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VECTOR FIELD CONSTRUCTION OF SEGRE SETS

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Abstract.

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The metric and differentiable structures called *SubRiemannian* (SR) in the anglo-saxon literature and *Carnot-Carathéodory* in France, which are associated to $\mathcal{C}^\infty$-smooth real manifolds equipped with systems of vector fields satisfying the so-called *Chow's accessibility condition* appear naturally in various domains of mathematics. The consideration of SR structures finds motivations and ramifications notably in Control Theory (with concrete applications in robotics), in the Analysis of Partial Differential Equations (mainly in the study of hypoelliptic operators and of subelliptic estimates), in Probability (study of diffusion on manifolds and sublaplacian), in Hamiltonian Mechanics, in Contact Geometry, in Real Algebraic Geometry, and finally in *Cauchy-Riemann* geometry. Before becoming a geometric field in its own, the study of subRiemannian structures has been mainly impulsed by the celebrated theorem of Hörmander about the $\mathcal{C}^\infty$ hypoellipticity [Hö] of the sums of squares type operators $X = \sum_{j=1}^k L_j^2$, where the vector fields L_j with $\mathcal{C}^\infty$ coefficients over $\mathbb{R}^n$ satisfy Chow's condition. Subsequent developements in the analysis of PDE's have been conducted by Métivier [Met], by Rothschild-Stein [RS] (who introduced the notions of dilatations, of weight, and the structure of nilpotent Lie group on the

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tangent space to a SR structure at a regular point, *see* also [Mit]), by Fefferman-Phong, by Varopoulos, by Jerison-Sanchez-Sanchez-Calle, and others. The reader may consult the extensive Bourbaki survey by Kupka [K] for further information and more complete references. The paradigmatic examples of SR structures are the Grušin plane and the Heisenberg group, which is of contact type. The interest of contact structures (a particular case of nonholonomy) is motivated by their link with the symplectic structures introduced by Lagrange, Souriau and Weinstein as a geometric frame for classical mechanics, and later highly developed in the Russian school, notably after Arnold, Gromov and Eliashberg. Quite recently, some fundamentals of the geometric aspects of SR structures have been developed in a long article by Gromov [Gro], in which it is addressed an impressive number of open questions, conjectures, *etc.* and among other topics, the ball-box theorem, the Hausdorff dimension of SR structures, their imbedding, the inequalities of Sobolev type, the disc theorem, *etc.* The interested reader may consult the introductory article by Bellaïche [Bell]. Such SR structures appear also quite naturally in real analytic (or algebraic) Cauchy-Riemann geometry, a field where the questions of hypoellipticity are motivated concretely, *see* [Trv], [Trp]. Since 1996, some more algebraic aspects of such structures in the real analytic case and especially some new invariants called *Segre sets* have been introduced by Baouendi-Ebenfelt-Rothschild [BER1,2], notably in order to establish some CR regularity theorems about CR mappings between CR manifolds. The present article is exclusively devoted to a geometric and detailed exposition of these local CR algebraic invariants and aims to tend towards a presentation of their properties which is as elementary as possible. A more general approach of algebraic and analytic aspects of real analytic complexifiable SR structures underlies our considerations, but we shall concentrate the exposition on real analytic CR manifolds.

§1. Geometry of finite type and review of CR-extension theory

1.1. Real analytic CR structures and CR orbits. A CR-generic real analytic ($\mathcal{C}^\omega$) submanifold M of $\mathbb{C}^n$ carries two fundamental geometric invariants:

1. The *complex tangent bundle* $T^cM = \mathrm{Re}(T^{1,0}M) = \mathrm{Re}(T^{0,1}M)$, and:
2. The so-called *family of Segre varieties*.

Our main goal in this article is to explain in expository style how these two invariants can be identified, geometrically. But let us begin first with a quick historical review of these two objects. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For the discussion which we shall endeavour here, we assume that the reader is familiar with the following *Orbit Theorem*, see Nagano [N], Sussmann [Sus], Baouendi-Ebenfelt-Rothschild [BER] or §9 below where we reprove it. Let $\mathbb{S} = \{X_\alpha\}$, $1 \leq \alpha \leq a$, $a \in \mathbb{N}_*$ be a system of vector fields *with* $\mathbb{K}$*-analytic coefficients* which is defined over a small $\mathbb{K}$-analytic manifold M. We denote by $\mathrm{Lie}(\mathbb{S})$ the Lie algebra generated by $\mathbb{S}$, *i.e.* consisting of all the multiple Lie brackets of any length of elements of $\mathbb{S}$. Let $p \in M$ be arbitrary.

Orbit Theorem. ([N], [Sus]) *There exists a unique germ* $\mathcal{O}_{\mathbb{S}}(M,p)$ *of manifold-piece passing through* p*, called the* $\mathbb{S}$-orbit of p in M*, satisfying*:

(1) Minimality property: *Every manifold-piece* Λ_p *through* p *with the property that each vector field* $X_\alpha \in \mathbb{S}$ *is tangent to* Λ_p *must* contain $\mathcal{O}_{\mathbb{S}}(M,p)$.

(2) Reachability : *Each point $q \in \mathcal{O}_{\mathbb{S}}(M,p)$ is the endpoint of a piecewise smooth integral curve of various elements of $\mathbb{S}$ in a finite number.*

(3) Dimensionality : *The dimension of this manifold-piece is equal to the dimension of the Lie algebra* $\mathrm{Lie}(\mathbb{S})$ *at p. Furthermore, the dimension of* $\mathrm{Lie}(\mathbb{S})$ *is* constant *and equal to* $\dim_{\mathbb{K}} \mathcal{O}_{\mathbb{S}}(M,p)$ *at each point $q \in \mathcal{O}_{\mathbb{S}}(M,p)$.*

By definition, the CR orbits $\mathcal{O}_{CR}(M,p)$ of M are the $\mathbb{S}$-orbits with $\mathbb{S} =: T^cM$.

1.2. Smooth CR-extension theory. In the last decade, it appeared that the CR-orbits of M adequately govern the analytic extension of CR functions, in all the regularity categories $\mathcal{C}^\omega$, $\mathcal{C}^\infty$ and $\mathcal{C}^{2,\alpha}$ (*see* [Trp], [Tu1,2], [J], [M1], and [BER2]). In fact, some important preliminary results in the $\mathcal{C}^\omega$ case have first focused attention on Lie brackets and finite type conditions, not on CR orbits. But after the works of Treves [Trv], Trépreau [Trp], Tumanov [Tu1,2], Jöricke [J] and the author [M1], it became clear that there exists a *one-to-one correspondence* between the CR-orbits $\mathcal{O}_{CR} \subset M$ of M and the *analytic wedges* attached to the orbits $\mathcal{O}_{CR}$, to which CR functions on M admit a holomorphic extension. These analytic wedges are simply complex manifolds with edge, attached to the CR-orbit and they have dimension $\dim_{\mathbb{C}} \mathcal{W}^{an} = \dim_{\mathbb{R}} \mathcal{O}_{CR} - \dim_{CR} M$ (*see* [Tu2] and [J] for the precise definitions). In case $\dim_{\mathbb{R}} \mathcal{O}_{CR} = \dim_{\mathbb{R}} M$, this statemement is the theorem on global minimality of [J] and [M1]. A large number of families of analytic discs of Bishop's type, which are attached to $\mathcal{O}_{CR}$ and to some subsequent deformations of $\mathcal{O}_{CR}$, fill in $\mathcal{W}^{an}$ by propagation and deformation [Tu2], [J], [M1]. In summary, CR-orbits are the good objects for understanding CR-extension theory.

Also, let us give here a brief review of some of these works on CR extension. Holomorphic extendability to one side of CR functions defined over a strongly pseudoconvex hypersurface in $\mathbb{C}^2$ was discovered by Hans Lewy in 1956 and subsequently worked out by many mathematicians : Hill-Taiani, Boggess-Polking, Bedford, Fornaess-Rea, Boggess-Pitt, Baouendi-Chang-Treves, Baouendi-Rothschild, Hanges-Treves and others (we refer the reader to the books [Bo], [BER2] for bibliographical data). In the early eighties, Treves pointed out the importance of CR orbits and of Sussmann's construction [Sus]. In 1988, Tumanov proved his celebrated extension theorem [Tu1]. In 1990, after the work of Hanges and Treves [HaTr], Trépreau identified the fundamental correspondence between CR orbits and CR extension by establishing propagation of wedge extendability using FBI (Fourier-Bros-Iagolnizer) transform. Then Tumanov in [Tu2] pushed forward the extension theory by establishing propagation of *CR extendability* by means of deformations of analytic discs.

1.3. Real analytic CR manifolds. The second invariant, called *Segre varieties*, disappears in the $\mathcal{C}^\infty$ category. Segre varieties indeed arise from complexification of the defining functions of M, which must therefore be real analytic. Nevertheless, it is well known that the Segre varieties are important tools in the reflection principle and in the study of CR mappings between $\mathcal{C}^\omega$ CR manifolds. They were introduced in a short note by Segre in 1931. Motivated by Poincaré's equivalence problem, Segre sought local differential invariants attached to M. Much later in 1977, exhuming Segre varieties, Webster proved a celebrated theorem : a local biholomorphic map between two strongly pseudoconvex real algebraic hypersurfaces in $\mathbb{C}^n$, $n \geq 2$, must be algebraic [W]. Two other related grounding articles were

written independently by Pinchuk [P] and Lewy [L]. Since then, this area became an intensive subject of research. The popularity of this field is certainly due to its connections with Algebra, Analysis and Geometry as well as on the fact that a great number of variations on the theme leave open many attractive questions (*cf.* the celebrated conjectures of Diederich and Pinchuk). The reader may consult [DP] or the book [BER2] for the classical presentation and also [Suk] for a differential Lie geometric viewpoint, as in Cartan's and Segre's original articles. As a matter of fact, Segre varieties have been developed substantially since 1996, in the works of Baouendi-Ebenfelt-Rothschild, especially about CR manifolds of high codimension.

§2. Identification of CR vector fields and Segre varieties

2.1. Segre sets and minimality criterion. Indeed, in the study of the algebraic regularity mapping problem, Segre varieties have been used by Sharipov-Sukhov, who introduced a geometric condition called "Segre-transversality", and then by Baouendi-Ebenfelt-Rothschild, to whom is due the definition of "*Segre sets*", a quite important novelty in the subject. Basically, the Segre sets of M are simple unions of Segre varieties as follows: $Q_p^1 := Q_p$ is the Segre variety through $p \in M$, with $p \in Q_p^1$; Q_p^2 is the union of Segre varieties Q_q as q ranges over Q_p^1; Q_p^3 is the union of Segre varieties Q_q as q ranges over Q_p^2, *and so on.* Clearly, by the (easy) property $r \in Q_q \Longleftrightarrow q \in Q_r$, one has $Q_p^k \subset Q_p^{k+1}$. Such Segre sets Q_p^k are also biholomorphically invariant, as the Q_p's are. As a matter of fact, we intend to discuss in this paper this construction of Segre sets and the celebrated "*minimality criterion*" proved in [BER1,2] and slightly modified after the contribution of Zaitsev [Z]. Thus, let us recall that M is called of *finite type* in the sense of Kohn and Bloom Graham at p if the Lie algebra generated by the complex tangent bundle T^cM spans T_pM at p (*cf.* [BG]). Equivalently, the CR orbit of p contains a neighborhood of p in M, by Chow's accessibility theorem [Cho] (*cf.* **(2)** of the Orbit Theorem). Using a now well established terminology, will say that M is *orbit-minimal* at p, or shortly "minimal" at p, "in the sense of Tumanov", *cf.* [Tu1]. As we have mentioned, the article [BER1] have provided a preliminary insight about the link between Segre varieties and finite-typeness of CR vector fields by establishing the following

Minimality Criterion. *M is minimal at p if and only if there is an integer $\nu_p \leq d+1$ such that the $(2\nu_p)$-th Segre set $Q_p^{2\nu_p}$ contains a neighborhood of p in $\mathbb{C}^n$.*

Remark. In this precise form, the statement is in fact due to Zaitsev, *see* [Z] or [BER2]. In [BER1], the authors have established that $Q_p^{\nu_p}$ contains an open set $\mathcal{V}$ of $\mathbb{C}^n$ with $\overline{\mathcal{V}} \ni p$ (but $p \notin \mathcal{V}$). The least such integer ν_p is a biholomorphic invariant of (M, p). Briefly, the core of the proof in [BER1,2] is to set up normal forms for generic $\mathcal{C}^\omega$ manifolds as follows from the work of Bloom-Graham [BG], to introduce the classical Hörmander numbers of M at p, to discuss then the homogeneous algebraic Taylor approximation M^{app}, to show that $Q(M^{app})_p^{2\nu_p}$ contains a neighborhood of p in $\mathbb{C}^n$ and to conclude by a perturbation argument that it is so for $Q(M)_p^{2\nu_p}$. We claim that such a proof can be substantially modified and that the whole theory of Segre sets can be geometrized.

2.2. Flows of complexified vector fields and Segre chains. Indeed, our main result in this article lies in *two canonical observations* which will fill up an ap-

propriate understanding of some canonical (and still missing) links between the complex tangent bundle T^cM *and* the Segre varieties. In truth, *these links become clearly visible only after complexifying* M. To offer a complete presentation of these ideas, we need at first to introduce a good deal of notation (*see* also §3, §4, §5, and §6 below). Thus, let $\mathcal{M} \subset \mathbb{C}^{2n}$ denote the extrinsic complexification of $M \subset \mathbb{C}^n$. This complexification $\mathcal{M}$ lives in $\mathbb{C}^n_t \times \mathbb{C}^n_\tau$, where $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{C}^n$ and is naturally equipped with two projections $\pi_t : \mathcal{M} \to \mathbb{C}^n_t$ and $\pi_\tau : \mathcal{M} \to \mathbb{C}^n_\tau$, the term t denoting some coordinates in $\mathbb{C}^n$ near (M, p) and the term $\tau := (\bar{t})^c$ being the "complexification coordinate" of $\bar{t}$ (*see* §3.5). Set $m := \dim_{CR} M$, $d := \operatorname{codim}_{\mathbb{R}} M$, $m + d = n$. Now, let $\mathcal{L} := (\mathcal{L}^1, \dots, \mathcal{L}^m)$ and $\underline{\mathcal{L}} := (\underline{\mathcal{L}}^1, \dots, \underline{\mathcal{L}}^m)$ be some two complexified *commuting* bases of the two complexied bundles $(T^{1,0}M)^c$ and $(T^{0,1}M)^c$ respectively, which have both holomorphic coefficients and are both tangent to $\mathcal{M}$. A complete description in local coordinates of $\mathcal{L}$ and of $\underline{\mathcal{L}}$ is provided in §5.9 below. As each one of the two distributions they define is Frobenius-integrable, we call $\mathcal{L}$ and $\underline{\mathcal{L}}$ "*m-vector fields*". We notice that, on the contrary, the distribution defined by the pair $\{\mathcal{L}, \underline{\mathcal{L}}\}$ is *not* Frobenius-integrable, *except* only in the Levi-flat case! Now, let $p \in M$ and set $p^c := (p, \bar{p}) \in \mathcal{M}$. Let $\mathcal{L}_w(p^c) := \mathcal{L}^1_{w^1}(\dots \mathcal{L}^m_{w^m}(p^c))$ denote the m-flow of $\mathcal{L}$, $w \in \mathbb{C}^m$, and *idem* for $\underline{\mathcal{L}}_\zeta(p^c)$, $\zeta \in \mathbb{C}^m$. We follow a notation like in [Sus], but in other texts, such m-flow would be denoted for instance by $\exp(w\mathcal{L})(p^c)$. The alternated concatenations of such flows will be called *Segre k-chains*, for instance, for $k = 2j$, it is the holomorphic map $(w_1, \dots, w_{2j}) \mapsto \underline{\mathcal{L}}_{w_{2j}}(\mathcal{L}_{w_{2j-1}}(\dots \underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(p^c)))) \in \mathcal{M}$, where $w_1 \in \mathbb{C}^m, \dots, w_{2j} \in \mathbb{C}^m$, or by induction: $\mathcal{L}_{w_1}(p^c)$, $\underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(p^c))$, $\mathcal{L}_{w_3}(\underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(p^c)))$, *etc.* We refer the reader to §6.1 below for further explanations. For short, we shall denote by $w_{(k)} := (w_1, \dots, w_k) \in \mathbb{C}^{mk}$ such a k-uple and we shall denote these above concatenated flow maps by the abbreviated expression $w_{(k)} \mapsto \Gamma_k(w_{(k)})$. Clearly, in place of the above maps, we could have chosen the collection of maps $w_1 \mapsto \underline{\mathcal{L}}_{w_1}(p^c)$, $w_{(2)} \mapsto \mathcal{L}_{w_2}(\underline{\mathcal{L}}_{w_1}(p^c))$, $w_{(3)} \mapsto \underline{\mathcal{L}}_{w_3}(\mathcal{L}_{w_2}(\underline{\mathcal{L}}_{w_1}(p^c)))$, *etc. Both the two choices are valuable.* In fact, we see no reason why there there should be a preferred choice here. We therefore introduce a corresponding notation $\underline{\Gamma}_k(w_{(k)})$ for these second k-concatenated flow maps. Then the two maps Γ_k and $\underline{\Gamma}_k$ are different, except in the Levi-flat case. In fact, it will appear that these two maps correspond to each other under the well known special symmetry $\sigma(t, \tau) := (\bar{t}, \bar{\tau})$ of $\mathcal{M}$ which is re-defined in §3.6 below, as follows: $\sigma(\Gamma_k(w_{(k)})) = \underline{\Gamma}_k(\overline{w_{(k)}})$. Finally, let us define the sets $\mathcal{S}^k_{\bar{p}} := \{\Gamma_k(w_{(k)}) \colon ||w_{(k)}|| \leq \delta\}$, which will be called *complexified Segre k-chains* and $\underline{\mathcal{S}}^k_p := \{\underline{\Gamma}_k(w_{(k)}) \colon ||w_{(k)}|| \leq \delta\}$, called *complexified conjugate Segre k-chain*, where $\delta > 0$. By slight abuse of terminology in the paper, we shall call *Segre k-chains* either the holomorphic maps Γ_k or the sets $\mathcal{S}^k_{\bar{p}}$. As we work locally, the value of $\delta > 0$ appears to be unimportant. Notice that in our terminology, the *Segre sets* of [BER1,2] live in ambient space whereas our *Segre chains* live in the complexification $\mathcal{M}$. In fact, these Segre chains can be identified with the "orbit-chains" in the spirit of Sussmann [Sus], which appear in the step by step construction of the orbit of the pair of integrable distributions $\{\mathcal{L}, \underline{\mathcal{L}}\}$ on $\mathcal{M}$, when one constructs their orbits, as we shall do in §6 and §7. Accordingly, the complexification $\mathcal{M}$ will be called *orbit-minimal at* p^c if $\mathcal{S}^k_{\bar{p}}$ contains a neighborhood of p^c in $\mathcal{M}$ for some $k \in \mathbb{N}_*$. This full neighborhood covering property holds equivalently for $\underline{\mathcal{S}}^k_p = \sigma(\mathcal{S}^k_{\bar{p}})$. In general,

we denote by $\mathcal{O}_{CR}(M,p) \subset M$ the CR-orbit of p in M and by $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$ the $\{\mathcal{L},\underline{\mathcal{L}}\}$-orbit of p^c in $\mathcal{M}$. Our first main observation can be summarized as follows:

Theorem 2.3. *Let $\mathcal{M} := (M)^c$, where M is $\mathcal{C}^\omega$ as above. Then*

- **(a)** *Segre sets arise from Segre k-chains: $Q_p^{2j} = \pi_t(\underline{\mathcal{S}}_p^{2j})$ and $Q_p^{2j+1} = \pi_\tau(\mathcal{S}_{\bar{p}}^{2j+1})$.*
- **(b)** *The generic ranks $r_k = \underline{r}_k$ of the maps Γ_k, $\underline{\Gamma}_k$ are biholomorphic invariants.*
- **(c)** *The complexification of CR-orbits satisfies $[\mathcal{O}_{CR}(M,p)]^c = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$.*

Remarks. **1.** Part **(c)** appears already in [BER1,2], with a technically very different proof. We believe that our proof given in §6 below is elementary and natural in itself, because by definition the two generators L and $\bar{L}$ for $T^{1,0}M$ and $T^{0,1}M$ which produce the CR-orbits admit as complexifications $L^c := \mathcal{L}$ and $(\bar{L})^c = \underline{\mathcal{L}}$. Then $[\mathcal{O}_{L,\bar{L}}(M,p)]^c = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M},p^c)$ should hold almost tautologically, if our notations are appropriate. We will show that such notations are consistent with the proof that we will conduct in §6.1.

2. Clearly, part **(a)** can be taken as a (new) definition of Segre sets. One can even forget Segre sets and consider only Segre chains in $\mathcal{M}$, after working only with complexified objects. This is in fact what the author does in [M2,3], when considering various algebraic or formal CR-mapping problems. We shall provide in §11 a summary of what is essential in the study of these CR-regularity problems.

3. Clearly also, looking at **(a)** we see that we could have also defined what could be called "*conjugate Segre sets* $\overline{Q_p^k}$" (not explicitely defined in [BER1,2] or [Z]) as follows: $\overline{Q_p^{2j}} := \pi_\tau(\mathcal{S}_{\bar{p}}^{2j})$ and $\overline{Q_p^{2j+1}} := \pi_t(\underline{\mathcal{S}}_p^{2j+1})$. Such a terminology is explained by the fact that they appear to be simply the conjugate sets $\{t \in \mathbb{C}^n : \bar{t} \in Q_p^k\} = \overline{\{t \in Q_p^k\}}$. This remark is not innocent.

2.4. Necessity of conjugate Segre varieties. In fact, inspired by this observation of the existence of "symmetric" Segre sets obtained by complex conjugation, we will be engaged to *define a* (*new*) *family of "conjugate Segre varieties* $\overline{Q_p}$*" inside* $\mathbb{C}^n$ (before complexification) and to provide them with the status of interesting biholomorphically invariant objects which are "symmetric" to the usual Segre varieties. We know for a fact that, although nowhere published, this tentative idea is very well known in the folklore: if $M = \{t \in \mathbb{C}^n : \rho(t,\bar{t}) = 0\}$, why not put $\rho(p,\bar{t}) = 0$ instead of $\rho(t,\bar{p}) = 0$ to define Segre varieties? This would yield $Q_p = \{t \in \mathbb{C}^n : \rho(t,\bar{p}) = 0\}$ and $\overline{Q_p} = \{\bar{t} \in \mathbb{C}^n : \rho(p,\bar{t}) = 0\}$ (warning: by Lemma 3.4 **(c)** below, we have in fact, $\{t \in \mathbb{C}^n : \rho(p,\bar{t}) = 0\} = Q_p$–nothing new!). In fact, because all the properties of the $\overline{Q_p}$ are obviously the same (*modulo* complex conjugation) as those of the Q_p, it may perharps seem to be superfluous to endeavour a task of duplication. However, inspired by the vision that the pair of m-vector fields $\{\mathcal{L},\underline{\mathcal{L}}\}$ induce two different symmetric intrinsic complex analytic foliation of $\mathcal{M}$ (*see* Theorem 2.6 and §5.13 below), the author feels that the correct point of view has to go further. To support these declarations, he claims two general ideas:

- **I.** *The good geometric objects appear after complexification.*
- **II.** *It is important to consider simultaneously the pairs $\{Q_p, \overline{Q_p}\}$.*

Argumentation of Claim I. Claim I bears on an imprecise but well established general philosophy about the study of real analytic objects that has been carried out intensively since the grounding works of Webster. □

Argumentation of Claim II. It is fairly well known that in principle, there is no preferred choice between holomorphic and antiholomorphic functions: both enjoy the same properties and are "the same" type of objects. Analogously, there is no preferred choice between the Q_p and the $\overline{Q_p}$, between $T^{1,0}M$ and $T^{0,1}M$, between L-derivations and $\overline{L}$-derivations in all the CR-mapping problems. But contrary to the holomorphic category, where a choice between t and $\bar{t}$ can be decided *once for all* at the beginning, in the CR category, *no choice can be really made once for all.* In fact, the simultaneous consideration of both t and $\bar{t}$ is crucial and any of the two choices would be in a certain sense erroneous by incompleteness. Indeed, since some real structure is added, the members of the above pairs are not "one and the same object": the Q_p really differ from the $\overline{Q_p}$ and they *come together in the Cauchy-Riemann geometry of* M. Any biholomorphic or anti-biholomorphic change of coordinates leaves invariant *the two families* Q_p *and* $\overline{Q_p}$. Therefore, there would perharps exist a sort of general "*principle of symmetry*" in analytic CR geometry, represented by the bar symmetry $\overline{(\bullet)}$. Correspondingly, we will also observe that *taking complex conjugates has a real geometric signification*, since it permutes the two foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ of Theorem 2.6 below. $\square$

Remarks. **1.** We have wished to insist on this matter because these "philosophical" ideas about complex conjugation in analytic CR geometry are not yet currently argued in the literature and because conjugate Segre varieties appear nowhere with an explicit status.

