ON THE LOCAL GEOMETRY OF GENERIC SUBMANIFOLDS OF $\mathbb{C}^n$ AND THE ANALYTIC REFLECTION PRINCIPLE (PART I)

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Chapter 1: General introduction for Part I

§1.1. Resumed historical background

1.1.1. Local Lie groups and the no Riemann mapping theorem at the boundary. Inspired by the general idea that, in analogy with É. Galois's group theory of algebraic equations, group analysis of differential equations should provide precious information about their solvability, S. Lie began around 1873–80 the classification of all continuous local groups of transformation acting on $\mathbb{C}^n$. He quickly succeeded for $n = 1$ and achieved the case $n = 2$ (*see* [18]), but the unavoidable complexity and richness for $n = 3$ exhausted his efforts; moreover, after more than one century, the task has never been achieved. Nevertheless, especially for $n = 2$, Lie's classificationf had the enormous power of providing any possible application to the study of transformations preserving arbitrary types of geometric structures. Thanks to the influence of G. Darboux, the works of S. Lie were rapidly known to French mathematicians. Based on the general approach of S. Lie, H. Poincaré (*see* [24]) discovered in 1907 that the automorphism groups of the two-dimensional unit ball $\mathbb{B}^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ and of the bidisc $\Delta^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ are represented by rational, but not isomorphic transformations and he deduced immediately that $\mathbb{B}^2$ and Δ^2 are not biholomorphically equivalent. This discovery was the starting point of the *no Riemann mapping theorem* in several complex variables.

In the beginning of the twentieth century, the birth of pluricomplex geometry also coincided with two other fundamental memoirs of F. Hartogs [13] (1906) and of E.E. Levi [16] (1910). However, whereas this direction had important ramifications in the years 1930-50, especially with the works of W. Osgood, of H. Kneser, of R. Fueter, of E. Martinelli, of K. Behnke, of F. Sommer, of S. Bochner, and culminated in the complete solution of the so-called *problem of Levi* given by K. Oka in 1951–52, the direction initiated by H. Poincaré in 1907 lay dormant for approximatively sixty years, with the major exception of four consecutive and historically isolated memoirs of B. Segre [25], [26] and of É. Cartan [3], [4] in the years 1931-32. Based on works of S. Lie, of A. Tresse (a french student of S. Lie), and of the young mathematician B. Segre, É. Cartan (who also had defended his thesis under the direction of S. Lie) provided an essentially complete classification of all Levi nondegenerate real analytic local hypersurfaces

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in $\mathbb{C}^2$, which ultimately relies on S. Lie's far reaching works about the classification of second order ordinary differential equations. Approximatively thirty years later, in 1974, É. Cartan's equivalence algorithm has been conducted in $\mathbb{C}^n$ for $n \geq 2$ by S.S. Chern and J.K. Moser in [5] to provide an *a priori* complete list of differential invariants for Levi nondegenerate real analytic hypersurface in $\mathbb{C}^n$ for $n \geq 2$, *see* also further developments by A.G. Vitushkin [33] and N.G. Kruzhillin [15]. However, the classification problem (in the sense of S. Lie) for real analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^n$ for $n \geq 3$ is essentially left incomplete by the analysis in [5], because the list of differential invariants does not provide immediately a list of all possible automorphisms groups.

1.1.2. Reflection principle and regularity of CR mappings. The real birth of Cauchy-Riemann geometry occured in the beginning of the years 1970, especially when in 1974, C. Fefferman (*see* [10]) established that every biholomorphism between two smoothly bounded strongly pseudoconvex domains extend smoothly as a CR diffeomorphism between their boundaries. It is still conjectured, but up to now unproved, that the result remains true without any pseudoconvexity assumption. Thus, the classification of bounded domains up to biholomorphisms reduces to the classification of boundaries up to CR diffeomorphisms. At the same time, S. Pinchuk discovered in [21], [22], [23] an important local extension theorem for CR mappings between real analytic hypersurfaces, in which both the Schwarz reflection principle phenomenon and the Hartogs extension phenomenon contribute to the analytic continuation of CR mappings. In 1977, generalizing H. Poincaré's grounding result, S.M. Webster established in [34] a general result according to which local CR mappings are complex algebraic as soon as the source and target hypersurfaces are algebraic. Thus, in the aim of generalizing Carathéodory's theorems about the boundary regularity of conformal maps in the complex plane, the grounding works of C. Fefferman, of S. Pinchuk and of S.M. Webster initiated a completely new subject about the regularity (or the analytic continuation) of biholomorphic (or proper) mappings (or of local CR mappings). Since then, this subject has been very active during almost thirty years and a substantial amount of efforts by numerous mathematicians has led to some remarkable refinements of the original statements[1].

§1.2. Conceptional description of the topics adressed in Part I of this memoir

1.2.1. Division in two parts. This memoir is devoted to a synthetic exposition of some recent results in the direction of the so-called *analytic reflection principle*. This terminology justifies by the fact that most arguments and proofs are based on Taylor series considerations. The main topics adressed in Part I of this memoir is to study *ab initio* the local geometry of arbitrary real algebraic or analytic submanifolds of $\mathbb{C}^n$ which are *generic*, namely which satisfy $T_pM + iT_pM = T_p\mathbb{C}^n$ for every $p \in M$. Our main goal is to explain how to go beyond the classical notion of Levi nondegeneracy, taking account of the complexity due to arbitrary dimension and codimension, in order to formulate appropriate generalizations of the reflection principle. In a forthcoming volume of the same collection, Part II of this memoir, which will be accompagnied by its own conceptional introduction, will be specifically devoted to the study of the analytic reflection principle. Thus the present Part I is a kind of thorough preparation, of which we can now present a quick description.

1.2.2. Canonical pair of foliations attached to the extrinsic complexification of a local generic submanifold. As will be established in Theorem 2.1.22, a given real analytic generic submanifold M in $\mathbb{C}^n$ of codimension d and of CR dimension $m = n-d$ may be locally

[1]However, one should remind of the historical bifurcation between the classification problem and the reflection principle. It is probable that too much emphasis has been put in the last decade on the reflection principle, which occulted in part the original motivation of classifying domains. Whereas this memoir is exclusively devoted to the so-called *analytic reflection principle*, we believe that it is time to come back to the original program of research hidden in the mathematical treasures of S. Lie and of É. Cartan, as suggested for instance in the recent works [29], [31].

represented, in a neighborhood of one of its points p, thanks to some appropriate coordinates $t = (z, w) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at p, by means of d complex defining fundamental equations of the form

$$\bar{w}_j = \Theta_j(\bar{z}, z, w), \qquad j = 1, \dots, d. \tag{1.2.3}$$

Here, we assume that the Taylor series of the complex analytic functions $\Theta_j(\bar{z}, z, w) = \sum_{\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^d} \Theta_{j,\beta,\gamma,\delta}\, \bar{z}^\beta\, z^\gamma\, w^\delta$ converge normally in some polydisc centered at the origin in $\mathbb{C}^{m+m+d}$. As will be more evident in the sequel, one finds many advantages to deal with complex defining equations instead of real defining equations. Of course, the conjugate defining equations $w_j = \overline{\Theta}_j(z, \bar{z}, \bar{w}) = \sum_{\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^d} \overline{\Theta}_{j,\beta,\gamma,\delta}\, z^\beta\, \bar{z}^\gamma\, \bar{w}^\delta$ must define the same generic real submanifold M, and the ambiguity due to complex defining equations disappears thanks to the existence of the following fundamental functional equations, obtained in Theorem 2.1.32:

$$\bar{w}_j \equiv \Theta_j(\bar{z}, z, \overline{\Theta(\bar{z}, z, w)}), \qquad j = 1, \dots, d. \tag{1.2.4}$$

We say that M is *algebraic* (in the sense of J. Nash) if the series $\Theta_j(\bar{z}, z, w)$ are algebraic (a power series $\varphi(x) \in \mathbb{C}\{x\}$ is (Nash) *algebraic* if there exists a nonzero polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(x, \varphi(x)) \equiv 0$).

Following S.M. Webster's general philosophy (*cf.* [37]), let $\tau = (\zeta, \xi) \in \mathbb{C}^m \times \mathbb{C}^d$ be new independent coordinates corresponding to $\bar{t} = (\bar{z}, \bar{w})$ and define the *extrinsic complexification* of M to be the d-codimensional complex submanifold $\mathcal{M}$ of $\mathbb{C}^{2n}$ defined by the equations

$$\xi_j = \Theta_j(\zeta, z, w), \qquad j = 1, \dots, d, \tag{1.2.5}$$

or equivalently by $w_j = \overline{\Theta}_j(z, \zeta, \xi)$, $j = 1, \dots, d$. Notice that the expressions $\Theta_j(\zeta, z, w) = \sum_{\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m, \delta \in \mathbb{N}^d} \Theta_{j,\beta,\gamma,\delta}\, \zeta^\beta\, z^\gamma\, w^\delta$ are meaningful only because the Θ_j are converging power series. This submanifold $\mathcal{M}$ comes immediately equipped with two foliations $\mathcal{F} =: \mathcal{M} \cap \{\tau = ct.\}$ and $\underline{\mathcal{F}} = \mathcal{M} \cap \{t = ct.\}$ by m-dimensional complex submanifolds, which were essentially discovered by B. Segre in [25], [26] (*see* also [3], [34]). We call the leaves of $\mathcal{F}$ the *complexified Segre varieties* and the leaves of $\underline{\mathcal{F}}$ the *conjugate complexified Segre varieties*. As we shall argue throughout this memoir, the main features of the geometry of M are hidden behind the interweaving of this pair of foliations $(\mathcal{F}, \underline{\mathcal{F}})$ lying on its complexification $\mathcal{M}$.

Since we are mainly interested in the study of mappings, let M' be a second generic submanifold of codimension d' in $\mathbb{C}^{n'}$ defined similarly by complex defining equations $\bar{w}'_{j'} = \Theta'_{j'}(\bar{z}', z', w')$, $j' = 1, \dots, d'$, where $m' = n' - d'$ and $t' = (z', w') \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$, and let a local mapping $t' = h(t) = (h_1(t), \dots, h_{n'}(t))$ from $\mathbb{C}^n$ to $\mathbb{C}^{n'}$ be a *local power series CR mapping from M to M'*. By this, we mean precisely that there exists a $d' \times d$ matrix of power series $a(t, \bar{t})$ such that if we split $h(t) = (f(t), g(t)) \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$, then the following vectorial formal power series holds in $\mathbb{C}[\![t, \bar{t}]\!]^{d'}$:

$$\bar{g}(\bar{t}) - \Theta'(\bar{f}(\bar{t}), f(t), g(t)) \equiv a(t, \bar{t})\, [\bar{w} - \Theta(\bar{z}, z, w)]. \tag{1.2.6}$$

In this memoir, we shall always assume that M and M' are real algebraic or analytic and we shall consider three different regularity classes for h, namely either $h(t)$ is a purely formal power series, or it is complex analytic, or it is complex algebraic. By complexifying (1.2.6), we trivially obtain the following identity in $\mathbb{C}[\![t, \tau]\!]^{d'}$:

$$\bar{g}(\tau) - \Theta'(\bar{f}(\tau), f(t), g(t)) \equiv a(t, \tau)\, [\xi - \Theta(\zeta, z, w)], \tag{1.2.7}$$

which means precisely that the power series mapping $(t', \tau') = (h(t), \bar{h}(\tau))$ maps the complexifications $\mathcal{M}$ into the complexification $\mathcal{M}'$. We shall denote by $h^c(t, \tau) := (h(t), \bar{h}(\tau))$ this complexified mapping. A straightforward but crucial observation is that h^c stabilizes the two pairs of foliations, namely it satisfies $h^c(\mathcal{F}) \subset \mathcal{F}'$ and $h^c(\underline{\mathcal{F}}) \subset \underline{\mathcal{F}}'$. The following symbolic figure is an attempt to illustrate this stabilization property[2].

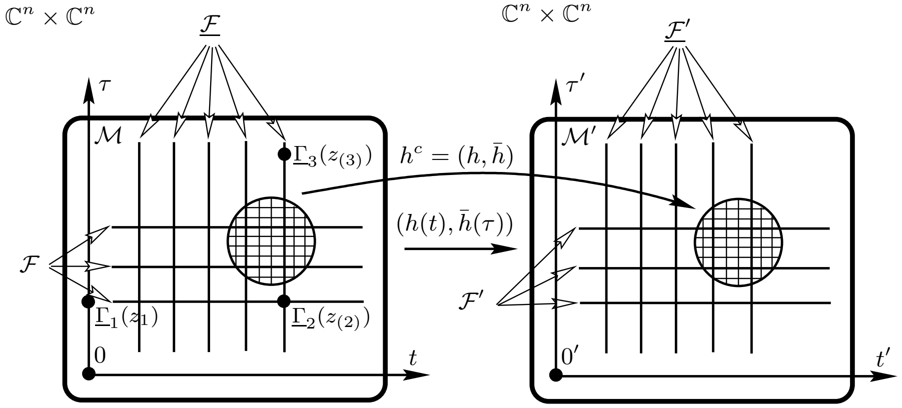

FIGURE 1.2.8: COMPLEXIFIED MAPPINGS RESPECT PAIRS OF FOLIATIONS

Some strong rigidity properties are due to the fact that $h^c = (h, \bar{h})$ must respect these two pairs of foliations. For instance, a theorem due to S.M. Webster in [34] states that every local biholomorphism $h : M \to M'$ between two *Levi nondegenerate* real algebraic hypersurfaces must be a complex algebraic mapping. This theorem may be interpreted intuitively by thinking that h^c (which is *a priori* only complex analytic) is forced to be as smooth as the two pairs of foliations $(\mathcal{F}, \underline{\mathcal{F}})$ and $(\mathcal{F}', \underline{\mathcal{F}}')$ are, namely to be complex algebraic.

1.2.9. Beyond Levi nondegeneracy: Minimality and finite nondegeneracy. In S.M. Webster's theorem, behind Levi nondegeneracy are hidden two highly different and independent concepts, the notion of *minimality* (in the sense of J.-M. Trépreau and A.E. Tumanov, following the general approach of H.J. Sussmann in [32], *see* also [2]) and the notion of *finite nondegeneracy* (introduced for the first time by K. Diederich and S.M. Webster in [9], and by C.K. Han in [12] and then studied by S.M. Baouendi, P. Ebenfelt and L.P. Rothschild in [1]).

The first main concept of minimality is of geometric nature and may easily be described in terms of the pair of foliations $(\mathcal{F}, \underline{\mathcal{F}})$. Let $z_1 \in \mathbb{C}^m$. We denote by $\underline{\Gamma}_1(z_1)$ the point located in the (vertical) m-dimensional complex leaf $\underline{\mathcal{F}}_0$ passing through the origin which lies at distance z_1 from the origin, *see* FIGURE 1.2.8. Of course, $\underline{\Gamma}_1(z_1)$ belongs to $\mathcal{M}$. In other words, we move vertically from the origin up to distance $z_1 \in \mathbb{C}^m$. Let $z_2 \in \mathbb{C}^m$. From this point $\underline{\Gamma}_1(z_1)$, we then move horizontally up to distance z_2, namely following the m-dimensional complex leaf $\mathcal{F}_{\underline{\Gamma}_1(z_1)}$. We denote by $\underline{\Gamma}_2(z_{(2)})$ the resulting point, *see* again FIGURE 1.2.8, where we use the notation $z_{(2)} := (z_1, z_2) \in \mathbb{C}^{2m}$. Of course, the point $\underline{\Gamma}_2(z_{(2)})$ also belongs to $\mathcal{M}$. Let $z_3 \in \mathbb{C}^m$. We further move vertically up to distance z_3 and we denote the resulting point by $\underline{\Gamma}_3(z_{(3)})$, where $z_{(3)} = (z_1, z_2, z_3) \in \mathbb{C}^{3m}$. More generally, by following alternately the two foliations $\underline{\mathcal{F}}$ and $\mathcal{F}$, we may define for every $k \in \mathbb{N}$ a point $\underline{\Gamma}_k(z_{(k)})$, where $z_{(k)} = (z_1, \ldots, z_k) \in \mathbb{C}^{km}$, which belongs to $\mathcal{M}$. It is easy to see that the mapping $z_{(k)} \mapsto \underline{\Gamma}_k(z_{(k)})$ satisfies $\underline{\Gamma}_k(0) = 0$ and has the same regularity as $\mathcal{M}$, namely it is complex algebraic or analytic. We call this map the *k-th conjugate Segre chain*. The precise construction of $\underline{\Gamma}_k$ is presented in Chapter 2 and there are combinatorial formulas which yield the complete expression of $\underline{\Gamma}_k(z_{(k)})$ by means of the fundamental power series $\Theta_j(\zeta, z, w)$.

[2]However, we warn the reader that the representation is slightly incorrect, because the ambient codimensions d and d' in $\mathcal{M}$ and in $\mathcal{M}'$ of the unions of foliations $\mathcal{F} \cup \underline{\mathcal{F}}$ and $\mathcal{F}' \cup \underline{\mathcal{F}}'$ is invisible in this picture. One should imagine for instance that $\mathcal{M}$ and $\mathcal{M}'$ are three-dimensional spaces equipped with pairs of foliations by horizontal orthogonal real lines.

The complexification $\mathcal{M}$ is then called *minimal at the origin* if there exists an integer k such that for every neighborhood $\mathcal{V}_k$ of the origin in $\mathbb{C}^{km}$, its image $\underline{\Gamma}_k(\mathcal{V}_k)$ contains a neighborhood of the origin in $\mathcal{M}$. Intuitively, the concept of minimality says that one can reach every point in a neighborhood of the origin in $\mathcal{M}$ by following alternately the two canonical foliations. Since the two foliations $\mathcal{F}$ and $\underline{\mathcal{F}}$ are biholomorphically invariant, the notion of minimality at one point $p \in M$ so defined is independent of the choice of coordinates vanishing at p. It is elementary to see that a Levi nondegenerate real analytic hypersurface in $\mathbb{C}^n$ $(n \geq 2)$ is minimal at every point.

The second main concept of finite nondegeneracy is of analytic nature and it may be easily described by means of a development in power series of the defining equations of $\mathcal{M}$:

$$\xi_j = \sum_{\beta \in \mathbb{N}^m} \zeta^\beta \, \Theta_{j,\beta}(t). \tag{1.2.10}$$

Here, the $\Theta_{j,\beta}(t) = \Theta_{j,\beta}(z,w)$ are complex algebraic or analytic power series converging normally in a uniform polydisc centered at the origin. Let $k \in \mathbb{N}$. By the *k-th Segre mapping* we mean the local complex algebraic or analytic mapping

$$\mathcal{Q}_k : \mathbb{C}^n \ni t \longmapsto (\Theta_{j,\beta}(t))_{1 \leq j \leq d, \, |\beta| \leq k} \in \mathbb{C}^{N_{d,n,k}}, \tag{1.2.11}$$

where the integer $N_{d,n,k}$ denotes the total number of k-th jets of a d-vectorial mapping of n independent variables, namely $N_{d,n,k} = d \, \frac{(n+k)!}{n! \; k!}$. The generic submanifold M is then called *finitely nondegenerate at the origin* if there exists an integer k such that the k-th Segre mapping is of (maximal possible) rank n at the origin. Although the mapping $\mathcal{Q}_k$ is defined in terms of a system of coordinates and although it seems to depend on the choice of complex defining equations for M, it may be established that its properties are essentially biholomorphically and invariantly attached to M, and in particular, the notion of finite nondegeneracy at a point $p \in M$ so defined is independent of the choice of coordinates vanishing at p. One can show that Levi nondegeneracy of M at the origin (in the sense that the kernel of the vector-valued Levi form of M is zero) is equivalent to the fact that the mapping $\mathcal{Q}_1$ is of rank n at the origin, hence the notion of finite nondegeneracy is a generalization of the notion of Levi nondegeneracy. More generally, M is called *holomorphically nondegenerate at the origin* (in the sense of N. Stanton, *cf.* [28]) if there exists an integer k such that the generic rank of $\mathcal{Q}_k$ is equal to n. Further study of nondegeneracy conditions on the mapping $\mathcal{Q}_k$ are presented in Chapter 3. Since this has been suggested in [8] and [9], we also endeavour a self-contained study of *jets of Segre varieties*, a fundamental topic for which we know no complete background reference.

1.2.12. Local geometry at a Zariski-generic point. Why are minimality and finite nondegeneracy adequate concepts from the point of view of local Cauchy-Riemann geometry ? Firstly, because it may be established that attached to a given arbitrary connected real agebraic or analytic generic submanifold M in $\mathbb{C}^n$, there exists an invariant integer $d_{2,M}$ and a proper real algebraic or analytic subvariety $E \subset M$ such that for every point $p \in M \backslash E$, there exists a neighborhood V_p of p in $\mathbb{C}^n$ and a system of complex algebraic or analytic coordinates $(t_1, \ldots, t_n)$ centered at p such $M \cap V_p$ is contained in the transverse intersection of $d_{2,M}$ Levi flat hypersurfaces defined by $\{\bar{t}_1 = t_1, \ldots, t_{d_{2,M}} = t_{d_{2,M}}\}$ and such that, moreover, for every constant $(c_1, \ldots, c_{d_{2,M}}) \in \mathbb{R}^{d_{2,M}}$, the intersection $M_c := M \cap \{t_1 = c_1, \ldots, t_{d_{2,M}} = c_{d_{2,M}}\} \cap V_p$ is minimal at every point (Corollary 2.8.5). Here, the M_c are elementary “bricks” and there is no “complex link” between them. Hence one may think that minimality is a good “general” assumption.

Secondly, it may be furthermore established that there exists an invariant integer n_M with $d \leq n_M \leq n$ and another proper subvariety $F \subset M$ such that for every point $p \in M \backslash F$, there exists a neighborhood V_p of p in $\mathbb{C}^n$ and a system of coordinates centered at p in which $M \cap V_p$ is a product $\underline{M}'_p \times \Delta^{n-n_M}$ of a d-codimensional generic submanifold $\underline{M}'_p$ in $\mathbb{C}^{n_M}$ by a complex polydisc Δ^{n-n_M}, such that moreover $\underline{M}'_p$ is finitely nondegenerate at its central point

(Theorem 3.5.48). In particular, M is holomorphically nondegenerate if and only if $n = n_M$, in which case M is finitely nondegenerate at every point of $M \backslash F$. Generally speaking, from the point of view of CR geometry where "Complex" and "Real" concepts should be truly associated, the factor Δ^{n-n_M} is essentially superfluous, hence one should think that finite nondegeneracy is a good "general" assumption.

Whereas minimality and finite nondegeneracy do not impose dimensional restrictions, it is well known that the assumption of Levi nondegeneracy requires that $d \leq m^2$. In addition, there exist some classes of hypersurfaces in $\mathbb{C}^3$ whose Levi form is of rank one at every point and which are finitely nondegenerate at every point (*see* Examples 3.2.15 and 3.2.20). In sum, we believe that minimality and finite nondegeneracy are adequate assumptions.

1.2.13. Nondegeneracy conditions for power series CR mappings. In Chapter 4 of part I of this memoir, we shall introduce various nondegeneracy conditions for power series CR mappings. As in §1.2.2, let h be a power series CR mapping from M to M', whose complexification $h^c = (h, \bar{h})$ satisfies the fundamental identities (1.2.7), which yields after replacing ξ by $\Theta(\zeta, t)$ the following formal identities in $\mathbb{C}[\![\zeta, t]\!]$:

$$\bar{g}_{j'}(\zeta, \Theta(\zeta, t)) \equiv \Theta_{j'}(\bar{f}(\zeta, \Theta(\zeta, t)), h(t)), \qquad j' = 1, \dots, d'. \tag{1.2.14}$$

We consider the following pairwise commuting m vector fields tangent to $\mathcal{M}$

$$\underline{\mathcal{L}}_k := \frac{\partial}{\partial \zeta_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial \zeta_k}(\zeta, t)\, \frac{\partial}{\partial \xi_j}, \tag{1.2.15}$$

which span the leaves of $\underline{\mathcal{F}}$ at every point. For every $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$, we introduce the derivation $\underline{\mathcal{L}}^\beta := \underline{\mathcal{L}}_1^{\beta_1} \cdots \underline{\mathcal{L}}_m^{\beta_m}$ of order $|\beta|$, which we apply to the equations (1.2.14). After some computations, this yields an expression of the form $R'_{j',\beta}(t, \tau, (\partial_\tau^\alpha \bar{h}(\tau))_{|\alpha| \leq |\beta|} : h(t))$, where $R'_{j',\beta}$ is a certain analytic expression in its variables. Based on the properties of the infinite collection of functions $R'_{j',\beta}$, we shall formulate five technical nondegeneracy conditions about h. For further intuitive explanation, we refer to the beginning of the conceptional introduction of the forthcoming Part II of this memoir.

`Note to the Russian translator(s):` Since my English has probably some deficiencies, please, do not hesitate to arrange the translation in classical Russian style. For any question about the meaning of a phrase or of a paragraph which would be difficult to understand and difficult to translate, please, do not hesitate to contact me by e-mail, or by regular mail, asking me to rewrite phrases or paragraphs in a better, more understandable style, which I would diligently do.

Chapter 2: Geometry of complexified generic submanifolds and Segre chains

2.1. Elementary local geometry of Cauchy-Riemann submanifolds

2.1.1. Formal, Analytic, Algebraic. As we shall essentially deal in the two parts of this memoir with local power series centered at the origin, we start with classical definitions. Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $n \in \mathbb{N}$ be a positive integer. Let $x = (x_1, \dots, x_n) \in \mathbb{K}^n$. We denote $|x| := \max\{|x_1|, \dots, |x_n|\}$. Let $\mathbb{K}[\![x]\!]$ denote the local ring of formal power series in the n variables $(x_1, \dots, x_n)$. By definition, an element $\varphi(x) \in \mathbb{K}[\![x]\!]$ writes in the form $\varphi(x) = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha\, x^\alpha$, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\varphi_\alpha \in \mathbb{K}$ for every multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We say that φ is a $\mathbb{K}$-*formal* power series. Such a power series $\varphi(x) = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha\, x^\alpha$ is *identically zero* if all its coefficients φ_α are zero. We write this property $\varphi(x) \equiv 0$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex in $\mathbb{N}^n$, we denote its *length* by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and the corresponding partial derivative of a power series by $\partial_x^\alpha \varphi(x) := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \varphi(x)$. Sometimes, we use also the equivalent notation $\partial^{|\alpha|}\varphi(x)/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. We have $\varphi_\alpha = [1/\alpha!]\, \partial_x^\alpha \varphi(x)|_{x=0}$. If the coefficients satisfy a Cauchy estimate like $|\varphi_\alpha| \leq C\,\rho^{-|\alpha|}$, where $C > 0$ and $\rho > 0$, the series *converges normally* in the polydisc $\Delta_n(\rho) = \{x \in \mathbb{K}^n : |x| < \rho\}$. We say that φ is $\mathbb{K}$-*analytic* and we write $\varphi(x) \in \mathbb{K}\{x\}$. If there exists moreover a nonzero polynomial $P(X_1, \dots, X_n, \Phi) \in \mathbb{K}[X_1, \dots, X_n, \Phi]$ such that $P(x_1, \dots, x_n, \varphi(x_1, \dots, x_n)) \equiv 0$ for all $x \in \Delta_n(\rho)$, we say that φ is $\mathbb{K}$-*algebraic* (in the sense of J. Nash) and we write $\varphi(x) \in \mathcal{A}_{\mathbb{K}}\{x\}$. By classical elimination theory it follows that if, more generally, a power series $\varphi(x) \in \mathbb{K}\{x\}$ satisfies a polynomial equation $P(\varphi(x)) \equiv 0$, where $P(\Phi) \in \mathcal{A}_{\mathbb{K}}\{x\}[T]$ is a polynomial in the indeterminate Φ with coefficients in $\mathcal{A}_{\mathbb{K}}\{x\}$, then $\varphi(x)$ is $\mathbb{K}$-algebraic. We have the following inclusions:

$$\mathbb{K}[\![x]\!] \supset \mathbb{K}\{x\} \supset \mathcal{A}_{\mathbb{K}}\{x\}, \tag{2.1.2}$$

which are all strict. The three rings $\mathbb{K}[\![x]\!]$, $\mathbb{K}\{x\}$ and $\mathcal{A}_{\mathbb{K}}\{x\}$ are local, noetherian and they enjoy the Weierstrass division property.

2.1.3. Composition, differentiation and implicit function theorem. Furthermore, the rings $\mathbb{K}[\![x]\!]$, $\mathbb{K}\{x\}$ and $\mathcal{A}_{\mathbb{K}}\{x\}$ are stable under elementary algebraic operations, under composition, under differentiation and the implicit function theorem holds true. Only $\mathcal{A}_{\mathbb{K}}\{x\}$ is dramatically unstable under integration. The following known theorem, that we shall admit, summarizes these properties.

Theorem 2.1.4. *The following three statements hold true:*

- **(1)** *Let n and d be positive integers, let $x \in \mathbb{K}^n$, let $y \in \mathbb{K}^d$, let $\varphi(x)$ belong to $\mathbb{K}[\![x]\!]$, to $\mathbb{K}\{x\}$ or to $\mathcal{A}_{\mathbb{K}}\{x\}$, let $\psi_1(y), \dots, \psi_n(y)$ belong to $\mathbb{K}[\![y]\!]$, to $\mathbb{K}\{y\}$, or to $\mathcal{A}_{\mathbb{K}}\{y\}$ and vanish at the origin. Then $\varphi(\psi_1(y), \dots, \psi_n(y))$ belongs $\mathbb{K}[\![y]\!]$, to $\mathbb{K}\{y\}$, or to $\mathcal{A}_{\mathbb{K}}(y)$.*
- **(2)** *Let n be a positive integer and let $x \in \mathbb{K}^n$. If a power series $\varphi(x)$ belongs to $\mathbb{K}[\![x]\!]$, to $\mathbb{K}\{x\}$, or to $\mathcal{A}_{\mathbb{K}}\{x\}$, then for every multiindex $\alpha \in \mathbb{N}^n$, the partial derivative $\partial_x^\alpha \varphi(x)$ also belongs to $\mathbb{K}[\![x]\!]$, to $\mathbb{K}\{x\}$, or to $\mathcal{A}_{\mathbb{K}}\{x\}$.*
- **(3)** *Let n and d be positive integers, let $x \in \mathbb{K}^n$, $y \in \mathbb{K}^d$ and let $H_1(x,y), \dots, H_d(x,y)$ be a collection of formal, analytic or algebraic power series vanishing at the origin, namely the $H_j(x,y)$ belongs to $\mathbb{K}[\![x,y]\!]$, to $\mathbb{K}\{x,y\}$, or to $\mathcal{A}_{\mathbb{K}}\{x,y\}$ and they satisfy $H_j(0,0) = 0$ for $j = 1, \dots, d$. Assume that the functional determinant $(\partial H_{j_1}/\partial y_{j_2}(0))_{1\leq j_1, j_2 \leq d}$ does not vanish. Then there exists a unique $\mathbb{K}^d$-valued power series mapping $\varphi(x) = (\varphi_1(x), \dots, \varphi_d(x))$, where the $\varphi_j(x)$ belong to $\mathbb{K}[\![x]\!]$, to $\mathbb{K}\{x\}$, or to $\mathcal{A}_{\mathbb{K}}\{x\}$ and vanish at the origin, such that $H_j(x, \varphi(x)) \equiv 0$ for $j = 1, \dots, d$.*

2.1.5. Local submanifolds and their mappings. By definition, a *local submanifold* M of $\mathbb{K}^n$ is identified with the data of $d \leq n$ power series $r_1(x), \dots, r_d(x)$ vanishing at the origin and which belong to $\mathbb{K}[\![x]\!]$, to $\mathbb{K}\{x\}$, or to $\mathcal{A}_{\mathbb{K}}\{x\}$ such that the linear forms $dr_1(0), \dots, dr_d(0)$ are linearly independent. Two data $r(x) = (r_1(x), \dots, r_d(x))$ and $\widehat{r}(x) = (\widehat{r}_1(x), \dots, \widehat{r}_d(x))$ define the *same submanifold* if there exists an invertible $d \times d$ matrix $a(x) = (a_{j_1,j_2}(x))_{1\leq j_1,j_2\leq d}$ of power series in $\mathbb{K}[\![x]\!]$, in $\mathbb{K}\{x\}$, or in $\mathcal{A}_{\mathbb{K}}\{x\}$, such that $\widehat{r}_j(x) \equiv \sum_{l=1}^d a_{j,l}(x)\, r_l(x)$, or in matrix notation $\widehat{r}(x) \equiv a(x)\, r(x)$. Clearly this defines an equivalence relation between d-tuples of power series $r(x) = (r_1(x), \dots, r_d(x))$ whose differentials are independent at the origin. A submanifold identifies with an equivalence class. We shall write $M : r_1(x) = \cdots = r_d(x) = 0$, keeping in mind that the identification of M with its "zero set" is meaningless in the formal category. We call d the *codimension* of M. Let $x' = \Phi(x)$ be a formal, algebraic or analytic invertible change of coordinates centered at the origin and let $x = \Phi'(x')$ denote its inverse. The *transformed submanifold* $M' := \Phi(M)$ is defined by the collection $r'(x') := (r_1(\Phi'(x')), \dots, r_d(\Phi'(x')))$. It follows from the formal, analytic or algebraic implicit function theorem, that there exists a local invertible transformation $x' = \Phi(x)$ such that $r_1(\Phi'(x')) = x_1', \dots, r_d(\Phi'(x')) = x_d'$, so the image $M' := \Phi(M)$ writes $M' : x_1' = \cdots = x_d' = 0$.

Let n and n' be two positive integers. A formal, analytic or algebraic *local mapping from* $\mathbb{K}^n$ *to* $\mathbb{K}^{n'}$ consists of the datum of a n'-tuple $h(x) = (h_1(x), \dots, h_{n'}(x))$ of power series $h_{i'}(x)$ in $\mathbb{K}[\![x]\!]$, in $\mathbb{K}\{x\}$, or in $\mathcal{A}_{\mathbb{K}}\{x\}$, with $h_{i'}(0) = 0$ for $i' = 1, \dots, n'$. We write $x' = h(x)$. If

n'' is a third positive integer and if $x'' = g(x')$ is a second formal, analytic or algebraic mapping, the *composition* $x'' = g(h(x))$ is the collection of power series $(g_1(h(x)), \dots, g_{n''}(h(x)))$, which is a local mapping from $\mathbb{K}^n$ to $\mathbb{K}^{n''}$. If $\widetilde{x} = \Phi(x)$ and $\widetilde{x}' = \Psi(x')$ are changes of coordinates in $\mathbb{K}^n$ and in $\mathbb{K}^{n'}$, the *transformed mapping* $\widetilde{h}$ is the mapping $\widetilde{x}' = \widetilde{h}(\widetilde{x})$, where $\widetilde{h}(\widetilde{x}) := \Psi(h(\Phi^{-1}(\widetilde{x})))$. The *rank at the origin* of h is the rank of the Jacobian matrix $(\partial h_{i'}(0)/\partial x_i)_{1\leq i'\leq n', \, 1\leq i\leq n}$. We denote it by $\mathrm{rk}_0(h)$. The *generic rank* of h is the largest integer $e \leq \min(n, n')$ such that there exists an $e \times e$ minor of the Jacobian matrix $\mathrm{Jac}\, h(x)$ which does not vanish identically, but all $(e+1)\times(e+1)$ minors do vanish identically. We denote it by $\mathrm{genrk}_{\mathbb{K}}(h)$.

Let now d and d' be two positive integers and let $M : r_1(x) = \cdots = r_d(x) = 0$ and $M' : r_1'(x') = \cdots = r_{d'}'(x') = 0$ be two formal, analytic or algebraic submanifolds. We say that *h maps M into M'* if there exists a $d' \times d$ matrix $b(x) = (b_{j',j}(x))_{1\leq j'\leq d', \, 1\leq j\leq d}$ of formal, analytic or algebraic power series such that $r_{j'}'(h(x)) \equiv \sum_{j=1}^d b_{j',j}\, r_j(x)$, or in matrix notation $r'(h(x)) \equiv b(x)\, r(x)$. This definition is meaningful, since if $\widehat{r}(x) = a(x)\, r(x)$ and $\widehat{r}'(x') = a'(x')\, r'(x')$ denote two equivalent defining formal, analytic or algebraic defining power series for M and for M', then $\widehat{r}'(h(x)) \equiv a'(h(x))\, r'(h(x)) \equiv a'(h(x))\, b(x)\, [a(x)]^{-1}\, \widehat{r}(x)$, so we have $\widehat{r}'(h(x)) \equiv \widehat{b}(x)\, \widehat{r}(x)$ with $\widehat{b}(x) := a'(h(x))\, b(x)\, [a(x)]^{-1}$.

2.1.6. Cauchy-Riemann submanifolds of $\mathbb{C}^n$. We want to study some aspects of the geometry of real submanifolds of $\mathbb{C}^n$. Most often in this memoir, we shall mainly be concentrated on the local study of pieces of submanifolds centered at one point. *However, we stress that we shall never use the language of germs, because it might sometimes be confusing.* Hence we have to work within precise neighborhoods of central points.

Without loss of generality, we can assume that the center point is the origin in suitable coordinates $z = x + iy \in \mathbb{C}^n$. Thus, we consider a (local) *real d*-codimensional submanifold M of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ passing through the origin which defined by equations $r_1(x,y) = \cdots = r_d(x,y) = 0$ where the differentials $dr_1, \dots, dr_d$ are linearly independent at the origin. If $z = x + iy \in \mathbb{C}$ or equivalently $(x,y) \in \mathbb{R}^{2n}$, we use the cube norms $|x| = \max_{1\leq k\leq n} |x_k|$, $|y| = \max_{1\leq k\leq n} |y_k|$ and the polydisc norm $|z| = \max_{1\leq k\leq n} |z_k|$, where $|z_k| = (x_k^2 + y_k^2)^{1/2}$. For given $\nu \in \mathbb{N}$ with $\nu \geq 1$ and $\rho \in \mathbb{R}$ with $\rho > 0$ we denote by $\mathbb{I}_\nu(\rho)$ the real cube $(-\rho, \rho)^\nu$ in $\mathbb{R}^\nu$. If $\rho > 0$, we denote by $\Delta_n(\rho) = \{z \in \mathbb{C}^n : |z| < \rho\}$ the open polydisc of radius ρ centered at the origin. *Throughout this memoir, we shall always work with cubes and polydiscs.*

For useful and complete background about Cauchy-Riemann (CR for short) structures, we refer the reader to [1], [7]. Here, we only give quick definitions for the purpose of being self-contained. Let J denote the complex structure of $T\mathbb{C}^n$, acting on real vectors as if it were multiplication by $\sqrt{-1}$, hence satisfying $J^2 = -\mathrm{Id}$. Let M be a connected local real algebraic or analytic submanifold of $\mathbb{C}^n$ of codimension d. For $p \in M$, the smallest J-invariant subspace of the tangent space T_pM is given by $T_p^cM := T_pM \cap JT_pM$ and is called the *complex tangent space to M at p*.

Definition 2.1.7. The submanifold M is called

- **(1)** *Holomorphic* if $T_p^cM = T_pM$ at every point $p \in M$;
- **(2)** *Totally real* if $T_p^cM = \{0\}$ at every point $p \in M$;
- **(3)** *Generic* if $T_pM + JT_pM = T_p\mathbb{C}^n$ at every point $p \in M$;
- **(4)** *Cauchy-Riemann* (CR for short) if the dimension of T_p^cM is equal to a fixed constant at every point $p \in M$.

In particular, holomorphic and totally real submanifolds are obviously CR. The generic submanifolds are also CR (and in fact of minimal possible CR dimension), because by the dimension formula $\dim_{\mathbb{R}}(E+F) = \dim_{\mathbb{R}} E + \dim_{\mathbb{R}} F - \dim_{\mathbb{R}}(E \cap F)$ for real vector subspaces, we deduce from $\dim_{\mathbb{R}}(T_pM + JT_pM) = 2n$ that $\dim_{\mathbb{R}}(T_pM \cap JT_pM) = 2n - 2d$, which is constant. We shall remember that for generic submanifolds, the CR dimension is given by $m = n - d$.

By means of the dimension formula, we also see that if M is totally real, then $\dim_{\mathbb{R}} M \leq n$; also, if M is generic, then $\dim_{\mathbb{R}} M \geq n$. If M is totally real and generic, then $\dim_{\mathbb{R}} M = n$. In this case, we call M *maximally real*.

The two J-invariant spaces $T_pM \cap JT_pM$ and $T_pM + JT_pM$ are clearly of even real dimension. We denote by m_p the integer $\frac{1}{2}\dim_{\mathbb{R}}(T_pM \cap JT_pM)$ and call it the *CR dimension of* M *at* p. We denote by c_p the integer $n - \frac{1}{2}\dim_{\mathbb{R}}(T_pM + JT_pM)$ and call it the *holomorphic codimension of* M *at* p. Of course, we have $c_p = d - n + m_p$. In terms of these two integers m_p and c_p, we may rephrase the above definition as follows.

Definition 2.1.8. The d-codimensional real submanifold $M \subset \mathbb{C}^n$ is

- **(1')** Holomorphic if $2n - d = \dim_{\mathbb{R}} M = 2m_p$ at every point $p \in M$;
- **(2')** Totally real if $m_p = 0$ at every point $p \in M$;
- **(3')** Generic if $m_p = n - d$ at every point $p \in M$; in this case, m_p is as small as possible and we call the integer $m := n - d$ the *CR dimension* of M;
- **(4')** CR if m_p is equal to a fixed constant m at every point $p \in M$; in this case, we call the integer m the *CR dimension* of M; also, it follows that the holomorphic codimension $c_p := d - n + m_p = d - n + m$ is constant and we call it the *holomorphic codimension* of M.

For the proof of the following local graph representation theorem, we refer to [1], [7].

Theorem 2.1.9. *Let M be a real algebraic or analytic submanifold of $\mathbb{C}^n$ of codimension d.*

- **(1)** *Assume that M is holomorphic, let $m = \frac{1}{2}\dim_{\mathbb{R}} M$ be the CR dimension of M and let $d_1 := \frac{1}{2}d$. Then for every point $p_0 \in M$, there exist local complex algebraic or analytic coordinates $(z, w) \in \mathbb{C}^m \times \mathbb{C}^{d_1}$ vanishing at p_0 and there exists $\rho_1 > 0$ such that $M \cap \Delta_n(\rho_1)$ is given by the d_1 complex equations $w_j = 0$, $j = 1, \ldots, d_1$ or equivalently by the d real equations $\operatorname{Re} w_j = \operatorname{Im} w_j = 0$, $j = 1, \ldots, d_1$.*
- **(2)** *Assume that M is totally real, let $c = d - n \geq 0$ be the holomorphic codimension of M and let $d_1 := d - 2c$. Then for every point $p_0 \in M$, there exist complex algebraic or analytic coordinates $(w, v) \in \mathbb{C}^{d_1} \times \mathbb{C}^c$ centered at p_0 and there exists $\rho_1 > 0$ such that $M \cap \Delta_n(\rho_1)$ is given by the d real equations*

$$\begin{cases} \operatorname{Im} w_j = 0, & j = 1, \ldots, d_1, \\ \operatorname{Re} v_k = \operatorname{Im} v_k = 0, & k = 1, \ldots, c. \end{cases} \tag{2.1.10}$$

- **(3)** *Assume that M is generic and let $m = d - n$ be the CR dimension of M. Then for every point $p_0 \in M$ and for every choice of complex affine coordinates $(z, w) \in \mathbb{C}^m \times \mathbb{C}^d$ centered at p_0 such that $T^c_{p_0}M \cap \{w = 0\} = \{0\}$, there exists $\rho_1 > 0$ and there exist uniquely defined real algebraic or analytic functions φ_j, $j = 1, \ldots, d$, converging normally in the cube $\mathbb{I}_{2m+d}(2\rho_1)$ and vanishing at the origin such that $M \cap \Delta_n(\rho_1)$ is given by the d real equations*

$$\operatorname{Im} w_j = \varphi_j(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w), \qquad j = 1, \ldots, d. \tag{2.1.11}$$

  We can in addition choose the coordinates in order that T_0M is given by the equations $\operatorname{Im} w_j = 0$, $j = 1, \ldots, d$, in which case we have moreover $d\varphi_j(0) = 0$, for $j = 1, \ldots, d$.
- **(4)** *Assume that M is CR, let m be the CR dimension of M, let $c = d - n + m$ be the holomorphic codimension of M and let $d_1 := d - 2c \geq 0$. Then for every point $p_0 \in M$, there exist local complex algebraic or analytic coordinates $(z, w, v) \in \mathbb{C}^m \times \mathbb{C}^{d_1} \times \mathbb{C}^c$ centered at p_0 with $T^c_{p_0}M \cap \{w = v = 0\} = \{0\}$ and there exist real algebraic or analytic functions φ_j converging normally in the cube $\mathbb{I}_{2m+d_1}(2\rho_1)$ for some $\rho_1 > 0$ and vanishing at the origin such that $M \cap \Delta_n(\rho_1)$ is given by the d real equations*

$$\begin{cases} \operatorname{Im} w_j = \varphi_j(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w), & j = 1, \ldots, d_1, \\ \operatorname{Re} v_k = \operatorname{Im} v_k = 0, & k = 1, \ldots, c. \end{cases} \tag{2.1.12}$$

In particular, M is contained and generic in the complex linear subspace $(\mathbb{C}^m \times \mathbb{C}^{d_1} \times \{0\}) \cap \Delta_n(\rho_1)$, which we call the intrinsic complexification of M. We can in addition choose the coordinates in order that $T_0 M$ is given by the equations $\operatorname{Im} w_j = 0$, $j = 1, \dots, d_1$, $\operatorname{Re} v_k = \operatorname{Im} v_k = 0$, $k = 1, \dots, c$, in which case we have moreover $d\varphi_j(0) = 0$ for $j = 1, \dots, d_1$.

2.1.13. Complex defining equations. We now consider a real algebraic or analytic *generic* submanifold M of $\mathbb{C}^n$ given as in Theorem 2.1.9 by real defining equations $v_j = \varphi_j(x, y, u)$, $j = 1, \dots, d$, where $(z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$. Unless the contrary is explicitely mentioned, *our generic submanifolds will always be of positive codimension $d \geq 1$ and of positive CR dimension $m \geq 1$*. Without loss of generality, we can assume that $d\varphi_j(0) = 0$ for $j = 1, \dots, d$. Replacing x by $(z + \bar{z})/2$, y by $(z - \bar{z})/2i$, u by $(w + \bar{w})/2$ and v by $(w - \bar{w})/2i$ in the defining equations of M, which yields

$$\frac{w_j - \bar{w}_j}{2i} = \varphi_j\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}, \frac{w + \bar{w}}{2}\right), \tag{2.1.14}$$

for $j = 1, \dots, d$, then by means of the algebraic or analytic implicit function theorem, we can solve the $\bar{w}_j$ in terms of $(\bar{z}, z, w)$, which yields

$$\bar{w}_j = \Theta_j(\bar{z}, z, w), \qquad j = 1, \dots, d, \tag{2.1.15}$$

for some complex algebraic or analytic functions Θ_j, which vanish at the origin and which are defined in a neighborhood of the origin in $\mathbb{C}^{2m+d}$. Shrinking $\rho_1 > 0$ if necessary, we can assume that the Θ_j converge normally in $\Delta_{2m+d}(2\rho_1)$. We call these new equations *complex defining equations* for M and we want to compare them with the real defining equations.

Generally speaking, given an arbitrary series $\Phi(t) = \sum_{\gamma \in \mathbb{N}^n} \Phi_\gamma \, t^\gamma$ with complex coefficients $\Phi_\gamma \in \mathbb{C}$, we are led to define the series $\overline{\Phi}(t) := \sum_{\gamma \in \mathbb{N}^n} \overline{\Phi}_\gamma \, t^\gamma$ by conjugating only its complex coefficients. With this definition, the conjugation operator (overline) can be applied independently over functions and over variables, as shown by the functional equation $\overline{\Phi(t)} \equiv \overline{\Phi}(\bar{t})$. We shall use this property very frequently.

