j}^{p_{j_1}p_{j_2}}$, which, by definition, are the coefficients of the basis elements:

$$\mathsf{p}_{j_1}^* \wedge \mathsf{p}_{j_2}^* \otimes \mathsf{q}_j, \quad (\mathsf{p}_{j_1}, \mathsf{p}_{j_2} \in \mathfrak{g}_-, \ \mathsf{q}_j \in \mathfrak{g})$$

of the vector space $\mathrm{Lin}\big(\Lambda^2\mathfrak{g}_-, \mathfrak{g}\big)$, in the expression of κ. Here is our main result, the proof of which will be concluded at the end of the paper, in section 7.

Theorem 1.1. *Associated to any $\mathscr{C}^6$-smooth Levi-nondegenerate real 3-dimensional hypersurface $M^3 \subset \mathbb{C}^2$, represented in coordinate $(z, w) := (x + iy, u + iv)$ as a graph:*

$$v = \varphi(x, y, u) = x^2 + y^2 + \mathrm{O}(3),$$

there is a unique $\mathfrak{g}$-valued Cartan connection which is normal and regular in the sense of Tanaka. Its curvature function reduces to:

$$\begin{aligned}
\kappa(p) = \kappa_{i_1}^{h_1t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 +\\
+ \kappa_{i_2}^{h_2t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1t}(p)\,\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2t}(p)\,\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},
\end{aligned}$$

where the two main curvature coefficients, having homogeneity 4, are of the form:

$$\begin{aligned}
\kappa_{i_1}^{h_1t}(p) &= -\,\boldsymbol{\Delta_1}\,c^4 - 2\,\boldsymbol{\Delta_4}\,c^3d - 2\,\boldsymbol{\Delta_4}\,cd^3 + \boldsymbol{\Delta_1}\,d^4,\\
\kappa_{i_2}^{h_1t}(p) &= -\,\boldsymbol{\Delta_4}\,c^4 + 2\,\boldsymbol{\Delta_1}\,c^3d + 2\,\boldsymbol{\Delta_1}\,cd^3 + \boldsymbol{\Delta_4}\,d^4,
\end{aligned}$$

in which the two functions $\mathbf{\Delta_1}$ *and* $\mathbf{\Delta_4}$ *of only the three variables* (x, y, u) *are* explicitly *given by:*

$$\begin{aligned}
\mathbf{\Delta_1} = \tfrac{1}{384}\big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\, H_1(H_2(H_1(\Phi_2))) - 11\, H_2(H_1(H_2(\Phi_1))) + \\
&+ 6\,\Phi_2\, H_2(H_1(\Phi_1)) - 6\,\Phi_1\, H_1(H_2(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_2)) + 3\,\Phi_1\, H_2(H_2(\Phi_1)) - \\
&- 3\,\Phi_1\, H_1(H_1(\Phi_1)) + 3\,\Phi_2\, H_2(H_2(\Phi_2)) - \big[H_1(\Phi_1)\big]^2 + \big[H_2(\Phi_2)\big]^2 - \\
&- 2\,(\Phi_2)^2\, H_1(\Phi_1) + 2\,(\Phi_1)^2\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_2(\Phi_2) + 2\,(\Phi_1)^2\, H_1(\Phi_1)\big], \\
\mathbf{\Delta_4} = \tfrac{1}{384}\big[& - 3\, H_2(H_1(H_2(\Phi_2))) - 3\, H_1(H_2(H_1(\Phi_1))) + 5\, H_1(H_2(H_2(\Phi_2))) + 5\, H_2(H_1(H_1(\Phi_1))) + \\
&+ 4\,\Phi_1\, H_1(H_1(\Phi_2)) + 4\,\Phi_2\, H_2(H_1(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_1)) - 3\,\Phi_1\, H_2(H_2(\Phi_2)) - \\
&- 7\,\Phi_2\, H_1(H_2(\Phi_2)) - 7\,\Phi_1\, H_2(H_1(\Phi_1)) - 2\, H_1(\Phi_1)\, H_1(\Phi_2) - 2\, H_2(\Phi_2)\, H_2(\Phi_1) + \\
&+ 4\,\Phi_1\Phi_2\, H_1(\Phi_1) + 4\,\Phi_1\Phi_2\, H_2(\Phi_2)\big],
\end{aligned}$$

and where the remaining four secondary curvature coefficients are given by:

$$\begin{aligned}
\kappa_{i_1}^{h_2 t} &= \kappa_{i_2}^{h_1 t}, \\
\kappa_{i_2}^{h_2 t} &= -\,\kappa_{i_1}^{h_1 t}, \\
\kappa_{j}^{h_1 t} &= \widehat{H}_1\big(\kappa_{i_2}^{h_2 t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1 t}\big), \\
\kappa_{j}^{h_2 t} &= -\widehat{H}_1\big(\kappa_{i_1}^{h_2 t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1 t}\big).
\end{aligned}$$

Corollary 1.6. *A* $\mathscr{C}^6$*-smooth Levi nondegenerate local hypersurface* $M^3 \subset \mathbb{C}^2$ *is biholomorphic to* $\mathbb{H}^3$*, namely is spherical, if and only if* $0 \equiv \mathbf{\Delta_1} \equiv \mathbf{\Delta_4}$*, identically as functions of* (x, y, u)*.*

The proof is just an application of the Frobenius theorem ([24]), real analyticity of M being forced by these two zero curvature equations.

In continuation to [6, 13] and to the present work, we shall construct in [21] Cartan connections on geometry-preserving deformations of the 3-codimensional model:

$$\begin{cases}
w_1 - \overline{w}_1 = 2iz\overline{z}, \\
w_2 - \overline{w}_2 = 2iz\overline{z}(z + \overline{z}), \\
w_3 - \overline{w}_3 = 2z\overline{z}(z - \overline{z}),
\end{cases}$$

in $\mathbb{C}^4$ equipped with coordinates (z, w_1, w_2, w_3).

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2. Infinitesimal CR Automorphisms and Tanaka Algebra

We start by proving Proposition 1.2 in this section. A local $(1, 0)$ vector field:

$$\mathsf{X} = Z(z, w)\,\frac{\partial}{\partial z} + W(z, w)\,\frac{\partial}{\partial w}$$

defined in a neighborhood of the origin and having *holomorphic* coefficients $Z(z, w)$ and $W(z, w)$ is an *infinitesimal CR automorphism* of the Heisenberg sphere $\mathbb{H}^3$ if and only if $X + \overline{X}$ is tangent to the zero-set $\big\{(z, w) \in \mathbb{C}^2 \colon\ w - \overline{w} = 2i\, z\overline{z}\big\}$. Performing a standard extrinsic complexification ([11, 7, 5, 20, 18, 19]) which replaces the conjugate variables $(\overline{z}, \overline{w})$ by new independent variables $(\underline{z}, \underline{w})$, this tangency holds if and only if the equation:

$$0 \equiv \big[W - 2i\underline{z}\, Z - \overline{W} - 2iz\, \overline{Z}\big]_{w = \underline{w} + 2i\, z\underline{z}}$$

holds identically in $\mathbb{C}\{z, \underline{z}, \underline{w}\}$, that is to say if and only if:

$$(1) \qquad 0 \equiv W(z,\, \underline{w} + 2i\, z\underline{z}) - 2i\underline{z}\, Z(z,\, \underline{w} + 2i\, z\underline{z}) - \overline{W}(\underline{z},\, \underline{w}) - 2iz\, \overline{Z}(\underline{z}, \underline{w}).$$

The holomorphicity of the coefficients Z and W of X enables one to expand them with respect to the powers of z. After replacing the complex variable w by $\underline{w}+2i\, z\underline{z}$ in such expansions, we get:

$$\begin{aligned} Z(z, \underline{w} + 2i\, z\underline{z}) &= \sum_{k\in\mathbb{N}} z^k\, Z_k(\, \underline{w} + 2i\, z\underline{z}) \qquad \text{and} \\ W(z, \underline{w} + 2i\, z\underline{z}) &= \sum_{k\in\mathbb{N}} z^k\, W_k(\, \underline{w} + 2i\, z\underline{z}), \end{aligned}$$

hence the fundamental equation (1) changes into the following one:

$$(2) \qquad \begin{aligned} 0 \equiv \sum_{k\in\mathbb{N}} z^k\, W_k(\underline{w} + 2i\, z\underline{z}) - 2i\underline{z} \sum_{k\in\mathbb{N}} z^k\, Z_k(\underline{w} + 2i\, z\underline{z}) - \\ - \sum_{k\in\mathbb{N}} \underline{z}^k\, \overline{W}^k(\underline{w}) - 2iz \sum_{k\in\mathbb{N}} \underline{z}^k\, \overline{Z}_k(\underline{w}). \end{aligned}$$

Now we can expand functions Z_k and W_k according to their complex Taylor series near the origin. More precisely, for any holomorphic function $A = A(\underline{w})$ defined near the origin, one has $A(\underline{w}) = \sum_{l\in\mathbb{N}} A_{\underline{w}^l}(0) \frac{1}{l!} \underline{w}^l$, where $A_{\underline{w}^l}$ denote the l-th derivative of $A(\underline{w})$, and thus:

$$(3) \qquad A(\underline{w} + 2i\, z\underline{z}) = \sum_{l\in\mathbb{N}} A_{\underline{w}^l}(\underline{w})\, (2i\, z\underline{z})^l\, \frac{1}{l!}.$$

Replacing similar expansions of holomorphic functions $W_k(\underline{w}+2iz\underline{z})$ and $Z_k(\underline{w}+2iz\underline{z})$ in the equation (2) gives:

$$(4) \qquad \begin{aligned} 0 \equiv \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} \Big(z^k\, (2i\, z\underline{z})^l\, \frac{1}{l!}\, W_{k,\underline{w}^l}(\underline{w}) - 2i\, \underline{z}\, z^k\, (2i\, z\underline{z})^l\, \frac{1}{l!}\, Z_{k,\underline{w}^l}(\underline{w}) \Big) - \\ - \sum_{k\in\mathbb{N}} \underline{z}^k \Big(\overline{W}_k(\underline{w}) + 2i\, z\, \overline{Z}_k(\underline{w}) \Big). \end{aligned}$$

To fulfill this last equation, the coefficients of the monomials $\underline{z}^k$ for all $k \geqslant 2$ and of the monomials $z\, \underline{z}^{k'}$ for all $k' \geqslant 3$ must vanish, identically. Then one easily shows (full details are available in the expanded electronic version [2]) that the two functions Z and W are in fact of the following truncated form:

$$\left[\begin{aligned} Z(z, w) &= Z_0(w) + z\, Z_1(w) + z^2\, Z_2(w) \\ W(z, w) &= W_0(w) + z\, W_1(w). \end{aligned} \right.$$

Taking account of these expressions, equation (4) changes into the following more simplified form:

$$
\begin{aligned}
0 \equiv\, & W_0(\underline{w}) + 2iz\underline{z}\, W_{0,\underline{w}}(\underline{w}) - 4z^2\underline{z}^2\,\tfrac{1}{2!}\, W_{0,\underline{w}^2}(\underline{w}) - 8iz^3\underline{z}^3\,\tfrac{1}{3!}\, W_{0,\underline{w}^3}(\underline{w}) + \cdots \\
& + z\, W_1(\underline{w}) + 2iz^2\underline{z}\, W_{1,\underline{w}}(\underline{w}) - 4z^3\underline{z}^2\,\tfrac{1}{2!}\, W_{1,\underline{w}^2}(\underline{w}) - \cdots \\
& -\, 2i\underline{z}\, Z_0(\underline{w}) + 4\, z\underline{z}^2\, Z_{0,\underline{w}}(\underline{w}) + 8iz^2\underline{z}^3\,\tfrac{1}{2!}\, Z_{0,\underline{w}^2}(\underline{w}) + \cdots \\
& -\, 2i\underline{z}z\, Z_1(\underline{w}) + 4z^2\underline{z}^2\, Z_{1,\underline{w}}(\underline{w}) + 8iz^3\underline{z}^3\tfrac{1}{2!}\, Z_{1,\underline{w}^2}(\underline{w}) + \cdots \\
& -\, 2i\underline{z}z^2 Z_2(\underline{w}) + 4z^3\underline{z}^2\, Z_{2,\underline{w}} + \cdots\,, \\
& -\, \overline{W}_0(\underline{w}) - \underline{z}\,\overline{W}_1(\underline{w}) - 2iz\,\overline{Z}_0(\underline{w}) - 2iz\underline{z}\,\overline{Z}_1(\underline{w}) - 2iz\underline{z}^2\,\overline{Z}_2(\underline{w}).
\end{aligned}
$$

Now, we extract the coefficients of the monomials $z^\mu \underline{z}^\nu$ of this form for small values of μ and ν, and these coefficients must vanish identically in $\mathbb{C}\{\underline{w}\}$.

Here, let us consider such values of the non-negative integers μ and ν which will help us to compute the desired functions Z and W — for the sake of clarity in what follows, we put the pair (μ, ν) in front the coefficient of the monomial $z^\mu \underline{z}^\nu$:

$$(0,0): \quad 0 \equiv W_0(\underline{w}) - \overline{W}_0(\underline{w}), \tag{5}$$

$$(1,0): \quad 0 \equiv W_1(\underline{w}) - 2i\,\overline{Z}_0(\underline{w}), \tag{6}$$

$$(1,1): \quad 0 \equiv 2i\, W_{0,\underline{w}}(\underline{w}) - 2i\, Z_1(\underline{w}) - 2i\,\overline{Z}_1(\underline{w}), \tag{7}$$

$$(2,1): \quad 0 \equiv 2i\, W_{1,\underline{w}}(\underline{w}) - 2i\, Z_2(\underline{w}), \tag{8}$$

$$(2,2): \quad 0 \equiv -4\,\tfrac{1}{2!}\, W_{0,\underline{w}^2}(\underline{w}) + 4\, Z_{1,\underline{w}}(\underline{w}), \tag{9}$$

$$(2,3): \quad 0 \equiv 8i\,\tfrac{1}{2!}\, Z_{0,\underline{w}^2}(\underline{w}), \tag{10}$$

$$(3,2): \quad 0 \equiv -4\,\tfrac{1}{2!}\, W_{1,\underline{w}^2}(\underline{w}) + 4\, Z_{2,\underline{w}}(\underline{w}), \tag{11}$$

$$(3,3): \quad 0 \equiv -8i\,\tfrac{1}{3!}\, W_{0,\underline{w}^3}(\underline{w}) - 8i\,\tfrac{1}{2!}\, Z_{1,\underline{w}^2}(\underline{w}). \tag{12}$$

Among these equalities, (10) clearly implies that Z_0 is affine, namely:

$$Z_0(w) := \mathsf{z}_{0,0} + \mathsf{z}_{0,1} w,$$

for two *complex* constants $\mathsf{z}_{0,0} = \mathsf{x}_{0,0} + i\mathsf{y}_{0,0}$ and $\mathsf{z}_{0,1} = \mathsf{x}_{0,1} + i\mathsf{y}_{0,1}$ in $\mathbb{C}$. Then (6) immediately gives the expression of W_1 as follows:

$$W_1(w) = 2i\overline{\mathsf{z}}_{0,0} + 2i\overline{\mathsf{z}}_{0,1} w. \tag{13}$$

Next, differentiating (9) once with respect to $\underline{w}$ and comparing to (12), we get:

$$0 \equiv W_{0,\underline{w}^3}(\underline{w}) \qquad \text{and} \qquad 0 \equiv Z_{1,\underline{w}}(\underline{w}).$$

It follows firstly that W_0 is quadratic:

$$W_0(w) = \mathsf{u}_{0,0} + \mathsf{u}_{0,1}\, w + \mathsf{u}_{0,2}\, w^2, \tag{14}$$

but taking account of (5), we see moreover that the three appearing coefficients $\mathsf{u}_{0,0}$, $\mathsf{u}_{0,1}$, $\mathsf{u}_{0,2}$ must be *real*. Furthermore, it follows that $Z_1(w) = \mathsf{z}_{1,0} + \mathsf{z}_{1,1}\, w$ is affine and even more precisely, taking in addition account of (7) and of (9), we have:

$$Z_1(w) = \tfrac{1}{2}\,\mathsf{u}_{0,1} + i\,\mathsf{y}_{1,0} + \mathsf{u}_{0,2}\, w. \tag{15}$$

Finally, (8) and (13) immediately imply that $Z_2(w)$ is constant, namely:

$$Z_2(w) = 2\,\mathsf{y}_{0,1} + 2i\,\mathsf{x}_{0,1}.$$

Now we have found the expressions of the desired functions $Z_i(w)$ for $i = 0, 1, 2$ and $W_j(w)$ for $j = 0, 1$ in terms of the eight real constants:

$$\mathsf{x}_{0,0},\quad \mathsf{y}_{0,0},\quad \mathsf{x}_{0,1},\quad \mathsf{y}_{0,1},\quad \mathsf{u}_{0,0},\quad \mathsf{u}_{0,1},\quad \mathsf{u}_{0,2},\quad \mathsf{y}_{1,0},$$

which give us eight $\mathbb{R}$-linearly independent infinitesimal automorphisms of the Heisenberg sphere $\mathbb{H}^3$. In summary:

Proposition 2.1. *The Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ of infinitesimal CR automorphisms of the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$ is 8-dimensional and is generated by the following eight $\mathbb{R}$-linearly independent holomorphic vector fields:*

$$\mathsf{H}_1 := \partial_z + 2iz\,\partial_w, \quad \mathsf{H}_2 := i\,\partial_z + 2z\,\partial_w, \quad \mathsf{I}_1 := (w + 2iz^2)\,\partial_z + 2izw\,\partial_w,$$
$$\mathsf{I}_2 := (iw + 2z^2)\,\partial_z + 2zw \quad \mathsf{T} := \partial_w, \quad \mathsf{D} := z\,\partial_z + 2w\,\partial_w,$$
$$\mathsf{J} := zw\,\partial_z + w^2\,\partial_w, \quad \mathsf{R} := iz\,\partial_z,$$

with the following table of Lie brackets:

	T	H_1	H_2	D	R	I_1	I_2	J
T	0	0	0	$2\,\mathsf{T}$	0	H_1	H_2	D
H_1	$*$	0	$4\,\mathsf{T}$	H_1	H_2	$6\,\mathsf{R}$	$2\,\mathsf{D}$	I_1
H_2	$*$	$*$	0	H_2	$-\mathsf{H}_1$	$-2\,\mathsf{D}$	$6\,\mathsf{R}$	I_2
D	$*$	$*$	$*$	0	0	I_1	I_2	$2\,\mathsf{J}$
R	$*$	$*$	$*$	$*$	0	$-\mathsf{I}_2$	I_1	0
I_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{J}$	0
I_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

Inspecting the above table shows that the Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ is 2-graded, in the sense of Tanaka. More precisely, setting $\mathfrak{l}_{-2} := \mathrm{Span}_{\mathbb{R}}(\mathsf{T}), \mathfrak{l}_{-1} := \mathrm{Span}_{\mathbb{R}}(\mathsf{H}_1, \mathsf{H}_2), \mathfrak{l}_0 := \mathrm{Span}_{\mathbb{R}}(\mathsf{D}, \mathsf{R}), \mathfrak{l}_1 := \mathrm{Span}_{\mathbb{R}}(\mathsf{I}_1, \mathsf{I}_2)$ and $\mathfrak{l}_2 := \mathrm{Span}_{\mathbb{R}}(\mathsf{J})$, then we have:

$$\mathfrak{hol}(\mathbb{H}^3) = \mathfrak{l}_{-2} \oplus \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2,$$

together with the property, for any $i, j \in \mathbb{Z}$:

$$[\mathfrak{l}_i, \mathfrak{l}_j] \subseteq \mathfrak{l}_{i+j},$$

on the agreement that $\mathfrak{l}_k \equiv 0$ for $k \leqslant -3$ or $k \geqslant 3$. Here, the subalgebra $\mathfrak{l}_- := \mathfrak{l}_{-2} \oplus \mathfrak{l}_{-1}$ is called the *Levi-Tanaka algebra* of the Heisenberg sphere $\mathbb{H}^3$. Of course, it is well known that this Lie algebra is simple, isomorphic to $\mathfrak{sl}_3(\mathbb{R})$.