2. However, we mention that in a *non-CR context*, Moser and Webster introduced a pair of involutions $\{\tau_1, \tau_2\}$ attached to a 2-surface with isolated complex tangency in $\mathbb{C}^2$ (defined in terms of the two projections π_t and π_τ of the complexification Σ^c of Σ), which are intertwined by the complexification $\sigma(t,\tau) := (\bar{\tau}, \bar{t})$ of the complex conjugation operator. These involutions exchange the (generically of cardinal two) fibers of $\pi_t|_{\Sigma^c}$ and $\pi_\tau|_{\Sigma^c}$. In the CR context, one remark in [MW] p. 262 reminds something corresponding to **(k)** and **(l)** of Theorem 2.10 below, although it is not formulated there *with pairs* as we shall do. The two involutions of Moser and Webster should correspond to our pair of flows maps $w \mapsto \mathcal{L}_w(p^c)$ and $\zeta \mapsto \mathcal{L}_\zeta(p^c)$. On the other hand, these two involutions are considered *as symmetric pairs* in [MW], as we aim to do in this article. Although the (more elementary) CR case is not studied by Moser and Webster, there is a strong structural analogy between their constructions and ours below. In conclusion, we would like to say that our geometric constructions are deeply related to the ones in [MW].

3. Furthermore in his recent works [M3], the author has observed that working out simultaneously *reflection identities* and (new) *conjugate reflection identities* in the formal CR-mapping problem enables to establish a well known conjecture, the convergence of formal equivalences between two real analytic minimal and holomorphically nondegenerate CR manifolds. This illustrates the interest of studying real analytic CR objects *as conjugate pairs.* The content of such pairs of reflection identities will be exposed briefly in §11 below.

In summary, in real analytic CR geometry, the complex conjugation operator has an important signification and it deserves to be thematized as an independent geometric object linking the geometric and the algebraic objects *as conjugate pairs.*

2.5. Segre type. From Theorem 2.3, we will easily derive as a particular case:

Theorem 2.6. *The following properties are equivalent:*

- **(d)** *The CR-generic $\mathcal{C}^\omega$ manifold (M,p) is $\{L,\bar{L}\}$-orbit-minimal at p.*
- **(e)** *Its complexification $(\mathcal{M},p^c)$ is $\{\mathcal{L},\underline{\mathcal{L}}\}$-orbit-minimal at $p^c=(p,\bar{p})$.*
- **(f)** *There exists an integer $\mu_p\leq d+2$ such that the $(2\mu_p-1)$-th Segre chain $\mathcal{S}_{\bar{p}}^{2\mu_p-1}$ (or equivalently, its conjugate $\underline{\mathcal{S}}_p^{2\mu_p-1}$) contains a small open set $\mathcal{U}\subset\mathcal{M}$ with $\mathcal{U}\ni p^c$.*
- **(g)** *There exist $w^*_{(2\mu_p-1)}\in\mathbb{C}^{m(2\mu_p-1)}$ arbitrarily close to the origin such that $\Gamma_{2\mu_p-1}\colon(\mathbb{C}^{m(2\mu_p-1)},w^*_{(2\mu_p-1)})\to(\mathcal{M},p^c)$ is a submersion (same property for the map $\underline{\Gamma}_{2\mu_p-1}\colon(\mathbb{C}^{m(2\mu_p-1)},\overline{w^*_{(2\mu_p-1)}})\to(\mathcal{M},p^c)$).*
- **(h)** *With this integer μ_p, the Segre sets $Q_p^{2\mu_p-2}$ and $\overline{Q_p^{2\mu_p-2}}$ both contain a neighborhood of p.*

Furthermore, we have $\nu_p=\mu_p-1$ (where ν_p is as in the Minimality Criterion).

Remarks. **1.** As ν_p, the least such integer μ_p is a biholomorphic invariant of $\mathcal{M}$, called in this article the *Segre type of $\mathcal{M}$ at p.*

2. Notice that we state that $Q_p^{2\nu_p}$ contains a neighborhood of p, so we recover the same minimality criterion as in [Z], [BER2]. This integer $2\nu_p$ cannot be improved (*e.g.* as an *odd integer* $2\nu_p-1$) as Example 7.21 will show.

Correspondingly, we can reformulate a (new) minimality criterion in the complexification $\mathcal{M}$ by using Segre chains instead of Segre sets as follows.

Minimality criterion 2.7. *(M,p) is orbit-minimal if and only if there is an integer $\mu_p\leq d+2$ such that the $(2\mu_p-1)$-th Segre chain $\mathcal{S}_{\bar{p}}^{2\mu_p-1}$ contains a neighborhood of p^c in $\mathcal{M}$.*

We shall give a complete proof of it in §7. For the moment, as a matter of fact, and quite naturally, we claim that this (new) minimality criterion becomes (almost) immediate (hence better than the previous one), because we can check the following elementary arguments:

- **1:** **(d)** $\Longleftrightarrow\mathbb{C}\otimes T_pM=\text{Lie algebra}_p(\{L,\bar{L}\})$. We assume that this equivalence is well known, *i.e.* we assume that the reader is familiar with Chow's and Nagano's classical theorems (*cf.* the Orbit Theorem above).
- **2:** The complexification $[\text{Lie algebra}_p(\{L,\bar{L}\})]^c=\text{Lie algebra}_p(\{\mathcal{L},\underline{\mathcal{L}}\})$.
- **3:** Hence **(d)** $\Longleftrightarrow$ **(e)**, because $(T_pM)^c=T_{p^c}\mathcal{M}$.
- **4:** Finally, the equivalence **(e)** $\Longleftrightarrow$ **(f)** is by definition, *q.e.d.*

Remarks. **1.** The equivalence **(f)** $\Longleftrightarrow$ **(g)** follows from Sussmann's construction of orbits. We will recover it in §7-8 below, where we refine the constructions of [Sus].

2. In fact, it is also easy to check **(f)** $\Longleftrightarrow$ **(h)**, hence to recover the (first) minimality criterion of [BER1,2], [Z] from the Minimality Criterion 2.7. This equivalence can be quickly established by observing that $\pi_t\colon(\mathcal{M},p^c)\to(\mathbb{C}^n_t,p)$ and $\pi_\tau\colon(\mathcal{M},p^c)\to(\mathbb{C}^n_\tau,\bar{p})$ are submersions with fibers satisfying $\pi_t^{-1}(p)\cap\pi_\tau^{-1}(\bar{p})=p^c$.

2.8. The geometry of the complexification. Because the minimality criterion becomes canonical in the complexification, it seems to be more natural to consider and to work out Segre chains in the complexification $\mathcal{M}$ instead of considering Segre sets in $\mathbb{C}^n$. Recently, it also became clear that the geometry of the pair $\{\mathcal{L},\underline{\mathcal{L}}\}$ is well

adapted to the CR mapping problems, *see* §11 below. A final argumentation to give evidence to this conclusion will be provided by the next fundamental observation.

2.9. Double foliation by complexified Segre varieties. In fact, the Segre varieties Q_p and the conjugate Segre varieties $\overline{Q_p}$ also admit canonical extrinsic complexifications, which are m-dimensional complex manifolds contained in $\mathcal{M}$. By definition, these complexifications will be $\mathcal{S}_{\tau_p} := \{(t, \tau_p) \colon \rho(t, \tau_p) = 0\}$, where $\tau_p \in \mathbb{C}^n$ is *fixed*, and $\underline{\mathcal{S}}_{t_p} := \{(t_p, \tau) \colon \rho(t_p, \tau) = 0\}$, where $t_p \in \mathbb{C}^n$ is *fixed* (*see* §5.12 for further precisions). Now, we can state our second main geometrical observation (**(l)** below). A very quick proof of this "observational" theorem is provided in §5.6, §5.9, §5.12 below and can be read independently of the rest of the article.

Theorem 2.10. *Let $\delta > 0$ be small. Recall that $\mathcal{L}$ and $\underline{\mathcal{L}}$ are Frobenius-integrable.*

- **(i)** *$\mathcal{L}$ and $\underline{\mathcal{L}}$ induce naturally two local flow foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ of $\mathcal{M}$.*
- **(j)** *$\sigma(\mathcal{F}_{\mathcal{L}}) = \mathcal{F}_{\underline{\mathcal{L}}}$ and their two leaves through p^c satisfy $\mathcal{F}_{\mathcal{L}}(p^c) \cap \mathcal{F}_{\underline{\mathcal{L}}}(p^c) = p^c$.*
- **(k)** *The fibers of the projections π_t and π_τ also coincide with the leaves of the flow foliations $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$, respectively.*
- **(l)** *The leaves of the foliation $\mathcal{F}_{\mathcal{L}}$ are the Segre varieties $\mathcal{S}_{\tau_p}$ and the leaves of the foliation $\mathcal{F}_{\underline{\mathcal{L}}}$ are the conjugate Segre varieties $\underline{\mathcal{S}}_{t_p}$:*

$$\mathcal{F}_{\mathcal{L}} = \bigcup_{\tau_p \in \mathbb{C}^n, ||\tau_p|| < \delta} \mathcal{S}_{\tau_p} \quad \text{and} \quad \mathcal{F}_{\underline{\mathcal{L}}} = \bigcup_{t_p \in \mathbb{C}^n, ||t_p|| < \delta} \underline{\mathcal{S}}_{t_p} \tag{2.11}$$

In other words, the leaves of these two flow foliations are the two families of complexified (conjugate) Segre varieties. In symbolic representation, for these two foliations, we have the correspondence:

(2.12) *CR-flow foliations of $\mathcal{M}$ $\iff$ Foliations by complexified Segre varieties.*

Remarks. **1.** Observe that in the foliated unions of eqs. (2.11), the dimensions correspond: the dimension of $\mathcal{M}$ is equal to the leaf dimension m plus n, which is equal to the dimension of the parameter space $||\tau_p|| < \delta$ (or $||t_p|| < \delta$).

2. It is well known however that classical Segre varieties *do not* in general foliate a neighborhood of p in $\mathbb{C}^n$. For instance, think of the simplest quadric $z_2 = \bar{z}_2 + i z_1 \bar{z}_1$ in $\mathbb{C}^2$. However, *when we pass to the complexification, we blow-up $\bigcup_{q \in \mathcal{V}_{\mathbb{C}^n}(p)} Q_q$ and $\bigcup_{q \in \mathcal{V}_{\mathbb{C}^n}(p)} \overline{Q_q}$ in a double foliation of $\mathcal{M}$ by complex m-dimensional Segre surfaces.*

2.13. Summary. Our second observation, valuable only in $\mathcal{C}^\omega$ category, is clear:

(2.14)

Complexified Segre varieties $\iff$ Complexified CR vector fields foliation.

Remark. Again, we should say at the informal level that the above equivalence makes very transparent and canonical the Minimality Criterion of [BER1,2], [Z] expressed by means of Segre sets. To the knowledge of the author, although very preliminary in the subject, this interpretation of the two complexified Segre varieties as <u>foliation</u> leaves of the two flow <u>foliations</u> by complexified CR m-vector fields (*i.e.* m-dimensional Frobenius-integrable distributions) is also new.

2.15. Propagation philosophy. We notice furthermore that in the $\mathcal{C}^\infty$ case, while Segre varieties disappear, it was known that the remaining $\mathcal{C}^\infty$ sections of T^cM propagate some structural analytic properties of CR distributions: vanishing in a neighborhood of a point, being extendable to a wedge, are propagating properties along CR orbits [Trv], [Trp], [Tu2], [J], [Me1]. *It is therefore natural that in the $\mathcal{C}^\omega$ minimal case, Segre varieties become the support of propagating properties for CR mappings*, like algebraicity, jet solvability, analytic regularity, which are propagating properties along Segre chains, *see* [BER1,2], [M2,3], [Z]. The chronology of this discovery is amazingly inversed: the propagation phenomena were first better understood in the $\mathcal{C}^\infty$ category than in the $\mathcal{C}^\omega$ one, perharps because the ideas and tools came from Analysis and more specifically from Partial Differential Equations. By the way, the author wonders whether there exist smooth geometrical objects which fill in the disappearing of Segre varieties in the various CR-mapping problems between smooth CR manifolds, in analogy with Bishop's discs in the smooth CR-extension theory.

2.16. Organisation of the remainder of the article. In §3 and in §4, we summarize the classical presentation of Segre sets due to Baouendi-Ebenfelt-Rothschild. The core of the article begins in §5.4 where the geometry in the complexification $\mathcal{M}$ starts. In §5, we establish Theorem 2.10 by an inspection of some equations of $\mathcal{M}$ in coordinates. Then in §6 and in §7, we study the (extrinsic or intrinsic) complexifications of CR orbits and we establish Theorem 2.6. Numerous examples illustrating the (rather dry) general theory are provided in §8, taking inspiration from works of Freeman, of Loboda and of Ebenfelt. Finally, we construct a slight refinement of Sussmann's constructions in §9 to prove Nagano's theorem using flows of vector fields instead of Lie algebras.

2.18. Closing remark. None of the result presented here is really new and most of the theorems that we (re)prove are in fact originally due to Baouendi-Ebenfelt-Rothschild. Only our geometrical viewpoint makes a difference.

2.19. Acknowledgement. The author is grateful to an anonymous referee who suggested to rewrite the manuscript in an expository style and to render the topic the more attractive and the less technical possible.

§3. Segre varieties and extrinsic complexification

3.1. Real analytic CR-generic manifolds. To begin with, we need a good deal of preliminary material. It is well known that real analytic CR manifolds are generic in their (complex analytic) semi-local intrinsic complexification. Accordingly, the local study of $\mathcal{C}^\omega$ CR manifolds reduces to the study of the CR-generic ones. Thus, here and in the sequel, let M be a piece through the origin of a $\mathcal{C}^\omega$ CR-*generic* manifold in $\mathbb{C}^n$ and set $m := \dim_{CR} M$, $d := \operatorname{codim}_{\mathbb{R}} M$, with $m + d = n$. Then there exists a system of d-vectorial defining functions for M: $\rho = (\rho_1, \dots, \rho_d)$, which are $\mathcal{C}^\omega$ in a neighborhood U of 0 in $\mathbb{C}^n$ such that $M \cap U$ coincides with the zero-set $\{t \in U \colon \rho(t, \bar{t}) = 0\}$, where $\rho(t, \bar{t}) \in \mathbb{R}^d$ satisfies $\rho(0) = 0$ and $\partial\rho_1 \wedge \cdots \wedge \partial\rho_m \neq 0$ over $M \cap U$, by genericity of M. Of course, as a germ at 0 of a real analytic subset, $(M, 0)$ is independent of the choice of such defining equations. We can assume that the linear coordinates $t \in \mathbb{C}^n$ are chosen in order that $T_0M = \mathbb{C}^m \times \mathbb{R}^d$. We can also

expand $\rho(t,\bar t)=\sum_{\mu,\nu\in\mathbb{N}^n_*}\rho_{\mu,\nu}\,t^\mu\,\bar t^\nu\in\mathbb{C}\{t,\bar t\}^d$, where $\rho_{\mu,\nu}=\bar\rho_{\nu,\mu}\in\mathbb{C}^d$, $\forall\ \mu,\nu\in\mathbb{N}^n_*$ (*cf.* Lemma 3.4 below). Naturally, we shall assume that this d-vectorial power series converges *normally* in an open polydisc U through 0, a polydisc which we can (and will) assume to be "big" after an eventual dilatation of coordinates, say $U:=2^n\Delta^n=:\Delta^n_2$.

3.2. Segre varieties. Now, let us denote by t_p the coordinates of a fixed point $p\in\Delta^n$. Then the *Segre variety* $Q_p:=\{t\in\Delta^n_2\colon\rho(t,\bar t_p):=\sum_{\mu,\nu\in\mathbb{N}^n_*}\rho_{\mu,\nu}\,t^\mu\,\bar t_p^\nu=0\}$, obtained by a *polarization* of ρ and by a substitution of $\bar t$ by the constant $\bar t_p$, is a *connected smooth complex m-dimensional submanifold of* Δ^n_2, for all $\bar t_p\in\Delta^n$ (possibly after a supplementary dilatation of coordinates). It is well known that the definition of the Segre variety Q_p does not depend on the choice of local defining functions of M, that Segre varieties are biholomorphically invariant, *i.e.* $h(Q_p)=Q'_{h(p)}$ for every biholomorphism h of Δ^n_2 where $M':=h(M)$ and the $Q'_{p'}$ are associated with p', that $q\in Q_p$ iff $p\in Q_q$ and that $p\in Q_p$ iff $p\in M$. Usually, these properties are established thanks to a complexification of ρ as follows.

3.3. Complexification of the defining equations. To begin with, let us define the complexification $\rho(t,\bar t)^c:=\rho(t,\tau):=\sum_{\mu,\nu\in\mathbb{N}^n}\rho_{\mu,\nu}\,t^\mu\,\tau^\nu$, after replacing $\bar t$ by an independent variable τ in the series defining ρ and $\bar\rho(t,\tau):=\sum_{\mu,\nu\in\mathbb{N}^n}\bar\rho_{\mu,\nu}\,t^\mu\,\tau^\nu$, so $\overline{\rho(t,\tau)}=\bar\rho(\bar t,\bar\tau)$. We can think that τ is the "complexified variable of $\bar t$" and write $\tau=(\bar t)^c$ with superscript $(\bullet)^c$ for "complexified". There are also complexifications of $\mathcal{C}^\omega$ vector fields, of $\mathcal{C}^\omega$ differential forms. We write: $[\chi(t,\bar t)]^c=\chi(t,\tau)$, if χ is $\mathcal{C}^\omega$ and $[\sum_{j=1}^n a_j(t,\bar t)\partial/\partial t_j+\sum_{j=1}^n b_j(t,\bar t)\partial/\partial\bar t_j]^c:=\sum_{j=1}^n a_j(t,\tau)\partial/\partial t_j+\sum_{j=1}^n b_j(t,\tau)\partial/\partial\tau_j$, whence $(Lf)^c=L^cf^c$. Now, we can state the elementary reality properties from which follow easily the basic properties of Segre varieties quoted above.

Lemma 3.4. *As $\rho(t,\bar t)=\sum_{\mu,\nu\in\mathbb{N}^n}\rho_{\mu,\nu}\,t^\mu\,\bar t^\nu\in\mathbb{R}^d$ is $\mathcal{C}^\omega$ and real-valued, one has:*

- **(a)** $\bar\rho_{\mu,\nu}=\rho_{\nu,\mu}$, $\forall\ \mu,\nu\in\mathbb{N}^n_*$;
- **(b)** $\rho(t,\tau)\equiv\bar\rho(\tau,t)$;
- **(c)** $\rho(t,\tau)=0$ *if and only if* $\rho(\bar\tau,\bar t)=0$.

Proof. By reality, $\rho(t,\bar t)\equiv\bar\rho(\bar t,t)$, whence **(a)**. Clearly, **(a)** $\Rightarrow$ **(b)** $\Rightarrow$ **(c)**. □

3.5. The extrinsic complexification of M. We now introduce the most important object, thanks to which the geometrical aspects of the CR geometry of M will be clearly more appearant. To the complexification $\rho(t,\tau)$ of $\rho(t,\bar t)$ is canonically (and classically) associated an *extrinsic complexification* M^c of M defined by:

$$(M)^c:=\mathcal{M}:=\{(t,\tau)\in\Delta^{2n}\colon\rho(t,\tau)=0\}\subset\mathbb{C}^n_t\times\mathbb{C}^n_\tau,\tag{3.6}$$

which is a *connected complex $2m+d$-dimensional submanifold of* Δ^{2n}. As we can embed $\mathbb{C}^n$ in $\mathbb{C}^n_t\times\mathbb{C}^n_\tau$ to be the totally real plane, often called the *antidiagonal*, $\underline{\Lambda}:=\{(t,\tau)\in\mathbb{C}^{2n}\colon\tau=\bar t\}$, *i.e.* to be the manifold graph of the map $t\mapsto\bar t$, we can also embed M in $\underline{\Lambda}\subset\mathbb{C}^n_t\times\mathbb{C}^n_\tau$, as being $M=\{(t,\bar t)\in\Delta^n\times\Delta^n\}$, whence M appears to be a *maximally real submanifold of* $\mathcal{M}$ (*cf.* [BER1,2]). Finally, if $p\in M$, *i.e.* $t_p\in M$, we denote by p^c the point $(p,\bar p)$, *i.e.* with coordinates $(t_p,\bar t_p)\in\mathcal{M}$. Then $p^c=\pi_t^{-1}(\{p\})\cap\underline{\Lambda}$, where $\pi_t:\mathbb{C}^n_t\times\mathbb{C}^n_\tau\to\mathbb{C}^n_t$, is the projection $(t,\tau)\mapsto t$.

We also put $\pi_\tau : \mathbb{C}^n_t \times \mathbb{C}^n_\tau \to \mathbb{C}^n_\tau$, $(t,\tau) \mapsto \tau$, so that $p^c = \pi_\tau^{-1}(\{\bar{p}\}) \cap \underline{\Lambda}$. We will keep these notations throughout the article.

3.7. Antiholomorphic involution. Now, as in [MW], [BER], let us define the antiholomorphic self-map $\sigma(t,\tau) := (\bar{\tau},\bar{t})$ of $\mathbb{C}^{2n}$. Clearly, σ is involutive, *i.e.* $\sigma^2 = \mathrm{Id}$. Notice also that $\sigma(t,\bar{t}) = (t,\bar{t})$, which means that σ fixes $\underline{\Lambda}$ point by point, so in particular $\sigma|_M = I_M$. By Lemma 3.4 **(c)**, we have:

Lemma 3.8. *The complex manifold $\mathcal{M}$ is fixed by σ, i.e. $\sigma(\mathcal{M}) = \mathcal{M}$.*

In fact, σ appears to be the fundamental symmetry of $\mathcal{M}$ which corresponds by complexification to the complex conjugation $t \mapsto \bar{t}$ in $\mathbb{C}^n$. Furthermore, it is easy to observe that this invariance property of $\mathcal{M}$ under σ is a characterizing property of the fact that $\mathcal{M} = M^c$ is a complexification.

3.9. Characterization of complexifications of submanifolds. More generally, we have the following lemma, which we will need in §6.1. For its proof, which we leave as an (easy) exercise, one has to use Lemma 3.4 **(c)**.