Let $(z, w) \in M$. Conjugating the defining equations of M, we get $w_j = \overline{\Theta}_j(z, \bar{z}, \bar{w})$ and replacing the $\bar{w}_l$ by their value Θ_l, we get the following equation, valuable for all (z, w) belonging to M:

$$w_j = \overline{\Theta}_j(z, \bar{z}, \Theta(\bar{z}, z, w)), \qquad j = 1, \dots, d. \tag{2.1.16}$$

But as we may write $(z, w) = (z, u + i\varphi(x, y, u)) \in M$, where $u = (u_1, \dots, u_d) = \operatorname{Re} w$, we can replace in (2.1.16), which yields a power series identity in terms of the variables (x, y, u) for all $(x, y, u) \in \mathbb{I}_{2m+d}(\rho_1)$. As the $(2m + d)$-dimensional real algebraic or analytic submanifold $\{(x, y, u + i\varphi(x, y, u))\}$ of $\mathbb{C}^{2m+d}$ is maximally real, by an application of the generic uniqueness principle, we get the power series identity

$$w_j \equiv \overline{\Theta}_j(z, \bar{z}, \Theta(\bar{z}, z, w)), \qquad j = 1, \dots, d. \tag{2.1.17}$$

in $\mathbb{C}\{z, \bar{z}, \bar{w}\}$ or for all $(\bar{z}, z, w) \in \Delta_{2m+d}(\rho_1)$.

Conversely, suppose that these power series identities (2.1.17) holds. By the implicit function theorem, there exists unique complex algebraic or analytic solutions $\varphi_j((z + \zeta)/2, (z - \zeta)/2i, w)$, $j = 1, \dots, d$, $z \in \mathbb{C}^m$, $\zeta \in \mathbb{C}^m$, $w \in \mathbb{C}^d$ of the functional equations

$$\begin{cases} w_j - i\varphi_j((z + \zeta)/2, (z - \zeta)/2i, \zeta, w) \equiv \\ \qquad\qquad \equiv \Theta_j(\zeta, z, w + i\varphi((z + \zeta)/2, (z - \zeta)/2i, w)), \end{cases} \tag{2.1.18}$$

for $j = 1, \dots, d$. We then claim that $\varphi_j(x, y, u)$ is real for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$. Indeed, by replacing first w by $u + i\varphi((z + \bar{z})/2, (z - \bar{z})/2i, u)$ in the functional equations (2.1.17), we get

$$\begin{cases} u_j + i\varphi_j((z + \bar{z})/2, (z - \bar{z})/2i, u) \equiv \\ \qquad\qquad \overline{\Theta}_j(z, \bar{z}, \Theta(\bar{z}, z, u + i\varphi((z + \bar{z})/2, (z - \bar{z})/2i, u))), \end{cases} \tag{2.1.19}$$

for $j = 1, \dots, d$. Using the implicit equations (2.1.18) which define φ with ζ replaced by $\bar{z}$ and w replaced by u, we may then simplify the terms behind Θ in (2.1.19), wich yields

$$u_j + i\varphi_j(x, y, u) \equiv \overline{\Theta}_j(z, \bar{z}, u - i\varphi(x, y, u)), \qquad j = 1, \dots, d. \tag{2.1.20}$$

Conjugating these identities, we get

$$u_j - i\overline{\varphi}_j(x, y, u) \equiv \Theta_j(\bar{z}, z, u + i\overline{\varphi}(x, y, u)), \qquad j = 1, \dots, d. \tag{2.1.21}$$

Comparing with the implicit equations (2.1.18) with ζ replaced by $\bar{z}$ and w replaced by u, we see that $\varphi(x, y, u)$ and $\overline{\varphi}(x, y, u)$ are solutions of the same implicit equations. By uniqueness in the implicit function theorem, we obtain $\overline{\varphi}(x, y, u) \equiv \varphi(x, y, u)$, as claimed. Finally, the identities $u_j - i\varphi_j(x, y, u) \equiv \Theta_j(z, \bar{z}, u + i\varphi(x, y, u))$ show that the set of points (z, w) satisfying $\bar{w}_j = \Theta_j(\bar{z}, z, w)$, $j = 1, \dots, d$, coincides with the real algebraic or analytic generic submanifold of equations $v_j = \varphi_j(x, y, u)$, for $j = 1, \dots d$. In conclusion, we have established the following important theorem which we shall use very often in the sequel.

Theorem 2.1.22. *Let M be a real algebraic or analytic generic submanifold of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$ in $\mathbb{C}^n$. Then for every point $p_0 \in M$, and for every choice of complex affine coordinates $t = (z, w) \in \mathbb{C}^m \times \mathbb{C}^d$ centered at p_0 such that $T^c_{p_0}M \cap \{w = 0\} = \{0\}$, there exists $\rho_1 > 0$ and there exist uniquely defined complex algebraic or analytic functions Θ_j, $j = 1, \dots, d$, vanishing at the origin, defined and converging normally in $\Delta_{2m+d}(2\rho_1)$ such that $M \cap \Delta_n(\rho_1)$ is given by the d complex defining equations*

$$\bar{w}_j = \Theta_j(z, \bar{z}, w), \qquad j = 1, \dots, d, \tag{2.1.23}$$

or equivalently by the d conjugate complex defining equations

$$w_j = \overline{\Theta}_j(z, \bar{z}, \bar{w}), \qquad j = 1, \dots, d. \tag{2.1.24}$$

Here, the vector-valued mapping $\Theta := (\Theta_1, \dots, \Theta_d)$ satisfies the two conjugate vectorial functional equations

$$\begin{cases} \bar{w} \equiv \Theta(\bar{z}, z, \overline{\Theta}(z, \bar{z}, \bar{w})), \\ w \equiv \overline{\Theta}(z, \bar{z}, \Theta(\bar{z}, z, w)). \end{cases} \tag{2.1.25}$$

Conversely, given a collection $\Theta = (\Theta_1, \dots, \Theta_d)$ of complex algebraic or analytic functions vanishing at the origin, converging normally in $\Delta_{2m+d}(2\rho_1)$ for some $\rho_1 > 0$ and satisfying the functional equations (2.1.25)*, then the set $M := \{(z, w) \in \Delta_n(\rho_1) : \bar{w}_j = \Theta_j(\bar{z}, z, w), j = 1, \dots, d\}$ is a real generic submanifold of codimension d. Finally, with these equations, a basis of $(0, 1)$ vector fields tangent to M is given for $k = 1, \dots, m$ by*

$$\overline{L}_k := \frac{\partial}{\partial \bar{z}_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial \bar{z}_k}(\bar{z}, z, w) \frac{\partial}{\partial \bar{w}_j}. \tag{2.1.26}$$

Let $\tau = (\zeta, \xi) \in \mathbb{C}^m \times \mathbb{C}^d$ be new independent complex variables. As in §2.2.10 below, we define the *extrinsic complexification* $\mathcal{M}$ *of* M to be the complex analytic or algebraic d-codimensional submanifold of $\mathbb{C}^n \times \mathbb{C}^n$ defined by the equations $\xi_j - \Theta_j(\zeta, t) = 0$, $j = 1, \dots, d$. The following lemma, which is equivalent to the functional equations (2.1.25), will also be very useful.

Lemma 2.1.27. *There exists an invertible $d \times d$ matrix $a(t, \tau)$ of algebraic or analytic power series such that*

$$\xi - \Theta(\zeta, t) \equiv a(t, \tau)\, [w - \overline{\Theta}(w, \tau)]. \tag{2.1.28}$$

Proof. We consider the involution σ defined by $\sigma(t, \tau) := (\tau, t)$. Let us say that an ideal $\mathcal{J}$ of $\mathcal{A}_{\mathbb{C}}\{t, \tau\}$ or of $\mathbb{C}\{t, \tau\}$ is *invariant under the involution* σ if for every element $\psi(t, \tau) \in \mathcal{J}$, we have $\overline{\psi}(\sigma(t, \tau)) = \overline{\psi}(\tau, t) \in \mathcal{J}$. Then the ideal $\mathcal{J}$ generated by the functions $(w_j - \xi_j)/2i - \varphi_j((z + \zeta)/2, (z - \zeta)/2i, (w + \xi)/2)$, for $j = 1, \dots, d$, is clearly invariant under the involution

σ, since the φ_j are real functions. By the implicit function theorem, we can solve as above with respect to ξ, and we have

$$\mathcal{J} = \langle \xi_j - \Theta(\zeta, t)\rangle_{1\leq j\leq d} = \sigma_*(\mathcal{J}) = \left\langle w_j - \overline{\Theta}(z,\tau)\right\rangle_{1\leq j\leq d}, \tag{2.1.29}$$

from which the existence of the matrix $a(t,\tau)$ follows. □

2.1.30. Existence of normal coordinates. For notational convenience, it is often more appropriate in the real algebraic or analytic categories to consider power series not with respect to $(x,y)\in\mathbb{R}^m\times\mathbb{R}^m$ but with respect to $(z,\bar z)\in\mathbb{C}^m\times\mathbb{C}^m$, which is equivalent because $(x,y) = ((z+\bar z)/2, (z-\bar z)/2i)$ and $(z,\bar z) = (x+iy, x-iy)$.

Convention 2.1.31. As the local generic submanifold M is algebraic, analytic or formal, we shall write its defining equations $v_j = \varphi_j(z,\bar z, u)$, for $j = 1,\dots,d$, in coordinates $(z,w) = (x+iy, u+iv)\in\mathbb{C}^m\times\mathbb{C}^d$, where the φ_j are power series with respect to $(z,\bar z,u)$ centered at the origin. In such a representation, we mix the real and the complex variables.

We can now state the existence of *normal coordinates*, which are coordinates in which the conditions (2.1.34) below are satisfied. Such coordinates are not unique, as may easily be verified.

Theorem 2.1.32. *Let M be as in Theorem 2.1.22. Then there exists a complex algebraic or analytic change of coordinates $t' = h(t)$ of the special form*

$$z' = z, \qquad w' = g(z,w), \tag{2.1.33}$$

such that the image $M' := h(M)$ has real defining equations of the form $v_j' = \varphi_j'(z',\bar z', u')$, $j = 1,\dots,d$ and complex defining equations of the form $\bar w_j' = \Theta_j'(\bar z', z', w')$, $j = 1,\dots,d$, satisfying

$$\begin{cases} \varphi_j'(0,\bar z', u') \equiv \varphi_j'(z',0,u') \equiv 0, \\ \Theta_j'(0,z',w') \equiv \Theta_j'(\bar z', 0, w') \equiv w_j'. \end{cases} \tag{2.1.34}$$

Proof. After a linear transformation of the form (2.1.33), we can assume that $T_0M = \{v = 0\}$, hence $d\varphi_j(0) = 0$, $j = 1,\dots,d$. Next, the local transformation defined by $z = z'$, $w = w' + i\varphi(0,0,w')$ straightens the maximally real d-dimensional submanifold

$$\{(0, v + i\varphi(0,0,v)) : j = 1,\dots,d\} \subset \{0\}\times\Delta_d(\rho_1) \tag{2.1.35}$$

to the d-dimensional plane $\{(0,v')\}$. Thus, we can also assume that $\varphi_j(0,0,v)\equiv 0$, $j = 1,\dots,d$. It follows that $\Theta_j(0,0,w)\equiv w_j$. We continue the proof with the complex defining equations of M.

But before proceeding further, we remark firstly that by reality of the power series φ_j', we have $\varphi_j'(z',\bar z', u') \equiv \overline{\varphi}_j'(\bar z', z', u')$, whence the collection of relations $\varphi_j'(0,\bar z', u')\equiv 0$, $j = 1,\dots,d$, is equivalent to the collection of relations $\varphi_j'(z',0,u')\equiv 0$, $j = 1,\dots,d$. Secondly, using the functional relations (2.1.25), we see immediately that the collection of relations $\Theta_j'(0,z',w')\equiv w_j'$, $j = 1,\dots,d$, is also equivalent to the collection of relations $\Theta_j'(\bar z',0,w')\equiv w_j'$, $j = 1,\dots,d$. Thirdly, by inspecting the way how the real and the complex defining equations of M' are related (*see* especially the proof of Theorem 2.1.22), we observe easily that the collection of relations in the first line of (2.1.34) is equivalent to the collection of relation in the second line of (2.1.34). Consequently, it suffices to find a change of coordinates of the form (2.1.33) such that $\Theta_j'(\bar z', 0, w')\equiv w_j'$ for $j = 1,\dots,d$.

We then claim that the transformation $(z',w') := (z, \Theta(0,z,w))$ is appropriate. Indeed, working with the extrinsic complexifications $\mathcal{M}$ and $\mathcal{M}'$, we have $(z,w,\zeta,\xi)\in\mathcal{M}$ if and only if $(z,\Theta(0,z,w),\zeta,\overline{\Theta}(0,\zeta,\xi))\in\mathcal{M}'$, which yields

$$\overline{\Theta}(0,\zeta,\xi) = \xi' = \Theta'(\zeta, z, \Theta(0,z,w)), \tag{2.1.36}$$

again for $(z, w, \zeta, \xi) \in \mathcal{M}$. Replacing ξ by its value $\Theta(\zeta, z, w)$ on $\mathcal{M}$ and setting $z = 0$, we obtain the power series identity

$$\overline{\Theta}(0, \zeta, \Theta(\zeta, 0, w)) \equiv \Theta'(\zeta, 0, \Theta(0, 0, w)). \tag{2.1.37}$$

But we remember that the functional equations (2.1.25) hold, which enables us to simplify the left hand side and we remember that we have already the relation $\Theta(0, 0, w) \equiv w$, which enables us to simplify the right hand side and we obtain the desired power series identity

$$w \equiv \Theta'(\zeta, 0, w). \tag{2.1.38}$$

This completes the proof of Theorem 2.1.32. □

2.1.39. The formal case. All the previous computations are meaningful in the purely formal case. Especially, Theorems 2.1.22 and 2.1.32 hold true in the formal category.

2.1.40. Conclusion. As we shall observe and confirm in the sequel, the representation of M by complex defining equations is substantially more convenient and more tractable than the representation by real defining equations. We remind that, unless the contrary is explicitely mentioned, *our generic submanifolds will always be of positive codimension $d \geq 1$ and of positive CR dimension $m \geq 1$.*

Notation 2.1.41. Throughout this memoir, we shall fix the following notations:

(1) The generic submanifold M of $\mathbb{C}^n$ will be of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$. The coordinates on $\mathbb{C}^n$ will be denoted by

$$t = (t_1, \dots, t_n) = (z_1, \dots, z_k, w_1, \dots, w_d) \in \mathbb{C}^m \times \mathbb{C}^d = \mathbb{C}^n \tag{2.1.42}$$

and the complex defining equations of M by $\bar{w}_j = \Theta_j(\bar{z}, z, w)$, $j = 1, \dots, d$.

(2) The index $i \in \mathbb{N}$ will run from 1 to n, namely $i = 1, \dots, n$, for instance in the denotation of a vector field $L = \sum_{i=1}^n a_i(t)\, \partial_{t_i}$. The letter i will also be used to denote $\sqrt{-1}$.

(3) The index $j \in \mathbb{N}$ will run from 1 to d, namely $j = 1, \dots, d$.

(4) The index $k \in \mathbb{N}$ will run from 1 to m, namely $k = 1, \dots, m$. The letter k will also often be used to denote another integer varying in $\mathbb{N}$.

(5) Also, we denote $z_k = x_k + iy_k$, $w_j = u_j + iv_j$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$.

2.1.43. Precise definition of local generic submanifolds. Throughout this memoir, we shall often need to localize our geometric constructions. It is therefore necessary to formulate once for all times a firm and precise choice of local representation.

Definition 2.1.44. A *local generic submanifold M of $\mathbb{C}^n$* of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$ is defined in coordinates $t = (z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at a point $p_0 \in M$ as a graph

$$M = \{(z, w) \in \Delta_n(\rho_1) : v_j = \varphi_j(x, y, u),\ j = 1, \dots, d\}, \tag{2.1.45}$$

where the functions φ_j are real algebraic or analytic for $|(x, y, u)| < 2\rho_1$. We also require that for all ρ with $0 \leq \rho \leq \rho_1$, we have $|\varphi(x, y, u)| < \rho$ if $|(x, y, u)| < \rho$, namely M is a "good graph", as shown in FIGURE 2.1.47 below. Of course, after peharps shrinking $\rho_1 > 0$, this condition is automatically satisfied if we adjust the coordinates in order that $T_0 M = \{v = 0\}$. In fact, we shall often prefer the representation by complex defining equations

$$M = \{(z, w) \in \Delta_n(\rho_1) : \bar{w}_j = \Theta_j(\bar{z}, z, w),\ j = 1, \dots, d\}, \tag{2.1.46}$$

where M is again a good graph and the Θ_j converge normally for $|(\bar{z}, z, w)| < 2\rho_1$.

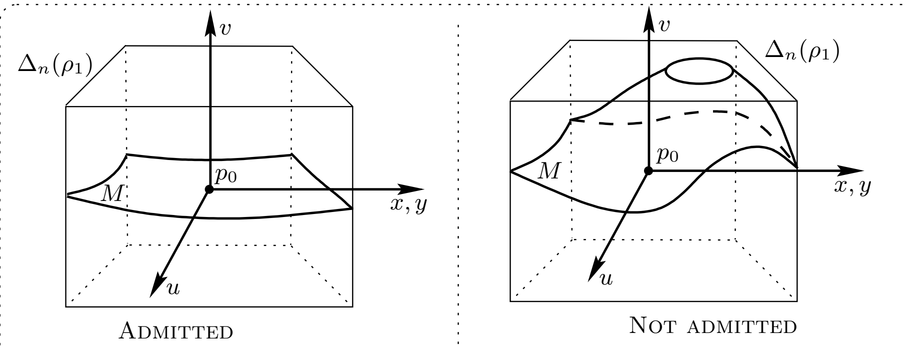

Figure 2.1.47: Local generic submanifold

The *process of localization* consists in choosing subsequent smaller polydiscs $\Delta_n(\rho_2)$, $\Delta_n(\rho_3)$, $\Delta_n(\rho_4)$, $\dots$, with $0 < \cdots < \rho_4 < \rho_3 < \rho_2 < \rho_1$, where the choice of smaller radii ρ_2, ρ_3, ρ_4 depends on the construction of further geometric objects related to M. We say that p_0 is the *central point*. This process may be illustrated symbolically as follows:

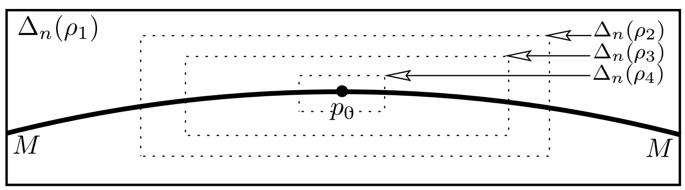

Figure 2.1.48: A local generic submanifold M in the polydisc $\Delta_n(\rho_1)$

If a globally defined connected generic submanifold M is given, for every point $p_0 \in M$, we can obviously *localize M at p_0* by choosing complex affine coordinates vanishing at p_0 such that M in a neighborhood of p_0 is represented as in Definition 2.1.44.

§2.2. Segre varieties and extrinsic complexification

2.2.1. Reality condition. Although we shall mainly work with the complex defining equations, it will be useful on occasion to work with arbitrary real defining equations. So we consider an arbitrary set of d real defining equations for M which we denote by $\rho_j(t, \bar{t}) = 0$, $j = 1, \dots, d$, for instance $\rho_j(t, \bar{t}) := v_j - \varphi_j(z, \bar{z}, u)$, with $\rho_j(0) = 0$. Here, we assume that the complex differentials $\partial\rho_1, \dots, \partial\rho_d$ are linearly independent at the origin, so that M is generic. By reality of the ρ_j, we have $\rho_j(t, \bar{t}) \equiv \overline{\rho_j(t, \bar{t})}$. Developping the ρ_j in power series, we may write $\rho_j(t, \bar{t}) \equiv \sum_{\mu, \, \nu \in \mathbb{N}^n} \rho_{j,\mu,\nu} \, t^\mu \, \bar{t}^\nu$, with $\rho_{j,\mu,\nu} \in \mathbb{C}$. From this functional equation, we deduce that $\rho_{j,\mu,\nu} = \overline{\rho_{j,\nu,\mu}}$ for all j, μ, ν. Conversely, any such converging power series with complex coefficients satisfying $\rho_{j,\mu,\nu} = \overline{\rho_{j,\nu,\mu}}$ takes only real values. As an application, we may write $\overline{\rho_j(t, \bar{t})} \equiv \bar{\rho}_j(\bar{t}, t)$, so the reality condition on ρ_j is simply

$$\rho_j(t, \bar{t}) \equiv \bar{\rho}_j(t, \bar{t}), \qquad j = 1, \dots, d. \tag{2.2.2}$$

Now, let $\tau \in \mathbb{C}^n$ be a new independent variable corresponding to the extrinsic complexification of the variable $\bar{t}$. We shall write symbolically $\tau := (\bar{t})^c$, where the letter “c” stands for the word “complexified”. As (2.2.2) is equivalent to $\rho_{j,\mu,\nu} = \overline{\rho_{j,\nu,\mu}}$, we observe that the complexified series $\rho_j(t, \tau)$ satisfy the important symmetry functional equation

$$\rho_j(t, \tau) \equiv \bar{\rho}_j(\tau, t), \qquad j = 1, \dots, d. \tag{2.2.3}$$

which is simply obtained by replacing $\bar{t}$ by τ in (2.2.2). We can summarize these observations.

Lemma 2.2.4. *As the defining functions $\rho_j(t, \bar{t}) = \sum_{\mu,\nu \in \mathbb{N}^n} \rho_{j,\mu,\nu} \, t^\mu \, \bar{t}^\nu$, $j = 1, \dots, d$ are real power series, we have $\rho_{j,\mu,\nu} = \overline{\rho_{j,\nu,\mu}}$ for all j, μ, ν and*

(1) $\rho_j(t,\tau) \equiv \bar\rho_j(\tau,t)$.
(2) $\rho_j(t,\tau)=0$ *if and only if* $\rho_j(\bar\tau,\bar t)=0$.

Property **(2)** follows trivially from **(1)** and will be useful later.

2.2.5. Classical Segre varieties and conjugate Segre varieties. Let $\rho_1>0$ such that the ρ_j converge normally in $\Delta_{2n}(2\rho_1)$ and consider the zero-set $M:=\{t\in\Delta_n(\rho_1):\ \rho_j(t,\bar t)=0\}$. Let $\rho_j'(t,\bar t)=0$ be an another choice of defining equations for the same generic submanifold. It follows that there exists an invertible $d\times d$ matrix $a(t,\bar t)$ of real power series (of the same regularity as M, namely algebraic or analytic) such that $\rho'(t,\bar t)\equiv a(t,\bar t)\,\rho(t,\bar t)$. With this relation, we observe easily that the (classical) *Segre variety* associated to a point $p\in\Delta_n(\rho_1)$ with coordinates $t_p=(t_{1p},\dots,t_{np})\in\mathbb{C}^n$, defined as in [34] by

$$S_{\bar t_p}:=\{t\in\Delta_n(\rho_1):\ \rho_j(t,\bar t_p)=0,\ j=1,\dots,d\}, \tag{2.2.6}$$

does not depend on the choice of defining equations for M, namely we also have $S_{\bar t_p}=\{t\in\Delta_n(\rho_1):\ \rho'(t,\bar t_p)=0\}$. In the litterature, this Segre variety is usually denoted by Q_p, *cf.* [1], [8], [9], [12], [22], [23], [27], [30], [31], [34], [35], [36]. Here, we choose instead the letter "S", because it is the initial of the name Segre. More importantly, we stress the notation $S_{\bar t_p}$ or $S_{\bar p}$, and not S_p, *with the bar of complex conjugation over* t_p, as in the expresson $\rho_j(t,\bar t_p)$.

In fact, for reasons of symmetry, we are also led to define the *conjugate Segre variety* by

$$\overline{S}_{t_p}:=\{\bar t\in\Delta_n(\rho_1):\ \rho_j(t_p,\bar t)=0,\ j=1,\dots,d\}. \tag{2.2.7}$$

To the author's knowledge, conjugate Segre varieties are not considered in the literature. As a matter of fact, if for an arbitrary subset $E\subset\mathbb{C}^n$ we define the set of conjugate points of E by $\overline{E}:=\{\bar t:\ t\in E\}$, it follows that $\overline{S}_{t_p}$ is just the set of conjugate points of $S_{\bar t_p}$, as the reader may verify thanks to Lemma 2.2.4. It follows that we can write

$$\overline{S}_{t_p}=\overline{S}_{\bar{\bar t}_p}=\overline{S_{\bar t_p}} \qquad \text{and} \qquad S_{\bar t_p}=\overline{\overline{S}}_{\bar t_p}=\overline{\overline{S}_{t_p}}. \tag{2.2.8}$$

with the complex conjugation operator acting separately as an involution over the letter S and over its argument t_p. Finally, we would like to observe that the third (tempting) definition $\{t\in\Delta_n(\rho_1):\ \rho_j(t_p,\bar t)=0,\ 1,\dots,d\}$ (instead of (2.2.7)) does not provide us with the correct definition of conjugate Segre variety, because Lemma 2.2.4 implies that this set coincides in fact with $S_{\bar t_p}$.

We claim that both Segre and conjugate Segre varieties are biholomorphic invariants of M. Indeed, let $t'=h(t)$ be a local biholomorphic change of coordinates and denote by $t=h'(t')$ its inverse, by $M':=h(M)$ and by $\rho_j'(t',\bar t'):=\rho_j(h'(t'),\bar h'(\bar t'))$ the defining equations of M', for $j=1,\dots,d$. Since h maps M into M', according to §2.1.5, there exists an invertible $d\times d$ matrix $a(t,\bar t)$ of power series such that $\rho'(h(t),\bar h(\bar t))\equiv a(t,\bar t)\,\rho(t,\bar t)$. From this relation, it follows easily that $h(S_{\bar t_p})=S'_{\bar h(\bar t_p)}$ and $h(\overline{S}_{t_p})=\overline{S}'_{\bar h(\bar t_p)}$, which proves the claim. Finally, we collect some classical properties.

Lemma 2.2.9. *The following four properties are satisfied:*

(1) $q\in S_{\bar t_p}$ *if and only if* $p\in S_{\bar t_q}$.
(2) $p\in S_{\bar t_p}$ *if and only if* $p\in M$.
(3) $\bar q\in\overline{S}_{t_p}$ *if and only if* $\bar p\in\overline{S}_{t_q}$.
(4) $\bar p\in\overline{S}_{t_p}$ *if and only if* $p\in M$.

Proof. Indeed, applying Lemma 2.2.4 **(2)**, we have $\rho(t_q,\bar t_p)=0$ if and only if $\rho(t_p,\bar t_q)=0$, which yields **(1)** and **(3)**. Also, we have $\rho(t_p,\bar t_p)=0$ if and only if $t_p\in M$, which yields **(2)** and **(4)**. □

2.2.10. Extrinsic complexification. Now, let $\zeta \in \mathbb{C}^m$ and $\xi \in \mathbb{C}^d$ denote some new independent coordinates corresponding to the complexification of the variables $\bar{z}$ and $\bar{w}$, which we denote symbolically by $\zeta := (\bar{z})^c$ and $\xi := (\bar{w})^c$, where the letter "c" stands for the word "complexified". We also write $\tau := (\bar{t})^c$, so $\tau = (\zeta, \xi) \in \mathbb{C}^n$. The *extrinsic complexification* $\mathcal{M} := (M)^c$ of M is the complex d-codimensional submanifold defined precisely by

$$\mathcal{M} := \{(z, w, \zeta, \xi) \in \Delta_n(\rho_1) \times \Delta_n(\rho_1) : \xi_j = \Theta_j(\zeta, z, w), \ j = 1, \dots, d\}. \tag{2.2.11}$$

Let σ denote the antiholomorphic involution defined by $\sigma(t, \tau) := (\bar{\tau}, \bar{t})$. Since by Lemma 2.1.27, there exists an invertible $d \times d$ matrix $a(t, \tau)$ of power series such that $w - \overline{\Theta}(z, \zeta, \xi) \equiv a(t, \tau)[\xi - \Theta(\zeta, z, w)]$, we see that σ maps $\mathcal{M}$ bi-antiholomorphically onto $\mathcal{M}$. In this chapter, we shall essentially deal with $\mathcal{M}$ instead of dealing with M. In fact, M clearly imbeds in $\mathcal{M}$ as the intersection of $\mathcal{M}$ with the antiholomorphic diagonal defined by $\underline{\Lambda} := \{(t, \tau) \in \mathbb{C}^n \times \mathbb{C}^n : \tau = \bar{t}\}$. Also, we shall very frequently use the fact that $\mathcal{M}$ can be represented by the following two equivalent families of d complex defining equations:

$$\mathcal{M} : \quad w_j = \overline{\Theta}_j(z, \zeta, \xi), \qquad j = 1, \dots, d, \qquad \text{or} \qquad \xi_j = \Theta_j(\zeta, z, w), \qquad j = 1, \dots, d. \tag{2.2.12}$$

§2.3. Complexified Segre varieties and complexified CR vector fields

2.3.1. Complexified Segre varieties. Next, for $\tau_p \in \Delta_n(\rho_1)$ fixed, we define the associated *complexified Segre variety* by

$$\mathcal{S}_{\tau_p} := \{(t, \tau) \in \Delta_{2n}(\rho_1) : \tau = \tau_p, \, w_j = \overline{\Theta}_j(z, \tau_p), \, j = 1, \dots, d\}. \tag{2.3.2}$$

We shall write symbolically $\mathcal{S}_{\tau_p} = (S_{\bar{t}_p})^c$. Clearly, $\mathcal{S}_{\tau_p}$ is an m-dimensional submanifold contained in $\mathcal{M}$ and it coincides in fact with the intersection of $\mathcal{M}$ with the horizontal slice $\{(t, \tau) : \tau = \tau_p\}$. Analogously, for $t_p \in \Delta_n(\rho_1)$ fixed, we define the associated *conjugate complexified Segre variety* by

$$\underline{\mathcal{S}}_{t_p} := \{(t, \tau) \in \Delta_{2n}(\rho_1) : t = t_p, \, \xi_j = \Theta_j(\zeta, t_p), \, j = 1, \dots, d\}. \tag{2.3.3}$$

Clearly again, $\underline{\mathcal{S}}_{t_p}$ is an m-dimensional submanifold contained in $\mathcal{M}$ and it coincides in fact with the intersection of $\mathcal{M}$ with the vertical slice $\{(t, \tau) : t = t_p\}$.

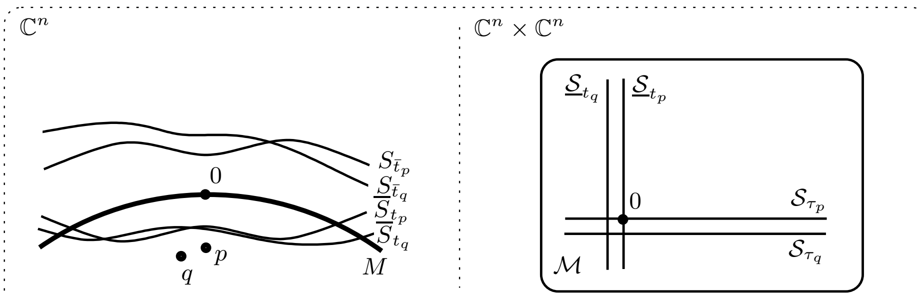

Figure 2.3.4: Non foliation property in ambient space
Complexification as a blowing up yielding two foliations

It is very important to notice that the ambient Segre varieties $S_{\bar{t}_p}$ and $\overline{S}_{t_p}$ are extrinsic to M: they lie in general outside M, even in the Levi-flat case. Moreover, the union of $\cup_{p \in \Delta_n(\rho_1)} S_{\bar{p}}$ never makes a foliation by m-dimensional submanifolds. These assertions may easily be checked by inspecting the Levi-flat hyperplane $\{\operatorname{Im} w = 0\}$ in $\mathbb{C}^n$ and the Heisenberg sphere $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$ in $\mathbb{C}^n$. Fortunately, by the strange miracle of extrinsic complexification, *we blow-up the two unions $\cup_p S_{\bar{p}}$ and $\cup_p \overline{S}_p$ in a double foliation of $\mathcal{M}$ by complex m-dimensional Segre varieties* (as explained in Theorem 2.3.9 below). This geometric observation is of utmost importance, is illustrated symbolically in Figure 2.3.4 and will be explained more closely in the next subparagraphs.

2.3.5. Complexified CR vector fields. We consider the following "natural" basis of $(1,0)$ vector fields tangent to M:

$$L_k := \frac{\partial}{\partial z_k} + \sum_{j=1}^{d} \frac{\partial \overline{\Theta}_j}{\partial z_k}(z,\bar{z},\bar{w})\,\frac{\partial}{\partial w_j}, \qquad k=1,\dots,m. \tag{2.3.6}$$

One verifies immediately that $L_k(w_j - \overline{\Theta}_j(z,\bar{z},\bar{w})) \equiv 0$ for $k=1,\dots,m$ and $j=1,\dots,d$. We also consider the conjugates of these vector fields, which form a basis of the $(0,1)$ vector fields tangent to M:

$$\overline{L}_k := \frac{\partial}{\partial \bar{z}_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial \bar{z}_k}(\bar{z},z,w)\,\frac{\partial}{\partial \bar{w}_j}, \qquad k=1,\dots,m. \tag{2.3.7}$$

Again obviously, we verify that $\overline{L}_k(\bar{w}_j - \Theta_j(\bar{z},z,w)) \equiv 0$ for $k=1,\dots,m$ and $j=1,\dots,d$. Of course, this second system of relations is the conjugate of the first.

By complexification, the vector fields behave as follows: we write $[\chi(t,\bar{t})]^c = \chi(t,\tau)$, if $\chi(t,\bar{t})$ is a real analytic function of $(t,\bar{t})$ and $\left[\sum_{j=1}^{n} a_j(t,\bar{t})\,\partial/\partial t_j + \sum_{j=1}^{n} b_j(t,\bar{t})\,\partial/\partial \bar{t}_j\right]^c := \sum_{j=1}^{n} a_j(t,\tau)\,\partial/\partial t_j + \sum_{j=1}^{n} b_j(t,\tau)\,\partial/\partial \tau_j$. It follows that $(L\chi)^c = L^c\chi^c$.

Consequently, we can complexify the pair of conjugate generating families of CR vector fields tangent to M given by (2.3.6) and (2.3.7), namely the vector fields $L_1,\dots,L_m$ and their conjugates $\overline{L}_1,\dots,\overline{L}_m$ above. Their complexification yields a pair of collections of m vector fields defined explicitely over $\Delta_n(\rho_1)\times\Delta_n(\rho_1)$ by

$$\begin{cases} \mathcal{L}_k := \dfrac{\partial}{\partial z_k} + \displaystyle\sum_{j=1}^{d} \frac{\partial \overline{\Theta}_j}{\partial z_k}(z,\zeta,\xi)\,\frac{\partial}{\partial w_j}, & k=1,\dots,m, \\ \underline{\mathcal{L}}_k := \dfrac{\partial}{\partial \zeta_k} + \displaystyle\sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial \zeta_k}(\zeta,z,w)\,\frac{\partial}{\partial \xi_j}, & k=1,\dots,m. \end{cases} \tag{2.3.8}$$

We write $\mathcal{L}_k = (L_k)^c$ and $\underline{\mathcal{L}}_k = (\overline{L}_k)^c$. The reader may check directly that $\mathcal{L}_k(w_j - \overline{\Theta}_j(z,\zeta,\xi)) \equiv 0$ (this relation also holds by complexication), which shows that the vector fields $\mathcal{L}_k$ are tangent to $\mathcal{M}$. Similarly, $\underline{\mathcal{L}}_k(\xi_j - \Theta_j(\zeta,z,w)) \equiv 0$, so the vector fields $\underline{\mathcal{L}}_k$ are also tangent to $\mathcal{M}$. Of course, all of this is obvious, but we prefer to start slowly. Furthermore, we may check the commutation relations $[L_{k_1}, L_{k_2}] = 0$, $[\overline{L}_{k_1}, \overline{L}_{k_2}] = 0$, $[\mathcal{L}_{k_1}, \mathcal{L}_{k_2}] = 0$ and $[\underline{\mathcal{L}}_{k_1}, \underline{\mathcal{L}}_{k_2}] = 0$ for all $k_1, k_2 = 1,\dots,m$. By the theorem of Frobenius, it follows that the two m-dimensional distributions spanned by the two collections of m vector fields $\{\mathcal{L}_k\}_{1\le k\le m}$ and $\{\underline{\mathcal{L}}_k\}_{1\le k\le m}$ have the integral manifold property. This is not astonishing, due to the fact that the vector fields $\mathcal{L}_k$ are just the vector fields tangent to the intersection of $\mathcal{M}$ with the sets $\{\tau = \tau_p = ct.\}$, which are the m-dimensional complexified Segre varieties $\mathcal{S}_{\tau_p}$ already defined above. Similarly, the $\underline{\mathcal{L}}_k$ have the conjugate complexified Segre varieties $\underline{\mathcal{S}}_{t_p}$ as integral manifolds. Hence in fact, we do not have to appeal to the theorem of Frobenius.

All the geometric observations which we have done so far may be gathered in the following statement just below. We shall frequently use the abbreviations $\mathcal{L} = \{\mathcal{L}_k\}_{1\le k\le m}$ and $\underline{\mathcal{L}} = \{\underline{\mathcal{L}}_k\}_{1\le k\le m}$. We denote by $\pi_t : (t,\tau) \mapsto t$ and $\pi_\tau : (t,\tau) \mapsto \tau$ the two canonical projections.

Theorem 2.3.9. *Let $\mathcal{M} = (M)^c$ be as above and let $\mathcal{L}_k$, $k=1,\dots,m$ be a basis of complexified $(1,0)$ vector fields tangent to M and let $\underline{\mathcal{L}}_k$, $k=1,\dots,m$, be their complexified conjugates. Recall that $\{\mathcal{L}_k\}_{1\le k\le m}$ and $\{\underline{\mathcal{L}}_k\}_{1\le k\le m}$ are Frobenius-integrable. Then the following four properties hold true:*

- **(1)** *$\mathcal{L}$ and $\underline{\mathcal{L}}$ induce naturally two local flow foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ of $\mathcal{M}$.*
- **(2)** *If $\sigma(t,\tau) := (\bar{\tau},\bar{t})$, then $\sigma(\mathcal{F}_{\mathcal{L}}) = \mathcal{F}_{\underline{\mathcal{L}}}$ and their two leaves passing through a point $p^c = (t_p,\bar{t}_p) \in \mathbb{C}^n\times\mathbb{C}^n$ satisfy $\mathcal{F}_{\mathcal{L}}(p^c)\cap\mathcal{F}_{\underline{\mathcal{L}}}(p^c) = p^c$.*

(3) *The fibers of the projections* π_t *and* π_τ *also coincide with the leaves of the flow foliations* $\mathcal{F}_{\underline{\mathcal{L}}}$ *and* $\mathcal{F}_{\mathcal{L}}$*, respectively.*

(4) *The leaves of the foliation* $\mathcal{F}_{\mathcal{L}}$ *are the Segre varieties* $\mathcal{S}_{\tau_p}$ *and the leaves of the foliation* $\mathcal{F}_{\underline{\mathcal{L}}}$ *are the conjugate Segre varieties* $\underline{\mathcal{S}}_{t_p}$:

$$\mathcal{F}_{\mathcal{L}} = \bigcup_{\tau_p \in \Delta_n(\rho_1)} \mathcal{S}_{\tau_p} \quad \text{and} \quad \mathcal{F}_{\underline{\mathcal{L}}} = \bigcup_{t_p \in \Delta_n(\rho_1)} \underline{\mathcal{S}}_{t_p}. \tag{2.3.10}$$

In other words, the leaves of these two flow foliations are the two families of complexified (*conjugate*) *Segre varieties. In symbolic representation, for these two foliations, we have the correspondence*:

$$\textit{CR-flow foliations of } \mathcal{M} \iff \textit{Foliations by complexified Segre varieties.} \tag{2.3.11}$$

2.3.12. Conclusion. The following symbolic picture summarizes this geometrical theorem. However, we warn the reader that the codimension $d \geq 1$ of the union of the two foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ in $\mathcal{M}$ is not rendered visible in this two-dimensional figure. A three-dimensional FIGURE 2.3.3 will be provided below.

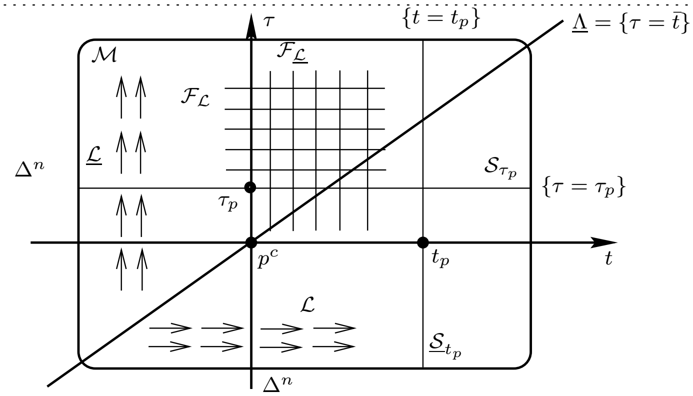

FIGURE 2.3.13: GEOMETRY OF THE COMPLEXIFICATION $\mathcal{M}$

2.4. MULTIPLE FLOWS AND SEGRE CHAINS

2.4.1. Pair of complex flows. Now, we introduce the "multiple" flows of the two collections of conjugate vector fields $(\mathcal{L}_k)_{1 \leq k \leq m}$ and $(\underline{\mathcal{L}}_k)_{1 \leq k \leq m}$. This multiple flow will be used frequently throughout the next chapters of Part II. Precisely, for an arbitrary point $p = (w_p, z_p, \zeta_p, \xi_p) \in \mathcal{M}$ and for an arbitrary complex "multitime" parameter $z_1 = (z_{1,1}, \ldots, z_{1,m}) \in \mathbb{C}^m$, we define

$$\begin{cases} \mathcal{L}_{z_1}(z_p, w_p, \zeta_p, \xi_p) & := \exp(z_1 \mathcal{L})(p) := \exp(z_{1,1} \mathcal{L}_1(\cdots(\exp(z_{1,m}\mathcal{L}_m(p)))\cdots)) := \\ & := \big(z_p + z_1,\, \overline{\Theta}(z_p + z_1, \zeta_p, \xi_p),\, \zeta_p,\, \xi_p\big). \end{cases} \tag{2.4.2}$$

With this formal definition, there exists a maximal connected open subset Ω of $\mathcal{M} \times \mathbb{C}^m$ containing $\mathcal{M} \times \{0\}$ such that $\mathcal{L}_{z_1}(p) \in \mathcal{M}$ for all $(z_1, p) \in \Omega$. Analogously, for (ζ_1, p) running in a similar open subset $\underline{\Omega}$, we may define the map

$$\underline{\mathcal{L}}_{\zeta_1}(z_p, w_p, \zeta_p, \xi_p) := \big(z_p,\, w_p,\, \zeta_p + \zeta_1,\, \Theta(\zeta_p + \zeta_1, z_p, w_p)\big). \tag{2.4.3}$$

Of course, the two mappings (2.4.2) and (2.4.3) are of the same regularity as M, namely they are algebraic or analytic.

2.4.4. Segre chains. Now, let us start from the point p being the origin and let us move alternately in the (horizontal) direction of $\mathcal{F}_{\mathcal{L}}$ (namely the direction of $\mathcal{S}$) and in the (vertical) direction of $\mathcal{F}_{\underline{\mathcal{L}}}$ (namely the direction of $\underline{\mathcal{S}}$). More precisely, we consider the two maps $\Gamma_1(z_1) := \mathcal{L}_{z_1}(0)$ and $\underline{\Gamma}_1(z_1) := \underline{\mathcal{L}}_{z_1}(0)$, where $z_1 \in \mathbb{C}^m$. Next, we start from these endpoints and we move in the other direction. More precisely, we consider the two maps

$$\Gamma_2(z_1, z_2) := \underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0)), \qquad \underline{\Gamma}_2(z_1, z_2) := \mathcal{L}_{z_2}(\underline{\mathcal{L}}_{z_1}(0)), \tag{2.4.5}$$

where $z_1, z_2 \in \mathbb{C}^m$. Also, we define $\Gamma_3(z_1, z_2, z_3) := \mathcal{L}_{z_3}(\underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0)))$, *etc.* For the sake of concreteness, let us exhibit the complete expressions of Γ_1, Γ_2 and Γ_3, which follows by a repeated application of formulas (2.4.2) and (2.4.3):
(2.4.6)

$$\begin{cases} \mathcal{L}_{z_1}(0) = \big(z_1, \, \overline{\Theta}(z_1, 0, 0), \, 0, \, 0\big). \\ \underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0)) = \big(z_1, \, \overline{\Theta}(z_1, 0, 0), \, z_2, \, \Theta(z_2, z_1, \overline{\Theta}(z_1, 0, 0))\big). \\ \mathcal{L}_{z_3}(\underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(0))) = \big(z_1 + z_3, \, \overline{\Theta}(z_1 + z_3, z_2, \Theta(z_2, z_1, \overline{\Theta}(z_1, 0, 0))), \, z_2, \, \Theta(z_2, z_1, \overline{\Theta}(z_1, 0, 0))\big). \end{cases}$$

By induction, for every integer $k \in \mathbb{N}$ with $k \geq 1$, we obtain two maps $\Gamma_k(z_1, \dots, z_k)$ and $\underline{\Gamma}_k(z_1, \dots, z_k)$, where $z_1, \dots, z_k \in \mathbb{C}^m$. Clearly, there are precise combinatorial formulas generalizing (2.4.6). In the sequel, we shall often use the notation $z_{(k)} := (z_1, \dots, z_k) \in \mathbb{C}^{mk}$. We shall call the map Γ_k the *k-th Segre chain* and the map $\underline{\Gamma}_k$ the *conjugate k-th Segre chain*. Since $\Gamma_k(0) = \underline{\Gamma}_k(0) = 0$, for every $k \in \mathbb{N}_*$, there exists a sufficiently small open polydisc $\Delta_{mk}(\delta_k)$ centered at the origin in $\mathbb{C}^{mk}$ with $\delta_k > 0$ such that $\Gamma_k(z_{(k)})$ and $\underline{\Gamma}_k(z_{(k)})$ belong to $\mathcal{M}$ for all $z_{(k)} \in \Delta_{mk}(\delta_k)$.

We also exhibit a simple link between the maps Γ_k and $\underline{\Gamma}_k$. Let σ be the antiholomorphic involution defined by $\sigma(t, \tau) := (\bar{\tau}, \bar{t})$. Since $w = \overline{\Theta}(z, \zeta, \xi)$ if and only if $\xi = \Theta(\zeta, z, w)$, this involution maps $\mathcal{M}$ onto $\mathcal{M}$ and it also fixes the antidiagonal $\underline{\Lambda}$ pointwise. Using the definitions (2.4.2) and (2.4.3), we see readily that $\sigma(\mathcal{L}_{z_1}(0)) = \underline{\mathcal{L}}_{\bar{z}_1}(0)$. It follows generally that $\sigma(\Gamma_k(z_{(k)})) = \underline{\Gamma}_k(\overline{z_{(k)}})$. To give a concrete illustration, we may compute the explicit expressions of $\underline{\Gamma}_1$, $\underline{\Gamma}_2$ and $\underline{\Gamma}_3$ and compare with (2.4.6):
(2.4.7)

$$\begin{cases} \underline{\mathcal{L}}_{z_1}(0) = (0, \, 0, \, z_1, \, \Theta(z_1, 0, 0)). \\ \mathcal{L}_{z_2}(\underline{\mathcal{L}}_{z_1}(0)) = \big(z_2, \, \overline{\Theta}(z_2, z_1, \Theta(z_1, 0, 0)), \, z_1, \, \Theta(z_1, 0, 0)\big). \\ \underline{\mathcal{L}}_{z_3}(\mathcal{L}_{z_2}(\underline{\mathcal{L}}_{z_1}(0))) = \big(z_2, \, \overline{\Theta}(z_2, z_1, \Theta(z_1, 0, 0)), \, z_1 + z_3, \, \Theta(z_1 + z_3, z_2, \overline{\Theta}(z_2, z_1, \Theta(z_1, 0, 0)))\big) \end{cases}$$

Also, we observe that $\Gamma_{k+1}(z_{(k)}, 0) = \Gamma_k(z_{(k)})$, since $\mathcal{L}_0$ and $\underline{\mathcal{L}}_0$ coincide with the identity map by (2.4.2) and (2.4.3). So, for k increasing, the ranks at the origin of the maps Γ_k are increasing. We now introduce the following important definition.

Definition 2.4.8. The generic submanifold M is said to be *minimal* at p if the maps Γ_k are of (maximal possible) rank equal to $2m + d = \dim_{\mathbb{C}} \mathcal{M}$ at the origin in $\Delta_{mk}(\delta_k)$ for all k large enough.

In other words, M is minimal at p if and only if sufficiently high order Segre chains are submersive. Equivalently, sufficiently high order conjugate Segre chains are submersive. In the next Section 2.5, we shall show that minimality is characterized by the fact that the maps are of generic rank equal to $2m + d$ for all k large enough. First of all, in order to enlighten this definition, we shall prove a general *local orbit theorem* in the spirit of H.J. Sussmann [32].

