LOW POLE ORDER FRAMES ON VERTICAL JETS OF THE UNIVERSAL HYPERSURFACE

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ABSTRACT. For low order jets, it is known how to construct meromorphic frames on the space of the so-called *vertical k-jets* $J^k_{\mathsf{vert}}(\mathcal{X})$ of the universal hypersurface $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}$ parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree d. In 2004, for $k = n$, Siu announced that there exist two constants $c_n \geqslant 1$ and $c'_n \geqslant 1$ such that the twisted tangent bundle:

$$T_{J^n_{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(c'_n)$$

is generated at every point by its global sections. In the present article, we establish this property outside a certain exceptional algebraic subset $\Sigma \subset J^n_{\mathsf{vert}}(\mathcal{X})$ defined by the vanishing of certain Wronskians, with the *effective* pole order $c_n = \frac{n^2+5n}{2}$, thus recovering $c_2 = 7$ (Paŭn), $c_3 = 12$ (Rousseau), and with $c'_n = 1$.

Moreover, at the cost of raising c_n up to $c_n = n^2+2n$, the same generation property holds outside the smaller set $\widetilde{\Sigma} \subset \Sigma \subset J^n_{\mathsf{vert}}(\mathcal{X})$ which is defined by the vanishing of all first order jets. Applications to *weak* (with Σ) and to *strong* (with $\widetilde{\Sigma}$) algebraic degeneracy of entire holomorphic curves $\mathbb{C} \to X$ are upcoming.

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§1. INTRODUCTION

The Kobayashi hyperbolicity conjecture (1970), in optimal degree and taking account of Brody's theorem (1978), expects that all entire holomorphic curves $f : \mathbb{C} \to X$ into a complex projective (algebraic, smooth) hypersurface $X \subset \mathbb{P}^{n+1}$ must be constant if $\deg X \geqslant 2n + 1$, provided X is generic.

In 2004, Siu [18] announced a strategy of proof, valid in arbitrary dimensions for (extremely) high (noneffective) degrees $d \gg n$. Two major techniques are used.

Inspired by Bloch's ideas, one looks firstly for global sections of the Green-Griffiths bundle $E^{GG}_{k,m}T^*_X$ of jet differentials of order k and weighted degree m (*cf.* [9]), which vanish on some ample divisor; an Ahlfors-Schwarz-type theorem then forces every entire curve $f : \mathbb{C} \to X$ to satisfy the corresponding differential equation ([3]), a first step toward algebraic degeneracy. In 1997, Demailly introduced a refined subbundle $E_{k,m}T^*_X$ having better positivity properties which consists of jet differentials that are invariant under (local) reparametrizations of the source $\mathbb{C}$. In dimension $n = 3$ for jets of order $k = 3$, Rousseau ([14]) completely described the algebraic structure of $E_{k,m}T^*_X$ in its fibers, decomposed it in direct sums of Schur bundles $\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T^*_X$, computed its Euler

Date: 2014-11-1.

2000 *Mathematics Subject Classification.* Primary: 32Q45; Secondary: 14N05, 14J70.

Key words and phrases. Multivariate Faà di Bruno formula, Projective algebraic hypersurfaces, Jets of holomorphic curves, Weak and strong Green-Griffiths algebraic degeneracy.

characteristic $\chi\big(X,\, E_{k,m}T_X^*\big)$, majorated from above $h^2\big(X,\, E_{k,m}T_X^*\big)$ (*see* [15]), and established existence of global algebraic differential equations in degree $d \geqslant 97$.

In [11, 12], one finds a *complete algorithm* to generate all Demailly-Semple invariants in arbitrary dimension $n \geqslant 1$ and for jets of any order $k \geqslant 1$. In particular, for $n = k = 4$, there are 16 fundamental, mutually independent bi-invariant polynomials generating the Demailly-Semple (unipotent-invariant) algebra sharing 41 (gröbnerized) syzygies, and one deduces by polarization that the algebra of all invariants for $n = k = 4$ is generated by 2835 polynomials. Nonconstant entire holomorphic curves valued in an algebraic 3-fold (resp. 4-fold) $X^3 \subset \mathbb{P}^4(\mathbb{C})$ (resp. $X^4 \subset \mathbb{P}^5(\mathbb{C})$) of degree d satisfy ([12]) global differential equations as soon as $d \geqslant 72$ (resp. $d \geqslant 259$).

In [4], for dimensions $n = 2, 3, 4, 5$ and for jet orders $k = 3, 4, 5, \underline{5}$, resp., it is shown that asymptotically as $m \to \infty$:

$$H^0\big(X,\, E_{k,m}T_X^* \otimes A^{-1}\big) \neq 0,$$

in degrees $d \geqslant 16$, 74, 298, 1222 resp., where $A \to X$ is any auxiliary ample line bundle. But it is also shown that $H^0\big(X,\, E_{k,m}T_X^*\big) = 0$, for all jet orders $k \leqslant \dim X - 1$, generalizing a theorem of Rousseau ([15]) in dimension 3. Furthermore, for jet order k equal to the dimension n, with n arbitrary, Diverio shows in [5] that there exists an integer $\delta_n \gg n$ (up to now not effective) insuring existence of global sections of $E_{n,m}T_X^* \otimes A^{-1}$ in degree $d \geqslant \delta_n$.

The second technique, initiated by Clemens [1], Ein [8], Voisin [20] and pushed further by Siu [18], Paŭn [13], Rousseau [16], consists in constructing meromorphic frames on the space of the so-called *vertical* k*-jets* $J^k_{\mathsf{vert}}(\mathcal{X})$ in the universal hypersurface $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\, d!}-1}$ parametrizing all $X \subset \mathbb{P}^{n+1}$ of degree d, so as to produce, by frame differentiations, enough *independent* algebraic differential equations from just one global section of $E_{k,m}T_X^* \otimes A^{-1}$.

In [18], p. 557, Siu announced that, for $k = n$, there exist two constants $c_n \geqslant 1$ and $c_n' \geqslant 1$ such that the twisted tangent bundle:

$$T_{J^n_{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\, d!}-1}}\big(c_n'\big)$$

is generated at every point by its global sections (frame property). In the present article, we establish this property outside a certain exceptional algebraic subset $\Sigma \subset J^n_{\mathsf{vert}}(\mathcal{X})$ defined by the vanishing of certain Wronskians, with the effective pole order $c_n = \frac{n^2+5n}{2}$, recovering $c_2 = 7$ (Paŭn [13]), $c_3 = 12$ (Rousseau [16]), and with $c_n' = 1$.

Moreover, at the cost of raising c_n up to $c_n = n^2 + 2n$, the same generation property holds outside the smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of all first order jets. Applications to *weak* (with Σ) and to *strong* (with $\widetilde{\Sigma}$) algebraic degeneracy of entire holomorphic curves are given in [6], following Rousseau's Schur bundle decomposition strategy in dimension $n = 4$, and also in higher dimensions, thanks to Diverio's use ([4, 5]) of the algebraic version of Demailly's Morse inequalities due to Trapani ([19]).

Acknowledgments. During the author's stay at the Mittag-Leffler Institute (1–21 April 2008), Yum-Tong Siu and Mihai Paŭn have provided helpful oral explanations of [18, 13]. The stronger property of generation at every point of $J^n_{\mathsf{vert}}(\mathcal{X})\backslash\widetilde{\Sigma}$ was obtained thanks to fruitful exchanges joint with Simone Diverio and Erwan Rousseau during the *Workshop Complex Hyperbolic Geometry and Related Topics* at the Fields Institute, Toronto, Canada, 17-21 November 2008.

§2. UNIVERSAL HYPERSURFACE AND VERTICAL JETS

Representation in coordinates. Consider the *universal hypersurface* $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}$ parametrizing all complex n-dimensional algebraic hypersurfaces of fixed degree $d \geqslant 1$ in $\mathbb{P}^{n+1}$ which is defined, in two collections of homogeneous coordinates:

$$[Z] = [Z_0 : Z_1 : \cdots : Z_n : Z_{n+1}] \in \mathbb{P}^{n+1}$$
$$[A] = \big[(A_\alpha)_{\alpha\in\mathbb{N}^{n+2},\,|\alpha|=d}\big] \in \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1},$$

as the zero-set locus:

$$\mathcal{X} \;: \qquad 0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+2}\\ |\alpha|=d}} A_\alpha\, Z^\alpha$$

of the general homogeneous degree d polynomial. Here of course, a multiindex $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ has *length* defined by $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_{n+1}$ and we abbreviate $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} \cdots Z_{n+1}^{\alpha_{n+1}}$.

Our goal is to perform, for jets of order κ equal to the dimension n of hypersurfaces $\mathcal{X}(A) \subset \mathbb{P}^{n+1}$, a construction of meromorphic vector fields on the space of jets of holomorphic discs (or entire maps) valued in $\mathcal{X}$ which was initiated by Clemens [1], Ein [8], Voisin [20] for $\kappa = 1$, $n \geqslant 1$, then announced for higher κ's by Siu [18] and recently detailed by Paŭn [13] for $n = \kappa = 2$ and by Rousseau [16] for $n = \kappa = 3$. For general $\kappa = n$, a concise book-keeping of indices appears to be available here.

As in [13, 16], we shall mainly work in *in*homogeneous coordinates on the chart $\{Z_0 \neq 0\} \times \{A_{0d0\cdots0} \neq 0\}$, a copy of $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}}$. Dividing by $(Z_0)^d$ and by $A_{0d0\cdots0}$, and setting $z_i := Z_i/Z_0$, the equation of $\mathcal{X}$ then transfers to:

$$\mathcal{X}_0 \;: \qquad 0 = z_1^d + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\,\alpha_1<d}} a_\alpha\, z^\alpha,$$

with new coefficients $a_{\alpha_1\cdots\alpha_{n+1}} := \frac{A_{\alpha_0\alpha_1\cdots\alpha_{n+1}}}{A_{0d0\cdots0}}$ in which $\alpha_0 := d - \alpha_1 - \cdots - \alpha_{n+1}$. By convention, we shall set $a_{d0\cdots0} = 1$.

In view of applications to the Green-Griffiths algebraic degeneracy conjecture ($d \geqslant n+3$) or to the Kobayashi hyperbolicity conjecture ($d \geqslant 2n+1$), it will, without loss of generality, be assumed that $d > n$ throughout.

Defining equations for the space of vertical jets. To settle Kobayashi hyperbolicity or Green-Griffiths algebraic degeneracy, the strategy initiated by Bloch and pursued by Green-Griffiths [9], Siu [18], Demailly [3] consists in producing enough (global, algebraic) differential equations that every entire map $\mathbb{C} \ni \zeta \longmapsto \big(z_1(\zeta), \dots, z_{n+1}(\zeta)\big)$ valued in an algebraic variety $\mathcal{X}(A)$ for (very) generic fixed coefficients A_α should satisfy. Accordingly, if one introduces independent coordinates corresponding to derivatives with respect to ζ:

$$\Big(z_i,\, a_\alpha,\, z'_{j_1},\, z''_{j_2}, \dots, z^{(\kappa)}_{j_\kappa}\Big) \in \mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}} \times \underbrace{\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^{n+1}}_{\kappa \text{ times}},$$

the manifold of κ-jets of such entire maps has equations obtained by just formally differentiating the monomials z^α with respect to the variable $\zeta \in \mathbb{C}$, the a_α being constant.

