On Transfer of Biholomorphisms Across Nonminimal Loci

Joël MERKER

Abstract. A connected real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ whose Levi form is nondegenerate in at least one point — hence at every point of some Zariski-dense open subset — is locally biholomorphic to the model Heisenberg quadric pseudosphere of signature $(k, n-k)$ in one point if and only if, at every other Levi nondegenerate point, it is also locally biholomorphic to some Heisenberg pseudosphere, possibly having a different signature $(l, n-l)$. Up to signature, pseudo-sphericity then jumps across the Levi degenerate locus, and in particular, across the nonminimal locus, if there exists any.

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1. INTRODUCTION

The goal of this paper is to provide the complete details of an alternative direct proof of a recent theorem due to Kossovskiy and Shafikov ([3]) which relies on the explicit zero-curvature equations obtained in [7, 8], following the lines of a clever suggestion of Beloshapka. In fact, the proof we give here freely brings a more general statement.

Let $M \subset \mathbb{C}^{n+1}$ be a connected real analytic hypersurface with $n \geqslant 1$. One says that M is $(k, n-k)$ *pseudo-spherical* at one of its points p if it is locally near p biholomorphic to some Heisenberg $(k, n-k)$-pseudo-sphere having, in coordinates $(z_1, \dots, z_n, w) \in \mathbb{C}^{n+1}$, the model quadric equation:

$$w = \overline{w} + 2i\big(-\, z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n \big),$$

for some integer k with $0 \leqslant k \leqslant n-k$; when $n = 1$, one simply says that M is *spherical*.

It is known that a connected real analytic hypersurface of $\mathbb{C}^{n+1}$ which is Levi nondegenerate at every point is $(k, n-k)$-pseudospherical at one point if and only if it is $(k, n-k)$-pseudospherical at every point. More generally, we establish that propagation of $(k, n-k)$ pseudo-sphericality also holds in presence of Levi degenerate points of arbitrary kind.

Theorem 1.1. *Let $M \subset \mathbb{C}^{n+1}$ be a connected real analytic geometrically smooth hypersurface which is Levi nondegenerate in at least one point (hence in some nonempty open subset). Then:*

(a) *The set of Levi nondegenerate points of M is a Zariski open subset of M in the sense that there exists a certain* proper — i.e. *having dimension*

$\leqslant \dim M - 1 = 2n$ — *locally closed global real analytic subset* $\Sigma_{\mathsf{LD}} \subset M$ *locating exactly the Levi degenerate points of* M*:*

$$p \in M \backslash \Sigma_{\mathsf{LD}} \iff \text{Levi form of } M \text{ at } p \text{ is nondegenerate.}$$

(b) *If* M *is locally biholomorphic, in a neighborhood of one of its points* $p \in M$*, to some Heisenberg* $(k, n-k)$*-pseudo-sphere having equation:*

$$w = \overline{w} + 2i\big(- z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n\big),$$

for some integer k *with* $0 \leqslant k \leqslant n-k$ *(so that* $p \in M \backslash \Sigma_{\mathsf{LD}}$ *necessarily is a Levi nondegenerate point of* M *too), then locally at every other Levi nondegenerate point* $q \in M \backslash \Sigma_{\mathsf{LD}}$*, the hypersurface* M *is also locally biholomorphic to some Heisenberg* $(l, n-l)$*-pseudo-sphere, with, possibly* $l \neq k$*.*

Surprisingly, Example 6.2 in [3] shows that $l \neq k$ may occur, in the case of a *nonminimal* hypersurface of $\mathbb{C}^{n+1}$ with $n \geqslant 2$, a local example for which the Levi degenerate locus Σ_{LD} consists precisely of a complex n-dimensional hypersurface contained in M.

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2. PROOF IN $\mathbb{C}^2$

Let $M \subset \mathbb{C}^2$ be a connected real analytic hypersurface. Pick a point:

$$p \in M,$$

and choose some affine coordinates centered at p:

$$(z, w) = \big(x + iy,\, u + iv\big)$$

satisfying:

$$T_0M = \big\{\mathsf{Im}\, w = 0\big\},$$

so that the implicit function theorem represents M as:

$$u = \varphi(x, y, v),$$

in terms of some *graphing function* φ which is expandable in convergent Taylor series in some (possibly small) open bidisc:

$$\square_{\rho_0}^2 := \big\{(z,w)\in\mathbb{C}^2\colon\ |z|<\rho_0,\ |w|<\rho_0\big\},$$

with of course $\rho_0>0$.