2.4.9. Local orbit theorem in the $\mathbb{K}$-analytic category. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\Delta := \{x \in \mathbb{K} : |x| < 1\}$ and $r\Delta := \{x \in \mathbb{K} : |x| < r\}$. Let $n \in \mathbb{N}$ with $n \geq 1$. In Δ^n equipped with coordinates $x = (x_1, \dots, x_n)$, we consider the origin as a center point. Let $\mathbb{L} = \{L^a\}_{1 \leq a \leq A}$, $A \geq 1$, be a *finite* system of nonzero vector fields defined all over Δ^n. *We do not require that this set is stable under taking linear combinations with coefficients being analytic or algebraic functions over* Δ^n. Let $L \in \mathbb{L}$. As previously, we shall simply denote the flow map of L by $(t, x) \mapsto L_t(x) \equiv \exp(tL)(x)$.

We recall the defining properties of the flow map: $L_0(x) = x$ and $\frac{d}{dt}(L_t(x)) = L(L_t(x))$, where $L(x')$ denotes the value of L at x'. As is known, the $\mathbb{K}$-algebraic case is exceptional because the flow of a vector field L having $\mathbb{K}$-algebraic coefficients is in general transcendent. This is why we shall in fact make two kinds of precise regularity assumptions:

`(Algebraic)`: The coefficients of all elements of $\mathbb{L}$ are $\mathbb{K}$-algebraic *and moreover* their flow if also $\mathbb{K}$-algebraic. In fact, the $\mathbb{K}$-algebraicity of the flow implies the $\mathbb{K}$-algebraicity of the coefficients.

`(Analytic)`: The coefficients of all elements of $\mathbb{L}$ are $\mathbb{K}$-analytic power series centered at the origin converging in Δ^n, whence their flow is $\mathbb{K}$-analytic.

Choose now r with $0 < r \leq 1/2$. We first define finite concatenations of flow mappings of vector fields in $\mathbb{L}$ as follows. If $k \in \mathbb{N}_*$, $L = (L^1, \dots, L^k) \in \mathbb{L}^k$, $t = (t_1, \dots, t_k) \in \mathbb{K}^k$ and $x \in (r\Delta)^n$, we use again the notation $L_t(x) = L^k_{t_k}(\cdots(L^1_{t_1}(x))\cdots)$ whenever the composition is defined. Anyway, after bounding $k \leq 3n$, it is clear that there exists $\delta > 0$ such that all maps $(t, x) \mapsto L_t(x)$ are well-defined for $t \in (2\delta\Delta)^k$, $x \in (\frac{r}{2}\Delta)^n$ and satisfy $L_t(x) \in (r\Delta)^n$. By definition, the point $x' = L^k_{t_k} \circ \cdots \circ L^1_{t_1}(x)$ is the endpoint of a piecewise smooth algebraic or analytic curve with origin x: it consists in following L^1 during time t_1, following L^2 during time t_2 ... and following L^k during time t_k.

Next, we shall say that an embedded small piece of $\mathbb{K}$-manifold $N \subset \Delta^n$ passing through the origin (which is either $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic) is a *weak* $\mathbb{L}$*-integral manifold* if $T_xN \supset \mathbb{L}(x)$ for all $x \in N$. In the formal case, however, this condition is meaningless. Equivalently, we mean that for each $L \in \mathbb{L}$, $L|_N$ is tangent to N. Now, this new condition makes sense in the formal case. We shall in fact consider the formal case afterwards, as a generalization of the algebraic and analytic cases. In particular, it clearly ensues from the tangency of $L|_N$ to N that any integral curve of an element $L \in \mathbb{L}$ with origin a point $x \in N$ which belongs to N stays in N.

Now, we introduce the following special definitions. The $\mathbb{L}$*-orbit of* 0 *in* Δ^n, denoted by $\mathcal{O}_{\mathbb{L}}(0)$, is the set of all points $L_t(0) \in (r\Delta)^n$ for all $t \in (\delta\Delta)^k$, $k \leq 3n$. The reason why we bound $k \leq 3n$ will be clear afterwards. We shall say that the open set $(r\Delta)^n$ is $\mathbb{L}$*-minimal at* 0 if $\mathcal{O}_{\mathbb{L}}(0)$ contains a polydisc $(\varepsilon\Delta)^n$, where $\varepsilon > 0$. If $L = (L^1, \dots, L^k) \in \mathbb{L}^k$ with $k \leq 3n$, we shall denote by $\Gamma_L(t)$ the mapping $t \mapsto L^k_{t_k}(\cdots(L^1_{t_1}(0))\cdots)$.

We can now state the *local orbit theorem*. By induction, we shall construct a special sequence of vector fields $L^{*k} := (L^{*1}, \dots, L^{*k}) \in \mathbb{L}^k$, $k \in \mathbb{N}_*$. We state a long but progressively explained theorem with the purpose of exhibiting all the relevant informations.

Theorem 2.4.10. *As above, let* $\mathbb{L} = \{L^a\}_{1 \leq a \leq A}$, $A \in \mathbb{N}_*$, *be a finite nonempty set of vector fields which are defined over* Δ^n *and satisfy one of the two regularity assumptions* `(Algebraic)` *or* `(Analytic)`. *Then there exists an integer* $e \geq 1$ *and a multiplet of vector fields* $L^* = (L^{*1}, \dots, L^{*e}) \in \mathbb{L}^e$ *such that the following seven properties hold*:

- **(1)** *For every* $k = 1, \dots, e$, *the map* $(t_1, \dots, t_k) \mapsto L^{*k}_{t_k}(\cdots(L^{*1}_{t_1}(0))\cdots)$ *is of generic rank equal to* k.
- **(2)** *For every arbitrary element* $L' \in \mathbb{L}$, *the map* $(t_1, \dots, t_e, t') \mapsto L'_{t'}(L^{*e}_{t_e}(\cdots(L^{*1}_{t_1}(0))\cdots))$ *is of generic rank* e, *hence* e *is the maximal possible generic rank.*
- **(3)** *There exists an element* $t^* \in \Delta^e$ *arbitrarily close to the origin which is of the special form* $(t_1^*, \dots, t_{e-1}^*, 0)$, *namely with* $t_e^* = 0$, *and there exists an open connected neighborhood* ω^* *of* t_* *in* Δ^e *such that the map* $\Gamma_{L^*} : t \mapsto L^{*e}_{t_e}(\cdots(L^{*1}_{t_1}(0)))$ *is of constant rank* e *over* ω^*.
- **(4)** *After setting* $L^* := (L^{*1}, \dots, L^{*e})$, $K^* := (L^{*e-1}, \dots, L^{*1})$ *and* $s^* := (-t_{e-1}^*, \dots, -t_1^*)$, *we then have* $K^*_{s^*} \circ L^*_{t^*}(0) = 0$. *Furthermore, the map* $\psi : \omega^* \to \Delta^n$ *defined by* $\psi : t \mapsto K^*_{s^*} \circ L^*_t(0)$ *is of constant rank equal to* e *over the domain* ω^*.
- **(5)** *Then the image* $\psi(\omega^*)$ *is a piece of* $\mathbb{K}$*-manifold passing through the origin which is either* $\mathbb{K}$*-algebraic or* $\mathbb{K}$*-analytic, because the flows of the elements of* $\mathbb{L}$ *are* $\mathbb{K}$*-algebraic or* $\mathbb{K}$*-analytic.*

(6) *This piece of $\mathbb{K}$-manifold is a weak $\mathbb{L}$-integral manifold. Furthermore, very weak $\mathbb{L}$-integral manifold passing through* 0 *must contain* $\mathcal{O}_{\mathbb{L}}(0)$ *in a neighborhood of* 0.

(7) *There exists* $\varepsilon > 0$ *such that* $\mathcal{O}_{\mathbb{L}}(0) \cap (\varepsilon\Delta)^n = \psi(\omega^*)$.

In conclusion, the local orbit $\mathcal{O}_{\mathbb{L}}(0)$ *is represented by the small* $\mathbb{K}$*-manifold* $\psi(\omega^*)$ *and its dimension* e *is characterized by the generic rank properties* **(1)** *and* **(2)**.

Proof. If all vector fields in $\mathbb{L}$ vanish at the origin, then $\mathcal{O}_{\mathbb{L}}(0) = \{0\}$. We now exclude this possibility. Choose a vector field $L^{*1} \in \mathbb{L}$ which does not vanish at 0. The map $t_1 \mapsto L^{*1}_{t_1}(0)$ is of generic rank one. If there exists $L' \in \mathbb{L}$ such that the map $(t_1, t') \mapsto L'_{t'}(L^{*1}_{t_1}(0))$ is of generic rank two, we choose one such L' and we denote it by L^{*2}. Continuing this process, we get vector fields $L^{*1}, \dots, L^{*e}$ satisfying properties **(1)** and **(2)**. Since the generic rank of the map $\Gamma_{L^*} : (t_1, \dots, t_e) \mapsto L^{*e}_{t_e}(\cdots(L^{*1}_{t_1}(0))\cdots)$ equals e, and since this map is either $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic, there exist elements $t^* \in \Delta^e$ arbitrarily close to the origin such that its rank at t_* equals e. We claim that we can moreover choose t^* to be of the special form $(t^*_1, \dots, t^*_{e-1}, 0)$, *i.e.* with $t^*_e = 0$. This is a consequence of the following lemma.

Lemma 2.4.11. *Let* $n \in \mathbb{N}_*$, $e \in \mathbb{N}_*$, $t \in \mathbb{K}^e$ *and* $t \mapsto \varphi(t) = (\varphi_1(t), \dots, \varphi_n(t)) \in \mathbb{K}^n$ *be a mapping of generic rank equal to* $(e-1)$ *which is either* $\mathbb{K}$*-algebraic or* $\mathbb{K}$*-analytic. Let* $L' \in \mathbb{L}$ *and assume that the mapping* $\psi : (t, t') \mapsto L'_{t'}(\varphi(t))$ *has generic rank* e. *Then there exists a point* $(t^*, 0)$ *at which the rank of* ψ *is equal to* e.

Proof. Suppose on the contrary that at all points $(t^*, 0)$, the map $(t, t') \mapsto L'_{t'}(\varphi(t))$ has rank $\leq e-1$. Choose t^* arbitrarily close to zero at which φ has maximal, hence locally constant rank $(e-1)$. By the rank theorem, there exists a neighborhood ω^* of t^* in $\mathbb{K}^e$ such that $N := \varphi(\omega^*)$ is a small piece of $\mathbb{K}$-manifold which is either $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic. Let $L' \in \mathbb{L}$. Since by assumption the rank of $(t, t') \mapsto L'_{t'}(\varphi(t))$ equals $(e-1)$ at every point $(t^*, 0) \in \omega^* \times \mathbb{K}$, it follows necessarily that L' is tangent to N. Consequently, the flow of L' stabilizes N. Finally, for $\varphi(t) \in N$, we have $L'_{t'}(\varphi(t)) \in N$ also, whence the rank of $(t, t') \mapsto L'_{t'}(\varphi(t))$ is less than or equal to $\dim_{\mathbb{K}} N = e-1$ at every point in a neighborhood of $(t^*, 0)$ in $\mathbb{K}^n \times \mathbb{K}$. We have proved that the mapping $(t, t') \mapsto L'_t(\varphi(t))$ is of generic rank $(e-1)$ in a neighborhood of $(t^*, 0)$ in $\mathbb{K}^e \times \mathbb{K}$, hence everywhere by the principle of analytic continuation, which contradicts the assumption that it has generic rank equal to e. This completes the proof. □

End of the proof of Theorem 2.4.9. Now, we choose a point $(t^*_1, \dots, t^*_{e-1}, 0) \in \Delta^e$ arbitrarily close to 0 at which the rank of $t \mapsto L^{*e}_{t_e}(\cdots(L^{*1}_{t_1}(0))\cdots)$ is maximal and equals e, so we get **(3)**. In **(4)**, the property $K^*_{s^*} \circ L^*_{t^*}(0)$ is obvious, since the mapping:

$$L^{*1}_{-t^*_1} \circ \cdots \circ L^*_{-t^*_{e-1}} \circ L^{*e}_0 \circ L^*_{t^*_{e-1}} \circ \cdots \circ L^*_{t^*_1}(\cdot) = \mathrm{Id} \tag{4.5.7}$$

is the indentity. By **(2)**, the mapping $(t, s) \mapsto K^*_s \circ L^*_t(0)$ is also of generic rank e, whence its restriction to $\omega^* \times \{s^*\}$ is of constant rank e since the mapping $K^*_{s^*}(\cdot)$ is a local diffeomorphism. We get **(4)** and then **(5)** obviously. So we have constructed a piece N of $\mathbb{K}$-manifold passing through the origin. Let $L' \in \mathbb{L}$. We claim that L' is tangent to N. Otherwise, if L' would not be tangent, the mapping $(t, s, t') \mapsto L'_{t'}(K_s(L_t(0)))$ would be of generic rank $\geq e+1$, contrarily to the definition of e. This completes the proof of Theorem 2.4.9. □

§2.5. Segre type and Segre multitype

We now apply the general considerations of Theorem 2.4.9 to the specific situation where $\mathbb{K} = \mathbb{C}$, where Δ^n is replaced by $\mathcal{M}$ and where the collection $\mathbb{L}$ of vector fields is our previous complexified vector fields $\{\mathcal{L}_k, \underline{\mathcal{L}}_k\}_{1 \leq k \leq m}$. We also bring some refinements.

2.5.1. Increasing generic ranks. Let us denote by $\text{genrk}_{\mathbb{C}}(\Phi)$ the *generic rank* of a $\mathbb{C}$-algebraic or $\mathbb{C}$-analytic map $\Phi : X \to Y$ of connected complex manifolds. Here of course, we have $\text{genrk}_{\mathbb{C}}(\Gamma_1) = \text{genrk}_{\mathbb{C}}(\underline{\Gamma}_1) = m$ and $\text{genrk}_{\mathbb{C}}(\Gamma_2) = \text{genrk}_{\mathbb{C}}(\underline{\Gamma}_2) = 2m$, which is evident in equations (2.4.6) and (2.4.7). We set $e_1 := m$ and $e_2 := m$. Next, we set $e_3 := \text{genrk}_{\mathbb{C}}(\Gamma_3) - 2m$ and, by induction $e_{k+1} := \text{genrk}_{\mathbb{C}}(\Gamma_{k+1}) - e_3 - \cdots - e_k - 2m$, whence $\text{genrk}_{\mathbb{C}}(\Gamma_k) = 2m + e_3 + \cdots + e_k$ if $k \geq 3$, and similarly, we can define the sequence $\underline{e}_k$ for $\underline{\Gamma}_k$. We notice at once that we have $\underline{e}_k = e_k$, since $\sigma(\Gamma_k(z_{(k)})) = \underline{\Gamma}_k(\overline{z_{(k)}})$.

We claim that $e_l = 0$ for all $l \geq k+1$ if $e_{k+1} = 0$ and $e_k \neq 0$. In other words, the generic rank enjoys a stabilization property. Indeed, we first choose a point $z_{(k)}^*$ arbitrarily close to the origin in $\mathbb{C}^{mk}$ such that Γ_k has (necessarily locally constant) rank equal to $2m + e_3 + \cdots + e_k$ at $z_{(k)}^*$ and we set $q := \Gamma_k(z_{(k)}^*) \in \mathcal{M}$. Then by the rank theorem, the image $\mathcal{H}$ of a neighborhood $\mathcal{W}^*$ of $z_{(k)}^*$ is a submanifold of $\mathcal{M}$ of dimension $2m + e_3 + \cdots + e_k$. We claim that the vector fields $\mathcal{L}_k$ and $\underline{\mathcal{L}}_k$ are all tangent to $\mathcal{H}$. For instance, to fix ideas, we assume that k is even (the odd case will be similar). Thus we can write $\Gamma_k(z_{(k)}) = \underline{\mathcal{L}}_{z_k}(\cdots(\mathcal{L}_{z_1}(0))\cdots)$, *i.e.* the chain Γ_k ends-up with a $\underline{\mathcal{L}}$. This shows that $\mathcal{H}$ is fibered by the leaves of $\mathcal{F}_{\underline{\mathcal{L}}}$, so the $\underline{\mathcal{L}}_k$ are already tangent to $\mathcal{H}$ at every point. By differentiating $\Gamma_{k+1} = \mathcal{L}_{z_{k+1}}(\Gamma_k(z_{(k)}))$ with respect to z_{k+1} at $z_{k+1} = 0$, we obtain the m-dimensional space $\mathcal{L}(\Gamma_k(z_{(k)}))$, namely the tangent space to the foliation $\mathcal{F}_{\mathcal{L}}$ at the point $\Gamma_k(z_{(k)})$. Then the assumption $e_{k+1} = 0$ entails that this space $\mathcal{L}(\Gamma_k(z_{(k)}))$ is necessarily contained in the tangent space to $\mathcal{H}$ at $\Gamma_k(z_{(k)})$, which proves the claim. Finally, as the $\mathcal{L}_k$ and the $\underline{\mathcal{L}}_k$ are all tangent to $\mathcal{H}$, it follows that their local flow at q is contained in $\mathcal{H}$, whence the range of the subsequent Γ_l, $l \geq k+1$, is locally contained in $\mathcal{H}$. Because they are either algebraic or analytic, this shows that their generic rank does not go beyond $2m + e_3 + \cdots + e_k$, which proves the claim.

In conclusion, there exists a well-defined integer $\mu_0 \geq 2$ with $\mu_0 \leq d+2$ such that $e_3 > 0, \dots, e_{\mu_0} > 0$ and $e_l = 0$, for all $l \geq \mu_0 + 1$. We call the integer μ_0 the *Segre type* of $\mathcal{M}$ at the origin and we call the μ_0-tuple $(m, m, e_3, \dots, e_{\mu_0})$ the *Segre multitype* of $\mathcal{M}$ at the origin. This *Segre multitype* simply recollects all the jumps of generic ranks of the Γ_k. It is clear that Segre type and multitype are biholomorphic invariants, because the Segre foliations defined by $\mathcal{L}$ and $\underline{\mathcal{L}}$ are so. To summarize, we have:

- **(a)** $\text{genrk}_{\mathbb{C}}(\Gamma_k) = 2m + e_3 + \cdots + e_k = \text{genrk}_{\mathbb{C}}(\underline{\Gamma}_k)$, for $2 \leq k \leq \mu_0$,
- **(b)** $\text{genrk}_{\mathbb{C}}(\Gamma_k) = 2m + e_3 + \cdots + e_{\mu_0} = \text{genrk}_{\mathbb{C}}(\underline{\Gamma}_k)$, for $k \geq \mu_0$.

The main advantage of dealing with $\mathcal{M}$, $\mathcal{L}$, $\underline{\mathcal{L}}$, Γ_k and $\underline{\Gamma}_k$ lies in the fact that all these objects may be understood in a coordinate-free way. Even the two projections π_t and π_τ can be defined abstractly, because their fibers are the leaves of the Segre foliations. As §2.4.9, We may define the orbit $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, 0)$ and we have the following theorem.

Theorem 2.5.2. *Assume that M is a real algebraic or analytic generic submanifold of $\mathbb{C}^n$, let $p_0 \in M$, let $\mathcal{M} := (M)^c$ be its complexification and let $(p_0)^c := (p_0, \bar{p}_0)$, identified with the origin in coordinates (t, τ). Let $\mathcal{L}_k$ and $\underline{\mathcal{L}}_k$, $k = 1, \dots, n$, be vector fields generating the pair of Segre foliations. Then there exists $z_{(\mu_0)}^* \in \mathbb{C}^{m\mu_0}$ arbitrarily close to the origin of the form $z_{(\mu_0)}^* = (z_1^*, \dots, z_{\mu_0-1}^*, 0)$ and a small neighborhood $\mathcal{W}^*$ of $z_{(\mu_0)}^*$ in $\Delta_{m\mu_0}(\delta_{\mu_0})$ such that, if we denote $\omega_{(\mu_0-1)}^* := (-z_{\mu_0-1}^*, \dots, -z_1^*)$, then we have*:

- **(c)** *The complex algebraic or analytic map Γ_{μ_0} is of rank $2m + e_3 + \cdots + e_{\mu_0}$ at $z_{(\mu_0)}^*$.*
- **(d)** $\Gamma_{2\mu_0-1}(z_{(\mu_0)}^*, \omega_{(\mu_0-1)}^*) = 0$.
- **(e)** *The restricted map $\Gamma_{2\mu_0-1} : \mathcal{W}^* \times \omega_{(\mu_0-1)}^* \to \mathcal{M}$ is of constant rank equal to $2m + e_3 + \cdots + e_{\mu_0}$.*
- **(f)** *The image $\Gamma_{2\mu_0-1}(\mathcal{W}^* \times \omega_{(\mu_0-1)}^*)$ is a complex algebraic or analytic manifold-piece, denoted by $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, 0)$ and called the $\{\mathcal{L}, \underline{\mathcal{L}}\}$-orbit of the origin with the property that the vector fields $\mathcal{L}_k$ and $\underline{\mathcal{L}}_k$ are tangent to it.*
- **(g)** *This integer $2m + e_3 + \cdots + e_{\mu_0}$ is equal to* $\dim_{\mathbb{C}} \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, 0)$.

Of course, the same statement holds with $\underline{\Gamma}_{2\mu_0-1}$ instead of $\Gamma_{2\mu_0-1}$.

Proof. The proof is similar to the proof of Theorem 2.4.10, with minor modifications. According to **(b)**, Γ_{μ_0} is of generic rank $2m + e_3 + \cdots + e_{\mu_0}$. Consequently, for every point $z^*_{(\mu_0)} \in \Delta_{m\mu_0}(\delta_{\mu_0})$ outside of some proper complex subvariety, the map Γ_{μ_0} is of rank $2m + e_3 + \cdots + e_{\mu_0}$ *at* $z^*_{(\mu_0)}$. In fact, we claim that we can even choose such a $z^*_{(\mu_0)}$ of the form $(z^*_1, \ldots, z^*_{\mu_0-1}, 0)$, *i.e.* with $z^*_{\mu_0} = 0$. Indeed, as $\Gamma_{(k)}(z_{(k)}) = [\mathcal{L} \,\text{or}\, \underline{\mathcal{L}}]_{z_k}(\Gamma_{(k-1)}(z_{(k-1)}))$, we can apply the following lemma, which follows directly form Lemma 2.4.11.

Lemma 2.5.3. *Let $z \in \mathbb{C}^m$, $z' \in \mathbb{C}^{m'}$, let $\Gamma(z') \in \mathcal{M}$ be a formal, algebraic or analytic map of z' with $\Gamma(0) = 0$ and let $\varphi(z, z') := \mathcal{L}_z(\Gamma(z'))$ or $\varphi(z, z') := \underline{\mathcal{L}}_z(\Gamma(z'))$. Then φ attains its maximal generic rank at some points of the form $(0, z'^*) \in \mathbb{C}^m \times \mathbb{C}^{m'}$.*

Now, we fix such a $z^*_{(\mu_0)}$ of the form $(z^*_1, \ldots, z^*_{\mu_0-1}, 0)$, which satisfies **(c)** and we check that it satisfies the other claims. Let $\omega^*_{(\mu_0-1)} := (-z^*_{\mu_0-1}, \ldots, -z^*_1)$. First, **(d)** is easy: suppose for instance μ_0 is even, then we have $\Gamma_{2\mu_0-1}(z^*_{(\mu_0)}, \omega^*_{(\mu_0-1)}) = \mathcal{L}_{-z^*_1} \circ \cdots \circ \mathcal{L}_{-z^*_{\mu_0-1}} \circ \underline{\mathcal{L}}_0 \circ \mathcal{L}_{z^*_{\mu_0-1}} \circ \cdots \circ \mathcal{L}_{z^*_1}(0) = 0$, because $\underline{\mathcal{L}}_0(q) = q$, $\mathcal{L}_{-w} \circ \mathcal{L}_w = \text{Id}$ and $\underline{\mathcal{L}}_{-\zeta} \circ \underline{\mathcal{L}}_\zeta = \text{Id}$. The odd case is similar. Now, we prove **(e)**. The restricted map $z_{(\mu_0)} \mapsto \mathcal{L}_{-z^*_1} \circ \cdots \circ \mathcal{L}_{-z^*_{\mu_0-1}} \circ \Gamma_{\mu_0}(z_{(\mu_0)})$ (again written in case μ_0 is even), is clearly of rank $2m + e_3 + \cdots + e_{\mu_0}$ at the point $z^*_{(\mu_0)}$, because the map $q \mapsto \mathcal{L}_{-z^*_1} \circ \cdots \circ \mathcal{L}_{-z^*_{\mu_0-1}}(q)$ is a local biholomorphism, by definition of flows. Notice that $\Gamma_{2\mu_0-1}$ is then of *constant rank* equal to $2m + e_3 + \cdots + e_{\mu_0}$ in a neighborhood of $(z^*_{(\mu_0)}, \omega^*_{(\mu_0-1)})$ in $\mathcal{W}^* \times \omega^*_{(\mu_0-1)}$, since, by **(b)**, $2m + e_3 + \cdots + e_{\mu_0}$ is already the maximum value of all the generic ranks of the Γ_k. This proves **(e)**. By definition, the orbit $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, 0)$ is the union of the ranges of the maps Γ_k and $\underline{\Gamma}_k$. It is easy to check that this double union coincides in fact with the union of only the Γ_k (or of only the $\underline{\Gamma}_k$), simply because, setting $z_1 = 0$, we have $\Gamma_k(0, z_2, \ldots, z_k) \equiv \underline{\Gamma}_{k-1}(z_2, \ldots, z_k)$. Thanks to the constant rank property **(e)**, we already know that this orbit contains the $(2m + e_3 + \cdots + e_{\mu_0})$-dimensional manifold-piece passing through 0: $\mathcal{N} := \Gamma_{2\mu_0-1}(\mathcal{W}^* \times \omega^*_{(\mu_0-1)})$. Because by **(b)** the next generic ranks for $k \geq 2\mu_0 - 1$ do not increase and because of the principle of analytic continuation, we then deduce that all the ranges of the subsequent Γ_k are contained in this manifold piece $\mathcal{N}$ and it follows that $\mathcal{L}$ and $\underline{\mathcal{L}}$ are tangent to this manifold-piece. In conclusion, we get **(f)** and **(g)**, which completes the proof of Theorem 2.5.2. □

In the hypersurface case, we have the following simple criterion of minimality, left to the reader.

Corollary 2.5.4. *If M is a real algebraic or analytic hypersurface,* i.e. *if $d = 1$, then*

(h) *M is minimal at* $0 \iff \mu_0 = 3$.
(i) *M is nonminimal at* $0 \iff \mu_0 = 2$.

2.5.7. Example. We now give a simple example in the hypersurface case which illustrates statements **(e)** and **(f)** of Theorem 2.5.2 in a very concrete way. We let M be the hypersurface of $\mathbb{C}^2$ of equation $z = \bar{z} + iw^2\bar{w}^2$. We choose $p = 0$ and here $2\mu_0 - 1 = 5$. We compute:

$$
(2.5.8) \qquad \begin{cases}
\Gamma_1(z_1) = (z_1,\, 0,\, 0,\, 0) \\
\Gamma_2(z_1, z_2) = (z_1,\, 0,\, z_2,\, -iz_1^2 z_2^2) \\
\Gamma_3(z_1, z_2, z_3) = (z_1 + z_3,\, iz_2^2[z_3^2 + 2z_1z_3],\, z_2,\, -iz_1^2 z_2^2) \\
\Gamma_4(z_1, z_2, z_3, z_4) = (z_1 + z_3,\, iz_2^2[z_3^2 + 2z_1z_3],\, z_2 + z_4, \\
\qquad\qquad iz_2^2[z_3^2 + 2z_1z_3] - i[(z_2 + z_4)(z_1 + z_3)]^2) \\
\Gamma_5(z_1, z_2, z_3, z_4, z_5) = (z_1 + z_3 + z_5,\, iz_2^2[z_3^2 + 2z_1z_3] - \\
\qquad - i[(z_2 + z_4)(z_1 + z_3)]^2) + i[(z_1 + z_3 + z_5)(z_2 + z_4)]^2, \\
\qquad z_2 + z_4,\, iz_2^2[z_3^2 + 2z_1z_3] - i[(z_2 + z_4)(z_1 + z_3)]^2).
\end{cases}
$$

The maps Γ_k have range in $\mathcal{M}$, on which either the coordinates (z, w, ζ) or (ζ, ξ, z) can be chosen. We do the first choice for k even and the second choice for k odd. Thus, we view Γ_5 as a map $\mathbb{C}^5 \to \mathbb{C}^3_{(z,\zeta,\xi)}$, *i.e.* we forget the second w-coordinate in the above expression of Γ_5. Now, computing the 3×5 Jacobian matrix of Γ_5 at the point $(z^*_{(3)}, \omega^*_{(2)})$ as in Theorem 2.5.2 which is necessarily of the form $(z_1^*, z_2^*, 0, -z_2^*, -z_1^*)$, and for which we clearly have $\Gamma_5(z_1^*, z_2^*, 0, -z_2^*, -z_1^*) = 0$, we see that the determinant of the first 3×3 submatrix is equal to $2iz_1^*(z_2^*)^2$. Thus, it is nonzero for an arbitrary choice of $z_1^* \neq 0$ and $z_2^* \neq 0$. By the way, the question arises whether the integer $(2\mu_0 - 1)$ in Theorem 2.5.2 is optimal. Incidentally, this example shows that it is optimal. Indeed, if we ask whether there exists $z^*_{(4)} = (z_1^*, z_2^*, z_3^*, z_4^*)$ such that $\Gamma_4(z^*_{(4)}) = 0$ and the rank at $z^*_{(4)}$ of the differential of Γ_4 equals 3 (the dimension of $\mathcal{M}$), then looking at eqs (2.5.8), we get first $z_1^* + z_3^* = 0$, $(z_2^*)^2 z_3^* [z_3^* + 2z_1^*] = 0$ and $z_2^* + z_4^* = 0$, thus $z^*_{(4)}$ is necessarily of the form $(0, z_2^*, 0, -z_2^*)$ or $(z_1^*, 0, -z_1^*, 0)$. Viewing now Γ_4 as a map $\mathbb{C}^4 \to \mathbb{C}^3_{(z,w,\zeta)}$, and computing its 3×4 Jacobian matrix at such points, one sees that it is of rank 2, which proves the claim.

2.5.9. Segre geometry in the formal category. Replacing the complex defining equations (2.1.15) by purely formal power series, the reader may verify that all the previous results are meaningful and may be expressed in terms of purely formal power series.

§2.6. Extrinsic complexification of CR orbits

2.6.1. Intrinsic CR orbits and their smoothness. On M the complex tangent bundle T^cM is generated by the $2m$ sections $\operatorname{Re} L_1, \operatorname{Im} L_1, \ldots, \operatorname{Re} L_m, \operatorname{Im} L_m$. In the formal case, their flow is formal. In the analytic case, their flow is analytic. However, a very subtle point occurs in the algebraic category. We have seen that the complex flows of the complexified vector fields $\mathcal{L}_k$, given by (2.4.2) and (2.4.3), are algebraic. This is untrue about the real flows of the real and imaginary parts of the vector fields L_k, as shows the following example.

Example 2.6.2. Let M be the real analytic hypersurface passing through the origin in $\mathbb{C}^2$ defined by $\operatorname{Im} w = \sqrt{1 + z\bar{z}} - 1$. The vector field $\overline{L} := \partial_{\bar{z}} + iz(1 + z\bar{z})^{-1/2}\, \partial_{\bar{w}}$ generates $T^{0,1}M$. Its real and imaginary parts are given by

$$\text{(2.6.3)} \qquad \begin{cases} 2\operatorname{Re}\overline{L} = \partial_x - y(1 + x^2 + y^2)^{-1/2}\,\partial_u + x(1 + x^2 + y^2)^{-1/2}\,\partial_v, \\ 2\operatorname{Im}\overline{L} = \, -\partial_y + x(1 + x^2 + y^2)^{-1/2}\,\partial_u + y(1 + x^2 + y^2)^{-1/2}\,\partial_v. \end{cases}$$

We claim that the flow of $2\operatorname{Re}\overline{L}$ is not algebraic. Indeed, let s denote a real time parameter and let $(x(s), y(s), u(s), v(s))$ be the unique integral curve of $2\operatorname{Re}\overline{L}$ with $x(0) = x_0$, $y(0) = y_0$, $u(0) = u_0$ and $v(0) = v_0$ with $(x_0 + iy_0, u_0 + iv_0) \in M$. We compute first $x(s) = x_0 + s$, $y(s) = y_0$, $v(s) = (1 + y_0^2 + (x_0 + s)^2)^{1/2} - 1$ and then $u(s)$ satisifies the ordinary differential equation

$$\text{(2.6.4)} \qquad du(s)/ds = -y_0(1 + y_0^2(x_0 + s)^2)^{-1/2},$$

which may be integrated as

$$\text{(2.6.5)} \qquad u(s) = u_0 - y_0 \left[\operatorname{Arcsh}\left(\frac{x_0 + s}{\sqrt{1 + y_0^2}} \right) - \operatorname{Arcsh}\left(\frac{x_0}{\sqrt{1 + y_0^2}} \right) \right]$$

Consequently, the flow of $2\operatorname{Re}\overline{L}$ is not algebraic. However, *we stress that the complex flow of the complexified vector field* $\underline{\mathcal{L}} = \partial_\zeta + iz(1 + z\zeta)^{-1/2}\,\partial_\xi$ *is complex algebraic*, as shown by the general expression (2.4.3), which yields in this particular case $\underline{\mathcal{L}}_\zeta(z_p, w_p, \zeta_p, \xi_p) = (z_p, w_p, \zeta_p + \zeta, w_p + 2i(\sqrt{1 + z\zeta} - 1)))$: indeed, this expression is clearly algebraic !

We now consider the set $\mathbb{L} := \{\operatorname{Re} L_1, \operatorname{Im} L_1, \ldots, \operatorname{Re} L_m, \operatorname{Im} L_m\}$ of $2m$ real vector fields generating T^cM. Applying Theorem 2.4.10, we may construct the *local CR orbits of points* p *in* M, which we denote by $\mathcal{O}_{CR}(M, p)$. Since they are weak T^cM-integral manifolds, they are of the same CR dimension as M. According to Example 2.6.2 just above, we only know that $\mathcal{O}_{CR}(M,p)$ is a real analytic submanifold passing through p. Algebraicity seems to be lost, but naturally, according to Leibniz' principle of sufficient reason, one expects however that $\mathcal{O}_{CR}(M,p)$ is algebraic if M is algebraic

The above Example 2.6.2 shows that the algebraicity of the CR orbits of a real algebraic generic manifold cannot be established by means of sections of the complex tangent bundle $T^cM = \mathrm{Re}\, T^{0,1}M$. Fortunately, by passing to the extrinsic complexification, we may avoid this difficulty.

Theorem 2.6.6. *There is a one-to-one correspondence between the CR orbits and their extrinsic complexifications, namely for all p in a neighborhood of the origin, we have:*

(1) $\mathcal{O}_{CR}(M,p)^c = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$, *and*
(2) $\mathcal{O}_{CR}(M,p) = \pi_t(\underline{\Lambda} \cap \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c))$,

where we remind the notation $p^c = (p,\bar{p}) \in \mathbb{C}^n \times \mathbb{C}^n$. *In particular, if M is a real algebraic generic submanifold, the complexified orbit $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$ is complex algebraic, whence by* **(2)** *the CR orbit $\mathcal{O}_{CR}(M,p)$ is real algebraic.*

Proof. By Theorem 2.4.10, $\mathcal{O}_{CR}(M,p)$ is a real analytic closed submanifold of M passing through p. Even if M is real analytic, the flows of elements of $\mathbb{L}$ are not algebraic in general, as shows Example 2.6.2, so we do not know more than the analyticity of $\mathcal{O}_{CR}(M,p)$. Thus, let $\mathcal{O}$ be a small open connected manifold-piece of $\mathcal{O}_{CR}(M,p)$ through p, and let $\mathcal{O}^c$ be its extrinsic complexification. Because $L_k|_{\mathcal{O}}$ and $\bar{L}_k|_{\mathcal{O}}$ are tangent to $\mathcal{O}$, the *generic uniqueness principle* (*via* $\mathcal{O} \subset \underline{\Lambda}$, where $\underline{\Lambda}$ is maximally real, so $\mathcal{O}$ is generic in $\mathcal{O}^c$) entails that $\mathcal{L}_k|_{\mathcal{O}^c}$ and $\underline{\mathcal{L}}_k|_{\mathcal{O}^c}$ are tangent to $\mathcal{O}^c$. Therefore $\mathcal{O}^c$ is an integral manifold for $\{\mathcal{L}, \underline{\mathcal{L}}\}$ through p^c, whence $\mathcal{O}^c \supset \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$, since a characterizing property of the orbit $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$ is to say that it is the *smallest integral manifold-piece* for $\{\mathcal{L}, \underline{\mathcal{L}}\}$ through p^c.

Conversely, Let $\mathcal{N}$ be a manifold-piece of $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$ through p^c. We have just shown that $\mathcal{N} \subset \mathcal{O}^c$, hence to finish the proof, we want to show that $\mathcal{N} \supset \mathcal{O}^c$. We claim that we have $\sigma(\mathcal{N}) = \mathcal{N}$ in a neighborhood of p^c. Indeed, By definition, the orbit is the following set of endpoints of concatenations of flows of $\mathcal{L}$ and of flows of $\underline{\mathcal{L}}$ (notice that because $\mathcal{L}_{z_2} \circ \mathcal{L}_{z_1} = \mathcal{L}_{z_1+z_2}$ and $\underline{\mathcal{L}}_{\zeta_2} \circ \underline{\mathcal{L}}_{\zeta_1} = \underline{\mathcal{L}}_{\zeta_1+\zeta_2}$ but $\mathcal{L}$ and $\underline{\mathcal{L}}$ do not commute, there can be only two different kinds of concatenated flow maps; we do not use the abbreviated notation Γ_k here):

$$\text{(2.6.7)} \qquad \begin{cases} \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c) = \{\mathcal{L}_{z_k} \circ \cdots \circ \mathcal{L}_{z_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{z_1}(p^c) : \\ \qquad\qquad : z_1, \zeta_1, z_2, \dots, z_k \in \mathbb{C} \text{ small}, k \in \mathbb{N}\} \bigcup \\ \qquad \bigcup \{\underline{\mathcal{L}}_{\zeta_k} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_2} \circ \mathcal{L}_{z_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p^c) : \\ \qquad\qquad : \zeta_1, z_1, \zeta_2, \dots, \zeta_k \in \mathbb{C} \text{ small}, k \in \mathbb{N}\} := E \cup F. \end{cases}$$

Now an examination of (2.4.2) and (2.4.3) shows that we have $\sigma(\mathcal{L}_w(q)) = \underline{\mathcal{L}}_{\bar{w}}(\sigma(q))$ and $\sigma(\underline{\mathcal{L}}_\zeta(q)) = \mathcal{L}_{\bar{\zeta}}(\sigma(q))$, for each $q \in \mathcal{M}$. Consequently:

$$\text{(2.6.8)} \qquad \begin{cases} \sigma(\mathcal{L}_{z_k} \circ \underline{\mathcal{L}}_{\zeta_{k-1}} \circ \cdots \circ \mathcal{L}_{z_1}(p^c)) = \underline{\mathcal{L}}_{\bar{w}_k}(\sigma(\underline{\mathcal{L}}_{\zeta_{k-1}} \circ \cdots \circ \mathcal{L}_{z_1}(p^c))) = \\ = \underline{\mathcal{L}}_{\bar{w}_k} \circ \mathcal{L}_{\bar{\zeta}_{k-1}} \circ \cdots \circ \sigma(\mathcal{L}_{z_1}(p^c)) = \underline{\mathcal{L}}_{\bar{w}_k} \circ \mathcal{L}_{\bar{\zeta}_{k-1}} \circ \cdots \circ \underline{\mathcal{L}}_{\bar{w}_1}(p^c), \end{cases}$$

since $\sigma(p^c) = p^c$. This proves $F = \sigma(E)$, hence $\sigma(\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)) = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$, so we have $\sigma(\mathcal{N}) = \mathcal{N}$ as announced. By Theorem 2.4.10, $\mathcal{N}$ is smooth at p^c and satisfies $\sigma(\mathcal{N}) = \mathcal{N}$. To conclude, we need the following lemma.

Lemma 2.6.9. *There is a one-to-one correspondence between real analytic subsets $\Sigma \subset M$ and complex analytic subvarieties Σ_1 of $\mathcal{M}$ satisfying $\sigma(\Sigma_1) = \Sigma_1$ given by $\Sigma \mapsto \Sigma^c =: \Sigma_1$, with inverse $\Sigma_1 \mapsto \pi_t(\Sigma_1 \cap \underline{\Lambda}) =: \Sigma$. Furthermore, Σ is a smooth submanifold if and only if Σ_1 is smooth.*

Proof. Let $\Sigma \subset M$ be given by real analytic equations $\chi_l(t, \bar{t}) = 0$, $l = 1, \dots, c$. If Σ is smooth, we can assume that $d\rho_1 \wedge \cdots \wedge d\rho_d \wedge d\chi_1 \wedge \cdots \wedge d\chi_l(0) \neq 0$. Let $\Sigma^c \subset \mathcal{M}$ be defined by $\chi_l(t, \tau) = 0$. Clearly, Σ^c is again smooth and $\Sigma = \pi_t(\Sigma^c \cap \underline{\Lambda})$.

Conversely, let $\Sigma_1 \subset \mathcal{M}$ be given by $\chi_l(t, \tau) = 0$, $l = 1, \dots, c$. If Σ_1 is smooth, we can assume that $d\rho_1 \wedge \cdots \wedge d\rho_d \wedge d\chi_1 \wedge \cdots \wedge d\chi_l(0) \neq 0$. By assumption, Σ_1 is σ-invariant, *i.e.* we have $\chi_l(t, \tau) = 0$, $l = 1, \dots, c$ if and only if $\chi_l(\bar{\tau}, \bar{t}) = 0$, $l = 1, \dots, c$. This implies that if we set $\Sigma := \{t \in M : \chi_l(t, \bar{t}) = 0, \, l = 1, \dots, c\} = \pi_t(\Sigma_1 \cap \underline{\Lambda})$, then Σ is real, *i.e.* satisfies $t \in \Sigma$ if and only if $\bar{t} \in \Sigma$ and satisfies $(\Sigma)^c = \Sigma_1$. Finally, Σ is smooth if Σ_1 is. □

Continuing the proof of Theorem 2.6.6, we now know that there exists $N := \pi_t(\mathcal{N} \cap \underline{\Lambda})$ a unique piece of a real analytic submanifold $N \subset M$ passing through p such that $N^c = \mathcal{N}$.

Let us denote $\mathcal{N} = \{\rho(t, \tau) = 0, \chi(t, \tau) = 0\}$, so that $N = \{\rho(t, \bar{t}) = 0, \chi(t, \bar{t}) = 0\}$. Then $\mathcal{L}_k \rho = 0$, $\underline{\mathcal{L}}_k \rho = 0$, $\mathcal{L}_k \chi = 0$, $\underline{\mathcal{L}}_k \chi = 0$ on $\{\rho = \chi = 0\}$, since $\mathcal{N}$ is an $\{\mathcal{L}, \underline{\mathcal{L}}\}$-integral manifold. Therefore, after restriction to the antidiagonal $\{\tau = \bar{t}\} = \underline{\Lambda}$, we have $L_k \rho = 0$, $\bar{L}_k \rho = 0$, $L_k \chi = 0$ and $\bar{L}_k \chi = 0$ on $\{\rho(t, \bar{t}) = 0, \chi(t, \bar{t}) = 0\} = N$, so that N is an $\{L, \bar{L}\}$-integral manifold. Thus by the minimality property of CR-orbits, we have $N \supset \mathcal{O}$ as germs at p. By complexifying, we get $\mathcal{N} \supset \mathcal{O}^c$, as desired. □

Thanks to Theorem 4, an equivalent definition of minimality is as follows (*cf.* Definition 2.4.8):

Definition 2.6.10. The generic submanifold M is called *minimal at* $p \in M$ if the CR orbit $\mathcal{O}_{CR}(M, p)$ is of maximal dimension equal to $\dim_{\mathbb{R}} M$.

2.6.11. Segre type of M. Now, let us define the maps $\psi^1(z_1) := \pi_t(\Gamma_1(z_1))$, $\psi^2(z_1, z_2) := \pi_\tau(\Gamma_2(z_1, z_2))$ and more generally:

$$\psi^{2j}(z_{(2j)}) := \pi_\tau(\Gamma_{2j}(z_{(2j)})) \quad \text{and} \quad \psi^{2j+1}(z_{(2j+1)}) := \pi_t(\Gamma_{2j+1}(z_{(2j+1)})). \tag{2.6.12}$$

Notice that by the definitions (2.4.2) and (2.4.3)), the action of the flow of $\mathcal{L}$ leaves unchanged the (ζ, ξ)-coordinates, and vice versa, the action of the flow of $\underline{\mathcal{L}}$ leaves unchanged the (z, w)-coordinates. Similarly also, we can define the maps $\underline{\psi}^k$ by $\underline{\psi}^{2j}(z_{(2j)}) := \pi_t(\underline{\Gamma}_{2j}(z_{(2j)}))$ and $\underline{\psi}^{2j+1}(z_{(2j+1)}) := \pi_\tau(\underline{\Gamma}_{2j+1}(z_{(2j+1)}))$. We need the following lemma (of course, a similar statement also holds with $\underline{\Gamma}_{k+2}$ and $\underline{\psi}^{k+1}$ instead):

Lemma 2.6.13. *For $0 \leq k \leq \mu_0$, we have*:

$$m + \mathrm{genrk}_{\mathbb{C}}(\psi^{k+1}) = \mathrm{genrk}_{\mathbb{C}}(\Gamma_{k+2}) = 2m + e_3 + \cdots + e_k, \tag{2.6.14}$$

and $\mathrm{genrk}_{\mathbb{C}}(\psi^{k+1}) = m + e_3 + \cdots + e_{\mu_0}$ *for* $k \geq \mu_0$.

Proof. For $k = 0$, we have $\psi^1(z_1) = (z_1, i\bar{\Theta}(z_1, 0, 0))$, whence $\mathrm{genrk}_{\mathbb{C}}(\psi^1) = m$ obviously. Recall that, by (2.4.6), we have

$$\Gamma_2(z_1, z_2) = \left(z_1, \, \overline{\Theta}(z_1, 0, 0), \, z_2, \, \Theta(z_2, z_1, \overline{\Theta}(z_1, 0, 0))\right), \tag{2.6.15}$$

so $m + \mathrm{genrk}_{\mathbb{C}}(\psi^1) = \mathrm{genrk}_{\mathbb{C}}(\Gamma_2) = 2m$. More generally, for $k = 2j$, we have:

$$\begin{cases} \mathcal{L}_{z_{2j+1}}(\Gamma_{2j}(z_{(2j)})) = \mathcal{L}_{z_{2j+1}}(z(z_{(2j)}), w(z_{(2j)}), \zeta(z_{(2j)}), \xi(z_{(2j)})) \\ \qquad = \left(z_{2j+1} + z(z_{(2j)}), \, \bar{\Theta}(z_{2j+1} + z(z_{(2j)}), \zeta(z_{(2j)}), \xi(z_{(2j)})), \, \zeta(z_{(2j)}), \, \xi(z_{(2j)})\right). \end{cases} \tag{2.6.16}$$

We choose the coordinates (z, ζ, ξ) on $\mathcal{M}$, whence we consider the map $\Gamma_{2j+1}(z_{(2j+1)})$ in (2.6.16) to have range in $\mathbb{C}^{2m+d}_{(z,\zeta,\xi)}$. It is then the map $(z_{(2j)}, z_{2j+1}) \mapsto \left(z_{2j+1} + z(z_{(2j)}), \zeta(z_{(2j)}), \, \xi(z_{(2j)})\right)$. It follows immediately that

$$\mathrm{genrk}_{\mathbb{C}}(\Gamma_{2j+1}) = m + \mathrm{genrk}_{\mathbb{C}}[z_{(2j)} \mapsto (\zeta(z_{(2j)}), \xi(z_{(2j)}))] = m + \mathrm{genrk}_{\mathbb{C}} \psi^{2j}. \tag{2.6.17}$$

This completes the proof of Lemma 2.6.13. □

We now define the *Segre type of M at $p \in M$* (not to be confused with μ_0) to be the smallest integer ν_0 satisfying $\text{genrk}_{\mathbb{C}}(\psi^{\nu_0}) = \text{genrk}_{\mathbb{C}}(\psi^{\nu_0+1})$. By (2.6.17), we readily observe that in fact, we have $\nu_0 = \mu_0 - 1$. The Segre type *of* M can be related to its CR orbits as will be explained in the next paragraph.