3. Tanaka Prolongation

Although it is known (*see* [26, 13]) that the Levi-Tanaka algebra of any Levi-nondegenerate $M^3 \subset \mathbb{C}^2$ is isomorphic to the Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$, in this section we aim to confirm this fact by direct computation.

Consider a finite-dimensional graded real Lie algebra indexed by negative integers:

$$\mathfrak{g}_- = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

satisfying $[\mathfrak{g}_{-l_1}, \mathfrak{g}_{-l_2}] \subset \mathfrak{g}_{-l_1-l_2}$ with the convention that $\mathfrak{g}_k = 0$ for $k \leqslant -\mu - 1$. Following [25], $\mathfrak{g}_-$ will be said to be *of μ-th kind.* Assume that there is a complex structure $J\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ such that $J^2 = -\mathrm{Id}$, whence $\mathfrak{g}_{-1}$ is even-dimensional and bears a natural structure of a complex vector space. Tanaka's prolongation of $\mathfrak{g}_-$ is an algebraic procedure which generates a certain larger graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots$$

inductively as follows.

By definition, the order-zero component $\mathfrak{g}_0$ consists of all linear endomorphisms $\mathsf{d}\colon \mathfrak{g}_- \to \mathfrak{g}_-$ which preserve grading: $\mathsf{d}(\mathfrak{g}_k) \subset \mathfrak{g}_k$, which commute with the complex structure: $\mathsf{d}(J\,\mathsf{x}) = J\,\mathsf{d}(\mathsf{x})$ for all $\mathsf{x} \in \mathfrak{g}_{-1}$ and which are *derivations*, namely satisfy:

$$\mathsf{d}([\mathsf{y},\, \mathsf{z}]) = [\mathsf{d}(\mathsf{x}),\, \mathsf{y}] + [\mathsf{x},\, \mathsf{d}(\mathsf{y})], \tag{16}$$

for every $\mathsf{y}, \mathsf{z} \in \mathfrak{g}_-$. Then the bracket between a $\mathsf{d} \in \mathfrak{g}_0$ and an $\mathsf{x} \in \mathfrak{g}_-$ is simply defined by $[\mathsf{d},\, \mathsf{x}] := \mathsf{d}(\mathsf{x})$, while the bracket between *two* elements $\mathsf{d}', \mathsf{d}'' \in \mathfrak{g}_0$ is defined to be the commutator $\mathsf{d}' \circ \mathsf{d}'' - \mathsf{d}'' \circ \mathsf{d}'$ between endomorphisms.

More generally, the l-th component $\mathfrak{g}_l$ of the prolonged Lie algebra $\mathfrak{g}$ is defined as follows, inductively:

$$\mathfrak{g}_l = \Big\{\mathsf{d} \in \bigoplus_{k \leqslant -1} \mathrm{Lin}(\mathfrak{g}_k,\, \mathfrak{g}_{k+l})\colon \mathsf{d}([\mathsf{y},\, \mathsf{z}]) = [\mathsf{d}(\mathsf{y}),\, \mathsf{z}] + [\mathsf{y},\, \mathsf{d}(\mathsf{z})], \quad \forall\, \mathsf{y},\, \mathsf{z} \in \mathfrak{g}_-\Big\}. \tag{17}$$

Now, for arbitrary $\mathsf{d} \in \mathfrak{g}_k$ and $\mathsf{e} \in \mathfrak{g}_l$, by induction on the integer $k + l \geqslant 0$, one defines the bracket $[\mathsf{d},\, \mathsf{e}] \in \mathfrak{g}_{k+l} \otimes \mathfrak{g}_-^*$ by:

$$[\mathsf{d},\, \mathsf{e}](\mathsf{x}) = \big[[\mathsf{d},\, \mathsf{x}],\, \mathsf{e}\big] + \big[\mathsf{d},\, [\mathsf{e},\, \mathsf{x}]\big] \qquad \text{for } \mathsf{x} \in \mathfrak{g}_-. \tag{18}$$

3.1. **Heisenberg algebra.** The symbol Lie algebra $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ associated to any Levi nondegenerate CR manifold $M^3 \subset \mathbb{C}^2$ is three-dimensional, with $\mathfrak{g}_{-2} = \mathbb{R}\,\mathsf{x}_1$, with $\mathfrak{g}_{-1} = \mathbb{R}\,\mathsf{x}_2 \oplus \mathbb{R}\mathsf{x}_3$ with $\mathsf{x}_3 = J(\mathsf{x}_2)$ and with only nonzero Lie bracket $[\mathsf{x}_2,\, \mathsf{x}_3] = 4\,\mathsf{x}_1$ (*cf* [5, 13]).

In this case, each element of the order-zero component $\mathfrak{g}_0$ of the prolonged Lie algebra $\mathfrak{g}$ is a derivation of the form $\mathsf{d} := (\mathsf{d}_1, \mathsf{d}_2)$ with $\mathsf{d}_i : \mathfrak{g}_{-i} \longrightarrow \mathfrak{g}_{-i}$, $i = 1, 2$. Let us set $\mathsf{d}_1(\mathsf{x}_2) := r_1\mathsf{x}_2 + r_2\mathsf{x}_3$, $\mathsf{d}_1(\mathsf{x}_3) := r_3\mathsf{x}_2 + r_4\mathsf{x}_3$ and $\mathsf{d}_2(\mathsf{x}_1) := k\mathsf{x}_1$ for some five real coefficients r_1, r_2, r_3, r_4, k. The preservation of the complex structure J by d means that $\mathsf{d}_1(J\mathsf{x}_2) = J\mathsf{d}_1(\mathsf{x}_2)$, and this reads as:

$$r_3\mathsf{x}_2 + r_4\mathsf{x}_3 = r_1\mathsf{x}_3 - r_2\mathsf{x}_2,$$

whence $r_1 = r_4$ and $r_2 = -r_3$. Furthermore, the derivation property (16) of d with $\mathsf{y} = \mathsf{x}_2$ and $\mathsf{z} = \mathsf{x}_3$ reads as:

$$\begin{aligned} 4k\mathsf{x}_1 &= [r_1\mathsf{x}_2 + r_2\mathsf{x}_3, \mathsf{x}_3] - [r_3\mathsf{x}_2 + r_4\mathsf{x}_3, \mathsf{x}_2] = \\ &= 4r_1\mathsf{x}_1 + 4r_4\mathsf{x}_1, \end{aligned}$$

whence $k = r_1 + r_4$. Now, taking account of the three obtained relations between the five real coefficients r_1, r_2, r_3, r_4, k, one can express them as some combinations of only two coefficients r_1 and r_2. This means that the Lie algebra $\mathfrak{g}_0$ is two-dimensional and generated by derivations x_3 and x_4, obtained by putting two choices $(\mathsf{r}_1, \mathsf{r}_2) = (-1, 0)$ and $(\mathsf{r}_1, \mathsf{r}_2) = (0, -1)$ in the general definition of d. Here, one can check that the commutator $\mathsf{x}_4 \circ \mathsf{x}_5 - \mathsf{x}_5 \circ \mathsf{x}_4 = 0$ vanishes, and at this stage, the Lie brackets between the obtained x_k read as follows, if listed by increasing homogeneity:

$$\boxed{-3}\colon\ \big\{\ [\mathsf{x}_1, \mathsf{x}_2] = 0, \qquad [\mathsf{x}_1, \mathsf{x}_3] = 0,$$
$$\boxed{-2}\colon\ \big\{\ [\mathsf{x}_2, \mathsf{x}_3] = 4\,\mathsf{x}_1, \quad [\mathsf{x}_1, \mathsf{x}_4] = 2\,\mathsf{x}_1, \qquad [\mathsf{x}_1, \mathsf{x}_5] = 0,$$
$$\boxed{-1}\colon\ \big\{\ [\mathsf{x}_2, \mathsf{x}_4] = \mathsf{x}_2, \qquad [\mathsf{x}_3, \mathsf{x}_4] = \mathsf{x}_3, \qquad [\mathsf{x}_2, \mathsf{x}_5] = \mathsf{x}_3, \qquad [\mathsf{x}_3, \mathsf{x}_5] = -\mathsf{x}_2$$
$$\boxed{0}\colon\ \big\{\ [\mathsf{x}_4, \mathsf{x}_5] = 0.$$

Computation of the next components of the prolonged algebra $\mathfrak{g}$ is very similar to the computation of $\mathfrak{g}_0$ (full details typed in [2]). At the end, one gets two other components $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which means that the Lie algebra $\mathfrak{g}$ is two-graded of the form:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Here, $\mathfrak{g}_1$ is a two-dimensional subalgebra, generated over $\mathbb{R}$ by two derivations x_6 and x_7 and $\mathfrak{g}_2$ is one-dimensional generated by x_8. One moreover obtains the following supplementary brackets, listed by increasing homogeneity:

$$\boxed{-1}\colon\ \big\{\ [\mathsf{x}_1, \mathsf{x}_6] = \mathsf{x}_2, \qquad [\mathsf{x}_1, \mathsf{x}_7] = \mathsf{x}_3,$$

$$\boxed{0}\colon\ \begin{cases} [\mathsf{x}_2, \mathsf{x}_6] = 6\,\mathsf{x}_5, \quad [\mathsf{x}_3, \mathsf{x}_6] = -2\,\mathsf{x}_4, \quad [\mathsf{x}_2, \mathsf{x}_7] = 2\,\mathsf{x}_4, \\ [\mathsf{x}_3, \mathsf{x}_7] = 6\,\mathsf{x}_5, \quad [\mathsf{x}_1, \mathsf{x}_8] = \mathsf{x}_4, \end{cases}$$

$$\boxed{1}\colon\ \begin{cases} [\mathsf{x}_4, \mathsf{x}_6] = \mathsf{x}_6, \qquad [\mathsf{x}_4, \mathsf{x}_7] = \mathsf{x}_7, \qquad [\mathsf{x}_5, \mathsf{x}_6] = -\mathsf{x}_7, \\ [\mathsf{x}_5, \mathsf{x}_7] = \mathsf{x}_6, \qquad [\mathsf{x}_2, \mathsf{x}_8] = \mathsf{x}_6, \qquad [\mathsf{x}_3, \mathsf{x}_8] = \mathsf{x}_7, \end{cases}$$

$$\boxed{2}\colon\ \big\{\ [\mathsf{x}_4, \mathsf{x}_8] = 2\,\mathsf{x}_8, \qquad [\mathsf{x}_5, \mathsf{x}_8] = 0, \qquad [\mathsf{x}_6, \mathsf{x}_7] = 4\,\mathsf{x}_8.$$

Now let us define a trivial map from $\mathfrak{g}$ to $\mathfrak{hol}(\mathbb{H}^3)$:

$$\mathsf{x}_1 \to \mathsf{T},\ \ \mathsf{x}_2 \to \mathsf{H}_1,\ \ \mathsf{x}_3 \to \mathsf{H}_2,\ \ \mathsf{x}_4 \to \mathsf{D},\ \ \mathsf{x}_5 \to \mathsf{R},\ \ \mathsf{x}_6 \to \mathsf{I}_1,\ \ \mathsf{x}_7 \to \mathsf{I}_2,\ \ \mathsf{x}_8 \to \mathsf{J},$$

and extend it by linearity. By inspecting all commutators of these two Lie algebras, one easily verifies that this yields an isomorphism. Thus as announced, the Lie algebra $\mathfrak{hol}(\mathbb{H}^3)$ of infinitesimal automorphisms of the Heisenberg sphere and the Tanaka Lie algebra $\mathfrak{g}$ of every Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ are isomorphic. For more clarity, we will employ from now on the letters t, h_1, h_2, d, r, i_1, i_2, j instead of $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4, \mathsf{x}_5, \mathsf{x}_6, \mathsf{x}_7, \mathsf{x}_8$ as generators of $\mathfrak{g}$. For later use, we draw up the full commutator table between these eight generators of the abstract Lie algebra $\mathfrak{g}$:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

4. Second Cohomology and its Relation to Cartan Geometry

For two Lie algebras $\mathfrak{g}_- \subseteq \mathfrak{g}$ and for a non-negative integer k, the vector space $\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$ of *k-cochains* is defined by:

$$\mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g}) := \mathrm{Lin}\big(\Lambda^k \mathfrak{g}_-,\, \mathfrak{g}\big) \cong \Lambda^k \mathfrak{g}_-^* \otimes \mathfrak{g}.$$

When $\mathfrak{g}$ is equipped with the structure of a μ-graded Lie algebra:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\mu,$$

a k-cochain $\Phi \in \mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$ is said to be *of homogeneity* $i \in \mathbb{Z}$ whenever for arbitrary elements $\mathsf{z}_{i_1} \in \mathfrak{g}_{i_1}, \ldots\ldots, \mathsf{z}_{i_k} \in \mathfrak{g}_{i_k}$, its value $\Phi(\mathsf{z}_{i_1}, \ldots \mathsf{z}_{i_k})$ belongs to $\mathfrak{g}_{i_1+\cdots+i_k+i}$. In fact, one easily convinces oneself that any k-cochain Φ splits up as a direct sum of k-cochains of fixed homogeneity:

$$\Phi = \cdots + \Phi^{(i-1)} + \Phi^{(i)} + \Phi^{(i+1)} + \cdots$$

where we denote the i-th component of Φ just by $\Phi^{(i)}$.

Classically ([26, 28, 8, 6, 9, 13]), there are for each k linear *differential operators* ∂^k:

$$\partial^k\colon \quad \mathscr{C}^k\big(\mathfrak{g}_-,\, \mathfrak{g}\big) \longrightarrow \mathscr{C}^{k+1}\big(\mathfrak{g}_-,\, \mathfrak{g}\big)$$

which map any k-cochain $\Phi \in \mathscr{C}^k(\mathfrak{g}_-, \mathfrak{g})$, to the $(k+1)$-cochain $\partial^k\Phi$ defined on any $(k+1)$-tuple $\big(\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_k\big) \in \Lambda^k \mathfrak{g}_-$ through the specific formula:

$$\begin{aligned}(\partial^k\Phi)\big(\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_k\big) := & \sum_{i=0}^{k} (-1)^i \big[\mathsf{z}_i,\, \Phi(\mathsf{z}_0, \ldots, \widehat{\mathsf{z}}_i, \ldots, \mathsf{z}_k)\big]_{\mathfrak{g}} + \\ & + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\, \Phi\big([\mathsf{z}_i, \mathsf{z}_j]_{\mathfrak{g}}, \mathsf{z}_0, \ldots, \widehat{\mathsf{z}}_i, \ldots, \widehat{\mathsf{z}}_j, \ldots, \mathsf{z}_k\big),\end{aligned}$$

where as usual, $\widehat{\mathsf{z}}_l$ means removal of the term z_l. This action is multilinear and the composition $\partial^k \circ \partial^{k-1}$ vanishes for each $k \in \mathbb{N}$. Thus, one gets a *cochain complex*:

$$0 \stackrel{\partial^0}{\longrightarrow} \mathscr{C}^1 \stackrel{\partial^1}{\longrightarrow} \mathscr{C}^2 \stackrel{\partial^2}{\longrightarrow} \cdots \stackrel{\partial^{n-2}}{\longrightarrow} \mathscr{C}^{n-1} \stackrel{\partial^{n-1}}{\longrightarrow} \mathscr{C}^n \stackrel{\partial^n}{\longrightarrow} 0,$$

and one defines the k-th cohomology space $H^k(\mathfrak{g}_-, \mathfrak{g})$ as the quotient:

$$H^k\big(\mathfrak{g}_-, \mathfrak{g}\big) := \mathscr{Z}^k \big/ \mathscr{B}^k,$$

where $\mathscr{Z}^k := \ker\big(\partial^k\big)$ and $\mathscr{B}^k := \mathrm{im}\big(\partial^{k-1}\big)$.