Classically, writing:

$$\tfrac{w+\overline{w}}{2}=\varphi\big(\tfrac{z+\overline{z}}{2},\ \tfrac{z-\overline{z}}{2i},\ \tfrac{w-\overline{w}}{2i}\big)$$

and using the analytic implicit function theorem, one solves w in terms of $z,\overline{z},\overline{w}$ getting a representation of M as:

$$w=\Theta\big(z,\overline{z},\overline{w}\big);$$

recall that implicitly, when one does this, one must considers $(z,w,\overline{z},\overline{w})$ as 4 independent complex variables, which amounts to *complexify* them, namely to introduce the *complexified variables*:

$$\big(z,w,\underline{z},\underline{w}\big)\in\mathbb{C}^4;$$

in what follows, we will work with $(z,w,\overline{z},\overline{w})$-variables, keeping in mind that they can be replaced by $(z,w,\underline{z},\underline{w})$ *since all objects are convergent Taylor series*; so here, $\Theta(z,\underline{z},\underline{w})$ is a convergent Taylor series of $(z,\underline{z},\underline{w})$ for $|z|<\rho_0$, $|\underline{z}|<\rho_0$, $|\underline{w}|<\rho_0$, after shrinking $\rho_0>0$ if necessary.

Moreover, since:

$$0=\varphi(0)=\varphi_x(0),=\varphi_y(0)=\varphi_u(0),$$

one has:

$$\Theta=\overline{w}+\mathrm{O}(2).$$

Now, it is known — or it could be taken here as a definition — that M is *Levi nondegenerate* at $0\in M$ when the local holomorphic map:

$$\begin{aligned}\mathbb{C}^2&\longrightarrow\ \mathbb{C}^2\\ \big(\overline{z},\overline{w}\big)&\longmapsto\ \Big(\Theta\big(0,\overline{z},\overline{w}\big),\ \Theta_z\big(0,\overline{z},\overline{w}\big)\Big)\end{aligned}$$

is of rank 2 at $(\overline{z},\overline{w})=(0,0)$; of course, one would better think in terms of $(\underline{z},\underline{w})$-variables here.

More generally, M is *Levi nondegenerate* at an arbitrary point close to the origin:

$$(z_p,w_p)\in M\cap\square_{\rho_0}^2$$

when the map:

$$\begin{aligned}\mathbb{C}^2&\longrightarrow\ \mathbb{C}^2\\ \big(\overline{z},\overline{w}\big)&\longmapsto\ \Big(\Theta\big(z_p,\overline{z},\overline{w}\big),\ \Theta_z\big(z_p,\overline{z},\overline{w}\big)\Big)\end{aligned}$$

is of rank 2 at $(\overline{z}_p,\overline{w}_p)$, which precisely means the nonvanising of the Jacobian determinant:

$$0\neq\det\begin{pmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{pmatrix}=\Theta_{\overline{z}}\,\Theta_{z\overline{w}}-\Theta_{\overline{w}}\,\Theta_{z\overline{z}}$$

at:

$$(z, \overline{z}, \overline{w}) = (z_p, \overline{z}_p, \overline{w}_p).$$

One may either show-check that such a definition re-gives the standard definition of Levi nondegeneracy (*cf.* [4, 5, 12]), or prove directly that as it stands, it really is independent of the choice of coordinates ([11]).

Although we could then spend time to re-prove it properly, the following fact — here admitted — is well known.

Proposition 2.1. *If a connected real analytic hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is Levi nondegenerate in at least one point, then the set of Levi degenerate points of M is a proper real analytic subset:*

$$\Sigma_{\sf LD} \subsetneqq M. \quad \square$$

In Theorem 1.1, we indeed make the assumption that $\Sigma_{\sf LD}$ is proper, since otherwise, the real analytic M would be *Levi-flat*, hence as is known, everywhere locally biholomorphic to $\mathbb{C}^n \times \mathbb{R}$.