2.6.18. Intrinsic complexification of CR-orbits. By the *intrinsic complexification N^{i_c} of a real CR manifold N*, we understand the smallest complex algebraic or analytic manifold containing N in $\mathbb{C}^n$. It exists and satisfies $\dim_{\mathbb{C}} N^{i_c} = \text{CRdim}\, N + \text{Hcodim}\, N$ (*holomorphic codimension*, *cf.* §2.1.6). Let $\mathcal{O}$ denote a manifold-piece of $\mathcal{O}_{CR}(M,p)$ through p and let $\mathcal{O}^{i_c}$ be its *intrinsic complexification*, namely the smallest complex manifold of the ambient space $\mathbb{C}^n$ containing $\mathcal{O}$. By construction, the ranges of the ψ^{2j} are contained in $\mathbb{C}^n_\tau$, but we will prefer to work in $\mathbb{C}^n_t$ (although it is equivalent in principle to work in $\mathbb{C}^n_\tau$), hence we shall consider the $\underline{\psi}^{2j}$ instead. We can now establish that $\text{genrk}_{\mathbb{C}}(\underline{\psi}^{\nu_0}) = \dim_{\mathbb{C}} \mathcal{O}^{i_c}$ and that the range of $\underline{\psi}^{2\nu_0}$ contains a manifold-piece of $\mathcal{O}^{i_c}$ through p.

Theorem 2.6.19. *There exist some points $\underline{z}^*_{(2\nu_0)} \in \mathbb{C}^{m2\nu_0}$ arbitrarily close to the origin and small neighborhoods $\mathcal{V}^*$ of $\underline{z}^*_{(2\nu_0)}$ in $(\delta\Delta^m)^{2\nu_0}$ such that we have:*

- **(j)** $\underline{\psi}^{2\nu_0}(\underline{z}^*_{(2\nu_0)}) = p$.
- **(k)** *The map $\underline{\psi}^{2\nu_0}$ is of constant rank $m + e_3 + \cdots + e_{\mu_0}$ in $\mathcal{V}^*$.*
- **(l)** *$\underline{\psi}^{2\nu_0}(\mathcal{V}^*)$ is a manifold-piece $\mathcal{O}^{i_c}$ of the intrinsically complexified CR orbit of M through p.*
- **(m)** $m + e_3 + \cdots + e_{\mu_0} = \dim_{\mathbb{C}} \mathcal{O}^{i_c} = \text{CRdim}\, \mathcal{O} + \text{Hcodim}\, \mathcal{O}$.

Proof. Recall that in view of Theorem 2.5.2, there exists $\underline{z}^*_{(2\mu_0-1)} \in (\delta\Delta^m)^{2\mu_0-1}$ with $\underline{\Gamma}_{2\mu_0-1}(\underline{z}^*_{(2\mu_0-1)}) = p^c$, such that $\underline{\Gamma}_{2\mu_0-1}$ is of rank $2m + e_3 + \cdots + e_{\mu_0}$ at $\underline{z}^*_{(2\mu_0-1)}$. Looking again at (2.6.17) (for $k = 2j+1$ odd, which we have not written, but the corresponding equation is similar), and using the chain rule, we deduce that $\underline{\psi}^{2\mu_0-2}$ is of rank $m + e_3 + \cdots + e_{\mu_0}$ at the point $\underline{z}^*_{2\nu_0} := (\underline{z}^*_1, \dots, \underline{z}^*_{2\mu_0-2})$ and that $\underline{\psi}^{2\nu_0}(\underline{z}^*_{(2\nu_0)}) = p$ (recall $\nu_0 = \mu_0 - 1$). This yields **(j)** and **(k)**. For reasons of dimension, we already know that $\dim_{\mathbb{C}} \mathcal{O}^{i_c}$ must be equal to $m + e_3 + \cdots + e_{\mu_0}$, since $\text{CRdim}\, \mathcal{O} = m$ and $\dim_{\mathbb{C}} \mathcal{O}^c = m + \dim_{\mathbb{C}} \mathcal{O}^{i_c}$. This yields **(m)**. Finally, to deduce **(l)**, we claim that it can be observed that the range of $\underline{\psi}^{2\nu_0}$ is *a priori* contained in $\mathcal{O}^{i_c}$, and afterwards for dimensional reasons, the image $\underline{\psi}^{2\nu_0}(\mathcal{V}^*)$ will necessarily be a manifold-piece of $\mathcal{O}^{i_c}$ through p. To complete this observation, we introduce holomorphic coordinates $(z, w_1, w_2) \in \mathbb{C}^m \times \mathbb{C}^{e_3+\cdots+e_{\mu_0}} \times \mathbb{C}^{n-m-e_3-\cdots-e_{\mu_0}}$ vanishing at p in which the equation of $\mathcal{O}^{i_c}$ is $\{w_2 = 0\}$, which is possible. Using the assumption that $M \cap \{w_2 = 0\}$ is smooth and of CR dimension m, one shows that the equations of $\mathcal{M}$ can then be written in the form $w_1 = \overline{\Theta}_1(z, \zeta, \xi_1, \xi_2)$ and $w_2 = \overline{\Theta}_2(z, \zeta, \xi_1, \xi_2)]$ with $\overline{\Theta}_2(z, \zeta, \xi_1, 0) \equiv 0$. Then an inspection of the inductive construction of the maps $\underline{\Gamma}_k$ shows that they have range contained in $\{w_2 = 0, \xi_2 = 0\}$, whence the maps $\underline{\psi}_{2j}$ have range in $\{w_2 = 0\}$, as announced. The proof of Theorem 2.6.19 is complete. □

Example 2.6.20. Looking at the map Γ_4 in (2.5.8), we see that the integer $2\nu_0 = 2\mu_0 - 2$ satisfying the assertions **(j)** and **(k)** of Theorem 2.6.19 is in general optimal.

§2.7. Segre chains and Segre sets in ambient space

2.7.1. Segre chains as k-th orbit chains of vector fields. In this section, we shall define certain subsets of $\mathcal{M}$ which are the images of the Segre chains Γ_k. These last results will close up our presentation of the general theory of Segre chains. Although we shall not use them in the sequel, their definition seems to be interesting geometrically speaking. Here is an illustration:

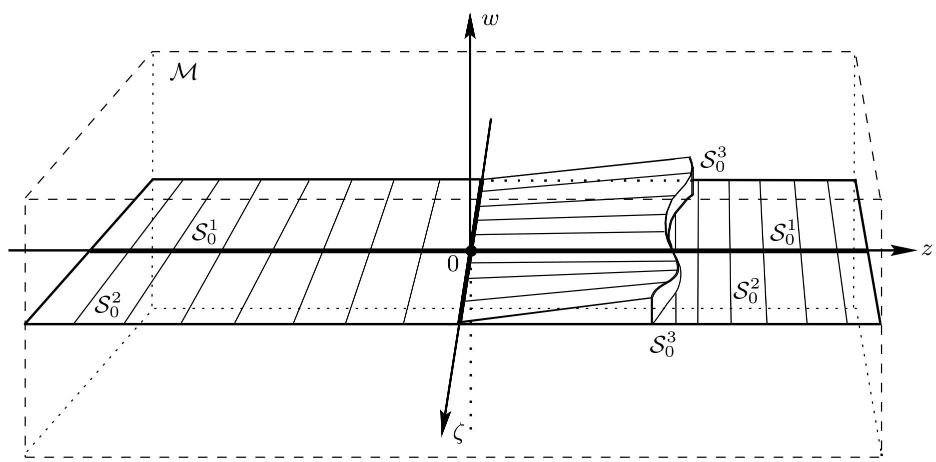

FIGURE 2.7.2: EMERGENCE OF SEGRE CHAINS

At first, we come back to the concatenated flow maps in (2.4.2) and (2.4.3). For each $k \in \mathbb{N}$, we choose in advance $\delta_k > 0$ so small that $[\mathcal{L} \,\text{or}\, \underline{\mathcal{L}}]_{z_k}(\cdots(\mathcal{L}_{z_1}(p^c))\cdots)$ belongs to $\Delta_{2n}(\rho_1)$ for all $z_{(k)} \in \Delta_{mk}(\delta_k)$ and all $(t_p, \tau_p) \in \mathcal{M}$ with $(t_p, \tau_p) \in \Delta_{2n}(\rho_1/2)$. Looking at (2.5.2) and (2.5.3), we see that (up to a shrinking) the complexified Segre varieties of a point $(t_p, \tau_p) \in \mathcal{M}$ can be defined by $\mathcal{S}_{\tau_p} := \{\mathcal{L}_{z_1}(t_p, \tau_p) \in \Delta_{2n}(\rho_1) : z_1 \in \Delta_m(\delta_1)\}$ and $\underline{\mathcal{S}}_{t_p} := \{\underline{\mathcal{L}}_{z_1}(t_p, \tau_p) \in \Delta_{2n}(\rho_1) : z_1 \in \Delta_m(\delta_1)\}$. At order $k = 2$, we can define:

$$
(2.7.3) \qquad \begin{cases} \mathcal{S}_{\tau_p}^2 = \{\underline{\mathcal{L}}_{z_2}(\mathcal{L}_{z_1}(t_p, \tau_p)) \in \Delta_{2n}(\rho_1) : (z_1, z_2) \in \Delta_{2m}(\delta_2)\}, \\ \underline{\mathcal{S}}_{t_p}^2 = \{\mathcal{L}_{z_2}(\underline{\mathcal{L}}_{z_1}(t_p, \tau_p)) \in \Delta_{2n}(\rho_1) : (z_1, z_2) \in \Delta_{2m}(\delta_2)\}. \end{cases}
$$

More generally, we want to define the sets $\mathcal{S}_{\tau_p}^k$ and $\underline{\mathcal{S}}_{t_p}^k$. By a slight abuse of language, we shall also call these sets the k-th *Segre chain* and the k-th *conjugate Segre chain*. Since we shall only use the mappings Γ_k and $\underline{\Gamma}_k$ and not their images, there will be no risk of confusion.
At first, we remind that, because only two "starting actions" $\mathcal{L}_{z_1}(t_p, \tau_p)$ and $\underline{\mathcal{L}}_{\zeta_1}(t_p, \tau_p)$ can make a difference in a concatenation of flows of $\mathcal{L}$ and of $\underline{\mathcal{L}}$, there can exist only two different families of k-th Segre chains. The precise definition of $\mathcal{S}_{\tau_p}^k$ and of $\mathcal{S}_{t_p}^k$ is as follows:

$$
(2.7.4) \qquad \begin{cases} \mathcal{S}_{\tau_p}^{2j} := \{\underline{\mathcal{L}}_{z_{2j}} \circ \cdots \circ \mathcal{L}_{z_1}(t_p, \tau_p) : \, (z_1, \dots, z_{2j}) \in \Delta_{2mj}(\delta_{2j})\}, \\ \mathcal{S}_{\tau_p}^{2j+1} := \{\mathcal{L}_{z_{2j+1}} \circ \cdots \circ \mathcal{L}_{z_1}(t_p, \tau_p) : \, (z_1, \dots, z_{2j+1}) \in \Delta_{2mj+m}(\delta_{2j+1})\}, \\ \underline{\mathcal{S}}_{t_p}^{2j} := \{\mathcal{L}_{z_{2j}} \circ \cdots \circ \underline{\mathcal{L}}_{z_1}(t_p, \tau_p) : \, (z_1, \dots, z_{2j}) \in \Delta_{2mj}(\delta_{2j})\}, \\ \underline{\mathcal{S}}_{t_p}^{2j+1} := \{\underline{\mathcal{L}}_{z_{2j+1}} \cdots \circ \underline{\mathcal{L}}_{z_1}(t_p, \tau_p) : \, (z_1, \dots, z_{2j+1}) \in \Delta_{2mj+m}(\delta_{2j+1})\}, \end{cases}
$$

for $k = 2j$ or $k = 2j + 1$, where $j \in \mathbb{N}$. Clearly, we have $\mathcal{S}_{\tau_p}^k \subset \mathcal{M}$ and $\underline{\mathcal{S}}_{t_p}^k \subset \mathcal{M}$. As $\sigma(\mathcal{L}_w(q)) = \underline{\mathcal{L}}_{\bar{w}}(\sigma(q))$, we have $\sigma(\mathcal{S}_{\tau_p}^k) = \underline{\mathcal{S}}_{\bar{t}_p}^k$.

2.7.5. Segre sets in ambient space. We can now define Segre sets in ambient space as certain projections of Segre chains. The following picture gives an idea of the definition of Segre sets as unions of Segre varieties in the case of a minimal hypersurface in $\mathbb{C}^2$.

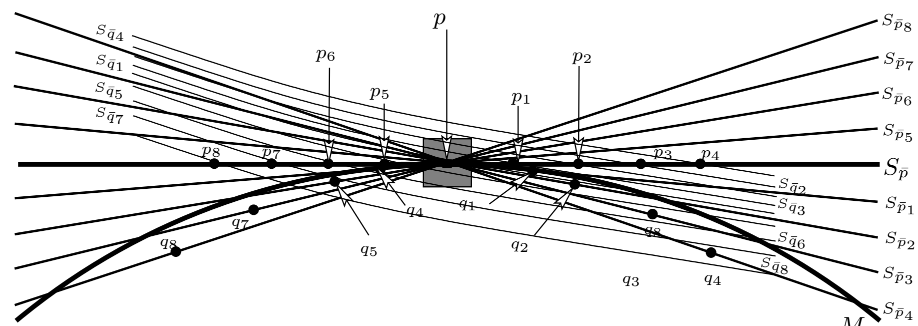

Figure 2.7.6: Segre sets as unions of Segre varieties
$p \in M$, $p_1 \in S_{\bar p}$, $\ldots$, $p_8 \in S_{\bar p}$, $q_1 \in S_{\bar p_1}$, $\ldots$, $q_8 \in S_{\bar p_8}$

The sets $S^{2j+1}_{\bar t_p} := \pi_t(\mathcal{S}^{2j+1}_{\bar t_p}) \subset \Delta_n(\rho_1)$, $\overline{S}^{2j+1}_{t_p} := \pi_\tau(\underline{\mathcal{S}}^{2j+1}_{t_p}) \subset \Delta_n(\rho_1)$, $S^{2j}_{\bar t_p} := \pi_\tau(\mathcal{S}^{2j}_{\bar t_p}) \subset \Delta_n(\rho_1)$ and $\overline{S}^{2j}_{t_p} := \pi_t(\underline{\mathcal{S}}^{2j}_{t_p}) \subset \Delta_n(\rho_1)$ will be called the *k-th Segre sets and the k-th conjugate Segre sets*, with $k = 2j$ or $k = 2j+1$. Notice that by the definitions (2.4.2) and (2.4.3)), the action of the flow of $\mathcal{L}$ leaves unchanged the (ζ, ξ)-coordinates, and vice versa, the action of the flow $\underline{\mathcal{L}}$ leaves unchanged the (z, w)-coordinates. This is why in the definition of Segre sets, we alternately project in the $\mathbb{C}^n_t$-space and in the $\mathbb{C}^n_\tau$-space.

An equivalent, purely set-theoretic definition of Segre sets is as follows. We define: $S^0_{\bar t_p} := \{\bar t_p\}$, and $S^1_{\bar t_p} := S_{\bar t_p} = \bigcup_{\bar t \in S^0_{\bar t_p}} S_{\bar t}$, $S^2_{\bar t_p} = \bigcup_{t \in S^1_{\bar t_p}} \overline{S}_t$, and then inductively, for $j \in \mathbb{N}_*$, $S^{2j}_{\bar t_p} = \bigcup_{t \in S^{2j-1}_{\bar t_p}} \overline{S}_t$ and $S^{2j+1}_{\bar t_p} = \bigcup_{\bar t \in S^{2j}_{\bar t_p}} S_{\bar t}$. On the other hand, we also define $\overline{S}^0_{t_p} := \{t_p\}$, and $\overline{S}^1_{t_p} := \overline{S}_t = \bigcup_{t \in \overline{S}^0_{t_p}} \overline{S}_t$, $\overline{S}^2_{t_p} := \bigcup_{\bar t \in \overline{S}^1_{t_p}} S_{\bar t}$, and inductively, for $j \in \mathbb{N}_*$, $\overline{S}^{2j}_{t_p} := \bigcup_{\bar t \in \overline{S}^{2j-1}_{t_p}} S_{\bar t}$, and $\overline{S}^{2j+1}_{t_p} = \bigcup_{t \in \overline{S}^{2j}_{t_p}} \overline{S}_t$. Finally, we mention the following two elementary properties:

(1) $\overline{S^k_{\bar t_p}} = \overline{S}^k_{t_p}$ and $S^k_{\bar t_p} = \overline{\overline{S}^k_{t_p}}$, $k \in \mathbb{N}$.

(2) $\overline{h}(S^{2j}_{\bar t_p}) = S'^{2j}_{\overline{h}(\bar t_p)}$, $h(S^{2j+1}_{\bar t_p}) = S'^{2j+1}_{\overline{h}(\bar t_p)}$, $h(\overline{S}^{2j}_{t_p}) = \overline{S}'^{2j}_{h(t_p)}$, $\overline{h}(\overline{S}^{2j+1}_{t_p}) = \overline{S}'^{2j+1}_{h(t_p)}$.

§2.8. Generic Segre multitype

2.8.1. Segre chains with varying base point. As before, let M be a connected generic real algebraic or analytic submanifold of $\mathbb{C}^n$. Let $p_0 \in M$. In a neighborhood of p_0, we can consider the pair of Segre foliations of the local complexification $\mathcal{M}$ of M. Let $p = (t_p, \tau_p) \in \mathcal{M}$. Identifying the point p_0 with the origin in some system of coordinates, we have denoted by $\Gamma_k(z_{(k)})$ the mapping $[\mathcal{L} \, \text{or} \, \underline{\mathcal{L}}]_{z_k} \circ \cdots \circ \mathcal{L}_{z_1}(p_0, \bar p_0)$, namely the Segre chain with base point $(p_0)^c = (0,0) \in \mathbb{C}^n \times \mathbb{C}^n$. We want to let the base point vary, so we need a new notation. For $p = (t_p, \tau_p)$ in a neighborhood of $(p_0)^c$ in $\mathcal{M}$, we define

$$\Gamma_k(z_{(k)}, t_p, \tau_p) := [\mathcal{L} \, \text{or} \, \underline{\mathcal{L}}]_{z_k} \circ \cdots \circ \mathcal{L}_{z_1}(t_p, \tau_p). \tag{2.8.2}$$

Similarly, we define $\underline{\Gamma}_k(z_{(k)}, t_p, \tau_p)$. In order to indicate the dependence of the Segre type with respect to p, we shall denote it by μ_p. Also, we shall denote the Segre multitype at p by $(m, m, e_{3,p}, \ldots, e_{\mu_p,p})$.

By Theorem 2.5.2., the generic rank of Γ_k stabilizes when $k \geq \mu_p$. We know that $\mu_p \leq d+2$ for all p in a neighborhood of p_0 and that the mapping $z_{(k)} \mapsto \Gamma_k(z_{(k)}, t_p, \tau_p)$ provides an open piece of the $\{\mathcal{L}, \underline{\mathcal{L}}\}$-orbit through $p \in \mathcal{M}$ for all $k \geq 2\mu_p - 1$. As $2\mu_p - 1 \leq 2d+3$, and as $2d+3 \leq 3(m+d) = 3n$, because $m \geq 1$ and $d \geq 1$ by assumption, we observe that the mapping $\Gamma_{3n}(z_{(3n)}, t_p, \tau_p)$ with the uniform integer $k = 3n$ suffices to construct the $\{\mathcal{L}, \underline{\mathcal{L}}\}$-orbits of all points p in a neighborhood of $(p_0)^c$ in $\mathcal{M}$.

Thanks to this observation and thanks to the algebraicity or the analyticity of the mapping $\Gamma_{3n}(z_{(3n)}, t_p, \tau_p)$, it is easy to see that there is a proper complex algebraic or complex analytic subvariety $\mathcal{E}$ of $\mathcal{M}$ with the property that the Segre type and multitype of $\mathcal{M}$ are constant at every point $p \in \mathcal{M}\backslash\mathcal{E}$. We shall denote these constants by $(m, m, e_{3,M}, \dots, e_{\mu_M,M})$, where μ_M is the constant Segre type of $\mathcal{M}$ outside $\mathcal{E}$. In particular, the orbit dimension $2m + e_{1,M} + \cdots + e_{\mu_M,M}$ is constant in a neighborhood of p. Moreover, it also follows from the algebraicity or analyticity of the mappings $\Gamma_{3n}(z_{(3n)}, t_p, \tau_p)$ that the functions $p \mapsto \mu_p$ and $p \mapsto e_{3,p}, \dots, p \mapsto e_{\mu_p,p}$ are lower semi-continuous. Finally, using the σ-invariance of the CR orbits of complexified points $p^c = (t_p, \bar{t}_p) \in \mathcal{M} \cap \underline{\Lambda}$, with $p \in M$, we get the following theorem, which states that the Segre geometry possesses constant invariants over a Zariski open subset of M.

Theorem 2.8.3. *Let M be a connected real algebraic or analytic generic submanifold of $\mathbb{C}^n$ of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$. Then there is a proper real algebraic or analytic subvariety E of M such that for every point $p \in M\backslash E$, the Segre type of M at p is constant equal to an integer $\nu_M = \mu_M - 1 \leq d + 1$ and the Segre multitype of M at p is also constant equal to the multiplet $(m, e_{3,M}, \dots, e_{M,\mu_M})$. In particular, the CR-orbit dimension $\dim_{\mathbb{R}} \mathcal{O}_{CR}(M, p)$ is constant equal to $2m + e_{3,M} + \cdots + e_{\mu_M,M}$ for all $p \in M\backslash E$.*

We call μ_M the *generic Segre type of $\mathcal{M}$* and the multiplet $(m, m, e_{3,M}, \dots, e_{\mu_M,M})$ the *generic Segre multitype of $\mathcal{M}$*. Let

$$d_M := e_{3,M} + \cdots + e_{\mu_M,M}. \tag{2.8.4}$$

We call the integer $2m + d_M$ the *Zariski-generic orbit dimension* of M. We call the integer $d - d_M$ the *Zariski-generic orbit codimension* of M. Then using again the mapping $\Gamma_{3n}(z_{(3n)}, t_p, \tau_p)$, we can derive the following algebraic or analytic CR foliation theorem which shows that $d - d_M$ coincides with the Zariski-generic holomorphic codimension of the intrinsic complexification of CR orbits.

Corollary 2.8.5. *Let $p \in M\backslash E$ and set $d_{2,M} := d - e_{3,M} - \cdots - e_{\mu_M,M}$. Then a neighborhood of p in M is real algebraically or analytically foliated by CR orbits, namely there exist $d_{2,M}$ complex algebraic or analytic functions $h_1, \dots, h_{d_{2,M}}$ with $\partial h_1 \wedge \cdots \wedge \partial h_{d_{2,M}}(p) \neq 0$ such that*

(1) *M is contained in $\{h_1 = \bar{h}_1, \dots, h_{d_{2,M}} = \bar{h}_{d_{2,M}}\}$. In other words, M is contained in a transverse intersection of $d_{2,M}$ Levi flat hypersurfaces in general position.*

(2) *For every $c = (c_1, \dots, c_{d_{2,M}}) \in \mathbb{R}^{d_{2,M}}$, the manifold $M_c := M \cap \{h_1 = c_1, \dots, h_{d_{2,M}} = c_{d_{2,M}}\}$ is a CR orbit of M.*

Proof. Thanks to the mapping $\Gamma_{3n}(z_{(3n)}, t_p, \tau_p)$, we find real algebraic or analytic functions $h_1, \dots, h_{d_{2,M}}$ with linearly independent real differentials such that the level sets $\{h_1 = c_1, \dots, h_{d_{2,M}} = c_{d_{2,M}}\}$ are the CR orbits of M in a neighborhood of p. Since the functions $h_1, \dots, h_{d_{2,M}}$ are constant in each CR orbit, they are in particular trivially CR. By the Severi-Tomassini extension theorem, they extend complex algebraically or analytically to a neighborhood of p in $\mathbb{C}^n$. This proves the corollary. □

Taking the functions $h_1, \dots, h_{d_{2,M}}$ as part of a system of complex coordinates and applying Theorem 2.1.32, we deduce:

Corollary 2.8.6. *For every point $p \in M\backslash E$, there exist complex algebraic or analytic local normal coordinates $(z, w_1, w_2) \in \mathbb{C}^m \times \mathbb{C}^{d-d_{2,M}} \times \mathbb{C}^{d_{2,M}}$ vanishing at p such that the complex defining equations of M are of the form*

$$\begin{cases} 0 & = \bar{w}_2 - w_2, \\ 0 & = \bar{w}_1 - \Theta_1(\bar{z}, z, w_1, w_2), \end{cases} \tag{2.8.7}$$

where $\Theta_1(0, z, w_1, w_2) \equiv w_1$, where for $u_{2,q} \in \mathbb{R}^{d_{2,M}}$ sufficiently small, the sets $M \cap \{w_2 = u_{2,q} = ct.\}$ coincide with the local CR orbit of the points $q = (0, 0, u_{2,q}) \in M$ and where the *generic submanifold of* $\mathbb{C}^{m+d-d_{2,M}}$ *defined by the equations*

$$0 = \bar{w}_1 - \Theta_1(\bar{z}, z, w_1, u_{2,q}) \tag{2.8.8}$$

is minimal at $(z, w_1) = (0, 0)$ *for every* $u_{2,q}$.

§2.9. Local representation of nonminimal generic submanifolds

As a conclusion, we can now produce a general summary of important results which we will use constantly in the sequel. In advance, we formulate them in the most appropriate way for later use. As in Definition 2.1.44, let $M \subset \mathbb{C}^n$ be a local generic submanifold.

2.9.1. Minimal generic submanifolds. The following theorem is a corollary of Theorem 2.5.2 and Theorem 2.6.19 in the minimal case where $2m + e_2 + \cdots + e_{\mu_0} = 2m + d$. For later applications to the study of CR mappings, it is more convenient to state it with the conjugate Segre chain $\underline{\Gamma}_{2\nu_0}$.

Theorem 2.9.2. *If* M *is minimal at* p_0*, there exists a positive integer* $\nu_0 \leq d + 1$*, the Segre type of* M *at* p_0*, there exists an element* $\underline{z}^*_{(2\nu_0)} \in \mathbb{C}^{2m\nu_0}$ *arbitrarily close to the origin, there exists a* n*-dimensional complex affine subspace* H^* *passing through* $\underline{z}^*_{(2\nu_0)}$ *in* $\mathbb{C}^{2m\nu_0}$ *and there exists a complex affine parametrization* $s \mapsto \underline{z}_{(2\nu_0)}(s)$ *of* H^* *with* $\underline{z}_{(2\nu_0)}(0) = \underline{z}^*_{(2\nu_0)}$ *such that the mapping defined by composing the projection onto the first factor with the* $(2\nu_0)$*-th Segre chain, namely the mapping*

$$\mathbb{C}^n \ni s \longmapsto \pi_t(\underline{\Gamma}_{2\nu_0}(\underline{z}_{(2\nu_0)}(s))) \in \mathbb{C}^n \tag{2.9.3}$$

is of rank n *and vanishes at* $s = 0$.

We shall use this formulation very frequently in Part II of this memoir.

2.9.4. General generic submanifolds. In the case where M is not necessarily minimal, the holomorphic codimension in $\mathbb{C}^n$ of the local CR orbit $\mathcal{O}_{CR}(M, p_0)$ is an arbitrary integer d_2 with $0 \leq d_2 \leq d$ and $d_2 = 0$ if and only if M is minimal at p_0. We set $d_1 := d - d_2$, so $\mathcal{O}_{CR}(M, p_0)$ is of dimension $2m + d_1$. By Theorem 2.6.20, the intrinsic complexification $[\mathcal{O}_{CR}(M, p_0)]^{i_c}$ is a complex algebraic or analytic CR submanifold of $\mathbb{C}^n$ passing through p_0 which is of complex codimension d_2. After straightening it, we can assume that in coordinates $(z, w_1, w_2) \in \mathbb{C}^m \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$, it coincides with $\{z_2 = 0\}$, so there are local defining equations for M of the form

$$\begin{cases} \bar{w}_{1,j_1} & = \Theta_{1,j_1}(\bar{z}, z, w_1, w_2), \qquad j_1 = 1, \ldots, d_1, \\ \bar{w}_{2,j_2} & = \Theta_{2,j_2}(\bar{z}, z, w_1, w_2), \qquad j_2 = 1, \ldots, d_2, \end{cases} \tag{2.9.5}$$

where $\Theta_{2,j_2}(\bar{z}, z, w_1, 0) \equiv 0$ and where the generic submanifold M_1 of $\mathbb{C}^{m+d_1}$ defined by

$$M_1 := M \cap \{w_2 = 0\} \tag{2.9.6}$$

is minimal at the origin, with Segre type equal to ν_0. Complexifying $(M)^c := \mathcal{M}$, we obtain the equations

$$\begin{cases} \xi_{1,j_1} & = \Theta_{1,j_1}(\zeta, z, w_1, w_2), \qquad j_1 = 1, \ldots, d_1, \\ \xi_{2,j_2} & = \Theta_{2,j_2}(\zeta, z, w_1, w_2), \qquad j_2 = 2, \ldots, d_2, \end{cases} \tag{2.9.7}$$

for $k = 1, \ldots, m$ and the complexified $(1, 0)$ vector fields

$$\begin{cases} \mathcal{L}_k := \dfrac{\partial}{\partial z_k} + \displaystyle\sum_{j_1=1}^{d_1} \dfrac{\partial \Theta_{1,j_1}}{\partial z_k}(\zeta, z, w_1, w_2)\, \dfrac{\partial}{\partial w_{1,j_1}} + \\ \qquad\qquad + \displaystyle\sum_{j_2=1}^{d_2} \dfrac{\partial \Theta_{2,j_2}}{\partial z_k}(\zeta, z, w_1, w_2)\, \dfrac{\partial}{\partial w_{2,j_2}}, \end{cases} \tag{2.9.8}$$

for $k = 1, \dots, m$ and the complexified $(1,0)$ vector fields

$$\text{(2.9.9)} \qquad \begin{cases} \underline{\mathcal{L}}_k := \frac{\partial}{\partial \zeta_k} + \sum_{j_1=1}^{d_1} \frac{\partial \overline{\Theta}_{1,j_1}}{\partial \zeta_k}(z,\zeta,\xi_1,\xi_2)\, \frac{\partial}{\partial \xi_{1,j_1}} + \\ \qquad\qquad + \sum_{j_2=1}^{d_2} \frac{\partial \overline{\Theta}_{2,j_2}}{\partial \zeta_k}(z,\zeta,\xi_1,\xi_2)\, \frac{\partial}{\partial \xi_{2,j_2}}, \end{cases}$$

In the ambient space $\mathbb{C}^{2n}$ of the complexification $\mathcal{M}$, we shall denote the six coordinates in $\mathbb{C}^m \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^m \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ by

$$\text{(2.9.10)} \qquad (z, w_1, w_2, \zeta, \xi_1, \xi_2).$$

In $\mathbb{C}^{2n}$, the set

$$\text{(2.9.11)} \qquad \mathcal{T} := \{(0,0,w_2,0,\Theta_1(0,0,0,w_2),\Theta_2(0,0,0,w_2),w_2)\}$$

is a transversal in $\mathcal{M}$ to the complexification $\mathcal{M}_1 := (M_1)^c$ given by

$$\text{(2.9.12)} \qquad \mathcal{M}_1 : \quad w_2 = \xi_2 = 0, \qquad \xi_1 = \Theta_1(\zeta, w_1, 0),$$

namely we have $T_0\mathcal{M}_1 \oplus T_0\mathcal{T} = T_0\mathcal{M}$. Of course, this transversal depends on the choice of coordinates. To simplify a bit the expression of a choice of $\mathcal{T}$, we can (without loss of generality) assume that the coordinates (z, w_1, w_2) are normal, as described in Theorem 2.1.32, hence $\Theta_1(0, z, w_1, w_2) \equiv w_1$ and $\Theta_2(0, z, w_1, w_2) \equiv w_2$. Then

$$\text{(2.9.13)} \qquad \mathcal{T} = \{(0,0,w_2,0,0,w_2)\}.$$

With this choice, we may now generalize the definition of Segre chains by including the transversal parameter w_2 as follows. Firstly, for $z_{(1)} \in \mathbb{C}^m$, we set

$$\text{(2.9.14)} \qquad \begin{cases} \underline{\Gamma}_1(z_{(1)} : w_2) := \underline{\mathcal{L}}_{z_1}(0,0,w_2,0,0,w_2) \\ \qquad\qquad = (0,0,w_2,z_1,\Theta_1(z_1,0,0,w_2),\Theta_2(z_1,0,0,w_2)). \end{cases}$$

Secondly, for $z_{(2)} = (z_1, z_2) \in \mathbb{C}^{2m}$, we set

$$\text{(2.9.15)} \qquad \begin{cases} \underline{\Gamma}_2(z_{(2)} : w_2) \quad := \mathcal{L}_{z_2}(\Gamma_1(z_1 : w_2)), \\ \underline{\Gamma}_3(z_{(3)} : w_2) \quad := \underline{\mathcal{L}}_{z_3}(\underline{\Gamma}_2(z_{(2)} : w_2)), \end{cases}$$

and so on by induction. As a slight generalization of Theorem 2.5.2, we have the following theorem which describes the local Segre chain geometry in a neighborhood of an arbitrary point p_0 of M, without any nondegeneracy condition on M, in the most general setting.

Theorem 2.9.16. *If d_2 denotes the holomorphic codimension of the CR orbit of p_0 in $\mathbb{C}^n$ and if the coordinates $(z, w_1, w_2) \in \mathbb{C}^m \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ are chosen such that $M_1 := M \cap \{w_2 = 0\}$ is the CR orbit of p_0, if the integer ν_0 with $\nu_0 \leq d_2 + 1$ denotes the Segre type of M at p_0, then there exists an element $\underline{z}^*_{(2\nu_0)} \in \mathbb{C}^{2m\nu_0}$ arbitrarily close to the origin, there exists a $(n-d_2)$-dimensional complex affine subspace H^* passing through $\underline{z}^*_{(2\nu_0)}$ in $\mathbb{C}^{2m\nu_0}$, there exists a complex affine parametrization $s \mapsto \underline{z}_{(2\nu_0)}(s)$ of H^* with $\underline{z}_{(2\nu_0)}(0) = \underline{z}^*_{(2\nu_0)}$ such that the projection of the conjugate Segre chains with origin the transversal $\mathcal{T}$ to the complexification $\mathcal{M}_1$, namely*

$$\text{(2.9.17)} \qquad \mathbb{C}^{n-d_2} \times \mathbb{C}^{d_2} \ni (s, w_2) \longmapsto \pi_t(\underline{\Gamma}_{2\nu_0}(\underline{z}_{(2\nu_0)}(s) : w_2)) =: t \in \mathbb{C}^n$$

is of rank n and vanishes at $(s, w_2) = (0,0)$.

In particular, if M is real algebraic or real analytic, if the local CR orbits of points p varying in a neighborhood of p_0 are all of holomorphic codimension equal to d_2, it follows from Corollary 2.8.6 that we can represent M in a neighborhood of p_0 by the equations

$$\text{(2.9.18)} \qquad \begin{cases} \bar{w}_{1,j_1} \quad = \Theta_{1,j_1}(\bar{z}, z, w_1, w_2), \qquad j_1 = 1, \dots, d_1, \\ \bar{w}_{2,j_2} \quad = w_{2,j_2}, \qquad\qquad\qquad j_2 = 1, \dots, d_2, \end{cases}$$

where the last two equations represent the transversal intersection of d_2 Levi-flat hyperplanes in general position.

Corollary 2.9.19. *If M is real algebraic or analytic and if the orbit codimension is constant in a neighborhood of p_0, then for every $u_2 \in \mathbb{R}^{d_2}$ fixed, the image of the mapping*

$$\mathbb{C}^{n-d_2} \ni s \longmapsto \pi_t(\underline{\Gamma}_{2\nu_0}(\underline{z}_{(2\nu_0)}(s) : u_2)) =: t \in \mathbb{C}^n \tag{2.9.20}$$

covers a local piece of the intrinsic complexification of the CR orbit of the point in M with coordinates $(0, 0, u_2)$.

Chapter 3: Nondegeneracy conditions for generic submanifolds

§3.1. Segre mapping

3.1.1. Definition. Let M be a connected generic submanifold of $\mathbb{C}^n$ of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$ and let $p_0 \in M$. As provided by Theorem 2.1.9, we choose coordinates $t = (t_1, \dots, t_n) = (z_1, \dots, z_m, w_1, \dots, w_d) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at p_0 in which M is represented by the d complex defining equations

$$\bar{w}_j = \Theta_j(\bar{z}, t) = \Theta_j(\bar{z}, z, w), \qquad j = 1, \dots, d. \tag{3.1.2}$$

We remind that for every choice of coordinates (z, w) vanishing at p_0 such that $T_{p_0}^c M \cap (\{0\} \times \mathbb{C}_w^d) = \{0\}$, there exists a unique collection power series $\Theta_j(\bar{z}, t)$ such that M is represented by (3.1.2). Here, we shall assume that the powers series $\Theta_j(\bar{z}, z, w)$ are complex algebraic or analytic, namely they belong to $\mathcal{A}_{\mathbb{C}}\{\bar{z}, z, w\}$ or to $\mathbb{C}\{\bar{z}, z, w\}$. In this section, we shall only work at the central point p_0, which is the origin in these coordinates.

By developing the series $\Theta_j(\bar{z}, t)$ in powers of $\bar{z}$, we may write $\bar{w}_j = \sum_{\beta \in \mathbb{N}^m} \bar{z}^\beta \, \Theta_{j,\beta}(t)$. In terms of such a development, the *infinite Segre mapping of M* is defined to be the mapping

$$\mathcal{Q}_\infty : \quad \mathbb{C}^n \ni t \longmapsto (\Theta_{j,\beta}(t))_{1 \leq j \leq d, \, \beta \in \mathbb{N}^m} \in \mathbb{C}^\infty. \tag{3.1.3}$$

Let $k \in \mathbb{N}$. For finiteness reasons, it is convenient to truncate this infinite collection and to define the *k-th Segre mapping of M* by

$$\mathcal{Q}_k : \quad \mathbb{C}^n \ni t \longmapsto (\Theta_{j,\beta}(t))_{1 \leq j \leq d, \, |\beta| \leq k} \in \mathbb{C}^{N_{d,n,k}}, \tag{3.1.4}$$

where the integer $N_{d,n,k}$ denotes the number of k-th jets of a d-vectorial mapping of n independent variables $(t_1, \dots, t_n)$, namely $N_{d,n,k} = d \frac{(n+k)!}{n! \; k!}$. Let $k_2 \geq k_1$ and let π_{k_2,k_1} denote the canonical projection $\mathbb{C}^{N_{d,n,k_2}} \to \mathbb{C}^{N_{d,n,k_1}}$. Then we obviously have $\pi_{k_2,k_1}(\mathcal{Q}_{k_2}(t)) = \mathcal{Q}_{k_1}(t)$.

We shall see that these Segre mappings $\mathcal{Q}_k$ and $\mathcal{Q}_\infty$ are of utmost importance among the biholomorphically invariant objects attached to a real algebraic or analytic generic submanifold M.

3.1.5. Transformation of the Segre mapping under a change of coordinates. Apparently, the definition of the mappings $\mathcal{Q}_k$ strongly depends on the choice of coordinates and so the $\mathcal{Q}_k$ do not seem to represent an invariant analytico-geometric concept. However, we shall establish some canonical transformation rules which will show that all the definitions provided in this chapter are biholomorphically invariant.

The necessary ingredients for a biholomorphic transformation are as follows. Let $t' = h(t) = (h_1(t), \dots, h_n(t))$ be an invertible transformation, where the series $h_i(t)$ belong to $\mathcal{A}_{\mathbb{C}}\{t\}$ or to $\mathbb{C}\{t\}$, satisfy $h_i(0) = 0$ and $\det([\partial h_{i_1}/\partial t_{i_2}](0)_{1 \leq i_1, i_2 \leq n}) \neq 0$. Let $t = h'(t')$ denote the inverse mapping and split the mapping $h' = (f', g') = (f'_1, \dots, f'_m, g'_1, \dots, g'_d)$ according to the splitting $t = (z, w)$ of the coordinates t. Furthermore, substitute t by $h'(t')$ and $\bar{t}$ by $\bar{h}'(\bar{t}')$ in (3.1.2), which yields

$$\bar{g}'_j(\bar{t}') = \Theta_j(\bar{f}'(\bar{t}'), h'(t')), \qquad j = 1, \dots, d. \tag{3.1.6}$$

If necessary, we renumber the coordinates in order that after splitting $t' = (z'_1, \dots, z'_m, w'_1, \dots, w'_d)$, after applying the algebraic or the analytic implicit function theorem, we can solve $\bar{w}'$ in terms of $(\bar{z}', z', w')$ in the following form which is analogous to (3.1.2):

$$\bar{w}'_j = \Theta'_j(\bar{z}', t') = \Theta'_j(\bar{z}', z', w'), \qquad j = 1, \dots, d. \tag{3.1.7}$$

This yields an algebraic or an analytic generic submanifold M' of $\mathbb{C}^n$. Finally, we develope these series in powers of $\bar{z}'$, which yields $\bar{w}'_j = \sum_{\beta \in \mathbb{N}^m} (\bar{z}')^\beta \, \Theta'_{j,\beta}(t')$, and we define the *transformed k-th Segre mapping of* M' by

$$\mathcal{Q}'_k : \quad \mathbb{C}^n \ni t' \longmapsto (\Theta'_{j,\beta}(t'))_{1 \le j \le d, \, |\beta| \le k} \in \mathbb{C}^{N_{d,n,k}}, \tag{3.1.8}$$

and we also define the *transformed infinite Segre mapping* $\mathcal{Q}'_\infty$ *of* M'. With such notations at hand, we can now state the following important transformation rules which we will prove in Section 3.6 below.

Theorem 3.1.9. *For every $j = 1, \dots, d$ and every $\beta \in \mathbb{N}^m$, there exists a mapping $R_{j,\beta}$ which is complex algebraic or analytic in its variables such that*

$$\begin{cases} \Theta'_{j,\beta}(h(t)) + \displaystyle\sum_{\gamma \in \mathbb{N}^m \setminus \{0\}} \frac{(\beta+\gamma)!}{\beta! \; \gamma!} \, (\bar{f}(0, \Theta(0,t)))^\gamma \, \Theta'_{j,\beta+\gamma}(h(t)) \equiv \\ \qquad\qquad \equiv Q_{j,\beta}(\{\Theta_{j_1,\beta_1}(t)\}_{1 \le j_1 \le d, \, |\beta_1| \le |\beta|}). \end{cases} \tag{3.1.10}$$

Here, the left hand side is the power series developement of $\frac{1}{\beta!} \, [\partial^\beta_{\zeta'} \Theta'_{j,\beta}](\bar{f}(0, \Theta(0,t)), h(t))$*. A collection of relations equivalent to the collection* (3.1.10) *is as follows*

(7.1.11)

$$\begin{cases} \Theta'_{j,\beta}(h(t)) \equiv \displaystyle\sum_{\gamma \in \mathbb{N}^m} (-1)^\gamma \, \frac{(\beta+\gamma)!}{\beta! \; \gamma!} \, (\bar{f}(0, \Theta(0,t)))^\gamma \, Q_{j,\beta+\gamma}(\{\Theta_{j_1,\beta_1}(t)\}_{1 \le j_1 \le d, \, |\beta_1| \le |\beta|+|\gamma|}) \\ \qquad\qquad \equiv R_{j,\beta}(\{\Theta_{j_1,\beta_1}(t)\}_{1 \le j_1 \le d, \, \beta_1 \in \mathbb{N}^m}). \end{cases}$$

for every $j = 1, \dots, d$ and every $\beta \in \mathbb{N}^m$.

Admitting this theorem, we shall verify in Section 3.3 below that every nondegeneracy condition on M which is defined in terms of the Segre mapping $\mathcal{Q}_\infty$ is invariant under changes of coordinates, namely such a condition is satisfied for $(M, \mathcal{Q}_\infty)$ if and only if it is satisfied for $(M', \mathcal{Q}'_\infty)$. First of all, we present five such nondegeneracy conditions.

3.2. Five pointwise nondegeneracy conditions

3.2.1. Levi nondegeneracy. To begin with, we shall say that M is *Levi nondegenerate at* p_0 if the first Segre mapping $\mathcal{Q}_1$ is of rank n at the origin.

We verify that this definition coincides with the usual one, firstly in the case $d = 1$. After diagonalizing the Hermitian matrix of its Levi form, a given hypersurface M passing through the origin in $\mathbb{C}^n$ may be represented by the equation $\bar{w} = w + i \sum_{k=1}^r \varepsilon_k \, |z_k| + \mathrm{O}(3)$, where $0 \le r \le n-1$ is the rank of the Levi form of M at the origin and where $\varepsilon_k = \pm 1$. Then $\mathcal{Q}_1(t) = (w, i\varepsilon_1 z_1, \dots, i\varepsilon_r z_r, 0, \dots, 0) + \mathrm{O}(2)$. Clearly, the rank of $\mathcal{Q}_1$ at 0 equals n if and only if $r = n-1$, as announced.

Secondly, in the case $d \ge 2$, a generic submanifold of codimension d may be represented by equations $\bar{w}_j = w_j + i \sum_{k_1,k_2=1}^m a_{j,k_1,k_2} \, z_{k_1} \, \bar{z}_{k_2} + \mathrm{O}(3)$, where $a_{j,k_1,k_2} = \overline{a_{j,k_2,k_1}} \in \mathbb{C}$. Introducing the Hermitian forms $\langle A_j(z), \bar{z} \rangle := \sum_{k_1,k_2=1}^m a_{j,k_1,k_2} \, z_{k_1} \, \bar{z}_{k_2}$, where the $A_j(z) = (\sum_{k_1=1}^m a_{j,k_1,1} \, z_{k_1}, \dots, \sum_{k_1=1}^m a_{j,k_1,m} \, z_{k_1})$ are complex linear endomorphisms of $\mathbb{C}^m$, we know by definition that M is Levi nondegenerate at the origin if and only if $\bigcap_{j=1}^d \mathrm{Ker} \, A_j = \{0\}$. On the other hand, we may write

$$\mathcal{Q}_1(t) = (w_j, iA_j(z))_{1 \le j \le d} + \mathrm{O}(2), \tag{3.2.2}$$

hence the rank of $\mathcal{Q}_1$ at the origin equals n if and only if $\bigcap_{j=1}^d \operatorname{Ker} A_j = \{0\}$ again. In conclusion, the two definitions of Levi nondegeneracy are equivalent.

Equivalently, we remind that Levi nondegeneracy of M can be expressed directly by means of a collection of d defining equations $r_1(t,\bar{t}) = \cdots = r_d(t,\bar{t}) = 0$ for M in a neighborhood of p_0, simply by the condition that the intersection of the kernels of the Levi forms of the defining functions $r_1, \dots, r_d$ at the origin reduces to $\{0\}$. In Lemma 3.2.6 below, this criterion is generalized.

3.2.3. Finite nondegeneracy. More generally, we shall say that M is *finitely nondegenerate at p_0* if there exists an integer k such that the k-th Segre mapping $\mathcal{Q}_k$ is of rank n at the origin. Of course, this implies that $\mathcal{Q}_l$ is also of rank n for all $l \geq k$. If ℓ_0 denotes the smallest integer k such that $\mathcal{Q}_k$ is of rank n at the origin, we shall say that M is *ℓ_0-finitely nondegenerate at p_0*. Evidently, M is 1-finitely nondegenerate at p_0 if and only if it is Levi nondegenerate. For the moment, we shall admit that ℓ_0-nondegeneracy is independent of the choice of coordinates and of defining equations (this will be proved in Section 3.3 below and it follows in addition from the proof of Lemma 3.2.6 below). By the definition of $\mathcal{Q}_k$, the following characterization is immediate.

Lemma 3.2.4. *The generic submanifold M is ℓ_0-finitely nondegenerate at the origin if and only if there exists multiindices $\beta_*^1, \dots, \beta_*^n \in \mathbb{N}^m$ with $|\beta_*^i| \leq \ell_0$, $i = 1, \dots, n$, and integers $j_*^1, \dots, j_*^n$ with $1 \leq j_*^i \leq d$, $i = 1, \dots, n$, such that the rank at the origin of the mapping*

$$t \mapsto (\Theta_{j_*^1, \beta_*^1}(t), \dots, \Theta_{j_*^n, \beta_*^n}(t)) \tag{3.2.5}$$

is equal to n, but such a property is impossible with multiindices satisfying $|\beta_*^i| \leq \ell_0 - 1$.