The basic chain rule yields the first five equations, up to $\kappa = 4$:

$$0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+1} \\ |\alpha|\leqslant d,\, a_{d0\cdots 0}=1}} a_\alpha\, z^\alpha$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z_{j_1}' \bigg)$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z_{j_1}'' + \sum_{j_1,\, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}\, z_{j_1}' z_{j_2}' \bigg)$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z_{j_1}''' + \sum_{j_1,\, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}\, 3\, z_{j_1}' z_{j_2}'' + \sum_{j_1,\, j_2,\, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}}\, z_{j_1}' z_{j_2}' z_{j_3}' \bigg)$$

$$\begin{aligned} 0 = \sum_\alpha a_\alpha \bigg(& \sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z_{j_1}'''' + \sum_{j_1,\, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} \big(4\, z_{j_1}' z_{j_2}''' + 3\, z_{j_1}'' z_{j_2}'' \big) + \\ & + \sum_{j_1,\, j_2,\, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}}\, 6\, z_{j_1}' z_{j_2}' z_{j_3}'' + \sum_{j_1,\, j_2,\, j_3,\, j_4} \frac{\partial^4(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}\partial z_{j_4}}\, z_{j_1}' z_{j_2}' z_{j_3}' z_{j_4}' \bigg), \end{aligned}$$

on understanding that $a_{d0\cdots 0} = 1$ and that all summations $\sum_{j_1}$, $\sum_{j_1,\, j_2}$ *etc.* are performed for the indices j_i running from 1 to $n+1$. Equivalently, this submanifold of $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\, d!}} \times \mathbb{C}^{\kappa(n+1)}$ may be be defined as the submanifold of the full κ-jet manifold $J^\kappa(\mathbb{C},\, \mathcal{X}_0)$ consisting of only the jets tangent to the fibers of the projection $\mathcal{X}_0 \to \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\, d!}-1}$ onto the second factor. They are called *vertical jets* in [18, 13, 16] and will be denoted by $J^\kappa_{\mathsf{vert}}(\mathcal{X}_0)$.

Formally differentiating any polynomial in the jet variables amounts to applying the *total differentiation operator*:

$$\mathsf{D}(\bullet) := \sum_{\lambda\in\mathbb{N}} \sum_{k=1}^{n+1} \frac{\partial(\bullet)}{\partial z_k^{(\lambda)}} \cdot z_k^{(\lambda+1)},$$

and above, it is clear that each next equation is obtained from the previous one by applying D to it so that, for jets of arbitrary order κ up to κ equal to the dimension n, the $(n+1)$ defining equations of $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ happen to be:

$$0 = \sum_\alpha a_\alpha z^\alpha = \mathsf{D}\Big(\sum_\alpha a_\alpha z^\alpha \Big) = \cdots = \mathsf{D}^n \Big(\sum_\alpha a_\alpha z^\alpha \Big).$$

Then a suitable multivariate version of the classical Faà di Bruno formula provides a *closed, explicit formula* for all such equations.

Lemma 1. ([2, 10]) *The $(n+1)$ defining equations of $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ write as follows, where $\kappa = 0, 1, 2, \ldots, n$:*

$$\begin{aligned} 0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+1} \\ |\alpha|\leqslant d,\, a_{d0\cdots 0}=1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant 1,\ldots,\mu_e\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!} \\ \sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n+1} \cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}}\, z_{j_1^1}^{(\lambda_1)}\cdots z_{j_{\mu_1}^1}^{(\lambda_1)}\cdots z_{j_1^e}^{(\lambda_e)}\cdots z_{j_{\mu_e}^e}^{(\lambda_e)}. \end{aligned}$$

To read this general formula with the help of the formulas specialized above, we comment it backwards from its end.

The general monomial $\prod z_\bullet^{(\lambda_1)} \prod z_\bullet^{(\lambda_2)} \cdots \prod z_\bullet^{(\lambda_e)}$ in the jet variables gathers derivatives of increasing orders $\lambda_1 < \lambda_2 < \cdots < \lambda_e$, with $\mu_1, \mu_2, \dots, \mu_e$ counting their respective numbers. Then each monomial z^α is subjected to a partial derivative of order $\mu_1 + \mu_2 + \cdots + \mu_e$, the total number of $z_j^{(\lambda_i)}$ in the monomial in question. Since there are $n+1$ variables z_i, the dots in the $z_\bullet^{(\lambda_i)}$ should receive indices, and in fact, there appear general sums $\sum_{j_1^i,\dots,j_{\mu_i}^i=1}^{n+1}$ over *all possible* such indices. Notice that these observations are confirmed by the formulas developed above up to $\kappa = 4$.

In the sequel, we will in fact not need all the information of such a precise, explicit formula, but it will suffice to know that, among the $(n+1)$ defining equations, the equation numbered κ is a certain finite sum with certain integer coefficients of terms of the form:

$$\boxed{\sum_{\substack{\beta\in\mathbb{N}^{n+1} \\ |\beta|\leqslant d,\, a_{d0\cdots 0}=1}} a_\beta \bigg(\sum_{j_1,\dots,j_e=1}^{n+1} \frac{\partial^e (z^\beta)}{\partial z_{j_1}\cdots \partial z_{j_e}} \cdot z_{j_1}^{(\nu_1)} \cdots z_{j_e}^{(\nu_e)} \bigg)},$$

where the derivative orders $\nu_i \geqslant 1$ of the jet monomial $z_{j_1}^{(\nu_1)} \cdots z_{j_e}^{(\nu_e)}$ are nondecreasing and where $\nu_1 + \cdots + \nu_e = \kappa$. The reader unacquainted with the Faà di Bruno combinatorics could readily prove this less informative representation by reasoning inductively on κ.

Frames and generation by global sections. Now, a globally defined vector field on the ambient space $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\, d!}} \times \mathbb{C}^{n(n+1)}$ writes under the general form:

$$\mathsf{T} = \sum_{i=1}^{n+1} \mathsf{Z}_i \frac{\partial}{\partial z_i} + \sum_{\substack{\alpha\in\mathbb{N}^{n+1} \\ |\alpha|\leqslant d,\, \alpha_1<d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha} + \sum_{k=1}^{n+1} \mathsf{Z}_k' \frac{\partial}{\partial z_k'} + \sum_{k=1}^{n+1} \mathsf{Z}_k'' \frac{\partial}{\partial z_k''} + \cdots + \sum_{k=1}^{n+1} \mathsf{Z}_k^{(n)} \frac{\partial}{\partial z_k^{(n)}}.$$

We shall seek vector fields of this form which should extend meromorphically to the full space of vertical jets and which should make a spanning frame of vectors tangent to $J^n_{\mathsf{vert}}(\mathcal{X})$ at almost every point, say outside a certain exceptional set. After twisting by $(\bullet)\otimes\mathcal{O}_{\mathbb{P}^{n+1}}(c_n)\otimes\mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\, d!}-1}}(c_n')$ for some two suitable constants $c_n \geqslant 1$ and $c_n' \geqslant 1$, one may in fact erase the appearing poles of the meromorphic coefficients, so that one may speak of global *holomorphic* sections instead of meromorphic sections.

Theorem. *Let $\widetilde{\Sigma}$ be the closure, in $J^n_{\mathsf{vert}}(\mathcal{X})$, of the Zariski closed subset of the space $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ of vertical* affine *jets defined by requiring that all* first order *jet vanish:*

$$\widetilde{\Sigma}_0 := \Big\{ \big(z_i, a_\alpha, z_{j_1}', \dots, z_{j_n}^{(n)}\big) : \; z_1' = z_2' = \cdots = z_{n+1}' = 0 \Big\},$$

so that in any other standard affine chart $(t_0, \dots, t_{v-1}, t_{v+1}, \dots, t_{n+1}) \in \mathbb{C}^{n+1}$ on $\mathbb{P}^{n+1}(\mathbb{C})$, the representation of $\widetilde{\Sigma}$ is yielded by exactly the same equations $0 = t_0' = \cdots = t_{v-1}' = t_{v+1}' = \cdots = t_{n+1}'$. Then the following two properties hold true.

- *$J^n_{\mathsf{vert}}(\mathcal{X})\backslash\Sigma$ is smooth of pure codimension equal to $n+1$ at every point, namely, it is of pure dimension equal to:*

$$\begin{aligned} j_n^d &:= n + 1 + \tfrac{(n+1+d)!}{(n+1)!\, d!} + n(n+1) - (n+1) \\ &= \tfrac{(n+1+d)!}{(n+1)!\, d!} + n(n+1). \end{aligned}$$

- *The twisted tangent bundle:*

$$T_{J^n_{\rm vert}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}\big(n^2+2n\big) \otimes \mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(1)$$

is generated by its global sections on $J^n_{\sf vert}(\mathcal{X})\backslash\widetilde{\Sigma}$*, that is to say: at every point* $p^{[n]} \in J^n_{\sf vert}(\mathcal{X})\backslash\widetilde{\Sigma}$ *not lying in* $\widetilde{\Sigma}$*, one may find* j^d_n global *sections* $\mathsf{T}_1, \dots, \mathsf{T}_{j^d_n}$ *over* X *of this twisted tangent bundle such that:*

$$\mathbb{C}\mathsf{T}_1(p^{[n]}) \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j^d_n}(p^{[n]}) = T_{J^n_{\sf vert}(\mathcal{X}),\,p^{[n]}}.$$

Comments about applications. The simplicity of the defining equations of the avoided exceptional set $\widetilde{\Sigma} = \{z_i' = 0\}$ has considerable advantages in the study of Green-Griffiths algebraic degeneracy of entire holomorphic curves $f : \mathbb{C} \to X$.

Indeed, by employing jet differentials, one shows in a first moment that the n-jet $j^n f$ of any such an f must satisfy[1] at least one nontrivial global algebraic differential equation $P(j^n f) = 0$. Then in a second moment, following Siu's strategy (*see* [18, 13, 16, 6]) which consists in applying some well chosen multi-derivations $(\mathsf{T}_1)^{\nu_1} \cdots (\mathsf{T}_{j^d_n})^{\nu_{j^d_n}}$ to $P(j^n f) = 0$ so as to get sufficiently many *supplementary* differential equations, one comes down to distinguishing two cases:

- ☐ either $j^n f(\mathbb{C}) \not\subset \widetilde{\Sigma}$; in this first case, one is then able to show ([16, 6]) that $f(\mathbb{C})$ is contained in a certain proper algebraic subvariety $Y \subsetneqq X$ which is independent of f, and this yields *strong* algebraic degeneracy;
- ☐ or else $j^n f(\mathbb{C}) \subset \widetilde{\Sigma}$ fully; in this second case, one cannot apply any derivation $\mathsf{T}_1, \dots, \mathsf{T}_{j^d_n}$, but then the condition $j^n f(\mathbb{C}) \subset \Sigma$ simply reads $0 \equiv f_1'(\zeta) \equiv f_2'(\zeta) \equiv \cdots \equiv f_n'(\zeta)$, hence f is *constant* and strong degeneracy again holds *gratuitously*.