Suppose to begin with for $M^3 \subset \mathbb{C}^2$ that:

$$0 \notin \Sigma_{\sf LD}.$$

Then the above map being of rank 2 at $(\overline{z}, \overline{w}) = (0, 0)$, one can solve, following [7], the following two equations:

$$w(z) = \Theta\big(z, \overline{z}, \overline{w}\big),$$
$$w_z(z) = \Theta_z\big(z, \overline{z}, \overline{w}\big),$$

for the two variables $(\overline{z}, \overline{w})$, and then insert the latter in:

$$w_{zz}(z) = \Theta_{zz}\big(z, \overline{z}, \overline{w}\big),$$

to get a complex second-order ordinary differential equation:

$$w_{zz}(z) = \Phi\big(z, w(z), w_z(z)\big).$$

One should notice that the possibility of solving $(\overline{z}, \overline{w})$ is expressed by the nonvanishing of *exactly and precisely the same* Jacobian determinant as the one expressing Levi nondegeneracy.

In the article [7], one deduces from an explicitly known condition on Φ for this second-order equation $w_{zz} = \Phi(z, w, w_z)$ to be pointwise equivalent to the free particle Newtonian equation:

$$w'_{z'z'} = 0,$$

that a real analytic hypersurface $M \subset \mathbb{C}^2$ which is Levi nondegenerate at $0 \in M$ as above is *spherical* in the sense — recall the definition — that it is locally biholomorphic to:

$$w' = \overline{w}' + 2i\, z'\overline{z}',$$

if and only if its complex graphing function Θ satisfies an explicit (not completely developed) equation which we now present.

Introduce the expression:

$$
\begin{aligned}
\mathsf{AJ}^4(\Theta) := \frac{1}{[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}]^3} \bigg\{ & \Theta_{zz\overline{z}\overline{z}} \bigg(\Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) - \\
& - 2\Theta_{zz\overline{z}\overline{w}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) + \Theta_{zz\overline{w}\overline{w}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix} \bigg) + \\
& + \Theta_{zz\overline{z}} \bigg(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{w}\overline{w}} \end{vmatrix} - 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{w}} \end{vmatrix} + \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{z}} \end{vmatrix} \bigg) + \\
& + \Theta_{zz\overline{w}} \bigg(- \Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}\overline{w}} \end{vmatrix} + 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{w}} \end{vmatrix} - \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{z}} \end{vmatrix} \bigg) \bigg\},
\end{aligned}
$$

noticing that its denominator:

$$
\big[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}\big]^3
$$

does not vanish at the origin since $0 \in M$ was assumed (temporarily) to be a Levi nondegenerate point. Introduce also the vector field derivation:

$$
\mathscr{D} := \frac{-\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}} \frac{\partial}{\partial \overline{z}} + \frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}} \frac{\partial}{\partial \overline{w}}.
$$

Then the main and unique theorem of [7] states that M is spherical at 0 if and only if:

$$
0 \equiv \mathscr{D}\big(\mathscr{D}\big(\mathsf{AJ}^4(\Theta)\big)\big),
$$

identically in $\mathbb{C}\{z, \overline{z}, \overline{w}\}$.

Unfortunately, it is essentially impossible to print in a published article what one obtains after a full expansion of these two derivations.

Nevertheless, by thinking a bit, one convinces oneself that after full expansion, and reduction to a common denominator, one obtains a kind of expression that we will denote in summarized form as:

$$
\frac{\mathsf{polynomial}\big(\big(\Theta_{z^j\overline{z}^k\overline{w}^l}\big)_{1\leqslant j+k+l\leqslant 6}\big)}{\big[\Theta_{\overline{z}}\,\Theta_{z\overline{w}} - \Theta_{\overline{w}}\,\Theta_{z\overline{z}}\big]^7},
$$

and hence instantly, sphericity of M is characterized by:

$$
\boxed{0 \equiv \mathsf{polynomial}\Big(\big(\Theta_{z^j\overline{z}^k\overline{w}^l}(z, \overline{z}, \overline{w})\big)_{1\leqslant j+k+l\leqslant 6}\Big).}
$$

One notices that the complex graphing function Θ is differentiated always at leat once.