Finite nondegeneracy of M at the origin can be expressed directly by means of a collection of d defining equations $r_1(t,\bar{t}) = \cdots = r_d(t,\bar{t}) = 0$ for M in a neighborhood of p_0. Indeed, let $\overline{L}_1, \dots, \overline{L}_m$ be a basis of $(0,1)$ vector fields tangent to M in a neighborhood of p_0. If $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$, we use the abbreviated notation $\overline{L}^\beta$ for the derivation of order $|\beta|$ defined by $\overline{L}_1^{\beta_1} \cdots \overline{L}_m^{\beta_m}$. Let $\nabla_t(r_j)(t,\bar{t})$ denote the complex gradient of $r_j(t,\bar{t})$, namely $([\partial r_j/\partial t_1](t,\bar{t}), \dots, [\partial r_j/\partial t_n](t,\bar{t}))$.

Lemma 3.2.6. *The generic submanifold M is ℓ_0-finitely nondegenerate at p_0 if and only if the complex linear span of all derivatives $\overline{L}^\beta[\nabla_t(r_j)(t,\bar{t})]|_{t=p_0}$, for $|\beta| \leq \ell_0$ and for $1 \leq j \leq d$, generates $\mathbb{C}^n$, a property which we may write symbolically as follows*

$$\operatorname{Span}_{\mathbb{C}}\{\overline{L}^\beta[\nabla_t(r_j)(t,\bar{t})]|_{t=0} : \beta \in \mathbb{N}^m, |\beta| \leq \ell_0, j = 1, \dots, d\} = \mathbb{C}^n, \tag{3.2.7}$$

but such a property cannot be satisfied with multiindices satisfying $|\beta| \leq \ell_0 - 1$.

Proof. Firstly, we can check directly that this new condition does not depend on the choice of a basis of $(0,1)$ vector fields tangent to M. Indeed, if $\overline{L}'_1, \dots, \overline{L}'_m$ is another basis, there exists an invertible $m \times m$ matrix of power series $b(t,\bar{t})$ such that $\overline{L}'_k = \sum_{l=1}^m b_{k,l}(t,\bar{t}) \, \overline{L}_l$. Then computing $(\overline{L}')^\beta \, [\nabla_t(r_j)(t,\bar{t})]|_{t=0}$, we obtain a linear combination with constant coefficients of the vectors $(\overline{L})^{\beta_1} \, [\nabla_t(r_j)(t,\bar{t})]|_{t=0}$, where $|\beta_1| \leq |\beta|$. It follows that we have the inclusion relation

$$\begin{cases} \operatorname{Span}_{\mathbb{C}}\{\overline{L'}^\beta[\nabla_t(r_j)(t,\bar{t})]|_{t=0} : |\beta| \leq \ell_0, j = 1, \dots, d\} \subset \\ \subset \operatorname{Span}_{\mathbb{C}}\{\overline{L}^\beta[\nabla_t(r_j)(t,\bar{t})]|_{t=0} : |\beta| \leq \ell_0, j = 1, \dots, d\}. \end{cases} \tag{3.2.8}$$

As b is invertible, we can reverse the rôles of the vector fields $\overline{L}_k$ and of the vector fields $\overline{L}'_k$, which yields the opposite inclusion relation, hence an equality in (3.2.8).

Secondly, we verify that the new condition (3.2.7) neither depends on the choice of defining equations for M, nor on the choice of coordinates. This is a little bit tedious, but the principle of proof is also quite simple. Indeed, let $t' = h(t)$ be a change of coordinates with $h(0) = 0$ and assume that the image $M' := h(M)$ is represented by equations $r'_1(t',\bar t') = \cdots = r'_d(t',\bar t') = 0$. Equivalently, there exists an invertible $d\times d$ matrix $a(t,\bar t)$ of complex algebraic or analytic power series such that $r'(h(t),\bar h(\bar t)) \equiv a(t,\bar t)\, r(t,\bar t)$. As we have already checked that the condition (3.2.7) does not depend on the choice of a basis of $(0,1)$ vector fields tangent to M', we may choose the basis $\overline{L}'_k := (\bar h)_*(\overline{L}_k)$, namely $\overline{L}'_k := \sum_{i=1}^n \overline{L}_k(\bar h_i)\, \partial_{\bar t'_i}$. If we denote the complex gradient $\nabla_t(r_j)$ by $([\partial r_j/\partial t_i](t,\bar t))_{1\leq i\leq n}$, we may compute the gradient of both sides of the vector identity $r'(h(t),\bar h(\bar t)) \equiv a(t,\bar t)\, r(t,\bar t)$, which yields for $j=1,\dots,d$:

$$
(3.2.9)\qquad \begin{cases} \left(\sum_{i'=1}^n \frac{\partial h_{i'}}{\partial t_i}(t)\, \frac{\partial r'_j}{\partial t'_{i'}}(h(t),\bar h(\bar t))\right)_{1\leq i\leq n} \equiv \\ \equiv \left(\sum_{l=1}^d \frac{\partial a_{j,l}}{\partial t_i}(t,\bar t)\, r_l(t,\bar t) + \sum_{l=1}^d a_{j,l}(t,\bar t)\, \frac{\partial r_l}{\partial t_i}(t,\bar t)\right)_{1\leq i\leq n}. \end{cases}
$$

Next, we apply the derivations $(\overline{L})^\beta$ to this identity. Taking into account that the holomorphic terms are not differentiated, using the definition of $\overline{L}'_k$ and using the rule of Leibniz for the differentiation of a product, we get the following expression

$$
(3.2.10)\qquad \begin{cases} \left(\sum_{i'=1}^n \frac{\partial h_{i'}}{\partial t_i}(t)\, (\overline{L'}^\beta)\left(\frac{\partial r'_j}{\partial t'_{i'}}\right)(h(t),\bar h(\bar t))\right)_{1\leq i\leq n} \equiv \\ \quad\equiv \left(\sum_{\gamma\leq\beta} \frac{\beta!}{\gamma!\,(\beta-\gamma)!}\left(\sum_{l=1}^d \left(\overline{L}^\gamma\left(\frac{\partial a_{j,l}}{\partial t_i}\right)(t,\bar t)\ \overline{L}^{\beta-\gamma}(r_l)(t,\bar t) + \right.\right.\right. \\ \qquad\qquad \left.\left.\left. +\overline{L}^\gamma(a_{j,l})(t,\bar t)\ \overline{L}^{\beta-\gamma}\left(\frac{\partial r_l}{\partial t_i}\right)(t,\bar t)\right)\right)\right)_{1\leq i\leq n}. \end{cases}
$$

Here, for a multiindex $\gamma\in\mathbb{N}^m$, we write $\gamma\leq\beta$ if $\gamma_k\leq\beta_k$ for $k=1,\dots,m$. In this expression, we set $t=0$. Since the vector fields $\overline{L}_k$ are tangent to M, all the expressions $\overline{L}^{\beta-\gamma}(r_l)(0,0)$ in the right hand side vanish. Using the invertibility of the Jacobian matrix $(\partial h_{i'}/\partial t_i(0))_{1\leq i,i'\leq n}$, we then see that the vectors $\left((\overline{L'})^\beta\left(\partial r'_j/\partial t'_{i'}\right)(0,0)\right)_{1\leq i'\leq n}$ are linear combinations with constant coefficients of the vectors $\left((\overline{L}^{\beta_1})\,(\partial r_{j_1}/\partial t_i)\,(0,0)\right)_{1\leq i\leq n}$, where $j_1 = 1,\dots,d$ and $|\beta_1|\leq|\beta|$. This entails that we have the inclusion relation

$$
(3.2.11)\qquad \begin{cases} \mathrm{Span}_{\mathbb{C}}\{(\overline{L'})^\beta[\nabla_{t'}(r'_j)(t',\bar t')]|_{t'=0} : \ |\beta|\leq\ell_0,\ j=1,\dots,d\} \subset \\ \qquad \subset \mathrm{Span}_{\mathbb{C}}\{\overline{L}^\beta[\nabla_t(r_j)(t,\bar t)]|_{t=0} : \ |\beta|\leq\ell_0,\ j=1,\dots,d\}. \end{cases}
$$

Let $t = h'(t')$ denote the inverse of $t'=h(t)$. Reasoning as above, we get the opposite inclusion relation, hence an equality in (3.2.11), as desired. We notice that the preceding reasoning also shows that for a fixed coordinate system, the condition (3.2.7) does not depend on the choice of d defining equations for M.

Finally, we check that this definition of finite nondegeneracy coincides with the first one given in the beginning of §3.2.3. As we have shown that the second definition of ℓ_0-finite nondegeneracy does not depend on the choice of defining equations, we can assume that $r_j(t,\bar t) := \Theta_j(\bar z,t) - \bar w_j$. Then $\nabla_t(r_j)(t,\bar t) = ([\partial\Theta_j/\partial t_1](\bar z,t),\dots,[\partial\Theta_j/\partial t_n](\bar z,t))$. We can also assume that the basis of $(0,1)$ vector fields tangent to M is the usual one, as in Chapter 2, which is given by $\overline{L}_k := \partial_{\bar z_k} + \sum_{j=1}^d \Theta_{j,\bar z_k}(\bar z,t)\,\partial_{\bar w_j}$, for $k=1,\dots,m$. Then using the development $\Theta_j(\bar z,t) = \sum_{\gamma\in\mathbb{N}^m}(\bar z)^\gamma\,\Theta_{j,\gamma}(t)$, we may compute

$$
(3.2.12)\qquad \overline{L}^\beta[\nabla_t(r_j)(t,\bar t)]|_{t=0} = \beta!\,([\partial\Theta_{j,\beta}/\partial t_1](0),\dots,[\partial\Theta_{j,\beta}/\partial t_n](0)).
$$

As the right hand side coincides up to a nonzero factor with the (j,β)-th column of the Jacobian matrix of the Segre mapping $\mathcal{Q}_{\ell_0} : t\mapsto(\Theta_{j,\beta}(t))_{1\leq j\leq d,\,|\beta|\leq\ell_0}$, we see immediately that $\mathcal{Q}_{\ell_0}$ is of rank n at the origin if and only if (3.2.7) holds. The proof of Lemma 3.2.6 is complete. □

Example 3.2.13. We provide some elementary examples of finitely nondegenerate hypersurfaces at the origin:

(1) $\bar{w} = w + i[z^5\bar{z} + \bar{z}^5 z]$ in $\mathbb{C}^2$ is 5-finitely nondegenerate.
(2) $\bar{w} = w + i[z_1\bar{z}_1 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2]$ in $\mathbb{C}^3$ is 2-finitely nondegenerate.
(3) $\bar{w} = w + i[z_1\bar{z}_1 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2 + z_1^3\bar{z}_3 + \bar{z}_1^3 z_3]$ in $\mathbb{C}^4$ is 3-finitely nondegenerate.

More generally, let $\varphi_1(z), \dots, \varphi_m(z)$ be a collection of holomorphic functions vanishing at the origin in $\mathbb{C}^m$ such that the mapping $z \mapsto (\varphi_1(z), \dots, \varphi_m(z))$ is of rank m at the origin. Let $\psi_1(z), \dots, \psi_m(z)$ be an arbitrary collection of nonconstant holomorphic functions with different order of vanishing at 0 and let ℓ_0 be the highest order of vanishing of the ψ_k. Then the hypersurface

$$\bar{w} = w + i\left[\varphi_1(z)\overline{\psi_1(z)} + \overline{\varphi_1(z)}\psi_1(z) + \cdots + \varphi_m(z)\overline{\psi_m(z)} + \overline{\varphi_m(z)}\psi_m(z)\right] \tag{3.2.14}$$

is ℓ_0-finitely nondegenerate at the origin. On the contrary, $\bar{w} = w + iz^2\bar{z}^2$ is not finitely nondegenerate at the origin (it is in fact essentially finite, *see* §3.2.25 below).

Finite nondegeneracy of M is not a gratuitous generalization of the notion of Levi nondegeneracy, which would be simply something like a folklore "higher order Levi form". On the contrary, it will appear to be a very natural nondegeneracy condition. In particular, we shall establish that an arbitrary real algebraic or analytic generic submanifold M is, locally in a neighborhood of a Zariski-generic point $p \in M$, biholomorphic to a product $\underline{M}_p' \times \Delta^\kappa$ of finitely nondegenerate generic submanifold $\underline{M}_p' \subset \mathbb{C}^{n-\kappa}$ with a certain polydisc Δ^κ. This property says that up to neglecting the "flat part" Δ^κ, every real algebraic or analytic generic submanifold is finitely nondegenerate at a Zariski-generic point. Furthermore, some classical CR manifolds (as the tube over the two-dimensional light cone for instance), are Levi degenerate at every point, but are finitely nondegenerate.

Example 3.2.15. The tube over the two-dimensional light cone $\Gamma_{\mathbb{C}}$ in $\mathbb{C}^3$ is the singular hypersurface defined by $u^2 = x_1^2 + x_2^2$, where $u = \operatorname{Re} w$ and $x_k = \operatorname{Re} z_k$, $k = 1, 2$. We consider the regular part of $\Gamma_{\mathbb{C}}$, which coincides with $\Gamma_{\mathbb{C}} \cap \{(x_1, x_2) \neq (0, 0)\}$. In a neighborhood of the smooth point $(1, 0, 1)$, we check that $\Gamma_{\mathbb{C}}$ is Levi degenerate. The $(0,1)$ vector fields tangent to $\Gamma_{\mathbb{C}}$ are generated by

$$\begin{cases} \overline{L}_1 := \dfrac{\partial}{\partial \bar{z}_1} + \dfrac{x_1}{u}\dfrac{\partial}{\partial \bar{w}}, \\ \overline{L}_2 := \dfrac{\partial}{\partial \bar{z}_2} + \dfrac{x_2}{u}\dfrac{\partial}{\partial \bar{w}}. \end{cases} \tag{3.2.16}$$

We observe that the dilatation vector field

$$\overline{T} := x_1\frac{\partial}{\partial \bar{z}_1} + x_2\frac{\partial}{\partial \bar{z}_2} + u\frac{\partial}{\partial \bar{w}}, \tag{3.2.17}$$

which coincides with $x_1\,\overline{L}_1 + x_2\,\overline{L}_2$ on M, lies in the kernel of the Levi form, since $[\overline{L}_1, T] = \frac{1}{2}\,L_1$ and $[\overline{L}_2, T] = \frac{1}{2}\,L_2$. We also observe that, according to [7], [11], $\Gamma_{\mathbb{C}}$ is necessarily foliated by complex curves. In fact, the regular locus of $\Gamma_{\mathbb{C}}$ is globally foliated by complex lines as follows: $z_1 := (r + is)\cos\theta + i\lambda$, $z_2 := (r + is)\sin\theta + i\mu$, $z_3 := r + is$, where $r, s, \theta, \lambda, \mu$ are real parameters. Finally, applying Lemma 3.2.6, we may check rapidly that $\Gamma_{\mathbb{C}}$ is 2-finitely nondegenerate at every point.

Example 3.2.18. A generalization of this example is the regular part M of M. Freeman's cubic $x_1^3 + x_2^3 - u^3 = 0$, *cf.* [11]. On $\{u \neq 0\}$, the $(0,1)$ vector fields tangent to M are generated by

$$\begin{cases} \overline{L}_1 := \frac{\partial}{\partial \bar{z}_1} + \frac{x_1^2}{u^2}\,\frac{\partial}{\partial \bar{w}}, \\ \overline{L}_2 := \frac{\partial}{\partial \bar{z}_2} + \frac{x_2^2}{u^2}\,\frac{\partial}{\partial \bar{w}}. \end{cases} \tag{3.2.19}$$

Again, the dilatation vector field lies in the kernel of the Levi form, M is foliated by complex lines, but M is (2- or 3-) finitely nondegenerate at every point.

Example 3.2.20. Another example of everywhere Levi degenerate real algebraic hypersurface in $\mathbb{C}^3$ is the hypersurface M_0 defined in the domain $\{(z_1, z_2, w) \in \mathbb{C}^3 : |z_2| < 1\}$ by the equation

$$\bar{w} = w + i\left[\frac{2z_1\bar{z}_1 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2)}{(1 - z_2\bar{z}_2)}\right]. \tag{3.2.21}$$

The $(0,1)$ vector fields tangent to M_0 are generated by

$$\begin{cases} \overline{L}_1 := \frac{\partial}{\partial \bar{z}_1} + i\left[\frac{2z_1 + 2\bar{z}_1 z_2}{1 - z_2\bar{z}_2}\right]\,\frac{\partial}{\partial \bar{w}}, \\ \overline{L}_2 := \frac{\partial}{\partial \bar{z}_2} + i\left[\frac{(z_1 + \bar{z}_1 z_2)^2}{(1 - z_2\bar{z}_2)^2}\right]\,\frac{\partial}{\partial \bar{w}}. \end{cases} \tag{3.2.22}$$

The kernel of the Levi form is generated by the vector field

$$\overline{T} := -\left[\frac{z_1 + \bar{z}_1 z_2}{1 - z_2\bar{z}_2}\right]\,\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2} - i\left[\frac{(z_1 + \bar{z}_1 z_2)^2}{(1 - z_2\bar{z}_2)^2}\right]\,\frac{\partial}{\partial \bar{w}}. \tag{3.2.23}$$

Indeed, we compute:

$$[\overline{L}_1, T] = -\left(\frac{1}{1 - z_2\bar{z}_2}\right)\,L_1, \qquad [\overline{L}_2, T] = -\left(\frac{z_1 + \bar{z}_1 z_2}{(1 - z_2\bar{z}_2)^2}\right)\,L_1. \tag{3.2.24}$$

Finally, according to [7], [11], M_0 is necessarily foliated by complex curves. In fact, M_0 is foliated by the complex lines $z_1 := z_0 - \bar{z}_0\zeta$, $z_2 := \zeta$, $w := -iz_0\bar{z}_0 + i\zeta\bar{z}_0^2 + \lambda$, where $z_0 \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and where the complex variable $\zeta \in \mathbb{C}$ satisfies $|\zeta| < 1$. Finally, by a direct application of Lemma 3.2.4, we see that M_0 is 2-finitely nondegenerate at every point.

3.2.25. Essential finiteness. More generally, we shall say that M is *essentially finite at* p_0 if there exists an integer k such that the Segre mapping $\mathcal{Q}_k$ is locally finite in a neighborhood of the origin. Of course, this implies that $\mathcal{Q}_l$ is also locally finite for all $l \geq k$. Moreover, finite nondegeneracy implies trivially essential finiteness. For the moment, we shall admit that essential finiteness is independent of the choice of coordinates and of defining equations (this will be proved in Section 3.3.). We say that M is ℓ_0-*essentially finite at* p_0 if ℓ_0 is the smallest such integer. By D. Hilbert's Nullstellensatz, the following characterization is immediate.

Lemma 3.2.26. *The generic submanifold M is ℓ_0-essentially finite at the origin if and only if the complex algebraic or analytic set defined by*

$$\Theta_{j,\beta}(t) - \Theta_{j,\beta}(0) = 0, \qquad j = 1, \ldots, d, \quad |\beta| \leq \ell_0, \tag{3.2.27}$$

is zero-dimensional at 0*, but the same complex algebraic or analytic subset defined with* $|\beta| \leq \ell_0 - 1$ *is positive-dimensional at* 0.

Classically, it is known that this property is equivalent to the fact that the ideal generated by the functions in the left hand side of (3.2.27) is of finite codimension in $\mathcal{A}_{\mathbb{C}}\{t\}$ or in $\mathbb{C}\{t\}$. More precisely, we define the integers ℓ_1 and ε_1 as follows: $\ell_1 \geq \ell_0$ is the smallest integer such that the ideal generated by all $\Theta_{j,\beta}(t) - \Theta_{j,\beta}(0)$ coincides with the ideal generated by the $\Theta_{j,\beta}(t) - \Theta_{j,\beta}(0)$ with $|\beta| \leq k$. This integer exists, by noetherianity of $\mathcal{A}_{\mathbb{C}}\{t\}$ or of $\mathbb{C}\{t\}$. Also, we define the integer ε_1 to be the codimension of the ideal $\langle \Theta_{j,\beta}(t) - \Theta_{j,\beta}(0) \rangle_{1 \leq j \leq d,\, |\beta| \leq \ell_1}$ in $\mathcal{A}_{\mathbb{C}}\{t\}$ or in $\mathbb{C}\{t\}$. By essential finiteness, $\varepsilon_1 < \infty$. We shall also observe in Section 3.3

below that ε_1 is a biholomorphic invariant of M at p_0. We call ε_1 the *essential type of M at p_0* and we denote it by $\mathrm{Ess\,Type}(M, p_0)$.

Essential finiteness of M at the origin can be expressed geometrically as follows. We introduce the locus of coincidence of Segre varieties

$$\mathbb{A}_{p_0} := \{t \in \Delta_n(\rho_1) : S_{\bar{t}} = S_{\bar{p}_0}\}. \tag{3.2.28}$$

Lemma 3.2.29. *The set $\mathbb{A}_{p_0}$ is a complex algebraic or analytic subset of a neighborhood of p_0 in $\mathbb{C}^n$ which is contained in M and which is described in local coordinates by $\mathbb{A}_{p_0} = \{t \in \Delta_n(\rho_1) : \Theta_{j,\beta}(t) = \Theta_{j,\beta}(0),\ j = 1, \dots, d,\ \beta \in \mathbb{N}^m\}$.*

Proof. Let $t \in \mathbb{A}_{p_0}$. Since $p_0 \in M$, we have $p_0 \in S_{\bar{p}_0}$ by Lemma 2.2.9. So $p_0 \in S_{\bar{t}}$, whence again by Lemma 2.2.9, $t \in S_{\bar{p}_0} = S_{\bar{t}}$, whence $t \in M$. This shows that $\mathbb{A}_{p_0}$ is contained in M. Next, in coordinates, we represent $S_{\bar{p}_0} = \{(z, w) \in \Delta_n(\rho_1) : w_j = \sum_{\beta \in \mathbb{N}^m} z^\beta\, \overline{\Theta}_{j,\beta}(0),\ j = 1, \dots, d\}$ and $S_{\bar{t}} = \{(z, w) \in \Delta_n(\rho_1) : w_j = \sum_{\beta \in \mathbb{N}^m} z^\beta\, \overline{\Theta}_{j,\beta}(\bar{t}),\ j = 1, \dots, d\}$. The two m-dimensional complex manifolds $S_{\bar{p}_0}$ and $S_{\bar{t}}$ coincide if and only if all the coefficients of their graphing functions coincide, namely $\overline{\Theta}_{j,\beta}(\bar{t}) = \overline{\Theta}_{j,\beta}(0)$ for all j and all β, which completes the proof. □

Example 3.2.30. We provide some elementary examples of essentially finite hypersurfaces at 0.

(1) $\bar{w} = w + i[z^N \bar{z}^N]$ in $\mathbb{C}^2$ has essential type equal to N.
(2) $\bar{w} = w + i[z_1^3 \bar{z}_1^3 + z_2^4 \bar{z}_2^4]$ in $\mathbb{C}^3$ has essential type equal to 12.

More generally, let $\varphi_1(z), \dots, \varphi_m(z)$, be an arbitrary collection of holomorphic functions with

$$\dim\left(\mathbb{C}\{z\}/\left\langle(\varphi_k(z))_{1 \le k \le m}\right\rangle\right) =: \varepsilon_1 < \infty. \tag{3.2.31}$$

Then the essential type at the origin of $w = \bar{w} + i[\varphi_1(z)\overline{\varphi_1(z)} + \cdots + \varphi_m(z)\overline{\varphi_m(z)}]$ is equal to ε_1. On the contrary, the hypersurface $\bar{w} = w + i[z_1 \bar{z}_1(1 + z_2 \bar{z}_2)]$ is not essentially finite at the origin (it is in fact Segre nondegenerate, *see* §3.2.32 just below).

3.2.32 Segre nondegeneracy. More generally, we shall say that M is *Segre nondegenerate at p_0* if there exists an integer k such that the restriction to $S_{\bar{p}_0}$ of the k-th Segre mapping is of maximal possible generic rank equal to m:

$$\mathrm{genrk}_{\mathbb{C}}\left(S_{\bar{p}_0} \ni t \mapsto \mathcal{Q}_k(t)\right) = m, \tag{3.2.33}$$

which means more precisely that

$$\mathrm{genrk}_{\mathbb{C}}\left(z \mapsto \left(\Theta_{j,\beta}(z, \overline{\Theta}(z, 0))_{1 \le j \le d,\, |\beta| \le k}\right)\right) = m. \tag{3.2.34}$$

We say that M is *ℓ_0-Segre nondegenerate at p_0* if ℓ_0 is the smallest such integer. Then the following characterization is immediate

Lemma 3.2.35. *The generic submanifold M is ℓ_0-Segre nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \le \ell_0$, $k = 1, \dots, m$, and integers $j_*^1, \dots, j_*^m$ with $1 \le j_*^k \le d$, $k = 1, \dots, m$, such that the determinant*

$$\det\left(\left([L_{k_1} \Theta_{j_*^{k_2}, \beta_*^{k_2}}](z, \overline{\Theta}(z, 0))\right)_{1 \le k_1, k_2 \le m}\right) \tag{3.2.36}$$

does not vanish identically as a power series in z, but such a property is impossible for multiindices satisfying $|\beta_*^k| \le \ell_0 - 1$.

Example 3.2.37. Let $\varphi_1(z), \dots, \varphi_m(z)$ be a collection of holomorphic functions defined in a neighborhood of the origin $\mathbb{C}^m$ such that the generic rank of $z \mapsto (\varphi_1(z), \dots, \varphi_m(z))$ is equal to m. Then the hypersurface $\bar{w} = w + i[\varphi_1(z)\overline{\varphi_1(z)} + \cdots + \varphi_m(z)\overline{\varphi_m(z)}]$ is Segre nondegenerate at the origin. On the contrary, the real algebraic hypersurface M in $\mathbb{C}^3$ of equation

$$\mathrm{Im}\, w = \frac{|z_1|^2 |1 + z_1 \bar{z}_2|^2}{1 + \mathrm{Re}\,(z_1 \bar{z}_2)} - \mathrm{Re}\, w\, \frac{\mathrm{Im}\,(z_1 \bar{z}_2)}{1 + \mathrm{Re}(z_1 \bar{z}_2)} \tag{3.2.38}$$

is not Segre nondegenerate at the origin (it is in fact holomorphically nondegenerate, *see* §3.2.40 just below). Indeed, solving with respect to $\bar{w}$, we may compute its complex defining equation, which yields

$$\bar{w} = -2i\, z_1 \bar{z}_1 (1 + z_1 \bar{z}_2) + w\, (1 + z_1 \bar{z}_2)/(1 + \bar{z}_1 z_2). \tag{3.2.39}$$

Then $S_{\bar{0}} = \{(z, 0)\}$ and the mappings $\mathcal{Q}_k|_{S_{\bar{0}}}$, for $k \geq 2$ are (up to zero terms) equal to $(z_1, z_2) \mapsto (-2iz_1, -2iz_1^2)$, hence of generic rank $1 < 2$.

3.2.40. Holomorphic nondegeneracy. More generally, we shall say that M is *holomorphically nondegenerate at p_0* if there exists an integer k such that the k-th Segre mapping is of maximal generic rank equal to n. This means that

$$\operatorname{genrk}_{\mathbb{C}} \left(t \mapsto (\Theta_{j,\beta}(t))_{1 \leq j \leq d,\, |\beta| \leq k} \right) = n. \tag{3.2.41}$$

We say that M is ℓ_0-holomorphically nondegenerate at p_0 if ℓ_0 is the smallest possible such integer. Then the following characterization is immediate

Lemma 3.2.42. *The generic submanifold M is holomorphically nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \ldots, \beta_*^n \in \mathbb{N}^n$ with $|\beta_*^i| \leq \ell_0$, $i = 1, \ldots, n$, and integers $j_*^1, \ldots, j_*^n$ with $1 \leq j_*^i \leq d$, $i = 1, \ldots, n$, such that the determinant*

$$\det \left(\left(\frac{\partial \Theta_{j_*^{i_1}, \beta_*^{i_1}}}{\partial t_{i_2}}(t) \right)_{1 \leq i_1, i_2 \leq m} \right) \tag{3.2.43}$$

does not vanish identically as a power series in t, but such a property is impossible for multiindices satisfying $|\beta_*^i| \leq \ell_0 - 1$.

3.2.44. Links between the five nondegeneracy conditions. Finally, we shall establish the following hierarchy between the five nondegeneracy conditions presented in this chapter.

Theorem 3.2.45. *Let M be a real algebraic or analytic generic submanifold in $\mathbb{C}^n$ and let $p_0 \in M$. Then the following four implications hold:*

$$\begin{cases} \quad M \text{ is } \ell_0\text{-holomorphically nondegenerate at } p_0 \Longleftarrow \\ \Longleftarrow M \text{ is } \ell_0\text{-Segre nondegenerate at } p_0 \Longleftarrow \\ \Longleftarrow M \text{ is } \ell_0\text{-essentially finite at } p_0 \Longleftarrow \\ \Longleftarrow M \text{ is } \ell_0\text{-finitely nondegenerate at } p_0 \Longleftarrow \\ \Longleftarrow M \text{ is Levi nondegenerate at } p_0. \end{cases} \tag{3.2.46}$$

Proof. We prove the first implication, considering that the other three follow from known results in local complex analytic geometry (*see* especially Lemma 4.1.4 below). By specializing the functional equations (2.1.25), we obtain $\Theta(0, z, \overline{\Theta}(z, 0, 0)) \equiv 0$ and $w \equiv \overline{\Theta}(0, 0, \Theta(0, 0, w))$. Remind the notational coincidence $\Theta_{j,0}(z, w) \equiv \Theta_j(0, z, w)$. It follows that the mapping $(0, w) \mapsto (\Theta_{j,0}(0, w))_{1 \leq j \leq d}$ is already of rank d at the origin. Furthermore, the restriction to $S_{\bar{0}}$ of the zeroth Segre mapping is identically zero: $\mathcal{Q}_0(z, \overline{\Theta}(z, 0)) = (\Theta_{j,0}(z, \overline{\Theta}(z, 0)))_{1 \leq j \leq d} \equiv 0$.

Suppose now that M is ℓ_0-Segre nondegenerate, hence ℓ_0 is the smallest integer such that the generic rank of the mapping $z \mapsto (\Theta_{j,\beta}(z, \overline{\Theta}(z, 0))_{1 \leq j \leq d,\ 1 \leq |\beta| \leq \ell_0})$ is equal to m (notice that we have written $1 \leq |\beta| \leq \ell_0$ and not $|\beta| \leq \ell_0$). Let $k \in \mathbb{N}$ with $k \geq 1$. It follows immediately that the generic rank of the mapping

$$(z, w) \mapsto \left((\Theta_{j,0}(0, w))_{1 \leq j \leq d}, (\Theta_{j,\beta}(z, \overline{\Theta}(z, 0)))_{1 \leq j \leq d,\ 1 \leq |\beta| \leq k} \right) \tag{3.2.47}$$

is equal to n if only if $k \geq \ell_0$, hence M is ℓ_0-holomorphically nondegenerate. This completes the proof of Theorem 3.2.45. □

3.2.48. Expression of the five nondegeneracy conditions in normal coordinates. Assume now that the coordinates (z, w) are normal, namely we have $\Theta_j(0, z, w) \equiv w_j$ and $\Theta_j(\bar{z}, 0, w) \equiv w_j$, *cf.* Theorem 2.1.32. It follows that in the development $\bar{w}_j = \sum_{\beta \in \mathbb{N}^m} (\bar{z})^\beta \, \Theta_{j,\beta}(t)$ of the defining equations of M, we have $\Theta_{j,0}(t) \equiv w_j$ and $\Theta_{j,\beta}(0) = 0$ for all j and all β. Then we can simplify a little bit the expression of the five nondegeneracy conditions, which is sometimes useful in applications.

Lemma 3.2.49. *In normal coordinates, we have the following characterizations:*

(1) *M is Levi nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| = 1$, $k = 1, \dots, m$, and integers $j_*^1, \dots, j_*^m$ with $1 \leq j_*^k \leq d$, $k = 1, \dots, m$, such that* $\det \left([\partial \Theta_{j_*^{k_1}, \beta_*^{k_1}} / \partial z_{k_2}](0) \right)_{1 \leq k_1, k_2 \leq m} \neq 0$.

(2) *M is ℓ_0-finitely nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \leq \ell_0$, $k = 1, \dots, m$, and integers $j_*^1, \dots, j_*^m$ with $1 \leq j_*^k \leq d$, $k = 1, \dots, m$, such that* $\det \left([\partial \Theta_{j_*^{k_1}, \beta_*^{k_1}} / \partial z_{k_2}](0) \right)_{1 \leq k_1, k_2 \leq m} \neq 0$, *but such a property is impossible for multiindices β_*^k satisfying $|\beta_*^k| \leq \ell_0 - 1$.*

(3) *M is essentially finite at the origin if and only if there exists an integer ℓ_0 such that the ideal generated by the $\Theta_{j,\beta}(z, 0)$ for $j = 1, \dots, d$ and $|\beta| \leq \ell_0$ is of finite codimension in $\mathcal{A}_{\mathbb{C}}\{z\}$ or in $\mathbb{C}\{z\}$, but the same ideal for $|\beta| \leq \ell_0$ is of infinite codimension.*

(4) *M is Segre nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \leq \ell_0$, $k = 1, \dots, m$, and integers $j_*^1, \dots, j_*^m$ with $1 \leq j_*^k \leq d$, $k = 1, \dots, m$, such that* $\det \left([\partial \Theta_{j_*^{k_1}, \beta_*^{k_1}} / \partial z_{k_2}](z, 0) \right)_{1 \leq k_1, k_2 \leq m} \not\equiv 0$ *in $\mathcal{A}_{\mathbb{C}}\{z\}$ or in $\mathbb{C}\{z\}$, but such a property is impossible for multiindices β_*^k satisfying $|\beta_*^k| \leq \ell_0 - 1$.*

(5) *M is holomorphically nondegenerate at the origin if and only if there exist multiindices $\beta_*^1, \dots, \beta_*^m \in \mathbb{N}^m$ with $|\beta_*^k| \leq \ell_0$, $k = 1, \dots, m$, and integers $j_*^1, \dots, j_*^m$ with $1 \leq j_*^k \leq d$, $k = 1, \dots, m$, such that* $\det \left([\partial \Theta_{j_*^{k_1}, \beta_*^{k_1}} / \partial z_{k_2}](z, w) \right)_{1 \leq k_1, k_2 \leq m} \not\equiv 0$ *in $\mathcal{A}_{\mathbb{C}}\{z, w\}$ or in $\mathbb{C}\{z, w\}$, but such a property is impossible for multiindices β_*^k satisfying $|\beta_*^k| \leq \ell_0 - 1$.*

Proof. Thanks to $\Theta_{j,0}(t) \equiv w_j$, we observe that the zero-th order Segre mapping already provides a rank d subset of power series. Then each one of the five characterizations may be checked easily. □

§3.3. Biholomorphic invariance

3.3.1. Finite nondegeneracy. Let $t' = h(t)$ be a change of coordinates centered at the origin and let $M' := h(M)$ as in §3.1.5. Our purpose is to verify that the above five nondegeneracy conditions are biholomorphically invariant. Then the following lemma is relatively crucial.

Lemma 3.3.2. *For every $k \in \mathbb{N}$, the ranks at the origin of the two Segre mappings $t \mapsto \mathcal{Q}_k(t)$ and $t' \mapsto \mathcal{Q}'_k(t')$ are equal.*

Proof. Let $\mathbf{1}_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^m$ denote the multiindex with 1 at the k-th place and zero elsewhere. By differentiating (3.1.10) with respect to t_i at $t_i = 0$, we have the following relations for all $j = 1, \dots, d$ and all $\beta \in \mathbb{N}^m$:

$$\text{(3.3.3)} \qquad \left\{ \begin{aligned} & \sum_{i'=1}^{n} \frac{\partial \Theta'_{j,\beta}}{\partial t'_{i'}}(0) \, \frac{\partial h_{i'}}{\partial t_i}(0) + \sum_{k=1}^{m} \sum_{l=1}^{d} (\beta_k + 1) \, \frac{\partial \bar{f}_k}{\partial w_l}(0) \, \frac{\partial \Theta_{0,l}}{\partial t_i}(0) \, \Theta'_{j,\beta+\mathbf{1}_k}(0) = \\ & \qquad = \sum_{j_1=1}^{d} \sum_{|\beta_1| \leq |\beta|} \frac{\partial Q_{j,\beta}}{\partial \Theta_{j_1,\beta_1}} (\{\Theta_{j_1,\beta_1}(0)\}_{1 \leq j_1 \leq d, \, |\beta_1| \leq |\beta|}) \, \frac{\partial \Theta_{j_1,\beta_1}}{\partial t_i}(0). \end{aligned} \right.$$

Since the Jacobian matrix $([\partial h_{i'} / \partial t_i](0))_{1 \leq i, i' \leq n}$ is invertible, we deduce immediately from this relation that each partial derivative $[\partial \Theta'_{j,\beta} / \partial t'_{i'}](0)$ is a linear combination with constant coefficients of the partial derivatives $[\partial \Theta_{j_1,\beta_1}/\partial t_i](0)$, with $i = 1, \dots, n$, $j_1 = 1, \dots, d$ and $|\beta_1| \leq |\beta|$. Consequently, for all $k \in \mathbb{N}$, we have the following inequality

$$\operatorname{rk}_0 \left(t' \mapsto (\Theta'_{j,\beta}(t'))_{1\leq j\leq d,\, |\beta|\leq k}\right) \leq \operatorname{rk}_0 \left(t \mapsto (\Theta_{j,\beta}(t))_{1\leq j\leq d,\, |\beta|\leq k}\right). \tag{3.3.4}$$

Applying the same reasoning to the inverse transformation $t = h'(t')$, we also obtain the reverse inequality. In conclusion, we have the equality of ranks

$$\operatorname{rk}_0 \left(t' \mapsto (\Theta'_{j,\beta}(t'))_{1\leq j\leq d,\, |\beta|\leq k}\right) = \operatorname{rk}_0 \left(t \mapsto (\Theta_{j,\beta}(t))_{1\leq j\leq d,\, |\beta|\leq k}\right), \tag{3.3.5}$$

which completes the proof of Lemma 3.3.2. □

In particular, it follows immediately that M is ℓ_0-finitely nondegenerate at p_0 if and only if M' is ℓ_0-finitely nondegenerate at $p'_0 = h(p_0)$. This proves that the definition given in §3.2.3 is invariant under complex algebraic or analytic changes of coordinates.

3.3.6. Levy multitype at the origin. More generally, since the ranks of the Segre mappings $\mathcal{Q}_k$ are invariant, we may introduce the successive invariant integers $\lambda_{k,0}$ such that the rank at the origin of $\mathcal{Q}_k$ equals $\lambda_{0,0} + \lambda_{1,0} + \cdots + \lambda_{k,0}$. Since the mapping $w \mapsto \Theta(0, 0, w)$ is invertible, we have $\lambda_{0,0} = d$. Since M is ℓ_0-finitely nondegenerate at the origin, we have

$$d + \lambda_{1,0} + \cdots + \lambda_{\ell_0,0} = n. \tag{3.3.7}$$

If M is ℓ_0-finitely nondegenerate at p_0, we call the multiplet $(d, \lambda_{1,0}, \dots, \lambda_{\ell_0,0})$ the *Levi multitype of M at p_0*.

3.3.8. Essential finiteness. Now, we check that essential finiteness is a biholomorphically invariant property. It suffices to prove the following lemma.

Lemma 3.3.9. *If $p'_0 = h(p_0)$, we have $h(\mathbb{A}_{p_0}) = \mathbb{A}'_{p'_0}$.*

Proof. Let $t \in \mathbb{A}_{p_0}$, namely $\Theta_{j,\beta}(t) = \Theta_{j,\beta}(0)$ for all $j = 1, \dots, d$ and all $\beta \in \mathbb{N}^m$. The recipe is again to look at (3.1.10). As $\Theta_0(t) = \Theta_0(0) = 0$ and as $\bar{f}(0) = 0$, we obtain from (3.1.10)

$$\begin{cases} \Theta'_{j,\beta}(h(t)) \equiv R_{j,\beta}(\{\Theta_{j_1,\beta_1}(t)\}_{1\leq j_1\leq d,\, |\beta_1|\leq|\beta|}) \\ \qquad\qquad \equiv R_{j,\beta}(\{\Theta_{j_1,\beta_1}(0)\}_{1\leq j_1\leq d,\, |\beta_1|\leq|\beta|}) \equiv \Theta'_{j,\beta}(h(0)), \end{cases} \tag{3.3.10}$$

so $h(t) \in \mathbb{A}'_{p'_0}$, where $p'_0 = h(p_0)$ is the origin in the coordinates t'. In other words, we have shown that $h(\mathbb{A}_{p_0}) \subset \mathbb{A}'_{p'_0}$. Since h is invertible, we also get similarly $h'(\mathbb{A}'_{p'_0}) \subset \mathbb{A}_{p_0}$. In conclusion, $h(\mathbb{A}_{p_0}) = \mathbb{A}'_{p'_0}$, as desired. This completes the proof. □

3.3.11. Segre nondegeneracy. Next, we check that Segre nondegeneracy is a biholomorphically invariant property. At first, we observe that it follows from the first functional equation in (2.1.25) that $\Theta(0, z, \overline{\Theta}(z, 0)) \equiv 0$, or equivalently $\Theta_0(z, \overline{\Theta}(z, 0)) \equiv 0$. The recipe is again to look at (3.1.10), replacing t by $(z, \overline{\Theta}(z, 0))$, which yields

$$\Theta'_{j,\beta}(h(z, \overline{\Theta}(z, 0))) \equiv R_{j,\beta}(\{\Theta_{j_1,\beta_1}(z, \overline{\Theta}(z, 0))\}_{1\leq j_1\leq d,\, |\beta_1|\leq|\beta|}). \tag{3.3.12}$$

Since h is invertible and maps the Segre variety $S_{\bar{p}_0}$ onto the Segre variety $S'_{\bar{p}'_0}$, we get the inequality

$$\operatorname{genrk}_{\mathbb{C}} \left(z' \mapsto \mathcal{Q}'_k(z', \overline{\Theta}'(z', 0))\right) \leq \operatorname{genrk}_{\mathbb{C}} \left(z \mapsto \mathcal{Q}_k(z, \overline{\Theta}(z, 0))\right). \tag{3.3.13}$$

Reversing the rôles of t and of t', we also get the opposite inequality, hence an equality. In particular, M is Segre nondegenerate at p_0 if and only if M' is Segre nondegenerate at p'_0.

3.3.14. Holomorphic nondegeneracy. Using again (3.1.10) and the analogous relation in which the rôles of t and of t' are reversed, we may establish that for $k \in \mathbb{N}$, we have

$$\operatorname{genrk}_{\mathbb{C}} (t' \mapsto \mathcal{Q}'_k(t')) = \operatorname{genrk}_{\mathbb{C}} (z \mapsto \mathcal{Q}_k(t)). \tag{3.3.15}$$

In particular, M is holomorphically nondegenerate at p_0 if and only if M' is holomorphically nondegenerate at p'_0.

§3.4. Manifolds without nondegeneracy conditions

3.4.1. Reflection mapping. It is also interesting to study generic submanifolds without requiring any nondegeneracy condition on them, as we shall see in our study of the generalized reflection principle in Part II of this memoir. Let $t' = h(t)$ be a local complex algebraic or analytic equivalence defined in a neighborhood of $p_0 \in M$ satisfying the same conditions as in §3.1.5. We do not require that h is invertible. Let $\bar{w}'_j = \Theta'_j(\bar{z}', t')$, $j = 1, \ldots, d$ be the equations of M'. Let $\bar{\nu}' := (\bar{\lambda}', \bar{\mu}') \in \mathbb{C}^m \times \mathbb{C}^d$. We define the *reflection mapping* associated to such defining equations to be the vectorial power series

$$\mathcal{R}'_h(t, \bar{\nu}') := \left(\bar{\mu}'_j - \Theta'_j(\bar{\lambda}', h(t))\right)_{1 \leq j \leq d}, \tag{3.4.2}$$

which belongs to $\mathcal{A}_{\mathbb{C}}\{t, \bar{\nu}'\}^d$ or to $\mathbb{C}\{t, \bar{\nu}'\}^d$. By developing the right hand side in powers of $\bar{\lambda}'$, we can write more explicitely

$$\mathcal{R}'_h(t, \bar{\nu}') = \left(\bar{\mu}'_j - \sum_{\beta \in \mathbb{N}^m} (\bar{\lambda}')^\beta \, \Theta'_{j,\beta}(h(t))\right)_{1 \leq j \leq d}. \tag{3.4.3}$$

The datum of $\mathcal{R}'_h$ is essentially equivalent to the datum of the infinite collection of complex algebraic or analytic functions $\Theta'_{j,\beta}(h(t))$, or equivalently to the composition of the infinite Segre mapping of M' with h, namely the mapping $t \mapsto \mathcal{Q}'_\infty(h(t))$.

We observe that $\mathcal{R}'_h$ is biholomorphic invariant in the following sense. Let $t'' = h'(t')$ be a second local mapping, which we assume to be invertible (do not confuse here h' with the notation used in §3.1.5 for the inverse of h). Let $M'' := h'(M')$ and assume that its equations are $\bar{w}''_j = \Theta''_j(\bar{z}'', z'', w'')$, $j = 1, \ldots, d$. We consider the composition $t'' = h'(h(t))$. Applying Theorem 3.1.9 and (3.1.11), we know that $\Theta''_{j,\beta}(h'(t')) \equiv R'_{j,\beta}(\{\Theta'_{j,\beta}(t')\}_{1 \leq j \leq d, \, \beta \in \mathbb{N}^m})$, where the $R'_{j,\beta}$ are certain algebraic or analytic expressions which depend only on h'. It follows that we can write

$$\begin{cases} \mathcal{R}''_{h' \circ h}(t, \bar{\nu}'') = \bar{\mu}'' - \displaystyle\sum_{\beta \in \mathbb{N}^m} (\bar{\lambda}'')^\beta \, \Theta''_{j,\beta}(h'(h(t))) \\ \qquad\qquad\qquad = \bar{\mu}'' - \displaystyle\sum_{\beta \in \mathbb{N}^m} (\bar{\lambda}'')^\beta \, R'_{j,\beta}(\{\Theta'_{j,\beta}(h(t))\}_{1 \leq j \leq d, \, \beta \in \mathbb{N}^m}). \end{cases} \tag{3.4.4}$$

So we can essentially express the reflection mapping $\mathcal{R}''_{h' \circ h}(t, \bar{\nu}'')$ by means of the reflection mapping $\mathcal{R}'_h(t, \bar{\nu}')$, modulo algebraic or analytic expressions $R'_{j,\beta}$ which only depend on the change of coordinates $t'' = h'(t')$. In particular, the $R'_{j,\beta'}$ are algebraic if M', h' and M'' are algebraic and the relation (3.4.4) shows that if $\mathcal{R}'_h(t, \bar{\nu}')$ was also algebraic at the beginning, then its algebraicity is preserved after the algebraic change of coordinates $t'' = h'(t')$. It is in this sense that we say that $\mathcal{R}'_h$ is a biholomorphic invariant object. We shall study the reflection mapping thoroughly in Part II of this memoir.

3.4.5. Tangent holomorphic vector fields. It is desirable to obtain a rank property analogous to (3.3.5) for points t_p close to the origin. Whereas for a fixed integer k, it is in general untrue that (3.3.5) holds with the first rk_0 replaced by $\mathrm{rk}_{h(t_p)}$ and the second rk_0 replaced by rk_{t_p} (*see* Example 3.5.16 below), the corresponding property for $k = \infty$ is true.