Quite differently, in [13, 16] and in a preliminary version of the present article as well, the exceptional set Σ that one had to avoid was substantially larger than $\widetilde{\Sigma}$. Then as a consequence in these references, the condition $j^n f \subset \Sigma$ in the second case above only meant that $j^n f$ was contained in the intersection of X with some one-codimensional linear subspace H of $\mathbb{P}^{n+1}(\mathbb{C})$ which in general depended upon f, so that only *weak* algebraic degeneracy of $f(\mathbb{C})$ could be deduced[2]. Here is the weaker statement which we generalize in arbitrary dimension $n \geqslant 2$.

Theorem'. *Let* Σ *be the closure, in* $J^n_{\sf vert}(\mathcal{X})$*, of the Zariski closed subset of the space* $J^n_{\sf vert}(\mathcal{X}_0)$ *of vertical* affine *jets defined by requiring that all* $n \times n$ *Wronskians vanish:*

$$\Sigma_0 := \Big\{ \big(z_i, a_\alpha, z'_{j_1}, \dots, z^{(n)}_{j_n}\big) :\ 0 = \det\, \big(z_i^{(\lambda_j)}\big)^{1\leqslant j\leqslant n}_{1\leqslant i\leqslant n+1} \quad \text{for all } \lambda_1, \dots, \lambda_n \text{ with } 1 \leqslant \lambda_j \leqslant n \Big\}.$$

Then the twisted tangent bundle:

$$T_{J^n_{\rm vert}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}\big(\tfrac{n^2+5n}{2}\big) \otimes \mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(1)$$

is generated by its global sections at every point of $J^n_{\sf vert}(\mathcal{X})\backslash\Sigma$*.*

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[1] *see* [9, 3, 18, 14, 15, 4]; we only summarize very briefly the ideas here.

[2] A more careful inspection shows that in fact, H is two-codimensional (Simone Diverio).

Notice that the twisting order $\frac{n^2+5n}{2}$ along the z-direction is smaller than the one n^2+2n of the preceding theorem: a certain price has to be "paid" in order to shrink the exceptional set, and to thereby gain strong degeneracy.

As said, one may verify that the vanishing of all $n \times n$ minors of the $n \times (n+1)$ Wronskian-like matrix $\big(f_j^{(\lambda)}(\zeta)\big)_{1 \leqslant j \leqslant n+1}^{1 \leqslant \lambda \leqslant n}$ implies that the components $f_1(\zeta), \dots, f_{n+1}(\zeta)$ satisfy at least two linearly independent linear relations:

$$0 \equiv \sum_{i=1}^{n+1} a_i\, f_i(\zeta) \equiv \sum_{i=1}^{n+1} b_i\, f_i(\zeta),$$

for $\zeta \in \mathbb{C}$, with no universal control on the coefficients a_i, b_i.

Before proceeding to establishing the two theorems, let us check that the set Σ is represented by the same kind of equations $0 = t_1' = \cdots = t_{v-1}' = t_{v+1}' = \cdots = t_{n+1}'$ in any other standard chart $\{Z_v \neq 0\}$ on $\mathbb{P}^{n+1}(\mathbb{C})$ in which the affine coordinates are defined just by:

$$t_0 = \frac{Z_0}{Z_v}, \dots, t_{v-1} = \frac{Z_{v-1}}{Z_v},\ t_{v+1} = \frac{Z_{v+1}}{Z_v}, \dots, t_{n+1} = \frac{Z_{n+1}}{Z_v}.$$

Indeed, coming back to the definition $z_i = \frac{Z_i}{Z_0}$ of the z_i, $i = 1, \dots, n+1$, the change of chart $\{Z_0 \neq 0\} \to \{Z_v \neq 0\}$ is given by the well known basic formulas:

$$t_0 = \frac{1}{z_v}, \dots, t_{v-1} = \frac{z_{v-1}}{z_v},\ t_{v+1} = \frac{z_{v+1}}{z_v}, \dots, t_{n+1} = \frac{z_{n+1}}{z_v},$$

whence by differentiating the right-hand sides as if they virtually depended upon a variable $\zeta \in \mathbb{C}$, we get the transformation rules for the first order jets:

$$t_0' = -\frac{z_v'}{z_v^2}, \dots, t_{v-1}' = \frac{z_{v-1}'}{z_v} - \frac{z_{v-1} z_v'}{z_v^2},\ t_{v+1}' = \frac{z_{v+1}'}{z_v} - \frac{z_{v+1} z_v'}{z_v^2}, \dots, t_{n+1}' = \frac{z_{n+1}'}{z_v} - \frac{z_{n+1} z_v'}{z_v^2}.$$

Then visibly, the two representations $\{0 = z_1' = \cdots = z_{n+1}'\}$ and $\{0 = t_0' = \cdots = t_{v-1}' = t_{v+1}' = \cdots = t_{n+1}'\}$ of the set Σ coincide coherently on the intersection $\{Z_0 \neq 0\} \cap \{Z_v \neq 0\}$ of the two affine charts. One may verify that $\widetilde{\Sigma}$ also enjoys a similar invariance property.

Organization. The remainder of the paper is entirely devoted to the proof of the first theorem. When necessary, we shall briefly indicate which mild modifications suffice to gain the second theorem at the same time.

§3. FIRST PACKAGE OF COEFFICIENT VECTOR FIELDS

First family of global sections. We begin by seeking tangent vector fields globally defined over $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\, d!}} \times \mathbb{C}^{n(n+1)}$ of the specific, short form:

$$\mathsf{T} = \sum_{|\alpha| \leqslant n} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha}$$

in the space of only the coefficient variables a_α, up to length n. Afterwards, we shall deal with $\sum_{\substack{n \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha}$, and in Section 4, the remaining directions $\partial / \partial z_i$ and $\partial / \partial z_j^{(\lambda)}$ will complete the sought generating tangent vector fields.

Any arbitrary point $p^{[n]} \in J_{\mathsf{vert}}^n(\mathcal{X}_0)$ not in $\widetilde{\Sigma}$ lies in at least one of the open sets $\{z_i' \neq 0\}$. Fixing such an index i with $1 \leqslant i \leqslant n+1$, we shall construct a collection of vector fields of the above form that are defined in $\{z_i' \neq 0\}$ and that extend meromorphically to $J_{\mathsf{vert}}^n(\mathcal{X})$. To this aim, let us rewrite the defining equations of $J_{\mathsf{vert}}^n(\mathcal{X}_0)$ under the following convenient form, in which we denote by $\epsilon_i = (0, \dots, 1, \dots, 0)$ the i-th basic multiindex having 1 at the i-th place and 0 elsewhere, whence $n\epsilon_i = (0, \dots, n, \dots, 0)$:

$$\begin{cases} 0 = a_0 + a_{\epsilon_i} z_i + \cdots\cdots\cdots + a_{n\epsilon_i} z_i^n + \sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, z^\beta \\ 0 = \qquad a_{\epsilon_i} \mathsf{D}(z_i) + \cdots\cdots + a_{n\epsilon_i} \mathsf{D}(z_i^n) + \sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, \mathsf{D}(z^\beta) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad a_{\epsilon_i} \mathsf{D}^n(z_i) + \cdots + a_{n\epsilon_i} \mathsf{D}^n(z_i^n) + \sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, \mathsf{D}^n(z^\beta). \end{cases}$$

Here in the last n lines, we emphasize a generalized $n \times n$ Wronskian-like matrix, about which the next lemma states that its determinant is nonzero *if and only if* $z_i' \neq 0$, an assumption we made. For short in the sequel, we shall write $(z_i^k)^{(\kappa)}$ instead of $D^\kappa(z_i^k)$, since it is now clear and unambiguous that primes denote abstract jet variables.

Lemma. *For every* $i = 1, 2, \dots, n+1$*, one has:*

$$\begin{vmatrix} z_i' & (z_i^2)' & \cdots & (z_i^n)' \\ z_i'' & (z_i^2)'' & \cdots & (z_i^n)'' \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ z_i^{(n)} & (z_i^2)^{(n)} & \cdots & (z_i^n)^{(n)} \end{vmatrix} = 1!2! \cdots n! \cdot z_i' \, (z_i')^2 \cdots (z_i')^n$$

$$= 1!2! \cdots n! \cdot (z_i')^{\frac{n(n+1)}{2}}.$$

Our appendix is devoted to the proof of this elementary, but not straightforward, determinantal identity. As a result, we immediately deduce that $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ is smooth of pure codimension $(n+1)$ at each one of its points which lies in $\{z_i' \neq 0\}$, for the last n defining equations written above can at first be solved with respect to $a_{\epsilon_i}, \dots, a_{n\epsilon_i}$ thanks to Cramer's rule, while the first defining equation is trivially solvable with respect to a_0. In other words, in our open set $\{z_i' \neq 0\}$, the vertical affine jet manifold may be represented as a plain semi-global *graph*:

$$a_0, a_{\epsilon_i}, \dots, a_{n\epsilon_i} = \text{certain functions of } \big(z, z', \dots, z^{(n)}, \widetilde{a}_i\big),$$

where $\widetilde{a}_i := \big(a_\beta\big)^{1 \leqslant |\beta| \leqslant d}_{\beta \neq 0, \epsilon_i, \dots, n\epsilon_i}$ gathers all the other coefficients of the universal hypersurface.

Now, we seek vector fields of the form $\mathsf{T} = \sum_{|\alpha| \leqslant n} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha}$ which would be tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ with the length of the appearing multiindices being bounded by n. For this reason, and because the equations of $J^n_{\mathsf{vert}}(\mathcal{X}_0)$ are *linear* with respect to the coefficients a_β, when one applies such a derivation T to the equations in question, every monomial z^β with $n+1 \leqslant |\beta| \leqslant d$ disappears automatically, hence we come down to solving the following linear system:

$$\begin{cases} 0 = \mathsf{A}_0 + \mathsf{A}_{\epsilon_i} z_i + \cdots\cdots + \mathsf{A}_{n\epsilon_i} z_i^n + \sum\limits_{\substack{\alpha\neq\epsilon_i,\dots,n\epsilon_i \\ 1\leqslant|\alpha|\leqslant n}} \mathsf{A}_\alpha\, z^\alpha \\ 0 = \qquad \mathsf{A}_{\epsilon_i} z_i' + \cdots + \mathsf{A}_{n\epsilon_i} (z_i^n)' + \sum\limits_{\substack{\alpha\neq\epsilon_i,\dots,n\epsilon_i \\ 1\leqslant|\alpha|\leqslant n}} \mathsf{A}_\alpha\, (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad \mathsf{A}_{\epsilon_i} z_i^{(n)} + \cdots + \mathsf{A}_{n\epsilon_i} (z_i^n)^{(n)} + \sum\limits_{\substack{\alpha\neq\epsilon_i,\dots,n\epsilon_i \\ 1\leqslant|\alpha|\leqslant n}} \mathsf{A}_\alpha (z^\alpha)^{(n)}, \end{cases}$$

having the A_α as unknowns, where notably, $|\alpha| \leqslant n$ everywhere.