Interpretation. *Then the true thing is: after erasing the Levi determinant lying at denominator place, if this explicit equation vanishes in some very small neighborhood of some point:*

$$
(z_p, \overline{z}_p, \overline{w}_p) \;\in\; \square_{\rho_0}^3
$$

of the threedisc of convergence of Θ, then by the uniqueness principle for analytic functions, the concerned polynomial numerator:

$$
\mathsf{polynomial}\big(\big(\Theta_{z^j\overline{z}^k\overline{w}^l}(z, \overline{z}, \overline{w})\big)_{1\leqslant j+k+l\leqslant 6}\big)
$$

immediately vanishes identically allover $\square^3_{\rho_0}$*, so that sphericity at one point should freely propagate to all other Levi nondegenerate points* $q \in M \cap \square^2_{\rho_0}$.

Before providing rigorous details to explain the latter assertion, a further comment is in order.

Speculative intuitive thought. *The explicit sphericity formula brings the important information that denominator places are occupied by nondegeneracy conditions, so that division is allowed only at points where these conditions are satisfied, but numerator places happen to be polynomial, a computional fact which hence enables one to jump across degenerate points through the 'bridge-numerator' from one nondegenerate point to another nondegenerate point.*

Now, the local version of Theorem 1.1 is as follows. Notice that from now on, one does not assume anymore that the origin $0 \in M \backslash \Sigma_{\mathsf{LD}}$ be a Levi nondegenerate point.

Proposition 2.2. *With* $M^3 \subset \mathbb{C}^2$, $\square^2_{\rho_0}$, $(z, \overline{z}, \overline{w})$, φ, Θ *as above, assuming that the real analytic subset* Σ_{LD} *of Levi degenerate points is proper, if* M *is spherical at one Levi nondegenerate point:*

$$p \in \big(M \backslash \Sigma_{\mathsf{LD}}\big) \cap \square^2_{\rho_0},$$

then M *is also spherical at every other Levi nondegenerate point:*

$$q \in \big(M \backslash \Sigma_{\mathsf{LD}}\big) \cap \square^2_{\rho_0}.$$

Proof. Take a (possibly much) smaller bidisc:

$$p + \square^2_{\rho'} \subset\subset \square^2_{\rho_0},$$

with $0 < \rho' \ll \rho_0$ to be chosen below, and center new coordinates at:

$$p = (z_p, w_p),$$

that is to say, introduce the new translated coordinates:

$$z' := z - z_p, \qquad\qquad w' := w - w_p.$$

The graphed complex equation:

$$w = \Theta\big(z, \overline{z}, \overline{w}\big)$$

then becomes:

$$w' + w_p = \Theta\big(z' + z_p,\, \overline{z}' + \overline{z}_p,\, \overline{w}' + \overline{w}'_p\big).$$

Of course, the fact that $p \in M$ reads:

$$w_p = \Theta\big(z_p, \overline{z}_p, \overline{w}_p\big),$$

and hence, in the new coordinates (z', w') centered at p, the equation of M becomes:

$$\boxed{\begin{aligned} w' &= \Theta\big(z' + z_p,\, \overline{z}' + \overline{z}_p,\, \overline{w}' + \overline{w}_p\big) - \Theta\big(z_p, \overline{z}_p, \overline{w}_p\big) \\ &=: \Theta'\big(z', \overline{z}', \overline{w}'\big), \end{aligned}}$$

in terms of a new graphing function Θ' that visibly satisfies:

$$\Theta'\big(0',0',0'\big) = 0.$$

Observation 2.3. *For any integers:*

$$(j,k,l)\in\mathbb{N}^3,$$

with:

$$j+k+l\geqslant 1,$$

one has:

$$\boxed{\Theta'_{z'^j\overline{z}'^k\overline{w}'^l}(0',0',0') = \Theta_{z^j\overline{z}^k\overline{w}^l}\big(z_p,\overline{z}_p,\overline{w}_p\big).}$$

Proof. Indeed, the constant $-\Theta\big(z_p,\overline{z}_p,\overline{w}_p\big)$ disappears after just a single differentiation. □

Now, the Levi nondegeneracy of M at p which reads according to what precedes as:

$$0\neq\big[\Theta_{\overline{z}}\,\Theta_{z\overline{w}}-\Theta_{\overline{w}}\,\Theta_{z\overline{z}}\big]\big(z_p,\overline{z}_p,\overline{w}_p\big),$$

reads in the new coordinates as:

$$0\neq\big[\Theta'_{\overline{z}'}\,\Theta'_{z'\overline{w}'}-\Theta'_{\overline{w}'}\,\Theta_{z'\overline{z}'}\big](0',0',0'),$$

which means as we know Levi nondegeneracy at $(z',\overline{z}',\overline{w}')=(0',0',0')$.