From now on, we assume that $\mathfrak{g}$ is *semi-simple* and μ-graded and that $\mathfrak{h}$ is the subalgebra:

$$\mathfrak{h} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\mu.$$

Among the cohomology spaces of various orders, the second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is most interesting for us, since it can encode deformations of Lie algebras and since it intervenes in Cartan connections. More precisely, it is known that the curvature function $\kappa\colon \mathscr{G} \longrightarrow \mathrm{Lin}\big(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}\big)$ of a Cartan geometry of type (G, H) has the property that for any point $p \in \mathscr{G}$, the linear map $\kappa(p)$ is in fact a 2-cochain of $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$, where $\mathfrak{g}_- \cong \mathfrak{g}/\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Furthermore, there are several interesting properties of the curvature function κ which can be found by inspecting the associated second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$. For example, it is known (*see* [26]) that the number of essential curvature components — which will be defined in Section 7 — of the Cartan geometry is equal to the dimension of $H^2(\mathfrak{g}_-, \mathfrak{g})$. In fact, one has the following useful equalities, known as *Bianchi identities* (*cf.* [8, 13] and *see* Section 8, pp. 64–65 in [2] for expanded details):

(19)
$$\partial\kappa^{(i)}(\mathsf{x},\mathsf{y},\mathsf{z}) = -\sum_{cycl}\sum_{j=1}^{i-1}\Big\{\kappa^{(i-j)}(\kappa_-^{(j)}(\mathsf{x},\mathsf{y}),\mathsf{z}) + \big(\omega^{-1}(\mathsf{z})\cdot\kappa^{(i+|\mathsf{z}|)}\big)(\mathsf{x},\mathsf{y})\Big\},$$

which enable one to express the i-th homogeneous component of κ in terms of its lower homogeneity components; in this formula, κ_- is the composition of the curvature function κ with the projection map of $\mathfrak{g}$ onto $\mathfrak{g}_-$ and the vector field $\omega^{-1}(\mathsf{z})$ on $\mathscr{G}$ acts as a derivation '$\cdot$' on curvature components $\kappa^{(l)}$.

Moreover, when $\mathfrak{g}$ is semi-simple, as is our $\mathfrak{hol}(\mathbb{H}^3) \cong \mathfrak{sl}_3(\mathbb{R})$, an adjoint co-differential operator ∂^* exists (*cf.* [8, 13] and *see* [2], pp. 37–38 and pp. 65–66 for related explanations), and one has a Hodge-type decomposition:

$$\mathscr{Z}^2(\mathfrak{g}_-, \mathfrak{g}) = \mathscr{B}^2(\mathfrak{g}_-, \mathfrak{g}) \oplus \big(\ker\partial \cap \ker\partial^*\big),$$

whence the interesting cohomology space:

$$H^2(\mathfrak{g}_-, \mathfrak{g}) = \ker\partial \cap \ker\partial^*$$

consists of so-called *harmonic* 2-cochains. A Cartan connection associated to such a pair $(\mathfrak{g}_-, \mathfrak{g})$ is said to be *normal* if $\partial^*\kappa^{(i)} = 0$ for all i.

Proposition 4.1. (*See* [8, 13]) *Let κ be the curvature function of a normal Cartan connection modeled on the pair (G, H) with the corresponding algebra $\mathfrak{g}$ being semi-simple. Then, the homogeneity of the first non-zero homogeneous component of κ is larger than the homogeneity of the first non-zero homogeneous component of $H^2(\mathfrak{g}_-, \mathfrak{g})$.* □

4.1. **Computation of the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$.** Now let us turn back to our 2-graded semi-simple Lie algebra $\mathfrak{g} \cong \mathfrak{hol}(\mathbb{H}^3)$ dealt with in the former section. Let $\mathsf{t}^*, \mathsf{h}_1^*, \mathsf{h}_2^*$ be the basis elements of the dual space $\mathfrak{g}_-^*$ of $\mathfrak{g}_-$. Furthermore, put the order $h_1 \prec h_2 \prec t$ on the corresponding typefaces. Then, we have 24 basis elements of the vector space $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g}) \cong \Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}$:

$$\mathsf{x}^* \wedge \mathsf{y}^* \otimes \mathsf{v},$$

where $\mathsf{x} \prec \mathsf{y}$ are any two elements of $\mathfrak{g}_-$ and where v belongs to $\mathfrak{g}$. With such bases, a general 2-cochain writes under the shape:

$$\Phi = \sum_{x\prec y}\sum_{v}\phi_v^{xy}\,\mathsf{x}^* \wedge \mathsf{y}^* \otimes \mathsf{v},$$

where the real coefficients ϕ_v^{xy} are arbitrary.

To compute the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$, we apply a Maple-based computational algorithm that we introduced recently in the article [1] jointly with Aghasi and Alizadeh. (Fully detailed computations achieved by hand are also typed in Section 5 of [2].) According to the result gained, we have the following basis elements of the desired second cohomology:

$$\mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_2 - 2\,\mathsf{h}_1^* \wedge \mathsf{h}_2^* \otimes \mathsf{j} \quad \text{and} \quad \mathsf{t}^* \wedge \mathsf{h}_2^* \otimes \mathsf{i}_1 - \mathsf{t}^* \wedge \mathsf{h}_1^* \otimes \mathsf{i}_2.$$

Furthermore, the following table displays the dimensions of different homogeneous components of $\mathscr{C}^2, \mathscr{Z}^2, \mathscr{B}^2$ and H^2.

Homogeneity	$\dim \mathscr{C}^2$	$\dim \mathscr{Z}^2$	$\dim \mathscr{B}^2$	$\dim H^2$
0	1	1	1	0
1	4	4	4	0
2	6	5	5	0
3	6	4	4	0
4	5	3	1	2
5	2	0	0	0

Corollary 4.2. *The first non-zero homogeneous component of the Cartan curvature corresponding to some normal Cartan connection ω associated to a Levi nondegenerate (geometry-preserving) deformation $M^3 \subset \mathbb{C}^2$ of the Heisenberg sphere $\mathbb{H}^3$ should occur in homogeneity $\geqslant 4$.* □

5. Initial Frame

Consider an arbitrary real $\mathscr{C}^6$-smooth hypersurface $M \subset \mathbb{C}^2$ represented as a graph $v = \varphi(x, y, u)$ with $0 \in M$ and $T_0 M = \{\mathrm{Im}\, w = 0\}$ so that $0 = \varphi(0) = \varphi_x(0) = \varphi_y(0) = \varphi_u(0)$. A $(0,1)$ vector field of the form:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \overline{\mathtt{A}}\,\frac{\partial}{\partial \overline{w}},$$

is tangent to M if and only if $0 = \frac{\overline{\mathtt{A}}}{2i} + \varphi_{\overline{z}} + \frac{\overline{\mathtt{A}}}{2}\,\varphi_u$, or equivalently:

$$\overline{\mathtt{A}} = \frac{2\,\varphi_{\overline{z}}}{i - \varphi_u}.$$

Here $\overline{\mathscr{L}}$ is written *extrinsically*, namely it involves the extra coordinate v and it lives in a neighborhood of M, in $\mathbb{C}^2$, while M itself, which is three-dimensional, is naturally equipped with the three real coordinates (x, y, u). Since we want two real independent sections of $T^c M = \mathrm{Re}\, T^{0,1} M$, we need at first to pullback this $\overline{\mathscr{L}}$ to M, which simply means dropping the basic field $\frac{\partial}{\partial v}$ and replacing v by $\varphi(x, y, u)$ in the coefficient functions. Thus:

$$\overline{\mathscr{L}}\big|_M = \frac{1}{2}\,\frac{\partial}{\partial x} + \frac{i}{2}\,\frac{\partial}{\partial y} + \bigg(\frac{2\,\varphi_{\overline{z}}}{i - \varphi_u}\bigg)\bigg(\frac{1}{2}\,\frac{\partial}{\partial u}\bigg),$$

and it generates $T^{0,1} M$, *intrinsically* (*cf.* also the basic first chapters of [7]). So it only remains to decompose the coefficient $\overline{\mathtt{A}}$ in real and imaginary parts:

$$\overline{\mathtt{A}} = \frac{\varphi_x + i\,\varphi_y}{i - \varphi_u} = \frac{\varphi_y - \varphi_x\,\varphi_u}{1 + \varphi_u^2} + i\,\frac{-\varphi_x - \varphi_y\,\varphi_u}{1 + \varphi_u^2}.$$

Lemma 5.1. *For any local $\mathscr{C}^6$ hypersurface $M^3 \subset \mathbb{C}^2$ which is represented as:*

$$v = \varphi(x, y, u)$$

in coordinates $(z, w) = (x + iy,\, u + iv)$, the complex tangent bundle $T^c M = \mathrm{Re}\, T^{0,1} M$ is generated by the two explicit vector fields:

$$\begin{cases} H_1 := \dfrac{\partial}{\partial x} + \bigg(\dfrac{\varphi_y - \varphi_x\,\varphi_u}{1 + \varphi_u^2}\bigg)\dfrac{\partial}{\partial u} \\[2ex] H_2 := \dfrac{\partial}{\partial y} + \bigg(\dfrac{-\varphi_x - \varphi_y\,\varphi_u}{1 + \varphi_u^2}\bigg)\dfrac{\partial}{\partial u}. \end{cases}$$

For later use let us introduce three abbreviations for the appearing functions:

$$\Delta := 1+\varphi_u^2, \qquad \Lambda_1 := \varphi_y - \varphi_x\,\varphi_u, \qquad \Lambda_2 := -\varphi_x - \varphi_y\,\varphi_u.$$

Our next focus will be to compute the Lie bracket $[H_1, H_2]$. A direct computation (either by hand or with the help of a computer) yields:

$$[H_1, H_2] = \frac{\Upsilon}{\Delta^2}\frac{\partial}{\partial u},$$

where:

$$\begin{aligned}\Upsilon := & -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + \\ & + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy}.\end{aligned}$$

Now, assume that M is Levi nondegenerate at the origin, so that second order terms can be assumed to be normalized as:

$$v = \varphi(x,y,u) = x^2 + y^2 + \mathrm{O}(3).$$

Thanks to this assumption, we have $[H_1, H_2]\big|_0 = -4\frac{\partial}{\partial u}$. On the other hand, it is clear that $H_1\big|_0 = \frac{\partial}{\partial x}$ and $H_2\big|_0 = \frac{\partial}{\partial y}$. Hence, the three vector fields H_1, H_2 and $[H_1, H_2]$ remain linearly independent in some neighborhood of the origin and since M is 3-dimensional, one immediately concludes that they form an initial frame for M around this point. In summary, we have:

$$[H_1, H_2] = 4\,T \quad \text{if one sets} \quad T := \frac{1}{4}\,\Upsilon\,\frac{\partial}{\partial u},$$

5.1. **length-three brackets.** Consider the two brackets of length three $[H_1, T] = \frac{1}{4}\big[H_1, [H_1, H_2]\big]$ and $[H_2, T] = \frac{1}{4}\big[H_2, [H_1, H_2]\big]$. According to the explicit expressions of the vector fields H_1, H_2 and T, obtained just above, T is a multiple of $\frac{\partial}{\partial u}$, H_1 is a combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial u}$ and H_2 is a combination of $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial u}$, and in addition both the coefficients of $\frac{\partial}{\partial x}$ in H_1 and $\frac{\partial}{\partial y}$ in H_2 are exactly equal to 1. Therefore, similar to T, the two bracket fields $[H_1, T]$ and $[H_2, T]$ are just multiples of the standard field $\frac{\partial}{\partial u}$. More precisely:

Lemma 5.2. *Allowing the two notational coincidences $x_1 \equiv x$, $x_2 \equiv y$, one has:*

$$[H_1, T] = \Phi_1 T \qquad \text{and} \qquad [H_2, T] = \Phi_2 T,$$

where:

$$\Phi_1 = \frac{A_1}{\Delta^2\Upsilon}, \qquad \Phi_2 = \frac{A_2}{\Delta^2\Upsilon}$$

and where:

$$A_i := \Delta^2\,\Upsilon_{x_i} + \Delta\big(-2\,\Delta_{x_i}\,\Upsilon + \Lambda_i\,\Upsilon_u - \Upsilon\,\Lambda_{i,u}\big) - \Lambda_i\,\Upsilon\,\Delta_u \qquad (i=1,2).$$

In the expanded memoir [2], the one-page long explicit expressions of both Φ_1 and Φ_2 are provided in terms of the defining function $\varphi(x,y,u)$ and its partial derivatives up to order $\leqslant 3$ (*cf.* also the Maple worksheet [4]). It is important to notice that such kinds of expressions are necessary to build the sought Cartan connection explicitly in terms of $\varphi(x,y,u)$.

Lemma 5.3. *The two functions $H_1(\Phi_2) = H_2(\Phi_1)$ are equal.*

Proof. By what has been seen at the moment, we have by definition:

$$[H_1, T] = \Phi_1 \, T, \qquad\qquad [H_2, T] = \Phi_2 \, T,$$

whence, by bracketing the second (resp. first) equation with $[H_1, \cdot]$ (resp. $[H_2, \cdot]$):

$$\begin{aligned} \big[H_1, [H_2, T]\big] &= \big[H_1, \Phi_2 \, T\big] = H_1(\Phi_2) \, T + \Phi_2 \Phi_1 \, T, \\ \big[H_2, [H_1, T]\big] &= \big[H_2, \Phi_1 \, T\big] = H_2(\Phi_1) \, T + \Phi_1 \Phi_2 \, T. \end{aligned}$$

On the other hand, the Jacobi identity enables us to realize that these two iterated brackets are in fact equal:

$$\big[H_1, [H_2, T]\big] - \big[H_2, [H_1, T]\big] = -\big[T, [H_1, H_2]\big] = \big[T, \, 4 \, T\big] = 0,$$

so that we deduce at once $H_1(\Phi_2) = H_2(\Phi_1)$, as was claimed. □

6. Free Lie Algebras of Rank Two and Relations Between Brackets of Length $\leqslant 6$

In this section, we seek relations between iterated Lie brackets of length $\leqslant 6$ between the two generators H_1 and H_2 of $T^c M$. The use of these relations will be necessary to simplify the computations of the next section and also to normalize the obtained curvatures. To our knowledge, this computational aspect is absent in all existing constructions of Cartan connections associated to Levi nondegenerate $M^3 \subset \mathbb{C}^2$ ([10, 11, 16, 22, 17, 13, 15]).

6.1. **Free Lie algebras of rank two.** At first, let us introduce briefly the concept of free Lie algebras in our context (*see* [20], pp. 9–11 for a survey and also Section 7 in [2] for more details).

For two linearly independent elements h_1 and h_2 of a certain vector space over $\mathbb{R}$, the *free Lie algebra* $\mathscr{F}$ of rank two is the smallest $\mathbb{R}$-algebra having h_1 and h_2 as elements, with bilinear multiplication:

$$(h, h') \longmapsto [h, h'] \in \mathscr{F} \qquad\qquad {\scriptstyle (h, h' \in \mathscr{F})}$$

satisfying skew-symmetry:

$$0 = [h, \, h'] + [h', \, h] \qquad\qquad {\scriptstyle (h, h' \in \mathscr{F})}$$

and the Jacobi identity:

$$0 = \big[h, \, [h', \, h'']\big] + \big[h'', \, [h, \, h']\big] + \big[h', \, [h'', \, h]\big] \qquad\qquad {\scriptstyle (h, h' \, h'' \in \mathscr{F})}.$$

We call an arbitrarily iterated Lie bracket of h_1 and h_2 a *word of length* l when we see these two elements exactly for l times inside the bracket. Furthermore, a word is called *simple* if it has the following form:

$$\big[h_{i_1}, \, \big[h_{i_2}, \, \big[\dots \big[h_{i_{\ell'-1}}, \, h_{i_{\ell'}}\big] \dots\big]\big]\big],$$

where $1 \leqslant i_1, i_2, \dots, i_{\ell'-1}, i_{\ell'} \leqslant 2$. We denote by $\mathscr{F}_l$, the $\mathbb{R}$-vector space generated by all simple words of the above form with $\ell' \leqslant \ell$. It is known (*see e.g.* [20], p. 11) that all words express $\mathbb{Z}$-linearly in terms of simple words and that for an arbitrary integer ℓ, the dimension $n_\ell - n_{\ell-1}$ of each quotient space $\mathscr{F}_\ell / \mathscr{F}_{\ell-1}$ satisfies the inductive relation:

$$n_\ell - n_{\ell-1} = \frac{1}{\ell} \sum_{d \text{ divides } \ell} \mu(d) \, 2^{\frac{\ell}{d}},$$

where μ is the Möbius function:

$$\mu(d) = \begin{cases} 1, \text{ if } d = 1; \\ 0, \text{ if } d \text{ contains square integer factors;} \\ (-1)^\nu, \text{ if } d = p_1 \cdots p_\nu \text{ is the product of } \nu \text{ distinct prime numbers.} \end{cases}$$

For what interests us, this formula simply gives the following relations:

$$n_2 - n_1 = 1, \qquad n_3 - n_2 = 2, \qquad n_4 - n_3 = 3,$$
$$n_5 - n_4 = 6, \qquad n_6 - n_5 = 9.$$

6.2. **Relations up to length six.** Let us start with the length $l = 2$. As one would expect, there are only one — up to skew symmetry — simple word:

$$[h_1, h_2]$$

of this length which also confirms the above equality $n_2 - n_1 = 1$.