Noticing that the number of directions $\frac{\partial}{\partial a_\alpha}$ equals $\frac{(n+1+n)!}{(n+1)!\,n!}$ while the number of equations above equals $(n+1)$, we may now claim that for every $\alpha \neq 0, \epsilon_i, \dots, n\epsilon_i$ with $|\alpha| \leqslant n$, there are $\frac{(n+1+n)!}{(n+1)!\,n!} - (n+1)$ linearly independent vector fields of the specific form:

$$\frac{\partial}{\partial a_\alpha} - \mathsf{B}_{\alpha,0}^i \frac{\partial}{\partial a_0} - \mathsf{B}_{\alpha,1}^i \frac{\partial}{\partial a_{\epsilon_i}} - \cdots - \mathsf{B}_{\alpha,n}^i \frac{\partial}{\partial a_{n\epsilon_i}}$$

that are tangent to the (semi-global) graph $J^n_{\mathsf{vert}}(\mathcal{X}_0) \cap \{z_i' \neq 0\}$, that is to say, the coefficients of which satisfy the written linear system. In order to insure meromorphic prolongation to projective spaces (*see* below), it is convenient to multiply in advance the basic vector field $\frac{\partial}{\partial a_\alpha}$ of such a kind of sought vector field by the Wronskian-like determinant $\Delta(z_i') := 1!2!\cdots n! \cdot (z_i')^{\frac{n(n+1)}{2}}$ that the lemma computed. In sum, for any α with $1 \leqslant |\alpha| \leqslant n$ which is different from $0, \epsilon_i, \dots, n\epsilon_i$, the vector field:

$$\mathsf{T}_\alpha := \Delta(z_i') \frac{\partial}{\partial a_\alpha} - \mathsf{B}_{\alpha,0}^i \frac{\partial}{\partial a_0} - \mathsf{B}_{\alpha,1}^i \frac{\partial}{\partial a_{\epsilon_i}} - \cdots - \mathsf{B}_{\alpha,n}^i \frac{\partial}{\partial a_{n\epsilon_i}}$$

is tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0) \cap \{z_i' \neq 0\}$ if and only if its unknown coefficients $\mathsf{B}_{\alpha,k}^i$ satisfy the following linear system:

$$\begin{cases} 0 = -\mathsf{B}_{\alpha,0}^i - \mathsf{B}_{\alpha,1}^i\, z_i - \cdots\cdots - \mathsf{B}_{\alpha,n}^i\, z_i^n + \Delta(z_i') \cdot z^\alpha \\ 0 = \qquad -\mathsf{B}_{\alpha,1}^i\, z_i' - \cdots\cdots - \mathsf{B}_{\alpha,n}^i\, (z_i^n)' + \Delta(z_i') \cdot (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad -\mathsf{B}_{\alpha,1}^i\, z_i^{(n)} - \cdots - \mathsf{B}_{\alpha,n}^i\, (z_i^n)^{(n)} + \Delta(z_i') \cdot (z^\alpha)^{(n)}. \end{cases}$$

A basic application of Cramer's rule now enable us to solve the last n equations, and afterwards, we may then substitute the obtained solutions in the first equation:

$$\left[\begin{aligned} \mathsf{B}_{\alpha,k}^i &:= \begin{vmatrix} z_i' & \cdots & (z^\alpha)' & \cdots & (z_i^n)' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ z_i^{(n)} & \cdots & (z^\alpha)^{(n)} & \cdots & (z_i^n)^{(n)} \end{vmatrix} \qquad (\kappa\text{-th column},\ 1 \leqslant k \leqslant n) \\ \mathsf{B}_{\alpha,0}^i &:= -\mathsf{B}_{\alpha,1}^i\, z_i - \cdots - \mathsf{B}_{\alpha,n}^i\, z_i^n + \Delta(z_i') \cdot z^\alpha. \end{aligned}\right.$$

Clearly, the so obtained vector fields T_α with $\alpha \neq 0, \epsilon_i, \dots, n\epsilon_i$ are linearly independent at every point of $J^n_{\mathsf{vert}}(\mathcal{X}_0) \cap \{z_i' = 0\}$.

Meromorphic prolongation and computation of pole orders. Recall that any polynomial $P(t_0, \dots, t_{v-1}, t_{v+1}, \dots, t_{n+1})$ of degree $e \geqslant 1$ on an affine $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, when viewed as a meromorphic map $\mathbb{P}^{n+1} \to \mathbb{P}^1$, has pole order equal to e, for a change of standard affine chart:

$$t_0 = \frac{1}{z_v}, \dots, t_{v-1} = \frac{z_{v-1}}{z_v},\; t_{v+1} = \frac{z_{v+1}}{z_v}, \dots, t_{n+1} = \frac{z_{n+1}}{z_v},$$

transfers P to $P\big(\frac{1}{z_v}, \dots, \frac{z_{v-1}}{z_v}, \frac{z_{v+1}}{z_v}, \dots, \frac{z_{n+1}}{z_v}\big)$. Through such an inversion map, the first-order jets, second-order jets, *etc*, are transferred to:

$$\frac{z_i'}{z_v} - \frac{z_i z_v'}{z_v^2}, \qquad \frac{z_i''}{z_v} - 2\,\frac{z_i' z_v'}{z_v^2} - \frac{z_i z_v''}{z_v^2} + 2\,\frac{z_i z_v' z_v'}{z_v^3}, \qquad \textit{etc.},$$

hence by just looking at the maximal power of z_v at the denominator, one easily observes by induction that:

$$\mathsf{Pole\text{-}order}\Big[z^\alpha\, (z')^{\alpha^1} \cdots (z^{(n)})^{\alpha^n}\Big] = |\alpha| + |\alpha^1| + \cdots + |\alpha^n| + n,$$

and furthermore, one differentiation of such a monomial increases its pole order by just one unit. Now, we claim that:

$$\left[\begin{aligned} \mathsf{Pole\text{-}order}\big[\Delta(z_i')\big] &= n^2 + n, \\ \mathsf{Pole\text{-}order}\big[\mathsf{B}^i_{\alpha,k}\big] &= |\alpha| + n^2 + n - k \\ \mathsf{Pole\text{-}order}\big[\mathsf{B}^i_{\alpha,0}\big] &= |\alpha| + n^2 + n, \end{aligned}\right.$$

so that the highest pole order occurs to be the coefficient $\mathsf{B}^i_{\alpha,0}$ of $\frac{\partial}{\partial a_0}$ in each T_α.

Indeed, replacing the entries of the determinant $\Delta(z_i')$ plainly by the nonnegative integers which indicate the pole orders, we may write symbolically:

$$\mathsf{Pole\text{-}order}\big[\Delta(z_i')\big] = \mathsf{Pole\text{-}order\ of} \begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ 4 & 5 & 6 & \cdots & n+3 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ n+1 & n+2 & n+3 & \cdots & 2n \end{vmatrix}.$$

When one expands the determinant as a sum of monomials with $\pm$ signs, pole orders are just added, symbolically speaking. Then one easily convinces oneself that *each one* of the obtained monomials has the *same* pole order, hence it suffices to compute the pole order of the monomial of the main diagonal, which is equal to: $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

Next, $\mathsf{B}^i_{\alpha,k}$ is obtained from $\Delta(z_i')$ by replacing the k-th column of $\Delta(z_i')$ by the new column of pole order entries $|\alpha|+1$, $|\alpha|+2$, ..., $|\alpha|+n$. The pole order $|\alpha|$ being "factorizable", we get:

$$\mathsf{Pole\text{-}order}\big[\mathsf{B}^i_{\alpha,k}\big] = |\alpha| + \mathsf{Pole\text{-}order\ of} \begin{vmatrix} 2 & 3 & \cdots & 1 & \cdots & n+1 \\ 3 & 4 & \cdots & 2 & \cdots & n+2 \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdots & \cdot\cdot \\ n+1 & n+2 & \cdots & n & \cdots & 2n \end{vmatrix},$$

where the central-looking column is the k-th, the only which differs from $\Delta(z_i')$. Again, one easily convinces oneself that the pole order of *each one* of the monomials obtained after expansion is the *same*, so that by looking again at the main diagonal:

$$\begin{aligned}\mathsf{Pole\text{-}order}\big[\mathsf{B}^i_{\alpha,k}\big] &= |\alpha| + 2 + \cdots + 2(k-1) + k + 2(k+1) + \cdots + 2n \\ &= |\alpha| + n(n+1) - k.\end{aligned}$$

Finally, coming back to the definition of $\mathsf{B}^i_{\alpha,0}$, one then immediately sees that each term in $\mathsf{B}^i_{\alpha,0}$ has the same pole order, equal to $|\alpha| + n^2 + n$.

In conclusion, the maximal pole order is reached by any $\mathsf{B}^i_{\alpha,0}$ with $|\alpha| = n$, and is equal to $n^2 + 2n$, as it appears in the statement of the main theorem. Clearly, the poles are compensated by the twisting $(\bullet) \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(n^2 + 2n)$. Notice that the coefficients of the constructed T_α's depend only on the jet variables $(z, z', \dots, z^{(n)})$, absolutely not on the coefficients a_β.

The other vector fields that we will construct in the remainder of the article, so as to complete a true framing, will all have pole order in the z-direction smaller than $n^2 + 2n$, and will all have pole order at most 1 in the a-direction.