Corollary 3.4.6. *For all t_p in a neighborhood of the origin, the rank at t_p of the infinite Segre mapping $t \mapsto \mathcal{Q}_\infty(t)$ of M coincides with the rank at $t'_{p'} = h(t_p)$ of the infinite Segre mapping $t' \mapsto \mathcal{Q}'_\infty(t')$ of M'.*

Proof. There exists an integer ℓ_p such that the rank n_p at t_p of the infinite Segre mapping $t \mapsto \mathcal{Q}_\infty(t)$ coincides with the rank at t_p of the ℓ_p-th Segre mapping $t \mapsto \mathcal{Q}_{\ell_p}(t)$. Hence for every $l = 1, \ldots, d$ and every $|\beta| \geq \ell_p + 1$, the gradient of $\Theta_{j,\beta}(t)$ at t_p is a linear combination of the columns of the Jacobian matrix of $\mathrm{Jac}\, \mathcal{Q}_{\ell_p}(t_p)$. Using this fact, using the invertibilty of h and differentiating the two sides of (3.1.10), we deduce that the rank $n'_{p'}$ at $t'_{p'}$ of the infinite Segre mapping $t' \mapsto \mathcal{Q}'_\infty(t')$ is less than or equal to n_p, namely $n'_{p'} \leq n_p$. Since h is invertible, by considering the inverse $t = h'(t')$, we can reverse the rôles of t_p and of $t'_{p'}$ and we get the opposite inequality $n_p \leq n'_{p'}$, which completes the proof. □

In particular, the generic rank $n_M = \max_{p\in M} n_p$ of the infinite Segre mapping of M is a biholomorphic invariant of M. We call this integer the *essential holomorphic dimension of M*. Concretely, n_M is the smallest integer such that there exists a $n_M \times n_M$ minor of the infinite Jacobian matrix

$$\mathrm{Jac}\,\mathcal{Q}_\infty(t) = ([\partial \Theta_{j,\beta}/\partial t_i](t))^{1\leq i\leq n}_{1\leq j\leq d,\, \beta\in\mathbb{N}^m} \tag{3.4.7}$$

that does not vanish identically, but all its $(n_M+1)\times(n_M+1)$ minors do vanish identically.

Theorem 3.4.8. *With this integer n_M, there exist $(n-n_M)$ holomorphic vector fields*

$$\mathcal{T}_k = \sum_{i=1}^n a_i(t)\,\frac{\partial}{\partial t_i}, \qquad k = 1,\dots,n-n_M, \tag{3.4.9}$$

which have complex algebraic or analytic coefficients $a_i(t)$, which are defined in a neighborhood V_0 of the origin, which are tangent to $M\cap V_0$ and which are linearly independent at a Zariski-generic point $p\in V_0$. Conversely, the integer $(n-n_M)$ is the maximal number of holomorphic vector fields with complex algebraic or analytic coefficients defined in a neighborhood V_0 of the origin which are tangent to $M\cap V_0$ and linearly independent at a Zariski-generic point.

Proof. We choose integers $j_*^1,\dots,j_*^{n_M}$ with $1\leq j_*^i\leq d$, $i=1,\dots,n_M$, and multiindices $\beta_*^1,\dots,\beta_*^{n_M}\in\mathbb{N}^m$ such that the generic rank of the mapping $t\mapsto (\Theta_{j_*^l,\beta_*^l}(t))_{1\leq l\leq n_M}$ is equal to n_M. In other words, the Jacobian matrix $([\partial\Theta_{j_*^l,\beta_*^l}(t)/\partial t_i](t))_{1\leq l\leq n_M,\,1\leq i\leq n}$ possesses a $n_M\times n_M$ minor which does not vanish identically. If $n-n_M>0$, applying classical linear algebra (Cramer's rule for solving systems of linear equations), we see that there exist $(n-n_M)$ independent power series vectorial solutions $(a_{k,1}(t),\dots,a_{k,n}(t))$, $k=1,\dots,n-n_M$, of the system of n_M equations

$$\sum_{i=1}^n a_{k,i}(t)\,\frac{\partial\Theta_{j_*^l,\beta_*^l}}{\partial t_i}(t)\equiv 0, \qquad l=1,\dots,n_M. \tag{3.4.10}$$

Equivalently, the $(n-n_M)$ vector fields

$$\mathcal{T}_k := \sum_{i=1}^n a_{k,i}(t)\,\frac{\partial}{\partial t_i}, \tag{3.4.11}$$

$k=1,\dots,n-n_M$, are linearly independent at a Zariski-generic point of V_0 and they satisfy $\mathcal{T}_k\Theta_{j_*^l,\beta_*^l}(t)\equiv 0$ for $k=1,\dots,n-n_M$ and $l=1,\dots,n_M$. Since M is generic, the restriction to M of the vector fields $\mathcal{T}_k$ are also linearly independent at a Zariski-generic point.

Let now $(j,\beta)\neq(j_*^l,\beta_*^l)$, for $l=1,\dots,n_M$. By assumption, we also have

$$\mathrm{genrk}_{\mathbb{C}}\left(t\longmapsto\left((\Theta_{j_*^l,\beta_*^l}(t))_{1\leq l\leq n_M},\,\Theta_{j,\beta}(t)\right)\right)=n_M. \tag{3.4.12}$$

In a neighborhood V_p of a point $t_p\in V_0$ at which this rank is equal to its maximum n_M, there exists a complex algebraic or analytic mapping $R_{j,\beta}$ such that we can write

$$\Theta_{j,\beta}(t)\equiv R_{j,\beta}(\Theta_{j_*^1,\beta_*^1}(t),\dots,\Theta_{j_*^{n_M},\beta_*^{n_M}}(t)) \tag{3.4.13}$$

for all $t\in V_p$. Since $\mathcal{T}_k\Theta_{j_*^l,\beta_*^l}(t)\equiv 0$, it follows that $\mathcal{T}_k\Theta_{j,\beta}(t)\equiv 0$ for $t\in V_p$, hence for all $t\in V_0$ thanks to the principle of analytic continuation. In summary, we have shown that $\mathcal{T}_k\Theta_{j,\beta}(t)\equiv 0$ for all $k=1,\dots,n-n_M$, all $j=1,\dots,d$ and all $\beta\in\mathbb{N}^m$. We conclude immediately that the $\mathcal{T}_k$ are tangent to M, since

$$\mathcal{T}_k(\bar w_j-\Theta_j(\bar z,t))\equiv\sum_{\beta\in\mathbb{N}^m}(\bar z)^\beta\,\mathcal{T}_k\Theta_{j,\beta}(t)\equiv 0. \tag{3.4.14}$$

Conversely, suppose that there exist χ holomorphic vector fields $\mathcal{T}_k$, $k = 1, \dots, \chi$, like (3.4.9) with complex algebraic or analytic coefficients which are linearly independent at a Zariski-generic point of V_0 and such that $\mathcal{T}_k$ is tangent to $M \cap V_0$. By the tangency condition (3.4.14), we get $\mathcal{T}_k \Theta_{j,\beta}(t) \equiv 0$ for all $k = 1, \dots, \chi$, all $j = 1, \dots, d$ and all $\beta \in \mathbb{N}^m$. By considering these equations at a point at which the vector fields $\mathcal{T}_k$ are linearly independent, we deduce the inequality $\chi \leq n - n_M$. The proof of Theorem 3.4.8 is complete. □

Corollary 3.4.15. *The real algebraic or analytic generic submanifold M is holomorphically nondegenerate at* 0*, i.e. $n_M = n$, if and only if there does not exist any nonzero holomorphic vector field defined in a neighborhood V_0 of* 0 *which is tangent to $M \cap V_0$.*

This property may be considered as an equivalent definition of holomorphic nondegeneracy, as was done by N. Stanton in [28]. However, we believe that the previous definition in terms of the generic rank of the Segre mapping is more adequate.

3.4.16. Exceptional locus of M. We define the *extrinsic exceptional locus of M* to be the proper complex analytic set $\mathcal{E}^{\text{exc}}$ which is the zero locus of all $n_M \times n_M$ minors of the Jacobian matrix $\text{Jac}\, \mathcal{Q}_\infty(t)$. By Corollary 3.4.6, $\mathcal{E}^{\text{exc}}$ is an invariant complex algebraic or analytic set which is independent of coordinates. We define the *intrinsic exceptional locus* of M to the proper real algebraic or analytic subset $M \cap \mathcal{E}^{\text{exc}}$. By definition, the rank of $t \mapsto \mathcal{Q}_\infty(t)$ at $t_p \in V_0$ equals n_M if and only if $t_p \in M \backslash \mathcal{E}^{\text{exc}}$. We shall come back to $\mathcal{E}^{\text{exc}}$ in Corollary 3.5.53 below.

§3.5. Jets of Segre varieties and global nondegeneracy conditions

3.5.1. Fundamental definitions. Let $\mathcal{M}$ be the extrinsic complexification of M given by the equations $\xi_j = \Theta_j(\zeta, t)$, $j = 1, \dots, d$. We shall assume that $\mathcal{M}$ is complex algebraic or analytic and defined in the polydisc $\Delta_{2n}(\rho_1)$, as in Definition 2.1.44. Recall that the conjugate complexified Segre variety $\underline{\mathcal{S}}_t$ is the m-dimensional complex algebraic or analytic submanifold defined by the equations $\xi_j = \Theta_j(\zeta, t)$, $j = 1, \dots, d$, where t is fixed. We consider the mapping of k-th order jets of $\underline{\mathcal{S}}_t$ at one of its point $(\zeta, \Theta(\zeta, t))$ which is defined by

$$J_\tau^k \underline{\mathcal{S}}_t := \left(\zeta, \left(\frac{1}{\beta!} \, \partial_\zeta^\beta \Theta_j(\zeta, t) \right)_{1 \leq j \leq d,\, |\beta| \leq k} \right). \tag{3.5.2}$$

In this section, we shall study the mapping $J_\tau^k \underline{\mathcal{S}}_t$ thoroughly. It is complex algebraic or analytic with values in $\mathbb{C}^{m+N_{d,m,k}}$. If $k_2 \geq k_1$ and if π_{k_2,k_1} denotes the canonical projection $\mathbb{C}^{m+N_{d,m,k_2}} \to \mathbb{C}^{m+N_{d,m,k_1}}$, we obviously have $\pi_{k_2,k_1}(J_\tau^{k_2} \underline{\mathcal{S}}_t) = J_\tau^{k_1} \underline{\mathcal{S}}_t$.

Compared to the Segre mapping introduced in §3.1.1, here we notice that the term ζ is present and we may obviously identify $J_\tau^k \underline{\mathcal{S}}_t|_{\zeta=0}$ with $\mathcal{Q}_k(t)$.

We shall sometimes denote this mapping by $(t, \tau) \mapsto J_\tau^k \underline{\mathcal{S}}_t$, where we implicitely mean that $(t, \tau) \in \mathcal{M}$.

Analogously, we may also consider the mapping of k-th order jets of the complexified Segre variety $\mathcal{S}_\tau$ defined by the equations $w_j = \overline{\Theta}_j(z, \tau)$, $j = 1, \dots, d$, where τ is fixed. Its explicit expression is similar:

$$J_t^k \mathcal{S}_\tau := \left(z, \left(\frac{1}{\beta!} \, \partial_z^\beta \overline{\Theta}_j(z, \tau) \right)_{1 \leq j \leq d,\, |\beta| \leq k} \right). \tag{3.5.3}$$

The link between these two mappings is very simple:

$$J_{\bar{t}}^k \underline{\mathcal{S}}_{\bar{\tau}} \equiv \overline{J_t^k \mathcal{S}_\tau} \tag{3.5.4}$$

In other words, the following diagram is commutative.

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\ \sigma\ } & \mathcal{M} \\
{\scriptstyle J^k_\bullet \mathcal{S}_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle J^k_\bullet \underline{\mathcal{S}}_\bullet} \\
\mathbb{C}^{m+N_{d,m,k}} & \xrightarrow{\ (\bar{\bullet})\ } & \mathbb{C}^{m+N_{d,m,k}}
\end{array},
$$

where $(\bar{\bullet})$ denotes the complex conjugation operator. Since the two jet mappings are therefore essentially equivalent, we shall only study the nondegeneracy conditions for the mapping $J^k_\tau \underline{\mathcal{S}}_t$.

3.5.5. Invariance under changes of coordinates. As in §3.1.5, let $t' = h(t)$ be a change of complex algebraic or analytic coordinates. We shall prove the following theorem in Section 3.6 below. Notice that for $\zeta = 0$, we recover Theorem 3.1.9.

Theorem 3.5.6. *For every $j = 1, \dots, d$ and every $\beta \in \mathbb{N}^m$, there exists a complex algebraic or analytic mapping in its variables $Q_{j,\beta}$ such that*

$$
\frac{1}{\beta!}\,\frac{\partial^{|\beta|}\Theta'_j}{\partial(\zeta')^\beta}\left(\bar f(\zeta,\Theta(\zeta,t)),h(t)\right) \equiv Q_{j,\beta}\left(\zeta,\ \left(\frac{1}{\beta_1!}\,\partial_\zeta^{\beta_1}\Theta_{j_1}(\zeta,t)\right)_{1\leq j_1\leq d,\,|\beta_1|\leq|\beta|}\right). \tag{3.5.7}
$$

Here, the points (ζ, t) belong to a neighborhood of the origin, say to the polydisc $\Delta_{2m+d}(\rho_1)$. Fix $p = (t_p, \tau_p) \in \mathcal{M}$ which we identify with $(\zeta_p, t_p) \in \mathbb{C}^{2m+d}$ and denote $t'_{p'} := h(t_p)$ and $\zeta'_{p'} := \bar f(\tau_p)$. Locally in a neighborhood of (ζ_p, t_p), the mapping

$$
(\zeta,t)\mapsto(\bar f(\zeta,\Theta(\zeta,t)),h(t)) =: (\zeta',t') \tag{3.5.8}
$$

is invertible by assumption (remind that $\zeta \mapsto \bar f(\zeta, \Theta(\zeta, 0))$ is invertible at $\zeta = 0$). Applying Theorem 3.5.6, we may deduce:

Corollary 3.5.9. *For every $k \in \mathbb{N}$, the following equality of ranks holds*

$$
\left\{
\begin{aligned}
&\mathrm{rk}_{(\zeta_p,t_p)}\left((\zeta,t)\mapsto(\zeta,\ (1/\beta!)\,(\partial_\zeta^\beta\Theta_j(\zeta,t))_{1\leq j\leq d,\,|\beta|\leq k})\right) = \\
&\qquad = \mathrm{rk}_{(\zeta'_{p'},t'_{p'})}\left((\zeta',t')\mapsto(\zeta',\ (1/\beta!)\,(\partial_{\zeta'}^\beta\Theta'_j(\zeta',t'))_{1\leq j\leq d,\,|\beta|\leq k})\right).
\end{aligned}
\right. \tag{3.5.10}
$$

Proof. Indeed, using the change of coordinates (3.5.8), it suffices to show that the rank at (ζ_p, t_p) of the mapping $(\zeta,t)\mapsto\left(\bar f(\zeta,\Theta(\zeta,t)),\ \frac{1}{\beta!}\,\partial_{\zeta'}^\beta\Theta'_j(\bar f(\zeta,\Theta(\zeta,t)),h(t))\right)_{1\leq j\leq d,\,|\beta|\leq k}$ is less than or equal to the rank at (ζ_p, t_p) of the mapping $(\zeta,t)\mapsto\left(\zeta,\ \frac{1}{\beta!}\,\partial_\zeta^\beta\Theta_j(\zeta,t)\right)_{1\leq j\leq d,\,|\beta|\leq k}$, because after reversing the rôles of t and of t', we also get the opposite inequality. But this inequality follows directly by differentiating the two sides of (3.5.7) with respect to (ζ, t) at (ζ_p, t_p). □

Let $p \in M$. The ranks at $(p, \bar p)$ of the mappings $(t,\tau) \mapsto J^k_\tau \underline{\mathcal{S}}_t$ are invariant under changes of coordinates. Thus, we may introduce several pointwise invariants of M at p as follows. We denote by $m + n_p \leq m + n$ the maximal rank of the mapping $(t,\tau) \mapsto J^k_\tau \underline{\mathcal{S}}_t$ at $(t_p, \bar t_p)$ for $k = 0, 1, \dots$ and by ℓ_p the smallest integer k such that the rank at $(t_p, \bar t_p)$ of the mapping $(t,\tau) \mapsto J^k_\tau \underline{\mathcal{S}}_t$ is equal to $m + n_p$. More generally, for $k = 0, \dots, \ell_M$, we denote by $\lambda_{k,p}$ the nonnegative integers satisfying

$$
\mathrm{rk}_{(t_p,\bar t_p)}\left((t,\tau)\mapsto J^k_\tau\underline{\mathcal{S}}_t\right) = m+\lambda_{0,p}+\cdots+\lambda_{k,p}. \tag{3.5.11}
$$

Obviously, the functions $p \mapsto n_p$, $p \mapsto \ell_p$, $p \mapsto \lambda_{k,p}$ are lower semicontinuous in the Zariski topology.

3.5.12. Generic ranks. To begin with, we observe that it follows from Corollary 3.5.9 that the generic ranks of the mappings $(t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t$, which increase with k, are invariant under changes of coordinates. We need the following stabilization result.

Lemma 3.5.13. *If* $\mathrm{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^{k+1}_\tau \underline{\mathcal{S}}_t\right) = \mathrm{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^{k}_\tau \underline{\mathcal{S}}_t\right)$, *then for all* $l\geq 1$, *we also have* $\mathrm{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^{k+l}_\tau \underline{\mathcal{S}}_t\right) = \mathrm{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^{k}_\tau \underline{\mathcal{S}}_t\right)$.

Proof. By assumption, in a neighborhood $\mathcal{V}_p$ of a point $(t_p,\tau_p)\in\mathcal{M}$ at which the ranks of the first two mappings are equal to their generic rank, hence maximal and locally constant, it follows from the constant rank theorem that for every $j=1,\dots,d$ and for every multiindex β with $|\beta|=k+1$, there exists a complex algebraic or analytic function $R_{j,\beta}$ such that we can write

$$\frac{1}{\beta!}\,\partial_\zeta^\beta \Theta_j(\zeta,t) = R_{j,\beta}\left(\zeta,\,\left(\frac{1}{\beta_1!}\,\partial_\zeta^{\beta_1}\Theta_{j_1}(\zeta,t)\right)_{1\leq j_1\leq d,\,|\beta_1|\leq|k|}\right), \tag{3.5.14}$$

for all $(t,\tau)\in\mathcal{V}_p$. Differentiating these relations with respect to ζ and making substitutions, we obtain that for every $j=1,\dots,d$ and every multiindex β with $|\beta|=k+l$, $l\geq 1$, there exists a complex algebraic or analytic function $R_{j,\beta}$ satisfying a relation like (3.5.14). This implies that the generic rank of the mapping $\mathcal{V}_p\ni(t,\tau)\mapsto J^{k+l}_\tau\underline{\mathcal{S}}_t$ is the same as the generic rank of the mapping $\mathcal{V}_p\ni(t,\tau)\mapsto J^{k}_\tau\underline{\mathcal{S}}_t$. As the generic rank propagates by the principle of analytic continuation, the lemma follows. □

3.5.15. Reformulation of the five nondegeneracy conditions. We shall now observe that the five nondegeneracy conditions introduced in Section 3.2 may also be expressed by means of the morphism of jets of Segre varieties. This formulation will be much better than the formulation in terms of the Segre mapping given in Section 3.2, because it will be valuable not only for the central point p_0, but also for an arbitrary point p varying in a neighborhood of p_0. Before stating the theorem, we observe that, on the contrary, the k-th Segre mappings $t\mapsto\mathcal{Q}_k(t)$ are not appropriate to express the nondegeneracy conditions for other points than the origin.

Example 3.5.16. The first Segre mapping $\mathcal{Q}_1$ of the real algebraic hypersurface M of $\mathbb{C}^3$ given by the equation (which is the cubic tangent to the Example 3.2.20)

$$\bar{w} = w + i[2z_1\bar{z}_1 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2] \tag{3.5.17}$$

is the map

$$(z_1,z_2,w)\longmapsto(w,\,2iz_1,\,iz_1^2), \tag{3.5.18}$$

which is only of rank 2 at every point. This shows that the rank of the Levi form of M at the origin is equal to 1, which is true. Does this imply that the rank of the Levi form of M is equal to 1 at every point ? Of course not, because the Levi matrix

$$\mathcal{H}(\varphi)(z,\bar{z}) = \begin{pmatrix} \frac{\partial^2\varphi}{\partial z_1\partial\bar{z}_1} & \frac{\partial^2\varphi}{\partial z_2\partial\bar{z}_1} \\ \frac{\partial^2\varphi}{\partial z_1\partial\bar{z}_2} & \frac{\partial^2\varphi}{\partial z_2\partial\bar{z}_2}\end{pmatrix} = \begin{pmatrix} 2 & 2z_1 \\ 2\bar{z}_1 & 0\end{pmatrix} \tag{3.5.19}$$

is of rank 2 at every point with $z_1\neq 0$, hence M is Levi nondegenerate outside $\{z_1=0\}$. This example shows that the k-th Segre mappings $\mathcal{Q}_k$ are appropriate to define nondegeneracy conditions only at the origin.

We shall therefore make some translations. Let $p=t_p=(z_p,w_p)$ be a point varying in a neighborhood of the central point p_0. To express the five nondegeneracy conditions using the original definitions given in Section 3.2, we must choose coordinates vanishing at p. Since we have already argued that the nondegeneracy conditions are then independent of the choice of coordinates vanishing at p, we can simply make a translation of coordinates by setting

$$t_1 := t - t_p, \qquad \text{or equivalently} \qquad t = t_p + t_1. \tag{3.5.20}$$

We shall assume that t_p belongs to M, hence $p^c = (t_p, \bar{t}_p)$ belongs to $\mathcal{M}$. The precise changes of coordinates will therefore be

$$\left\{ \begin{array}{rclrcl} z_1 & = & z - z_p, & z & = & z_1 + z_p, \\ w_1 & = & w - w_p, & w & = & w_1 + w_p, \\ \zeta_1 & = & \zeta - \bar{z}_p, & \zeta & = & \zeta_1 + \bar{z}_p, \\ \xi_1 & = & \xi - \bar{w}_p, & \xi & = & \xi_1 + \bar{w}_p. \end{array} \right. \tag{3.5.21}$$

In the coordinates $t_1 = (z_1, w_1)$ vanishing at t_p, we can represent the translation $\mathcal{M}_1$ of $\mathcal{M}$ by complex defining equations

$$\xi_{1,j} = \Theta_{1,j}(\zeta_1, t_1), \qquad j = 1, \dots, d. \tag{3.5.22}$$

Of course, we may compute the $\Theta_{1,j}(\zeta_1, t_1)$ in terms of the $\Theta_j(\zeta, t)$ as follows. Since $\bar{w}_{j,p} = \Theta_j(\bar{z}_p, t_p)$, we have

$$\Theta_{1,j}(\zeta_1, t_1) = \xi_{1,j} = \xi_j - \bar{w}_{j,p} = \Theta_j(\zeta, t) - \Theta_j(\bar{z}_p, t_p), \tag{3.5.23}$$

which yields for $j = 1, \dots, d$:

$$\Theta_{1,j}(\zeta_1, t_1) = \Theta_j(\zeta_1 + \bar{z}_p, t_1 + t_p) - \Theta_j(\bar{z}_p, t_p). \tag{3.5.24}$$

If we now develope $\Theta_{1,j}(\zeta_1, t_1)$ in powers of ζ_1 (as we did for $\Theta(\zeta, t)$)

$$\Theta_{1,j}(\zeta_1, t_1) = \sum_{\beta \in \mathbb{N}^m} (\zeta_1)^\beta \, \Theta_{1,j,\beta}(t_1), \tag{3.5.25}$$

then putting $\zeta_1 = 0$ in (3.5.24), we obtain for $\beta = 0 \in \mathbb{N}^m$ the formula

$$\Theta_{1,j,0}(t_1) = \sum_{\beta \in \mathbb{N}^m} (\bar{z}_p)^\beta \left[\Theta_{j,\beta}(t_1 + t_p) - \Theta_{j,\beta}(t_p) \right], \tag{3.5.26}$$

and differentiating (3.5.24) at $\zeta_1 = 0$, we also obtain for all nonzero multiindices $\beta \in \mathbb{N}^m \backslash \{0\}$ the important general explicit formulas

$$\left\{ \begin{aligned} \Theta_{1,j,\beta}(t_1) &= \sum_{\gamma \in \mathbb{N}^m} \frac{(\beta + \gamma)!}{\beta! \; \gamma!} \, (\bar{z}_p)^\gamma \, \Theta_{j,\beta+\gamma}(t_1 + t_p) \\ &= \frac{1}{\beta!} \left[\partial_\zeta^\beta \Theta_j(\zeta, t) \right]_{\zeta = \bar{z}_p, \, t = t_1 + t_p}. \end{aligned} \right. \tag{3.5.27}$$

By means of these formulas, we have thus expressed $\Theta_{1,j}(\zeta_1, t_1)$ in terms of $\Theta(\zeta, t)$.

Consequently, we can introduce the k-th Segre mapping in the coordinates t_1

$$\mathcal{Q}_{1,k} : \quad \mathbb{C}^n \ni t_1 \longmapsto (\Theta_{1,j,\beta}(t_1))_{1 \leq j \leq d, \, |\beta| \leq k} \in \mathbb{C}^{N_{d,n,k}}, \tag{3.5.28}$$

and speak of the five nondegeneracy conditions in terms of $\mathcal{Q}_{1,k}$, since the coordinates t_1 are centered at t_p.

However, a better way of considering the nondegeneracy conditions at points p in a neighborhood of p_0 would be to express them in a single system of coordinates.

The following theorem provides the desired characterization of the five nondegeneracy conditions by means of the morphism of jets of Segre varieties, expressed in a single system of coordinates.

Theorem 3.5.29. *Let M be a real algebraic or analytic local generic submanifold of $\mathbb{C}^n$ given as usual by the equations* (3.1.2) *and let $J_\tau^k \underline{\mathcal{S}}_t$ be the morphism of k-th jets of conjugate Segre* *varieties given explicitely by*

$$
\left\{
\begin{aligned}
(\zeta,t) \longmapsto J_\tau^k \underline{\mathcal{S}}_t :=& \left(\zeta,\ \left(\frac{1}{\beta!}\,\partial_\zeta^\beta \Theta_j(\zeta,t)\right)_{1\leq j\leq d,\,|\beta|\leq k}\right) \\
=& \left(\zeta,\ \left(\sum_{\gamma\in\mathbb{N}^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\,(\zeta)^\gamma\,\Theta_{j,\beta+\gamma}(t)\right)_{1\leq j\leq d,\,|\beta|\leq k}\right),
\end{aligned}
\right. \tag{3.5.30}
$$

which is a complex algebraic or analytic map defined in $\Delta_{m+n}(\rho_1)$ with values in $\mathbb{C}^{m+N_{d,m,k}}$. Let $t_p \in M$ with $|t_p| < \rho_1$, let $(t_p, \bar t_p) \in \mathcal{M}$ and let $\ell_0 \in \mathbb{N}$ with $\ell_0 \geq 1$. Then

- **(1)** *M is Levi nondegenerate at t_p if and only if $J_\tau^1 \underline{\mathcal{S}}_t$ is of rank equal to $m+n$ at $(\bar z_p, t_p)$.*
- **(2)** *M is ℓ_0-finitely nondegenerate at t_p if and only if ℓ_0 is the smallest integer k such that $J_\tau^k \underline{\mathcal{S}}_t$ is of rank equal to $m+n$ at $(\bar z_p, t_p)$.*
- **(3)** *M is ℓ_0-essentially finite at t_p if and only if ℓ_0 is the smallest integer k such that $J_\tau^k \underline{\mathcal{S}}_t$ is a locally finite complex algebraic or analytic map in a neighborhood of $(\bar z_p, t_p)$.*
- **(4)** *M is ℓ_0-Segre nondegenerate at t_p if and only if ℓ_0 is the smallest integer k such that the restriction of $J_\tau^k \underline{\mathcal{S}}_t$ to the second Segre chain, namely*

$$
\left\{
\begin{aligned}
\mathcal{S}_{\bar t_p}^2 =& \ \big\{\big(z_1+z_p,\ \overline{\Theta}(z_1+z_p,\bar t_p),\ \zeta_1+\bar z_p, \\
& \Theta(\zeta_1+\bar z_p, z_1+z_p, \overline{\Theta}(z_1+z_p,\bar t_p))\big) \in \Delta_{2n}(\rho_1) :\ z_1\in\mathbb{C}^m,\ \zeta_1\in\mathbb{C}^m\big\}
\end{aligned}
\right. \tag{3.5.31}
$$

  is of generic rank equal to $2m$ at $(z_1,\zeta_1)=(0,0)$, hence all over $\mathcal{S}_{\bar t_p}^2$.
- **(5)** *M is ℓ_0-holomorphically nondegenerate at t_p if and only if ℓ_0 is the smallest integer k such that $J_\tau^k \underline{\mathcal{S}}_t$ is of generic rank $m+n$ in a neighborhood of $(\bar z_p, t_p)$, hence all over $\Delta_{m+n}(\rho_1)$.*

Proof. To establish this theorem, it suffices to inspect the five definitions given in Section 3.2 in the convenient system of coordinate $t_1 = t - t_p$ vanishing at t_p which was introduced before stating the theorem.

Indeed, thanks to the expressions (3.5.26) and (3.5.27) of $\Theta_{1,j,\beta}(t_1)$, we observe that except for $\beta = 0$, the components of $J_\tau^k \underline{\mathcal{S}}_t$ in (3.5.30) after the first m components $(\zeta_1,\dots,\zeta_m)$ coincide with the components $\Theta_{1,j,\beta}(t_1)$ of $\mathcal{Q}_{1,k}(t_1)$, after $(\zeta)^\gamma$ has been replaced by $(\bar z_p)^\gamma$. For $\beta = 0$, according to (3.5.26), the difference between the two mappings is only the constant $-\sum_{\beta\in\mathbb{N}^m}(\bar z_p)^\beta\,\Theta_{j,\beta}(t_p) = -\bar w_p$, which disappears by differentiation.

Consequently, we verify the following relation between the Jacobian matrix of $J_\tau^k \underline{\mathcal{S}}_t$ at $(\zeta,t) = (\bar z_p, t_p)$ and the Jacobian matrix of $\mathcal{Q}_{1,k}$ at $t_1 = 0$, *i.e.* at $t = t_p$:

$$
\mathrm{Jac}(J_\tau^k \underline{\mathcal{S}}_t)(\bar z_p, t_p) = \begin{pmatrix} \mathrm{I}_{m\times m} & 0 \\ *\!*\!* & \mathrm{Jac}\,\mathcal{Q}_{1,k}(0) \end{pmatrix} \tag{3.5.32}
$$

where $\mathrm{I}_{m\times m}$ denotes the identity $m\times m$ matrix and $*\!*\!*$ some terms which we need not to compute. We deduce at once that

$$
\mathrm{rk}_{\bar z_p,t_p}\left((\zeta,t)\mapsto J_\tau^k \underline{\mathcal{S}}_t\right) = m + \mathrm{rk}_0\left(t_1 \mapsto \mathcal{Q}_{1,k}(t_1)\right). \tag{3.5.33}
$$

By expressing Levi nondegeneracy and ℓ_0-finite nondegeneracy in terms of $\mathcal{Q}_{1,k}(t_1)$ in the coordinates t_1 vanishing at t_p, we immediately get characterizations **(1)** and **(2)** of Theorem 3.5.29.

Since more generally, at a point $(z_1+\bar z_p, t_1+t_p)$ varying in a neighborhood of $(\bar z_p, t_p)$, we have

$$
\mathrm{Jac}(J_\tau^k \underline{\mathcal{S}}_t)(z_1+\bar z_p, t_1+t_p) = \begin{pmatrix} \mathrm{I}_{m\times m} & 0 \\ *\!*\!* & \mathrm{Jac}\,\mathcal{Q}_{1,k}(t_1) \end{pmatrix}, \tag{3.5.34}
$$

we deduce at once that

$$
\mathrm{genrk}_{\mathbb{C}}\left((\zeta,t)\mapsto J_\tau^k \underline{\mathcal{S}}_t\right) = m + \mathrm{genrk}_{\mathbb{C}}\left(t_1 \mapsto \mathcal{Q}_{1,k}(t_1)\right). \tag{3.5.35}
$$

By expressing holomorphic nondegeneracy in terms of $\mathcal{Q}_{1,k}(t_1)$ in the coordinates t_1 vanishing at t_p, we immediately get the characterization **(5)** of Theorem 3.5.29.

Next, we also observe that

$$\text{(3.5.36)}\qquad \begin{cases} \dim_{\mathbb{C}} \mathbb{C}\{t_1\}/\ \langle \Theta_{1,j,\beta}(t_1)\rangle_{1\leq j\leq d,\,|\beta|\leq k} = \\ \qquad\qquad = \dim_{\mathbb{C}} \mathbb{C}\{\zeta - \bar{z}_p,\, t-t_p\} \Big/\!\!\Big/ \Big\langle j_\tau^k \underline{\mathcal{S}}_t - j_{\bar{t}_p}^k \underline{\mathcal{S}}_{t_p} \Big\rangle . \end{cases}$$

We deduce the characterization **(3)** of Theorem 3.5.29.

For the last case **(4)** to be considered, we notice that the restriction of $\mathcal{Q}_{1,k}(t_1)$ to the Segre variety of M_1 passing through the origin in coordinates t_1, which is by definition the map

$$\text{(3.5.37)}\qquad z_1 \longmapsto \big(\Theta_{1,j,\beta}(z_1, \overline{\Theta}_1(z_1,0))\big)_{1\leq j\leq d,\,|\beta|\leq k},$$

coincides thanks to (3.5.27) (neglecting the constant $-\bar{w}_p$ which appears for $\beta=0$) with

$$\text{(3.5.38)}\qquad z_1 \longmapsto \left(\sum_{\gamma\in\mathbb{N}^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\,(\bar{z}_p)^\gamma\,\Theta_{j,\beta+\gamma}(z_1+z_p, \overline{\Theta}_1(z_1,0)+w_p)\right)_{1\leq j\leq d,\,|\beta|\leq k}.$$

But since we have by (3.5.23)

$$\text{(3.5.39)}\qquad \overline{\Theta}_1(z_1,0)+w_p = \overline{\Theta}(z_1+z_p, \bar{t}_p),$$

we can rewrite (3.5.38) as

$$\text{(3.5.40)}\qquad z_1 \longmapsto \left(\sum_{\gamma\in\mathbb{N}^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\,(\bar{z}_p)^\gamma\,\Theta_{j,\beta+\gamma}(z_1+z_p, \overline{\Theta}(z_1+z_p,\bar{t}_p))\right)_{1\leq j\leq d,\,|\beta|\leq k}.$$

We claim that this mapping coincides with the last components of the restriction of the mapping $J_\tau^k \underline{\mathcal{S}}_t$ to the second Segre chain (3.5.31). Indeed, computing explicitely this restriction, we get exactly

$$\text{(3.5.41)}\qquad \begin{cases} (z_1,\zeta_1) \longmapsto \Bigg(\zeta_1+\bar{z}_p, \Bigg(\displaystyle\sum_{\gamma\in\mathbb{N}^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!}\,(\zeta_1+\bar{z}_p)^\gamma \\ \qquad\qquad \Theta_{j,\beta+\gamma}(z_1+z_p, \overline{\Theta}(z_1+z_p,\bar{t}_p))\big)_{1\leq j\leq d,\,|\beta|<k}\Bigg). \end{cases}$$

Consequently, we deduce

$$\text{(3.5.42)}\qquad \mathrm{genrk}_{\mathbb{C}}\left((z_1,\zeta_1) \longmapsto J_\tau^k \underline{\mathcal{S}}_t|_{\mathcal{S}^2_{\bar{t}_p}}\right) = m + \mathrm{genrk}_{\mathbb{C}}\left(z_1 \mapsto \mathcal{Q}_{1,k}(z_1, \overline{\Theta}_1(z_1,0))\right).$$

By expressing Segre nondegeneracy in terms of $\mathcal{Q}_{1,k}(t_1)$ in the coordinates t_1 vanishing at t_p, we immediately get the characterization **(4)** of Theorem 3.5.29.

The proof of Theorem 3.5.29. is complete. □

3.5.43. Essential holomorphic dimension and Levi multitype. If M is a local piece of generic submanifold as above, we denote by ℓ_M the smallest integer k such that Lemma 3.5.13 holds and we call it the *Levi type of* M. We denote by $m+n_M \leq m+n$ the generic rank of the mapping $(t,\tau)\mapsto J_\tau^{\ell_M}\underline{\mathcal{S}}_t$ and we call n_M the *essential holomorphic dimension of* M. This terminology is justified by the fact that locally in a neighborhood of a Zariski-generic point $p\in M$, then M is biholomorphically equivalent to a product $\underline{M}'_p \times \Delta^{n-n_M}$, where $\underline{M}'_p$ is a generic submanifold of codimension d of $\mathbb{C}^{n_M}$ (*see* Theorem 3.5.48 below).

By a specialization of the second functional equation (2.1.25), which yields $w \equiv \overline{\Theta}(0,0,\Theta(0,0,w))$, we see that the mapping $w\mapsto \Theta(0,0,w)$ is invertible. Consequently, the rank and the generic rank of the zeroth jet mapping $\mathrm{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J_\tau^0\underline{\mathcal{S}}_t\right) = (\zeta, \Theta(\zeta,z,w))$ is equal to $m+d$. Thus, the integer n_M always satisfies the inequalities $d\leq n_M\leq n$.

Generally speaking, we may define $\lambda_{0,M} := \text{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^0_\tau \underline{\mathcal{S}}_t\right) = d$ and for every $k = 1, \ldots, \ell_M$,

$$\lambda_{k,M} := \text{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t\right) - \text{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^{k-1}_\tau \underline{\mathcal{S}}_t\right). \tag{3.5.44}$$

By Lemma 3.5.13, we have $\lambda_{1,M} \geq 1, \ldots, \lambda_{\ell_M,M} \geq 1$. With these definitions, we have the relations

$$\text{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t\right) = d + \lambda_{1,M} + \cdots + \lambda_{k,M}, \tag{3.5.45}$$

for $k = 0, 1, \ldots, \ell_M$ and

$$\text{genrk}_{\mathbb{C}}\left((t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t\right) = n_M = d + \lambda_{1,M} + \cdots + \lambda_{\ell_M,M}, \tag{3.5.46}$$

for all $k \geq \ell_M$. It follows that we have the inequality

$$\ell_M \leq \lambda_{1,M} + \cdots + \lambda_{\ell_M,M} = n_M - d. \tag{3.5.47}$$

We are now in position to state and to prove the main theorem of this chapter in which we remind all the essential assumptions. Up to now, we have worked locally in a neighborhood of a point $p_0 \in M$. In the following theorem, we observe that we may easily globalize our constructions, provided that M is connected.

Theorem 3.5.48. *Let M be a connected real algebraic or analytic generic submanifold in $\mathbb{C}^n$ of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$. Then there exist well defined integers n_M, ℓ_M and $\lambda_{0,M}, \lambda_{1,M}, \ldots, \lambda_{\ell_M,M}$ and a proper real algebraic or analytic subvariety E of M such that for every point $p \in M\backslash E$ and for every system of coordinates (z,w) vanishing at p in which M is represented by defining equations $\bar{w}_j = \Theta_j(\bar{z},t)$, $j = 1, \ldots, d$, then the following four properties hold*:

- **(1)** $\lambda_{0,M} = d$, $d \leq n_M \leq n$ *and* $\ell_M \leq n_M - d$.
- **(2)** *For every $k = 0, 1, \ldots, \ell_M$, the mapping of k-th order jets of the conjugate complexified Segre varieties $(t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t$ is of rank equal to $m + \lambda_{0,M} + \cdots + \lambda_{k,M}$ at $(t_p, \bar{t}_p) = (0,0)$.*
- **(3)** $n_M = d + \lambda_{1,M} + \cdots + \lambda_{\ell_M,M}$ *and for every $k \geq \ell_M$, the mapping of k-th order jets of the conjugate complexified Segre varieties $(t,\tau)\mapsto J^k_\tau \underline{\mathcal{S}}_t$ is of rank equal to n_M at $(t_p, \bar{t}_p) = (0,0)$.*
- **(4)** *There exists a complex algebraic or analytic change of coordinates $t' = h(t)$ vanishing at p and defined in a neighborhood of p such that the image $M'_p := h(M)$ is the product $\underline{M}'_p \times \Delta^{n-n_M}$ of a real algebraic or analytic generic submanifold of codimension d in $\mathbb{C}^{n_M}$ by a complex polydisc Δ^{n-n_M}. Furthermore, at the central point $\underline{p}' \in \underline{M}'_p \subset \mathbb{C}^{n_M}$, the generic submanifold $\underline{M}'_p$ is ℓ_M-finitely nondegenerate, hence in particular its essential holomorphic dimension $n_{\underline{M}'_p}$ coincides with n_M.*

Proof. We fix $p_0 \in M$ and coordinates (z,w) as above vanishing at p_0. Let V_{p_0} be small neighborhood of p_0 in M. We first define $E \cap V_{p_0}$: it consists of the set of points $p \in V_{p_0}$ at which ℓ_p is not minimal, n_p is not maximal and the $\lambda_{k,p}$ are not maximal. Clearly, this set may be described by the vanishing of a collection of minors of the Jacobian matrix of the jet mapping $J^k_\tau \underline{\mathcal{S}}_t$, so it is a proper real algebraic or analytic subvariety E_{p_0} of V_{p_0}. Next, we verify that the various E_{p_0} glue together. Indeed, let assume that V_{p_0} and V_{q_0} overlap. Let $p \in V_{p_0} \cap V_{q_0}$. In the intersection $V_{p_0} \cap V_{q_0}$, we have to compare three real algebraic or analytic subvarieties E_{p_0}, E_p and E_{q_0} defined in terms of three morphisms of k-th order jets of conjugate Segre varieties. Using the important relation given by Theorem 3.5.6 (*cf.* also Corollary 3.5.9), and using an explicit description of the above mentioned collection of minors we may establish easily that E_{p_0} and E_{q_0} coincide with E_p in $V_{p_0} \cap V_{q_0}$. Consequently, the various E_{p_0} glue together. Taking account of the considerations which precede the statement of Theorem 3.5.48, this proves properties **(1)**, **(2)** and **(3)**.

Let us now prove **(4)**. Let $p \in M\backslash E$. We choose coordinates $t = (z,w)$ vanishing at p. By assumption, for every $k \geq \ell_M$, the k-th order Segre mapping $(\zeta, t) \mapsto (\zeta, \frac{1}{\beta!}\,(\partial^\beta_\zeta \Theta_j(\zeta,t))_{1\leq j\leq d,\, |\beta|\leq k})$ is of constant rank $m + n_M$ in a neighborhood of the origin (the point p) in $\mathbb{C}^{m+n}$. In particular, this entails that at every point of the form $(0, t_p)$ in a neighborhood of the origin, the mapping $t \mapsto (\Theta_{j,\beta}(t))_{1\leq j\leq d,\, |\beta|\leq k}$ is of constant rank n_M. It follows from the constant rank theorem that there exists an open neighborhood V_0 of the origin in $\mathbb{C}^n$ such that the union of level sets $\mathcal{F}_q := \{t \in V_0 : \Theta_{j,\beta}(t) = \Theta_{j,\beta}(q),\, j = 1, \dots, d,\, \beta \in \mathbb{N}^m\}$, for q running in V_0, constitutes a local complex algebraic or analytic foliation of V_0 by $(n - n_M)$-dimensional complex manifolds. We can straighten this foliation to a product $\Delta^{n_M} \times \Delta^{n-n_M}$, where the second factor corresponds to the leaves of this foliation. Let $t' = h(t)$ denote such a straightening change of coordinates. Let $M_0' := h(M)$ be of equation $\bar{w}' = \Theta'(\bar{z}', t')$. Thanks to Theorem 3.1.9, we observe that this foliation is again defined by the level sets of the functions $\Theta'_{j,\beta}(t')$, namely $\mathcal{F}_{p'} = \{t' \in V_0' : \Theta'_{j,\beta}(t') = \Theta'_{j,\beta}(p'),\, j = 1, \dots, d,\, \beta \in \mathbb{N}^m\}$. To conclude that in these coordinates, M_0' is a product $\underline{M_0'} \times \Delta^{n-n_M}$, it suffices to establish the following lemma.

Lemma 3.5.49. *If a point $p' \in V_0'$ belongs to M_0', then its leaf $\mathcal{F}_{p'}$ is entirely contained in M_0'.*

Proof. Indeed, let $q' \in \mathcal{F}_{p'}$, *i.e.* $\Theta'_{j,\beta}(t'_{q'}) = \Theta'_{j,\beta}(t'_{p'})$ for all j and all β. It follows first that

$$0 = \bar{w}'_{p'} - \Theta'(\bar{z}'_{p'}, t'_{p'}) = \bar{w}'_{p'} - \Theta'(\bar{z}'_{p'}, t'_{q'}). \tag{3.5.50}$$

Next, thanks to the reality of M_0', by Lemma 2.1.27, there exists an invertible $d \times d$ matrix of power series $a'(t', \tau')$ such that $w' - \overline{\Theta}'(z', \tau') \equiv a'(t', \tau')\, [\xi' - \Theta'(\zeta', t')]$, so we deduce $0 = w'_{q'} - \overline{\Theta}'(z'_{q'}, \bar{t}'_{p'})$ and by conjugating

$$0 = \bar{w}'_{q'} - \Theta'(\bar{z}'_{q'}, t'_{p'}). \tag{3.5.51}$$

Finally, using again the property $\Theta'_{j,\beta}(t'_{q'}) = \Theta'_{j,\beta}(t'_{p'})$ for all j and all β, we deduce

$$0 = \bar{w}'_{q'} - \Theta'(\bar{z}'_{q'}, t'_{q'}), \tag{3.5.52}$$

which shows that $q' \in M_0'$. This completes the proof of Lemma 3.5.49. □

The proof of Theorem 3.5.48 is complete. □

At a Zariski-generic point, the generic submanifold M incorporates a factor Δ^{n-n_M} which is in a certain sense "flat" with respect to the point of view of CR geometry. After dropping this innocuous factor, we come down to the study of a finitely nondegenerate generic submanifold. Thus, in a certain sense, finitely nondegenerate submanifolds M are the "generic" models. This is why it is interesting to state a direct corollary of Theorem 3.5.48 about holomorphically nondegenerate submanifolds.

Corollary 3.5.53. *Let M be a connected real algebraic or analytic generic submanifold in $\mathbb{C}^n$ of codimension $d \geq 1$ and of CR dimension $m = n - d \geq 1$. Assume that M is holomorphically nondegenerate. Then*

- **(1)** *There exists an integer ℓ_M with $1 \leq \ell_M \leq m$ and a proper real algebraic or analytic subset E of M such that M is ℓ_M-finitely nondegenerate at every point of $M \backslash E$.*
- **(2)** *There exists a proper complex algebraic or analytic subset $\mathcal{E}^{\rm exc}$ defined in a neighborhood of M in $\mathbb{C}^n$ which depends only on M such that M is finitely nondegenerate at a point p if and only if $p \in M \backslash \mathcal{E}^{\rm exc}$.*
- **(3)** *In general, the inclusion $(\mathcal{E}^{\rm exc} \cap M) \subset E$ is strict.*

Proof. In a polydisc neighborhood $V_0 \subset \mathbb{C}^n$ of an arbitrary point $p_0 \in M$, we have defined in §3.4.16 a local exceptional locus $\mathcal{E}^{\rm exc}_{p_0} \subset V_0$ such that $M \cap \mathcal{E}^{\rm exc}_{p_0}$ consists exactly of finitely degenerate points. Thanks to their biholomorphic invariance, these local complex algebraic or analytic subsets $\mathcal{E}^{\rm exc}_{p_0}$ glue together in a well defined global exceptional locus $\mathcal{E}^{\rm exc}$ defined in a neighborhood of M in $\mathbb{C}^n$. Finally, $E \backslash (\mathcal{E}^{\rm exc} \cap M)$ consists of points which are k-finitely nondegenerate for some $k \geq \ell_M + 1$, and so is clearly nonempty in general. This completes the proof of Corollary 3.5.53. □

§3.6. Transformation rules for jets of Segre varieties

We now establish the biholomorphic invariance of the mapping of jets of Segre varieties. As in §3.1.5, let $t' = h(t)$ with inverse $t = h'(t')$ be a complex algebraic or analytic local biholomorphism fixing the origin and let $M' := h(M)$. Theorems 3.1.9 follows from the two relations (8.5.2) and (8.5.3) after putting $\zeta = 0$, taking account of the fact that $\Theta_j(0,t)$ coincides with $\Theta_{j,0}(t)$ (in the notation $\Theta_{j,\beta}(t)$). Theorem 3.5.6 follows immediately from the two relations (8.5.2) and (8.5.3) just below, if we decide to consider the last argument of $Q_{j,\beta}$ simply as functions of $(\zeta, (\Theta_j(\zeta,t))_{1\leq j\leq d})$.