Next, for the length $l = 3$, again it is clear that there are, up to skew symmetry, exactly two simple words, confirming $n_3 - n_2 = 2$:

$$[h_1, [h_1, h_2]], \qquad [h_2, [h_1, h_2]].$$

In fact, the first non-trivial case starts at length $l = 4$. Up to skew symmetry, we see here four simple words:

$$\big[h_1, \big[h_1, [h_1, h_2]\big]\big], \quad \big[h_1, \big[h_2, [h_1, h_2]\big]\big], \quad \big[h_2, \big[h_1, [h_1, h_2]\big]\big], \quad \big[h_2, \big[h_2, [h_1, h_2]\big]\big].$$

But we can erase one of these words from the collection of generators using the following Jacobi identity (underlining '$\underline{\ \ }_\circ$' indicates a term vanishes):

$$0 = \big[h_1, \big[h_2, [h_1, h_2]\big]\big] + \underline{\big[[h_1, h_2], \big[h_1, h_2\big]\big]}_\circ + \big[h_2, [[h_1, h_2], h_1\big],$$

whence:

$$\big[h_2, \big[h_1, [h_1, h_2]\big]\big] = \big[h_1, \big[h_2, [h_1, h_2]\big]\big]. \tag{20}$$

Hence the number of generators of $\mathscr{F}_4/\mathscr{F}_3$ equals 3, which is coherent with the equality $n_4 - n_3 = 3$.

In the case of $l = 5$, the equality $n_5 - n_4 = 6$ asserts that $\mathscr{F}_5/\mathscr{F}_4$ can be generated by exactly six simple words. Namely:

$$\big[h_1, \big[h_1, \big[h_1, [h_1, h_2]\big]\big]\big], \qquad \big[h_1, \big[h_1, \big[h_2, [h_1, h_2]\big]\big]\big], \qquad \big[h_1, \big[h_2, \big[h_2, [h_1, h_2]\big]\big]\big],$$
$$\big[h_2, \big[h_1, \big[h_1, [h_1, h_2]\big]\big]\big], \qquad \big[h_2, \big[h_1, \big[h_2, [h_1, h_2]\big]\big]\big], \qquad \big[h_2, \big[h_2, \big[h_2, [h_1, h_2]\big]\big]\big].$$

In addition, it is also important for later use to explicitly represent all multiple iterated brackets of length five as specific linear combinations of simple words. For instance, there are exactly two brackets between two basic words of lengths 2 and 3, and the Jacobi identity gives[1]:

$$0 = \big[[\underline{h_1}, \underline{h_2}], \underline{\big[h_1, [h_1, h_2]\big]}\big] + \big[\big[\big[h_1, [h_1, h_2]\big], h_1\big], h_2\big] + \big[\big[h_2, \big[h_1, [h_1, h_2]\big]\big], h_1\big],$$
$$0 = \big[[\underline{h_1}, \underline{h_2}], \underline{\big[h_2, [h_1, h_2]\big]}\big] + \big[\big[\big[h_2, [h_1, h_2]\big], h_1\big], h_2\big] + \big[\big[h_2, \big[h_2, [h_1, h_2]\big]\big], h_1\big].$$

Here, in each one of the two lines, the last two words happen, thanks to skew-symmetry, to be all simple, whence (using (20) for the last term of the first line):

$$\begin{aligned} \big[[h_1, h_2], \big[h_1, [h_1, h_2]\big]\big] &= -\big[h_2, \big[h_1, \big[h_1, [h_1, h_2]\big]\big]\big] + \big[h_1, \big[h_1, \big[h_2, [h_1, h_2]\big]\big]\big], \\ \big[[h_1, h_2], \big[h_2, [h_1, h_2]\big]\big] &= -\big[h_2, \big[h_1, \big[h_2, [h_1, h_2]\big]\big]\big] + \big[h_1, \big[h_2, \big[h_2, [h_1, h_2]\big]\big]\big]. \end{aligned} \tag{21}$$

[1] For clarity, we underline the three terms that are subjected to a circular permutation.

A higher complexity occurs at length $l = 6$. By applying $[h_i, \cdot]$, $i = 1, 2$, to the above six linearly independent simple words of length 5, we at first get the following twelve simple words:

$$\begin{array}{lll}
[h_1,[h_1,[h_1,[h_1,[h_1,h_2]]]]], & [h_1,[h_1,[h_1,[h_2,[h_1,h_2]]]]], & [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]], \\
[h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]], & [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]], & [h_1,[h_2,[h_2,[h_2,[h_1,h_2]]]]], \\
[h_2,[h_1,[h_1,[h_1,[h_1,h_2]]]]], & [h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]], & [h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]], \\
[h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]], & [h_2,[h_2,[h_1,[h_2,[h_1,h_2]]]]], & [h_2,[h_2,[h_2,[h_2,[h_1,h_2]]]]].
\end{array}$$

But the equality $n_6 - n_5 = 9$ says that $\mathscr{F}_6/\mathscr{F}_5$ is generated by only nine of the above simple words and hence three of them should be specified as linear combinations of the other ones. To find such words, we have in fact to explore the relations between the brackets of simple words of lower lengths by applying the Jacobi identity. While some of these identities are trivial, we get the following seven relations between simple and non-simple words of length six (we apply equations (20) and (21) in the expressions, *see* [2] for full details):

$$\begin{aligned}
0 \overset{1}{=} & \big[\underline{[h_1,[h_1,h_2]]},[\underline{h_2},\underline{[h_1,h_2]}]\big] + \big[[h_1,h_2],[[h_1,[h_1,h_2]],h_2]\big] + \big[h_2,[[h_1,h_2],[h_1,[h_1,h_2]]]\big] \\
= & \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[[h_1,h_2],[h_2,[h_1,[h_1,h_2]]]\big] + \big[h_2,[[h_1,h_2],[h_1,[h_1,h_2]]]\big] \\
= & \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] - \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] - [h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]] + \\
& + [h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{2}{=} & \big[[\underline{h_1},\underline{[h_1,h_2]}],\underline{[h_2,[h_1,h_2]]}\big] + \big[[[h_2,[h_1,h_2]],h_1],[h_1,h_2]\big] + \big[[[h_1,h_2],[h_2,[h_1,h_2]]],h_1\big] \\
= & \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] - \big[h_1,[[h_1,h_2],[h_2,[h_1,h_2]]]\big] \\
= & \big[[h_1,[h_1,h_2]],[h_2,[h_1,h_2]]\big] + \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] + [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - \\
& - [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{3}{=} & \big[\underline{[h_1,h_2]},[\underline{h_1},\underline{[h_1,[h_1,h_2]]}]\big] + \underline{\big[[h_1,[h_1,h_2]],[[h_1,h_2],h_1]\big]}_{\circ} + \big[h_1,[[h_1,[h_1,h_2]],[h_1,h_2]]\big] \\
= & \big[[h_1,h_2],[h_1,[h_1,[h_1,h_2]]]\big] - \big[h_1,[[h_1,h_2],[h_1,[h_1,h_2]]]\big] \\
= & \big[[h_1,h_2],[h_1,[h_1,[h_1,h_2]]]\big] + [h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]] - [h_1,[h_1,[h_1,[h_2,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{4}{=} & \big[[\underline{h_1},\underline{h_2}],\underline{[h_1,[h_1,[h_1,h_2]]]}\big] + \big[[[h_1,[h_1,[h_1,h_2]]],h_1],h_2\big] + \big[[h_2,[h_1,[h_1,[h_1,h_2]]]],h_1\big] \\
= & \big[[h_1,h_2],[h_1,[h_1,[h_1,h_2]]]\big] + [h_2,[h_1,[h_1,[h_1,[h_1,h_2]]]]] - [h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{5}{=} & \big[[\underline{h_1},\underline{h_2}],\underline{[h_1,[h_2,[h_1,h_2]]]}\big] + \big[[[h_1,[h_2,[h_1,h_2]]],h_1],h_2\big] + \big[[h_2,[h_1,[h_2,[h_1,h_2]]]],h_1\big] \\
= & \big[[h_1,h_2],[h_1,[h_2,[h_1,h_2]]]\big] + [h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - [h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{6}{=} & \big[\underline{[h_1,h_2]},[\underline{h_2},\underline{[h_2,[h_1,h_2]]}]\big] + \underline{\big[[h_2,[h_1,h_2]],[[h_1,h_2],h_2]\big]}_{\circ} + \big[h_2,[[h_2,[h_1,h_2]],[h_1,h_2]]\big] \\
= & \big[[h_1,h_2],[h_2,[h_2,[h_1,h_2]]]\big] - \big[h_2,[[h_1,h_2],[h_2,[h_1,h_2]]]\big] \\
= & \big[[h_1,h_2],[h_2,[h_2,[h_1,h_2]]]\big] + [h_2,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - [h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]];
\end{aligned}$$

$$\begin{aligned}
0 \overset{7}{=} & \big[[\underline{h_1},\underline{h_2}],\underline{[h_2,[h_2,[h_1,h_2]]]}\big] + \big[[[h_2,[h_2,[h_1,h_2]]],h_1],h_2\big] + \big[[h_2,[h_2,[h_2,[h_1,h_2]]]],h_1\big] \\
= & \big[[h_1,h_2],[h_2,[h_2,[h_1,h_2]]]\big] + [h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]] - [h_1,[h_2,[h_2,[h_2,[h_1,h_2]]]]].
\end{aligned}$$

Now, an inspection shows that there are three linearly independent relations between the above simple words, as was looked for. Indeed, subtracting the third equation to the fourth, we get the first desired linearly combination as follows:

$$\begin{aligned}
0 \overset{9}{=} & [h_1,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - 2[h_1,[h_2,[h_1,[h_1,[h_1,h_2]]]]] + \\
& + [h_2,[h_1,[h_1,[h_1,[h_1,h_2]]]]].
\end{aligned}$$

Secondly, subtracting the seventh equation to the sixth, we get the second, visibly independent relation:

$$0 \overset{10}{=} [h_2,[h_2,[h_1,[h_2,[h_1,h_2]]]]] - 2\times[h_2,[h_1,[h_2,[h_2,[h_1,h_2]]]]] + \\ + [h_1,[h_2,[h_2,[h_2,[h_1,h_2]]]]].$$

Thirdly and lastly, adding the fifth equation multiplied by 2 to the first one and subtracting the second one, we get the third independent relation between simple words:

$$0 \overset{11}{=} [h_1,[h_1,[h_2,[h_2,[h_1,h_2]]]]] - 3\times[h_1,[h_2,[h_1,[h_2,[h_1,h_2]]]]] + \\ + 3\times[h_2,[h_1,[h_1,[h_2,[h_1,h_2]]]]] - [h_2,[h_2,[h_1,[h_1,[h_1,h_2]]]]].$$

6.3. **Iterated brackets of H_1 and H_2 of length six.** Taking account of Lemma 1.4, one checks by induction that every iterated simple word of arbitrary length $\ell \geqslant 2$ must always be a multiple of $[H_1, H_2]$ by means of a certain function $\Phi_{i_1,\dots,i_\ell}$ which depends on Φ_1 and Φ_2:

$$\big[H_{i_1},\big[H_{i_2},\big[\cdots,\big[H_{i_{\ell-1}},H_{i_\ell}\big],\cdots\big]\big]\big] = \Phi_{i_1,i_2,\dots,i_{\ell-1},i_\ell}\,[H_1,H_2] \tag{22}$$

where $i_1,\dots,i_\ell = 1,2$.

For later use, we need only to compute these functions $\Phi_{i_1,i_2,\dots,i_{\ell-1},i_\ell}$ when the length l equals 6. But for this, at first, it necessary to also compute such functions of lower lengths $l \leqslant 5$. For $l = 3$ we have the following two equations from Lemma 1.4:

$$\big[H_1,[H_1,H_2]\big] = \Phi_1\,[H_1,H_2] \qquad \text{and} \qquad \big[H_2,[H_1,H_2]\big] = \Phi_2\,[H_1,H_2].$$

Now if:

$$\big[H_{i_2},\big[H_{i_3},\big[\cdots,\big[H_{i_{\ell-1}},H_{i_\ell}\big],\cdots\big]\big]\big] = \Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}\,[H_1,H_2],$$

then we have:

$$\begin{aligned}\Phi_{i_1,i_2,\dots,i_{\ell-1},i_\ell}[H_1,H_2] &= \big[H_{i_1},\big[H_{i_2},\big[\cdots,\big[H_{i_{\ell-1}},H_{i_\ell}\big],\cdots\big]\big]\big] = \\ &= \big[H_{i_1},\Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}[H_1,H_2]\big] = \\ &= \big(H_{i_1}(\Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}) + \Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}\Phi_{i_1}\big)\big[H_1,H_2\big],\end{aligned}$$

and consequently a general, useful induction formula reads:

$$\Phi_{i_1,i_2,\dots,i_{\ell-1},i_\ell} = H_{i_1}\big(\Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}\big) + \Phi_{i_2,i_3,\dots,i_{\ell-1},i_\ell}\Phi_{i_1}.$$

Now one gets the following expressions for the iterated Lie brackets of length six by applying the above recursive formula:

$$\begin{aligned}[H_1,[H_1,[H_1,[H_1,[H_1,H_2]]]]] \overset{1}{=} \big(&H_1(H_1(H_1(\Phi_1))) + 4\Phi_1H_1(H_1(\Phi_1)) + \\ &+ 3H_1(\Phi_1)H_1(\Phi_1) + 6(\Phi_1)^2H_1(\Phi_1) + (\Phi_1)^4\big)[H_1,H_2];\end{aligned}$$

$$\begin{aligned}[H_1,[H_1,[H_1,[H_2,[H_1,H_2]]]]] \overset{2}{=} \big(&H_1(H_1(H_1(\Phi_2))) + 3\Phi_1H_1(H_1(\Phi_2)) + \\ &+ \Phi_2H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) + \\ &+ 3\Phi_1\Phi_2H_1(\Phi_1) + 3(\Phi_1)^2H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2];\end{aligned}$$

$$\begin{aligned}[H_1,[H_1,[H_2,[H_2,[H_1,H_2]]]]] \overset{3}{=} \big(&H_1(H_1(H_2(\Phi_2))) + 2\Phi_1H_1(H_2(\Phi_2)) + \\ &+ 2\Phi_2H_1(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) + \\ &+ (\Phi_1)^2H_2(\Phi_2) + (\Phi_2)^2H_1(\Phi_1) + 4\Phi_1\Phi_2H_1(\Phi_2) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2];\end{aligned}$$

$$\begin{aligned}[H_1,[H_2,[H_1,[H_1,[H_1,H_2]]]]] \overset{4}{=} \big(&H_1(H_2(H_1(\Phi_1))) + 2\Phi_1H_1(H_1(\Phi_2)) + \\ &+ \Phi_2H_1(H_1(\Phi_1)) + \Phi_1H_2(H_1(\Phi_1)) + 3H_1(\Phi_1)H_2(\Phi_2) + \\ &+ 3\Phi_1\Phi_2H_1(\Phi_1) + 3(\Phi_1)^2H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2];\end{aligned}$$

$$
\begin{aligned}
[H_1,[H_2,[H_1,[H_2,[H_1,H_2]]]]] \overset{5}{=} \big(&H_1(H_2(H_1(\Phi_2))) + \Phi_1 H_1(H_2(\Phi_2)) + 2\Phi_2 H_1(H_1(\Phi_2)) + \\
&+ \Phi_1 H_2(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) + \\
&+ 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_1,[H_2,[H_2,[H_2,[H_1,H_2]]]]] \overset{6}{=} \big(&H_1(H_2(H_2(\Phi_2))) + 3\Phi_2 H_1(H_2(\Phi_2)) + \\
&+ \Phi_1 H_2(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + \\
&+ 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_1,[H_1,[H_1,[H_1,H_2]]]]] \overset{7}{=} \big(&H_2(H_1(H_1(\Phi_1))) + 3\Phi_1 H_2(H_1(\Phi_1)) + \\
&+ \Phi_2 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) + \\
&+ 3(\Phi_1)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_1(\Phi_1) + (\Phi_1)^3\Phi_2\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_1,[H_1,[H_2,[H_1,H_2]]]]] \overset{8}{=} \big(&H_2(H_1(H_1(\Phi_2))) + 2\Phi_1 H_2(H_1(\Phi_2)) + \Phi_2 H_2(H_1(\Phi_1)) + \\
&+ \Phi_2 H_1(H_1(\Phi_2)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_2(\Phi_2)H_1(\Phi_1) + \\
&+ 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_1,[H_2,[H_2,[H_1,H_2]]]]] \overset{9}{=} \big(&H_2(H_1(H_2(\Phi_2))) + 2\Phi_2 H_2(H_1(\Phi_2)) + \\
&+ \Phi_1 H_2(H_2(\Phi_2)) + \Phi_2 H_1(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + \\
&+ 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_2,[H_1,[H_1,[H_1,H_2]]]]] \overset{10}{=} \big(&H_2(H_2(H_1(\Phi_1))) + 2\Phi_1 H_2(H_1(\Phi_2)) + \\
&+ 2\Phi_2 H_2(H_1(\Phi_1)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_1(\Phi_1)H_2(\Phi_2) + \\
&+ 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_2,[H_1,[H_2,[H_1,H_2]]]]] \overset{11}{=} \big(&H_2(H_2(H_1(\Phi_2))) + \Phi_1 H_2(H_2(\Phi_2)) + \\
&+ 3\Phi_2 H_2(H_1(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + \\
&+ 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
[H_2,[H_2,[H_2,[H_2,[H_1,H_2]]]]] \overset{12}{=} \big(&H_2(H_2(H_2(\Phi_2))) + 4\Phi_2 H_2(H_2(\Phi_2)) + \\
&+ 3H_2(\Phi_2)H_2(\Phi_2) + 6(\Phi_2)^2 H_2(\Phi_2) + (\Phi_2)^4\big)[H_1,H_2].
\end{aligned}
$$

Lastly, one may compute the only plain Lie brackets between two simple words of length 3, and also, the three Lie brackets between the single simple word of length 2 and the three simple words of length 3:

$$
\begin{aligned}
\big[[H_1,[H_1,H_2]],[H_2,[H_1,H_2]]\big] \overset{13}{=} \big(&\Phi_1 H_1(H_2(\Phi_2)) - \Phi_1 H_2(H_1(\Phi_2)) - \\
&- \Phi_2 H_1(H_1(\Phi_2)) + \Phi_2 H_2(H_1(\Phi_1))\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
\big[[H_1,H_2],[H_1,[H_1,H_2]]\big] \overset{14}{=} \big(&H_1(H_2(H_1(\Phi_2))) - H_2(H_1(H_1(\Phi_1))) + \\
&+ 2\Phi_1 H_1(H_1(\Phi_2)) - 2\Phi_1 H_2(H_1(\Phi_1))\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
\big[[H_1,H_2],[H_1,[H_2,H_2]]\big] \overset{15}{=} \big(&H_1(H_2(H_1(\Phi_2))) - H_2(H_1(H_1(\Phi_2))) + \\
&+ \Phi_2 H_1(H_2(\Phi_1)) - \Phi_2 H_2(H_1(\Phi_1)) + \\
&+ \Phi_1 H_1(H_2(\Phi_2)) - \Phi_1 H_2(H_1(\Phi_2))\big)[H_1,H_2];
\end{aligned}
$$

$$
\begin{aligned}
\big[[H_1,H_2],[H_2,[H_2,H_2]]\big] \overset{16}{=} \big(&H_1(H_2(H_2(\Phi_2))) - H_2(H_1(H_2(\Phi_2))) + \\
&+ 2\Phi_2 H_1(H_2(\Phi_2)) - 2\Phi_2 H_2(H_1(\Phi_2))\big)[H_1,H_2].
\end{aligned}
$$

Proposition 6.1. *The two functions Φ_1 and Φ_2 identically satisfy the following five third-order relations:*

$$
\begin{aligned}
0 \overset{\text{I}}{\equiv} &- H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\
&- \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)),
\end{aligned}
$$

$$
\begin{aligned}
0 \overset{\text{II}}{\equiv} &- H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \\
&- \Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)),
\end{aligned}
$$

$$\begin{aligned}0 &\overset{\mathrm{III}}{\equiv} -H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \\ &+ \Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)),\end{aligned}$$

$$\begin{aligned}0 &\overset{\mathrm{IV}}{\equiv} H_2(H_2(H_1(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_1(H_2(H_2(\Phi_2))) - \\ &- \Phi_2\,H_2(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)),\end{aligned}$$

$$\begin{aligned}0 &\overset{\mathrm{V}}{\equiv} H_1(H_1(H_2(\Phi_2))) - 3\,H_1(H_2(H_1(\Phi_2))) + 3\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\ &- \Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)) - \Phi_1\,H_1(H_2(\Phi_2)) + \Phi_1\,H_2(H_1(\Phi_2)),\end{aligned}$$

the first four being linearly independent, while the fifth coincides with $\mathrm{I} - \mathrm{II}$.