Modifications needed for the second theorem. By assumption, at least one (classical) Wronskian $\det\big(z_i^{(\lambda_j)}\big)_{1\leqslant i\leqslant n+1}^{1\leqslant j\leqslant n}$ does not vanish, where, without loss of generality, we may assume that $1 \leqslant \lambda_1 < \cdots < \lambda_n \leqslant n+1$. To fix ideas, we shall work in the open set $\big\{ \det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n} \neq 0 \big\}$ and we shall denote: $\mathsf{W} := \det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n}$. Then again, the variety of vertical jets is checked to be, in the open set $\big\{ \det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n} \neq 0 \big\}$, a semi-global graph of equations:

$$\begin{cases} 0 = a_0 + a_{\epsilon_1} z_1 + \cdots\cdots + a_{\epsilon_n} z_n + \sum\limits_{\substack{\beta\neq\epsilon_1,\dots,\epsilon_n \\ 1\leqslant|\beta|\leqslant d}} a_\beta\, z^\beta \\ 0 = \qquad a_{\epsilon_1} z_1' + \cdots\cdots + a_{\epsilon_n} z_n' + \sum\limits_{\substack{\beta\neq\epsilon_1,\dots,\epsilon_n \\ 1\leqslant|\beta|\leqslant d}} a_\beta\, (z^\beta)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad a_{\epsilon_1} z_1^{(n)} + \cdots + a_{\epsilon_n} z_n^{(n)} + \sum\limits_{\substack{\beta\neq\epsilon_1,\dots,\epsilon_n \\ 1\leqslant|\beta|\leqslant d}} a_\beta\, (z^\beta)^{(n)}, \end{cases}$$

having transversal coordinates $(a_0, a_{\epsilon_1}, \dots, a_{\epsilon_n})$, the ones that are then clearly solvable here. Thus if, similarly as in the previous paragraphs, one seeks tangent vector fields of the specific form:

$$\mathsf{T}_\alpha := \mathsf{W}\,\frac{\partial}{\partial a_\alpha} - \mathsf{B}_{\alpha,0}\,\frac{\partial}{\partial a_0} - \mathsf{B}_{\alpha,1}\,\frac{\partial}{\partial a_{\epsilon_1}} - \cdots - \mathsf{B}_{\alpha,n}\,\frac{\partial}{\partial a_{\epsilon_n}},$$

for any $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \leqslant n$ and with $\alpha \neq 0, \epsilon_1, \dots, \epsilon_n$, then the system we now have to solve becomes:

$$\begin{cases} 0 = -\mathsf{B}_{\alpha,0} - \mathsf{B}_{\alpha,1}\, z_1 - \cdots\cdots - \mathsf{B}_{\alpha,n}\, z_n + \mathsf{W}\cdot z^\alpha \\ 0 = \qquad -\mathsf{B}_{\alpha,1}\, z_1' - \cdots\cdots - \mathsf{B}_{\alpha,n}\, z_n' + \mathsf{W}\cdot (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad -\mathsf{B}_{\alpha,1}\, z_1^{(n)} - \cdots - \mathsf{B}_{\alpha,n}\, z_n^{(n)} + \mathsf{W}\cdot (z^\alpha)^{(n)}. \end{cases}$$

The unique solution is then again yielded by Cramer's rule:

$$\left[\begin{aligned} \mathsf{B}_{\alpha,k} &:= \left|\begin{array}{ccccc} z_1' & \cdots & (z^\alpha)' & \cdots & z_n' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ z_1^{(n)} & \cdots & (z^\alpha)^{(n)} & \cdots & z_n^{(n)} \end{array}\right| \qquad (k\text{-th column},\ 1 \leqslant k \leqslant n) \\ \mathsf{B}_{\alpha,0} &:= -\mathsf{B}_{\alpha,1}\, z_1 - \cdots - \mathsf{B}_{\alpha,n}\, z_n + \mathsf{W}\cdot z^\alpha. \end{aligned}\right.$$

One may now verify that:

$$\begin{aligned} \text{Pole order}\big[\mathsf{W}\big] &= \tfrac{(n+1)(n+2)}{2} \\ \text{Pole order}\big[\mathsf{B}_{\alpha,k}\big] &= 2 + \cdots + (n+1) - (k+1) + |\alpha| + k \\ &= \tfrac{n^2+3n-2}{2} + |\alpha| \\ \text{Pole order}\big[\mathsf{B}_{\alpha,0}\big] &= \tfrac{n^2+3n}{2} + |\alpha|, \end{aligned}$$

so that the maximal pole order is reached by $\mathsf{B}_{\alpha,0}$ for any multiindex α with $|\alpha| = n$, and is equal to $\frac{n^2+5n}{2}$, as this appears in the second theorem.

The other vector fields that we will construct in the sequel will complete a generating set both for the first and for the second theorem and will have lower pole order in the z-direction.

Higher lengths. At present, we construct globally defined tangent vector fields which span the remaining directions $\bigoplus_{\substack{n+1 \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathbb{C} \cdot \frac{\partial}{\partial a_\alpha}$ in the space of coefficients a_α. For an arbitrary multiindex $\ell = (\ell_1, \ell_2, \dots, \ell_{n+1}) \in \mathbb{N}^{n+1}$ of length:

$$n + 1 = \ell_1 + \ell_2 + \cdots + \ell_{n+1},$$

we introduce the following family of vector fields living only in the space of a-variables:

$$\mathsf{T}_\alpha^{\ell_1,\ell_2,\dots,\ell_{n+1}} = \mathsf{T}_\alpha^\ell = \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\, \ell'' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \, \frac{\ell!}{\ell'!\ \ell''!} \, z^{\ell''} \, \frac{\partial}{\partial a_{\alpha-\ell''}},$$

where the indices α are all possible indices satisfying $\alpha_1 \geqslant \ell_1, \dots, \alpha_{n+1} \geqslant \ell_{n+1}$, $|\alpha| \leqslant d$ and $\alpha_1 < d$, and where the sum abbreviates $\sum_{\ell_1'+\ell_1''=\ell_1} \cdots \sum_{\ell_{n+1}'+\ell_{n+1}''=\ell_{n+1}}$. For instance, for $n+1 = 4$ and with the special choice $\ell_1 = \ell_2 = 2$ (whence necessarily $\ell_3 = \ell_4 = 0$), we get the following family of vector fields defined for all α with $\alpha_1 \geqslant 2$, $\alpha_2 \geqslant 2$, $\alpha_3 \geqslant 0$, $\alpha_4 \geqslant 0$ and $|\alpha| \leqslant d$, $\alpha_1 < d$ (compare [16], p. 373):

$$\begin{aligned} \mathsf{T}_\alpha^{2,2,0,0} = {} & \frac{\partial}{\partial a_\alpha} - 2z_1 \, \frac{\partial}{\partial a_{\alpha-\epsilon_1}} - 2z_2 \, \frac{\partial}{\partial a_{\alpha-\epsilon_2}} + \\ & + z_1^2 \, \frac{\partial}{\partial a_{\alpha-2\epsilon_1}} + 4z_1z_2 \, \frac{\partial}{\partial a_{\alpha-\epsilon_1-\epsilon_2}} + z_2^2 \, \frac{\partial}{\partial a_{\alpha-2\epsilon_2}} - \\ & - 2z_1^2 z_2 \, \frac{\partial}{\partial a_{\alpha-2\epsilon_1-\epsilon_2}} - 2z_1 z_2^2 \, \frac{\partial}{\partial a_{\alpha-\epsilon_1-2\epsilon_2}} + z_1^2 z_2^2 \, \frac{\partial}{\partial a_{\alpha-2\epsilon_1-2\epsilon_2}}. \end{aligned}$$

After a moment's reflection, one may convince oneself that as ℓ with $|\ell| = n+1$ runs and as α with $\alpha_i \geqslant \ell_i$ runs, the T_α^ℓ together with the vector fields of the previous paragraph do span $\bigoplus_{\substack{n+1 \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathbb{C} \cdot \frac{\partial}{\partial a_\alpha}$; there are in fact redundancies among the *triangular* system defined by the T_α^ℓ, whenever one has $\alpha \geqslant \ell_1$ and $\alpha \geqslant \ell_2$ for two distinct ℓ^1, ℓ^2 with $|\ell^1| = |\ell^2| = n+1$.

Lemma. *For every nonnegative integer $e \leqslant n$ and for arbitrary indices $j_1, \dots, j_e$ with $1 \leqslant j_i \leqslant n+1$, one has:*

$$0 \equiv \mathsf{T}_\alpha^\ell \bigg(\sum_{\substack{\beta \in \mathbb{N}^{n+1} \\ |\beta| \leqslant d,\, a_{d0\cdots 0}=1}} a_\beta \, \frac{\partial^e (z^\beta)}{\partial z_{j_1} \cdots \partial z_{j_e}} \bigg),$$

and as a result, T_α^ℓ identically annihilates all the defining equations of $J_{\mathsf{vert}}^n(\mathcal{X}_0)$, hence is tangent to $J_{\mathsf{vert}}^n(\mathcal{X}_0)$.

Proof. Let $w_1, w_2, \dots, w_{n+1}$ be auxiliary complex variables. For every derivation order $e \leqslant n$ strictly less than the vanishing order $\sum \ell_i = n+1$, we trivially have:

$$0 \equiv \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_e}} \Big([w_1 - z_1]^{\ell_1} \, [w_2 - z_2]^{\ell_2} \cdots [w_{n+1} - z_{n+1}]^{\ell_{n+1}} \Big)\Big|_{w=z}.$$

In other words, by expanding $[w-z]^\ell = \sum_{\ell' + \ell'' = \ell} (-1)^{|\ell''|} \frac{\ell!}{\ell'!\,\ell''!} \, w^{\ell'} \, z^{\ell''}$ thanks to the multinomial formula, by letting the derivation $\partial^e(\bullet)\big/\partial z_{j_1} \cdots \partial z_{j_e}$ act on this expansion, by setting $w = z$, and finally, by multiplying the result obtained by $z^{\alpha - \ell}$, we get the useful identities:

$$\boxed{0 \equiv \sum_{\substack{\ell' + \ell'' = \ell \\ \ell',\, \ell' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \, \frac{\ell!}{\ell'!\,\ell''!} \, z^{\alpha - \ell''} \cdot \frac{\partial^e \big(z^{\ell''} \big)}{\partial z_{j_1} \cdots \partial z_{j_e}}}.$$

On the other hand, by letting the derivation T_α^ℓ act as it should, the identities of the lemma that we have to check may be written:

$$\begin{aligned} 0 &\overset{?}{\equiv} \sum_{\substack{\ell' + \ell'' = \ell \\ \ell',\, \ell' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \, \frac{\ell!}{\ell'!\,\ell''!} \, \frac{\partial}{\partial a_{\alpha - \ell''}} \bigg(\sum_{\substack{\beta \in \mathbb{N}^{n+1} \\ |\beta| \leqslant d,\, a_{d0\cdots 0}=1}} a_\beta \, \frac{\partial^e (z^\beta)}{\partial z_{j_1} \cdots \partial z_{j_e}} \bigg) \cdot z^{\ell''} \\ &= \sum_{\substack{\ell' + \ell'' = \ell \\ \ell',\, \ell' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \, \frac{\ell!}{\ell'!\,\ell''!} \, \frac{\partial^e \big(z^{\alpha - \ell''} \big)}{\partial z_{j_1} \cdots \partial z_{j_e}} \cdot z^{\ell''}. \end{aligned}$$

Compared to the boxed, known identities, the derivation is now switched to the other monomial. Generally, we claim that for every $e = 0, 1, \dots, n$ and for every decomposition $e = e_1 + (e - e_1)$ with $0 \leqslant e_1 \leqslant e$, the expression:

$$\big(j_1, \dots, j_{e_1} \big| j_{e_1+1}, \dots, j_e \big) := \sum_{\substack{\ell' + \ell'' = \ell \\ \ell',\, \ell' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \, \frac{\ell!}{\ell'!\,\ell''!} \, \frac{\partial^{e_1} \big(z^{\alpha - \ell''} \big)}{\partial z_{j_1} \cdots \partial z_{j_{e_1}}} \cdot \frac{\partial^{e - e_1} \big(z^{\ell''} \big)}{\partial z_{j_{e_1+1}} \cdots \partial z_{j_e}}$$

vanishes identically, for all indices $j_1, \dots, j_e = 1, 2, \dots n+1$. We know that this assertion is true when $e_1 = 0$ for all $e = 0, 1, \dots, n$ and the lemma corresponds to $e - e_1 = 0$ for all $e_1 = 0, 1, \dots, n$.