Theorem 3.6.1. *For every $j = 1, \ldots, d$ and every $\beta \in \mathbb{N}^m$, there exists a universal rational mapping in its variables $Q_{j,\beta}$ such that*

$$\text{(3.6.2)}\quad \begin{cases} \dfrac{1}{\beta!}\,\dfrac{\partial^{|\beta|}\Theta'_j}{\partial(\zeta')^\beta}\left(\bar f(\zeta,\Theta(\zeta,t)), h(t)\right) \equiv \\ \qquad \equiv Q_{j,\beta}\left(\left(\partial_\zeta^{\beta_1}\Theta_{j_1}(\zeta,t)\right)_{1\leq j_1\leq d,\,|\beta_1|\leq|\beta|},\ \left(\partial_\tau^{\alpha_1}\bar h_{i_1}(\zeta,\Theta(\zeta,t))\right)_{1\leq i_1\leq n,\,|\alpha_1|\leq|\beta|}\right). \end{cases}$$

Here, the $Q_{j,\beta}$ are algebraic or analytic in a neighborhood of the constant jet $((\partial_\zeta^{\beta_1}\Theta_{j_1}(0,0))_{1\leq j_1\leq d,\,|\beta_1|\leq|\beta|},\ (\partial_t^{\alpha_1}\bar h_{i_1}(0,0))_{1\leq i_1\leq n,\,|\alpha_1|\leq|\beta|})$. Equivalently, we have the relations for all $j = 1, \ldots, d$ and all $\beta \in \mathbb{N}^m$:

$$\text{(3.6.3)}\quad \begin{cases} \Theta'_{j,\beta}(h(t)) \equiv \displaystyle\sum_{\gamma\in\mathbb{N}^m} (-1)^\gamma\,\frac{(\beta+\gamma)!}{\beta!\ \gamma!}\,\bar f(\zeta,\Theta(\zeta,t))^\gamma \\ Q_{j,\beta+\gamma}\left(\left(\partial_\zeta^{\beta_1}\Theta_{j_1}(\zeta,t)\right)_{1\leq j_1\leq d,\,|\beta_1|\leq|\beta|+|\gamma|},\ \left(\partial_\tau^{\alpha_1}\bar h_{i_1}(\zeta,\Theta(\zeta,t))\right)_{1\leq i_1\leq n,\,|\alpha_1|\leq|\beta|+|\gamma|}\right). \end{cases}$$

Proof. As in §3.1.5, we may assume that the complex defining equations of M' are of the form $\bar w'_j = \Theta'_j(\bar z', t')$, $j = 1, \ldots, d$ in coordinates $t' = (z', w') \in \mathbb{C}^m \times \mathbb{C}^d$. Geometrically speaking, this means that the linear mapping $\pi'_{z'} \circ dh : T_0^c M \to \mathbb{C}^m_{z'}$ is submersive, where $\pi'_{z'} : \mathbb{C}^n \to \mathbb{C}^m_{z'}$ is the natural projection onto the z'-space. We decompose the mapping $h(t) = (f(t), g(t)) \in \mathbb{C}^m \times \mathbb{C}^d$. By complexifying the fundamental relations $\bar g_j(\tau) = \Theta'_j(\bar f(\tau), h(t))$, $j = 1, \ldots, d$, which express that h maps M into M' and by replacing ξ by $\Theta(\zeta,\tau)$, we obtain the following power series identities

$$\text{(3.6.4)}\qquad \bar g_j(\zeta,\Theta(\zeta,t)) \equiv \Theta'_j(\bar f(\zeta,\Theta(\zeta,t)), h(t)), \qquad j = 1, \ldots, d.$$

We differentiate this relation with respect to ζ_k, for $k = 1, \ldots, m$. Remembering that the explicit expression of the natural basis of complexified $(0,1)$-vector fields is given by

$$\text{(3.6.5)}\qquad \underline{\mathcal{L}}_k = \frac{\partial}{\partial\zeta_k} + \sum_{j=1}^{1} \frac{\partial\Theta_j}{\partial\zeta_k}(\zeta,t)\,\frac{\partial}{\partial\xi_j},$$

for $k = 1, \ldots, m$, we immediately see that differentiation with respect to ζ_k of a power series $\psi(\zeta,\Theta(\zeta,t))$ is equivalent to applying the vector field $\underline{\mathcal{L}}_k$ to ψ, viewed as a derivation. So we get by the chain rule

$$\text{(3.6.6)}\qquad \underline{\mathcal{L}}_k \bar g_j(\zeta,\Theta(\zeta,t)) \equiv \sum_{l=1}^m \underline{\mathcal{L}}_k \bar f_l(\zeta,\Theta(\zeta,t))\,\frac{\partial\Theta'_j}{\partial\zeta'_l}(\bar f(\zeta,\Theta(\zeta,t)), h(t)).$$

Since $\pi'_{z'} \circ dh : T_0^c M \to \mathbb{C}^m_{z'}$ is submersive, we have the nonvanishing of the following determinant

$$\text{(3.6.7)}\qquad \det\left(\underline{\mathcal{L}}_{k_1} \bar f_{k_2}(0)\right)_{1\leq k_1,k_2\leq m} \neq 0.$$

Hence we can divide locally for (ζ,t) in a neighborhood of the origin by the determinant

$$\text{(3.6.8)}\qquad \mathcal{D}(\zeta,t) := \det\left(\underline{\mathcal{L}}_{k_1} \bar f_{k_2}(\zeta,\Theta(\zeta,t))\right)_{1\leq k_1,k_2\leq m}.$$

Viewing (3.6.6) as an inhomogeneous linear system and using the classical rule of Cramer, for every $j = 1, \dots, d$, we can solve the first partial derivatives $\partial \Theta_j' / \partial \zeta_k'$ with respect to the other terms, which yields expressions of the form

$$\frac{\partial \Theta_j'}{\partial \zeta_k'}(\bar{f}(\zeta, \Theta(\zeta, t)), h(t)) \equiv \frac{R_{j,k}\left((\underline{\mathcal{L}}_{k_1'} \bar{h}_{i_1}(\zeta, \Theta(\zeta, t)))_{1 \leq i_1 \leq n,\, 1 \leq k_1' \leq m} \right)}{\mathcal{D}(\zeta, t)}. \tag{3.6.9}$$

Here, by the very application of Cramer's rule, it follows that the terms $R_{j,k}$ are certain universal polyomials of determinant type (some minors). By differentiating again (3.6.9) with respect to the variables ζ_k, using again Cramer's rule, we get that for every pair of integers (k_1, k_2) with $1 \leq k_1,\, k_2 \leq m$ and for every $j = 1, \dots, d$, there exist a universal polynomial R_{j,k_1,k_2} such that we can write

$$\frac{\partial^2 \Theta_j'}{\partial \zeta_{k_1}' \partial \zeta_{k_2}'} \left(\bar{f}(\zeta, \Theta(\zeta, t)), h(t) \right) \equiv \frac{R_{j,k_1,k_2}\left((\underline{\mathcal{L}}_{k_1', k_2'} \bar{h}_{i_1}(\zeta, \Theta(\zeta, t)))_{1 \leq i_1 \leq n,\, 1 \leq k_1', k_2' \leq m} \right)}{\mathcal{D}(\zeta, t)^3}. \tag{3.6.10}$$

The reader should notice the exponent 3 in the denominator, with the decomposition "3"="2"+"1" where "2" comes from the derivatives of the quotient $R_{j,k}/\mathcal{D}$ in (3.6.9) and where "1" comes from the second application of Cramer's rule.

Remind that for an arbitrary multi-index $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$, we denote by $\underline{\mathcal{L}}^\beta$ the antiholomorphic derivation of order $|\beta|$ defined by $(\underline{\mathcal{L}}_1)^{\beta_1} \cdots (\underline{\mathcal{L}}_m)^{\beta_m}$.

Differentiating more generally the relations (3.6.4) with respect to $\zeta^\beta = \zeta_1^{\beta_1} \cdots \zeta_m^{\beta_m}$, we see by an easy induction that for every multindex $\beta \in \mathbb{N}^m$ and for every $j = 1, \dots, d$, there exists a complicated but universal polynomial $R_{j,\beta}$ such that the following identity holds:

$$\frac{1}{\beta!} \frac{\partial^{|\beta|} \Theta_j'}{\partial (\zeta')^\beta} \left(\bar{f}(\zeta, \Theta(\zeta, t)), h(t) \right) \equiv \frac{R_{j,\beta}\left((\underline{\mathcal{L}}^{\beta_1} \bar{h}_{i_1}(\zeta, \Theta(\zeta, t)))_{1 \leq i_1 \leq n,\, 1 \leq |\beta_1| \leq |\beta|} \right)}{[\mathcal{D}(\zeta, t)]^{2|\beta| - 1}}. \tag{3.6.11}$$

An important observation is in order. The composed derivations $\underline{\mathcal{L}}^{\beta_1}$ are certain differential operators with nonconstant coefficients. Using the explicit expressions of the $\underline{\mathcal{L}}_k$, we see that all these coefficients are certain universal polynomials of the collection of partial derivatives $(\partial^{|\beta_2|} \Theta_{j_2}(\zeta, t) / \partial z^{\beta_2})_{1 \leq j_2 \leq d,\, 1 \leq |\beta_2| \leq |\beta_1|}$. Thus the numerator of (3.6.11) becomes a certain universal (computable by means of combinatorial formulas) algebraic or analytic function of the collection

$$\left(\left(\partial_\zeta^{\beta_1} \Theta_{j_1}(\zeta, t) \right)_{1 \leq j_1 \leq d,\, |\beta_1| \leq |\beta|},\, \left(\partial_t^{\alpha_1} \bar{h}_{i_1}(\zeta, \Theta(\zeta, t)) \right)_{1 \leq i_1 \leq n,\, |\alpha_1| \leq |\beta|} \right) \tag{3.6.12}$$

A similar property holds for the denominator. In conclusion, we have constructed the rational mapping $Q_{j,\beta}$ satisfying (3.6.2).

For the second part of Theorem 3.6.1, let us rewrite the relations (3.6.2) in the following more explicit form, simply obtained by developing the left hand side with respect to the powers $(\bar{f})^\gamma$:

(3.6.13)

$$\begin{cases} \sum_{\gamma \in \mathbb{N}^m} \frac{(\beta + \gamma)!}{\beta!\, \gamma!}\; \bar{f}(\zeta, \Theta(\zeta, t))^\gamma\, \Theta_{j, \beta + \gamma}'(h(t)) \equiv \\ \qquad \equiv Q_{j,\beta}\left(\left(\partial_\zeta^{\beta_1} \Theta_{j_1}(\zeta, t) \right)_{1 \leq j_1 \leq d,\, |\beta_1| \leq |\beta|},\, \left(\partial_t^{\alpha_1} \bar{h}_{i_1}(\zeta, \Theta(\zeta, t)) \right)_{1 \leq i_1 \leq n,\, |\alpha_1| \leq |\beta|} \right). \end{cases}$$

We may interpret this infinite collection of identities as an infinite upper triangular inhomogeneous linear system with unknowns being the $\Theta_{j,\beta}'(h(t))$. The inversion of this infinite triangular matrix is in fact very elementary. Indeed, by interpreting Taylor's formula at a purely formal level, we see that if we are given an infinite collection of equalities with complex coefficients and with $\zeta \in \mathbb{C}^m$ which is of the form

$$\sum_{\gamma \in \mathbb{N}^m} \frac{(\beta+\gamma)!}{\beta!\ \gamma!} \zeta^\gamma\, \Theta'_{j,\beta+\gamma} = Q_{j,\beta}, \tag{3.6.14}$$

for all $j = 1, \dots, d$ and all $\beta \in \mathbb{N}^m$, then we can solve the unknowns $\Theta'_{j,\beta}$ in terms of the right hand side terms $Q_{j,\beta}$ by means of a totally similar formula, except for signs:

$$\sum_{\gamma \in \mathbb{N}^m} (-1)^\gamma \frac{(\beta+\gamma)!}{\beta!\ \gamma!} \zeta^\gamma\, Q_{j,\beta+\gamma} = \Theta'_{j,\beta}, \tag{3.6.15}$$

for all $j = 1, \dots, d$ and all $\beta \in \mathbb{N}^m$. Applying this observation to (3.6.13), we deduce the representation (3.6.3) in Theorem 3.6.1, which completes the proof. □

§3.7. Local geometry of CR submanifolds at a Zariski-generic point

Let $M \subset \mathbb{C}^n$ be a *not necessarily generic* connected real algebraic or analytic CR submanifold of codimension d, of CR dimension m and of holomorphic codimension $c = d - n + m$. Combining Theorem 2.1.9, Theorem 2.1.32, Corollary 2.8.6 and Theorem 3.5.48, we obtain the following local explicit coordinate representation of M locally in a neighborhood of a Zariski-generic point. This theorem will be useful in Part II of this memoir.

Theorem 3.7.1. *There exists a proper real algebraic or analytic subset E of M and integers m_1, m_2, d_1, d_2, c which depend only on M and which satisfy*

$$\begin{cases} d = d_1 + d_2 + 2c, \\ m = m_1 + m_2, \end{cases} \tag{3.7.2}$$

and in the case where $m_1 \geq 1$, there exist moreover two integers ℓ_M and ν_M which satisfy

$$\begin{cases} \ell_M \ \leq m_1, \\ \nu_M \ \leq d_1 + 1 \end{cases} \tag{3.7.3}$$

such that for every point $p_0 \in M \backslash E$, there exist local complex analytic or algebraic normal coordinates

$$(z_1, z_2, w_1, w_2, w_3) \in \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^c \tag{3.7.4}$$

vanishing at p_0 and complex algebraic or analytic defining functions $\Theta_{1,j_1}(\bar z_1, z_1, w_1, w_2)$, $j_1 = 1, \dots, d_1$ which converge normally in $\Delta_{2m_1+d_1+d_2}(2\rho_1)$ for some $\rho_1 > 0$, which satisfy $\Theta_{1,j_1}(0, z_1, w_1, w_2) \equiv 0$ for $j_1 = 1, \dots, d_1$, and which are independent of $(\bar z_2, z_2)$ such that M is represented locally in a neighborhood of p_0 by the complex defining equations

$$\begin{cases} 0 = w_3, \\ 0 = \bar w_2 - w_2, \\ 0 = \bar w_1 - \Theta_1(\bar z_1, z_1, w_1, w_2), \end{cases} \tag{3.7.5}$$

in the polydisc $\Delta_n(\rho_1)$ and such that, moreover, for every constant $u_{2,q} \in \mathbb{R}^{d_2}$, the generic submanifold $M_{1,u_{2_q}}$ of $\mathbb{C}^{m_1} \times \mathbb{C}^{d_1}$ defined by the complex equations

$$0 = \bar w_1 - \Theta_1(\bar z_1, z_1, w_1, u_{2,q}), \tag{3.7.6}$$

which identifies with the intersection of M with the complex subspace $\{w_2 = u_{2,q} = ct., \ w_3 = 0\}$, is minimal of Segre type ν_M and ℓ_M-finitely nondegenerate at $(z_1, w) = (0, 0)$. In the case where $m_1 = 0$, the third equation in (3.7.5) should be replaced by the simpler vectorial equation $\bar w_1 = w_1$, hence in this case M identifies in a neighborhood of p_0 with the intersection of $\Delta_n(\rho_1)$ with the Levi-flat product $\mathbb{C}^{m_2} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \{0\}$ in $\mathbb{C}^{m_2} \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^c$.

Chapter 4: Nondegeneracy conditions for power series CR mappings

§4.1. CR-horizontal nondegeneracy conditions for power series CR mappings

§4.1.1. Nondegeneracy conditions for power series mappings. The datum of a formal, or complex algebraic or complex analytic mapping $h : \mathbb{C}^n \to \mathbb{C}^{n'}$ with $h(0) = 0$ is equivalent to the datum of a collection of n' power series $(h_1(t), \dots, h_{n'}(t))$, where $t \in \mathbb{C}^n$, with the $h_{i'}(t)$ being scalar power series vanishing at the origin and belonging to $\mathbb{C}[\![t]\!]$, to $\mathbb{C}\{t\}$ or to $\mathcal{A}_{\mathbb{C}}\{t\}$. We introduce five classical nondegeneracy conditions, which we formulate in the case where $h(t) \in \mathbb{C}[\![t]\!]^{n'}$.

Definition 4.1.2. A formal power series mapping $h(t) = (h_1(t), \dots, h_{n'}(t))$, with components $h_{i'}(t) \in \mathbb{C}[\![t]\!]$, $i' = 1, \dots, n'$, is called

(1) *Invertible* if $n' = n$ and $\det\left([\partial h_{i_1}/\partial t_{i_2}](0)\right)_{1 \le i_1, i_2 \le n} \neq 0$.
(2) *Submersive* if $n \ge n'$ and there exist integers $1 \le i(1) < \cdots < i(n') \le n$ such that $\det\left([\partial h_{i'_1}/\partial t_{i(i'_2)}](0)\right)_{1 \le i'_1, i'_2 \le n'} \neq 0$.
(3) *Finite* if the ideal generated by the components $h_1(t), \dots, h_{n'}(t)$ is of finite codimension in $\mathbb{C}[\![t]\!]$. This implies $n' \ge n$.
(4) *Dominating* if $n \ge n'$ and there exist integers $1 \le i(1) < \cdots < i(n') \le n$ such that $\det([\partial h_{i'}/\partial t_{i(i'_2)}](t))_{1 \le i'_1, i'_2 \le n'} \not\equiv 0$ in $\mathbb{C}[\![t]\!]$.
(5) *Transversal* if there does not exist a nonzero power series $G(t'_1, \dots, t'_{n'}) \in \mathbb{C}[\![t'_1, \dots, t'_{n'}]\!]$ such that $G(h_1(t), \dots, h_{n'}(t)) \equiv 0$ in $\mathbb{C}[\![t]\!]$.

It is elementary to see that invertibility implies submersiveness which implies domination. Furthermore, we have:

Lemma 4.1.3. *If a formal power series is either invertible, submersive or dominating, then it is transversal.*

Proof. It suffices to prove the statement in the case where h is dominating. Suppose on the contrary that there exists a nonzero power series $G(t'_1, \dots, t'_{n'}) \in \mathbb{C}[\![t'_1, \dots, t'_{n'}]\!]$ such that $G(h_1(t), \dots, h_{n'}(t)) \equiv 0$ in $\mathbb{C}[\![t]\!]$. By differentiating this identity with respect to the t_i and looking at the linear homogeneous system so obtained, we deduce that all the $n' \times n'$ minors (of maximal dimension) of the Jacobian matrix $([\partial h_{i'}/\partial t_i](t))_{1 \le i \le n,\, 1 \le i' \le n'}$ vanish identically in $\mathbb{C}[\![t]\!]$, contradicting domination. □

Lemma 4.1.4. *In the equidimensional case $n' = n$, invertibility implies* (*and is in fact equivalent to*) *submersiveness, which implies finiteness, which implies domination, which finally implies transversality.*

Proof. Classical result from local complex analytic geometry, which may easily reproved by the reader or found for instance [1], [6] or in the references therein. □

4.1.5. Power series CR mappings and CR-horizontal nondegeneracy conditions. Let M and M' be two real algebraic or analytic generic submanifolds of $\mathbb{C}^n$ and of $\mathbb{C}^{n'}$. Let $r_j(t, \bar{t}) := \bar{w}_j - \Theta_j(\bar{z}, t)$, $j = 1, \dots, d$ and $r'_{j'}(t', \bar{t}') := \bar{w}'_{j'} - \Theta'_{j'}(\bar{z}', t')$, $j' = 1, \dots, d'$, be defining equations for M and for M'. Following the definition given in §2.1.5, we say that h *is a* (*local*) *power series CR mapping from M to M'* if there exists a $d' \times d$ matrix $a(t, \bar{t})$ of formal, analytic or algebraic power series such that, in vectorial notation

$$r'(h(t), \bar{h}(\bar{t})) \equiv a(t, \bar{t})\, r(t, \bar{t}). \tag{4.1.6}$$

Setting $\bar{t} = 0$ in this matrix identity, we get

$$r'(h(t), 0) \equiv a(t, 0)\, r(t, 0). \tag{4.1.7}$$

The set defined by the equations $r_j(t, 0) = 0$ is of course the Segre variety $S_{\bar{p}_0}$ passing through the origin. Similarly, the set defined by $r'_{j'}(t', 0) = 0$, $j' = 1, \dots, d'$, is the Segre variety $S_{\bar{p}'_0}$. Then (4.1.7) shows that h induces a power series mapping from the Segre variety $S_{\bar{p}_0}$ to $S'_{\bar{p}'_0}$. We can thus apply Definition 4.1.2 at the level of Segre varieties.

Definition 4.1.8. A power series CR mapping $h : M \to M'$ is called

- **(1)** *CR-invertible at* p_0 if $m = m'$ and if the induced formal mapping $h|_{S_{\bar{p}_0}} : S_{\bar{p}_0} \to S'_{\bar{p}'_0}$ between Segre varieties passing through the origin is a formal invertible mapping at 0.
- **(2)** *CR-submersive at* p_0 if $m \geq m'$ and if the induced formal mapping $h|_{S_{\bar{p}_0}} : S_{\bar{p}_0} \to S'_{\bar{p}'_0}$ between Segre varieties passing through the origin is a formal submersion at 0.
- **(3)** *CR-finite at* p_0 if the induced formal mapping $h|_{S_{\bar{p}_0}} : S_{\bar{p}_0} \to S'_{\bar{p}'_0}$ between Segre varieties passing through the origin is finite at 0.
- **(4)** *CR-dominating at* p_0 if the induced formal mapping $h|_{S_{\bar{p}_0}} : S_{\bar{p}_0} \to S'_{\bar{p}'_0}$ between Segre varieties passing through the origin is dominating at 0.
- **(5)** *CR-transversal at* p_0 if the induced formal mapping $h|_{S_{\bar{p}_0}} : S_{\bar{p}_0} \to S'_{\bar{p}'_0}$ between Segre varieties passing through the origin is transversal at 0.

4.1.9. Complexified mapping. We may reformulate these definitions in a more concrete way as follows. First of all, we may of course complexify the defining identities (4.1.6), which yields $r'(h(t), \bar{h}(\tau)) \equiv a(t,\tau)\, r(t,\tau)$. These complexified identities show that the *complexified mapping*

$$h^c(t,\tau) := (h(t), \bar{h}(\tau)) \in \mathbb{C}[\![t]\!]^n \times \mathbb{C}[\![\tau]\!]^n. \tag{4.1.10}$$

induces a power series mapping from $\mathcal{M}$ to $\mathcal{M}'$. As explained in §1.2.2, the complexified mapping h^c stabilizes the two pairs of foliations $(\mathcal{F}, \underline{\mathcal{F}})$ and $(\mathcal{F}', \underline{\mathcal{F}}')$, namely h^c sends (conjugate) complexified Segre varieties of $\mathcal{M}$ to (conjugate) complexified Segre varieties of $\mathcal{M}'$.

After replacing ξ by $\Theta(\zeta, t)$ in (4.1.10), we obtain the following power series identities in $\mathbb{C}[\![\zeta, t]\!]$:

$$\bar{g}_{j'}(\zeta, \Theta(\zeta,t)) \equiv \Theta'_{j'}(\bar{f}(\zeta, \Theta(\zeta,t)), h(t)), \qquad j' = 1, \dots, d'. \tag{4.1.11}$$

In fact, as may be easily established, these identities are equivalent to the existence of a $d' \times d$ matrix of formal power series $a(t,\tau)$ which satisfies (4.1.6). However, *throughout this memoir, we shall work only with the more convenient fundamental power series identities* (4.1.11).

Of course, the Segre variety passing through p_0 is represented by

$$S_{\bar{p}_0} : \quad w_j = \overline{\Theta}_j(z, 0), \qquad j = 1, \dots, d. \tag{4.1.12}$$

Similarly, the Segre variety passing through p'_0 is represented by

$$S'_{\bar{p}'_0} : \quad w'_{j'} = \overline{\Theta}'_{j'}(z', 0), \qquad j' = 1, \dots, d'. \tag{4.1.13}$$

If we split $h = (f, g) \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$, then the restriction of h to $S_{\bar{p}_0}$ coincide with the power series CR mapping

$$\mathbb{C}^m \ni z \longmapsto \Big(f(z, \overline{\Theta}(z,0)),\, \overline{\Theta}'(f(z, \overline{\Theta}(z,0)), 0) \Big) \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}. \tag{4.1.14}$$

By projection onto the $\mathbb{C}^{m'} \times \{0\}$, we may of course identify this mapping with the *CR-horizontal part* of the mapping defined by

$$\mathbb{C}^m \ni z \longmapsto f(z, \overline{\Theta}(z,0)) \in \mathbb{C}^{m'}. \tag{4.1.15}$$

In the special case where M' is given in normal coordinates, we have $\Theta'(z', 0) \equiv 0$, hence the last d' terms in (4.1.7) all vanish and the identification of $h|_{S_{\bar{p}_0}}$ with its CR-horizontal part is trivial in this case (but we shall avoid for the moment to introduce normal coordinates). With such notations, we can reformulate the above five nondegeneracy conditions more concretely.

Definition 4.1.16. Such a power series CR mapping $h : M \to M'$ is

- **(1)** *CR-invertible at* p_0 if its CR-horizontal part is a formal invertible map at 0.
- **(2)** *CR-submersive at* p_0 if its CR-horizontal part is a formal submersion at 0.
- **(3)** *CR-finite at* p_0 if its CR-horizontal part is finite at 0.
- **(4)** *CR-dominating at* p_0 if its CR-horizontal part is dominating at 0.
- **(5)** *CR-transversal at* p_0 if its CR-horizontal part is transversal at 0.

To conclude, as direct corollaries of Lemmas 4.1.3 and 4.1.4, we have

Lemma 4.1.17. *If a formal power series CR mapping $h : M \to M'$ is either CR-invertible, CR-submersive or CR-dominating, then it is CR-transversal. Furthermore, in the CR-equidimensional case $m' = m$, CR-invertibility implies (and is in fact equivalent to) CR submersiveness, which implies CR-finiteness, which implies CR-domination, which finally implies CR-transversality.*

§4.2. Segre nondegeneracy conditions for power series CR mappings

4.2.1. Preliminaries. After having introduced the five CR-horizontal nondegeneracy conditions on the power series CR mapping h, we may introduce *Segre nondegeneracy conditions* on h which are related to the reflection principle, which will be studied in the next chapters of Part II of this memoir. Consequently, *this Section 4.2 is of utmost importance to understand the whole memoir.*

As in §4.1.9, let $h^c : \mathcal{M} \to \mathcal{M}'$ be a complexified power series CR mapping, namely start with the fundamental power series identities in $\mathbb{C}[\![\zeta, t]\!]$:

$$\bar{g}_{j'}(\zeta, \Theta(\zeta, t)) - \Theta'_{j'}(\bar{f}(\zeta, \Theta(\zeta, t)), h(t)) \equiv 0, \tag{4.2.2}$$

for $j' = 1, \ldots, d'$. As in (2.3.8), let $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_m$ be the basis of complexified $(0,1)$ vector fields tangent to $\mathcal{M}$ given by

$$\underline{\mathcal{L}}_k = \frac{\partial}{\partial \zeta_k} + \sum_{j=1}^{d} \frac{\partial \Theta_j}{\partial \zeta_k}(\zeta, t)\, \frac{\partial}{\partial \xi_j}. \tag{4.2.3}$$

Let $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$. Applying the derivations $\underline{\mathcal{L}}^\beta = \underline{\mathcal{L}}_1^{\beta_1} \cdots \underline{\mathcal{L}}_m^{\beta_m}$ to (4.2.2) and without writing the arguments, we get

$$\underline{\mathcal{L}}^\beta \bar{g}_{j'} - \sum_{\gamma' \in \mathbb{N}^{m'}} \underline{\mathcal{L}}^\beta (\bar{f}^\gamma)\, \Theta'_{j', \gamma'}(h) \equiv 0, \tag{4.2.4}$$

for all $\beta \in \mathbb{N}^m$, for $j' = 1, \ldots, d'$ and for $(t, \tau) \in \mathcal{M}$. By developping the derivations $\underline{\mathcal{L}}^\beta$, we observe that

Lemma 4.2.5. *For every $i' = 1, \ldots, n'$ and every $\beta \in \mathbb{N}^m$, there exists a polynomial $P_{i', \beta}$ in the jet $J_\tau^{|\beta|} \bar{h}(\tau)$ with coefficients being power series in (t, τ) which depend only on the defining functions $\xi_j - \Theta_j(\zeta, t)$ of $\mathcal{M}$ and which can be computed by means of some combinatorial formula, such that*

$$\underline{\mathcal{L}}^\beta \bar{h}_{i'}(\tau) \equiv P_{i', \beta}(t, \tau, J_\tau^{|\beta|} \bar{h}(\tau)). \tag{4.2.6}$$

Proof. For $|\beta| = 1$, this follows by inspecting the coefficients of the vector fields $\underline{\mathcal{L}}_k$ in (4.2.4). By induction, assuming that such a formula holds for all $\beta \in \mathbb{N}^m$ with $|\beta| = k \geq 1$, applying the vector fields $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_m$ to (4.2.6) and using the chain rule, we get a similar type of formula for all $\beta \in \mathbb{N}^m$ with $|\beta| = k + 1$. Clearly, the coefficients of $P_{i', \beta}$ depend on the partial derivatives of the $\Theta_j(\zeta, t)$ and one can refine this statement by providing an explicit combinatorial formula (which we shall not need). □

In formula (4.2.6), we have written the first two arguments of $P_{i', \beta}$ to be (t, τ). In fact, by inspecting the coefficients of the vector fields $\underline{\mathcal{L}}_k$, these first two arguments are (ζ, t). However, since we are always considering the variables (t, τ) to belong to $\mathcal{M}$, we have to replace everywhere τ by $\Theta(\zeta, t)$ or w by $\overline{\Theta}(z, t)$. Consequently, we can identify a function of (t, τ) with a function of (ζ, t) or with a function of (z, τ). Before going further, we shall make the following convention which will allow us to make some slight abuse of notation on occasion.

Convention 4.2.7. Let $k, l \in \mathbb{N}$. On the complexification $\mathcal{M}$, we identify (notationally) a power series written under the complete form

$$R(t, \tau, J^k h(t), J^l \bar{h}(\tau)), \tag{4.2.8}$$

with a power series written under one of the following four forms

$$\begin{cases} (\mathbf{1}) \;\; R\left(t, \zeta, \Theta(\zeta, t), J^k h(t), J^l \bar{h}(\zeta, \Theta(\zeta, t))\right), \\ (\mathbf{2}) \;\; R\left(t, \zeta, J^k h(t), J^l \bar{h}(\zeta, \Theta(\zeta, t))\right), \\ (\mathbf{3}) \;\; R\left(z, \overline{\Theta}(z, \tau), \tau, J^k h(z, \overline{\Theta}(z, \tau)), J^l \bar{h}(\tau)\right), \\ (\mathbf{4}) \;\; R\left(z, \tau, J^k h(z, \overline{\Theta}(z, \tau)), J^l \bar{h}(\tau)\right). \end{cases} \tag{4.2.9}$$

Admitting this convention, applying Lemma 4.2.5 and using the chain rule, we deduce that for every $j' = 1, \dots, d'$ and every $\beta \in \mathbb{N}^m$, there exist formal power series $R'_{j',\beta} = R'_{j',\beta}(t, \tau, J_\tau^{|\beta|} : t')$ which depend only on the defining equations of $\mathcal{M}$ and on the defining equations of $\mathcal{M}'$ such that we can rewrite the equations (4.2.4) under the general form

$$\underline{\mathcal{L}}^\beta [\bar{g}_{j'}(\tau) - \Theta_{j'}(\bar{f}(\tau), h(t))] =: R'_{j',\beta}(t, \tau, J_\tau^{|\beta|} \bar{h}(\tau) : h(t)) \equiv 0, \tag{4.2.10}$$

for $j' = 1, \dots, d'$, where the identity "$\equiv 0$" is understood "on $\mathcal{M}$", namely as a formal power series identity in $\mathbb{C}[\![\zeta, t]\!]$ after replacing ξ by $\Theta(\zeta, t)$ or equivalently, as a formal power series identity in $\mathbb{C}[\![z, \tau]\!]$ after replacing w by $\overline{\Theta}(z, \tau)$. We shall constantly refer to these identities (4.2.10) in the sequel.

Importantly, the smoothness of the power series $R'_{j',\beta}$ is the minimum of the two smoothesses of M and of M'. This crucial remark will be the basis of all the various formal reflection principles developed in the next chapters of Part II.

For instance, the power series $R'_{j',\beta}$ are all complex analytic if M is real analytic and if M' is real algebraic, even if the power series CR mapping $h(t)$ was assumed to be purely formal and non convergent.

By a careful inspection of the application of the chain rule in the development of (4.2.5), we even see that each $R'_{j',\beta}$ is relatively polynomial with respect to the derivatives of positive order $(\partial_\tau^\alpha \bar{h}(\tau))_{1 \le |\alpha| \le |\beta|}$.

4.2.11. Segre nondegeneracy conditions for CR mappings. We are now ready to define nondegeneracy conditions for power series CR mappings which generalize the nondegeneracy conditions for generic submanifolds introduced in Chapter 3. In the equations (4.2.10), we replace $h(t)$ by a new independent variable $t' \in \mathbb{C}^{n'}$, we set $(t, \tau) = (0, 0)$, and we define the following collection of power series

$$\Psi'_{j',\beta}(t') := \left[\underline{\mathcal{L}}^\beta \bar{g}_{j'} - \sum_{\gamma' \in \mathbb{N}^{m'}} \underline{\mathcal{L}}^\beta (\bar{f}^{\gamma'}) \, \Theta'_{j',\gamma'}(t') \right]_{t=\tau=0}, \tag{4.2.12}$$

for $j' = 1, \dots, d'$ and $\beta \in \mathbb{N}^m$. Here, for $\beta = 0$, we mean that $\Psi'_{j',0}(t') = -\Theta'_{j'}(0, t')$. Clearly, an equivalent way of defining $\Psi'_{j',\beta}(t')$ is as follows

$$\Psi'_{j',\beta}(t') := R'_{j',\beta}(0, 0, J_\tau^{|\beta|} \bar{h}(0) : t'). \tag{4.2.13}$$

Now, just before introducing the desired five definitions, we make the following crucial heuristic remark. When $n = n'$, $M = M'$ and $h = \mathrm{Id}$, we drop the dashes and we denote by T (instead of t') the new independent variable, hence we may compute

$$
(4.2.14)\qquad \begin{cases} \Psi_{j,\beta}(T) & = \left[\underline{\mathcal{L}}^{\beta} \xi_j - \sum_{\gamma \in \mathbb{N}^m} \underline{\mathcal{L}}^{\beta} (\zeta)^{\gamma}\, \Theta_{j,\gamma}(T) \right]_{t=\tau=0} \\ & = \left[\underline{\mathcal{L}}^{\beta} \Theta_j(\zeta, t) - \beta!\, \Theta_{j,\beta}(T) \right]_{t=\tau=0} \\ & = \beta!\, [\Theta_{j,\beta}(0) - \Theta_{j,\beta}(T)]. \end{cases}
$$

Consequently, we see that up to an affine combination, we recover with $\Psi_{j,\beta}(T)$ the components of the infinite Segre mapping (3.1.3). Furthermore, using the computation (4.2.14), we may check easily that the following five nondegeneracy conditions just below are a generalization to CR mappings of the five nondegeneracy conditions introduced in Section 3.2 for generic submanifolds of $\mathbb{C}^n$.

Definition 4.2.15. The formal, algebraic or analytic power series CR mapping h is called

(1) *Levi-nondegenerate* at the origin if the mapping

$$
(4.2.16)\qquad t' \mapsto \left(R'_{j',\beta}(0, 0, J_{\tau}^{|\beta|} \bar{h}(0) : t') \right)_{1 \leq j' \leq d',\, |\beta| \leq 1}
$$

is of rank n' at $t' = 0$. This condition requires the dimensional inequality $d'(m+1) \leq n'$.

(2) ℓ_1-*finitely nondegenerate* at the origin if there exists an integer ℓ such that the mapping

$$
(4.2.17)\qquad t' \mapsto \left(R'_{j',\beta}(0, 0, J_{\tau}^{|\beta|} \bar{h}(0) : t') \right)_{1 \leq j' \leq d',\, |\beta| \leq \ell}
$$

is of rank n' at $t' = 0$, and if ℓ_1 is the smallest such integer ℓ.

(3) ℓ_1-*Segre finite* at the origin if there exists an integer ℓ such that the mapping

$$
(4.2.18)\qquad t' \mapsto \left(R'_{j',\beta}(0, 0, J_{\tau}^{|\beta|} \bar{h}(0) : t') \right)_{1 \leq j' \leq d',\, |\beta| \leq \ell}
$$

is locally finite at $t' = 0$, and if ℓ_1 is the smallest such integer.

(4) ℓ_1-*Segre nondegenerate* if there exist an integer ℓ, integers ${j'_*}^1, \dots, {j'_*}^{n'}$ with $1 \leq {j'_*}^{i'} \leq d'$ for $i' = 1, \dots, n'$ and multiindices $\beta_*^1, \dots, \beta_*^{n'}$ with $|\beta_*^{i'}| \leq \ell$ for $i' = 1, \dots, n'$, such that the determinant

$$
(4.2.19)\qquad \det \left(\frac{\partial R'_{{j'_*}^{i'_1}, \beta_*^{i'_1}}}{\partial t'_{i'_2}} \left(z, \overline{\Theta}(z,0), 0, 0, J^{|\beta_*^{i'_1}|} \bar{h}(0) : h(z, \overline{\Theta}(z,0)) \right) \right)_{1 \leq i'_1, i'_2 \leq n'}
$$

does not vanish identically in $\mathbb{C}[\![z]\!]$, and if ℓ_1 is the smallest such integer ℓ.

(5) ℓ_1-*holomorphically nondegenerate* at the origin if there exists an integer ℓ, integers ${j'_*}^1, \dots, {j'_*}^{n'}$ with $1 \leq {j'_*}^{i'} \leq d'$ for $i' = 1, \dots, n'$ and multiindices $\beta_*^1, \dots, \beta_*^{n'}$ with $|\beta_*^{i'}| \leq \ell$ for $i' = 1, \dots, n'$, such that the determinant

$$
(4.2.20)\qquad \det \left(\frac{\partial R'_{{j'_*}^{i'_1}, \beta_*^{i'_1}}}{\partial t'_{i'_2}} \left(0, 0, 0, 0, J^{|\beta_*^{i'_1}|} \bar{h}(0) : h(t) \right) \right)_{1 \leq i'_1, i'_2 \leq n'}
$$

does not vanish identically in $\mathbb{C}[\![t]\!]$, and if ℓ_1 is the smallest such integer ℓ.

These five nondegeneracy conditions for power series CR mappings are of utmost importance for the reflection principle and they will be studied in Part II of this memoir. Some of them are suggested by K. Diederich and S.M. Webster in [9]. The notion of ℓ_1-finite nondegeneracy unifies conditions which appear in the works [1], [8], [14], [17], [22], [23], [27], [30], [34], [35], [36].

We notice that Levi nondegeneracy implies finite nondegeneracy which implies Segre finiteness. However, Segre finiteness and Segre nondegeneracy are totally independent conditions, as shown by the following two examples.

Example 4.2.21. We already know that essential finiteness of M does not imply Segre nondegeneracy, hence simply by considering the identity map of the hypersurface $v = z_1\bar{z}_1(1 + z_1\bar{z}_2)$ of $\mathbb{C}^3$, we see that **(3)** does not imply **(4)** in Definition 4.2.15 just above.

On the other hand, it is easy to check that the mapping $(z_1, w) \mapsto (z_1, 0, w)$ from $M : \bar{w} = w + i\, z_1^2\bar{z}_1^2$ to $M' : \bar{w}' = w' + i[{z_1'}^2\bar{z_1'}^2 + z_1'\bar{z_2'}^2 + \bar{z}_1' {z_2'}^2]$ is Segre finite at the origin but it is not Segre nondegenerate.

We also mention that the above five nondegeneracy conditions are meaningful for sufficiently smooth local CR mappings, by considering the Taylor series of M at p_0, of h at p_0 and of M' at p_0'.

4.2.22. Necessary conditions. Coming back to (4.2.12), noticing that $\bar{f}(0) = 0$, we see that the constant $\underline{\mathcal{L}}^\beta(\bar{f}^\gamma)|_{t=\tau=0}$ vanishes for $|\gamma| > |\beta|$, whence every $\Psi'_{j',\beta}(t')$ is an affine combination with constant coefficients of some (but not all) of the power series $\Theta'_{j_1',\beta_1'}(t')$, for $1 \leq j_1' \leq d'$ and $|\beta_1'| \leq |\beta|$. Based on this observation, we deduce immediately properties **(1)**, **(2)**, **(3)** and **(5)** of Lemma 4.2.23 just below, which states necessary conditions for h to be nondegenerate in each one of the above five senses. The proof of **(4)** is also elementary.

Lemma 4.2.23. *Let $h : M \to M'$ be a power series CR mapping as above.*

- **(1)** *If h is Levi-nondegenerate at the origin, then M' is Levi-nondegenerate at the origin.*
- **(2)** *If h is ℓ_1-finitely nondegenerate at the origin, then M' is ℓ_0'-finitely nondegenerate at the origin for some $\ell_0' \leq \ell_1$.*
- **(3)** *If h is ℓ_1-essentially finite at the origin, then M' is ℓ_0'-essentially finite at the origin for some $\ell_0' \leq \ell_1$.*
- **(4)** *If h is ℓ_1-Segre nondegenerate at the origin, then M' is ℓ_0'-Segre nondegenerate at the origin for some $\ell_0' \leq \ell_1$.*
- **(5)** *If h is ℓ_1-holomorphically nondegenerate at the origin, then M' is ℓ_0'-holomorphically nondegenerate at the origin for some $\ell_0' \leq \ell_1$.*

§4.3. Study of CR-transversal power series CR mappings

We now show that CR-transversality of the mapping h insures that it enjoys exactly the same nondegeneracy condition as the target M'. The condition of CR transversality is much more general than the condition of domination, because for instance it does not impose any dimensional inequality between m and m' or between n and n'. We shall end this section with the proof of the following theorem, which is quite long and technical, but of central importance. Here, we assume that M and M' are either algebraic or analytic, and that h is algebraic, analytic or even formal. According to Lemma 4.1.17, the following lemma applies to many situtations.

Theorem 4.3.1. *Assume that h is CR-transversal at p_0. Then the following five properties hold:*

- **(1)** *If M' is Levi nondegenerate at p_0', then h is ℓ_1-finitely nondegenerate at p_0 for some $\ell_1 \geq 1$.*
- **(2)** *If M' is ℓ_0'-finitely nondegenerate at p_0', then h is ℓ_1-finitely nondegenerate at p_0 for some $\ell_1 \geq \ell_0'$.*
- **(3)** *If M' is ℓ_0'-essentially finite at p_0', then h is ℓ_1-Segre finite at p_0 for some $\ell_1 \geq \ell_0'$.*
- **(4)** *If M' is ℓ_0'-Segre nondegenerate at p_0', then h is ℓ_1-Segre nondegenerate at p_0 for some $\ell_1 \geq \ell_0'$.*
- **(5)** *If M' is ℓ_0'-holomorphically nondegenerate at p_0', and if moreover h is transversal at p_0, then h is ℓ_1-holomorphically nondegenerate at p_0 for some $\ell_1 \geq \ell_0'$.*

Proof. In order to simplify a little bit the notations and the computations, we shall assume that the coordinates (z, w) for M near p_0 and (z', w') for M' near p_0' are normal. Thus, the

Segre variety $S_{\bar{p}_0}$ is given by $\{(z,0)\}$ instead of $\{(z,\overline{\Theta}(z,0))\}$ and similarly for $S'_{\bar{p}'_0}$, which will slightly simplify the presentation of the formal calculations below.

We remind that by the fundamental definition (4.2.10), the functions $R'_{j',\beta}(t,\tau,J^{|\beta|}\bar{h}(\tau):t')$ are the power series development of $\underline{\mathcal{L}}^\beta r'_{j'}(t',\bar{h}(\tau))$. Here, the integer j' satisfies $1\le j'\le d'$ and the multiindex β belongs to $\mathbb{N}^m$.

We consider the gradient of $R'_{j',\beta}$ with respect to the distinguished variable t':

$$\nabla_{t'}\, R'_{j',\beta}(t,\tau,J^{|\beta|}\bar{h}(\tau):\, t') := \left(\frac{\partial R'_{j',\beta}}{\partial t'_i}(t,\tau,J^{|\beta|}\bar{h}(\tau):\, t')\right)_{1\le i'\le n'}, \tag{4.3.2}$$

considered as a vertical vector, *i.e.* as a $n'\times 1$ matrix. Also, we shall work in the sequel with a fixed j' and we shall consider the expression

$$\left(\nabla_{t'}\, R'_{j',\beta}(t,\tau,J^{|\beta|}\bar{h}(\tau):\, t')\right)_{\beta\in\mathbb{N}^m} \tag{4.3.3}$$

as an $n'\times\infty$ matrix. We introduce a new notation. Let $\nu\in\mathbb{N}$ with $\nu\ge 1$, let $x=(x_1,\dots,x_\nu)\in\mathbb{K}^\nu$, let $\mu\in\mathbb{N}$ with $\mu\ge 1$ and let $A(x)$ by a $\mu\times\infty$ matrix of power series. By genMrk $A(x)$, we denote the generic rank of the matrix $A(x)$, which is defined to be the largest integer $\kappa\le\mu$ such that there exists a $\kappa\times\kappa$ minor of $A(x)$ which does not vanish identically as a power series in x. The letter "M" in genMrk stands for the word "Matrix". The notation genMrk should not be confused with the notation $\text{genrk}_{\mathbb{C}}$ introduced in §2.1.5.

In the sequel, we shall in fact put $(t,\tau):=(0,0)$ in $R'_{j',\beta}$. Then, the generic rank of the matrix

$$\text{genMrk}\,\left(\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, t')\right)_{\beta\in\mathbb{N}^m} \tag{4.3.4}$$

is of course the largest integer κ' such that there exists a $\kappa'\times\kappa'$ minor of (4.3.4) which does not vanish identically as a power series in t'. In the case where we put $t'=0$, we even do not have to speak of generic rank, hence we simply denote by

$$\text{Mrk}\,\left(\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, 0)\right)_{\beta\in\mathbb{N}^m} \tag{4.3.5}$$

the rank of a $n'\times\infty$ constant matrix. Of course, we use the same notations genMrk and Mrk for the truncated matrices where we allow only $|\beta|\le k$, for some integer $k\in\mathbb{N}$.

First of all we prove part **(2)** of Theorem 4.3.1, which contains of course part **(1)**. We state a technical lemma which holds essentially in the case of codimension $d=1$.

Lemma 4.3.6. *Fix j' with $1\le j'\le d'$. Let $n'_{j'}$ be the integer defined by*

$$n'_{j'} := \text{Mrk}\,\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(0)\right)_{\gamma'\in\mathbb{N}^{m'}}. \tag{4.3.7}$$

Assume that h is CR-transversal at p_0. Then we also have

$$\text{Mrk}\,\left(\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, 0)\right)_{\beta\in\mathbb{N}^m} = n'_{j'}. \tag{4.3.8}$$

We first show that this technical lemma implies part **(2)** of the theorem. Indeed, by the definition of $R'_{j',\beta}$ (*see* the expression (4.2.5)) $\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\,0)$ is a linear combination with complex coefficients of the vectors $\nabla_{t'}\,\Theta'_{j',\gamma'}(0)$, for $|\gamma'|\le|\beta|$, namely more precisely

$$\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, 0) = -\sum_{\gamma'\in\mathbb{N}^{m'}} \underline{\mathcal{L}}^\beta(\bar{f}^{\gamma'})(0)\,[\nabla_{t'}\,\Theta'_{j',\gamma'}(0)]. \tag{4.3.9}$$

We notice that this sum is in fact truncated, with $|\gamma'|\le|\beta|$. Consequently, we deduce that

$$\text{Span}_{\mathbb{C}}\left\{\nabla_{t'}\, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, 0):\, \beta\in\mathbb{N}^m\right\} \tag{4.3.10}$$

is automatically contained in

$$\text{Span}_{\mathbb{C}} \left\{ \nabla_{t'} \, \Theta'_{j',\gamma'}(0) : \gamma' \in \mathbb{N}^{m'} \right\}. \tag{4.3.11}$$

But thanks to the rank condition stated in Lemma 4.3.6 (to be proved below), we deduce that these two subspaces must coincide. In other words, for each $j' = 1, \dots, d'$, there exists integers $\ell_{1,j'}$ and $\ell'_{0,j'}$ with $\ell_{1,j'} \geq \ell'_{0,j'}$ such that

$$\begin{cases} \text{Span}_{\mathbb{C}} \left\{ \nabla_{t'} \, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0) : 0) : |\beta| \leq \ell_{1,j'} \right\} = \\ \qquad\qquad \text{Span}_{\mathbb{C}} \left\{ \nabla_{t'} \, \Theta'_{j',\gamma'}(0) : |\gamma'| \leq \ell'_{0,j'} \right\}. \end{cases} \tag{4.3.12}$$

Finally, by assumption of ℓ'_0-finite nondegeneracy of M' at p'_0, the vector space spanned by the collection of terms in the right hand sides of (4.3.12), when $j' = 1, \dots, d'$, is equal to the whole of $\mathbb{C}^{n'}$, with of course $\ell'_0 = \max(\ell'_{0,1}, \dots, \ell'_{0,n'})$. It follows that the vector space spanned by the collection of terms in the left hand sides of (4.3.12), when $j' = 1, \dots, d'$, is also equal to the whole of $\mathbb{C}^{n'}$. In conclusion, setting of course $\ell_1 = \max(\ell_{1,1}, \dots, \ell_{1,n'}) \geq \ell'_0$, we have shown that h is ℓ_1-finitely nondegenerate at p_0. It remains to establish the technical lemma.