Lemma 3.10. *There is a one-to-one correspondence between real analytic subsets $\Sigma \subset M$ and complex analytic subvarieties Σ_1 of $\mathcal{M}$ satisfying $\sigma(\Sigma_1) = \Sigma_1$ given by $\Sigma \mapsto \Sigma^c =: \Sigma_1$, with inverse $\Sigma_1 \mapsto \pi_t(\Sigma_1 \cap \underline{\Lambda}) =: \Sigma$. Furthermore, Σ is a smooth submanifold if and only if Σ_1 is.*

§4. Segre sets and iterated complexifications

After these preliminaries, we can present the definitions of higher order Segre varieties. Following [BER1,2], [Z1], let $Q^0_p := \{p\}$, where $p \in \Delta^n$, let $Q^1_p := \bigcup_{q \in Q^0_p} Q_q \cap \Delta^n$, $Q^2_p := \bigcup_{q \in Q^1_p} Q_q \cap \Delta^n$ and $Q^{k+1}_p := \bigcup_{q \in Q^k_p} Q_q \cap \Delta^n$. Then Q^k_p is called the *k-th Segre set*. We would like to mention that in these definitions, some accuracy about sets upon which unions are taken is necessary, *cf.* [BER2], but we shall not enter into the details before §7 where we introduce the sightly different notion of *Segre chains*, from which we shall recover these Segre sets. An analytical definition using the defining equations $\rho_j(t,\bar{t}) = 0$, $1 \leq j \leq d$, of M, is presented as follows by these authors. Consider the *iterated complexifications* $\mathcal{M}_k$, $k \in \mathbb{N}$, of M (introduced by Zaitsev) to be the sets $\mathcal{M}_{2j}(p) := \{(p,\tau_1,t_1,\tau_2,t_2,\dots,t_{j-1},\tau_j) \in \Delta^{2jn} \colon \rho(p,\zeta_1) = \cdots = \rho(t_j,\tau_j) = 0,\ \rho(t_1,\tau_2) = \cdots = \rho(t_{j-1},\tau_j) = 0\}$ for $k = 2j$ and a similar definition for $\mathcal{M}_{2j+1}$, which are both *complex manifolds*. Using Lemma 3.4 *passim*, it is easy to see that the Segre sets are projections of the $\mathcal{M}_k$'s over their last $\mathbb{C}^n$ factor. Up to now, the Segre sets are just sets and they have few structure, but using the above projection, one sees that the Segre sets are the images of certain holomorphic maps v^k with source $\mathbb{C}^{mk}$. These maps can be described in coordinates as follows. Assume that the d scalar equations of M are given by $z_j = \bar{Q}_j(w,\zeta,\xi)$ in the notations of §5.6 below, where $\bar{Q}_j(w,\zeta,\xi) := \xi + i\bar{\Theta}(w,\zeta,\xi)$ and let $Q(w,\zeta,\xi)$ denote the $\mathbb{C}^d$-valued power series whose components are the $Q_j(w,\zeta,\xi)$. Notice that we do not assume that the coordinates are regular, *i.e.* that $\bar{Q}(w,0,\xi) \equiv 0$. Here, we follow the terminology of Ebenfelt and use the words "regular coordinates", instead of "normal coordinates". Following the presentation of [BER2,3], we can write for each integer $k \geq 1$ these holomorphic mappings $v^k \colon (\mathbb{C}^{mk},0) \to (\mathbb{C}^n,0)$ as follows. For $k = 0,1$, we define $v^0 := (0,0) \in \mathbb{C}^m \times \mathbb{C}^d$

and $v^1(w_1) := (w_1, \bar{Q}(w_1, 0, 0))$. Then $v^2(w_1, w_2) := (w_1, \bar{Q}(w_1, w_2, Q(w_2, 0, 0)))$ and $v^3(w_1, w_2, w_3) := (w_1, \bar{Q}(w_1, w_2, Q(w_2, w_3, \bar{Q}(w_3, 0, 0))))$. Furthermore, for $k = 2j, 2j+1$, where $j \geq 1$, define:

$$\begin{cases} v^{2j}(w_1, \dots, w_{2j}) := \big(w_1, \bar{Q}(w_1, w_2, Q(w_2, w_3, \bar{Q}(w_3, w_4, \dots \\ \qquad \dots, Q(w_{2j-2}, w_{2j-1}, \bar{Q}(w_{2j-1}, w_{2j}, Q(w_{2j}, 0, 0))) \dots)))\big), \\ v^{2j+1}(w_1, \dots, w_{2j+1}) := \big(w_1, \bar{Q}(w_1, w_2, Q(w_2, w_3, \bar{Q}(w_3, w_4, \dots \\ \qquad \dots, \bar{Q}(w_{2j-1}, w_{2j}, Q(w_{2j}, w_{2j+1}, \bar{Q}(w_{2j+1}, 0, 0))) \dots)))\big), \end{cases} \tag{4.1}$$

In this definition, the principle of iteration comes from the set-theoretic definition of the Q_k^p's. Whereas the geometric and set-theoretic definition of the Q_k^p's needs convergent defining equations $\rho_j(t, \bar{t})$, the maps v^k in eqs. (4.2) have sense both in the category of convergent and of formal mappings. In this article, we shall introduce a new geometric point of view on Segre varieties and as a consequence, we shall modify and geometrize the definitions of new maps $\psi^{2j} := \pi_\tau \circ \Gamma_{2j}$ and $\psi^{2j+1} := \pi_t \circ \Gamma_{2j+1}$ (*see* eq. (6.4) and eq. (7.9)), which are equal to the maps v^k up to a linear reparametrization with integer coefficients and up to complex conjugation.

§5. Segre varieties and conjugate Segre varieties.

5.1. Definitions and basic properties. Before entering into the geometry of the complexification, we can define the fundamental pair of *Segre varieties* and *conjugate Segre varieties*. As explained in §2.4, it is necessary to study *simultaneously* Segre varieties together with conjugate Segre varieties. Of course, there will be no real novelty with the properties of the second family. In fact, the main novelty of the point of view only lies in the simultaneous consideration of *both*. Thus, the Segre variety, usually denoted by $Q_p = \{t \in \Delta^n \colon \rho(t, \bar{t}_p) = 0\}$, will be denoted *in the remainder of this article* by

$$S_{\bar{t}_p} := \{t \in \Delta^n \colon \rho(t, \bar{t}_p) = 0\}. \tag{5.2}$$

We stress the notation $S_{\bar{t}_p}$ and not "S_{t_p}", nor "S_p", with the *bar of complex conjugation over t_p in the index*, as in the expression $\rho(t, \bar{t}_p)$. We could also have used the notation $S_{\bar{p}}$. In fact, this notation will be consistent with the following (new) definition of *conjugate Segre varieties*:

$$\overline{S}_{t_p} := \{\bar{t} \in \Delta^n \colon \rho(t_p, \bar{t}) = 0\}. \tag{5.3}$$

Indeed, thanks to Lemma 3.4 **(c)**, we clearly have $\overline{S}_{t_p} = \overline{S_{\bar{t}_p}} = \overline{S}_{\bar{\bar{t}}_p} = \overline{S}_{t_p}$, which shows the consistency of our notation, if, as usual, the set $\overline{E}$ denotes $\{\bar{t} \in \Delta^n \colon t \in E\}$ for an arbitrary set $E \subset \Delta^n$. Also, we have $\bar{t} \in \overline{S}_{t_p}$ if and only if $t \in S_{\bar{t}_p}$, *etc.* It is easy to check that the Segre and conjugate Segre varieties are complex analytic submanifolds closed and connected in $U = \Delta_2^n$, if $t_p \in \Delta^n$, and that they are independent of the choice of a $\mathcal{C}^\omega$ defining function for M. The $S_{\bar{t}_p}$ and the $\overline{S}_{t_p}$ are, moreover, biholomorphically invariant. We have indeed $h(S_{\bar{t}_p}) = S_{\bar{h}(\bar{t}_p)}$ and $\bar{h}(\overline{S}_{t_p}) = \overline{S}'_{h(t_p)}$, where h is as in §3.2. Now, as we suggested to duplicate

Segre varieties and sets for completeness, let us employ the notations $S_{\bar{t}_p}$ instead of Q_p. To mimic and duplicate the set-theoretic definitions given in §4 above, we can define: $S^0_{\bar{t}_p} := \{\bar{t}_p\}$, and $S^1_{\bar{t}_p} := S_{\bar{t}_p} = \bigcup_{\bar{t}\in S^0_{\bar{t}_p}} S_{\bar{t}}$, $S^2_{\bar{t}_p} = \bigcup_{t\in S^1_{\bar{t}_p}} \overline{S}_t$, and then inductively, for $j \in \mathbb{N}_*$, $S^{2j}_{\bar{t}_p} = \bigcup_{t\in S^{2j-1}_{\bar{t}_p}} \overline{S}_t$ and $S^{2j+1}_{\bar{t}_p} = \bigcup_{\bar{t}\in S^{2j}_{\bar{t}_p}} S_{\bar{t}}$. On the other hand, we can define $\overline{S}^0_{t_p} := \{t_p\}$, and $\overline{S}^1_{t_p} := \overline{S}_t = \bigcup_{t\in\overline{S}^0_{t_p}} \overline{S}_t$, $\overline{S}^2_{t_p} := \bigcup_{\bar{t}\in\overline{S}^1_{t_p}} S_{\bar{t}}$, and inductively, for $j \in \mathbb{N}_*$, $\overline{S}^{2j}_{t_p} := \bigcup_{\bar{t}\in\overline{S}^{2j-1}_{t_p}} S_{\bar{t}}$, and $\overline{S}^{2j+1}_{t_p} = \bigcup_{t\in\overline{S}^{2j}_{t_p}} \overline{S}_t$. As we shall introduce a more geometrical point of view in the complexification, these definitions are not very important for us. We just mention that this definition of Segre sets $S^k_{\bar{t}_p}$'s clearly coincides with the definition of the Q^k_p's and that we have the following elementary properties:

(1) $\overline{S^k_{\bar{t}_p}} = \overline{S}^k_{t_p}$ and $S^k_{\bar{t}_p} = \overline{\overline{S}^k_{t_p}}$, $k \in \mathbb{N}$.

(2) $\overline{h}(S^{2j}_{\bar{t}_p}) = S'^{2j}_{\overline{h}(\bar{t}_p)}$, $h(S^{2j+1}_{\bar{t}_p}) = S'^{2j+1}_{\overline{h}(\bar{t}_p)}$, $h(\overline{S}^{2j}_{t_p}) = \overline{S}'^{2j}_{h(t_p)}$, $\overline{h}(\overline{S}^{2j+1}_{t_p}) = \overline{S}'^{2j+1}_{h(t_p)}$.

Here, $h\colon M \to M'$ is a local biholomorphism. In principle, the above unions should be taken over $t \in S^k_p \cap (\delta\Delta)^n$, where $\delta > 0$ is sufficiently small and $k \leq 3n$, to be equivalent with the definition that we will give in §7.8 below after having analyzed the geometry of the complexification.

5.4. Complexications of Segre varieties. Here begins the main topic of the article: the appearance of *geometry in the extrinsic complexification*. We shall first establish our "observational" Theorem 2.10. So, in the sequel, we shall work *exclusively* in the complexification $\mathcal{M} \subset \mathbb{C}^n_t \times \mathbb{C}^n_\tau$. As in §2.9, let us define here two very important objects:

$$\begin{cases} \mathcal{S}_{\tau_p} := (S_{\bar{t}_p})^c := \{(t,\tau) \in \Delta^n \times \Delta^n \colon \tau = \tau_p,\ \rho(t,\tau_p) = 0\}, \\ \underline{\mathcal{S}}_{t_p} := (\overline{S}_{t_p})^c := \{(t,\tau) \in \Delta^n \times \Delta^n \colon t = t_p,\ \rho(t_p,\tau) = 0\}. \end{cases} \tag{5.5}$$

Here, $\tau_p \in \Delta^n$ and $t_p \in \Delta^n$ are both fixed points. These two m-dimensional complex submanifolds will be called *complexified Segre varieties* and *conjugate complexified Segre varieties*. Such a terminology is consistent with our previous definitions in §5.1, as shows an examination of eqs. (5.2) (5.3) (5.5). Both these two families $\mathcal{S}_{\tau_p}$ and $\underline{\mathcal{S}}_{t_p}$ are in fact contained in $\mathcal{M}$. Using Lemma 3.4 **(c)**, it is easy to check that $\sigma(\mathcal{S}_{\tau_p}) = \underline{\mathcal{S}}_{\bar{\tau}_p}$ and $\sigma(\underline{\mathcal{S}}_{t_p}) = \mathcal{S}_{\bar{t}_p}$, which shows a fundamental symmetry property, simply coming from the complexification of the relation $\overline{S_{\bar{t}_p}} = \overline{S}_{t_p}$. A particular case in eqs. (5.5) is when the fixed points τ_p (or t_p) belongs to M. Now, to check Theorem 2.10, we need the following precisions. We need explicit defining equations.

5.6. Coordinates. As usual, there exist holomorphic coordinates vanishing at $p \in M$, (w,z), where $w = (w_1,\dots,w_m)$ and $z = (z_1,\dots,z_d)$, $z = x + iy$, such that $T^c_0 M = \mathbb{C}^m_w \times \{0\}$, $T_0 M = \mathbb{C}^m_w \times \mathbb{R}^d_x$ and some d *real* equations of M are given by $y = h(w,\bar{w},x)$, where $h = \sum_{\alpha,\beta\in\mathbb{N}^m, k\in\mathbb{N}^d} h_{k,\beta,\alpha}\, x^k w^\beta \bar{w}^\alpha$, $h(0) = 0$, $dh(0) = 0$, $h_{k,\beta,\alpha} \in \mathbb{C}^d$, $\overline{h_{k,\beta,\alpha}} = h_{k,\alpha,\beta}$, $||h_{k,\beta,\alpha}|| \leq c\, 2^{-(|\alpha|+|\beta|+|k|)}$, $c > 0$ (after dilatation). If we replace $y = (z-\bar{z})/2i$, $x = (z+\bar{z})/2$ and solve in z or in $\bar{z}$ using the analytic implicit function theorem, we obtain as new equivalent equations for M and for its

extrinsic complexification:

$$\begin{cases} M: \quad z = \bar{z} + i\bar{\Theta}(w, \bar{w}, \bar{z}) \qquad \text{or} \qquad \bar{z} = z - i\Theta(\bar{w}, w, z), \\ \mathcal{M}: \quad z = \xi + i\bar{\Theta}(w, \zeta, \xi) \qquad \text{or} \qquad \xi = z - i\Theta(\zeta, w, z), \end{cases} \tag{5.7}$$

where $\Theta(\zeta, t) = \sum_{\beta \in \mathbb{N}^m, \mu \in \mathbb{N}^n} \theta_{\beta,\mu} \zeta^\beta t^\mu$, with $||\theta_{\beta,\mu}|| \leq c\, 2^{-(|\beta|+|\mu|)}$, $c > 0$. It is this explicit representation which will help us to define the invariant objects of Theorem 2.10. Sometimes we shall use eqs. (5.7) and sometimes we shall set $\bar{Q}(w, \bar{w}, \bar{z}) := \bar{z} + i\bar{\Theta}(w, \bar{w}, \bar{z})$, whence the equations of $\mathcal{M}$ should be written in a shorter way by $z = \bar{Q}(w, \tau)$ or equivalently by $\xi = Q(\zeta, t)$. Finally, we have the following relation

$$\Theta(\zeta, w, \xi + i\bar{\Theta}(w, \zeta, \xi)) \equiv \bar{\Theta}(w, \zeta, \xi), \tag{5.8}$$

which follows from the comparison of the two equivalent equations of $\mathcal{M}$. It is easy to see that conversely, to a complex analytic d-vectorial series Θ satisfying eq. (5.8) is associated a unique *real* analytic CR-generic manifold M, and indeed a series $h(w, \bar{w}, x)$, such that M^c is given by eq. (5.7) (this point is proved in the book [BER2], §4.2, but we shall not need it). Very important geometric objects are the CR vector fields.

5.9. CR (1,0) and anti-CR (0,1) vector fields. These *pairs* of m-vector fields and their complexifications with holomorphic coefficients are given in vectorial notation by:

$$\begin{cases} M: \quad L = \dfrac{\partial}{\partial w} + i\bar{\Theta}_w(w, \bar{w}, \bar{z})\dfrac{\partial}{\partial z} \qquad \text{and} \qquad \bar{L} = \dfrac{\partial}{\partial \bar{w}} - i\Theta_{\bar{w}}(\bar{w}, w, z)\dfrac{\partial}{\partial \bar{z}} \\ \mathcal{M}: \quad \mathcal{L} = \dfrac{\partial}{\partial w} + i\bar{\Theta}_w(w, \zeta, \xi)\,\dfrac{\partial}{\partial z} \qquad \text{and} \qquad \underline{\mathcal{L}} = \dfrac{\partial}{\partial \zeta} - i\Theta_\zeta(\zeta, w, z)\dfrac{\partial}{\partial \xi}. \end{cases} \tag{5.10}$$

In fact, these notations stand for $\mathcal{L} = (\mathcal{L}^1, \dots, \mathcal{L}^m)$ and $\underline{\mathcal{L}} = (\underline{\mathcal{L}}^1, \dots, \underline{\mathcal{L}}^m)$. Clearly, the second row is the complexification of the first. In other words, $L^c = \mathcal{L}$ and $\bar{L}^c = \underline{\mathcal{L}}$. Here, we drop the indices, for the reason that the behaviour of these (obviously) commuting $\mathcal{L}^j$ and commuting (too) $\underline{\mathcal{L}}^j$ is formally analogous to the behaviour of a single vector field. In fact, Frobenius integrable m-dimensional distributions behave like simple vector fields. We shall call them *m-vector fields*. We denote by $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\underline{\mathcal{L}}}$ the two flow foliations that they induce. Of course, the distribution defined by the pair $\{\mathcal{L}, \underline{\mathcal{L}}\}$ is *not* in general Frobenius integrable, unless M is Levi-flat. This is why Lie brackets and orbits come in consideration. At this subject, we shall need the following Lemma 5.10 below in §6.5. Granted the equivalences **(b)** $\Longleftrightarrow$ **(c)** and **(d)** $\Longleftrightarrow$ **(e)** which are explained in [Sus] (using Chow's theorem and Nagano's theorem) and that we shall admit, the following lemma is immediate:

Lemma 5.11. *Let $p \in M$. Then the following properties are equivalent*:

- **(a)** $\mathrm{Lie}_p\ (T^{1,0}M, T^{0,1}M) = \mathbb{C} \otimes T_pM$.
- **(b)** $\mathrm{Lie}_p\ (T^cM) = T_pM$.
- **(c)** (M, p) *is T^cM-orbit-minimal.*
- **(d)** $\mathrm{Lie}_{p^c}\ (\mathcal{L}, \underline{\mathcal{L}}) = T_{p^c}\mathcal{M}$.
- **(e)** $(\mathcal{M}, p^c)$ *is $\{\mathcal{L}, \underline{\mathcal{L}}\}$-orbit-minimal.* $\square$

Indeed, the only remaining equivalence to be checked: **(b)** $\Longleftrightarrow$ **(d)**, is trivial, because the complexification of an $\{L, \bar{L}\}$-Lie bracket of arbitrary length is the corresponding $\{\mathcal{L}, \underline{\mathcal{L}}\}$-Lie bracket of the complexified vector fields.

5.12. Complexified Segre varieties. Now, we achieve the proof of Theorem 2.10. At first, we observe that the two Segre varieties and conjugate Segre varieties, which can be rewritten as

$$S_{\bar{t}_p}: \ z = \bar{z}_p + i\bar{\Theta}(w, \bar{w}_p, \bar{z}_p) \qquad \text{and} \qquad \overline{S}_{t_p}: \ \bar{z} = z_p - i\Theta(\bar{w}_p, w_p, z_p) \tag{5.13}$$

admit two different complexifications in $\mathcal{M}$, $\mathcal{S}_{\tau_p}$ and $\mathcal{S}_{t_p}$, which now can be rewritten in terms of our choice of coordinates as

$$\begin{cases} \mathcal{S}_{\tau_p} = \mathcal{S}_{\zeta_p, \xi_p}: \quad \zeta = \zeta_p, \ \xi = \xi_p, \ z = \xi_p + i\bar{\Theta}(w, \zeta_p, \xi_p) \\ \underline{\mathcal{S}}_{t_p} = \underline{\mathcal{S}}_{w_p, z_p}: \quad w = w_p, \ z = z_p, \ \xi = z_p - i\Theta(\zeta, w_p, z_p). \end{cases} \qquad \text{and} \tag{5.14}$$

It is well known that the first vector fields L in (5.10) are tangent to the (classical) Segre varieties $S_{\bar{t}_p}$, whence $\mathcal{L}$ also is tangent to the complexification $\mathcal{S}_{\tau_p}$ (*cf.* [DW]). Indeed, one verifies immediately that $\mathcal{L}\mathcal{S}_{\tau_p} \equiv 0$. A similar relation holds between $\bar{L}$ and $\overline{S}_{t_p}$ and between $\underline{\mathcal{L}}$ and $\underline{\mathcal{S}}_{t_p}$, that is to say $\underline{\mathcal{L}}\underline{\mathcal{S}}_{t_p} \equiv 0$. More precisely, thanks to this complexification and for dimensional reasons, an inspection of eqs. (5.7), (5.10), (5.13) and (5.14) readily shows the following:

1. The $\mathcal{S}_{\tau_p}$ and the $\underline{\mathcal{S}}_{t_p}$ form *families of integral complex analytic manifolds for the m-vector fields $\mathcal{L}$ and $\underline{\mathcal{L}}$* respectively.
2. The $\mathcal{S}_{\tau_p}$ are the leaves of the flow foliation $\mathcal{F}_{\mathcal{L}}$ of $\mathcal{M}$ by $\mathcal{L}$ and the $\underline{\mathcal{S}}_{t_p}$ are the leaves of the flow foliation $\mathcal{F}_{\underline{\mathcal{L}}}$ of $\mathcal{M}$ by $\underline{\mathcal{L}}$.
3. Finally, we clearly have $\mathcal{S}_{\tau_p} = \mathcal{M} \cap \pi_\tau^{-1}(\tau_p)$ and $\underline{\mathcal{S}}_{t_p} = \mathcal{M} \cap \pi_t^{-1}(t_p)$. Equivalently, the complexified Segre varieties are simply the intersection of $\mathcal{M}$ with the coordinate planes.

In conclusion, Theorem 2.10 is established. There remain only elementary details to be checked and for which enough precisions are already given in the text. $\square$

Remark. Local foliations by Segre varieties in ambient space have been introduced by Sharipov and Sukhov for interesting application to the algebraic regularity mapping problem as follows (*see* [SS], [CMS,§4]). If H is a d-dimensional complex manifold through the origin with $T_0H \oplus T_0^cM = T_0\mathbb{C}^n$, it is easy to see that $\bigcup_{q \in H} Q_q$ makes a holomorphic foliation $\mathcal{F}_H$ by the m-dimensional complex leaves Q_q of a neighborhood of 0 in $\mathbb{C}^n$, but this foliation depends on the choice of H. On the other hand, the double foliation $\{\mathcal{F}_{\mathcal{L}}, \mathcal{F}_{\underline{\mathcal{L}}}\}$ is intrinsic and biholomorphically invariant. Finally, we mention that in codimension $d \leq 2$, minimality of M at p is equivalent to Segre transversality of M at p, *see* [M2].