Proof. Using the representations $\overset{1}{=}, \dots, \overset{16}{=}$ of the iterated brackets between H_1 and H_2 of lengths $\ell = 6$, one may substitute them in the eleven free Lie algebra relations $\overset{1}{=}, \dots, \overset{11}{=}$. Some of the obtained equations are redundant, and some reduce to the trivial identity $0 = 0$. □

Corollary 6.2. *The following two quantities are identically zero:*

$$\begin{aligned}0 &\overset{a}{\equiv} -H_2(H_2(H_1(\Phi_1))) + H_2(H_1(H_1(\Phi_2))) + H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) + \\ &+ \Phi_1\,H_1(H_2(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_2)) - \Phi_2\,H_1(H_1(\Phi_2)) + \Phi_2\,H_2(H_1(\Phi_1)),\end{aligned}$$

$$\begin{aligned}0 &\overset{b}{\equiv} H_1(H_2(H_2(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_2(H_2(H_1(\Phi_2))) - \\ &- H_2(H_1(H_1(\Phi_1))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_1(H_1(H_1(\Phi_2))) + \\ &\Phi_1\,H_1(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)) - \Phi_2\,H_2(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)).\end{aligned}$$

Proof. Indeed, using the proposition, the first identity is just $\mathrm{I}+\mathrm{II}$, while the second is just $\mathrm{III} + \mathrm{IV}$. □

7. Construction of a Cartan Curvature Connection and of its Associated Curvature Function

7.1. **General form of the Cartan connection frame.** Let H be the unique (local, connected) five-dimensional Lie group associated to the *abstract* Lie algebra $\mathfrak{h} \cong \mathfrak{g}/\mathfrak{g}_- = \mathrm{Span}_{\mathbb{R}}\big(\mathsf{d},\, \mathsf{r},\, \mathsf{i}_1,\, \mathsf{i}_2,\, \mathsf{j}\big)$ corresponding to the five generators D, R, I_1, I_2, J of the Lie isotropy algebra of the origin $0 \in \mathbb{H}^3$, and let G be the simply connected eight-dimensional Lie group corresponding to $\mathfrak{g} = \mathrm{Span}_{\mathbb{R}}\big(\mathsf{t},\, \mathsf{h}_1,\, \mathsf{h}_2,\, \mathsf{d},\, \mathsf{r},\, \mathsf{i}_1,\, \mathsf{i}_2,\, \mathsf{j}\big) \cong \mathfrak{aut}(\mathbb{H}^3)$.

In order to construct a regular normal Cartan connection ω of type (G, H) on an arbitrary Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$, we need at first to produce some five explicit left-invariant vector fields on the (local) Lie group H. Let H be equipped with five real coordinates denoted by (a, b, c, d, e) near its identity element. Then one can simply take exactly the same five left-invariant vector fields near the identity $(a_0, b_0, c_0, d_0, e_0) := (0, 0, 1, 1, 0)$ as those shown in [13]:

$$\begin{aligned}D &:= -a\,\tfrac{\partial}{\partial a} - b\,\tfrac{\partial}{\partial b} - c\,\tfrac{\partial}{\partial c} - d\,\tfrac{\partial}{\partial d} - 2e\,\tfrac{\partial}{\partial e}, \\ R &:= -b\,\tfrac{\partial}{\partial a} + a\,\tfrac{\partial}{\partial b} + d\,\tfrac{\partial}{\partial c} - c\,\tfrac{\partial}{\partial d}, \\ I_1 &:= \tfrac{\partial}{\partial a} - b\,\tfrac{\partial}{\partial e}, \\ I_2 &:= \tfrac{\partial}{\partial b} + a\,\tfrac{\partial}{\partial e}, \\ J &:= \tfrac{1}{2}\,\tfrac{\partial}{\partial e}.\end{aligned}$$

One verifies that the commutators of these vector fields are precisely the same as before, so that they indeed form a basis for the Lie algebra $\mathfrak{h}$ of the abstract Lie group H. Then according to the condition **(iii)** of Definition 1.1, we should have:

$$\widehat{X} = \omega^{-1}(\mathsf{x}) = X \qquad (X{=}D, R, I_1, I_2, J).$$

Then one sees that $[\widehat{X}, \widehat{Y}] = \widehat{[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}}$, for $X, Y = D, R, I_1, I_2, J$, which is consistent with the infinitesimal version of condition **(ii)** stated as condition **(c2)** just below.

Actually, to construct the desired Cartan connection, the only unknowns are the three *horizontal* vector fields:

$$\widehat{T} = \omega^{-1}(\mathsf{t}), \qquad \widehat{H}_1 = \omega^{-1}(\mathsf{h}_1), \qquad \widehat{H}_2 = \omega^{-1}(\mathsf{h}_2).$$

In order to perform the desired construction, we must therefore find certain functions $\alpha_{\bullet\bullet}$ as coefficients of these three lifted horizontal vector fields:

$$\begin{cases} \widehat{T} := \alpha_{tt}\, T + \alpha_{th_1}\, H_1 + \alpha_{th_2}\, H_2 + \alpha_{td}\, D + \alpha_{tr}\, R + \alpha_{ti_1}\, I_1 + \alpha_{ti_2}\, I_2 + \alpha_{tj}\, J \\ \widehat{H}_1 := \qquad \alpha_{h_1h_1}\, H_1 + \alpha_{h_1h_2}\, H_2 + \alpha_{h_1d}\, D + \alpha_{h_1r}\, R + \alpha_{h_1i_1}\, I_1 + \alpha_{h_1i_2}\, I_2 + \alpha_{h_1j}\, J \\ \widehat{H}_2 := \qquad \alpha_{h_2h_1}\, H_1 + \alpha_{h_2h_2}\, H_2 + \alpha_{h_2d}\, D + \alpha_{h_2r}\, R + \alpha_{h_2i_1}\, I_1 + \alpha_{h_2i_2}\, I_2 + \alpha_{h_2j}\, J, \end{cases}$$

such that the following four conditions would be satisfied at the end:

- **(c1)** at each point $p \in \mathscr{G}$, the linear map $\omega_p \colon T_p\mathscr{G} \to \mathfrak{g}$ should be an isomorphism;
- **(c2)** for any $X = D, R, I_1, I_2, J$ and for any $Y = H_1, H_2, T$ with corresponding $\mathsf{x} = \mathsf{d}, \mathsf{r}, \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}$ and $\mathsf{y} = \mathsf{h}_1, \mathsf{h}_2, \mathsf{t}$, one should have:

$$\big[\widehat{X}, \widehat{Y}\big] = \widehat{[\mathsf{x}, \mathsf{y}]_{\mathfrak{g}}}, \tag{23}$$

  and this condition is just the infinitesimal (equivalent) version of condition **(ii)** in Definition 1.1 (*cf* [8, 12, 9]);
- **(c3)** the connection form ω should be *normal*, namely the codifferential operator ∂^* (*see* [8, 9]) should annihilate all homogeneous curvature components, *i.e.* $\partial^*\kappa^{(i)} \equiv 0$ for all i;
- **(c4)** the connection should be *regular*, that is to say, all curvature components $\kappa^{(i)}$ of negative homogeneity $i \leqslant -1$ should vanish identically, but this is already so, because as does any element of $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$, the curvature function κ does not have any non-zero homogeneous component of negative homogeneity.

In fact, the process of construction (*cf.* [6, 13, 2]) will mainly consist in annihilating as many curvature components as possible, and without invoking ∂^*, we will be able to annihilate $\kappa^{(0)}$ (easiest thing), $\kappa^{(1)}$, $\kappa^{(2)}$ and $\kappa^{(3)}$ by an appropriate progressive building of ω which requires somewhat hard elimination computations. We shall then check at the end that $0 = \partial^*\kappa^{(4)} = \partial^*\kappa^{(5)}$, which will yield normality.

To specify appropriate expressions of the functions $\alpha_{\bullet\bullet}$, let us start by examining thoroughly **(c2)**, namely (23). Putting the above expressions of $\widehat{X} = \widehat{T}, \widehat{H}_1, \widehat{H}_2$ and $\widehat{Y} = \widehat{D}, \widehat{R}, \widehat{I}_1, \widehat{I}_2, \widehat{J}$ in (23) and subsequently equating the coefficients of the eight basic, framing sections $\big(T, H_1, H_2, D, R, I_1, I_2, J\big)$ of $\Gamma(T\mathscr{G})$ in the both sides of (23) gives a system of 110 partial differential equations, the unknowns of which are the 22 functions $\alpha_{\bullet\bullet}$. The solution set of this system of 110 equations is the following (*see* [2] for more details).

Lemma 7.1. *The general solution of the mentioned system of 110 partial differential equations is polynomial of degree* $\leqslant 4$ *with respect to the five vertical variables* a, b, c, d, e *of the principal bundle* H, *and it involves 22 coefficients* $\delta_1(x,y,z)$, $\ldots$, $\delta_{22}(x,y,z)$ *that are arbitrary functions of the horizontal variables* (x,y,z)*:*

$$
\begin{aligned}
\alpha_{tt} &= (c^2+d^2)\,\delta_{22},\\
\alpha_{th_1} &= -(ad+bc)\,\delta_{13}+(ac-bd)\,\delta_{14}+(c^2+d^2)\,\delta_{21},\\
\alpha_{th2} &= -(ad+bc)\,\delta_{11}+(ac-bd)\,\delta_{12}+(c^2+d^2)\,\delta_{20}\\
\alpha_{td} &= (-\tfrac14 bc-\tfrac14 ad)\,\delta_1+(\tfrac14 ac-\tfrac14 bd)\,\delta_2+(\tfrac14 c^2+\tfrac14 d^2)\,\delta_{15}-2e,\\
\alpha_{tr} &= (\tfrac14 c^2+\tfrac14 d^2)\,\delta_4+(\tfrac12 ac-\tfrac12 bd)\,\delta_7+(\tfrac12 c^2+\tfrac12 d^2)\,\delta_9-(\tfrac12 ad+\tfrac12 bc)\,\delta_{10}+\\
&\qquad +(\tfrac12 c^2+\tfrac12 d^2)\,\delta_{19}+3b^2+3a^2,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{ti_1} = {}& -(\tfrac14 a^2d+\tfrac14 abc)\,\delta_1+(-\tfrac14 abd+\tfrac14 a^2c)\,\delta_2+(\tfrac1{24}d^3+\tfrac1{24}c^2d)\,\delta_3+(\tfrac18 d^2b+\tfrac18 bc^2)\,\delta_4+\\
&+(\tfrac18 c^3+\tfrac18 cd^2)\,\delta_5+(\tfrac18 ac^2+\tfrac18 ad^2)\,\delta_6+(-\tfrac12 db^2+\tfrac12 bca)\,\delta_7-(\tfrac14 dcb+\tfrac14 ad^2)\,\delta_8+\\
&+(\tfrac14 bc^2+\tfrac12 d^2b-\tfrac14 dca)\,\delta_9-(\tfrac12 bda+\tfrac12 cb^2)\,\delta_{10}+(\tfrac14 d^2a+\tfrac14 ac^2)\,\delta_{15}+\\
&+(\tfrac14 c^3+\tfrac14 cd^2)\,\delta_{16}+(\tfrac14 c^2d+\tfrac14 d^3)\,\delta_{17}+(\tfrac12 d^2b+\tfrac12 bc^2)\,\delta_{19}+2a^2b+2b^3,\\
\alpha_{ti_2} = {}& -(\tfrac14 bda+\tfrac14 cb^2)\,\delta_1+(-\tfrac14 db^2+\tfrac14 bca)\,\delta_2+(\tfrac1{24}c^3+\tfrac1{24}cd^2)\,\delta_3+(-\tfrac18 ac^2-\tfrac18 ad^2)\,\delta_4+\\
&+(-\tfrac18 c^2d-\tfrac18 d^3)\,\delta_5+(\tfrac18 d^2b+\tfrac18 bc^2)\,\delta_6+(-\tfrac12 a^2c+\tfrac12 bda)\,\delta_7-(\tfrac14 dca+\tfrac14 bc^2)\,\delta_8+\\
&+(-\tfrac12 ac^2+\tfrac14 dcb-\tfrac14 d^2a)\,\delta_9+(\tfrac12 a^2d+\tfrac12 bca)\,\delta_{10}+(\tfrac14 bc^2+\tfrac14 d^2b)\,\delta_{15}-\\
&-(\tfrac14 c^2d+\tfrac14 d^3)\,\delta_{16}+(\tfrac14 c^3+\tfrac14 cd^2)\,\delta_{17}+(-\tfrac12 ac^2-\tfrac12 d^2a)\,\delta_{19}-2a^3-2ab^2,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{tj} = {}& -(acb+ade)\,\delta_1+(-edb+eca)\,\delta_2+(cb^2a+ca^3-db^3-dba^2)\,\delta_7+\\
&+(-d^2ab+c^2ab-dcb^2+dca^2)\,\delta_8+(-2dcba+c^2a^2+d^2b^2)\,\delta_9\\
&+(-da^3-db^2a-cb^3-cba^2)\,\delta_{10}+(ed^2+ec^2)\,\delta_{15}+\\
&+(ad^3+c^3b+dac^2+cbd^2)\,\delta_{16}+(-cad^2+bd^3+dbc^2-c^3a)\,\delta_{17}+\\
&+(d^4+c^4+2d^2c^2)\,\delta_{18}+(b^2c^2+d^2b^2+c^2a^2+a^2d^2)\,\delta_{19}+6a^2b^2-\\
&-4e^2+3a^4+3b^4,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{h_1h_1} &= (d)\,\delta_{13}-(c)\,\delta_{14},\\
\alpha_{h_1h_2} &= (d)\,\delta_{11}-(c)\,\delta_{12},\\
\alpha_{h_1d} &= (\tfrac14 d)\,\delta_1-(\tfrac14 c)\,\delta_2-2b,\\
\alpha_{h1r} &= -(\tfrac12 c)\,\delta_7+(\tfrac12 d)\,\delta_{10}-6a,\\
\alpha_{h_1i_1} &= (\tfrac14 ad)\,\delta_1-(\tfrac14 ac)\,\delta_2-(\tfrac18 c^2+\tfrac18 d^2)\,\delta_6-(\tfrac12 bc)\,\delta_7+(\tfrac14 d^2)\,\delta_8+(\tfrac14 cd)\,\delta_9+\\
&\quad+(\tfrac12 bd)\,\delta_{10}-4ab-2e,\\
\alpha_{h_1i_2} &= (\tfrac14 bd)\delta_1-(\tfrac14 bc)\,\delta_2-(\tfrac18 c^2+\tfrac18 d^2)\,\delta_4+(\tfrac12 ac)\,\delta_7+(\tfrac14 cd)\,\delta_8-(\tfrac14 d^2)\,\delta_9-\\
&\quad-(\tfrac12 ad)\,\delta_{10}+3a^2-b^2,\\
\alpha_{h_1j} &= (de)\,\delta_1-(ce)\,\delta_2-(\tfrac16 c^3+\tfrac16 cd^2)\,\delta_3+(\tfrac12 ac^2+\tfrac12 d^2a)\,\delta_4+(\tfrac12 d^3+\tfrac12 c^2d)\,\delta_5-\\
&\quad-(\tfrac12 d^2b+\tfrac12 bc^2)\,\delta_6-(a^2c+cb^2)\,\delta_7+(-dca+d^2b)\,\delta_8+(dcb+d^2a)\,\delta_9+\\
&\quad+(a^2d+db^2)\,\delta_{10}-8be-4a^3-4ab^2,\\
\alpha_{h_2h_1} &= (c)\,\delta_{13}+(d)\,\delta_{14},\\
\alpha_{h_2h_2} &= (c)\,\delta_{11}+(d)\,\delta_{12},\\
\alpha_{h_2d} &= (\tfrac14 c)\,\delta_1+(\tfrac14 d)\,\delta_2+2a,\\
\alpha_{h_2r} &= (\tfrac12 d)\,\delta_7+(\tfrac12 c)\,\delta_{10}-6b,\\
\alpha_{h_2i_1} &= (\tfrac14 ac)\,\delta_1+(\tfrac14 ad)\,\delta_2+(\tfrac18 c^2+\tfrac18 d^2)\,\delta_4+(\tfrac12 bd)\,\delta_7+(\tfrac14 cd)\,\delta_8+(\tfrac14 c^2)\,\delta_9+\\
&\quad+(\tfrac12 bc)\,\delta_{10}-3b^2+a^2,
\end{aligned}
$$

$$\alpha_{h_1 i_2} = (\tfrac{1}{4}cb)\,\delta_1 + (\tfrac{1}{4}bd)\,\delta_2 - (\tfrac{1}{8}c^2 + \tfrac{1}{8}d^2)\,\delta_6 - (\tfrac{1}{2}ad)\,\delta_7 + (\tfrac{1}{4}c^2)\,\delta_8 - (\tfrac{1}{4}dc)\,\delta_9 - \\ - (\tfrac{1}{2}ac)\,\delta_{10} + 4ab - 2e,$$

$$\alpha_{h_2 j} = (ce)\,\delta_1 + (ed)\,\delta_2 + (\tfrac{1}{6}d^3 + \tfrac{1}{6}c^2 d)\,\delta_3 + (\tfrac{1}{2}bc^2 + \tfrac{1}{2}d^2 b)\,\delta_4 + (\tfrac{1}{2}c^3 + \tfrac{1}{2}cd^2)\,\delta_5 + \\ + (\tfrac{1}{2}d^2 a + \tfrac{1}{2}ac^2)\,\delta_6 + (db^2 + da^2)\,\delta_7 + (-ac^2 + dcb)\,\delta_8 + (bc^2 + dca)\,\delta_9 + \\ + (ca^2 + cb^2)\,\delta_{10} - 4a^2 b + 8ae - 4b^3.$$