Precisely because in [7] one needs only this condition to hold in order to associate as was explained above a second-order complex ordinary differential equation:

$$w'_{z'z'} = \Phi'\big(z',w'(z'),w_{z'}(z')\big)$$

of course for some possibly very small:

$$|z'|<\rho', \qquad\qquad |w'|<\rho',$$

— this is where one has to choose ρ' with $0<\rho'\ll\rho_0$, the possible presence of rather close Levi degenerate points being a constraint in the needed application(s) of the implicit function theorem —, one has the impression that one can in principle only determine whether M is spherical restrictively in such a very narrow neighborhood $\square^2_{\rho'}$ of p in $\mathbb{C}^2$, when one applies the main result of [7].

But looking just at the numerator of the equation which expresses that M is spherical at p in the coordinates (z',w'):

$$0\equiv\frac{\overbrace{\mathsf{polynomial}}^{\text{same universal expression}}\Big(\big(\Theta'_{z'^j\overline{z}'^k\overline{w}'^l}\big)_{1\leqslant j+k+l\leqslant 6}\Big)}{\underbrace{\big[\Theta'_{\overline{z}'}\,\Theta'_{z'\overline{w}'}-\Theta'_{\overline{w}}\,\Theta'_{z'\overline{z}'}\big]^7}_{\text{nonvanishing at }(0',0',0')}},$$

if one takes account of the above observation, one readily realizes that:

$$\mathsf{polynomial}\Big(\big(\Theta'_{z'^j\underline{z}'^k\underline{w}'^l}(z',\underline{z}',\underline{w}')\big)_{1\leqslant j+k+l\leqslant 6}\Big) = \mathsf{polynomial}\Big(\big(\Theta_{z^j\underline{z}^k\underline{w}^l}(z,\underline{z},\underline{w})\big)_{1\leqslant j+k+l\leqslant 6}\Big),$$

so that the identical vanishing of the left-hand side for:

$$|z'| < \rho' \ll \rho_0, \qquad |\underline{z}'| < \rho' \ll \rho_0, \qquad |\underline{w}'| < \rho' \ll \rho_0,$$

means the identical vanishing of the right-hand side for:

$$|z - z_p| < \rho' \ll \rho_0, \qquad |\underline{z} - \overline{z}_p| < \rho' \ll \rho_0, \qquad |\underline{w} - \overline{w}_p| < \rho' \ll \rho_0,$$

which lastly yields *thanks to the uniqueness principle enjoyed by analytic functions* the identical vanishing of the original numerator *in the whole initial domain of convergence*:

$$\boxed{0 \equiv \mathsf{polynomial}\Big(\big(\Theta_{z^j\underline{z}^k\underline{w}^l}(z,\underline{z},\underline{w})\big)_{1\leqslant j+k+l\leqslant 6}\Big) \qquad (|z|<\rho_0,\ |\underline{z}|<\rho_0,\ |\underline{w}|<\rho_0).}$$

Take now any other Levi nondegenerate point:

$$q \in M \cap \square^2_{\rho_0}.$$

The goal is to prove that M is also spherical at q. Center similarly new coordinates at $q = (z_q, w_q)$:

$$z'' := z - z_q, \qquad w'' := w - w_q.$$

Introduce the new graphed equations:

$$\begin{aligned} w'' &= \Theta\big(z'' + z_q,\, \overline{z}'' + \overline{z}_q,\, \overline{w}'' + \overline{w}_q\big) - \Theta\big(z_q, \overline{z_q}, \overline{w}_q\big) \\ &=: \Theta''\big(z'', \overline{z}'', \overline{w}''\big). \end{aligned}$$

At such a point, since the Levi determinant is nonvanishing, one can for completeness construct the associated second-order complex ordinary differential equations:

$$w''_{z''z''}(z'') = \Phi''\big(z'', w''(z''),\, w''_{z''}(z'')\big),$$

or question directly whether local sphericity holds near q by plainly applying the main theorem of [7], namely question whether the following equation holds:

$$0 \overset{?}{\equiv} \frac{\overbrace{\mathsf{polynomial}}^{\text{again same universal expression}}\Big(\big(\Theta''_{z''^j\overline{z}''^k\overline{w}''^l}\big)_{1\leqslant j+k+l\leqslant 6}\Big)}{\underbrace{\big[\Theta''_{\overline{z}''}\,\Theta''_{z''\overline{w}''} - \Theta''_{\overline{w}}\,\Theta''_{z'\overline{z}''}\big]^7}_{\text{nonvanishing at } (0'',0'',0'')}}.$$