§6. Complexification of orbits of CR vector fields on M

6.1. Flows. In this paragraph, we now begin the (slightly longer) proof of Theorems 2.3 and 2.6. Let $w = (w^1, \dots, w^m) \in \mathbb{C}^m$ and $p^c \in \mathcal{M}$. We will denote by $\mathcal{L}_w(p^c)$ the composition of flows $\mathcal{L}^m_{w^m} \circ \cdots \circ \mathcal{L}^1_{w^1}(p^c)$ alluded to in §2.2. In fact,

for every permutation $\varpi : [\![1,m]\!] \to [\![1,m]\!]$, we have $\mathcal{L}^{\varpi(m)}_{w_{\varpi(m)}} \circ \cdots \circ \mathcal{L}^{\varpi(1)}_{w_{\varpi(1)}}(p^c) = \mathcal{L}^m_{w_m} \circ \cdots \circ \mathcal{L}^1_{w_1}(p^c)$, since the $\mathcal{L}^j$'s commute. The vectorial m-flows of $\mathcal{L}$ and of $\underline{\mathcal{L}}$ on $\mathcal{M}$ are simply given in coordinates by the following relations:

$$\begin{cases} \mathcal{L}_w(w_p,\, \bar{z}_p + i\bar{\Theta}(w_p, \bar{w}_p, \bar{z}_p),\, \bar{w}_p,\, \bar{z}_p) = \\ \qquad\qquad = \big(w_p + w,\, \bar{z}_p + i\bar{\Theta}(w_p + w, \bar{w}_p, \bar{z}_p),\, \bar{w}_p,\, \bar{z}_p\big)\,, \\ \underline{\mathcal{L}}_{\bar{w}}(w_p,\, z_p,\, \bar{w}_p,\, z_p - i\Theta(\bar{w}_p, w_p, z_p)) = \\ \qquad\qquad = (w_p,\, z_p,\, \bar{w} + \bar{w}_p,\, z_p - i\Theta(\bar{w} + \bar{w}_p,\, w_p,\, z_p))\,. \end{cases} \tag{6.2}$$

Applying σ to the eqs. (6.2), we get $\sigma(\mathcal{L}_w(p^c)) = \underline{\mathcal{L}}_{\bar{w}}(p^c)$ and $\sigma(\underline{\mathcal{L}}_\zeta(p^c)) = \mathcal{L}_{\bar{\zeta}}(p^c)$. We shall study closely the concatenated flow maps $\Gamma_k(w_{(k)})$ defined in §2.2. To understand them formally, let us write down these first three concatenated flows maps at $p^c = 0$, after choosing the (shorter) notation $\xi = Q(\zeta, t)$ instead of $\xi = z - i\Theta(\zeta, t)$ for the equations of $\mathcal{M}$. We express the right-hand sides in (w, z, ζ, τ) coordinates. Using (6.2), we can write these three first concatenated flow:

$$\begin{cases} \mathcal{L}_{w_1}(0) = \big(w_1,\, \bar{Q}(w_1, 0, 0),\, 0,\, 0\big)\,. \\ \underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(0)) = \big(w_1,\, \bar{Q}(w_1, 0, 0),\, w_2,\, Q(w_2, w_1, \bar{Q}(w_1, 0, 0))\big)\,. \\ \mathcal{L}_{w_3}(\underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(0)))) = \big(w_1 + w_3,\, \bar{Q}(w_1 + w_3, w_2, Q(w_2, w_1, \bar{Q}(w_1, 0, 0))), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , w_2,\, Q(w_2, w_1, \bar{Q}(w_1, 0, 0))\big)\,. \end{cases} \tag{6.3}$$

It is easy to check that the following relations hold between the Γ_k and the v^k:

$$\begin{cases} v^1(w_1) = \pi_t(\Gamma_1(w_1)), \\ \bar{v}^2(w_2, w_1) = \pi_\tau(\Gamma_2(w_1, w_2)), \\ v^3(w_3 + w_1,\, w_2, w_1) = \pi_t(\Gamma_3(w_1, w_2, w_3)) \\ \bar{v}^4(w_4 + w_2,\, w_3 + w_1,\, w_2, w_1) = \pi_\tau(\Gamma_4(w_1, w_2, w_3, w_4)), \\ v^5(w_5 + w_3 + w_1,\, w_4 + w_2,\, w_3 + w_1,\, w_2, w_1) = \\ \qquad\qquad\qquad\qquad\qquad\qquad = \pi_t(\Gamma_5(w_1, w_2, w_3, w_4, w_5)), \end{cases} \tag{6.4}$$

etc., which shows that the maps v^k (formal definition) and Γ_k (set-theoretic definition) coincide, up to a reparametrization with integer coefficients and up to complex conjugation. Let us now establish Theorem 2.3 **(c)**.

Proposition 6.5. *There is a one-to-one correspondence between the CR orbits and their extrinsic complexifications*:

(1) $\mathcal{O}_{CR}(M, p)^c = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$, *and*
(2) $\mathcal{O}_{CR}(M, p) = \pi_t(\underline{\Lambda} \cap \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c))$.

Proof. By the Orbit Theorem of §1.1, $\mathcal{O}_{CR}(M, p)$ is a $\mathcal{C}^\omega$ closed submanifold of M through p. Thus, let $\mathcal{O}$ be a small open connected manifold-piece of $\mathcal{O}_{CR}(M, p)$ through p, and let $\mathcal{O}^c$ be its extrinsic complexification. Because $L|_{\mathcal{O}}$ and $\bar{L}|_{\mathcal{O}}$ are tangent to $\mathcal{O}$, the *generic uniqueness principle* (*via* $\mathcal{O} \subset \underline{\Lambda}$, where $\underline{\Lambda}$ is maximally real) entails that $\mathcal{L}|_{\mathcal{O}^c}$ and $\underline{\mathcal{L}}|_{\mathcal{O}^c}$ are tangent to $\mathcal{O}^c$. Therefore $\mathcal{O}^c$ is an integral

manifold for $\{\mathcal{L}, \underline{\mathcal{L}}\}$ through p^c, whence $\mathcal{O}^c \supset \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$, since a characterizing property of the orbit $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$ is to say that it is the *smallest integral manifold-piece* for $\{\mathcal{L}, \underline{\mathcal{L}}\}$ through p^c.

Conversely, Let $\mathcal{N}$ be a manifold-piece of $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$ through p^c. We have just shown that $\mathcal{N} \subset \mathcal{O}^c$, hence to finish the proof, we want to show that $\mathcal{N} \supset \mathcal{O}^c$. We claim that we have $\sigma(\mathcal{N}) = \mathcal{N}$ as germs at p^c. Indeed, By definition, the orbit is the following set of endpoints of concatenations of flows of $\mathcal{L}$ and of flows of $\underline{\mathcal{L}}$ (notice that because $\mathcal{L}_{w_2} \circ \mathcal{L}_{w_1} = \mathcal{L}_{w_1+w_2}$ and $\underline{\mathcal{L}}_{\zeta_2} \circ \underline{\mathcal{L}}_{\zeta_1} = \underline{\mathcal{L}}_{\zeta_1+\zeta_2}$ but $\mathcal{L}$ and $\underline{\mathcal{L}}$ do not commute, there can be only two different kinds of concatenated flow maps ; we do not use the abbreviated notation Γ_k here) :

$$
\begin{cases}
\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c) = \{\mathcal{L}_{w_k} \circ \cdots \circ \mathcal{L}_{w_2} \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p^c) : \\
\qquad\qquad : \ w_1, \zeta_1, w_2, \ldots, w_k \in \mathbb{C} \text{ small}, k \in \mathcal{N}_*\} \bigcup \\
\qquad \bigcup \{\underline{\mathcal{L}}_{\zeta_k} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_2} \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p^c) : \\
\qquad\qquad : \zeta_1, w_1, \zeta_2, \ldots, \zeta_k \in \mathbb{C} \text{ small}, k \in \mathbb{N}_*\} := E \cup F.
\end{cases}
\tag{6.6}
$$

Now an examination of eqs. (6.2) shows that we have $\sigma(\mathcal{L}_w(q)) = \underline{\mathcal{L}}_{\bar{w}}(\sigma(q))$ and $\sigma(\underline{\mathcal{L}}_\zeta(q)) = \mathcal{L}_{\bar{\zeta}}(\sigma(q))$, for each $q \in \mathcal{M}$. Consequently :

$$
\begin{cases}
\sigma(\mathcal{L}_{w_k} \circ \underline{\mathcal{L}}_{\zeta_{k-1}} \circ \cdots \circ \mathcal{L}_{w_1}(p^c)) = \underline{\mathcal{L}}_{\bar{w}_k}(\sigma(\underline{\mathcal{L}}_{\zeta_{k-1}} \circ \cdots \circ \mathcal{L}_{w_1}(p^c))) = \\
= \underline{\mathcal{L}}_{\bar{w}_k} \circ \mathcal{L}_{\bar{\zeta}_{k-1}} \circ \cdots \circ \sigma(\mathcal{L}_{w_1}(p^c)) = \underline{\mathcal{L}}_{\bar{w}_k} \circ \mathcal{L}_{\bar{\zeta}_{k-1}} \circ \cdots \circ \underline{\mathcal{L}}_{\bar{w}_1}(p^c),
\end{cases}
\tag{6.7}
$$

since $\sigma(p^c) = p^c$. This proves $F = \sigma(E)$, hence $\sigma(\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)) = \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$, so we have $\sigma(\mathcal{N}) = \mathcal{N}$ as announced.

By the theorem of Nagano, $\mathcal{N}$ is smooth at p^c and satisfies $\sigma(\mathcal{N}) = \mathcal{N}$. By Lemma 3.10, there exists $N := \pi_t(\mathcal{N} \cap \underline{\Lambda})$ a unique germ at p of a $\mathcal{C}^\omega$ submanifold $N \subset M$ such that $N^c = \mathcal{N}$.

et us denote $\mathcal{N} = \{\rho(t, \tau) = 0, \chi(t, \tau) = 0\}$, so that $N = \{\rho(t, \bar{t}) = 0, \chi(t, \bar{t}) = 0\}$. Then $\mathcal{L}\rho = 0$, $\underline{\mathcal{L}}\rho = 0$, $\mathcal{L}\chi = 0$, $\underline{\mathcal{L}}\chi = 0$ on $\{\rho = \chi = 0\}$, since $\mathcal{N}$ is an $\{\mathcal{L}, \underline{\mathcal{L}}\}$-integral manifold. Therefore, after restriction to $\{\tau = \bar{t}\} = \underline{\Lambda}$, we have $L\rho = 0$, $\bar{L}\rho = 0$, $L\chi = 0$ and $\bar{L}\chi = 0$ on $\{\rho(t, \bar{t}) = 0, \chi(t, \bar{t}) = 0\} = N$, so that N is an $\{L, \bar{L}\}$-integral manifold. Thus by the minimality property of CR-orbits, we have $N \supset \mathcal{O}$ as germs at p. By complexifying, we get $\mathcal{N} \supset \mathcal{O}^c$, as desired. □

In conclusion, Theorem 2.3 is established. It remains only to check **(a)** and **(b)**, for which enough precisions are already given in the text. □

6.8. Summary. Of course, it follows at once from Proposition 6.5 that M is $\{L, \bar{L}\}$-minimal at p *if and only if* $\mathcal{M}$ is $\{\mathcal{L}, \underline{\mathcal{L}}\}$-minimal at p^c. This gives the equivalence **(d)** $\Longleftrightarrow$ **(e)** of Theorem 2.6. It remains to establish the remaining equivalences. Motivated by the construction of Segre sets by Baouendi-Ebenfelt-Rothschild, we shall introduce some special notation for the sets appearing in eq. (6.6), and call them *Segre k-chains*. A preliminary presentation of them was given in §2.2. Crucially, in their definition, *the Segre k-th chains will be endowed with holomorphic maps Γ_k coming from their geometric definition.*

§7. Complexified Segre k-chains and Segre sets

7.1. Segre chains as k-th orbits chains of vector fields. At first, we come back to the concatenated flow maps in eqs. (6.6) to analyze them. Looking at eqs. (5.14) and (6.2), we see that the complexified Segre varieties of a point $p^c \in \mathcal{M} \cap \underline{\Lambda}$, can be defined by $\mathcal{S}_{\tau_p} := \{\mathcal{L}_w(p^c) \in \Delta^{2n} \colon w \in \Delta^m\}$ and $\underline{\mathcal{S}}_{t_p} := \{\underline{\mathcal{L}}_\zeta(p^c) \in \Delta^{2n} \colon \zeta \in \Delta^m\}$. At order $k = 2$, we can define:

$$\begin{cases} \mathcal{S}^2_{\tau_p} = \{\underline{\mathcal{L}}_{\zeta_1}(\mathcal{L}_{w_1}(p^c)) \in \Delta^{2n} \colon w_1, \zeta_1 \in \delta\Delta^m\}, \\ \underline{\mathcal{S}}^2_{t_p} = \{\mathcal{L}_{w_1}(\underline{\mathcal{L}}_{\zeta_1}(p^c)) \in \Delta^{2n} \colon \zeta_1, w_1 \in \delta\Delta^m\}. \end{cases} \tag{7.2}$$

Here, $\delta > 0$ will be chosen in a while. More generally, let us define the $\mathcal{S}^k_{\tau_p}$ and $\underline{\mathcal{S}}^k_{t_p}$. We recall at first that, because only two "starting actions" $\mathcal{L}_{w_1}(p^c)$ and $\underline{\mathcal{L}}_{\zeta_1}(p^c)$ can make a difference in a concatenation of flows of $\mathcal{L}$ and of $\underline{\mathcal{L}}$, *there can exist only two different families of Segre k-chains.* We recall also that we have abbreviated in §2.2 by $w_{(k)} \mapsto \Gamma_k(w_{(k)})$ (resp. by $w_{(k)} \mapsto \underline{\Gamma}_k(w_{(k)})$) the concatenated flow maps starting by $\mathcal{L}_{w_1}$ (resp. starting by $\underline{\mathcal{L}}_{w_1}$). Here, Γ_k and $\underline{\Gamma}_k$ depend on p^c, but as we shall essentially fix p^c in a while, we shall not introduce a specific notational index for p^c in Γ_k and $\underline{\Gamma}_k$. Further, we have observed the relation $\sigma(\Gamma_k(w_{(k)})) = \underline{\Gamma}_k(\overline{w_{(k)}})$. Now, since we are interested in local considerations, let us choose $\delta > 0$ such that, for all $k \leq 3n$, all $w_{(k)} \in (\delta\Delta^m)^k$ and all $p \in ((\frac{1}{2}\Delta)^n \times (\frac{1}{2}\Delta)^n)) \cap \mathcal{M}$, then $\Gamma_k(w_{(k)}) \in \mathcal{M}$ and $\underline{\Gamma}_k(w_{(k)}) \in \mathcal{M}$ too. Thus, to be explicit, the *Segre k-chains* of $p^c \in \mathcal{M}$ are the sets:

$$\begin{cases} \mathcal{S}^{2j}_{\tau_p} := \{\underline{\mathcal{L}}_{\zeta_j} \circ \mathcal{L}_{w_j} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p^c) \colon\ w_1, \zeta_1, \ldots, w_j, \zeta_j \in \delta\Delta^m\} \\ \mathcal{S}^{2j+1}_{\tau_p} := \{\mathcal{L}_{w_{j+1}} \circ \underline{\mathcal{L}}_{\zeta_j} \circ \cdots \circ \underline{\mathcal{L}}_{\zeta_1} \circ \mathcal{L}_{w_1}(p^c) \colon\ w_1, \zeta_1, \ldots, w_{j+1} \in \delta\Delta^m\} \\ \underline{\mathcal{S}}^{2j}_{t_p} := \{\mathcal{L}_{w_j} \circ \underline{\mathcal{L}}_{\zeta_j} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p^c) \colon\ \zeta_1, w_1, \ldots, \zeta_j, w_j \in \delta\Delta^m\} \\ \underline{\mathcal{S}}^{2j+1}_{t_p} := \{\underline{\mathcal{L}}_{\zeta_{j+1}} \circ \mathcal{L}_{w_j} \circ \cdots \circ \mathcal{L}_{w_1} \circ \underline{\mathcal{L}}_{\zeta_1}(p^c) \colon\ \zeta_1, w_1, \ldots, \zeta_{j+1} \in \delta\Delta^m\}. \end{cases} \tag{7.3}$$

for $k = 2j$ or $k = 2j + 1$, where $j \in \mathbb{N}$ and $k \leq 3n$. Clearly, we have $\mathcal{S}^k_{\tau_p} \subset \mathcal{M}$ and $\underline{\mathcal{S}}^k_{t_p} \subset \mathcal{M}$. As $\sigma(\mathcal{L}_w(q)) = \underline{\mathcal{L}}_{\overline{w}}(\sigma(q))$, we have $\sigma(\mathcal{S}^k_{\tau_p}) = \underline{\mathcal{S}}^k_{\tilde{t}_p}$. We shall now endeavour a detailed study of these concatenated flow maps which will enable us to offer a slight refinement of the statements in Theorem 2.6.

7.4. Segre type and multitype. To begin with, we shall denote in the sequel by $\text{gen-rk}_{\mathbb{C}}(\varphi)$ the *generic* rank of a holomorphic map $\varphi \colon X \to Y$ of connected complex manifolds. Here of course, $\text{gen-rk}_{\mathbb{C}}(\Gamma_1) = \text{gen-rk}_{\mathbb{C}}(\underline{\Gamma}_1) = m$ and $\text{gen-rk}_{\mathbb{C}}(\Gamma_2) = \text{gen-rk}_{\mathbb{C}}(\underline{\Gamma}_2) = 2m$, which is also evident in eqs. (6.3). We set $e_1 := \text{gen-rk}_{\mathbb{C}}(\Gamma_3) - 2m$ and, by induction $e_{k+1} := \text{gen-rk}_{\mathbb{C}}(\Gamma_{k+1}) - e_k - 2m$, whence $\text{gen-rk}_{\mathbb{C}}(\Gamma_k) = m + m + e_1 + \cdots + e_k$ if $k \geq 3$, and similarly, we can define the sequence $\underline{e}_k$ for $\underline{\Gamma}_k$. We notice at once that we have $\underline{e}_k = e_k$, since $\sigma(\Gamma_k(w_{(k)})) = \underline{\Gamma}_k(\overline{w_{(k)}})$. We claim that $e_l = 0$ for all $l \geq k + 1$ if $e_{k+1} = 0$ and $e_k \neq 0$. Indeed, we first choose a point $w^*_{(k)}$ arbitrarily close to the origin in $\mathbb{C}^{mk}$ such that Γ_k has (necessarily locally constant) rank equal to $2m + e_1 + \cdots + e_k$ at $w^*_{(k)}$ and we set $q := \Gamma_k(w^*_{(k)}) \in \mathcal{M}$. Then by the rank theorem, the image H of a neighborhood $\mathcal{W}^*$ of $w^*_{(k)}$ is a complex manifold. We claim that $\mathcal{L}$ and $\underline{\mathcal{L}}$ are both tangent to H. For instance, to fix

ideas, we assume that k is even (the odd case will be similar). Thus we can write $\Gamma_k(w_{(k)}) = \underline{\mathcal{L}}_{w_k}(\cdots \mathcal{L}_{w_1}(0))$, *i.e.* the chain Γ_k ends-up with a $\underline{\mathcal{L}}$. This shows that H is fibered by the leaves of $\mathcal{F}_{\underline{\mathcal{L}}}$, so $\underline{\mathcal{L}}$ is already tangent to H at every point. On the other hand, if $\mathcal{L}$ were not tangent to H at every point, the chain $\Gamma_{k+1} = \mathcal{L}_{w_{k+1}}(\Gamma_k(w_{(k)}))$ would escape from H and we would have $r_{k+1} > r_k$, a contradiction. Finally, as $\mathcal{L}$ *and* $\underline{\mathcal{L}}$ are both tangent to H, it follows that their local flow at q is contained in H, whence the range of the subsequent Γ_l, $l \geq k+1$, is contained in H. Because they are holomorphic, this shows that their generic rank does not go beyond r_k, *q.e.d.*

Consequently, there exists a well-defined integer $\kappa_p \geq 0$ with $\kappa_p \leq d$ such that $e_1 > 0, \cdots, e_{\kappa_p} > 0$ and $e_l = 0$, for all $l \geq \kappa_p + 1$. We call the integer $\mu_p := 2 + \kappa_p$ the *Segre type* of $\mathcal{M}$ at p^c and we call the μ_p-tuple $(m, m, e_1, \dots, e_{\kappa_p})$ the *Segre multitype* of $\mathcal{M}$ at p^c. This *Segre multitype* simply recollects all the jumps of generic rank of the Γ_k's. It is clear that Segre type and multitype are biholomorphic invariants, because the Segre foliations defined by $\mathcal{L}$ and $\underline{\mathcal{L}}$ are so. To summarize, we have:

1) $\text{gen-rk}_{\mathbb{C}}(\Gamma_{k+2}) = 2m + e_1 + \cdots + e_k = \text{gen-rk}_{\mathbb{C}}(\underline{\Gamma}_{k+2})$, $0 \leq k \leq \kappa_p$,
2) $\text{gen-rk}_{\mathbb{C}}(\Gamma_{k+2}) = 2m + e_1 + \cdots + e_{\{\kappa_p\}} = \text{gen-rk}_{\mathbb{C}}(\underline{\Gamma}_{k+2})$, $\kappa_p \leq k \leq 3n-2$.

The main advantage of dealing with $\mathcal{M}$, $\mathcal{L}$, $\underline{\mathcal{L}}$, Γ_k lies in the fact that all these objects are coordinate-free. Even the two projections π_t and π_τ could be defined abstractly, because their fibers are the leaves of the Segre foliations. Correspondingly, the following geometric statement, which will be re-interpreted thanks to a more general construction in §9 below, should be understood in a coordinate-free style. This statement finishes the proof of Theorem 2.6, except of part **(h)**.

Theorem 7.5. *There exist some $w^*_{(\mu_p)} \in \mathbb{C}^{m\mu_p}$ arbitrarily close to the origin of the form $w^*_{(\mu_p)} = (w^*_1, \dots, w^*_{\mu_p-1}, 0)$ and small neighborhoods $\mathcal{W}^*$ of $w^*_{(\mu_p)}$ in $(\delta\Delta^m)^{\mu_p}$ such that, if we denote $\omega^*_{(\mu_p-1)} := (-w^*_{\mu_p-1}, \dots, -w^*_1)$, then we have:*

3) *The map Γ_{μ_p} is of rank $2m + e_1 + \cdots + e_{\kappa_p}$ at $w^*_{(\mu_p)}$.*
4) $\Gamma_{2\mu_p-1}(w^*_{(\mu_p)}, \omega^*_{(\mu_p-1)}) = p^c$.
5) *The restricted map $\Gamma_{2\mu_p-1} \colon \mathcal{W}^* \times \omega^*_{(\mu_p-1)} \to (\mathcal{M}, p^c)$ is of constant rank equal to $2m + e_1 + \cdots + e_{\kappa_p}$.*
6) *The image $\Gamma_{2\mu_p-1}(\mathcal{W}^* \times \omega^*_{(\mu_p-1)})$ is a manifold-piece $\mathcal{O}^c$ of the complexified CR orbit of M through p^c (or equivalently of $\mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c)$).*
7) $2m + e_1 + \cdots + e_{\kappa_p} = \dim_{\mathbb{C}} \mathcal{O}^c = \dim_{\mathbb{C}} \mathcal{O}_{\mathcal{L},\underline{\mathcal{L}}}(\mathcal{M}, p^c) = \dim_{\mathbb{R}} \mathcal{O}$.

As this statement is quite technical, it is perharps worth to see what happens in the (considerably more simple) hypersurface case and to give precise elementary examples before going into the details of the proof.

Corollary 7.6. *In particular, let M be a real $\mathcal{C}^\omega$ hypersurface,* i.e. *let $d = 1$. Then*

8) *M is minimal at p $\iff$ $\mu_p = 3$ $\iff$ $\kappa_p = 1$.*
9) *M is nonminimal at p $\iff$ $\mu_p = 2$ $\iff$ $\kappa_p = 0$.*

Proof. Firstly, we give the explanation in terms of the above definitions. Indeed, the $2m$-dimensional complex submanifold $\{\underline{\mathcal{L}}_{\zeta_1}(\mathcal{L}_{w_1}(p^c)) : w_1, \zeta_1 \in \delta\Delta^m\}$ is already of codimension only one in $\mathcal{M}$, *i.e.* of dimension $2m$. Then either $\text{gen-rk}_{\mathbb{C}}(\Gamma_3) =$

$2m+1$ or gen-rk$_{\mathbb{C}}(\Gamma_3) = 2m$, and the corollary follows. Secondly, we explain the same statement in a pedestrian way, using coordinates and relate it for completeness to this first proof of Corollary 7.6. We assume that $\mathcal{M}$ is given by eqs. (5.7), where $w \in \mathbb{C}^{n-1}$ and $z \in \mathbb{C}$, and that the Segre variety associated with the origin $S_{\bar{0}}$ is straightened to be the complex hyperplane $\{(w, 0)\}$. Of course, this can be achieved after an eventual biholomorphism. Since M is minimal if and only if $S_{\bar{0}}$ is not contained in M, then we have following dichotomy expressed simply in terms of the single ($d = 1$) defining equation:

8') M is minimal at $p \iff \Theta(\zeta, w, 0) \not\equiv 0$.
9') M is nonminimal at $p \iff \Theta(\zeta, w, 0) \equiv 0$.