Furthermore, the desired Cartan connection — obtained later from appropriate determinations of the three horizontal vector fields $\widehat{H}_1$, $\widehat{H}_2$, $\widehat{T}$ — satisfies the condition $(\mathbf{c2})$ *if and only if all the coefficient functions $\alpha_{\bullet\bullet}$ are in the above form.*

Fortunately, the choice of the seven coefficients $\alpha_{tt}, \alpha_{th_1}, \alpha_{th_2}, \alpha_{h_1h_1}, \alpha_{h_1h_2}$, $\alpha_{h_2h_1}$ and $\alpha_{h_2h_2}$ can be determined by the geometry[2] of the graded tangent bundle $T^cM \oplus (TM/T^cM)$. Indeed, inspecting the expressions of the two fixed complex-tangent local vector fields $H_1, H_2 \in \Gamma(T^cM)$ spanning T^cM, shows that one must choose:

$$\alpha_{tt} = c^2 + d^2, \quad \alpha_{th_1} = bd - ac, \quad \alpha_{th_2} = -ad - bc, \quad \alpha_{h_1h_1} = c, \\ \alpha_{h_1h_2} = d, \quad \alpha_{h_2h_1} = -d, \quad \alpha_{h_2h_2} = c.$$

Now we are ready to start the computation of the curvature function κ. At first recall that, as a function whose values belong to the vector space $\mathscr{C}^2(\mathfrak{g}_-, \mathfrak{g})$, the curvature function κ splits up into homogeneous components of homogeneities zero to five. Let $\kappa_{q_j}^{p_{j_1} p_{j_2}}(p)$ be the real-valued function defined on an arbitrary point p of $\mathscr{G}$ as the coefficient of q_j in $\kappa(p)(\mathsf{p}_{j_1}, \mathsf{p}_{j_2})$, where $\mathsf{p}_{j_1} \in \mathfrak{g}_{j_1}, \mathsf{p}_{j_2} \in \mathfrak{g}_{j_2}$, $\mathsf{q}_j \in \mathfrak{g}_j$, for $j_1, j_2 = -2, -1$ and for $j = -2, -1, 0, 1, 2$. One checks that if $s = j - (j_1 + j_2)$, then the s-th homogeneous component of κ can be expressed as:

$$\kappa^{(s)} = \sum_{s = j - (j_1 + j_2)} \kappa_{q_j}^{p_{j_1} p_{j_2}} \, \mathsf{p}_{j_1}^* \wedge \mathsf{p}_{j_2}^* \otimes \mathsf{q}_j. \tag{24}$$

We call $\kappa_{q_j}^{p_{j_1} p_{j_2}}$ a *curvature coefficient of homogeneity s*. Let $\widehat{T}^*, \dots, \widehat{J}^*$ be the 1-forms dual to $\widehat{T}, \dots, \widehat{J}$, respectively (for later use, we denote by β_{xy} the coefficient of Y^* in the expression of $\widehat{X}^*$ for $X, Y = T, \dots, J$). Then, according to the definition of κ, any curvature coefficient $\kappa_{q_j}^{p_{j_1} p_{j_2}}$ is equal to:

$$\kappa_{q_j}^{p_{j_1} p_{j_2}} = \widehat{Q}_j^* \big([\omega^{-1}(\mathsf{p}_{j_1}), \omega^{-1}(\mathsf{p}_{j_2})] - \omega^{-1}([\mathsf{p}_{j_1}, \mathsf{p}_{j_2}]) \big) = \\ = \widehat{Q}_j^* \big([\widehat{P}_{j_1}, \widehat{P}_{j_2}] - \widehat{[\mathsf{p}_{j_1}, \mathsf{p}_{j_2}]_{\mathfrak{g}}} \big).$$

Clearly if we find some 24 explicit curvature coefficients $\kappa_{\bullet}^{\bullet\bullet}$ that are consistent with the three conditions **(c1)**, **(c2)**, **(c3)**, then the explicit expression of the desired curvature function κ will follow. Here for example, **(c3)** enforces us to determine the functions $\alpha_{\bullet\bullet}$ so that the curvature coefficients of all homogeneities $i = 0, 1, 2, 3$ do vanish, identically.

Now let us present briefly the computations by organizing them according to the homogeneity.

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[2] The authors would like to thank Ben McLaughlin and Gerd Schmalz for their helpful explanations.

7.2. **Homogeneity zero.** The only curvature coefficient of this homogeneity is $\kappa_t^{h_1h_2}$. According to the definition we have:

$$\kappa_t^{h_1h_2} = \widehat{T}^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) = \frac{1}{\alpha_{tt}}(4\alpha_{h_1h_1}\alpha_{h_2h_2}-4\alpha_{h_1h_2}\alpha_{h_2h_1})-4.$$

All of the five functions $\alpha_{\bullet\bullet}$ appearing here were determined before, and their expressions automatically annihilate $\kappa_t^{h_1h_2}$, hence nothing has to be done in this homogeneity!

7.3. **Homogeneity one.** In this homogeneity, we encounter the following — after some simplifications, *cf.* [2] — four curvature coefficients:

$$\kappa_{h_1}^{h_1h_2} = \widehat{H}_1^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) = \big((a_{h_1r}+a_{h_2d})c^2+(a_{h_1r}+a_{h_2d})d^2-4a_{th_1}c-4a_{th_2}d\big)/\big(c^2+d^2\big),$$

$$\kappa_{h_2}^{h_1h_2} = \widehat{H}_1^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) = \big((a_{h_2r}-a_{h_1d})(c^2+d^2)-4a_{th_2}c+4a_{th_1}d\big)/\big(c^2+d^2\big),$$

$$\begin{aligned}\kappa_t^{h_1t} &= \widehat{T}^*([\widehat{H}_1,\widehat{T}])\\ &= \big(\Phi_1c^3+\Phi_2d^3+\Phi_2c^2d+\Phi_1cd^2-2a_{h_1d}c^2-2a_{h_1d}d^2+4a_{th_2}c-4a_{th_1}d\big)/\big(c^2+d^2\big),\end{aligned}$$

$$\begin{aligned}\kappa_t^{h_2t} &= \widehat{T}^*([\widehat{H}_2,\widehat{T}])\\ &= \big(\Phi_2c^3-\Phi_1d^3-\Phi_1c^2d+\Phi_2cd^2-2a_{h_2d}c^2-2a_{h_2d}d^2-4a_{th_1}c-4a_{th_2}d\big)/\big(c^2+d^2\big).\end{aligned}$$

Annihilating the above curvature coefficients — imposed by **(c3)** — is equivalent to the following determinations:

$$\begin{aligned}\alpha_{h_1d} &= -2b+\tfrac{1}{2}\Phi_1c+\tfrac{1}{2}\Phi_2d, &\quad \alpha_{h_2d} &= 2a+\tfrac{1}{2}\Phi_2c-\tfrac{1}{2}\Phi_1d,\\ \alpha_{h_1r} &= -6a-\tfrac{1}{2}\Phi_2c+\tfrac{1}{2}\Phi_1d, &\quad \alpha_{h_2r} &= -6b+\tfrac{1}{2}\Phi_1c+\tfrac{1}{2}\Phi_2d.\end{aligned}$$

Furthermore, comparing the above expressions with their corresponding ones in Lemma 7.1 implies that:

$$\delta_1 = 2\Phi_2, \qquad \delta_2 = -2\Phi_1, \qquad \delta_7 = \Phi_2, \qquad \delta_{10} = \Phi_1.$$

7.4. **Homogeneity two.** Here, we handle the following curvature coefficients that should be annihilated again in order to satisfy the normality condition **(c3)**:

$$\begin{aligned}\kappa_d^{h_1h_2} &= 2\alpha_{h_2i_2}-4\alpha_{td}+2\alpha_{h_1i_1}+\tfrac{1}{2}+2\big(\Phi_1bd-\Phi_2bc-\Phi_1ac-\Phi_2ad\big),\\ \kappa_r^{h_1h_2} &= -6\alpha_{h_1i_2}+6\alpha_{h_2i_1}-4\alpha_{tr}+\tfrac{1}{2}\big(H_2(\Phi_2)+H_1(\Phi_1)\big)c^2+\tfrac{1}{2}\big(H_2(\Phi_2)+(H_1)\Phi_1\big)d^2+\\ &\quad +24a^2+24b^2+2\Phi_2ac-2\Phi_1bc-2\Phi_1ad-2\Phi_2bd,\\ \kappa_{h_1}^{h_1t} &= \alpha_{td}-\alpha_{h_1i_1}+\Phi_1ac+\Phi_2ad-4ab,\\ \kappa_{h_2}^{h_1t} &= \alpha_{tr}-\alpha_{h_1i_2}+\Phi_1bc+\Phi_2bd-4b^2,\\ \kappa_{h_1}^{h_2t} &= -\alpha_{tr}-\alpha_{h_2i_1}+\Phi_2ac-\Phi_1ad+4a^2,\\ \kappa_{h_2}^{h_2t} &= \alpha_{td}-\alpha_{h_2i_2}+\Phi_2bc-\Phi_1bd+4ab.\end{aligned}$$

By looking at the above six expressions, we can see also six undetermined functions α_{td}, α_{tr}, $\alpha_{h_1i_1}$, $\alpha_{h_1i_2}$,$\alpha_{h_2i_1}$, $\alpha_{h_2i_2}$ and then we can expect to make them zero by appropriately determining these functions. Solving only the first five equations yields the unique solution:

$$\begin{aligned}
\alpha_{td} &= \tfrac{1}{2}(bd-ac)\Phi_1 - \tfrac{1}{2}\Phi_2(bc+ad)\Phi_2 - 2e,\\
\alpha_{tr} &= \tfrac{1}{32}\big(H_1(\Phi_1)H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 - \tfrac{1}{2}(ad+bc)\Phi_1 + \tfrac{1}{2}(ac-bd)\Phi_2 + 3a^2 + 3b^2,\\
\alpha_{h_1i_1} &= \tfrac{1}{2}(bd+ac)\Phi_1 - \tfrac{1}{2}(bc-ad)\Phi_2 - 4ab - 2e,\\
\alpha_{h_1i_2} &= \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)c^2 + \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + 3a^2 - b^2,\\
\alpha_{h_2i_1} &= -\tfrac{1}{32}H_1\big((\Phi_1)+H_2(\Phi_2)\big)c^2 - \tfrac{1}{32}\big(H_1(\Phi_1)+H_2(\Phi_2)\big)d^2 + \tfrac{1}{2}(bc-ad)\Phi_1 + \tfrac{1}{2}(ac+bd)\Phi_2 + a^2 - 3b^2.
\end{aligned}$$

But how can we find expression of the remaining function $\alpha_{h_2i_2}$? Let us look again at Lemma 7.1. Comparison between the above expressions and what we obtained in this lemma immediately implies the following expressions:

$$\delta_6 = \delta_8 = \delta_9 = \delta_{15} = 0, \quad \delta_4 = -\tfrac{1}{4}H_1(\Phi_1) - \tfrac{1}{4}H_2(\Phi_2), \quad \delta_{19} = \frac{3}{16}H_1(\Phi_1) + \frac{3}{16}H_2(\Phi_2).$$

Now, putting these coefficients in the expression of $\alpha_{h_2i_2}$ in Lemma 7.1 gives its explicit expression as follows:

$$\alpha_{h_2i_2} = -\tfrac{1}{2}(ac+bd)\Phi_1 - \tfrac{1}{2}(ad-bc)\Phi_2 + 4ab - 2e.$$

7.5. **Homogeneity three.** The homogeneity-three component of κ is the last component that we can predict to be vanishing, according to Corollary 4.2 — obtained under the assumption that a Cartan connection is already constructed and that it is normal. Here we have the following six curvature coefficients (computations become harder, *cf.* [2] for details):

$$\begin{aligned}
\kappa_{i_1}^{h_1h_2} &= \alpha_{h_2j} - 4\alpha_{ti_1} + \tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1 - H_1(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1))\big]c^3 +\\
&+ \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]d^3 + 12a^2b - 2\Phi_2ce + 12b^3 -\\
&- 3\Phi_1a^2c - 3\Phi_1b^2c + 2\Phi_1de - 3\Phi_2a^2d - 3\Phi_2b^2d - 8ae + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 +\\
&+ H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^2d + \tfrac{1}{32}\big[H_1(\Phi_1)\Phi_1 -\\
&- H_1(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1))\big]cd^2 + \tfrac{3}{8}\big[H_1\Phi_1 + H_2(\Phi_2)\big]bc^2 + \tfrac{3}{8}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]bd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_{i_2}^{h_1h_2} &= -\alpha_{h_1j} - 4\alpha_{ti_2} + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^3 -\\
&- \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2 + \tfrac{1}{32}\big[H_1(H_1(\Phi_1)) - H_2(\Phi_2)\Phi_1 - H_1(\Phi_1)\Phi_1 + H_1(H_2(\Phi_2))\big]d^3 +\\
&+ 2\Phi_1ce - 8be - 3\Phi_2a^2c - 3\Phi_2b^2c + 2\Phi_2de + 3\Phi_1a^2d + 3\Phi_1b^2d - 12a^3 - 12ab^2 -\\
&- \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ac^2 + \tfrac{1}{32}\big[H_1(H_1(\Phi_1)) - H_2(\Phi_2)\Phi_1 - H_1(\Phi_1)\Phi_1 + H_1(H_2)\Phi_2\big]c^2d +\\
&+ \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 - H_2(H_2(\Phi_2)) - H_2(H_1(\Phi_1))\big]cd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_1t} &= 2\alpha_{ti_2} - \alpha_{h_1j} - \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^3 + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]d^3 +\\
&+ 2\Phi_1ce - 8be + 2\Phi_2de + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]c^2d - \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]cd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_d^{h_2t} &= -2\alpha_{ti_1} - \alpha_{h_2j} + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]c^3 + \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]d^3 +\\
&+ 2\Phi_2ce - 2\Phi_1de + 8ae + \tfrac{1}{8}\big[H_1(H_1(\Phi_2)) - H_2(H_1(\Phi_1))\big]c^2d + \tfrac{1}{8}\big[H_2(H_1(\Phi_2)) - H_1(H_2(\Phi_2))\big]cd^2,
\end{aligned}$$

$$\begin{aligned}
\kappa_r^{h_1t} &= 6\alpha_{ti_1} + \tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 - 4H_2(H_1(\Phi_2)) + 5H_1(H_2(\Phi_2)) + \tfrac{1}{16}H_1(H_1(\Phi_1))\big]c^3 -\\
&- \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bd^2 - \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) -\\
&- H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]c^2d + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) -\\
&- H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]d^3 - 12a^2b - 12b^3 + 3\Phi_1a^2c + 3\Phi_1cb^2 + 3\Phi_2a^2d + 3\Phi_2b^2d +\\
&+ \tfrac{1}{32}\big[-H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 - 4H_2(H_1(\Phi_2)) + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]d^2c
\end{aligned}$$

$$\begin{aligned}\kappa_r^{h_2t} &= 6\alpha_{ti_2} - \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_2 + 4H_1(H_1(\Phi_2)) - H_2(H_2(\Phi_2))\big]c^3 + \\ &+ \tfrac{1}{32}\big[-5H_1(H_2(\Phi_2)) + 4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_1\big]d^3 + 3ca^2\Phi_2 + \\ &+ 3cb^2\Phi_2 - 3a^2d\Phi_1 - 3b^2d\Phi_1 + 12a^3 + 12ab^2 + \tfrac{3}{8}\big[H_2\Phi_2 + H_1(\Phi_1)big]c^2a + \tfrac{1}{32}\big[-5H_1(H_2(\Phi_2)) + \\ &+ 4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) + H_1(\Phi_1)\Phi_1\big]c^2d + \tfrac{1}{32}\big[-H_2(\Phi_2)\Phi_2 + 5H_2(H_1(\Phi_1)) - \\ &- H_1(\Phi_1)\Phi_2 - 4H_1(H_1(\Phi_2)) + H_2(H_2(\Phi_2))\big]cd^2 + \tfrac{3}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2.\end{aligned}$$