But then by exactly the same application of the above observation, we know that this last numerator satisfies:

$$\mathsf{polynomial}\Big(\big(\Theta''_{z''^j\underline{z}''^k\underline{w}''^l}(z'',\underline{z}'',\underline{w}'')\big)_{1\leqslant j+k+l\leqslant 6}\Big) = \mathsf{polynomial}\Big(\big(\Theta_{z^j\underline{z}^k\underline{w}^l}(z,\underline{z},\underline{w})\big)_{1\leqslant j+k+l\leqslant 6}\Big),$$

when:

$$\begin{gathered} |z''| < \rho'' \ll \rho_0, \qquad |\underline{z}''| < \rho'' \ll \rho_0, \qquad |\underline{w}'| < \rho'' \ll \rho_0, \\ |z - z_q| < \rho'' \ll \rho_0, \qquad |\underline{z} - \overline{z}_q| < \rho'' \ll \rho_0, \qquad |\underline{w} - \overline{w}_q| < \rho'' \ll \rho_0, \end{gathered}$$

and since we already know that the latter right-hand side vanishes, according to the last boxed equation, we conclude that M is indeed spherical at q. □

To finish the proof of Theorem 1.1 in the case $n = 1$, it remains only to *globalize* this local propagation of sphericity. One does this by means of standard arguments which consist to pick up one Levi nondegenerate point $p^\sim \in \square^2_{\rho_0}$ (possibly close to the boundary of the bidisc!), to center some affine coordinates at $p^\sim$, to use local real analytic equations for M expanded in some Taylor series which converge in some other bidisc $\square^2_{\rho_0^\sim}$ centered at $p^\sim$, and to apply the same reasonings as above to propagate sphericity from $p^\sim$ to any other Levi nondegenerate point $q^\sim \in M \cap \square^2_{\rho_0^\sim}$. By connectedness of M, one concludes.

3. Proof in $\mathbb{C}^{n+1}$ $(n \geqslant 2)$

We briefly summarize the quite similar arguments, relying upon [8].

The Levi determinant becomes:

$$\Delta := \begin{vmatrix} \Theta_{\overline{z}_1} & \cdots & \Theta_{\overline{z}_n} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \cdots & \Theta_{z_1\overline{z}_n} & \Theta_{z_1\overline{w}} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \Theta_{z_n\overline{z}_1} & \cdots & \Theta_{z_n\overline{z}_n} & \Theta_{z_n\overline{w}} \end{vmatrix}.$$

It is nonzero at one point:

$$p = \big(z_{1p}, \dots, z_{np}, w_p\big) \in M$$

if and only if M is Levi nondegenerate at p, and also, if and only if one can associate to M a completely integrable system of second-order partial differential equations:

$$w_{z_{k_1}z_{k_2}}(z) = \Phi_{k_1,k_2}\big(z,\, w(z),\, w_{z_1}(z), \dots, w_{z_n}(z)\big) \qquad {\scriptstyle (1 \leqslant k_1,\, k_2 \leqslant n)}.$$

Hachtroudi ([2]) established that such a system is pointwise equivalent to:

$$w'_{z'_{k_1}z'_{k_2}}(z') = 0 \qquad {\scriptstyle (1 \leqslant k_1,\, k_2 \leqslant n)}$$

if and only if:

$$\begin{aligned} 0 \equiv\ & \frac{\partial^2 \Phi_{k_1,k_2}}{\partial w_{z_{\ell_1}} w_{z_{\ell_2}}} - \\ & - \tfrac{1}{n+2} \sum_{\ell_3=1}^{n} \left(\delta_{k_1,\ell_1} \frac{\partial^2 \Phi_{\ell_3,k_2}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_2}}} + \delta_{k_1,\ell_2} \frac{\partial^2 \Phi_{\ell_3,k_2}}{\partial w_{z_{\ell_1}} \partial w_{z_{\ell_3}}} + \delta_{k_2,\ell_1} \frac{\partial^2 \Phi_{k_1,\ell_3}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_2}}} + \delta_{k_2,\ell_2} \frac{\partial^2 \Phi_{k_1,\ell_3}}{\partial w_{z_{\ell_1}} \partial w_{z_{\ell_3}}} \right) + \\ & + \tfrac{1}{(n+1)(n+2)} \big[\delta_{k_1,\ell_1}\delta_{k_2,\ell_2} + \delta_{k_2,\ell_1}\delta_{k_1,\ell_2}\big] \sum_{\ell_3=1}^{n} \sum_{\ell_4=1}^{n} \frac{\partial^2 \Phi_{\ell_3,\ell_4}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_4}}} \qquad {\scriptstyle (1 \leqslant k_1,\, k_2 \leqslant n) \atop (1 \leqslant \ell_1,\, \ell_2 \leqslant n)}. \end{aligned}$$

When one does apply Hachtroudi's results to CR geometry (instead of Chern-Moser's, which is up to now not sufficiently explicit to be applied), the signature of the Levi forms disappears for the following reason.

The infinite-dimensional local Lie (pseudo-)group of biholomorphic transformations:

$$\begin{aligned} \big(z_1, \dots, z_n, w\big) \longmapsto\ & \big(z'_1, \dots, z'_n, w'\big) \\ & = \big(z'_1(z_\bullet, w), \dots, z'_n(z_\bullet, w), w'(z_\bullet, w)\big) \end{aligned}$$

acts simultaneously on $(z_\bullet, w)$-variables and on $(\overline{z}_\bullet, \overline{w})$-variables as:

$$\begin{aligned}(\overline{z}_1, \dots, \overline{z}_n, \overline{w}) \longmapsto & \ (\overline{z}'_1, \dots, \overline{z}'_n, \overline{w}') \\ & = \big(\overline{z}'_1(\overline{z}_\bullet, \overline{w}), \dots, \overline{z}'_n(\overline{z}_\bullet, \overline{w}), \overline{w}'(\overline{z}_\bullet, \overline{w})\big).\end{aligned}$$

But when one passes to the extrinsic complexification, one replaces $(\overline{z}, \dots, \overline{z}_n, \overline{w})$-variables by new independent variables:

$$\big(\underline{z}_1, \dots, \underline{z}_n, \underline{w}\big),$$

considered as the *constants* of integration for the system of partial differential equations. Hence the local Lie (pseudo)-group considered by Hachtroudi becomes enlarged as the group of transformations:

$$\big(z_\bullet, w,\ \underline{z}_\bullet, \underline{w}\big) \longmapsto \Big(\text{holomorphic map}\big(z_\bullet, w\big),\ \text{other holomorphic map}\big(\underline{z}_\bullet, \underline{w}\big)\Big)$$

in which the transformations on the 'constant-of-integration' variables $(\underline{z}_1, \dots, \underline{z}_n, \underline{w})$ becomes completely dis-coupled from the group of transformations on the true variables $(z_1, \dots, z_n, w)$. By definition (*cf.* the explanations in [7]), transformations on differential equations, when viewed in the space of solutions, are always of this general form.

It is then clear that all *complexified* Heisenberg $(k, n-k)$ pseudospheres:

$$w = \underline{w} + 2i\big(-\ z_1\underline{z}_1 - \cdots - z_k\underline{z}_k + z_{k+1}\underline{z}_{k+1} + \cdots + z_n\underline{z}_n\big),$$

become all pairwise equivalent through such transformations, because one is allowed to replace $\underline{z}_1, \dots, \underline{z}_k$ by $-\underline{z}_1, \dots, -\underline{z}_k$ without touching $z_1, \dots, z_k$; even the factor i can be erased:

$$w = \underline{w} + z_1\underline{z}_1 + \cdots + z_k\underline{z}_k + z_{k+1}\underline{z}_{k+1} + \cdots + z_n\underline{z}_n$$

Therefore, when passing to systems of partial differential equations associated to CR manifolds, *Levi form signatures drop*.