Now, looking at eqs. (6.3), we can compute the flow maps explicitely:

$$\begin{cases} \mathcal{L}_{w_1}(0) = (w_1,\, 0,\, 0,\, 0), \\ \underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(0)) = (w_1,\, 0,\, w_2,\, -i\Theta(w_2, w_1, 0)), \\ \mathcal{L}_{w_3}(\underline{\mathcal{L}}_{w_2}(\mathcal{L}_{w_1}(0)))) = (w_1 + w_3,\, -i\Theta(w_2, w_1, 0) + i\bar{\Theta}(w_1 + w_3, w_2, \\ \qquad\qquad - i\Theta(w_2, w_1, 0)), w_2,\, -i\Theta(w_2, w_1, 0)). \end{cases} \tag{7.7}$$

Clearly, if $\Theta(\zeta, w, 0) \equiv 0$, then Γ_3 is of generic rank equal to $2n-2$, and so on for all the subsequent Γ_k's. We thus recover **9)** above. On the other hand, if $\Theta(\zeta, w, 0) \not\equiv 0$, it is also clear with this explicit representation after performing elementary verifications that Γ_3 is of (maximal possible) generic rank equal to $2n-1$. We thus recover **8)** above, which completes a second proof of Corollary 7.6. □

Examples 7.8. We now give a simple example in the hypersurface case which illustrates statements **5)** and **6)** of Theorem 7.5 in a very concrete way. We let M be the hypersurface of $\mathbb{C}^2$ of equation $z = \bar{z} + iw^2\bar{w}^2$. We choose $p = 0$ and here $2\mu_0 - 1 = 5$. We compute:

$$\begin{cases} \Gamma_1(w_1) = (w_1,\, 0,\, 0,\, 0) \\ \Gamma_2(w_1, w_2) = (w_1,\, 0,\, w_2,\, -iw_1^2 w_2^2) \\ \Gamma_3(w_1, w_2, w_3) = (w_1 + w_3,\, iw_2^2[w_3^2 + 2w_1w_3],\, w_2,\, -iw_1^2w_2^2) \\ \Gamma_4(w_1, w_2, w_3, w_4) = (w_1 + w_3,\, iw_2^2[w_3^2 + 2w_1w_3],\, w_2 + w_4, \\ \qquad\qquad iw_2^2[w_3^2 + 2w_1w_3] - i[(w_2 + w_4)(w_1 + w_3)]^2) \\ \Gamma_5(w_1, w_2, w_3, w_4, w_5) = (w_1 + w_3 + w_5,\, iw_2^2[w_3^2 + 2w_1w_3] - \\ \qquad - i[(w_2 + w_4)(w_1 + w_3)]^2) + i[(w_1 + w_3 + w_5)(w_2 + w_4)]^2, \\ \qquad w_2 + w_4,\, iw_2^2[w_3^2 + 2w_1w_3] - i[(w_2 + w_4)(w_1 + w_3)]^2). \end{cases} \tag{7.9}$$

The maps Γ_k have range in $\mathcal{M}$, on which either the coordinates (w, z, ζ) or (w, ζ, ξ) can be chosen. We do the first choice for k even and the second choice for k odd. Thus, we view Γ_5 as a map $\mathbb{C}^5 \to \mathbb{C}^3_{(w,\zeta,\xi)}$, *i.e.* we forget the second z-coordinate in the above expression of Γ_5. Now, computing the 3×5 Jacobian matrix of at the point $(w^*_{(3)}, \omega^*_{(2)})$ as in Theorem 7.5 which is necessarily of the form $(w_1^*, w_2^*, 0, -w_2^*, -w_1^*)$, and for which we clearly have $\Gamma_5(w_1^*, w_2^*, 0, -w_2^*, -w_1^*) = 0$, we see that the determinant of the first 3×3 submatrix is equal to $2iw_1^*(w_2^*)^2$. Thus,

it is nonzero for an arbitrary choice of $w_1^* \neq 0$ and $w_2^* \neq 0$. By the way, the question arises whether the integer $(2\mu_p - 1)$ in Theorem 7.5 is optimal. Incidentally, this example shows that it is optimal. Indeed, if we ask whether there exists $w_{(4)}^* = (w_1^*, w_2^*, w_3^*, w_4^*)$ such that $\Gamma_4(w_{(4)}^*) = 0$ and the rank at $w_{(4)}^*$ of the differential of Γ_4 equals 3 (the dimension of $\mathcal{M}$), then looking at eqs (7.9), we get first $w_1^* + w_3^* = 0$, $(w_2^*)^2 w_3^*[w_3^* + 2w_1^*] = 0$ and $w_2^* + w_4^* = 0$, thus $w_{(4)}^*$ is necessarily of the form $(0, w_2^*, 0, -w_2^*)$ or $(w_1^*, 0, -w_1^*, 0)$. Viewing now Γ_4 as a map $\mathbb{C}^4 \to \mathbb{C}^3_{(w,z,\zeta)}$, and computing its 3×4 Jacobian matrix at such points, one sees that it is of rank 2, which proves the claim.

Let us now give an example in codimension $d = 2$. To illustrate part **3)** of Theorem 7.5, we consider the generic manifold $M \subset \mathbb{C}^3$ given by $z_1 = \bar{z}_1 + iw\bar{w}$ and $z_2 = \bar{z}_2 + iw\bar{w}(w + \bar{w})$. The reader can check that the map $(w_1, w_2, w_3, w_4) \mapsto (w_1 + w_3,\, iw_2w_3,\, i(w_1 + w_3)(w_1 + w_2 + w_3) - iw_1w_2(w_1 + w_2),\, w_2 + w_4)$ is indeed of generic rank equal to 4. He can also compute Γ_7 to test the other statements.

Proof of Theorem 7.5. Now, we can proceed to the general arguments, which just use elementary properties of flows of vector fields, as in [Sus]. According to **2)**, Γ_{μ_p} is of generic rank $2m + e_1 + \cdots + e_{\kappa_p}$. Consequently, for every point $w_{(\mu_p)}^* \in (\delta\Delta^m)^{\mu_p}$ outside of some proper complex subvariety, then the map Γ_{μ_p} is of rank $2m + e_1 + \cdots + e_{\kappa_p}$ *at* $w_{(\mu_p)}^*$. In fact, we claim that we can even choose such a $w_{(\mu_p)}^*$ of the form $(w_1^*, \ldots, w_{\mu_p - 1}^*, 0)$, *i.e.* with $w_{\mu_p}^* = 0$. Indeed, as $\Gamma_{(k)}(w_{(k)}) = [\mathcal{L} \text{ or } \underline{\mathcal{L}}]_{w_k}(\Gamma_{(k-1)}(w_{(k-1)}))$, the following easy consequence of the fact that the flow maps are local biholomorphisms yields the existence of such a $w_{(\mu_p)}^*$ of this form. Let us state this property independently.

Lemma 7.10. *Let $w \in \mathbb{C}^m$, $w' \in \mathbb{C}^{m'}$, let $\Gamma(w') \in \mathbb{C}^\nu$ be holomorphic in w' with $\Gamma(0) = 0$ and let $\varphi(w, w') := \mathcal{L}_w(\Gamma(w'))$ or $\varphi(w, w') := \underline{\mathcal{L}}_w(\Gamma(w'))$. Then φ attains its maximal rank at some points of the form $(0, w'^*)$.* $\square$

Now, we fix such a $w_{(\mu_p)}^*$ of the form $(w_1^*, \ldots, w_{\mu_p - 1}^*, 0)$, which satisfies **3)** and we check that it satisfies the other claims. Let $\omega_{(\mu_p - 1)}^* := (-w_{\mu_p - 1}^*, \ldots, -w_1^*)$. First, **4)** is easy: suppose for instance μ_p is even, then we have $\Gamma_{2\mu_p - 1}(w_{(\mu_p)}^*, \omega_{(\mu_p - 1)}^*) = \mathcal{L}_{-w_1^*} \circ \cdots \circ \mathcal{L}_{-w_{\mu_p - 1}^*} \circ \underline{\mathcal{L}}_0 \circ \mathcal{L}_{w_{\mu_p - 1}^*} \circ \cdots \circ \mathcal{L}_{w_1^*}(p^c) = p^c$, because $\underline{\mathcal{L}}_0(q) = q$, $\mathcal{L}_{-w} \circ \mathcal{L}_w \equiv \mathrm{Id}$ and $\underline{\mathcal{L}}_{-\zeta} \circ \underline{\mathcal{L}}_\zeta = \mathrm{Id}$. The odd case is similar. Now, we proceed to **5)**. The restricted map $w_{(\mu_p)} \mapsto \mathcal{L}_{-w_1^*} \circ \cdots \circ \mathcal{L}_{-w_{\mu_p - 1}^*} \circ \Gamma_{\mu_p}(w_{(\mu_p)})$ (again written in case μ_p is even), is clearly of rank $2m + e_1 + \cdots + e_{\kappa_p}$ at the point $w_{(\mu_p)}^*$, because the maps $q \mapsto \mathcal{L}_{-w_1^*} \circ \cdots \circ \mathcal{L}_{-w_{\mu_p - 1}^*}(q)$ is a local biholomorphism, by definition of flows. Notice that $\Gamma_{2\mu_p - 1}$ is then of *constant rank* equal to $2m + e_1 + \cdots + e_{\kappa_p}$ in a neighborhood of $(w_{(\mu_p)}^*, \omega_{(\mu_p - 1)}^*)$ in $\mathcal{W}^* \times \omega_{(\mu_p - 1)}^*$, since, by **2)**, $2m + e_1 + \cdots + e_{\kappa_p}$ is already the maximum value of all the generic ranks of the Γ_k's. This proves **5)**. By definition, the orbit $\mathcal{O}_{\mathcal{L}, \underline{\mathcal{L}}}(\mathcal{M}, p^c)$ is the union of the ranges of the maps Γ_k and of the $\underline{\Gamma}_k$'s. It is easy to check that this double union coincides in fact with the union of only the Γ_k's (or of only the $\underline{\Gamma}_k$'s), simply because, setting $w_1 = 0$, we have $\Gamma_k(0, w_2, \ldots, w_k) \equiv \underline{\Gamma}_{k-1}(w_2, \ldots, w_k)$. Thanks to the constant rank property **5)**, we already know that this orbit contains the $(2m + e_1 + \cdots + e_{\kappa_p})$-dimensional manifold-piece passing through p^c: $\mathcal{N} := \Gamma_{2\mu_p - 1}(\mathcal{W}^* \times \omega_{(\mu_p - 1)}^*)$. Because by **2)**

the next generic ranks for $k \geq 2\mu_p - 1$ do not increase and because of the principle of analytic continuation, we then deduce that all the ranges of the subsequent Γ_k's are contained in this manifold piece $\mathcal{N}$ and it follows that $\mathcal{L}$ and $\underline{\mathcal{L}}$ are tangent to this manifold-piece. In conclusion, we get **6)** and **7)**, which completes the proof of Theorem 7.5. $\square$

Remarks. **1.** Of course, we get the same statements in Theorem 7.5 as well with $\underline{\Gamma}_{2\mu_p-1}$ instead of $\Gamma_{2\mu_p-1}$.

2. The above proof is a special application of Sussmann's construction ([Sus] studies mainly the $\mathcal{C}^\infty$ category) to the particular case where the distributions are holomorphic, which simplifies greatly the reasonings. A generalization of it is provided in §9 below.

We have argued of the simplicity of the hypersurface case. Another particular case is $m = \dim_{CR} M = 1$. We clearly have the following.

Corollary 7.11. *If* $\dim_{CR} M = 1$, *then* $e_1 = \ldots = e_{\kappa_p} = 1$. *Thus* $\mu_p = d + 2$ *and* $\kappa_p = d$ *if and only if* M *is minimal at* p.

Remark. The question arises whether one can produce examples of M's where the Segre multitype assumes arbitrary possible values $(m, m, e_1, \ldots, e_{\kappa_p})$ with $1 \leq e_j \leq m$, provided $e_1 + \cdots + e_{\kappa_p} = d$. We cannot answer this question.

7.12. Segre sets in ambient space. It is interesting to relate our construction with the construction of Segre sets given in [BER1,2] which was presented informally in §4. To this aim, it is convenient to define Segre sets in ambient space as certain projections of Segre chains. The germs $S^{2j+1}_{\bar{t}_p} := \pi_t(\mathcal{S}^{2j+1}_{\bar{t}_p}) \subset \Delta^n$, $\overline{S}^{2j+1}_{t_p} := \pi_\tau(\underline{\mathcal{S}}^{2j+1}_{t_p}) \subset \Delta^n$, $S^{2j}_{\bar{t}_p} := \pi_\tau(\mathcal{S}^{2j}_{\bar{t}_p}) \subset \Delta^n$ and $\overline{S}^{2j}_{t_p} := \pi_t(\underline{\mathcal{S}}^{2j}_{t_p}) \subset \Delta^n$ will be called the *Segre k-sets and conjugate Segre k-sets* ($k = 2j$ *or* $k = 2j+1$). Notice that because of eqs. (6.2) (see also eqs. (6.3), (7.7)), the action of the flow of $\mathcal{L}$ leaves unchanged the (ζ, ξ)-coordinates, and vice versa, the action of the flow $\underline{\mathcal{L}}$ leaves unchanged the (w, z)-coordinates. This is why in the definition of Segre k-sets, we alternately project in the $\mathbb{C}^n_t$-space and in the $\mathbb{C}^n_\tau$ space. Using eqs. (6.2-3-4), the reader can easily observe that this definition coincides with the definition of [BER2] in the form given in §4, *modulo* a correct correspondence of the ranges of the parameters $w_{(k)}$. Further, with this new definition, we recover the definition of *conjugate pairs* of Segre sets given in §5.2. Now, let us define the maps $\psi^1(w_1) := \pi_t(\Gamma_1(w_1))$, $\psi^2(w_1, w_2) := \pi_\tau(\Gamma_2(w_1, w_2))$ and more generally:

$$\psi^{2j}(w_{(2j)}) := \pi_\tau(\Gamma_{2j}(w_{(2j)})) \quad \text{and} \quad \psi^{2j+1}(w_{(2j+1)}) := \pi_t(\Gamma_{2j+1}(w_{(2j+1)})). \tag{7.13}$$

Similarly also, we can define the maps $\underline{\psi}^k$ by $\underline{\psi}^{2j}(w_{(2j)}) := \pi_t(\underline{\Gamma}_{2j}(w_{(2j)}))$ and $\underline{\psi}^{2j+1}(w_{(2j+1)}) := \pi_\tau(\underline{\Gamma}_{2j+1}(w_{(2j+1)}))$. Again, by an inspection of eqs. (6.2-3-4), the reader can observe that up to a reparametrization and up to complex conjugation, these maps are "the same" as the maps v^k, $\bar{v}^k$ of §4. We need the following.

Lemma 7.14. *For* $0 \leq k \leq \kappa_p$, *we have*:

$$m + \text{gen-rk}_{\mathbb{C}}(\psi^{k+1}) = \text{gen-rk}_{\mathbb{C}}(\Gamma_{k+2}) = 2m + e_1 + \cdots + e_k, \tag{7.15}$$

and $\text{gen-rk}_{\mathbb{C}}(\psi^{k+1}) = m + e_1 + \cdots + e_{\kappa_p}$ *for* $\kappa_p \leq k \leq 3n-2$.

Remark. Of course, this same statement also holds with $\underline{\Gamma}_{k+2}$ and $\underline{\psi}^{k+1}$ instead.

Proof. For $k = 0$, we have $\psi^1(w_1) = (w_1, i\bar{\Theta}(w_1, 0, 0))$, whence $\text{gen-rk}_{\mathbb{C}}(\psi^1) = m$ obviously. Recall that, by eq. (6.3), we have

$$\Gamma_2(w_1, w_2) = \big(w_1,\, i\bar{\Theta}(w_1, 0, 0),\, w_2,\, i\bar{\Theta}(w_1, 0, 0) - i\Theta(w_2, w_1, \bar{\Theta}(w_1, 0, 0))\big)\,, \tag{7.16}$$

so $m + \text{gen-rk}_{\mathbb{C}}(\psi^1) = \text{gen-rk}_{\mathbb{C}}(\Gamma_2) = 2m$. More generally, for $k = 2j$, we have:

$$\begin{cases} \mathcal{L}_{w_{2j+1}}(\Gamma_{2j}(w_{(2j)})) = \mathcal{L}_{w_{2j+1}}(w(w_{(2j)}), z(w_{(2j)}), \zeta(w_{(2j)}), \xi(w_{(2j)})) = \\ \qquad\qquad = \big(w_{2j+1} + w(w_{(2j)}),\, \xi(w_{(2j)}) + i\bar{\Theta}(w_{2j+1} + \\ \qquad\qquad\quad + w(w_{(2j)}), \zeta(w_{(2j)}), \xi(w_{(2j)})),\, \zeta(w_{(2j)}),\, \xi(w_{(2j)})\big)\,. \end{cases} \tag{7.17}$$

As in Example 7.8, we choose the coordinates (w, ζ, ξ) on $\mathcal{M}$, whence we consider the map Γ_{2j+1} in eq. (7.17) to have range in $\mathbb{C}^{2m+d}_{(w,\zeta,\xi)}$. It is then the map $(w_{(2j)}, w_{2j+1}) \mapsto (w_{2j+1} + w(w_{(2j)}), \zeta(w_{(2j)}),\, \xi(w_{(2j)})$. Then it follows immediately that

$$\text{gen-rk}_{\mathbb{C}}(\Gamma_{2j+1}) = m + \text{gen-rk}_{\mathbb{C}}[w_{(2j)} \mapsto (\zeta(w_{(2j)}), \xi(w_{(2j)}))] = m + \text{gen-rk}_{\mathbb{C}}\psi^{2j}. \tag{7.18}$$

This completes the proof of Lemma 7.14. □

We now define the *Segre type of* M *at* $p \in M$ (not to be confused with μ_p) to be the smallest integer ν_p satisfying $\text{gen-rk}_{\mathbb{C}}(\psi^{\nu_p}) = \text{gen-rk}_{\mathbb{C}}(\psi^{\nu_p+1})$. By eq. (7.15), we readily observe that in fact, we have $\nu_p = \kappa_p + 1$ and $\nu_p = \mu_p - 1$. The Segre type *of* M can be related to its CR orbits as follows in the next subparagraph. These last results will close up our presentation of the general theory of Segre chains.

7.19. Intrinsic complexification of CR-orbits. By the *intrinsic complexification* N^{i_c} *of a real CR manifold* N, we understand the smallest complex analytic manifold containing N in $\mathbb{C}^n$, which satisfies $\dim_{\mathbb{C}} N^{i_c} = \dim_{CR} N + \text{codim}_{CR} N$. Let $\mathcal{O}_p$ denote a manifold-piece of $\mathcal{O}_{CR}(M, p)$ through p and let $\mathcal{O}_p^{i_c}$ be its intrinsic complexification. By construction, the ranges of the ψ^{2j}'s are contained in $\mathbb{C}^n_\tau$, but we will prefer to work in $\mathbb{C}^n_t$ (although it is equivalent in principle to work in $\mathbb{C}^n_\tau$), hence we shall consider the $\underline{\psi}^{2j}$'s. We can now re-prove that $\text{gen-rk}_{\mathbb{C}}(\underline{\psi}^{\nu_p}) = \dim_{\mathbb{C}} \mathcal{O}_p^{i_c}$ (a theorem due to Baouendi-Ebenfelt-Rothschild, *see* [BER1,2]) and that the range of $\underline{\psi}^{2\nu_p}$ contains a manifold-piece of $\mathcal{O}^{i_c}$ through p, as announced in §2.5.

Theorem 7.20. *There exist some points* $\underline{v}^*_{(2\nu_p)} \in \mathbb{C}^{m2\nu_p}$ *arbitrarily close to the origin and small neighborhoods* $\mathcal{V}^*$ *of* $\underline{v}^*_{(2\nu_p)}$ *in* $(\delta\Delta^m)^{2\nu_p}$ *such that we have*:

- **10)** $\underline{\psi}^{2\nu_p}(\underline{v}^*_{(2\nu_p)}) = p$.
- **11)** *The map* $\underline{\psi}^{2\nu_p}$ *is of constant rank* $m + e_1 + \cdots + e_{\kappa_p}$ *near* $\underline{v}^*_{(2\nu_p)}$ *in* $\mathcal{V}^*$.
- **12)** $\underline{\psi}^{2\nu_p}(\mathcal{V}^*)$ *is a manifold-piece* $\mathcal{O}^{i_c}$ *of the intrinsically complexified CR orbit of* M *through* p.
- **13)** $m + e_1 + \cdots + e_{\kappa_p} = \dim_{\mathbb{C}} \mathcal{O}^{i_c} = \dim_{CR} \mathcal{O} + \text{codim}_{CR} \mathcal{O}$.

Proof. Recall that in view of Theorem 7.5, there exists $\underline{w}^*_{(2\mu_p-1)} \in (\delta\Delta^m)^{2\mu_p-1}$ with $\underline{\Gamma}_{2\mu_p-1}(\underline{w}^*_{(2\mu_p-1)}) = p^c$, such that $\Gamma_{2\mu_p-1}$ is of rank $2m+e_1+\cdots+e_{\kappa_p}$ at $\underline{w}^*_{(2\mu_p-1)}$. Looking again at eq. (7.17) (for $k = 2j+1$ odd, which we have not written, but the corresponding equation is similar), and using the chain rule, we deduce that $\underline{\psi}^{2\mu_p-2}$ is of rank $m+e_1+\cdots+e_{\kappa_p}$ at the point $\underline{v}^*_{2\nu_p} := (\underline{w}^*_1, \dots, \underline{w}^*_{2\mu_p-2})$ and that $\underline{\psi}^{2\nu_p}(\underline{v}^*_{(2\nu_p)}) = p$ (recall $\nu_p = \mu_p - 1$). This yields **10)** and **11)**. For reasons of dimension, we already know that $\dim_{\mathbb{C}} \mathcal{O}^{i_c}$ must be equal to $m+e_1+\cdots+e_{\kappa_p}$, since $\dim_{CR}\mathcal{O} = m$ and $\dim_{\mathbb{C}}\mathcal{O}^c = m + \dim_{\mathbb{C}}\mathcal{O}^{i_c}$, quite generally. This is **13)**. Finally, to deduce **12)**, we claim that it can be observed that the range of $\underline{\psi}^{2\nu_p}$ is *a priori* contained in $\mathcal{O}^{i_c}$, and afterwards for dimensional reasons, the image $\underline{\psi}^{2\nu_p}(\mathcal{V}^*)$ will necessarily be a manifold-piece of $\mathcal{O}^{i_c}$ through p. To complete this observation, we introduce holomorphic coordinates $(w, z_1, z_2) \in \mathbb{C}^m \times \mathbb{C}^{e_1+\cdots+e_{\kappa_p}} \times \mathbb{C}^{n-m-e_1-\cdots-e_{\kappa_p}}$ vanishing at p in which the equation of $\mathcal{O}^{i_c}$ is $\{z_2 = 0\}$ and $T_0M = \mathbb{C}^m_w \times \mathbb{R}^d_x$, which is possible. Using the assumption that $M \cap \{z_2 = 0\}$ is smooth and of CR dimension m, one shows that the equations of $\mathcal{M}$ can then be written in the form $z_1 = \xi_1 + i\bar{\Theta}_1(w, \zeta, \xi_1, \xi_2)$ and $z_2 = \xi_2[1 + i\bar{\Theta}_2(w, \zeta, \xi_1, \xi_2)]$. Then an inspection of the inductive construction of the maps $\underline{\Gamma}_k$ shows that they have range contained in $\{z_2 = 0, \xi_2 = 0\}$, whence the maps $\underline{\psi}_{2j}$ have range in $\{z_2 = 0\}$, as announced.

The proof of Theorem 7.20 is complete. □

Example 7.21. Looking at the map Γ_4 in eq. (7.9), we see that the integer $2\nu_p = 2\mu_p - 1$ satisfying the assertions **10)** and **11)** of Theorem 7.20 is in general optimal.