One sees here four undetermined functions α_{ti_1}, α_{ti_2}, α_{h_1j}, α_{h_2j} within these six expressions. Although the number (six) of equations is greater than the number (four) of unknowns, we can annihilate all the six curvature coefficients by making the following appropriate determinations:

$$\begin{aligned}\alpha_{ti_1} &= \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]d^3 + \\ &+ \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]c^3 + \\ &+ \tfrac{1}{192}\big[4H_2(H_1(\Phi_2)) + H_2(\Phi_2)\Phi_1 - H_1(H_1(\Phi_1)) - 5H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big]cd^2 + \\ &+ \tfrac{1}{16}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + \\ &+ H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^2d + \tfrac{1}{16}\big[H_2(\Phi_2) + H_1\Phi_1\big]bd^2 + \\ &+ \tfrac{1}{2}\big[-\Phi_1a^2c + 4b^3 - \Phi_1b^2c + 4ba^2 - \Phi_2b^2d - \Phi_2a^2d\big],\end{aligned}$$

$$\begin{aligned}\alpha_{ti_2} &= \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]c^3 - \\ &- \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ac^2 - \tfrac{1}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big]ad^2 + \tfrac{1}{192}\big[-H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2 - \\ &- 5H_2(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_2 + 4H_1(H_1(\Phi_2))\big]cd^2 + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - \\ &- H_2(\Phi_2)\Phi_1 + 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]c^2d + \tfrac{1}{192}\big[-4H_2(H_1(\Phi_2)) - H_1(\Phi_1)\Phi_1 - H_2(\Phi_2)\Phi_1 + \\ &+ 5H_1(H_2(\Phi_2)) + H_1(H_1(\Phi_1))\big]d^3 - \tfrac{1}{2}\big[\Phi_2a^2c + \Phi_2b^2c - \Phi_1b^2d - \Phi_1a^2d - 4ab^2 + 4a^3\big],\end{aligned}$$

$$\begin{aligned}\alpha_{h_1j} &= \tfrac{1}{96}\big[-H_2(H_2(\Phi_2) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + 7H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]c^3 - \\ &+ \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^2d + \\ &+ \tfrac{1}{96}\big[-H_2(H_2(\Phi_2)) + H_2(\Phi_2)\Phi_2 + H_1(\Phi_1)\Phi_2 + \tfrac{7}{16}H_2(H_1(\Phi_1)) - 8H_1(H_1(\Phi_2))\big]cd^2 + \\ &+ \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]d^3 - \\ &- \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]ac^2 - \tfrac{1}{8}\big[\frac{1}{8}H_2(\Phi_2) + H_1(\Phi_1)\big]ad^2 - \Phi_2a^2c - \Phi_2b^2c + 2\Phi_1ce - \\ &- 8be + 2\Phi_2de + \Phi_1b^2d + \Phi_1a^2d - 4ab^2 - 4a^3,\end{aligned}$$

$$\begin{aligned}\alpha_{h_2j} &= \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]c^3 + \\ &- \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + 8H_1(H_1(\Phi_2)) - \\ &- 7H_2(H_1(\Phi_1))\big]d^3 - \tfrac{1}{8}\big[H_2(\Phi_2) + H_1(\Phi_1)\big]bc^2 + \tfrac{1}{96}\big[-H_1(\Phi_1)\Phi_1 + 8H_2(H_1(\Phi_2)) - \\ &- 7H_1(H_2(\Phi_2)) - H_2(\Phi_2)\Phi_1 + H_1(H_1(\Phi_1))\big]cd^2 + \tfrac{1}{96}\big[-H_2(\Phi_2)\Phi_2 - H_1(\Phi_1)\Phi_2 + H_2(H_2(\Phi_2)) + \\ &+ 8H_1(H_1(\Phi_2)) - 7H_2(H_1(\Phi_1))\big]c^2d + \Phi_1a^2c - 2\Phi_1de + \Phi_2b^2d + \Phi_2a^2d + 8ae4b^3 + \Phi_1b^2c - \\ &- 4a^2b + 2\Phi_2ce.\end{aligned}$$

Similarly to what was done for lower homogeneities, we can again compare the expressions of the functions $\alpha_{\bullet\bullet}$ just obtained with the results of Lemma 7.1. This comparison immediately gives:

$$\begin{aligned}\delta_3 &= \tfrac{1}{2}H_1(H_1(\Phi_2)) - \tfrac{1}{16}\Phi_2H_2(\Phi_2) - \tfrac{7}{16}H_2(H_1(\Phi_1)) + \tfrac{1}{16}H_2(H_2(\Phi_2)) - \tfrac{1}{16}\Phi_2H_1(\Phi_1), \\ \delta_5 &= -\tfrac{1}{48}\Phi_1H_2(\Phi_2) - \tfrac{7}{48}H_1(H_2(\Phi_2)) + \tfrac{1}{48}H_1(H_1(\Phi_1)) - \tfrac{1}{48}\Phi_1H_1(\Phi_1) + \tfrac{1}{6}H_2(H_1(\Phi_2)), \\ \delta_{16} &= \tfrac{1}{32}\Phi_1H_1(\Phi_1) - \tfrac{1}{32}H_1(H_1(\Phi_1)) + \tfrac{1}{32}\Phi_1H_2(\Phi_2) - \tfrac{1}{32}H_1(H_2(\Phi_2)), \\ \delta_{17} &= \tfrac{1}{32}\Phi_2H_1(\Phi_1) - \tfrac{1}{32}H_2(H_2(\Phi_2)) + \tfrac{1}{32}\Phi_2H_2(\Phi_2) - \tfrac{1}{32}H_2(H_1(\Phi_1)).\end{aligned}$$

7.6. **Homogeneity four.** Until now, all functions $\alpha_{\bullet\bullet}$ have been determined except only one, namely α_{tj}, which is undetermined yet. In this homogeneity, we encounter five curvature coefficients $\kappa_j^{h_1h_2}$, $\kappa_{i_1}^{h_1t}$, $\kappa_{i_2}^{h_1t}$, $\kappa_{i_1}^{h_2t}$, $\kappa_{i_2}^{h_2t}$, and as one would expect, it is possible to make zero one of these curvature coefficients by an appropriate determination of α_{tj}. Let us pick $\kappa_j^{h_1h_2}$ with the following expression for this purpose:

$$
\begin{aligned}
\kappa_j^{h_1h_2} &= \widehat{J}^*([\widehat{H}_1,\widehat{H}_2]-4\widehat{T}) = -\widehat{H}_2(\alpha_{h_1j})+\alpha_{h_1h_2}H_2(\alpha_{h_2j})+\alpha_{h_1h_1}H_1(\alpha_{h_2j})+\\
&\quad+\beta_{jh_1}\big(\alpha_{h_2d}\alpha_{h_1h_1}+\alpha_{h_2r}\alpha_{h_2h_1}\big)+\beta_{jh_2}\big(\alpha_{h_2d}\alpha_{h_1h_2}+\alpha_{h_2r}\alpha_{h_2h_2}\big)+\\
&\quad+\beta_{jt}\big(4\alpha_{h_1h_1}\alpha_{h_2h_2}-4\alpha_{h_1h_2}\alpha_{h_2h_1}\big).
\end{aligned}
$$

Computations show that this curvature coefficient vanishes if and only if:

$$
\begin{aligned}
\alpha_{tj} &= 3a^4+3b^4-4e^2-\Phi_1a^2bc+ca\Phi_2b^2-\Phi_1ab^2d-\Phi_2a^2bd-2\Phi_2bce-2\Phi_1ace-2\Phi_2ade+2\Phi_1bde-\\
&-\Phi_1a^3d+\Phi_2a^3c-\Phi_1b^3c-\Phi_2b^3d+6a^2b^2+\big[\tfrac{3}{16}H_1(\Phi_1)+\tfrac{3}{16}H_2(\Phi_2)\big]b^2d^2+\\
&+\big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1-\tfrac{11}{3072}H_2({\Phi_2}^2)+\tfrac{1}{384}{\Phi_2}^2H_2(\Phi_2)-\tfrac{11}{3072}H_1(\Phi_1^2)+\\
&+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))+\tfrac{1}{384}{\Phi_2}^2H_1(\Phi_1)-\\
&-\tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{384}{\Phi_1}^2H_2(\Phi_2)-\\
&-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]d^4+\\
&+\big[-\tfrac{11}{768}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{7}{192}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{192}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{192}H_1(H_1(H_1(\Phi_1)))+\\
&+\tfrac{1}{24}H_1(H_2(H_1(\Phi_2)))-\tfrac{1}{96}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{32}H_1(H_2(\Phi_2))\Phi_1+\tfrac{1}{192}\Phi_2^2H_1(\Phi_1)-\tfrac{7}{192}H_1(H_1(H_2(\Phi_2)))+\\
&+\tfrac{1}{192}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{1536}H_1(\Phi_1^2)-\tfrac{1}{24}\Phi_2H_1(H_1(\Phi_2))-\tfrac{11}{1536}H_2(\Phi_2^2)+\tfrac{1}{32}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{96}H_1(H_1(\Phi_1))\Phi_1+\\
&+\tfrac{1}{192}\Phi_1^2H_2(\Phi_2)+\tfrac{1}{192}\Phi_1^2H_1(\Phi_1)-\tfrac{1}{24}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{24}H_2(H_1(H_1(\Phi_2)))\big]c^2d^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\\
&+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bcd^2+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\\
&-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]acd^2+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\\
&+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]ad^3+\big[\tfrac{1}{32}H_2(H_1(\Phi_1))+\tfrac{1}{32}H_2(H_2(\Phi_2))-\tfrac{1}{32}H_2(\Phi_2)\Phi_2-\tfrac{1}{32}H_1(\Phi_1)\Phi_2\big]ac^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2d^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]bd^3+\\
&+\big[-\tfrac{1}{32}H_1(H_1(\Phi_1))+\tfrac{1}{32}H_2(\Phi_2)\Phi_1-\tfrac{1}{32}H_1(H_2(\Phi_2))+\tfrac{1}{32}H_1(\Phi_1)\Phi_1\big]bc^3+\\
&+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]a^2c^2+\tfrac{3}{16}\big[H_1(\Phi_1)+H_2(\Phi_2)\big]b^2c^2+\tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2-H_2(H_1(\Phi_1))-\\
&-H_2(H_2(\Phi_2))+H_1(\Phi_1)\Phi_2\big]dbc^2+\tfrac{1}{32}\big[-H_1(H_1(\Phi_1))+H_2(\Phi_2)\Phi_1-H_1(H_2(\Phi_2))+H_1(\Phi_1)\Phi_1\big]ac^2d+\\
&+\big[-\tfrac{11}{1536}H_2(\Phi_2)H_1(\Phi_1)-\tfrac{1}{192}H_1(H_1(\Phi_1))\Phi_1-\tfrac{11}{3072}H_2(\Phi_2^2)+\tfrac{1}{384}\Phi_2^2H_2(\Phi_2)-\tfrac{11}{3072}H_1(\Phi_1^2)+\\
&+\tfrac{1}{384}\Phi_1^2H_1(\Phi_1)+\tfrac{1}{48}H_1(H_2(H_1(\Phi_2)))+\tfrac{1}{384}H_2(H_2(H_2(\Phi_2)))+\tfrac{1}{384}H_1(H_1(H_1(\Phi_1)))+\tfrac{1}{384}\Phi_2^2H_1(\Phi_1)-\\
&-\tfrac{1}{192}H_2(H_2(\Phi_2))\Phi_2+\tfrac{1}{48}H_2(H_1(H_1(\Phi_2)))+\tfrac{1}{64}H_2(H_1(\Phi_1))\Phi_2-\tfrac{1}{48}\Phi_1H_2(H_1(\Phi_2))+\tfrac{1}{384}\Phi_1^2H_2(\Phi_2)-\\
&-\tfrac{7}{384}H_2(H_2(H_1(\Phi_1)))+\tfrac{1}{64}H_1(H_2(\Phi_2))\Phi_1-\tfrac{7}{384}H_1(H_1(H_2(\Phi_2)))-\tfrac{1}{48}\Phi_2H_1(H_1(\Phi_2))\big]c^4.
\end{aligned}
$$

However, this choice does not annihilate the remaining four curvature coefficients. By a careful examination (either by hand or with the help of a computer), we realize that the remaining four curvature coefficients enjoy the following forms:

$$
\begin{aligned}
\kappa_{i_1}^{h_1t} &= \mathbf{\Delta_1}d^4+(\mathbf{\Delta_2}-\mathbf{\Delta_1})c^4+\mathbf{\Delta_2}c^2d^2+\mathbf{\Delta_3}c^3d+\mathbf{\Delta_3}cd^3,\\
\kappa_{i_2}^{h_1t} &= \mathbf{\Delta_4}d^4+(\mathbf{\Delta_3}+\mathbf{\Delta_4})c^4+(\mathbf{\Delta_3}+2\mathbf{\Delta_4})c^2d^2+(2\mathbf{\Delta_1}-\mathbf{\Delta_2})c^3d+(2\mathbf{\Delta_1}-\mathbf{\Delta_2})cd^3,
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{i_1}^{h_2t} &= -\kappa_{i_2}^{h_1t}-\widehat{R}(\kappa_{i_1}^{h_1t}),\\
\kappa_{i_2}^{h_2t} &= \kappa_{i_1}^{h_1t}-\widehat{R}(\kappa_{i_2}^{h_1t}),
\end{aligned}
$$

where:

$$\begin{aligned}\mathbf{\Delta_1} := \tfrac{1}{384}\big[& -20\Phi_2 H_1(H_1(\Phi_2)) - (H_1(\Phi_1))^2 - 2\Phi_2^2 H_1(\Phi_1) + 8H_1(H_2(H_1(\Phi_2))) + 2\Phi_1^2 H_1(\Phi_1) - \\ & - 7H_1(H_1(H_2(\Phi_2))) - 4\Phi_1 H_2(H_1(\Phi_2)) + H_1(H_1(H_1(\Phi_1))) + \Phi_1 H_1(H_2(\Phi_2)) + \\ & + 23\Phi_2 H_2(H_1(\Phi_1)) + (H_2(\Phi_2))^2 - 3\Phi_1 H_1(H_1(\Phi_1)) + 3\Phi_2 H_2(H_2(\Phi_2)) - 2\Phi_2^2 H_2(\Phi_2) - \\ & - 17H_2(H_2(H_1(\Phi_1))) + 2\Phi_1^2 H_2(\Phi_2) + 16H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_2(\Phi_2)))\big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_2} := \tfrac{1}{384}\big[& 24H_1(H_2(H_1(\Phi_2))) - 24\Phi_1 H_2(H_1(\Phi_2)) + 24\Phi_1 H_1(H_2(\Phi_2)) + 24H_2(H_1(H_1(\Phi_2))) - \\ & - 24H_1(H_1(H_2(\Phi_2))) - 24\Phi_2 H_1(H_1(\Phi_2)) + 24\Phi_2 H_2(H_1(\Phi_1)) - 24H_2(H_2(H_1(\Phi_1)))\big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_3} := \tfrac{1}{384}\big[& -2H_2(H_1(H_1(\Phi_1))) + 8H_1(H_1(H_1(\Phi_2))) - 2\Phi_1\Phi_2 H_1(\Phi_1) - 8\Phi_1\Phi_2 H_2(\Phi_2) - \\ & - 2H_1(H_2(H_2(\Phi_2))) - 10H_2(H_1(H_2(\Phi_2))) - 16\Phi_1 H_1(H_1(\Phi_2)) + 8H_1(\Phi_2)H_2(\Phi_2) + \\ & + 6\Phi_1 H_2(H_2(\Phi_2)) + 8H_2(H_2(H_1(\Phi_2))) + 22\Phi_2 H_1(H_2(\Phi_2)) - 16\Phi_2 H_2(H_1(\Phi_2)) + \\ & + 22\Phi_1 H_2(H_1(\Phi_1)) - 10H_1(H_2(H_1(\Phi_1))) + 4H_1(\Phi_1)H_1(\Phi_2) + 6\Phi_2 H_1(H_1(\Phi_1))\big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_4} := \tfrac{1}{384}\big[& 4\Phi_1 H_1(H_1(\Phi_2)) - 2H_1(\Phi_2)H_2(\Phi_2) - 2H_1(\Phi_1)H_1(\Phi_2) + 13H_2(H_1(H_2(\Phi_2))) - \\ & - 3H_1(H_2(H_2(\Phi_2))) - 3\Phi_2 H_1(H_1(\Phi_1)) - 15\Phi_2 H_1(H_2(\Phi_2)) + 4\Phi_1\Phi_2 H_1(\Phi_1) - \\ & - 8H_2(H_2(H_1(\Phi_2))) - 3H_1(H_2(H_1(\Phi_1))) + 12\Phi_2 H_2(H_1(\Phi_2)) - 3\Phi_1 H_2(H_2(\Phi_2)) - \\ & - 7\Phi_1 H_2(H_1(\Phi_1)) + 4\Phi_1\Phi_2 H_2(\Phi_2) + 5H_2(H_1(H_1(\Phi_1)))\big].\end{aligned}$$

The good news is that we can even express the above curvatures by means of just $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$. Indeed we have:

Lemma 7.2. *One in fact has, identically as functions of* (x, y, u)*:*

$$\boxed{0 \equiv \mathbf{\Delta_2}} \qquad \text{and} \qquad \boxed{0 \equiv \mathbf{\Delta_3} + 2\,\mathbf{\Delta_4}}.$$

Proof. These two nontrivial relations were already prepared in advance, *cf.* the Corollary 6.2. □

Furthermore, by taking account of the relations listed in Proposition 6.1, one sees that the expressions of the two remaining functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$ of (x, y, u) can be given better, completely symmetric forms, as is stated by the following summarizing proposition.