For $e = 0$, the assertion is thus known. Suppose it to be true at level e. Reasoning by induction, we then assume that:

$$0 \equiv \big(j_1, \dots, j_{e_1} \big| j_{e_1+1}, \dots, j_e \big),$$

for all $e_1 = 0, 1, \dots, e$ and all possible j_i, If $e+1$ is still $\leqslant n$, we differentiate all these identities with respect to z_k using Leibniz' rule and we organize the resulting equations as a convenient array:

$$\begin{aligned}
0 &\equiv \big(j_1, \dots, j_e, k \big| \emptyset\big) + \big(j_1, \dots, j_e \big| k\big) \\
0 &\equiv \big(j_1, \dots, j_{e-1}, k \big| j_e\big) + \big(j_1, \dots, j_{e-1} \big| j_e, k\big) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &\equiv \big(j_1, k \big| j_2, \dots, j_e\big) + \big(j_1 \big| j_2, \dots, j_e, k\big) \\
0 &\equiv \big(k \big| j_1, \dots, j_e\big) + \underline{\big(\emptyset \big| j_1, \dots, j_e, k\big)}_{\circ}.
\end{aligned}$$

We have underlined the last term, known to vanish. Then the first term of the last line vanishes, for all indices $k, j_1, \dots, j_e = 1, 2, \dots, n+1$. So the second term of the penultimate vanishes, *etc.*, and hence the very first term $\big(j_1, \dots, j_e, k \big| \emptyset\big)$ does vanish identically, as desired. □

§4. SECOND PACKAGE OF JET, COORDINATE VECTOR FIELDS

Spanning the $\frac{\partial}{\partial z_i}$-directions. To complete the framing, let us at first span all the $\frac{\partial}{\partial z_i}$ directions. By convention, $a_{d0\cdots 0} = 1$.

Lemma. *For $i = 1, 2, \dots, n+1$, the vector fields:*

$$\mathsf{T}_i := \frac{\partial}{\partial z_i} - \sum_{|\alpha| \leqslant d-1} a_{\alpha+\epsilon_i} (\alpha_i + 1) \frac{\partial}{\partial a_\alpha}$$

are all tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0)$.

Proof. Appying the derivation T_i to the first equation $0 = \sum_\alpha a_\alpha z^\alpha$ of $J^n_{\mathsf{vert}}(\mathcal{X}_0)$, we indeed get an identically vanishing result:

$$\sum_{|\alpha| \leqslant d} a_\alpha \frac{\partial (z^\alpha)}{\partial z_i} - \sum_{|\alpha| \leqslant d-1} a_{\alpha+\epsilon_i} (\alpha_i + 1)\, z^\alpha \equiv 0.$$

Since the T_i commute with the total differentiation operator D, it then follows immediately that T_i annihilates all the other defining equations:

$$0 \equiv \mathsf{T}_i\Big(\mathsf{D} \sum_\alpha a_\alpha z^\alpha\Big) \equiv \cdots \equiv \mathsf{T}_i\Big(\mathsf{D}^n \sum_\alpha a_\alpha z^\alpha\Big),$$

and this yields the tangency property claimed. □

Spanning the $\partial/\partial z_j^{(\lambda)}$ directions. For the last family of vector fields, we transfer to general $\kappa = n \geqslant 2$ the approach of [13] known for $\kappa = n = 2$ and also for $\kappa = n = 3$ [16], with few differences.

Let $\Lambda = \big(\Lambda^l_k\big)^{1 \leqslant l \leqslant n+1}_{1 \leqslant k \leqslant n+1}$ be a matrix in $\mathsf{GL}(n+1, \mathbb{C})$. To span the only remaining directions $\partial/\partial z_j^{(\lambda)}$, one seeks meromorphic vector fields tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0) \backslash \Sigma_0$ that are of the special form:

$$\begin{aligned}\mathsf{T}_\Lambda := \sum_{k=1}^{n+1}\Big(\textstyle\sum_{l=1}^{n+1}\Lambda_k^l\,z_l'\Big)\frac{\partial}{\partial z_k'}+\cdots+\sum_{k=1}^{n+1}\Big(\textstyle\sum_{l=1}^{n+1}\Lambda_k^l\,z_l^{(n)}\Big)\frac{\partial}{\partial z_k^{(n)}}+\\ +\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\big(z,a,\Lambda\big)\frac{\partial}{\partial a_\alpha},\end{aligned}$$

where, for various jet orders λ's, the coefficients $\mathsf{Z}_k^{(\lambda)}$ of the $\frac{\partial}{\partial z_k^{(\lambda)}}$, $k=1,\dots,n+1$, are defined a priori to be obtained by multiplying the jet matrix $\big(z_j^{(\lambda)}\big)_{1\leqslant j\leqslant n+1}^{1\leqslant\lambda\leqslant n}$ by such a matrix Λ:

$$\begin{pmatrix}\Lambda_1^1 & \cdots & \Lambda_1^n & \Lambda_1^{n+1}\\ \Lambda_2^1 & \cdots & \Lambda_2^n & \Lambda_2^{n+1}\\ \vdots & \ddots & \vdots & \vdots\\ \Lambda_{n+1}^1 & \cdots & \Lambda_{n+1}^n & \Lambda_{n+1}^{n+1}\end{pmatrix}\begin{pmatrix}z_1' & \cdots & z_1^{(n)}\\ z_2' & \cdots & z_2^{(n)}\\ \vdots & \ddots & \vdots\\ z_{n+1}' & \cdots & z_{n+1}^{(n)}\end{pmatrix}=\begin{pmatrix}\mathsf{Z}_1' & \cdots & \mathsf{Z}_1^{(n)}\\ \mathsf{Z}_2' & \cdots & \mathsf{Z}_2^{(n)}\\ \vdots & \ddots & \vdots\\ \mathsf{Z}_{n+1}' & \cdots & \mathsf{Z}_{n+1}^{(n)}\end{pmatrix},$$

and where the coefficients $\mathsf{A}_\alpha\big(z,a,\Lambda\big)$, to be computed shortly, should insure that T_Λ is effectively tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0)\backslash\Sigma_0$.

In fact, by plainly inspecting ranks of the matrix multiplication above, one easily sees that, at every point of our basic open set where at least one $n\times n$ (sub)Wronskian of the jet matrix $\big(z_j^{(\lambda)}\big)$ does not vanish, one has for Λ varying without restriction in $\mathsf{GL}(n+1,\mathbb{C})$:

$$\mathrm{Span}_\Lambda\bigg(\Lambda\,z'\,\frac{\partial}{\partial z'}+\cdots+\Lambda\,z^{(n)}\,\frac{\partial}{\partial z^{(n)}}\bigg)=\bigoplus_{1\leqslant k\leqslant n+1}\mathbb{C}\,\frac{\partial}{\partial z_k'}\;\cdots\;\bigoplus_{1\leqslant k\leqslant n+1}\mathbb{C}\,\frac{\partial}{\partial z_k^{(n)}}.$$

The following proposition will therefore complete the proof of the theorem.

Proposition. *There exist coefficients A_α for $\partial/\partial a_\alpha$ with $|\alpha|\leqslant d$, $\alpha_1<d$, which are polynomials in z of degree at most n:*

$$\mathsf{A}_\alpha\big(z,a,\Lambda\big)=\sum_{|\beta|\leqslant n}\mathcal{L}_\alpha^\beta\big(a,\Lambda\big)\,z^\beta$$

with coefficients $\mathcal{L}_\alpha^\beta\big(a,\Lambda\big)$ being bilinear in the variables $\big(a_\gamma,\,\Lambda_k^l\big)$ such that T_Λ is tangent to $J^n_{\mathsf{vert}}(\mathcal{X}_0)\backslash\Sigma_0$.

Proof. While writing down, say, the first two tangency equations, namely when applying the derivative T_Λ to the first two of the five big equations written at the beginning, one gets equations:

$$(\mathbf{0})\qquad 0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot z^\alpha$$

$$(\mathbf{1}_{j_1})\qquad 0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot\frac{\partial(z^\alpha)}{\partial z_{j_1}}+\sum_{\substack{|\alpha|\leqslant d\\ a_{d0\cdots0}=1}}a_\alpha\sum_{l=1}^{n+1}\frac{\partial(z^\alpha)}{\partial z_l}\,\Lambda_l^{j_1},$$

for which one is allowed to equate to zero the coefficient of each z_{j_1}', because the sought A_α should be independent of $z_{j_1}',z_{j_2}'',\dots,z_{j_n}^{(n)}$.

Next, when applying T_Λ to the third defining equation of $J^n_{\mathsf{vert}}(\mathcal{X}_0)$, one sees thanks to $(\mathbf{1})_{j_1}$ that the coefficient of each z''_{j_1} then automatically vanishes[3], hence we are left with just equating to zero the coefficients of the monomials $z'_{j_1} z'_{j_2}$, namely:

$$(\mathbf{2}_{j_1j_2}) \quad 0 = \sum_{\substack{|\alpha|\leqslant d \\ \alpha_1<d}} \mathsf{A}_\alpha \cdot \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} + \sum_{\substack{|\alpha|\leqslant d \\ a_{d0\cdots 0}=1}} a_\alpha \sum_{l=1}^{n+1} \bigg(\frac{\partial^2(z^\alpha)}{\partial z_l \partial z_{j_2}} \, \Lambda_l^{j_1} + \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_l} \, \Lambda_l^{j_2} \bigg).$$

By induction, such a simplification is easily seen to generalize and thus, the $(e+1)$-th condition of tangency, after taking account of the successive cancellations, is obtained by just looking at how T_Λ acts on the jet monomial $z'_{j_1}\cdots z'_{j_e}$, and the result then consists in the family of equations:

$(\mathbf{e}_{j_1\cdots j_e})$

$$0 = \sum_{\substack{|\alpha|\leqslant d \\ \alpha_1<d}} \mathsf{A}_\alpha \cdot \frac{\partial^e(z^\alpha)}{\partial z_{j_1}\cdots\partial z_{j_e}} + \sum_{\substack{|\alpha|\leqslant d \\ a_{d0\cdots 0}=1}} a_\alpha \sum_{l=1}^{n+1} \bigg(\bigg(\frac{\partial^e(z^\alpha)}{\partial z_l\partial z_{j_2}\cdots\partial z_{j_e}} \, \Lambda_l^{j_1} + \frac{\partial^e(z^\alpha)}{\partial z_{j_1}\partial z_l\cdots\partial z_{j_e}} \, \Lambda_l^{j_2} + \cdots + \frac{\partial(z^\alpha)}{\partial z_{j_1}\cdots\partial z_{j_{e-1}}\partial z_l} \, \Lambda_l^{j_e} \bigg),$$

where $j_1,\dots,j_e = 1,\dots,n+1$ are arbitrary.