This last observation finishes up our presentation of the general theory of Segre chains. Our next goal will be to give some nontrivial illustrations and examples of the theory, by specifying what are the constraints between the Segre multitype and various other CR invariant numbers in low codimension. Numerous refined examples of CR manifolds are spread over the literature and follow for instance from the study of infinitesimal CR automorphisms (especially in the Russian school) and from the study of normal forms (Chern-Moser, Loboda, Ezhov, Schmalz, Ebenfelt and others). In the next very concrete paragraph, we shall begin with a comparison between Hörmander numbers and Segre multitype.

§8. Hörmander numbers, Minimality, holomorphic degeneracy in low codimension and various examples

8.1. Preliminaries. As the Minimality Criterion was first established by means of the Hörmander numbers of the system $\{L^1, \dots, L^m, \bar{L}^1, \dots, \bar{L}^m\}$ of CR vector fields (which are equal to the Hörmander numbers of the complexified system $\{\mathcal{L}^1, \dots, \mathcal{L}^m, \underline{\mathcal{L}}^1, \dots, \underline{\mathcal{L}}^m\}$) and as an important part of the monograph [BER2] is devoted to the exposition of the proof of this criterion, we find enough motivation to ask whether or not these numbers are intrinsically related to the invariants κ_p and $e_1, \dots, e_{\kappa_p}$ of the pair of Segre foliations $\{\mathcal{F}_{\mathcal{L}}, \mathcal{F}_{\underline{\mathcal{L}}}\}$. One could say that the Hörmander numbers are *differential* in nature wheread the Segre multitype is *geometric* in nature. We need at first some definition and notation. If $\mathcal{D} = \{\mathcal{L}^1, \dots, \mathcal{L}^m, \underline{\mathcal{L}}^1, \dots, \underline{\mathcal{L}}^m\}$ is our collection of complexified CR 1-*vector fields* (which is of cardinal $2m$), we denote for $k \geq 1$ by $\mathcal{D}^k$ the set of vector fields including $\mathcal{D}$ and the multiple Lie brackets of length $\leq k$ of the $\mathcal{L}^i$ and the $\underline{\mathcal{L}}^j$. Let

$p \in M$. We assume that $\mathcal{M}$ is $\mathcal{D}$-minimal at p^c. Following [BER2,§3.4], we define the integers $h_p \in \mathbb{N}_*$ and $\mu_0, \mu_1, \dots, \mu_{h_p} \in \mathbb{N}_*$ called the *Hörmander numbers* of $\mathcal{D}$, together with their *multiplicities* $l_0, l_1, \dots, l_{h_p}$ as follows. The number μ_0 is equal to 1 and we define another number $l_0 := 2m$, the *multiplicity* of μ_0. The number μ_1 is the smallest integer for which there exists a multiple Lie bracket at p^c of length μ_1 which is not in the span of $\mathcal{D}(p^c) \subset T_{p^c}\mathcal{M}$. We define the subspace $\mathcal{D}^{\mu_1}(p^c) \subset T_{p^c}\mathcal{M}$ to be the linear span of $\mathcal{D}(p^c)$ and the values at p^c of all commutators of vector fields in $\mathcal{D}$ of length μ_1. We define l_1 to be

$$l_1 := \dim_{\mathbb{C}} \mathcal{D}^{\mu_1}(p^c) - 2m. \tag{8.2}$$

We define inductively the numbers $\mu_1 < \mu_2 < \cdots < \mu_{h_{p^c}}$ and special linear subspaces $\mathcal{D}(p^c) = \mathcal{D}^{\mu_0}(p^c) \subset \mathcal{D}^{\mu_1}(p^c) \subset \cdots \subset \mathcal{D}^{\mu_{h_p}}(p^c) \subset T_{p^c}\mathcal{M}$ as follows. The number μ_{k+1} is obtained as the smallest integer for which there exists a multiple Lie bracket at p^c of vector fields in $\mathcal{D}$ of length μ_{k+1} which does not belong to $\mathcal{D}^{\mu_k}(p^c)$. The subspace $\mathcal{D}^{\mu_{k+1}}$ is then defined as the span of $\mathcal{D}^{\mu_k}(p^c)$ and the values at p^c of all multiple Lie brackets of vector fields in $\mathcal{D}$ of length m_{k+1}. We define

$$l_{k+1} := \dim_{\mathbb{C}} \mathcal{D}^{\mu_{k+1}}(p^c) - \dim_{\mathbb{C}} \mathcal{D}^{\mu_k}(p^c) = \dim_{\mathbb{C}} \mathcal{D}^{\mu_{k+1}}(p^c) - 2m - \sum_{i=1}^{k} l_i. \tag{8.3}$$

It is clear that this process terminates after a finite number of steps and that $\mathcal{D}^{h_p}(p^c) = T_{p^c}\mathcal{M}$ by the minimality assumption. The number l_k is called the *multiplicity* of the Hörmander number μ_k. Now, the question arises whether there exist some peculiar links between the Hörmander numbers of $\mathcal{D}$ together with their multiplicities and the Segre multitype of $\mathcal{D}$. Of course, we must have

$$2m + l_1 + \cdots + l_{h_p} = 2m + e_1 + \cdots + e_{\kappa_p} = n. \tag{8.4}$$

Sometimes, in order to insist on the dependence of these numbers on the reference point p, where $p \in M$, we shall write them $e_k(p)$, $\mu_k(p)$, $l_k(p)$, as we have already written κ_p, μ_{h_p}. In fact, it can be established that for a Zariski-generic point $p \in M$, the Hörmander numbers $\mu_k(p)$ are all equal to k and also, their multiplicities $l_k(p)$ are constant in a neighborhood of p. We shall not enter into the details and simply admit here this property. Of course, the same statement holds for the numbers $e_k(p)$. Thus there exists a *common* Zariski-open subset U_{gen} of M such that for $p \in U_{gen}$, then all these numbers are constant equal to h_{gen}, κ_{gen}, $\mu_{k,gen}(= k)$, $l_{k,gen}$, $1 \leq k \leq h_{gen}$ and $e_{k,gen}$, $1 \leq k \leq \kappa_{gen}$. Finally, one can check that the dependence of all these numbers is lower semi-continuous with respect to the point $q \in M$. We shall use these properties in our discussion of some examples below. As a matter of fact, we shall assume throughout the remainder of §8 that all our manifolds M are *minimal at every point*. If (M, p) is minimal, this can always be assumed after picking a sufficiently small manifold-piece of M around p.

8.5. Comparison between Hörmander numbers and Segre multitype. After these preliminaries are finished, we now claim that, apart some very peculiar relations which hold in small CR dimension and small codimension, then eq. (8.4) is the only general relation between the Hörmander numbers of $\{\mathcal{L}, \underline{\mathcal{L}}\}$ together with

their multiplicities and the Segre multitype of $\{\mathcal{L}, \underline{\mathcal{L}}\}$, *even between the generic ones.* This will show that in the study of the CR invariants (and normal forms) of a real analytic CR manifold *at a generic point*, the generic Segre multitype should add some new independent information. Let us argue the assertion of independence with some examples. First, the minimal hypersurface in $\mathbb{C}^2$ given by $z = \bar{z} + iw^{\tilde{\mu}}\bar{w}^{\tilde{\mu}}$ shows that $\mu_1 = 2\tilde{\mu}$ can be arbitrarily large whereas $e_1(0) = e_{1,gen} = 1 = l_1(0) = l_{1,gen}$. As we have already noted, the hypersurface case ($d = 1$) is very particular. We have observed that $e_1(p) = 1$ for all $p \in M$ (recall that M is everywhere minimal) and of course also $l_1(p) = 1$, $\forall\, p \in M$, whence $e_{1,gen} = l_{1,gen} = 1$. In fact, this relation also follows immediately from equation (8.4). Now, the 2-codimensional generic manifold in $\mathbb{C}^4$ given by $z_1 = \bar{z}_1 + iw_1\bar{w}_1$ and $z_2 = \bar{z}_2 + iw_2^2\bar{w}_2^2$ has $m = 2$, $d = 2$, $e_1(0) = 2$ but $l_2(0) = 1 \neq e_1(0)$. However, $e_{1,gen} = l_{1,gen} = 2$ for it. In the sequel, we shall examine more thoroughly the case $m = d = 2$. After an inspection of the three different types of CR manifolds of codimension two in $\mathbb{C}^4$ with nondegenerate Levi-form given by Loboda [Lo] (*see* eqs. (8.18-19-20) below), ones sees that $e_{1,gen} = 2$ if $l_{1,gen} = 2$. To end-up the comparison between $l_{1,gen}$ and $e_{1,gen}$, we shall produce a nontrivial example with $l_{1,gen} = 1$ but $e_{2,gen} = 2$ (*see* §8.7).

On the other hand, in CR dimension $m = 1$ and general codimension d, we have seen that the numbers $e_k(p) = e_{k,gen}$ are all equal to 1 and that $\kappa_p = \kappa_{gen} = d$, whereas there is no reason why all the numbers l_k should be equal to 1. In CR dimension $m = 1$, after inspecting the multiple Lie brackets of lengths ≤ 3, one can easily see that $l_{2,gen} = 1$ and $l_{3,gen} = 1$ necessarily, because there are two Lie brackets of length 2 of the system $\{L^1, \bar{L}^1\}$ ($m = 1$) which are opposite to each other and there are four multiple Lie brackets of length 3 which are either opposite or conjugate to each other. However there may exist two multiple Lie brackets of lentgh 4 which can really differ from each other, thus making possible $l_{4,gen} = 2$. Consequently, the codimension $d \geq 4$ is necessary to construct an example with $l_{3,gen} = 2$ (whereas $e_{3,gen} = 1$). Here is such an example in $\mathbb{C}^5_{(w,z_1,z_2,z_3,z_4)}$:

$$M : \begin{cases} z_1 = \bar{z}_1 + iw\bar{w}, & z_2 = \bar{z}_2 + iw\bar{w}(w + \bar{w}) \\ z_3 = \bar{z}_3 + iw\bar{w}(w^2 + \bar{w}^2), & z_4 = \bar{z}_4 + iw^2\bar{w}^2. \end{cases} \tag{8.6}$$

8.7. Characterization of holomorphic degeneracy by Segre multitype in codimension 2 and CR dimension 2. As announced, we study in this paragraph the case $d = 2$, $m = 2$, with M minimal at *every point*. If M is connected (which we will also suppose), then there exists an integer κ_M with $0 \leq \kappa_M \leq 2$ such that M is biholomorphic to a product $\underline{M} \times \Delta^{\kappa_M}$ in a neighborhood of a Zariski-generic point $p \in M$, *see* [BER1,2], [M2]. We say that M is κ_M*-holomorphically degenerate*. If $\kappa_M = 2$, then $\underline{M}$ is a maximally real submanifold of $\mathbb{C}^2$ and M is biholomorphic to $\mathbb{R}^2 \times \Delta^2$ near such Zariski-generic points. In fact, this implies that M is Levi-flat, and equivalent to $\mathbb{R}^2 \times \Delta^2$ in a neighborhood of *every* point. Thus $e_1(p) = 0$, $\forall\, p \in M$, the CR-orbits are the leaves $x_p \times \Delta^2$ and M cannot be minimal, contradiction. On the other hand, if $\kappa_M = 1$, M can well be minimal at every point, and it is easy to see that for each Zariski-generic point $p \in M \cong \underline{M} \times \Delta$, then $e_1(p) = e_2(p) = 1$ and $e_3(p) = 0$, since $d = 2$. By the upper semi-continuity of the numbers $e_k(q)$ with respect to $q \in M$, it follows that $e_1(p) = e_2(p) = 1$ for *all* points

$p \in M$. In fact, we shall prove that the property $e_1(p) = 1 \; \forall \, p \in M$, for M minimal at every point of codimension $d = 2$ and CR dimension $m \geq 2$, *characterizes* 1-holomorphic degeneracy. This will show an interesting and unexpected illustration of the Segre multitype and will provide some new nontrivial examples. By the way, before proceeding to the codimension two case, we would like to remind that in codimension $d = 1$ and CR dimension $m = 1$, it is already well know that an everywhere minimal $M \subset \mathbb{C}^2$ is holomorphically nondegenerate ($\kappa_M = 0$). The following proposition generalizes such an observation to codimension 2 and CR dimension 2. Its interest lies in the analysis of the local form of an arbitrary generic six-dimensional $M \subset \mathbb{C}^4$ that we shall delineate during the course of its proof.

Proposition 8.8. *Let $M \subset \mathbb{C}^n$ be a connected CR-generic real analytic manifold of codimension $d = 2$ and CR dimension $m = 2$ which is minimal at every point. Then $1 \leq e_1(p) \leq 2$ for all $p \in M$ and the following properties are equivalent*

- **(a)** $e_{1,gen} = 1$.
- **(b)** $e_1(p) = 1$ *for all* $p \in M$.
- **(c)** M *is* 1*-holomorphically degenerate.*

Consequently M is holomorphically nondegenerate if and only if $e_{2,gen} = 2$.

Remark. According to [BER], M is holomorphically nondegenerate if and only if its *Levi type* ℓ is finite at a generic point and then $1 \leq \ell_{gen} \leq m$. By definition, the Levi-type at $p \in M$ is the smallest integer k such that

$$\text{Span}\,\{\bar{L}^\beta \nabla \rho_j(p, \bar{p}) \colon |\beta| \leq k, 1 \leq j \leq d\} = \mathbb{C}^n, \tag{8.9}$$

where $\nabla \rho_j$ denotes the holomorphic gradient $\partial \rho_j / \partial t$ of the j-th Cartesian equation of M. One says that M is *k-nondegenerate at p* if eq. (8.9) holds. Essentially two different classes of CR manifolds M arise in Proposition 8.8 according to whether $\ell_{gen} = 1$ or $\ell_{gen} = 2$. As in the study by Ebenfelt [E] of normal forms for hypersurfaces in $\mathbb{C}^3$, it holds for $m = d = 2$ that $\ell_{gen} = 1$ if and only if M is Levi degenerate at a generic point if and only if $l_{1,gen} = 1$.

Example 8.10. The generic manifold $2y_1 = w_1 \bar{w}_1$, $2y_2 = x_1^2 w_2 \bar{w}_2 + w_1^2 \bar{w}_1^2 w_2 \bar{w}_2 / 4$, or equivalently $M \colon \; z_1 = \bar{z}_1 + i w_1 \bar{w}_1, \; z_2 = \bar{z}_2 + i \bar{z}_1 [i w_1 \bar{w}_1 w_2 \bar{w}_2 + \bar{z}_1 w_2 \bar{w}_2]$, is minimal at 0, holomorphically nondegenerate, hence satisfies $e_{1,gen} = 2$ by the above Proposition 8.8 (this can be checked directly), but here we have $e_1(0) = 1$ (true semi-continuity). Consequently, the conditions **(a)** and **(b)** cannot be weakened. Inspired by eq. (8.28) below, we can further provide such an example with a *rigid* M which is 2-nondegenerate at 0, minimal at 0 and satisfies $e_{1,gen} = 2$ whereas $e_1(0) = 1$ only:

$$\begin{cases} z_1 = \bar{z}_1 + i[w_1 \bar{w}_1 + w_1^2 \bar{w}_2 + \bar{w}_1^2 w_2] \\ z_2 = \bar{z}_2 + i[w_1 \bar{w}_1 (w_1 + \bar{w}_1) - w_2 \bar{w}_1^2 (2 w_1 + \bar{w}_1) - \bar{w}_2 w_1^2 (2 \bar{w}_1 + w_1)]. \end{cases} \tag{8.11}$$

Example 8.12. Our goal now is to produce an example of M with $m = d = 2$ and $l_{1,gen} = 1$ which is minimal and holomorphically nondegenerate (exercise: verify that simple polynomial rigid examples exist if we drop one of the last two

assumptions). First, we would like to remind that, according to the simplification by Freeman of a theorem of Sommer, the kernel of the Levi form of any $\mathcal{C}^\omega$ (even $\mathcal{C}^2$) CR-generic M is *involutive* locally where it is of constant rank e, from which follows that M is locally *foliated* by complex manifolds of dimension e (*see* [F1], [Chi], p.152). An important example of a real analytic everywhere Levi degenerate but *not* holomorphically degenerate hypersurface is the set of regular points of the cubic discovered by Freeman [F]: $x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbb{C}^3_{z_1,z_2,z_3}$. Freeman sought hypersurfaces with everywhere degenerate Levi form which are not locally biholomorphic to a product by Δ, or equivalently in modern terms, which are not holomorphically degenerate. In fact, the same property holds true for the (quite simpler, because of degree two) hypersurface $x_1^2 + x_2^2 - x_3^2 = 0$, called the *light cone*, which is simply the tube manifold over the classical real cone in $\mathbb{R}^3$. The light cone is a homogeneous CR manifold, whereas Freeman's cubic is not. Both are rigid, because of the existence of the obvious CR-transversal infinitesimal CR automorphisms $\mathrm{Re}\,(i\partial/\partial z_j)$, $1 \leq j \leq 3$. Ebenfelt [E] shows that the light cone is finitely nondegenerate (and in fact 2-nondegenerate) at every point with exactly one nonzero eigenvalue for the Levi form and shows that its equations at a generic point (in fact at every point) of the form (5.7) can be written as: $z = \bar{z} + i[w_1\bar{w}_1 + w_1^2\bar{w}_2 + \bar{w}_1^2 w_2 + O_{weighted}(4)]$ ([E], p. 318, eq. (A.i.2); the same property holds true for Freeman's cubic outside the coordinate planes). Building on these two objects, we can construct as follows an example of $M = \Sigma_{reg} \subset \mathbb{C}^4_{z_1,z_2,z_3,z_4}$, with $l_{1,gen} = 1$ which is minimal and 2-nondegenerate at every point, where:

$$\Sigma : \begin{cases} x_1^2 + x_2^2 - x_3^2 = 0, \\ x_1^3 + x_2^3 - x_4^3 = 0. \end{cases} \tag{8.13}$$

Here, Σ has a complex foliation by complex lines described in terms of the holomorphic leaf variable $\zeta \in \mathbb{C}$ and of four real parameters a, b, c, d as follows:

$$(\zeta, a, b, c, d) \mapsto (a(1+\zeta)+ib, 1+\zeta, (1+a^2)^{1/2}(1+\zeta)+ic, (1+a^3)^{1/3}(1+\zeta)+id). \tag{8.14}$$

(*cf.* [F2]). Over the subset $M := \Sigma_{reg} = \Sigma \backslash \{0\}$ of regular points of Σ, the complex tangent bundle T^cM admits the following two spanning global sections (notice that $(x_1, x_2) \neq (0,0)$, $x_3 \neq 0$ and $x_4 \neq 0$ over M):

$$\begin{cases} L_1 = \dfrac{\partial}{\partial z_1} + \dfrac{x_1}{x_3}\dfrac{\partial}{\partial z_3} + \dfrac{x_1^2}{x_4^2}\dfrac{\partial}{\partial z_4}, \\ L_2 = \dfrac{\partial}{\partial z_2} + \dfrac{x_2}{x_3}\dfrac{\partial}{\partial z_3} + \dfrac{x_2^2}{x_4^2}\dfrac{\partial}{\partial z_4}. \end{cases} \tag{8.15}$$

An isotropic (for the $\mathbb{R}^2$-valued Levi form of M) vector field is simply the radial vector field, which is nowhere vanishing on M:

$$L = x_1\frac{\partial}{\partial z_1} + x_2\frac{\partial}{\partial z_2} + x_3\frac{\partial}{\partial z_3} + x_4\frac{\partial}{\partial z_4}, \qquad L|_M = x_1 L_1|_M + x_2 L_2|_M. \tag{8.16}$$

Obviously, the CR automorphism of M generated by L consists of the real dilatation $z \mapsto \lambda z$. By direct computations we have:

$$[L, \bar{L}] = -\bar{L} + L, \qquad [L, \bar{L}_1] = -L_1, \qquad [L, \bar{L}_2] = -L_2, \tag{8.17}$$

which shows that the kernel of the Levi form contains L. Then the computation of $[L_1, \bar{L}_1]$ and $[L_2, \bar{L}_2]$ shows that the Levi-form of M has exactly one zero and one nonzero eigenvalue at every point. Consequently, $l_{1,gen} = 1$. Further, it is easy to show that M is 2-nondegenerate at every point (*cf.* [E]) by checking that the five vectors ρ_1, $\bar{L}_1(\nabla\rho_1)$, $\bar{L}_2(\nabla\rho_1)$, $\bar{L}_1\bar{L}_1(\nabla\rho_1)$ and $\bar{L}_2\bar{L}_2(\nabla\rho_1)$ are linearly independent at every point of M, where $\nabla\rho_1 = (\rho_{1,z_1}, \rho_{1,z_2}, \rho_{1,z_3})$ is the complex gradient of $\rho_1 := x_1^2 + x_2^2 - x_3^2$. Finally, computing higher order Lie brackets, one sees that M is minimal at every point, and more precisely that $\mu_1(p) = 2$, $l_1(p) = 1$, $\mu_2(p) = 3$, $l_2(p) = 1$ if $x_{1,p}x_{2,p} \neq 0$ and that $\mu_1(p) = 2$, $l_1(p) = 1$, $\mu_2(p) = 4$, $l_2(p) = 1$ if $x_{1,p} = 0$ or $x_{2,p} = 0$ (but $(x_{1,p}, x_{2,p}) \neq (0,0)$).