Proposition 7.3. *The four remaining curvature coefficients of homogeneity* $i = 4$ *express explicitly as follows:*

$$\begin{aligned}\kappa_{i_1}^{h_1 t} &= -\,\mathbf{\Delta_1}\,c^4 - 2\,\mathbf{\Delta_4}\,c^3 d - 2\,\mathbf{\Delta_4}\,cd^3 + \mathbf{\Delta_1}\,d^4, \\ \kappa_{i_2}^{h_1 t} &= -\,\mathbf{\Delta_4}\,c^4 + 2\,\mathbf{\Delta_1}\,c^3 d + 2\,\mathbf{\Delta_1}\,cd^3 + \mathbf{\Delta_4}\,d^4, \\ \kappa_{i_1}^{h_2 t} &= \kappa_{i_2}^{h_1 t}, \\ \kappa_{i_2}^{h_2 t} &= -\,\kappa_{i_1}^{h_1 t},\end{aligned}$$

where the two functions $\mathbf{\Delta_1}$ *and* $\mathbf{\Delta_4}$ *of the three horizontal variables* (x, y, u) *have the following explicit expressions:*

$$\begin{aligned}
\mathbf{\Delta_1} = \tfrac{1}{384}\Big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\, H_1(H_2(H_1(\Phi_2))) - 11\, H_2(H_1(H_2(\Phi_1)))+ \\
& + 6\,\Phi_2\, H_2(H_1(\Phi_1)) - 6\,\Phi_1\, H_1(H_2(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_2)) + 3\,\Phi_1\, H_2(H_2(\Phi_1)) - \\
& - 3\,\Phi_1\, H_1(H_1(\Phi_1)) + 3\,\Phi_2\, H_2(H_2(\Phi_2)) - H_1(\Phi_1)\, H_1(\Phi_1) + H_2(\Phi_2)\, H_2(\Phi_2) - \\
& - 2\,(\Phi_2)^2\, H_1(\Phi_1) + 2\,(\Phi_1)^2\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_2(\Phi_2) + 2\,(\Phi_1)^2\, H_1(\Phi_1)\Big], \\
\mathbf{\Delta_4} = \tfrac{1}{384}\Big[& - 3\, H_2(H_1(H_2(\Phi_2))) - 3\, H_1(H_2(H_1(\Phi_1))) + 5\, H_1(H_2(H_2(\Phi_2))) + 5\, H_2(H_1(H_1(\Phi_1)))+ \\
& + 4\,\Phi_1\, H_1(H_1(\Phi_2)) + 4\,\Phi_2\, H_2(H_1(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_1)) - 3\,\Phi_1\, H_2(H_2(\Phi_2)) - \\
& - 7\,\Phi_2\, H_1(H_2(\Phi_2)) - 7\,\Phi_1\, H_2(H_1(\Phi_1)) - 2\, H_1(\Phi_1)\, H_1(\Phi_2) - 2\, H_2(\Phi_2)\, H_2(\Phi_1)+ \\
& + 4\,\Phi_1\Phi_2\, H_1(\Phi_1) + 4\,\Phi_1\Phi_2\, H_2(\Phi_2)\Big].
\end{aligned}$$

Proof. As said, one uses the relations listed in Proposition 6.1 until formal expressions show up symmetries. □

Although all the functions $\alpha_{\bullet\bullet}$ are determined at this stage, still there is one function of the type $\delta_\bullet$, namely δ_{18}, which is undetermined yet. Similarly to what was done in previous subsections, it is sufficient to have a comparison between the expressions of α_{tj}, mentioned just above and mentioned correspondingly in Lemma 7.1. This comparison yields the explicit expression of δ_{18} as follows:

$$\begin{aligned}
\delta_{18} = & \tfrac{1}{64}\Phi_2 H_2(H_1(\Phi_1)) - \tfrac{11}{1536} H_2(\Phi_2) H_1(\Phi_1) - \tfrac{1}{192}\Phi_1 H_1(H_1(\Phi_1)) - \tfrac{11}{3072} H_1(\Phi_1^2) + \tfrac{1}{48} H_1(H_2(H_1(\Phi_2)))- \\
& - \tfrac{7}{384} H_1(H_1(H_2(\Phi_2))) + \tfrac{1}{384}\Phi_2^2 H_2(\Phi_2) - \tfrac{1}{48}\Phi_2 H_1(H_1(\Phi_2)) - \tfrac{1}{192}\Phi_2 H_2(H_2(\Phi_2)) + \tfrac{1}{64}\Phi_1 H_1(H_2(\Phi_2))+ \\
& + \tfrac{1}{384}\Phi_1^2 H_1(\Phi_1) - \tfrac{11}{3072} H_2(\Phi_2^2) + \tfrac{1}{384}\Phi_2^2 H_1(\Phi_1) - \tfrac{1}{48}\Phi_1 H_2(H_1(\Phi_2)) + \tfrac{1}{48} H_2(H_1(H_1(\Phi_2)))+ \\
& + \tfrac{1}{384}\Phi_1^2 H_2(\Phi_2) + \tfrac{1}{384} H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{384} H_1(H_1(H_1(\Phi_1))) - \tfrac{7}{384} H_2(H_2(H_1(\Phi_1))).
\end{aligned}$$

7.7. **Homogeneity five.** In this homogeneity, there are two curvature coefficients $\kappa_j^{h_1 t}$ and $\kappa_j^{h_2 t}$. Fortunately here, the graded Bianchi identity (19) enables us to express these coefficients in terms of the obtained curvature coefficients of lower homogeneities.

We know that the fifth-homogeneous component $\kappa^{(5)}$ of the curvature function κ can be expressed under the following form:

$$\kappa^{(5)} = \kappa_j^{h_1 t}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_j^{h_2 t}\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j}. \tag{25}$$

Then, using just the direct definition of the differential operator ∂ gives:

$$\partial\kappa^{(5)}(\mathsf{h}_1, \mathsf{h}_2, \mathsf{t}) = \kappa_j^{h_2 t}\, \mathsf{i}_1 - \kappa_j^{h_1 t}\, \mathsf{i}_2, \tag{26}$$

while, on the other hand, the graded Bianchi identity (19) asserts that:

$$\begin{aligned}
\partial\kappa^{(5)}(\mathsf{h}_1, \mathsf{h}_2, \mathsf{t}) = & - \sum_{j=1}^{4} \kappa^{(5-j)}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa^{(j)}(\mathsf{h}_1, \mathsf{h}_2)\big), \mathsf{t}\Big) - \big(\widehat{T}\underline{\kappa^{(3)}}_{\circ}\big)(\mathsf{h}_1, \mathsf{h}_2) - \\
& - \sum_{j=1}^{4} \kappa^{(5-j)}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa^{(j)}(\mathsf{t}, \mathsf{h}_1)\big), \mathsf{h}_2\Big) - \big(\widehat{H}_2\kappa^{(4)}\big)(\mathsf{t}, \mathsf{h}_1) - \\
& - \sum_{j=1}^{4} \kappa^{(5-j)}\Big(\mathsf{proj}_{\mathfrak{g}_-}\big(\kappa^{(j)}(\mathsf{h}_2, \mathsf{t})\big), \mathsf{h}_1\Big) - \big(\widehat{H}_1\kappa^{(4)}\big)(\mathsf{h}_2, \mathsf{t}).
\end{aligned}$$

By the last equation, we can compute the value of $\partial\kappa^{(5)}$ on $(\mathsf{h}_1, \mathsf{h}_2, \mathsf{t})$ using the values of the lower homogeneous components $\partial\kappa^{(i)}$, $i = 1, 2, 3, 4$, on this triple. This equation reads as follows after the computation:

(27)
$$\partial\kappa^{(5)}(\mathsf{h}_1, \mathsf{h}_2, \mathsf{t}) = \big(\widehat{H}_2\big(\kappa_{i_1}^{h_1 t}\big) - \widehat{H}_1\big(\kappa_{i_1}^{h_2 t}\big)\big)\,\mathsf{i}_1 + \big(\widehat{H}_2\big(\kappa_{i_2}^{h_1 t}\big) - \widehat{H}_1\big(\kappa_{i_2}^{h_2 t}\big)\big)\,\mathsf{i}_2.$$

Now comparison of (26) and (27) immediately gives the explicit expressions of $\kappa_j^{h_1 t}$ and $\kappa_j^{h_2 t}$, namely we have:

$$\kappa_j^{h_1 t} = \widehat{H}_1\big(\kappa_{i_2}^{h_2 t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1 t}\big),$$
$$\kappa_j^{h_2 t} = -\widehat{H}_1\big(\kappa_{i_1}^{h_2 t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1 t}\big).$$

7.8. **Conclusion.** A review of the results obtained so far shows that the only non-zero curvature coefficients are:

$$\boxed{\text{Hom 4}} \quad \kappa_{i_1}^{h_1 t}, \quad \kappa_{i_2}^{h_1 t}, \quad \kappa_{i_1}^{h_2 t}, \quad \kappa_{i_2}^{h_2 t};$$
$$\boxed{\text{Hom 5}} \quad \kappa_{j}^{h_1 t}, \quad \kappa_{j}^{h_2 t}.$$

All these curvature coefficients can be expressed as certain combinations of $\kappa_{i_1}^{h_1 t}$ and $\kappa_{i_2}^{h_1 t}$ and of the values of the constant vector fields on them. These two curvature coefficients are called *essential curvatures*. Recall that the following are known to be equivalent statements about a Cartan geometry ([24]):

- **(i)** the curvature vanishes;
- **(ii)** the essential curvatures vanish (*see* [13] for more information).
- **(iii)** the geometry is locally isomorphic to the model geometry;
- **(iv)** the pseudogroup of local isomorphisms acts locally transitively on the total space of the bundle; moreover, the curvature of a parabolic geometry is constant if and only if it is zero (*see* [9] for a proof).

Hence a consequence of our results is the following:

Theorem 7.1. *The Cartan geometry associated to any $\mathscr{C}^6$-smooth Levi nondegenerate deformation $M^3 \subset \mathbb{C}^2$ of the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$ having curvature function:*

(28)
$$\begin{aligned}\kappa &= \kappa^{(4)} + \kappa^{(5)} = \\ &= \kappa_{i_1}^{h_1 t}\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1 t}\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2 t}\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \\ &\quad + \kappa_{i_2}^{h_2 t}\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1 t}\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t}\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},\end{aligned}$$

is equivalent to that of its model $\mathbb{H}^3$ if and only if its two essential curvatures $\kappa_{i_1}^{h_1 t}$ and $\kappa_{i_2}^{h_1 t}$ vanish identically; equivalently, the two explicit functions $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$ of only the three horizontal variables (x, y, u) vanish identically.

Up to now, since the expressions of the 22 functions $\alpha_{\bullet\bullet}$ are in the form of those presented in Lemma 7.1, we are sure that condition **(c2)** is fulfilled. Now, the next two propositions ensure us that the remaining two conditions **(c1)** and **(c3)** are also fulfilled.

Proposition 7.4. *For any element $p = (a, b, c, d, e, x, y, u)$ of $\mathscr{G}$, the $\mathfrak{g}$-valued 1-form $\omega_p \colon T_p\mathscr{G} \longrightarrow \mathfrak{g}$ dual to the constructed vector fields $\widehat{T}, \dots, \widehat{J}$ is an isomorphism.*

Proof. According to the expressions of the eight fields $\widehat{T}, \ldots, \widehat{J}$, the corresponding matrix of ω_p^{-1} is:

$$\begin{pmatrix} \alpha_{tt} & \alpha_{th_1} & \alpha_{th_2} & \alpha_{td} & \alpha_{tr} & \alpha_{ti_1} & \alpha_{ti_2} & \alpha_{tj} \\ 0 & \alpha_{h_1h_1} & \alpha_{h_1h_2} & \alpha_{h_1d} & \alpha_{h_1r} & \alpha_{h_1i_1} & \alpha_{h_1i_2} & \alpha_{h_1j} \\ 0 & \alpha_{h_2h_1} & \alpha_{h_2h_2} & \alpha_{h_2d} & \alpha_{h_2r} & \alpha_{h_2i_1} & \alpha_{h_2i_2} & \alpha_{h_2j} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and it visibly has as determinant:

$$\alpha_{tt}\big(\alpha_{h_1h_1}\alpha_{h_2h_2} - \alpha_{h_2h_1}\alpha_{h_1h_2}\big) = (c^2 + d^2)^2.$$

This expression is nonzero by our assumption that (a, b, c, d, e) lie near the neutral element $(a_0, b_0, c_0, d_0, e_0) = (0, 0, 1, 1, 0)$ of H, and hence, the linear map ω_p is an isomorphism. □

Proposition 7.5. *The Cartan connection constructed in the preceding paragraphs in a completely effective way is normal.*

Proof. According to (28), the t-, h_1-, h_2-, d- and r-components of the Cartan curvature κ vanish together. Vanishing of its t, h_1 and h_2-components means that this curvature is torsion free. Moreover, the d- and r-components of κ constitute its $\mathfrak{g}_0$-component and consequently $\kappa^{(0)} \equiv 0$ by construction. Therefore the Cartan connection is normal according to Definition 1.6.7 page 128 of [9]. □

8. Application to ellipsoids

Webster ([27]) emphasized that in higher-dimensional $\mathbb{C}^{n+1}$ with $n \geqslant 2$, the Hachtroudi-Chern curvature invariants of Levi nondegenerate real hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ have never been computed (except in the rigid case [15], while the general case is, to our knowledge, still open), but was nevertheless able to show that an ellipsoidal hypersurface:

$$1 = \sum_{k=1}^{n} \Big[z_k\overline{z}_k + A_j \operatorname{Re}\big(z_k z_k\big)\Big],$$

with $0 < A_1 < \cdots < A_n < \frac{1}{2}$ has no umbilic point, *i.e.* no point at which all curvatures vanish. But for an ellipsoid in $\mathbb{C}^2$:

$$1 = z'\overline{z}' + w'\overline{w}' + A \operatorname{Re}\big(z'z'\big) + B \operatorname{Re}\big(w'w'\big)$$

with $0 < A, B < 1$, it is only recently that Ezhov, McLaughlin and Schmalz showed that none of its vertices is umbilical. The explicit formulas provided here for the two main curvature invariants Δ_1 and Δ_4 confirm this result, as follows.

Naturally, by setting $u + iv := \sqrt{1-B}\big(u' + iv'\big)$, we come to a quadric of the plain form:

$$1 = a\,x^2 + b\,y^2 + c\,u^2 + v^2 \qquad \text{with } a = 1 + A, \quad b = 1 - A, \quad c = \frac{1+B}{1-B},$$

the vertex under consideration being $(0 + i0, 0 + i1)$. Applying then the above formulas to the obvious graphed form:

$$v = \sqrt{1 - a\,x^2 - b\,y^2 - c\,u^2},$$

we obtain after days of computed guiding that the second fundamental invariant $\Delta_4 = 0$ vanishes at this vertex (similarly as in [13]), whereas the first one:

$$\begin{aligned}\Delta_1 &= \frac{1}{384}\bigg[-\frac{3(c+1)(a-b)(7a^2c-3a^2+2abc-10ab-3b^2+7cb^2)}{a+b}\bigg]\\ &= -\frac{1}{384}\bigg[\frac{48\,A\,(2\,A^2+4\,B+A^2\,B)}{(1-B)^2}\bigg],\end{aligned}$$

clearly is nonzero.

We tried also to determine whether there exist umbilic points $(x_0+iy_0, u_0+i\,v_0)$ on the quadric, namely with $1 = a\,x_0^2+b\,y_0^2+c\,u_0^2+v_0^2$, but the numerators of both Δ_1 and Δ_4, which we were able to fully compute, contain $\sim 40\,000$ monomials not all of the same sign in $\mathbb{Z}\big[x_0, y_0, u_0, a, b, c\big]$, even after replacing $a = 1 + A$, $b = 1 - A$, $c = \frac{1+B}{1-B}$. We refer to [4] for publicly available cleaned up Maple worksheets.

9. Effectiveness

Lastly, for a general graphed $M^3 \subset \mathbb{C}^2$ represented by $v = \varphi(x, y, u)$ and in the notation of Proposition 1.5, the two numerators:

$$\begin{aligned}&A_{1111} - A_{222} + 11\,A_{2121} - 11\,A_{1212} +\\ &+ 6\,A_2\,A_{121} - 6\,A_1\,A_{212} - 3\,A_2\,A_{211} + 3\,A_1\,A_{122} - 3\,A_1\,A_{111} + 3\,A_2\,A_{222} -\\ &-\,A_{11}\,A_{11} + A_{22}\,A_{22} - 2\,A_2\,A_2\,A_{11} + 2\,A_1\,A_1\,A_{22} - 2\,A_2\,A_2\,A_{22} + 2\,A_1\,A_1\,A_{11}\end{aligned}$$

of Δ_1 and:

$$\begin{aligned}&-\,3\,A_{2212} - 3\,A_{1121} + 5\,A_{2122} + 5\,A_{1211} +\\ &+ 4\,A_1A_{211} + 4\,A_2\,A_{221} - 3\,A_2\,A_{111} - 3\,A_1\,A_{222} - 7\,A_2\,A_{212} - 7\,A_1\,A_{121} -\\ &-\,2\,A_{11}\,A_{21} - 2\,A_{22}\,A_{12} + 4\,A_1\,A_2\,A_{11} + 4\,A_1\,A_2\,A_{22},\end{aligned}$$

of Δ_4 incorporate, respectively:

1 553 198 and **1 634 457**

monomials in the differential ring in $\binom{6+3}{3} - 1 = 83$ variables:

$$\mathbb{Z}\big[\varphi_x, \varphi_y, \varphi_{x^2}, \varphi_{y^2}, \varphi_{u^2}, \varphi_{xy}, \varphi_{xu}, \varphi_{yu}, \ldots\ldots, \varphi_{x^6}, \varphi_{y^6}, \varphi_{u^6}, \ldots\big],$$

while the full expansion of the common numerator $\Delta^8\,\Upsilon^4$ of Δ_1 and Δ_4 contains **2 792** such monomials.

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