The equations for the unknows $\mathcal{L}_\alpha^\beta$ shall then be obtained by identifying the coefficients of the monomials z^ρ in the above equations $(\mathbf{0})$, $(\mathbf{1}_{j_1})$, ..., $(\mathbf{n}_{j_1\cdots j_n})$.

At first, we observe that since the degrees in z of the second terms of $(\mathbf{1}_{j_1})$, $(\mathbf{2}_{j_1j_2})$, *etc.* are at most $d-1$, $d-2$, etc., we can, without loss of generality, suppose that the $\mathcal{L}_\alpha^\beta$ are zero for $|\alpha|+|\beta| \geqslant d+1$, as it is written in the proposition. Next (cf. [13]), using the equation of $\mathcal{X}_0$, we may replace the occurence of z_1^d in the equation $(\mathbf{0})$ by $-\sum_{|\alpha|\leqslant d,\,\alpha_1<d} a_\alpha\, z^\alpha$, so that the degree in the z_1 variable is at most $d-1$ (as in [13], this will insure that the linear systems we have to solve are not overdetermined, and Cramer's basic rule will apply).

Now, the coefficient of each monomial z^ρ in the equation $(\mathbf{0})$ should vanish:

$$(\mathbf{0}_\rho) \qquad 0 = \sum_{\alpha+\beta=\rho} \mathcal{L}_\alpha^\beta.$$

Next, if as usual $\delta_{j_2}^{j_1}$ denotes the Kronecker symbol, equal to 1 if $j_1 = j_2$ and to 0 otherwise, we can shortly the various occuring partial derivatives of the monomial z^α as:

$$\frac{\partial(z^\alpha)}{\partial z_{j_1}} = \alpha_{j_1}\, z^{\alpha-\epsilon_{j_1}}, \quad \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} = \alpha_{j_1}\big(\alpha_{j_2} - \delta_{j_2}^{j_1}\big)\, z^{\alpha-\epsilon_{j_1}-\epsilon_{j_2}}, \ \dots\dots\, ,$$

$$\frac{\partial^e(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\cdots\partial z_{j_e}} = \alpha_{j_1}\big(\alpha_{j_2}-\delta_{j_2}^{j_1}\big)\cdots\big(\alpha_{j_e} - \delta_{j_e}^{j_1} - \cdots - \delta_{j_e}^{j_{e-1}}\big)\, z^{\alpha-\epsilon_{j_1}-\cdots-\epsilon_{j_e}}.$$

It follows that, for every $e \leqslant n$, the $(e+1)$-th family of equations, after equating to zero the coefficients of the monomial $z^{\rho-\epsilon_{j_1}-\cdots-\epsilon_{j_e}}$ and replacing α by $\rho-\beta$, identifies to

[3] This simplification trick justifies a posteriori, *cf.* [13], the *ad hoc*-looking assumption that the same matrix Λ appears in each jet vector field coefficient $\mathsf{Z}^{(\lambda)} = \Lambda \cdot z^{(\lambda)}$.

the collection:

$(\mathbf{e}_{j_1 j_2 \cdots j_e \rho})$

$$0 = \sum_{|\beta| \leqslant n} (\rho_{j_1} - \beta_{j_1})(\rho_{j_2} - \beta_{j_2} - \delta_{j_2}^{j_1}) \cdots (\rho_{j_e} - \beta_{j_e} - \delta_{j_e}^{j_1} - \cdots - \delta_{j_e}^{j_{e-1}}) \mathcal{L}_{\rho-\beta}^{\beta} + \\ + \mathsf{R}_{j_1 j_2 \cdots j_e \rho}(a, \Lambda),$$

where each second term $\mathsf{R}_{j_1 j_2 \cdots j_e \rho}(a, \Lambda)$, here considered as being just a remainder, is the coefficient of $z^{\rho - \epsilon_{j_1} - \cdots - \epsilon_{j_e}}$ in the second term of the equation $(\mathbf{e}_{j_1 j_2 \cdots j_e})$ and hence is clearly bilinear in (a_γ, Λ_k^l).

Thus, we have written a constant coefficient system of linear equations having the $\mathcal{L}_{\rho-\beta}^{\beta}$ as unknowns, $|\beta| \leqslant n$. As in [13, 16], we now claim that the determinant of its matrix is nonzero.

Indeed, for each fixed multiindex ρ, the matrix whose column C_β consists of the partial derivatives of order at most n of the monomial $z^{\rho-\beta}$ has the same determinant, at the point $(1, 1, \dots, 1)$ as the linear subsystem $(\mathbf{0}_\rho)$, $(\mathbf{1}_{j_1 \rho})$, ..., $(\mathbf{e}_{j_1 \cdots j_n \rho})$ we want to solve, where $j_1, \dots, j_n = 1, \dots, n+1$. Therefore, if the determinant would be zero, we would by linear combination, derive the existence of a *not* identically zero polynomial:

$$Q(z) := \sum_\beta c_\beta \, z^{\rho-\beta}$$

all of whose partial derivatives of order $\leqslant n$ vanish at $(1, \dots, 1)$. Hence the same would be true of:

$$P(z) := z^\rho \, Q(1/z_1, \dots, 1/z_{n+1}) = \sum_\beta c_\beta \, z^\beta,$$

and this would imply $P \equiv 0$, in contradiction to the assumption.

Thus for each fixed ρ, Cramer's rule solves the system for the $\mathcal{L}_\alpha^\beta$ with $\alpha + \beta = \rho$, and the solution is then obviously bilinear in (a, Λ). □

Invariance under reparametrization and logarithmic versions. We would like to make two final remarks, useful in applications. At first, similarly as it was pointed out in [13, 16], we claim that all vector fields constructed above are invariant under the group G_n of n-jets at the origin of local reparametrizations $\phi(\zeta) = \zeta + \phi''(0) \frac{\zeta^2}{2!} + \cdots + \phi^{(n)}(0) \frac{\zeta^n}{n!} + \cdots$ of $(\mathbb{C}, 0) \ni \zeta$ that are tangent to the identity[4], which acts on the jets $(z'_{i_1}, z''_{i_2}, z'''_{i_3}, \dots)$ by transforming them to:

$$w'_{i_1} := z'_{i_1}, \qquad w''_{i_2} := z''_{i_2} + \phi'' \, z'_{i_2}, \qquad w'''_{i_3} := z'''_{i_3} + 3\phi'' z''_{i_3} + \phi''' z'_{i_3}, \; \dots .$$

Such a transformation makes a diffeomorphism of $J^n_{\mathsf{vert}}(\mathcal{X})$, its inverse being associated to $\phi^{-1}(\zeta)$, and we must verify that our 4 families of tangent vector fields T_α, $\mathsf{T}_\alpha^{\ell_1, \dots, \ell_{n+1}}$, T_i and T_Λ are left unchanged under this diffeomorphism. Indeed, the claim trivially holds true for both the $\mathsf{T}_\alpha^{\ell_1, \dots, \ell_{n+1}}$ and the T_i, because they incorporate absolutely no $z'_{i_1}, z''_{i_2}, \dots, z^{(n)}_{i_n}$. Next, the $\frac{\partial}{\partial a_\beta}$ in the T_α are clearly left unchanged, while their coefficients are all, say in the case $n = 3$ to fix ideas, of the Wronskian-like form:

$$\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix} \equiv \begin{vmatrix} f' & g' & h' \\ f'' + \phi'' f' & g'' + \phi'' g' & h'' + \phi'' h' \\ f''' + 3\phi'' f'' + \phi''' f' & g''' + 3\phi'' g'' + \phi''' g' & h''' + 3\phi'' h'' + \phi''' h' \end{vmatrix},$$

[4] As a result, our two theorems can be applied in the framework of Demailly-Semple jets ([6]).

where $f, g, h \in \mathbb{C}[z_1, z_2, z_3]$ are some polynomials, but then such a determinant remains unchanged, thanks to obvious line manipulations. The general case $n \geqslant 3$ is similar. Finally, erasing indices and again for $n = 3$, we give the formal reason why the T_Λ are also invariant. The transformation is $w' = z'$, $w'' = z'' + \phi'' z'$, $w''' = z''' + 3\phi'' z'' + \phi''' z'$ and it replaces the basic vector fields by:

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial w'} + \phi'' \frac{\partial}{\partial w''} + \phi''' \frac{\partial}{\partial w'''}, \qquad \frac{\partial}{\partial z''} = \frac{\partial}{\partial w''} + 3\phi'' \frac{\partial}{\partial w'''}, \qquad \frac{\partial}{\partial z'''} = \frac{\partial}{\partial w'''},$$

so that $z' \frac{\partial}{\partial z'} + z'' \frac{\partial}{\partial z''} + z''' \frac{\partial}{\partial z'''} = w' \frac{\partial}{\partial w'} + w'' \frac{\partial}{\partial w''} + w''' \frac{\partial}{\partial w'''}$ is invariant.

Another argument (transmitted to us by Erwan Rousseau) for invariance under reparametrization would be to say that the system of linear equations that the coefficients Z_i, A_α, Z'_k, Z''_k, ..., $\mathsf{Z}_k^{(n)}$ of a general tangent vector field T have to satisfy:

$$0 = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha z^\alpha\big] = \mathsf{T}\big[\sum_\alpha a_\alpha (z^\alpha)'\big] = \mathsf{T}\big[\sum_\alpha a_\alpha (z^\alpha)''\big] = \mathsf{T}\big[\sum_\alpha a_\alpha (z^\alpha)'''\big] = \cdots,$$

is transformed, after reparametrization, into a system:

$$\begin{aligned} 0 &= \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha z^\alpha\big] = \mathsf{T}\big[\sum_\alpha a_\alpha (z^\alpha)'\big] = \mathsf{T}\big[\sum_\alpha a_\alpha (z^\alpha)'' + \phi'' \sum_\alpha a_\alpha (z^\alpha)'\big] \\ 0 &= \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)''' + 3\phi'' \sum_\alpha a_\alpha (z^\alpha)'' + \phi''' \sum_\alpha a_\alpha (z^\alpha)'\big] = \cdots \end{aligned}$$

which is completely equivalent to the first one, thanks to obvious linear combinations, so that any solution to this linear system is *a priori* forced to be invariant.

The second remark is that one may adapt the formalism provided here to show that the global generation property holds in a logarithmic setting with the *same* specific pole orders $c_n = \frac{n^2+5n}{2}$ (*cf.* [17] for $n = 3$) or $c_n = n^2 + 2n$. Application to effective algebraic degeneracy of entire holomorphic maps in the complement of a generic hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ are therefore also possible.