Proof of Proposition 8.8. We have already understood the implications **(a)** $\Longleftrightarrow$ **(b)** and **(c)** $\Longrightarrow$ **(a)**. Thus, it remains to establish **(b)** $\Longrightarrow$ **(c)**. We shall proceed to another implication, which we now know to be the contraposition of **(b)** $\Longrightarrow$ **(c)**, namely we shall prove that if M is holomorphically nondegenerate and minimal everywhere, then $e_{2,gen} = 2$. In fact, we shall see that the two hand-sides of the last equivalence stated in Proposition 8.8 come together in our discussion. To do so, we analyze first a generic point of minimality of M in terms of Lie brackets. By [BER1,2], the Levi-number of M is $\ell_M = 2$, whence a generic point of holomorphic nondegeneracy of M is 2-*nondegenerate* and morevover, it has the property that, in regular coordinates, the homogeneous Taylor approximation of M of order ≤ 3 with respect to $(w, \bar{w})$ is already 2-nondegenerate. We assume that such a choice of generic point for this property has been made for the reference point $0 \in M$ and we also assume that the Hörmander numbers of M at 0 are the generic ones. For combinatorial reasons, there are exactly two possibilities for the generic Hörmander numbers of M. Either they are equal to $\mu_{1,gen} = 2$, $\mu_{2,gen} = 3$, with multiplicities $l_{1,gen} = 1$ and $l_{2,gen} = 1$ or $\mu_{1,gen} = 2$ with multiplicity $l_{1,gen} = 2$. In the latter case, it is easy to deduce from $l_1(p) = 2$ that the two Levi forms of the two defining equations of M are linearly independent and that the intersection of their kernel is null. This assumption corresponds exactly to the *CR-generic manifolds with nondegenerate vector-valued Levi-form*, in the sense of the Russian school. It is known that such manifolds can be reduced to three (inequivalent) types (*see* [Lo], [Belo], [EIS]), the *elliptic* type:

$$M : \begin{cases} z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + O_w(3) + O_{\bar{z}}(1)] \\ z_2 = \bar{z}_2 + i[w_2\bar{w}_2 + O_w(3) + O_{\bar{z}}(1)], \end{cases} \tag{8.18}$$

or the *parabolic* type:

$$M : \begin{cases} z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + O_w(3) + O_{\bar{z}}(1)] \\ z_2 = \bar{z}_2 + i[w_1\bar{w}_2 + \bar{w}_1 w_2 + O_w(3) + O_{\bar{z}}(1)], \end{cases} \tag{8.19}$$

or the *hyperbolic* type:

$$M : \begin{cases} z_1 = \bar{z}_1 + i[w_1\bar{w}_1 - w_2\bar{w}_2 + O_w(3) + O_{\bar{z}}(1)] \\ z_2 = \bar{z}_2 + i[w_1\bar{w}_2 + \bar{w}_1 w_2 + O_w(3) + O_{\bar{z}}(1)]. \end{cases} \tag{8.20}$$

Direct computation shows that such manifolds are minimal, holomorphically nondegenerate (in fact their Taylor approximation of order 2 is already 1-nondegenerate) and that $e_{1,gen} = e_1(0) = 2$. Using the representation eq. (7.7), one sees that $e_1(0) = 2$ if and only if the following determinant of order one partial derivatives of the two defining functions Θ_1, Θ_2, restricted to the second Segre chain S_0^2:

$$\det \begin{pmatrix} \Theta_{1,w_1}(\zeta_1,\zeta_2,w_1,w_2,0,0) & \Theta_{1,w_2}(\zeta_1,\zeta_2,w_1,w_2,0,0) \\ \Theta_{2,w_1}(\zeta_1,\zeta_2,w_1,w_2,0,0) & \Theta_{2,w_2}(\zeta_1,\zeta_2,w_1,w_2,0,0) \end{pmatrix} \not\equiv 0, \tag{8.21}$$

does not vanish identically in $\mathbb{C}\{w_1,w_2,\zeta_1,\zeta_2\}$. In this case, the desired property $e_2(0) = 2$ comes immediately for each one of the three types (8.18-19-20) (just compute (8.21)). It remains therefore to study the (more degenerate) case where $\mu_{1,gen} = 2$, $l_{1,gen} = 1$, $\mu_{2,gen} = 3$, $l_{2,gen} = 1$. Centering the study at a generic point, this case can be reduced *in regular coordinates* to equations of the form

$$\begin{cases} z_1 = \bar z_1 + i[w_1\bar w_1 + \varepsilon_2 w_2\bar w_2 + H_1^3(w,\bar w) + O(|w|^4) + O(|\bar z|\,|w|^2)] \\ z_2 = \bar z_2 + i[H_2^3(w,\bar w) + O(|w|^4) + O(|\bar z|\,|w|^2)], \end{cases} \tag{8.22}$$

where H_1^3 and H_2^3 are homogeneous polynomial of degree 3 in $(w,\bar w)$ containing no pluriharmonic terms and where $\varepsilon_2 = 0, 1$ or -1, depending on the rank of the Levi-form of ρ_1 at 0. Because of regular coordinates, using eq. (5.8) one can see that H_1^3 and H_2^3 are *real* (however, the next terms homogeneous of order 4, $H_j^4(w,\bar w)$, are not in general real). Clearly, we have $l_1(0) = 1$ here. Now, computing multiple Lie brackets of length equal to 3, one sees that $l_2(0) = 1$ (which we assumed) if and only if $H_2^3(w,\bar w) \not\equiv 0$. So in the sequel, we will assume $H_2^3(w,\bar w) \not\equiv 0$. Consequently, we can assign weight 1 to w and to $\bar w$, weight 2 to z_1 and to $\bar z_1$ and finally weight 3 to z_2 and to $\bar z_2$. In the sequel, we shall work modulo $O_{weighted}(k)$, $k = 2,3,4,\dots$, say for short $O_{wt}(k)$. Now, since we have assumed $l_{1,gen} = 1$, the following 2×4 matrix of the vector-valued Levi form of M modulo $O_{wt}(2)$ must be of rank 1 everywhere:

$$\begin{pmatrix} 1 + H^3_{1,w_1\bar w_1} & \varepsilon_2 + H^3_{1,w_2\bar w_2} & H^3_{1,w_1\bar w_2} & H^3_{1,\bar w_1 w_2} \\ H^3_{2,w_1\bar w_1} & H^3_{2,w_2\bar w_2} & H^3_{2,w_1\bar w_2} & H^3_{2,\bar w_1 w_2} \end{pmatrix}, \tag{8.23}$$

again modulo $O_{wt}(2)$. If $\varepsilon_2 \neq 0$, we deduce $H^3_{2,w_1\bar w_1} \equiv H^3_{2,w_1\bar w_2} \equiv H^3_{2,\bar w_1 w_2} \equiv H^3_{2,w_2\bar w_2} \equiv 0$, whence $H_2^3(w,\bar w) \equiv 0$, a contradiction. Therefore, we must have $\varepsilon_2 = 0$ and we then deduce $H^3_{2,w_1\bar w_2} \equiv H^3_{2,w_1\bar w_2} \equiv H^3_{2,\bar w_1 w_2} \equiv 0$, whence $H_3^2(w,\bar w) = Aw_1\bar w_1(w_1+\bar w_1)$, with $A \neq 0$. After a complex scaling, we can assume $A = 1$ and the equations of M are thus in the form

$$\begin{cases} z_1 = \bar z_1 + i[w_1\bar w_1 + H_1^3(w,\bar w) + O_{wt}(4)] \\ z_2 = \bar z_2 + i[w_1\bar w_1(w_1+\bar w_1) + O_{wt}(4)]. \end{cases} \tag{8.24}$$

Now, we pay attention to the supplementary information coming from the Levi type ℓ_{gen} of M. The case $\ell_{gen} = 1$ *with M minimal* again implies Levi-nondegeneracy (in the sense of Beloshapka and Loboda) at a generic point (exercise), *i.e.* one of the three forms eq. (8.18-19-20). Thus, we can assume that $\ell_{gen} = 2$ and then that

M is 2-nondegenerate at 0 after a small shift (delocalization) of the origin. Then if we develope the homogeneous polynomial H_1^3 in the form

(8.25)
$$H_1^3 = 2\mathrm{Re}\left(Aw_1^2\bar{w}_1 + Bw_1^2\bar{w}_2 + Cw_1w_2\bar{w}_2 + Dw_1w_2\bar{w}_1 + Ew_2^2\bar{w}_1 + Fw_2^2\bar{w}_2\right),$$

we observe that M is 2-nondegenerate at 0 *if and only if* $B \neq 0$ or $C \neq 0$ or $F \neq 0$. Then coming back to the assumption $l_{1,gen} = 1$, we observe that this implies that the rank of the Levi-form of the first defining equation ρ_1 of M must be equal to 1 at every point of M in a neighborhood of 0 (otherwise, we come again to the case $\varepsilon_2 \neq 0$, which is excluded). Looking at the first row of matrix (8.23) (with now $\varepsilon_2 = 0$) and inspecting the Levi-form of ρ_1 modulo $O_{wt}(2)$, we see that necessarily $H^3_{1,w_2\bar{w}_2}(w,\bar{w}) = Cw_1 + \bar{C}\bar{w}_1 + 2Fw_2 + 2\bar{F}\bar{w}_2 \equiv 0$, whence $C = F = 0$. After a scaling and a linear change of coordinates in the z-space, we can assume that $A = 0$ and $B = 1$. Then replacing w_1 by $w_1 + Dw_1w_2 + Ew_2^2$, we can assume that $D = E = 0$ and in conclusion, the equations of M are reduced to the following form:

$$M : \begin{cases} z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + w_1^2\bar{w}_2 + \bar{w}_1^2 w_2 + O_{wt}(4)] \\ z_2 = \bar{z}_2 + i[w_1\bar{w}_1(w_1 + \bar{w}_1) + O_{wt}(4)]. \end{cases} \tag{8.26}$$

We notice that the cubic tangent to eqs. (8.26) is 2-nondegenerate, and minimal at 0. Further, it is in fact Levi-nondegenerate at a generic point (!). However, order 4 terms have an influence to make possible the property $l_{1,gen} = 1$, as in Example 8.12, a property which we are assuming since a while. Thus, continuing the proof of Proposition 8.8, we have to explicit the order 4 terms in eq. (8.26), which we now write as follows:

$$M : \begin{cases} z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + w_1^2\bar{w}_2 + \bar{w}_1^2 w_2 + \bar{H}_1^4(w,\bar{w}) + \\ \qquad\qquad + \bar{z}_1(\alpha_1 w_1\bar{w}_1 + \beta_1 w_1\bar{w}_2 + \gamma_1 w_2\bar{w}_1 + \delta_1 w_2\bar{w}_2) + O_{wt}(5)], \\ z_2 = \bar{z}_2 + i[w_1\bar{w}_1(w_1 + \bar{w}_1) + \bar{H}_2^4(w,\bar{w}) + \\ \qquad\qquad + \bar{z}_1(\alpha_2 w_1\bar{w}_1 + \beta_2 w_1\bar{w}_2 + \gamma_2 w_2\bar{w}_1 + \delta_2 w_2\bar{w}_2) + O_{wt}(5)], \end{cases} \tag{8.27}$$

where H_1^4 and H_2^4 are homogeneous polynomials of order four in $(w,\bar{w})$ containing no pluriharmonic terms. To finish the proof of Proposition 8.8, we claim that $e_2(0) = 2$ in the obtained form eq. (8.27) if furthermore $l_{1,gen} = 1$. We shall proceed by contradiction. Thus, we assume that $e_2(0) = 1$, *i.e.* that the determinant in eq. (8.21) vanishes identically. First, we deduce after inspecting the order three terms in this determinant that we have

$$(2w_1\zeta_1 + \zeta_1^2)\,\zeta_1 - \bar{H}^4_{2,w_2}(w,\zeta) \equiv 0. \tag{8.28}$$

However, we have not yet expressed the assumption $l_{1,gen} = 1$ (which will contradict eq. (8.28)). This assumption can be expressed by saying that the two Lie brackets $[L_1, \bar{L}_2]$ and $[L_2, \bar{L}_2]$ are colinear to the Lie bracket $[L_1, \bar{L}_1]$ at every nearby point. Let us compute these Lie brackets. We shall write them in the form:

$$\begin{cases} [L_1, \bar{L}_1] := P_1\dfrac{\partial}{\partial z_1} + Q_1\dfrac{\partial}{\partial z_2} + \bar{P}_1\dfrac{\partial}{\partial \bar{z}_1} + \bar{Q}_1\dfrac{\partial}{\partial \bar{z}_2} \\ [L_1, \bar{L}_2] := R\dfrac{\partial}{\partial z_1} + S\dfrac{\partial}{\partial z_2} + T\dfrac{\partial}{\partial \bar{z}_1} + U\dfrac{\partial}{\partial \bar{z}_2} \\ [L_2, \bar{L}_2] := P_2\dfrac{\partial}{\partial z_1} + Q_2\dfrac{\partial}{\partial z_2} + \bar{P}_2\dfrac{\partial}{\partial \bar{z}_1} + \bar{Q}_2\dfrac{\partial}{\partial \bar{z}_2} \end{cases} \tag{8.29}$$

From the colinearity of all these three Lie brackets at every point, we shall extract only one equation, namely: $\det\begin{pmatrix} P_1 & Q_1 \\ R & S \end{pmatrix} \equiv 0$. After inspection of eq. (8.27), one sees that $P_1 = -i + O_{wt}(1)$, $Q_1 = O_{wt}(1)$, $R = O_{wt}(1)$ and $S = O_{wt}(2)$. Since we want to extract from this equation only the order two terms, we need only to compute Q_1 and R to order 1 and S to order 2. We have the following complete expression for our two $(1,0)$ CR vector fields:

(8.30)
$$\begin{cases} L_1 = \dfrac{\partial}{\partial w_1} + i\left[\bar{w}_1 + 2w_1\bar{w}_2 + \bar{H}^4_{1,w_1}(w,\bar{w}) + \alpha_1\bar{z}_1\bar{w}_1 + \beta_1\bar{z}_1\bar{w}_2 + O_{wt}(4)\right]\dfrac{\partial}{\partial z_1} + \\ \qquad + i\left[2w_1\bar{w}_1 + \bar{w}_1^2 + \bar{H}^4_{2,w_1}(w,\bar{w}) + \alpha_2\bar{z}_1\bar{w}_1 + \beta_2\bar{z}_1\bar{w}_2 + O_{wt}(4)\right]\dfrac{\partial}{\partial z_2}, \\ L_2 = \dfrac{\partial}{\partial w_2} + i\left[\bar{w}_1^2 + \bar{H}^4_{1,w_2}(w,\bar{w}) + \gamma_1\bar{z}_1\bar{w}_1 + \delta_1\bar{z}_1\bar{w}_2 + O_{wt}(4)\right]\dfrac{\partial}{\partial z_1} + \\ \qquad + i\left[\bar{H}^4_{2,w_2}(w,\bar{w}) + \gamma_2\bar{z}_1\bar{w}_1 + \delta_2\bar{z}_1\bar{w}_2 + O_{wt}(4)\right]\dfrac{\partial}{\partial z_2} \end{cases}$$

Now, we can compute

$$\begin{aligned} &P_1 = -i + O_{wt}(1), && Q_1 = -i[2w_1 + 2\bar{w}_1 + O_{wt}(2)], \\ &R = -i[2w_1 + O_{wt}(2)], && S = -i[\bar{H}^4_{2,w_1\bar{w}_2}(w,\bar{w}) + \beta_2\bar{z}_1 + O_{wt}(3)], \end{aligned} \tag{8.31}$$

and the vanishing of the above 2×2 determinant yields the following supplementary partial differential equation for $\bar{H}^4_2$:

$$\bar{H}^4_{2,w_1\zeta_2}(w,\zeta) + \beta_2\bar{z}_1 - (2w_1 + 2\zeta_1)(2w_1) \equiv 0. \tag{8.32}$$

On one hand, we deduce from eq. (8.28) that the homogeneous polynomial $\bar{H}^4_2(w,\zeta)$ necessarily incorporates the monomial $w_2\zeta_1^3$. On the other hand, we deduce from eq. (8.32), that $\bar{H}^4_2(w,\zeta)$ contains necessarily the monomial $\frac{4}{3}\zeta_2 w_1^3$. The polynomial $\bar{H}^4_2(w,\zeta)$ is not real, but nevertheless, it satisfies the relation

$$\bar{H}^4_2(w,\zeta) - H^4_2(\zeta,w) - iw_1\zeta_1[\bar{\alpha}_2\zeta_1 w_1 + \bar{\beta}_2\zeta_1 w_2 + \bar{\gamma}_2\zeta_2 w_1 + \bar{\delta}_2\zeta_2 w_2] \equiv 0, \tag{8.33}$$

which follows from an inspection of the terms of weighted oder ≤ 4 in eqs. (8.27) and (5.8). Because of eq. (8.8), the monomial $\frac{4}{3}\zeta_2 w_1^3$ should be the conjugate of the monomial $w_2\zeta_1^3$, but $\frac{4}{3} \neq 1$. This is the desired contradiction, which completes the proof of Proposition 8.8 is complete. $\square$

Along with other results, Ebenfelt in [E] obtains essentially three forms *at a Zariski-generic point* for a minimal and holomorphically nondegenerate hypersurface in $\mathbb{C}^3$: $z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + w_2\bar{w}_2 + O_{wt}(3)]$ or $z_1 = \bar{z}_1 + i[w_1\bar{w}_1 - w_2\bar{w}_2 + O_{wt}(3)]$ in the classical Levi-nondegenerate cases, and the last form $z_1 = \bar{z}_1 + i[w_1\bar{w}_1 + w_1^2\bar{w}_2 + \bar{w}_1^2 w_2 + O_{wt}(4)]$ in the everywhere Levi-degenerate case $l_{1,gen} = 1$. Incidentally, our analysis has provided a similar list.

Corollary 8.34. *There are exactly four nonequivalent local Taylor representations at a Zariski-generic point for an arbitrary real analytic CR-generic manifold of*

codimension 2 *in* $\mathbb{C}^4$. *If* M *is Levi-nondegenerate, then it is either elliptic* (8.18) *or parabolic* (8.19) *or hyperbolic* (8.20). *If* M *is everywhere Levi-degenerate, then it is representable in the form* (8.26) *with further conditions as e.g. eq.* (8.32).

A quite general problem in CR geometry would be to devise a complete classification of local Taylor-approximated representations for arbitrary real analytic CR manifolds in any CR dimension and in any codimension. The author ignores whether or not this question if pure Utopia.

§9. Orbits of systems of holomorphic vector fields and a refinement of Sussmann's theorem

This appendix paragraph exhibits the central notion of *orbits of bundles of holomorphic vector fields* in a self-contained way. As we have been inspired by the general constructions in [Sus], we give here a brief generalization of Theorem 7.5. The notion of orbits of families of vector fields goes back to a well written paper of Sussmann [Sus], so we will refer the reader to it for background and further information. Sussmann considered only the $\mathcal{C}^\infty$ case, but his construction works as well in $\mathcal{C}^2$, $\mathcal{C}^k$, $\mathcal{C}^\omega$ or in the complex analytic category. Accordingly, we will in this section give a proof of the Orbit Theorem about integral submanifolds of bundles of holomorphic vector fields in the spirit of [Sus], but with a supplementary important simplification due to the principle of analytic continuation.

9.1. Flows of vector fields. Let Δ be the unit disc in $\mathbb{C}$ and $r\Delta = \{|z| < r\}$. Let $\mathbb{S} = \{L_\alpha\}_{\alpha\in A}$, $1 \leq \alpha \leq a$, $A = [\![1, a]\!]$, $a \in \mathbb{N}$, $a \geq 1$, be a *finite* system of nonzero m-vectorial holomorphic vector fields over Δ^n, $m \in \mathbb{N}$, $m \geq 1$. By m-vectorial, we mean that each L_α is a collection $(L_{\alpha 1}, \dots, L_{\alpha m})$ of m commuting linearly independent over Δ^n vector fields. Hence, considering the *multiple flow mapping* $\mathbb{C}^m \times \Delta^n \ni (s_1, \dots, s_m, z) \mapsto \exp(s_m L_{\alpha m}) \circ \cdots \circ \exp(s_1 L_{\alpha 1})(z) \in \Delta^n$ (which is defined on a certain domain), we have for every permutation $\varpi : [\![1, m]\!] \to [\![1, m]\!]$:

$$\exp(s_{\varpi(m)} L_{\alpha\varpi(m)}) \circ \cdots \circ \exp(s_{\varpi(1)} L_{\alpha\varpi(1)})(z) = \tag{9.2}$$
$$= \exp(s_m L_{\alpha m}) \circ \cdots \circ \exp(s_1 L_{\alpha 1})(z).$$

We shall simply denote this multiple flow map by $(s, z) \mapsto L_{\alpha s}(z)$ and we will work with multiple vector fields *formally* as if they were *usual* vector fields, *i.e.* as if $m = 1$. But s will be called an m-time and $s \mapsto L_{\alpha s}(z)$ an m-curve. We recall the defining properties of the flow map: $L_{\alpha 0}(z) = z$ and $\frac{d}{ds}(L_{\alpha s}(z)) = L_\alpha(L_{\alpha s}(z))$, where $L_\alpha(z')$ denotes the value of L_α at z', an m-vector in $T_{z'}\Delta^n$. Now, we assume that $\mathrm{rk}_{\mathbb{C}}(L_1(p), \dots, L_a(p)) = am$ all over Δ^n, in other words, we assume that the span over $\mathcal{O}(\Delta^n)$ of the L_α's generates a trivial am-dimensional holomorphic vector bundle on Δ^n. Put $d = n - am$, the *codimension* of $\mathbb{S}$. Let $0 < r \leq 1/2$ and $(r\Delta)^n \subset\subset \Delta^n$. Our aim is to define finite concatenation of flow mappings of such m-vector fields. If $k \in \mathbb{N}_*(:= \mathbb{N}\backslash\{0\})$, $\mathbb{L}^k = (L^1, \dots, L^k) \in \mathbb{S}^k$, $t_{(k)} = (t_1, \dots, t_k) \in \mathbb{C}^{mk}$, $z \in (r\Delta)^n$, we use the notation $\mathbb{L}^k_{t_{(k)}}(z) = L^k_{t_k} \circ \cdots \circ L^1_{t_1}(z)$ whenever the composition is defined. Anyway, after bounding $k \leq 3n$, it is clear that there exists $\delta > 0$ such that all maps $(t_{(k)}, z) \mapsto \mathbb{L}^k_{t_{(k)}}(z)$ are well-defined for $t_{(k)} \in (2\delta\Delta^m)^k$, $z \in (\frac{r}{2}\Delta)^n$ and satisfy $\mathbb{L}^k_{t_{(k)}}(z) \in (r\Delta)^n$, for all $t_{(k)} \in (2\delta\Delta^m)^k$, $z \in (\frac{r}{2}\Delta)^n$, $k \leq 3n$,

$\mathbb{L}^k \in \mathbb{S}^k$. The *geometric interpretation* is that the maps $t \mapsto L_{\alpha t}(z)$ are integral m-curves of the element $L_\alpha \in \mathbb{S}$ and more generally, that the maps $t_{(k)} \mapsto \mathbb{L}^k_{t_{(k)}}(z)$ can be visualized as multiple, *i.e.* composed, $\mathbb{L}$-integral m-curves with source the point z. Also, the point $z' = L^k_{t_k} \circ \cdots \circ L^1_{t_1}(z)$ is the endpoint of a piecewise smooth m-curve with origin z: follow L^1 during m-time $t_1, \ldots,$ follow L^k during m-time t_k. Now, following a well-established terminology, we shall say that a manifold $\Lambda \subset \Delta^n$ is called $\mathbb{S}$-*integral* if $T_z\Lambda \supset \mathrm{Span}_{\mathbb{C}}\mathbb{S}(z)$ for all $z \in \Lambda$. Then for each $L \in \mathbb{S}$, $L|_\Lambda$ is tangent to Λ. Thus, it is clear that any integral m-curve of an element $L \in \mathbb{S}$ with origin a point z of an $\mathbb{S}$-integral manifold Λ stays in Λ. Now, we introduce the following definitions. Let $z \in (\frac{r}{2}\Delta)^n$. The $\mathbb{S}$-*orbit of* z in Δ^n, $\mathcal{O}_{\mathbb{S}}(\Delta^n, z)$ is the set of all points $\mathbb{L}^k_{t_{(k)}}(z) \in (r\Delta)^n$ for all $t_{(k)} \in (\delta\Delta^m)^k$, $k \leq 3n$. Finally, As we will be interested only in the $\mathbb{S}$-orbit of 0, we can localize at $z = 0$. We shall say that the open set $(r\Delta)^n$ is $\mathbb{S}$-*minimal at* 0 if $\mathcal{O}_{\mathbb{S}}(\Delta^n, 0)$ contains a polydisc $(\varepsilon\Delta)^n$, $\varepsilon > 0$.

9.3. The orbit Theorem. Now, we propose a self-contained proof, inspired by [Sus] and which does not use Lie brackets, of the following special case of Nagano's theorem [N].

Theorem 9.4. ([N], [Sus]) *There exists $\varepsilon > 0$ such that the $\mathbb{S}$-orbit of 0 consists of a closed complex $\mathbb{S}$-integral manifold-piece $\mathcal{O}_0$ through 0 in the polydisc $(\varepsilon\Delta)^n$.*

It is easy to check that any $\mathbb{S}$-integral manifold Λ passing through the origin must contain the manifold-piece $\mathcal{O}_0$, so Theorem 9.4 explains what is the smallest such $\mathbb{S}$-integral manifold-piece. Also, as a coorollary, we see that the open set $(r\Delta)^n$ is $\mathbb{S}$-minimal at 0 if and only if the dimension of the $\mathbb{S}$-orbit is maximal, *i.e.* $\dim_{\mathbb{C}}\mathcal{O}_0 = n$. Let us introduce the following notation. By $\mathcal{V}_X(p)$, we shall denote a small open polydisc neighborhood of the point p in the complex manifold X.