Proof. So we fix j'. We proceed by contradiction. Assume that the rank κ' of the $n' \times \infty$ matrix

$$\left(\nabla_{t'} \, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0) : 0) \right)_{\beta \in \mathbb{N}^m} \tag{4.3.13}$$

is stricly smaller than $n'_{j'}$, namely $\kappa' \leq n'_{j'} - 1$. Equivalently, there exist multiindices $\beta_1, \dots, \beta_{\kappa'} \in \mathbb{N}^m$ such that for every multiindex $\beta \in \mathbb{N}^m$ different from $\beta_1, \dots, \beta_{\kappa'}$, there exist constanst $\Lambda^1_\beta, \dots, \Lambda^{\kappa'}_\beta$ such that we can write

$$\begin{cases} \nabla_{t'} \, R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0) : 0) = \Lambda^1_\beta \left[\nabla_{t'} \, R'_{j',\beta_1}(0,0,J^{|\beta|}\bar{h}(0) : 0) \right] + \cdots + \\ \qquad\qquad + \Lambda^{\kappa'}_\beta \left[\nabla_{t'} \, R'_{j',\beta_{\kappa'}}(0,0,J^{|\beta|}\bar{h}(0) : 0) \right]. \end{cases} \tag{4.3.14}$$

Now, replacing in these linear combinations the terms $\nabla_{t'} \, R'_{j',\beta}$ by their explicit expression (4.3.9), we obtain

$$\begin{cases} \displaystyle\sum_{\gamma' \in \mathbb{N}^{m'}} \underline{\mathcal{L}}^\beta(\bar{f}^{\gamma'})(0) \;\; [\nabla_{t'} \, \Theta'_{j',\gamma'}(0)] = \\ \qquad\qquad \Lambda^1_\beta \left(\displaystyle\sum_{\gamma' \in \mathbb{N}^{m'}} \underline{\mathcal{L}}^{\beta_1}(\bar{f}^{\gamma'})(0) \, [\nabla_{t'} \, \Theta'_{j',\gamma'}(0)] \right) + \cdots + \\ \qquad\qquad \Lambda^{\kappa'}_\beta \left(\displaystyle\sum_{\gamma' \in \mathbb{N}^{m'}} \underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar{f}^{\gamma'})(0) \, [\nabla_{t'} \, \Theta'_{j',\gamma'}(0)] \right). \end{cases} \tag{4.3.15}$$

As we are in normal coordinates, the conjugate complexified Segre variety passing through the origin is given by $\underline{\mathcal{S}}_0 = \{(0,0,\zeta,0)\}$ in (z,w,ζ,ξ) coordinates. Let $\gamma' \in \mathbb{N}^{m'}$. The restriction of $\bar{f}^{\gamma'}$ to $\underline{\mathcal{S}}_0$ is given by $\bar{f}^{\gamma'}(\zeta, 0)$. We develope its Taylor series with respect to the powers of ζ as follows

$$\bar{f}^{\gamma'}(\zeta, 0) = \sum_{\beta \in \mathbb{N}^m} \bar{f}_{\gamma',\beta} \, \frac{\zeta^\beta}{\beta!}, \tag{4.3.16}$$

where the $\bar{f}_{\gamma',\beta}$ are constants in $\mathbb{C}$. Since the coordinates are normal, with the usual vector fields $\underline{\mathcal{L}}_k$ given by

$$\underline{\mathcal{L}}_k = \frac{\partial}{\partial \zeta_k} + \sum_{j=1}^d \frac{\partial \Theta_j}{\partial \zeta_k}(\zeta, t) \, \frac{\partial}{\partial \xi_j}, \tag{4.3.17}$$

taking into account that $\Theta_j(\zeta, 0, w) \equiv \Theta_j(0, z, w) \equiv 0$ (*cf.* Theorem 2.1.32), we immediately see that the constants $\bar{f}_{\gamma',\beta}$ are simply given by

$$\bar{f}_{\gamma',\beta} = \underline{\mathcal{L}}^{\beta}(\bar{f}^{\gamma'})(0,0,0,0). \tag{4.3.18}$$

We may now rewrite the expression (4.3.15) as follows

$$\sum_{\gamma'\in\mathbb{N}^{m'}} \left(\bar{f}_{\gamma',\beta} - \Lambda_\beta^1\, \bar{f}_{\gamma',\beta_1} - \cdots - \Lambda_\beta^{\kappa'}\, \bar{f}_{\gamma',\beta_{\kappa'}}\right) \, \left[\nabla_{t'}\, \Theta'_{j',\gamma'}(0)\right] = 0. \tag{4.3.19}$$

Let us rewrite this expression temporarily as a linear system of the form

$$\sum_{\gamma'\in\mathbb{N}^{m'}} \overline{F}_{\beta,\gamma'} \, \left[\nabla_{t'}\, \Theta'_{j',\gamma'}(0)\right] = 0, \tag{4.3.20}$$

where the $\overline{F}_{\beta,\gamma'}$ are complex constant. Now, we use the assumption (4.3.7). Since the rank of the $n'\times\infty$ matrix $(\nabla_{t'}\,\Theta'_{j',\gamma'}(0))_{\gamma'\in\mathbb{N}^{m'}}$ is equal to $n'_{j'}$ it follows that after making some linear combinations between the lines of the system (4.3.20) that there exist $n'_{j'}$ distinct multiindices $\gamma'_1,\dots,\gamma'_{n'_{j'}} \in \mathbb{N}^{m'}$ such that for $i'=1,\dots,n'_{j'}$, we can solve

$$\overline{F}_{\beta,\gamma'_{i'}} = \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}\, \overline{F}_{\beta,\gamma'}, \tag{4.3.21}$$

where the $A'_{i',\gamma'}$ are complex constants.

We now replace the $\overline{F}_{\beta,\gamma'}$ by their values, which yields for $\beta\neq\beta_1,\dots,\beta_{\kappa'}$ and for $i' = 1,\dots,n'_{j'}$ the following equalities

$$\left\{\begin{aligned} & \bar{f}_{\gamma'_{i'},\beta} - \Lambda_\beta^1\, \bar{f}_{\gamma'_{i'},\beta_1} - \cdots - \Lambda_\beta^{\kappa'}\, \bar{f}_{\gamma'_{i'},\beta_{\kappa'}} = \\ & = \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'} \left[\bar{f}_{\gamma',\beta} - \Lambda_\beta^1\, \bar{f}_{\gamma',\beta_1} - \cdots - \Lambda_\beta^{\kappa'}\, \bar{f}_{\gamma',\beta_{\kappa'}}\right]. \end{aligned}\right. \tag{4.3.22}$$

We can now mix linearly the left and the right hand sides to obtain for $i'=1,\dots,n'_{j'}$ and for $\beta\neq\beta_1,\dots,\beta_{\kappa'}$

$$\left\{\begin{aligned} \bar{f}_{\gamma'_{i'},\beta} - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}\, \bar{f}_{\gamma',\beta} = {} & \Lambda_\beta^1 \left[\bar{f}_{\gamma'_{i'},\beta_1} - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}\, \bar{f}_{\gamma',\beta_1}\right] \\ & + \cdots + \\ & \Lambda_\beta^{\kappa'} \left[\bar{f}_{\gamma'_{i'},\beta_{\kappa'}} - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}\, \bar{f}_{\gamma',\beta_{\kappa'}}\right]. \end{aligned}\right. \tag{4.3.23}$$

On the other hand, by the definition (4.3.16) of the constants $\bar f_{\gamma',\beta}$, we can develope the following expression in power series of ζ

$$
(4.3.24)\qquad
\begin{cases}
\bar f^{\gamma_i'}(\zeta,0) - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f^{\gamma'}(\zeta,0) = \\[2mm]
\quad \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \left(\bar f_{\gamma_{i'}',\beta} - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f_{\gamma',\beta}\right)\frac{\zeta^\beta}{\beta!} + \\[2mm]
\quad + \left(\bar f_{\gamma_{i'}',\beta_1} - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f_{\gamma',\beta_1}\right)\frac{\zeta^{\beta_1}}{\beta_1!} + \cdots + \\[2mm]
\quad + \left(\bar f_{\gamma_{i'}',\beta_{\kappa'}} - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f_{\gamma',\beta_{\kappa'}}\right)\frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!}.
\end{cases}
$$

Now, we introduce the following new complex constants for $i'=1,\dots,n_{j'}'$

$$
(4.3.25)\qquad
\begin{cases}
\Pi_{i',1} := \bar f_{\gamma_{i'}',\beta_1} - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f_{\gamma',\beta_1}, \\
\qquad\cdots\cdots\cdots \\
\Pi_{i',\kappa'} := \bar f_{\gamma_{i'}',\beta_{\kappa'}} - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f_{\gamma',\beta_{\kappa'}},
\end{cases}
$$

and we use the preceding relations (4.3.23) to rewrite (4.3.24) more simply as follows, for $i'=1,\dots,n_{j'}'$:

(4.3.26)

$$
\begin{cases}
\bar f^{\gamma_{i'}'}(\zeta,0) - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{i',\gamma'}'\,\bar f^{\gamma'}(\zeta,0) = \\[2mm]
\quad = \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \left[\Lambda_\beta^1\,\Pi_{i',1} + \cdots + \Lambda_\beta^{\kappa'}\,\Pi_{i',\kappa'}\right]\frac{\zeta^\beta}{\beta!} + \Pi_{i',1}\frac{\zeta^{\beta_1}}{\beta_1!} + \cdots + \Pi_{i',\kappa'}\frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!} \\[2mm]
\quad = \Pi_{i',1}\left(\frac{\zeta^{\beta_1}}{\beta_1!} + \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \Lambda_\beta^1\frac{\zeta^\beta}{\beta!}\right) + \cdots + \Pi_{i',\kappa'}\left(\frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!} + \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \Lambda_\beta^{\kappa'}\frac{\zeta^\beta}{\beta!}\right) \\[2mm]
\quad =: \Pi_{i',1}\,G_1(\zeta) + \cdots + \Pi_{i',\kappa'}\,G_{\kappa'}(\zeta),
\end{cases}
$$

where the power series $G_1(\zeta),\dots,G_{\kappa'}(\zeta)$ are defined by the last equality. Now, we use the assumption $\kappa'\leq n_{j'}'-1$. Since there are striclty less than $n_{j'}'$ functions G in (4.3.26), it follows that there exist complex constants $\mu_1,\dots,\mu_{n_{j'}'}$ not all zero such that

$$
(4.3.27)\qquad
\begin{cases}
0 \equiv \mu_1\left(\bar f^{\gamma_1'}(\zeta,0) - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{1,\gamma'}\,\bar f^{\gamma'}(\zeta,0)\right) + \cdots + \\[2mm]
\qquad + \mu_{n_{j'}'}\left(\bar f^{\gamma_{n_{j'}'}'}(\zeta,0) - \sum_{\gamma'\neq\gamma_1',\dots,\gamma_{n_{j'}'}'} A_{n_{j'}',\gamma'}\,\bar f^{\gamma'}(\zeta,0)\right),
\end{cases}
$$

and this last equality clearly contradicts the assumption that h is CR-transversal at p_0.

The proofs of Lemma 4.3.6 and of parts **(1)** and **(2)** of Theorem 4.3.1 are complete. □

We now prove part **(3)** of Theorem 4.3.1. We also proceed by contradiction. Suppose that the ideal

$$\left\langle R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, t')\right\rangle_{1\leq j'\leq d',\,\beta\in\mathbb{N}^m} \tag{4.3.28}$$

is of infinite codimension in $\mathcal{A}_{\mathbb{C}}\{t'\}$ or in $\mathbb{C}\{t'\}$. By the Nullstellensatz, it follows that there exists a nonzero algebraic or analytic piece of curve passing through the origin

$$\mathbb{C}\ni s'\mapsto t'(s')\in\mathbb{C}^{n'}, \tag{4.3.29}$$

namely $t'(s')\in\mathcal{A}_{\mathbb{C}}\{s'\}^{n'}$ or $t'(s')\in\mathbb{C}\{s'\}^{n'}$, such that

$$R'_{j',\beta}(0,0,J^{|\beta|}\bar{h}(0):\, t'(s'))\equiv 0, \tag{4.3.30}$$

for all $j'=1,\dots,d'$ and all $\beta\in\mathbb{N}^m$. Replacing the $R'_{j',\beta}$ by their definition (4.2.10), this means that

$$\underline{\mathcal{L}}^{\beta}\,\bar{g}_{j'}(0)-\sum_{\gamma'\in\mathbb{N}^{m'}}\underline{\mathcal{L}}^{\beta}(\bar{f}^{\gamma'})(0)\,\Theta'_{j',\gamma'}(t'(s'))\equiv 0, \tag{4.3.31}$$

for $j'=1,\dots,d'$. But since the coordinates (z,w) and (z',w') are normal, we claim that we have

$$\underline{\mathcal{L}}^{\beta}\,\bar{g}_{j'}(0)=0 \tag{4.3.32}$$

for all $\beta\in\mathbb{N}^m$ and all $j'=1,\dots,d'$. Indeed, setting $t=0$ in the fundamental power series identities

$$\bar{g}_{j'}(\zeta,\Theta(\zeta,t))\equiv\Theta'_{j'}(\bar{f}(\zeta,\Theta(\zeta,t)),h(t)), \tag{4.3.33}$$

we get thanks to the normality of coordinates

$$\bar{g}_{j'}(\zeta,0)\equiv\Theta'_{j'}(\bar{f}(\zeta,0),0)\equiv 0, \tag{4.3.34}$$

hence

$$\underline{\mathcal{L}}^{\beta}\,\bar{g}_{j'}(0)=\partial_{\zeta}^{\beta}\bar{g}_{j'}(\zeta,0)|_{\zeta=0}=0, \tag{4.3.35}$$

as claimed. Developing

$$\bar{f}^{\gamma'}(\zeta,0)=\sum_{\beta\in\mathbb{N}^m}\bar{f}_{\gamma',\beta}\,\frac{\zeta^{\beta}}{\beta!}, \tag{4.3.36}$$

where (again thanks to normal coordinates)

$$\bar{f}_{\gamma',\beta}=\underline{\mathcal{L}}^{\beta}\,\bar{f}^{\gamma'}(0), \tag{4.3.37}$$

we can now rewrite the expression (4.3.31) in under the simpler form

$$\sum_{\gamma'\in\mathbb{N}^{m'}}\bar{f}_{\gamma',\beta}\,\Theta'_{j',\gamma'}(t'(s'))\equiv 0. \tag{4.3.38}$$

Because M' is essentially finite at p_0', there exists at smallest one integer j_0' with $1\leq j_0'\leq d'$ such that not all $\Theta'_{j_0',\gamma'}(t'(s'))$ vanish identically, for $\gamma'\in\mathbb{N}^{m'}$. Hence there exists $s_0'\in\mathbb{C}$ arbitrarily close to the origin such that the infinite family of complex constants $(\Theta'_{j_0',\gamma'}(t'(s_0')))_{\gamma'\in\mathbb{N}^{m'}}$ are not all zero. We put

$$\theta'_{\gamma'}:=\Theta'_{j_0',\gamma'}(t'(s_0'))\in\mathbb{C}. \tag{4.3.39}$$

Setting $s':=s_0'$ in (4.3.38), we therefore get for all $\beta\in\mathbb{N}^m$ a relation of the form

$$\sum_{\gamma'\in\mathbb{N}^{m'}}\bar{f}_{\gamma',\beta}\,\theta'_{\gamma'}=0, \tag{4.3.40}$$

which yields after integrating with respect to ζ the formal identity

$$\sum_{\gamma'\in\mathbb{N}^{m'}} \theta'_{\gamma'}\, \bar f^{\gamma'}(\zeta,0) \equiv 0. \tag{4.3.41}$$

But this clearly contradicts the CR-transversality of h at p_0.

The proof of part **(3)** of Theorem 4.3.1 is complete.

We now prove part **(4)** of Theorem 4.3.1. The proof has some similarities with the proof of part **(2)**, but is a little bit more technical. We use the notations introduced in the beginning of the proof of Theorem 4.3.1.

It follows directly from Section 3.2 (*cf.* especially (3.2.47) and Lemma 3.2.49) that in normal coordinates, and using the gradient notation $\nabla_{t'}$, then M' is Segre nondegenerate if and only if

$$\text{genMrk}\,\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(z',0)\right)_{1\leq j'\leq d',\,\gamma'\in\mathbb{N}^{m'}} = n'. \tag{4.3.42}$$

Notice that here we let the integer j' vary from 1 to d', but in the following lemma, we shall fix j'.

Lemma 4.3.43. *Fix j' with $1\leq j'\leq d'$. Let $n'_{j'}$ be the integer defined by*

$$n'_{j'} := \text{genMrk}\,\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(z',0)\right)_{\gamma'\in\mathbb{N}^{m'}}. \tag{4.3.44}$$

Assume that h is CR-transversal at p_0. Then we also have

$$\text{genMrk}\,\left(\nabla_{t'}\,R'_{j',\beta}(z,0,0,0,J^{|\beta|}\bar h(0) :\, h(z,0))\right)_{\beta\in\mathbb{N}^m} = n'_{j'}. \tag{4.3.45}$$

We first show that this technical lemma implies part **(4)** of Theorem 4.3.1. Indeed, by the definitions (4.2.5) and (4.2.10) of $R'_{j',\beta}$, after specifiying on the Segre chain $\mathcal{S}_0 = \{(z,0,0,0)\}$, we have

$$\begin{cases} \nabla_{t'}\,R'_{j',\beta}\ (z,0,0,0,J^{|\beta|}\bar h(0) :\, h(z,0)) = \\ \qquad\qquad = -\displaystyle\sum_{\gamma'\in\mathbb{N}^{m'}} \underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})(z,0,0,0)\,[\nabla_{t'}\,\Theta'_{j',\gamma'}(h(z,0))]. \end{cases} \tag{4.3.46}$$

Notice that we write here the first arguments of $R'_{j',\beta}$ and the arguments of the differentiated expression $\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})$ as (z,w,ζ,ξ), and not as (t,τ). Here, in the arguments $(z,0,0,0)$ of this expression $\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})$, the term z comes from the coefficients of the vector fields $\underline{\mathcal{L}}_k$, but $(\zeta,\xi)=(0,0)$. It follows that the sum in (4.3.46) is in fact truncated with $|\gamma'|\leq|\beta|$, since $\bar f^{\gamma'}$ vanishes to order $|\gamma'|$ at the origin. Thus, the columns of the matrix $(\nabla_{t'}R'_{j',\beta})_{\beta\in\mathbb{N}^m}$ on the left hand side of (4.3.46) are obtained as a linear combination (with coefficients being certain formal power series in z) of the columns of the matrix $(\nabla_{t'}\Theta'_{j',\gamma'})_{\gamma'\in\mathbb{N}^{m'}}$ on the right hand side. Thanks to Lemma 4.3.43 (to be proved below), we deduce that the formal linear space spanned by the columns of the matrix $(\nabla_{t'}R'_{j',\beta})_{\beta\in\mathbb{N}^m}$ on the left coincides with the formal linear space spanned by the columns of matrix $(\nabla_{t'}\Theta'_{j',\gamma'})_{\gamma'\in\mathbb{N}^{m'}}$ on the right. Finally, thanks to the Segre nondegeneracy assumption (4.3.42), we deduce

$$\text{genMrk}\,\left(\nabla_{t'}R'_{j',\beta}(z,0,0,0,J^{|\beta|}\bar h(0) :\, h(z,0))\right)_{1\leq j'\leq d',\,\beta\in\mathbb{N}^m} = n', \tag{4.3.47}$$

which shows that h is ℓ_1-Segre nondegenerate, for some $\ell_1\geq\ell'_0$. It remains to establish the technical Lemma 4.3.43.

Proof. So we fix j'. Again, we proceed by contradiction. Assume that the generic rank κ' of the $n'\times\infty$ matrix

$$\left(\nabla_{t'}\,R'_{j',\beta}(z,0,0,0,J^{|\beta|}\bar h(0) :\, h(z,0))\right)_{\beta\in\mathbb{N}^m} \tag{4.3.48}$$

is strictly smaller than $n'_{j'}$, namely $\kappa' \leq n'_{j'} - 1$. We choose κ' distinct multiindices $\beta_1, \dots, \beta_{\kappa'} \in \mathbb{N}^m$ such that the generic rank of the $n' \times \kappa'$ extracted matrix

$$\left(\nabla_{t'} R'_{j',\beta_{i'}}(z,0,0,0,J^{|\beta|}\bar{h}(0):\, h(z,0))\right)_{1\leq i'\leq \kappa'} \tag{4.3.49}$$

is equal to κ'. We let $\Lambda(z) \in \mathbb{C}[\![z]\!]$ denote a not identically zero $\kappa' \times \kappa'$ minor of this matrix. It then follows from Cramer's rule and from the rank assumption that for every multiindex $\beta \in \mathbb{N}^m$ different from $\beta_1, \dots, \beta_{\kappa'}$, there exist formal power series $\Lambda^1_\beta(z), \dots, \Lambda^{\kappa'}_\beta(z) \in \mathbb{C}[\![z]\!]$ such that we can write

$$\left\{\begin{aligned} &\Lambda(z)\,\nabla_{t'} R'_{j',\beta}\,(z,0,0,0,J^{|\beta|}\bar{h}(0):\, h(z,0)) \equiv \\ &\qquad\equiv \Lambda^1_\beta(z)\,\nabla_{t'} R'_{j',\beta_1}(z,0,0,0,J^{|\beta|}\bar{h}(0):\, h(z,0)) + \cdots + \\ &\qquad\quad + \Lambda^{\kappa'}_\beta(z)\,\nabla_{t'} R'_{j',\beta_{\kappa'}}(z,0,0,0,J^{|\beta|}\bar{h}(0):\, h(z,0)). \end{aligned}\right. \tag{4.3.50}$$

Replacing the $R'_{j',\beta}$ by their values given by (4.3.46), we get

$$\left\{\begin{aligned} &\Lambda(z)\sum_{\gamma'\in\mathbb{N}^{m'}} \underline{\mathcal{L}}^\beta(\bar{f}^{\gamma'})(z,0,0,0)\,[\nabla_{t'}\,\Theta'_{j',\gamma'}(h(z,0))] \equiv \\ &\qquad \equiv \sum_{\gamma'\in\mathbb{N}^{m'}} \Big(\Lambda^1_\beta(z)\,\underline{\mathcal{L}}^{\beta_1}(\bar{f}^{\gamma'})(z,0,0,0) + \cdots + \\ &\qquad\quad + \Lambda^{\kappa'}_\beta(z)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar{f}^{\gamma'})(z,0,0,0)\Big)\left[\nabla_{t'}\,\Theta'_{j',\gamma'}(h(z,0))\right]. \end{aligned}\right. \tag{4.3.51}$$

We remind that in normal coordinates, we have $g(z,0) \equiv 0$, hence

$$h(z,0) \equiv (f(z,0),0). \tag{4.3.52}$$

Before going further, we need the following elementary observation.

Lemma 4.3.53. *Assume that h is CR-transversal at p_0 and fix j' as in Lemma 4.3.43. Then*

$$\operatorname{genMrk}\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(f(z,0),0)\right)_{\gamma'\in\mathbb{N}^{m'}} = n'_{j'}. \tag{4.3.54}$$

Proof. Suppose on the contrary that this generic matrix rank is equal to an integer $\kappa' \leq n'_{j'} - 1$. By the definition (4.3.45) of $n'_{j'}$, there exists a $n'_{j'} \times n'_{j'}$ minor $d'(z')$ of the $n' \times \infty$ matrix $(\nabla_{t'}\,\Theta'_{j',\gamma'}(z',0))_{\gamma'\in\mathbb{N}^{m'}}$ which does not vanish identically as a power series of z'. We deduce that $d'(f(z,0)) \equiv 0$, contradicting the CR-transversality assumption on h, which completes the proof. □

After making a linear combination, we can rewrite temporarily the relations (4.3.51) under the form

$$\sum_{\gamma'\in\mathbb{N}^{m'}} \overline{F}_{\beta,\gamma'}(z)\;\left[\nabla_{t'}\,\Theta'_{j',\gamma'}(h(z,0))\right] \equiv 0. \tag{4.3.55}$$

By Lemma 4.3.53, there exist multiindices $\gamma'_1, \dots, \gamma'_{n'_{j'}}$ such that a $n'_{j'} \times n'_{j'}$ minor $A(z)$ of the $n' \times n'_{j'}$ matrix

$$\left(\nabla_{t'}\,\Theta'_{j',\gamma'_{i'}}(h(z,0))\right)_{1\leq i'\leq n'_{j'}} \tag{4.3.56}$$

does not vanish identically as a power series of z. We deduce that after making some linear combinations between the lines of the system (4.3.55) that for every $i' = 1, \dots, n'_{j'}$, there exist formal power series $A'_{i',\gamma'}(z)$ such that we can solve

$$A(z)\,\overline{F}_{\beta,\gamma'_{i'}}(z) \equiv \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\overline{F}_{\beta,\gamma'}(z). \tag{4.3.57}$$

We now replace the $\overline{F}_{\beta,\gamma'}$ by their values given by (4.3.51) and (4.3.55). We obtain the following formal equalities, valuable for $i'=1,\dots,n'_{j'}$ and for $\beta\neq\beta_1,\dots,\beta_{\kappa'}$:

$$\text{(4.3.58)}\qquad \left\{\begin{aligned} & A(z)\,\Lambda(z)\ \underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'_{i'}})(z,0,0,0)-A(z)\,\Lambda_\beta^1(z)\,\underline{\mathcal{L}}^{\beta_1}(\bar f^{\gamma'_{i'}})(z,0,0,0)-\cdots-\\ &\qquad -A(z)\,\Lambda_\beta^{\kappa'}(z)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar f^{\gamma'_{i'}})(z,0,0,0)\equiv\\ &\qquad\equiv\sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}}\Big(A'_{i',\gamma'}(z)\,\Lambda(z)\,\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})(z,0,0,0)-\\ &\qquad -A'_{i',\gamma'}(z)\,\Lambda_\beta^1(z)\,\underline{\mathcal{L}}^{\beta_1}(\bar f^{\gamma'})(z,0,0,0)-\cdots-\\ &\qquad -A'_{i',\gamma'}(z)\,\Lambda_\beta^1(z)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar f^{\gamma'})(z,0,0,0)\Big). \end{aligned}\right.$$

After making some linear combinations, we can rewrite this identity as

$$\text{(4.3.59)}\qquad \left\{\begin{aligned} & A(z)\,\Lambda(z)\ \underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'_{i'}})(z,0,0,0)-\sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}}A'_{i',\gamma'}(z)\,\Lambda(z)\,\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})(z,0,0,0)\equiv\\ &\quad\equiv\Lambda_\beta^1(z)\Big(A(z)\,\underline{\mathcal{L}}^{\beta_1}(\bar f^{\gamma'_{i'}})(z,0,0,0)-\\ &\qquad -\sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}}A'_{i',\gamma'}(z)\,\underline{\mathcal{L}}^{\beta_1}(\bar f^{\gamma'})(z,0,0,0)\Big)+\cdots+\\ &\quad +\Lambda_\beta^{\kappa'}(z)\Big(A(z)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar f^{\gamma'_{i'}})(z,0,0,0)-\\ &\qquad -\sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}}A'_{i',\gamma'}(z)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar f^{\gamma'})(z,0,0,0)\Big). \end{aligned}\right.$$

Next, we introduce the following new notation

$$\text{(4.3.60)}\qquad \bar f_{\gamma',\beta}(z):=\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})(z,0,0,0).$$

Of course, we have

$$\text{(4.3.61)}\qquad \bar f^{\gamma'}(\zeta,0)\equiv\sum_{\beta\in\mathbb{N}^m}\bar f_{\gamma',\beta}(0)\,\frac{\zeta^\beta}{\beta!}$$

and

$$\text{(4.3.62)}\qquad \left\{\begin{aligned} \bar f^{\gamma'}(\zeta,\Theta(\zeta,z,0)) &\equiv\sum_{\beta\in\mathbb{N}^m}\frac{\zeta^\beta}{\beta!}\left[\frac{\partial^{|\beta|}}{\partial\zeta^\beta}\Big(\bar f^{\gamma'}(\zeta,\Theta(\zeta,z,0))\Big)\right]_{\zeta=0}\\ &\equiv\underline{\mathcal{L}}^{\beta}(\bar f^{\gamma'})(z,0,0,0)\\ &\equiv\sum_{\beta\in\mathbb{N}^m}\frac{\zeta^\beta}{\beta!}\,\bar f_{\gamma',\beta}(z). \end{aligned}\right.$$

With this notation, we can rewrite (4.3.59) as follows, where $i' = 1, \dots, n'_{j'}$ and $\beta \neq \beta_1, \dots, \beta_{\kappa'}$:

$$
\tag{4.3.63}
\left\{
\begin{aligned}
& A(z)\,\Lambda(z)\ \bar f_{\gamma'_{i'}}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta}(z) \equiv \\
& \equiv \Lambda^1_\beta(z) \left(A(z)\, \bar f_{\gamma'_{i'},\beta}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta}(z) \right) + \cdots + \\
& + \Lambda^1_\beta(z) \left(A(z)\, \bar f_{\gamma'_{i'},\beta}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta}(z) \right).
\end{aligned}
\right.
$$

On the other hand, taking the definition (4.3.60) of $\bar f_{\gamma',\beta}(z)$ and (4.3.62) into account, we have the power series development

$$
\tag{4.3.64}
\left\{
\begin{aligned}
& \sum_{\beta\in\mathbb{N}^m} \frac{\zeta^\beta}{\beta!} \left(A(z)\,\Lambda(z)\, \bar f_{\gamma'_{i'},\beta}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f^{\gamma',\beta}(z) \right) \equiv \\
& \equiv A(z)\,\Lambda(z)\, \bar f^{\gamma'_{i'}}(\zeta, \Theta(\zeta, z, 0)) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f^{\gamma'}(\zeta, \Theta(\zeta, z, 0)).
\end{aligned}
\right.
$$

In this identity, we decompose the sum $\sum_{\beta\in\mathbb{N}^m}$ as the sum $\sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}}$ plus the κ' remaining terms corresponding to $\beta = \beta_1, \dots, \beta_{\kappa'}$ and we substitute the expressions obtained just previously in (4.3.63), which yields

$$
\tag{4.3.65}
\left\{
\begin{aligned}
& \sum_{\beta\in\mathbb{N}^m} \frac{\zeta^\beta}{\beta!} \left(A(z)\,\Lambda(z)\, \bar f_{\gamma'_{i'},\beta}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta}(z) \right) \equiv \\
& \equiv \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \frac{\zeta^\beta}{\beta!} \left(A(z)\,\Lambda(z)\, \bar f_{\gamma'_{i'},\beta}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta}(z) \right) + \\
& + \frac{\zeta^{\beta_1}}{\beta_1!} \left(A(z)\,\Lambda(z)\, \bar f_{\gamma'_{i'},\beta_1}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta_1}(z) \right) + \\
& + \cdots + \\
& + \frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!} \left(A(z)\,\Lambda(z)\, \bar f_{\gamma'_{i'},\beta_{\kappa'}}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\, A'_{i',\gamma'}(z)\, \bar f_{\gamma',\beta_{\kappa'}}(z) \right).
\end{aligned}
\right.
$$

Now, we make some linear combinations, which yields the following representation of the right hand side of (4.3.65)

(4.3.66)
$$\left\{\begin{aligned} & A(z)\,\Lambda(z)\ \bar f^{\gamma'_{i'}}(\zeta,\Theta(\zeta,z,0)) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\,A'_{i',\gamma'}(z)\,\bar f^{\gamma'}(\zeta,\Theta(\zeta,z,0)) \equiv \\ & \equiv \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \frac{\zeta^\beta}{\beta!} \left[\Lambda_\beta^1(z)\left(A(z)\,\bar f_{\gamma'_{i'},\beta_1}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_1}(z)\right) + \right. \\ & \qquad + \cdots + \\ & \qquad \left. + \Lambda_\beta^{\kappa'}(z)\left(A(z)\,\bar f_{\gamma'_{i'},\beta_{\kappa'}}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_{\kappa'}}(z)\right)\right] + \\ & \qquad + \frac{\zeta^{\beta_1}}{\beta_1!}\left[A(z)\,\bar f_{\gamma'_{i'},\beta_1}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_1}(z)\right] + \\ & \qquad + \cdots + \\ & \qquad + \frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!}\left[A(z)\,\bar f_{\gamma'_{i'},\beta_{\kappa'}}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_{\kappa'}}(z)\right]. \end{aligned}\right.$$

If we now set for $i'=1,\dots,n'_{j'}$:

$$\left\{\begin{aligned} \Pi_{i',1}(z) &:= A(z)\,\bar f_{\gamma'_{i'},\beta_1}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_1}(z), \\ & \cdots\cdots\cdots \\ \Pi_{i',\kappa'}(z) &:= A(z)\,\bar f_{\gamma'_{i'},\beta_{\kappa'}}(z) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} A'_{i',\gamma'}(z)\,\bar f_{\gamma',\beta_{\kappa'}}(z), \end{aligned}\right. \tag{4.3.67}$$

then we can rewrite (4.3.66) as follows

$$\left\{\begin{aligned} & A(z)\,\Lambda(z)\ \bar f^{\gamma'_{i'}}(\zeta,\Theta(\zeta,z,0)) - \sum_{\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}} \Lambda(z)\,A'_{i',\gamma'}(z)\,\bar f^{\gamma'}(\zeta,\Theta(\zeta,z,0)) \equiv \\ & \equiv \Pi_{i',1}(z)\left(\frac{\zeta^{\beta_1}}{\beta_1!}\,\Lambda(z) + \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \Lambda_\beta^1(z)\,\frac{\zeta^\beta}{\beta!}\right) + \\ & \quad + \cdots + \\ & \quad + \Pi_{i',\kappa'}(z)\left(\frac{\zeta^{\beta_{\kappa'}}}{\beta_{\kappa'}!}\,\Lambda(z) + \sum_{\beta\neq\beta_1,\dots,\beta_{\kappa'}} \Lambda_\beta^{\kappa'}(z)\,\frac{\zeta^\beta}{\beta!}\right) =: \\ & =: \Pi_{i',1}(z)\,G_1(z,\zeta) + \cdots + \Pi_{i',\kappa'}(z)\,G_{\kappa'}(z,\zeta). \end{aligned}\right. \tag{4.3.68}$$

We now set $C(z):=A(z)\,\Lambda(z)$ and for $i'=1,\dots,n'_{j'}$ and $\gamma'\neq\gamma'_1,\dots,\gamma'_{n'_{j'}}$,

$$B_{i',\gamma'} := \Lambda(z)\,A'_{i',\gamma'}(z). \tag{4.3.69}$$

In summary, we have obtained that for $i' = 1, \ldots, n'_{j'}$, we can write

$$\begin{cases} C(z)\, \bar{f}^{\gamma'_{i'}}(\zeta, \Theta(\zeta, z, 0)) - \sum_{\gamma' \neq \gamma'_1, \ldots, \gamma'_{n'_{j'}}} B_{i',\gamma'}(z)\, \bar{f}^{\gamma'}(\zeta, \Theta(\zeta, z, 0)) \equiv \\ \equiv \Pi_{i',1}(z)\, G_1(z, \zeta) + \cdots + \Pi_{i',\kappa'}(z)\, G_{\kappa'}(z, \zeta). \end{cases} \tag{4.3.70}$$

Because $\kappa' \leq n'_{j'} - 1$, hence there are less than $n'_{j'}$ functions $G_{i'}(z, \zeta)$ in the right hand side of (4.3.70), it follows that there exist power series $\mu_1(z), \ldots, \mu_{n'_{j'}}(z) \in \mathbb{C}[\![z]\!]$, not all zero, such that

$$\begin{cases} 0 \equiv \mu_1(z) \left(C(z)\, \bar{f}^{\gamma'_1}(\zeta, \Theta(\zeta, z, 0)) - \sum_{\gamma' \neq \gamma'_1, \ldots, \gamma'_{n'_{j'}}} B_{1,\gamma'}(z)\, \bar{f}^{\gamma'}(\zeta, \Theta(\zeta, z, 0)) \right) + \\ \qquad + \cdots + \\ \qquad + \mu_{n'_{j'}}(z) \left(C(z)\, \bar{f}^{\gamma'_{n'_{j'}}}(\zeta, \Theta(\zeta, z, 0)) - \sum_{\gamma' \neq \gamma'_1, \ldots, \gamma'_{n'_{j'}}} B_{n'_{j'},\gamma'}(z)\, \bar{f}^{\gamma'}(\zeta, \Theta(\zeta, z, 0)) \right). \end{cases} \tag{4.3.71}$$

Finally, we simplify a little bit this expression by writing it under the form

$$\begin{cases} 0 \equiv \mu_1(z)\, C(z)\, \bar{f}^{\gamma'_1}(\zeta, \Theta(\zeta, z, 0)) + \cdots + \mu_{n'_{j'}}(z)\, C(z)\, \bar{f}^{\gamma'_{n'_{j'}}}(\zeta, \Theta(\zeta, z, 0)) + \\ \qquad + \sum_{\gamma' \neq \gamma'_1, \ldots, \gamma'_{n'_{j'}}} E_{\gamma'}(z)\, \bar{f}^{\gamma'}(\zeta, \Theta(\zeta, z, 0)). \end{cases} \tag{4.3.72}$$

where the $E_{\gamma'}(z)$ are formal power series with respect to z. We are now in position to conclude the proof of Lemma 4.3.53, namely to come to an absurd as announced in the beginning of the proof.

Indeed, since $C(z) \not\equiv 0$ by construction, and since there exists at smallest one power series $\mu_{i'}(z)$ which does not vanish identically, we can apply the following elementary lemma to conclude that $\bar{f}(\zeta, 0)$ satisfies a nontrivial power series identity, which clearly contradicts the CR-transversality assumption.

Lemma 4.3.73. *Assume that there exists a collection of power series $E_{\gamma'}(z)$ indexed by $\gamma' \in \mathbb{N}^{m'}$ with the property that there exists at smallest one multiindex $\gamma'_0 \in \mathbb{N}^{m'}$ with $E_{\gamma'_0}(z) \not\equiv 0$ in $\mathbb{C}[\![z]\!]$ and assume that $\bar{f}(\zeta, \Theta(\zeta, z, 0))$ satisfies the formal power series identity*

$$\sum_{\gamma' \in \mathbb{N}^{m'}} E_{\gamma'}(z)\, \bar{f}^{\gamma'}(\zeta, \Theta(\zeta, z, 0)) \equiv 0 \tag{4.3.74}$$

in $\mathbb{C}[\![z, \zeta]\!]$. Then there exists a collection of constants $F_{\gamma'} \in \mathbb{C}$ indexed by $\gamma' \in \mathbb{N}^{m'}$ which do not all vanish *such that*

$$\sum_{\gamma' \in \mathbb{N}^{m'}} F_{\gamma'}\, \bar{f}^{\gamma'}(\zeta, 0) \equiv 0 \tag{4.3.75}$$

in $\mathbb{C}[\![\zeta]\!]$. In other words, the formal mapping $\zeta \mapsto \bar{f}(\zeta, 0)$ is not *transversal, in the sense of Definition 4.1.2* **(5)**.

Proof. We put $z = 0$ in (4.3.74). If there exists a multiindex $\gamma' \in \mathbb{N}^{m'}$ such that $E_{\gamma'}(0) \neq 0$, we are done. Otherwise, we differentiate (4.3.74) with respect to z and we put $z = 0$. If there exist an integer k with $1 \leq k \leq m$ and a multiindex $\gamma' \in \mathbb{N}^{m'}$ such that $[\partial E_{\gamma'}(0)/\partial z_k](0) \neq 0$, we are done. Otherwise, we again differentiate with respect to z and put $z = 0$. Since

$E_{\gamma_0'}(z) \not\equiv 0$ in $\mathbb{C}[\![z]\!]$, this process converges towards the conclusion after a finite number of steps, which completes the proof. □

The proofs of Lemma 4.3.53 and of part **(4)** of Theorem 4.3.1 are now complete. □

We now prove part **(5)** of Theorem 4.3.1. Proceeding as for parts **(2)** and **(4)**, we shall essentially reason by contradiction, but we shall summarize the main part of the proof (Lemma 4.3.79 below), because it is very similar to the proof of Lemma 4.3.53.

By assumption of ℓ_0'-holomorphic nondegeneracy of M' at p_0', we have

$$\text{genMrk}\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(t')\right)_{1\leq j'\leq d',\,\gamma'\in\mathbb{N}^{m'}} = n'. \tag{4.3.76}$$

Notice that in **(5)**, we make one supplementary assumption, by requiring in addition that h is transversal at p_0. This to insure that the following holds.

Lemma 4.3.77. *Assume that h is transversal at p_0 and that M' is holomorphically nondegenerate at p_0. Then*

$$\text{genMrk}\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(h(t))\right)_{1\leq j'\leq d',\,\gamma'\in\mathbb{N}^{m'}} = n'. \tag{4.3.78}$$

Proof. Suppose on the contrary that this generic matrix rank if equal to an integer $\kappa' \leq n'-1$. By hypothesis, there exists a $n'\times n'$ minor $d'(t')$ of the $n'\times\infty$ matrix (4.3.76) which does not vanish identically as a power series of t'. We deduce $d'(h(t))\equiv 0$, contradicting the transversality of h at p_0, which completes the proof. □

To establish part **(5)** of Theorem 4.3.1, we need the following technical lemma.

Lemma 4.3.79. *Fix j' with $1\leq j'\leq d'$. Let $n'_{j'}$ be the integer defined by*

$$n'_{j'} := \text{genMrk}\left(\nabla_{t'}\,\Theta'_{j',\gamma'}(t')\right)_{\gamma'\in\mathbb{N}^{m'}}. \tag{4.3.80}$$

Assume that h is CR-transversal at p_0. Then we also have

$$\text{genMrk}\left(\nabla_{t'}\,R_{j',\beta}(0,0,0,0,J^{|\beta|}\bar{h}(0):\,h(t))\right)_{\beta\in\mathbb{N}^m} = n'_{j'}. \tag{4.3.81}$$

As in the paragraph after the statement of Lemma 4.3.53, we can easily verify that this technical lemma implies part **(5)** of Theorem 4.3.1. It remains to prove Lemma 4.3.79.

Proof. So we fix j'. Again, we proceed by contradiction. We summarize the proof. Assuming that the generic rank of the $n'\times\infty$ matrix

$$\left(\nabla_{t'}\,R'_{j',\beta}(0,0,0,0,J^{|\beta|}\bar{h}(0):\,h(t))_{\beta\in\mathbb{N}^m}\right) \tag{4.3.82}$$

is equal to an integer $\kappa'\leq n'_{j'}-1$ and taking into account that

$$\text{genMrk}\left(\Theta'_{j',\gamma'}(h(t))\right)_{\gamma'\in\mathbb{N}^{m'}} = n'_{j'} \tag{4.3.83}$$

(by a slight generalization of Lemma 4.3.77) since h is transversal at p_0, we deduce that there exist κ' distinct multiindices $\beta_1,\dots,\beta_{\kappa'}\in\mathbb{N}^m$ and a not identically zero power series $\Lambda(t)\not\equiv 0$ and for every multiindex $\beta\neq\beta_1,\dots,\beta_{\kappa'}$, power series $\Lambda_\beta^1(t),\dots,\Lambda_\beta^{\kappa'}(t)$ such that we can write

$$\left\{\begin{aligned} 0\equiv \sum_{\gamma'\in\mathbb{N}^{m'}} & \Big(\Lambda(t)\,\underline{\mathcal{L}}^{\beta}(\bar{f}^{\gamma'})(0)\,-\Lambda_\beta^1(t)\,\underline{\mathcal{L}}^{\beta_1}(\bar{f}^{\gamma'})(0)\,-\cdots- \\ & -\Lambda_\beta^{\kappa'}(t)\,\underline{\mathcal{L}}^{\beta_{\kappa'}}(\bar{f}^{\gamma'})(0)\Big)\,\left[\nabla_{t'}\,\Theta'_{j',\gamma'}(h(t))\right]. \end{aligned}\right. \tag{4.3.84}$$

Reasoning as in the proof of Lemma 4.3.53, we deduce that there exist $n'_{j'}$ distinct multiindices $\gamma_1',\dots,\gamma_{j'}'\in\mathbb{N}^{m'}$, that there exists a not identically zero power series $A(t)\not\equiv 0$, that for $i'=1,\dots,n'_{j'}$ and for $\gamma'\neq\gamma_1',\dots,\gamma'_{n'_{j'}}$, there exist power series $A'_{i',\gamma'}(t)\in\mathbb{C}[\![t]\!]$, that there exist power series $\Pi_{i',1}(t), \dots, \Pi_{i',\kappa'}(t) \in \mathbb{C}[\![t]\!]$ and that there exist power series $G_1(t,\zeta), \dots, G_{\kappa'}(t,\zeta) \in \mathbb{C}[\![t,\zeta]\!]$ such that we can write for $i' = 1, \dots, n'_{j'}$:

$$\text{(4.3.85)} \qquad \begin{cases} A(t)\,\Lambda(t)\ \bar{f}^{\gamma'_{i'}}(\zeta,0) - \sum\limits_{\gamma' \neq \gamma'_1, \dots, \gamma'_{n'_{j'}}} \Lambda(t)\, A'_{i',\gamma'}(t)\, \bar{f}^{\gamma'}(\zeta,0) \equiv \\ \qquad \equiv \Pi_{i',1}(t)\, G_1(t,\zeta) + \cdots + \Pi_{i',\kappa'}(t)\, G_{\kappa'}(t,\zeta). \end{cases}$$

Since $\kappa' \leq n'_{j'} - 1$, we deduce that there exist power series $\mu_1(t), \dots, \mu_{n'_{j'}}(t) \in \mathbb{C}[\![t]\!]$ not all zero such that we can write, after setting $C(t) := A(t)\,\Lambda(t)$ and $B_{i',\gamma'}(t) := \Lambda(t)\, A'_{i',\gamma'}(t)$:

$$\text{(4.3.86)} \qquad \begin{cases} 0 \equiv \mu_1(t) \left(C(t)\, \bar{f}^{\gamma'_1}(\zeta,0) - \sum\limits_{\gamma' \neq \gamma'_1, \dots, \gamma'_{n'_{j'}}} B_{1,\gamma'}(t)\, \bar{f}^{\gamma'}(\zeta,0) \right) + \\ \qquad + \cdots + \\ \qquad + \mu_{n'_{j'}}(t) \left(C(t)\, \bar{f}^{\gamma'_{n'_{j'}}}(\zeta,0) - \sum\limits_{\gamma' \neq \gamma'_1, \dots, \gamma'_{n'_{j'}}} B_{n'_{j'},\gamma'}(t)\, \bar{f}^{\gamma'}(\zeta,0) \right). \end{cases}$$

Simplifying a little bit this expression by writing it under the form

$$\text{(4.3.87)} \qquad \begin{cases} 0 \equiv \mu_1(t)\, C(t)\, \bar{f}^{\gamma'_1}(\zeta,0) + \cdots + \mu_{n'_{j'}}(t)\, C(t)\, \bar{f}^{\gamma'_{n'_{j'}}}(\zeta,0) + \\ \qquad + \sum\limits_{\gamma' \neq \gamma'_1, \dots, \gamma'_{n'_{j'}}} E_{\gamma'}(t)\, \bar{f}^{\gamma'}(\zeta,0), \end{cases}$$

we deduce by applying the same reasoning as in Lemma 4.3.73 that there exists a not identically zero power series relation

$$\text{(4.3.88)} \qquad \sum_{\gamma' \in \mathbb{N}^{m'}} F_{\gamma'}\, \bar{f}^{\gamma'}(\zeta,0) \equiv 0,$$

contradicting the assumption of CR-transversality, which completes the proof of Lemma 4.3.79. □

This completes the proof of part **(5)**. In conclusion, the proof of Theorem 4.3.1 is complete. □

Bibliography for Part I

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