Consequently, when one applies the main theorem of [8], according to which a Levi nondegenerate $M \subset \mathbb{C}^{n+1}$ *having given Levi form signature* $(k, n-k)$ is pseudo-spherical if and only if (notation same as in [8]) its Hachtroudi system is equivalent to:

$$w'_{z'_{k_1} z'_{k_2}}(z') = 0 \qquad\qquad (1 \leqslant k_1,\ k_2 \leqslant n),$$

and moreover, if and only if — after translating back to the graphing function Θ the explicit condition of Hachtroudi — the following identical vanishing property holds:

$$
\begin{aligned}
0 \equiv \frac{1}{\Delta^3}\Bigg[\sum_{\mu=1}^{n+1}\sum_{\nu=1}^{n+1}\Bigg[\Delta^{\mu}_{[0_{1+\ell_1}]}\cdot\Delta^{\nu}_{[0_{1+\ell_2}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{k_1}\partial z_{k_2}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{k_1}\partial z_{k_2}\partial\overline{t}^\tau}\bigg\}-\\
-\tfrac{\delta_{k_1,\ell_1}}{n+2}\sum_{\ell_3=1}^{n}\Delta^{\mu}_{[0_{1+\ell_3}]}\cdot\Delta^{\nu}_{[0_{1+\ell_2}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{\ell_3}\partial z_{k_2}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{\ell_3}\partial z_{k_2}\partial\overline{t}^\tau}\bigg\}-\\
-\tfrac{\delta_{k_1,\ell_2}}{n+2}\sum_{\ell_3=1}^{n}\Delta^{\mu}_{[0_{1+\ell_1}]}\cdot\Delta^{\nu}_{[0_{1+\ell_3}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{\ell_3}\partial z_{k_2}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{\ell_3}\partial z_{k_2}\partial\overline{t}^\tau}\bigg\}-\\
-\tfrac{\delta_{k_2,\ell_1}}{n+2}\sum_{\ell_3=1}^{n}\Delta^{\mu}_{[0_{1+\ell_3}]}\cdot\Delta^{\nu}_{[0_{1+\ell_2}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{k_1}\partial z_{\ell_3}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{k_1}\partial z_{\ell_3}\partial\overline{t}^\tau}\bigg\}-\\
-\tfrac{\delta_{k_2,\ell_2}}{n+2}\sum_{\ell_3=1}^{n}\Delta^{\mu}_{[0_{1+\ell_1}]}\cdot\Delta^{\nu}_{[0_{1+\ell_3}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{k_1}\partial z_{\ell_3}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{k_1}\partial z_{\ell_3}\partial\overline{t}^\tau}\bigg\}+\\
+\tfrac{1}{(n+1)(n+2)}\cdot\big[\delta_{k_1,\ell_1}\delta_{k_2,\ell_2}+\delta_{k_2,\ell_1}\delta_{k_1,\ell_2}\big]\cdot\\
\cdot\sum_{\ell_3=1}^{n}\sum_{\ell_4=1}^{n}\Delta^{\mu}_{[0_{1+\ell_3}]}\cdot\Delta^{\nu}_{[0_{1+\ell_4}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{\ell_3}\partial z_{\ell_4}\partial\overline{t}_\mu\partial\overline{t}_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\overline{t}^\mu\overline{t}^\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{\ell_3}\partial z_{\ell_4}\partial\overline{t}^\tau}\bigg\}\Bigg]\Bigg],\\
(1\leqslant k_1,\,k_2\leqslant n;\ \ (1\leqslant \ell_1,\,\ell_2\leqslant n),
\end{aligned}
$$

one can reason exactly as in the preceding section for $M^3 \subset \mathbb{C}^2$ — noticing that the denominator is similarly $\frac{1}{\Delta^3}$, noticing that the numerator is similarly polynomial in the partial derivatives of the graphing function Θ —, but when one jumps from a Levi nondegenerate point $p \in M \cap \square^{n+1}_{\rho_0}$ to another Levi nondegenerate point $q \in M \cap \square^{n+1}_{\rho_0}$, from the property of local equivalence at q to:

$$
w''_{z''_{k_1} z''_{k_2}}(z'') = 0 \qquad\qquad (1 \leqslant k_1,\, k_2 \leqslant n),
$$

one can only conclude that the complexification of M near q is equivalent near q to:

$$
w'' = \underline{w}'' + z''_1\underline{z}''_1 + \cdots + z''_k\underline{z}''_k + z''_{k+1}\underline{z}''_{k+1} + \cdots + z''_n\underline{z}''_n
$$

so that the Levi form signature can in principle change — and really does in Example 6.3 of [3] — through Levi degenerate points. □

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