§5. APPENDIX: A DETERMINANTAL IDENTITY

Proof of the combinatorial lemma. At the beginning of Section 3, a determinant left aside had to be computed. We drop the index i and we denote it shortly by:

$$\Delta := \begin{vmatrix} z' & (z^2)' & \cdots & (z^n)' \\ z'' & (z^2)'' & \cdots & (z^n)'' \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ z^{(n)} & (z^2)^{(n)} & \cdots & (z^n)^{(n)} \end{vmatrix}.$$

On the first line, the entry of the k-th column is $(z^k)' = kz^{k-1}z'$. The trick is then to write the entry of the second line inside the same column as $k(z^{k-1}z')'$, *etc.*, and generally the entry of the κ-th line as $k(z^{k-1}z')^{(\kappa-1)}$, so that:

$$\Delta = \begin{vmatrix} z' & 2zz' & 3z^2z' & \cdots & nz^{n-1}z' \\ z'' & 2(zz')' & 3(z^2z')' & \cdots & n(z^{n-1}z')' \\ z''' & 2(zz')'' & 3(z^2z')'' & \cdots & n(z^{n-1}z')'' \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ z^{(n)} & 2(zz')^{(n-1)} & 3(z^2z')^{(n-1)} & \cdots & n(z^{n-1}z')^{(n-1)} \end{vmatrix}.$$

We see that $2 \cdot 3 \cdots n$ from the columns comes into factor. Next, using Leibniz's formula for the derivative of a product, we may expand $(z^{k-1}z')^{(\kappa-1)}$ just as $\sum_{0 \leqslant \lambda_1 \leqslant \kappa-1} \binom{\kappa-1}{\lambda_1} (z^{k-1})^{(\lambda_1)} z^{(1+\kappa-1-\lambda_1)}$. By subtracting to the k-th column the first one multiplied by z^{k-1}, the (κ, k)-entry then becomes $\sum_{1 \leqslant \lambda_1 \leqslant \kappa-1} \binom{\kappa-1}{\lambda_1} (z^{k-1})^{(\lambda_1)} z^{(\kappa-\lambda_1)}$, where now the sum starts from $\lambda_1 = 1$. In particular, the $(1, 2)$-, $(1, 3)$-, ..., $(0, n)$-entries all become null. By expanding the determinant along its first line, we are therefore left with an $(n-1)\times(n-1)$ determinant (notice the necessary shift of indices):

$$\frac{\Delta}{n!z'} = \left| \sum_{1\leqslant\lambda_1\leqslant\kappa} \binom{\kappa}{\lambda_1} \big(z^k\big)^{(\lambda_1)} z^{(1+\kappa-\lambda_1)} \right|_{1\leqslant\kappa\leqslant n-1}^{1\leqslant k\leqslant n-1}.$$

Iterating the trick, we again write:

$$\big(z^k\big)^{(\lambda_1)} = k\big(z^{k-1}z'\big)^{(\lambda_1-1)} = k\textstyle\sum_{0\leqslant\lambda_2\leqslant\lambda_1-1}\binom{\lambda_1-1}{\lambda_2}\big(z^{k-1}\big)^{(\lambda_2)} z^{(1+\lambda_1-1-\lambda_2)}.$$

We again see that $2\cdot 3\cdots(n-1)$ from the columns comes into factor, and then substituting the computed value of $\big(z^k\big)^{(\lambda_1)}$, we get:

$$\frac{\Delta}{n!z'} = (n-1)!\cdot \left| \sum_{1\leqslant\lambda_1\leqslant\kappa}\ \sum_{0\leqslant\lambda_2\leqslant\lambda_1-1} \binom{\kappa}{\lambda_1}\binom{\lambda_1-1}{\lambda_2}\cdot \right.$$
$$\left. \cdot \big(z^{k-1}\big)^{(\lambda_2)} z^{(1+\kappa-\lambda_1)} z^{(\lambda_1-\lambda_2)} \right|_{1\leqslant\kappa\leqslant n-1}^{1\leqslant k\leqslant n-1}.$$

The $(\kappa,1)$-entry inside the first colum is equal to $\sum_{1\leqslant\lambda_1\leqslant\kappa}\binom{\kappa}{\lambda_1} z^{(1+\kappa-\lambda_1)} z^{(\lambda_1)}$, because the terms $\big(z^0\big)^{(\lambda_2)}$ with $\lambda_2\geqslant 1$ are null. By subtracting to the k-th column the first one multiplied by z^{k-1}, the (κ,k)-th entry written above is slightly modified: the sum involving λ_2 is then just replaced by $\sum_{1\leqslant\lambda_2\leqslant\lambda_1-1}$. Moreover, the $(1,2)$-, $(1,3)$-, $\dots$, $(1,n-1)$- entries all become null, while the $(1,1)$ entry is $\binom{1}{1}z'z'$. By expanding the determinant along its first line, we are therefore left with an $(n-2)\times(n-2)$ determinant:

$$\frac{\Delta}{n!(n-1)!\,z'(z')^2} = \left| \sum_{1\leqslant\lambda_1\leqslant\kappa}\ \sum_{1\leqslant\lambda_2\leqslant\lambda_1-1} \binom{\kappa}{\lambda_1}\binom{\lambda_1-1}{\lambda_2}\cdot \right.$$
$$\left. \cdot \big(z^{k-1}\big)^{(\lambda_2)} z^{(1+\kappa-\lambda_1)} z^{(\lambda_1-\lambda_2)} \right|_{2\leqslant\kappa\leqslant n-1}^{2\leqslant k\leqslant n-1}.$$

We now have to change the indices. We at first set $k' := k-1$ and $\kappa' := \kappa-1$ and the determinant just obtained becomes:

$$\left| \sum_{1\leqslant\lambda_1\leqslant\kappa'+1}\ \sum_{1\leqslant\lambda_2\leqslant\lambda_1-1} \binom{\kappa'+1}{\lambda_1}\binom{\lambda_1-1}{\lambda_2}\cdot \right.$$
$$\left. \cdot \big(z^{k'}\big)^{(\lambda_2)} z^{(2+\kappa'-\lambda_1)} z^{(\lambda_1-\lambda_2)} \right|_{1\leqslant\kappa'\leqslant n-2}^{1\leqslant k'\leqslant n-2}.$$

Next, if we set $\lambda_1' := \lambda_1 - 1$ and if we observe the identification of sums:

$$\sum_{1\leqslant\lambda_1\leqslant\kappa'+1}\ \sum_{1\leqslant\lambda_2\leqslant\lambda_1-1}(\bullet) = \sum_{0\leqslant\lambda_1'\leqslant\kappa'}\ \sum_{1\leqslant\lambda_2\leqslant\lambda_1'}(\bullet) = \sum_{1\leqslant\lambda_2\leqslant\lambda_1'\leqslant\kappa'}(\bullet),$$

then our determinant simply becomes, after erasing the primes:

$$\left| \sum_{1\leqslant\lambda_2\leqslant\lambda_1\leqslant\kappa} \binom{\kappa+1}{\lambda_1+1}\binom{\lambda_1}{\lambda_2}\cdot z^{(\kappa-\lambda_1+1)} z^{(\lambda_1-\lambda_2+1)}\cdot \big(z^k\big)^{(\lambda_2)} \right|_{1\leqslant\kappa\leqslant n-2}^{1\leqslant k\leqslant n-2}.$$

Performing the same computational and transformational processes, the result of the next step will be:

$$\frac{\Delta}{n!(n-1)!(n-2)!\,z'(z')^2(z')^3} = \left| \sum_{1\leqslant\lambda_3\leqslant\lambda_2\leqslant\lambda_1\leqslant\kappa} \binom{\kappa+2}{\lambda_1+2}\binom{\lambda_1+1}{\lambda_2+1}\binom{\lambda_2}{\lambda_3}\cdot \right.$$
$$\left. \cdot z^{(\kappa-\lambda_1+1)} z^{(\lambda_1-\lambda_2+1)} z^{(\lambda_2-\lambda_3+1)}\cdot \big(z^k\big)^{(\lambda_3)} \right|_{1\leqslant\kappa\leqslant n-3}^{1\leqslant k\leqslant n-3}.$$

The induction is now clear, and at the end one obtains a 1×1 determinant $\left|(\bullet)\right|_{1\leqslant\kappa\leqslant 1}^{1\leqslant k\leqslant 1}$ with a sum $\sum_{1\leqslant\lambda_{n-1}\leqslant\cdots\leqslant\lambda_1\leqslant\kappa}(\bullet)$ inside which necessarily $k = \kappa = \lambda_1 = \cdots = \lambda_{n-1}$, so that this last 1×1 determinant equals:

$$\binom{n-1}{n-1}\cdots\binom{1}{1}\, z'\cdots z'\cdot z' = 1!\,(z')^n,$$

and this final observation completes the proof of the combinatorial lemma. □

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Repères méromorphes sur l'espace des jets verticaux de l'hypersurface universelle

[Repères méromorphes sur les jets de l'hypersurface universelle]

RÉSUMÉ. Pour des ordres de jets petits, on sait construire des repères méromorphes sur l'espace des des *jets verticaux* $J^k_{\mathsf{vert}}(\mathcal{X})$ de l'hypersurface universelle $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}}$ qui paramétrise toutes les hypersurfaces projectives $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré d. En 2004, pour $k = n$, Siu a annoncé qu'il existe deux constantes $c_n \geqslant 1$ et $c'_n \geqslant 1$ telles que le fibré tangent tensorisé:

$$T_{J^n_{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathcal{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}}}\big(c'_n\big)$$

est engendré par ses sections globales. Dans cet article, nous établissons cette propriété hors d'un certain ensemble algébrique exceptionnel $\Sigma \subset J^n_{\mathsf{vert}}(\mathcal{X})$ défini par l'annulation de certains wronskiens, avec l'ordre de pôles *effectif* $c_n = \frac{n^2+5n}{2}$, retrouvant ainsi $c_2 = 7$ (Paŭn), $c_3 = 12$ (Rousseau), et avec $c'_n = 1$.
De plus, quitte à augmenter c_n jusqu'à $c_n = n^2+2n$, la même propriété d'engendrement est satisfaite hors du plus petit sous-ensemble $\widetilde{\Sigma} \subset \Sigma \subset J^n_{\mathsf{vert}}(\mathcal{X})$ qui est défini par l'annulation de tous les jets d'ordre 1. Des applications à la dégénérescence algébrique *faible* (avec Σ) et *forte* (avec $\widetilde{\Sigma}$) des courbes holomorphes entières $\mathbb{C} \to X$ en découleront prochainement.

Mots et phrases-clés. Formule de Faà di Bruno à plusieurs variables, Hypersurfaces projectives algébriques, Jets de courbes holomorphes, Dégénérescence algébrique faible et forte au sens de Green-Griffiths.

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