Proof of Theorem 9.4. Notice that if $a = 1$, $\mathcal{O}_0$ is just the m-curve through 0 of the single element of $\mathbb{S}$, so we assume $a \geq 2$ in the sequel. The following definitions will generalize the notion of Segre multitype given in §7 above. At first, we need some preliminary. If $\mathbb{L}^k \in \mathbb{S}^k$ ($k \leq 3n$), we shall denote by $\Gamma_{\mathbb{L}^k}$ the holomorphic map $(\delta\Delta^m)^k \ni t_{(k)} \mapsto \mathbb{L}^k_{t_{(k)}}(0)$. By the definition of $\mathbb{S}$-orbits, it is clear that $\mathbb{L}^k_{t_{(k)}}(0) \in \mathcal{O}_{\mathbb{S}}(\Delta^n, 0)$. Also, let us recall that given $f : X \to Y$ a holomorphic map of *complex connected manifolds*, there exists a proper complex subvariety $Z \subset X$ with $\dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} X$ such that $\mathrm{rk}_{\mathbb{C},p}(f) = \max_{q\in X} \mathrm{rk}_{\mathbb{C},q}(f)$ for all $p \in X\backslash Z$. This integer is called the *generic rank of* f and we shall denote it by $\text{gen-rk}_{\mathbb{C}}(f)$. Of course, if U is an arbitrary open subset of X then we have $\text{gen-rk}_{\mathbb{C}}(f|_U) = \text{gen-rk}_{\mathbb{C}}(f)$, thanks to the principle of analytic continuation. Such properties of the generic rank of holomorphic maps will be of crucial importance in the construction of orbits and in the definition of the following integers. We construct indeed by induction a special sequence $\mathbb{L}^{*k} := (L^{*1}, \ldots, L^{*k})$, $k \in \mathbb{N}_*$, as follows. First, let us define $\mathbb{L}^{*a} = (L^{*1}, \ldots, L^{*a}) = (L_1, \ldots, L_a)$, *i.e.* $L^{*\alpha} = L_\alpha$ for $1 \leq \alpha \leq a$. By our assumption on $\mathbb{S}$ (shrinking δ if necessary), we have $\mathrm{rk}_{\mathbb{C},0}(\Gamma_{\mathbb{L}^{*a}}) = \text{gen-rk}_{\mathbb{C}}(\Gamma_{\mathbb{L}^{*a}}) = \mathrm{rk}_{\mathbb{C},t_{(a)}}(\Gamma_{\mathbb{L}^{*a}}) = am$, for all $t_{(a)} \in (\delta\Delta^m)^a$. Let $\alpha \in [\![1, a]\!]$. Given $\mathbb{L}^{*a}$ so defined, we define $\mathbb{L}^{*a}L_\alpha := (\mathbb{L}^{*a}, L_\alpha)$, an $(a+1)$-tuple of elements of $\mathbb{S}$, and we denote by $\Gamma_{\mathbb{L}^{*a}L_\alpha}$ the map $(\delta\Delta^m)^a \times (\delta\Delta^m) \ni (t_{(a)}, t_{a+1}) \mapsto L_{\alpha t_{a+1}} \circ \mathbb{L}^{*a}_{t_{(a)}}(0)$, which is consistent with our previous notations. Because $\Gamma_{\mathbb{L}^{*a}L_\alpha}((t_{(a)}, 0) \equiv \Gamma_{\mathbb{L}^{*a}}(t_{(a)})$, it is

clear that there exist well-defined integers $0 \leq e_1(\alpha) \leq e_1 \leq n - am$ satisfying

$$\text{gen-rk}_{\mathbb{C}}(\Gamma_{\mathbb{L}^{*a}L_\alpha}) := e_1(\alpha) + am, \qquad \text{and we set} \qquad e_1 := \sup_{1\leq\alpha\leq a} e_1(\alpha). \tag{9.5}$$

If $e_1 = 0$, our construction stops. If $e_1 > 0$, we choose an α with $e_1(\alpha) = e_1$ and we define the $(a+1)$-th vector field to be $L^{*a+1} := L_\alpha$ with this α. So we have completed the choice of $\mathbb{L}^{*a+1} := (\mathbb{L}^{*a}, L^{*a+1})$. Inductively now, we assume that $\mathbb{L}^{*k}$ is already defined, with corresponding integers $e_1 \geq 1, \dots, e_k \geq 1$ satisfying $\text{gen-rk}_{\mathbb{C}}(\Gamma_{\mathbb{L}^{*k}}) = am + e_1 + \cdots + e_k$. We shall write for short $e_{\{k\}} := e_1 + \cdots + e_k$. Let $\alpha \in [\![1, a]\!]$. To construct the $(k+1)$-th m-vector field L^{*k+1}, we define $\mathbb{L}^{*k}L_\alpha := (\mathbb{L}^{*k}, L_\alpha)$ and $\Gamma_{\mathbb{L}^{*k}L_\alpha} : (\delta\Delta^m)^k \times (\delta\Delta^m) \ni (t_{(k)}, t_{k+1}) \mapsto L_{\alpha t_{k+1}} \circ \mathbb{L}^{*k}_{t_{(k)}}(0)$. Again, because $\Gamma_{\mathbb{L}^{*k}L_\alpha}(t_{(k)}, 0) \equiv \Gamma_{\mathbb{L}^{*k}}(t_{(k)})$, it is clear that there are well-defined integers $0 \leq e_{k+1}(\alpha) \leq e_{k+1} \leq n - am - e_{\{k\}} \leq n - am - k$ satisfying

$$\text{gen-rk}_{\mathbb{C}}(\Gamma_{\mathbb{L}^{*k}L_\alpha}) := e_{k+1}(\alpha) + e_{\{k\}} + am, \qquad \text{and we set} \qquad e_{k+1} := \sup_{1\leq\alpha\leq a} e_{k+1}(\alpha). \tag{9.6}$$

If $e_{k+1} = 0$, our construction stops. If $e_{k+1} > 0$, we choose α with $e_{k+1}(\alpha) = e_{k+1}$ and we define the $(k+1)$-th vector field to be $L^{*k+1} = L_\alpha$, with this α. We define $e_{\{k+1\}} = e_{\{k\}} + e_{k+1}$. So we have completed the choice of $\mathbb{L}^{*k+1} := (\mathbb{L}^{*k}, L^{*k+1})$, hence also the choice of all the $\mathbb{L}^{*k+1}$. We can now define κ_0 to be the smallest integer such that $e_{\kappa_0+1} = 0$ and we set $\mu_0 := a+\kappa_0$. After $\Gamma_{\mathbb{L}^{*\mu_0}}$, *i.e.* for $k > \mu_0$, the generic ranks of the maps $\Gamma_{\mathbb{L}^k}$ cease to increase. Of course, we have the inequalities $\kappa_0 \leq n - am$ and $\mu_0 \leq n$ (since $m \geq 1$). Paralleling the definitions in §7, we shall call the integer μ_0 is a *minimality type* of $\mathbb{S}$ at 0. μ_0 depends on the subsequent choices of the α's. Although the integers μ_0 and e_k may very well depend on the subsequent choices of the α's which maximizes the subsequent generic ranks of the maps $\Gamma_{\mathbb{L}^{*k}L_\alpha}$, it will follow in the end of the proof of Theorem 9.6 that the integer $e = \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{S}}(\Delta^n, 0) = am + e_{\{\kappa_0\}}$ will not depend on the choices of the α's.

To summarize what we have done so far, we have constructed a μ_0-tuple of m-vector fields $\mathbb{L}^{*\mu_0} = (L_1, \dots, L_a, L^{*a+1}, \dots, L^{*\mu_0}) = (L^{*1}, \dots, L^{*\mu_0})$ which satisfies

$$m < 2m < \cdots < a(m-1) < am < am + e_1 < \cdots < am + e_{\{\kappa_0\}}. \tag{9.7}$$

We define $e := am + e_{\{\kappa_0\}}$ and we call the μ_0-tuple $(\overbrace{m, m, \dots, m}^{a \text{ times}}, e_1, \dots, e_{\kappa_0}) \in \mathbb{N}^{\mu_0}$ will be called a *minimality multitype* of $\mathbb{S}$ at 0, where $\mu_0 = a + \kappa_0$. We remark that if $a = 2$ as in the particular application to CR geometry given in §7, if we denote the doubleton $\mathbb{S} := \{L, \underline{L}\}$, then $\mathbb{L}^k_{t_{(k)}}(0)$ is written $\cdots \underline{L}_{t_2} \circ L_{t_1}(0)$ or $\cdots L_{t_2} \circ \underline{L}_{t_1}(0)$ and there is no other possibility. Therefore, if $a = 2$, there are exactly at most minimality multitypes. Furthermore, if the pair of vector fields $\{L, \underline{L}\}$ satisfies a particular symmetry condition $\sigma_*(L) = \underline{L}$, where σ is a biholomorphism, as the pair $\{\mathcal{L}, \underline{\mathcal{L}}\}$ in §5-7 does, then the two minimality multitypes must infact coincide. We believe that this explains a particular feature of the geometry of complexified Segre varieties. Now, we can state a slightly more precise version of Theorem 9.4 similar to that of Theorem 7.5 as follows.

Theorem 9.8. *Let $\mathbb{S} = \{L_\alpha\}_{1\leq\alpha\leq a}$ be as above be a system of m-vector fields over $\mathbb{C}^n$, with $n = am + d$, let μ_0 be a minimality type of $\mathbb{S}$ at 0, with $\mu_0 \leq a + d$ and let $(m, \dots, m, e_1, \dots, e_{\kappa_0})$ be the associated multitype, where $\mu_0 = a + \kappa_0$. Then there exists a μ_0-tuple of m-vector fields $\mathbb{L}^{*\mu_0} = (L^1, \dots, L^{*\mu_0}) \in \mathbb{S}^{\mu_0}$, there exists an element $t^*_{(\mu_0)} \in (\delta\Delta^m)^{\mu_0}$ arbitrarily close to the origin of the form $(t^*_1, \dots, t^*_{\mu_0-1}, 0)$, and there exists a neighborhood $\mathcal{W}^*$ of $t^*_{(\mu_0)}$ in $(\delta\Delta^m)^{\mu_0}$ such that after putting $\underline{\mathbb{L}}^{*\mu_0-1} := (L^{*\mu_0-1}, \dots, L^{*1})$ and $\underline{t}^*_{(\mu_0-1)} := (-t^*_{\mu_0-1}, \dots, -t^*_1)$, then we have:*

1) *The map $\Gamma_{\mathbb{L}^{*\mu_0}}$ is of rank $am + e_1 + \dots + e_{\kappa_0}$ at $t^*_{(\mu_p)}$.*

2) $\underline{\mathbb{L}}^{*\mu_0-1}_{\underline{t}^*_{(\mu_0-1)}} \circ \mathbb{L}^{*\mu_0}_{t^*_{\mu_0}}(0) = 0$.

3) *The map $\underline{\mathbb{L}}^{*\mu_0-1}_{\underline{t}^*_{(\mu_0-1)}} \circ \Gamma_{\mathbb{L}^{*\mu_0}} : \mathcal{W}^* \times \underline{t}^*_{(\mu_0-1)} \to \Delta^n$ is of constant rank equal to $am + e_1 + \dots + e_{\kappa_0}$.*

4) *Its image $\underline{\mathbb{L}}^{*\mu_0-1}_{\underline{t}^*_{(\mu_0-1)}} \circ \Gamma_{\mathbb{L}^{*\mu_0}}(\mathcal{W}^*) = \mathcal{O}_0$ is a manifold-piece through 0 of the $\mathbb{S}$ orbit $\mathcal{O}_{\mathbb{S}}(\Delta^n, 0)$. This means that every element of $\mathbb{S}$ is tangent to $\mathcal{O}_0$ and that every $\mathbb{S}$-integral manifold-piece Λ_0 through 0 must* contain *$\mathcal{O}_{\mathbb{S}}(\Delta^n, 0)$.*

5) $am + e_1 + \dots + e_{\kappa_0} = \dim_{\mathbb{C}} \mathcal{O}_0$.

As the proof of Theorem 9.8 now goes essentially the same way as the proof of Theorem 7.5, we shall omit to repeat the detailed arguments here. The proof of Theorem 9.6 is complete. □

Now, a final remark. As the above considerations are of course valuable in the $\mathbb{R}$-analytic category, we can deduce from Theorem 9.6 the theorem of Nagano about existence of CR orbits of a CR-generic $\mathcal{C}^\omega$ manifold M as in §5-7, which we recovered in Proposition 6.5 after passing to the extrinsic complexification.

§10. Segre geometry of formal CR manifolds

Some (straightforward) adaptations are needed to develope the theory of Segre chains in the *real and complex algebraic category*, in particular to establish the algebraicity of CR-orbits, *etc.* The reader may consult *e.g.* [M2] for a complete account about such modifications. Also, it is easy to observe that most of the theory of Segre chains extends to the category of formal CR manifolds. Such objects are given by the ring $\mathbb{C}[\![t, \bar{t}]\!]/((\rho_j(t, \bar{t}))_{1\leq j\leq d})$, where the d formal power series $\rho_j(t, \bar{t}) \in \mathbb{C}[\![t, \bar{t}]\!]$ satisfy $\rho_j(0) = 0$ and $\partial\rho_1(0) \wedge \dots \wedge \partial\rho_d(0) \neq 0$ and the theory builds up from the ρ_j. In a recent preprint, entitled *Dynamics of the Segre varieties of a real submanifold in complex space* (arXiv.org/abs/math/0008112), Baouendi-Ebenfelt-Rothschild endeavour such a study (but without the exposition of our geometric viewpoint). In fact, almost each one of the geometric concepts we have encountered in our route has an obvious *formal counterpart*: the *formal* extrinsic complexification, the *formal* (conjugate) Segre varieties, the antiholomorphic involution, the *formal* regular coordinates, the *formal* vector fields as in (5.10) and their *formal* flow as in (6.2) (one can even speak in a rigorous sense of formal local foliations...). Indeed, the definition in eq. (6.2) obviously has a formal meaning. The important fact is that in the definition of Segre chains, we indeed define inductively *formal* power series, as show eqs. (6.3), (6.4). The formal counterpart of the notion of generic rank is simply the nonvanishing of a suitable minor in the ring of formal power series. Thus, all our considerations extend straightforwardly to the formal

category, except perharps for the proof of the orbit Theorem 7.5, because we use a *nonzero* point $w^*_{(\mu_p)}$ at which the "value" of a formal power series would be senseless. However, the reader may check that if well interpreted in the formal category, the important map written in **5)** of Theorem 7.5 is in fact a formal power series vanishing at $p^c = 0$, whence the proof we give can easily be generalized, as desired.

Important remark. Does this theory have deep applications to the smooth category ? We believe not. Indeed, the reader should be aware that the formal orbits *at a point* of system of *smooth* vector fields *does not* represent appropriately the *smooth* orbit of that point, as shows the example in $\mathbb{R}^2$ with $L = \frac{\partial}{\partial x}$ and $L' = e^{\frac{-1}{x^2}} \frac{\partial}{\partial y}$. Here, the smooth orbit of 0 is a neighborhood of 0 in $\mathbb{R}^2$ whereas the formal orbit of the Taylor expansions of L and L' at 0 is the (formal) x-line at 0. To pursue our informal review of these questions, let us note that it is easy to show that in general, the formal orbit is "contained" in the smooth orbit (exercise : find the rigorous sense of this assertion and prove it). As expectable, this shows that the formal theory is much closer to the analytic one than to the smooth one. Further, it also explains why the naive theory of formal CR orbits should have very poor applications to solve the fine questions of regularity about *smooth* CR mappings.

§11. Application of the formalism to the regularity of CR mappings

We provide here a brief summary of what are the main properties to understand the CR mapping regularity problems in terms of our formalism and especially in terms of *the flows of* $\mathcal{L}$ *and of* $\underline{\mathcal{L}}$. The reader is referred to [BER2,3] and to [M3] for a complete account of how the methods work and what are the main results.

11.1. Holomorphic and formal maps of analytic CR manifolds. We consider a formal or holomorphic invertible mapping $h\colon (M,p) \to (M',p')$ between two real analytic CR manifolds. In coordinates $t \in \mathbb{C}^n$ vanishing at p and $t' \in \mathbb{C}^n$ vanishing at p', this map $h(t) = (h_1(t), \dots, h_n(t))$ is an n-tuple of power series with $h_j(t) \in \mathbb{C}\{t\}$ or $h_j(t) \in \mathbb{C}[\![t]\!]$ vanishing at 0, $h_j(0) = 0$, and its jacobian determinant $\det\,(\frac{\partial h_j}{\partial t_k}{}_{1\leq j,k\leq n}(0)) \neq 0$ is nonzero at 0. We denote by $z_j = \bar{Q}_j(w,\tau)$, $1 \leq j \leq d$ and $z'_j = \bar{Q}'_j(w',\tau')$ some real analytic equations of $\mathcal{M}$ and of $\mathcal{M}'$. According to the splitting of coordinates (w',z'), we split the map h in $h := (g,f)$. Then the assumption that h maps $\mathcal{M}$ into $\mathcal{M}'$ can be simply restated by saying that the d-vectorial power series (formal or converging) $\bar{f}(\tau) - Q'(\bar{g}(\tau), h(t)) \in \mathbb{C}\{t,\tau\}^d$ (or $\in \mathbb{C}[\![t,\tau]\!]^d$) is identically zero in $\mathbb{C}\{w,\zeta,\xi\}^d$ (or in $\mathbb{C}[\![w,\zeta,\xi]\!]^d$) after replacing z by $\bar{Q}(w,\tau)$ (or after replacing ξ by $Q(\zeta,t)$ in $\mathbb{C}\{w,z,\zeta\}^d$ or in $\mathbb{C}[\![w,z,\zeta]\!]^d$). Of course, we insist on the parallelism about the two (equivalent) possibilities of formulating this restatement in coordinates. For short, we shall say that $\bar{f}(\tau) \equiv \bar{\Theta}'(\bar{g}(\tau), h(t))$ on $\mathcal{M}$. We denote by $h^c := (h, \bar{h})$ the holomorphic (or formal) map $\mathcal{M} \to \mathcal{M}'$. Finally, we introduce the m-vector fields $\mathcal{L}$ and $\underline{\mathcal{L}}$ of §5 and we abbreviate their multiple concatenated flow maps (presented in §2.2 and in §6) by Γ_k and by $\underline{\Gamma}_k$.

11.2. Interest of flows of CR vector fields. In terms of the flows of the CR vector fields, the recipe for understanding the results of [BER2,3] and of [M3] is the following. One of the main assumption in the results therein is that (M,p) *is minimal*. Without entering into all considerations, we shall explain what is the

central role played by the maps Γ_k and we shall put in perspective the interest of considering *simultaneously* some *reflection identities* together with some *conjugate reflection identities*, as

(I) The first easy remark is the following. As $\Gamma_k(w_{(k)}) \in \mathcal{M}$ for all $k \in \mathbb{N}$, we have $h^c(\Gamma_k(w_{(k)})) \in \mathcal{M}'$ for all $k \in \mathbb{N}$. At the formal power series level, this property is expressed by the following equation:

$$\bar{f}(\Gamma_k(w_{(k)})) \equiv \Theta'(\bar{g}(\Gamma_k(w_{(k)})), h(\Gamma_k(w_{(k)}))) \quad \text{in} \quad \mathbb{C}[\![w_{(k)}]\!]^d. \tag{11.3}$$

(II) By Theorem 7.5, the minimality of (M, p) is equivalent to the fact that $\Gamma_{2\mu_p-1}$ is a submersion onto $(\mathcal{M}, p^c)$. Then the relation (11.3) for $k \geq 2\mu_p-1$ becomes equivalent to the fundamental identity $\bar{f}(\tau) \equiv Q'(\bar{g}(\tau), h(t))$ on $\mathcal{M}$. Incidentally, this shows how minimality plays its first role.

(III) Now, we come to the most important step. For all $\beta \in \mathbb{N}^m$, we consider the derivations $\underline{\mathcal{L}}^\beta := \underline{\mathcal{L}}^{1,\beta_1} \cdots \underline{\mathcal{L}}^{m,\beta_m}$ and we apply them to the fundamental equation $\bar{f}(\tau) \equiv \sum_{\gamma \in \mathbb{N}^m} \bar{g}(\tau)^\gamma \, Q'_\gamma(h(t))$. This process is very classical. We get an infinite family of equations, called *reflection identities*, of the form

$$\underline{\mathcal{L}}^\beta(\bar{f}) = \sum_{\gamma \in \mathbb{N}^m} \underline{\mathcal{L}}^\beta(\bar{g}(\tau)^\gamma) \; Q'_\gamma(h(t)), \qquad \forall \, \beta \in \mathbb{N}^m. \tag{11.4}$$

However, we could also have applied these derivations to the *conjugate* equation $f(t) \equiv \sum_{\gamma \in \mathbb{N}^m} g(t)^\gamma \, \bar{Q}'_\gamma(h(\tau)))$ on $\mathcal{M}$ *and since there is no reason why not to do it, we do it.*

(IV) Consequently, we obtain two *a priori* different families of equations:

$$\begin{cases} (*): \quad f \equiv \bar{Q}'(g, \bar{h}), \qquad 0 \;\equiv \displaystyle\sum_{\gamma \in \mathbb{N}^m} g^\gamma \; \underline{\mathcal{L}}^\beta(\bar{Q}'_\gamma(\bar{h})), \qquad \forall \, \beta \in \mathbb{N}^m_*. \\ (\underline{*}): \quad \bar{f} = Q'(\bar{g}, h), \quad \underline{\mathcal{L}}^\beta \bar{f} \;\equiv \displaystyle\sum_{\gamma \in \mathbb{N}^m} \underline{\mathcal{L}}^\beta(\bar{g}^\gamma) \; Q'_\gamma(h), \qquad \forall \, \beta \in \mathbb{N}^m_*. \end{cases} \tag{11.5}$$

These equations should be understood with $(t, \tau) \in \mathcal{M}$. Their respective bar-conjugates are the following:

$$\begin{cases} (\bar{*}): \quad \bar{f} \equiv Q'(\bar{g}, h), \qquad 0 \;\equiv \displaystyle\sum_{\gamma \in \mathbb{N}^m} \bar{g}^\gamma \; \mathcal{L}^\beta(Q'_\gamma(h)), \qquad \forall \, \beta \in \mathbb{N}^m_*. \\ (\underline{\bar{*}}): \quad f = \bar{Q}'(g, \bar{h}), \quad \mathcal{L}^\beta f \;\equiv \displaystyle\sum_{\gamma \in \mathbb{N}^m} \mathcal{L}^\beta(g^\gamma) \; \bar{Q}'_\gamma(\bar{h}), \qquad \forall \, \beta \in \mathbb{N}^m_*. \end{cases} \tag{11.6}$$

In summary, we obtain *four families of (conjugate) reflection identities.* To our knowledge, the system $(\underline{*})$ (or its bar-conjugate $(\underline{\bar{*}})$) is nowhere considered in the previous works on the subject, whereas $(*)$ and its bar-conjugate $(\bar{*})$ are always used. The interest of the system $(\underline{*})$ and the important information that it adds to study the regularity of the formal CR-reflection mapping is argued in [M3], but we cannot enter into all the details here. We just want to say here that according to our heuristic

principle of studying everything *as pairs* in analytic CR geometry, we have discovered new reflection identities which are *not completely equivalent* to the known ones.

(V) Finally, we would like to make an last important observation which shows how the flow maps $(w,q) \mapsto \mathcal{L}_w(q)$ and the reflection identities are linked together. Recall that if $q(x) \in \mathbb{C}\{x\}^{2n}$ is a series vanishing at 0 with $q(x) \in \mathcal{M}$, for $x \in \mathbb{C}^\nu$, then the derivative $[\partial_\zeta^\beta(\underline{\mathcal{L}}_\zeta(h^c(q(x)))]|_{w=0}$ is equal to $[\underline{\mathcal{L}}^\beta h^c](q(x))$ (just by definition of flows). Then to obtain eqs. (11.4), it is equivalent to differentiate the fundamental equation $\bar{f} = Q'(\bar{g}, h)$ with respect to $\underline{\mathcal{L}}^\beta$ as usual or to take the differentiations

$$\partial_\zeta^\beta|_{\zeta=0}[\bar{f}(\underline{\mathcal{L}}_\zeta(\tau))] = \sum_{\gamma\in\mathbb{N}^m} \partial_\zeta^\beta|_{\zeta=0}[\bar{g}(\underline{\mathcal{L}}_\zeta(\tau))^\gamma\ Q'_\gamma(h(\underline{\mathcal{L}}_\zeta(t)))], \qquad \forall\ \beta \in \mathbb{N}^m. \tag{11.7}$$

In particular, when differentiating eqs. (11.3) with respect to w_k at $w_k = 0$, we obtain either the system $(\underline{*})$ (if k is even) or the system $(\overline{\underline{*}})$ at the point $(t,\tau) = \Gamma_{k-1}(w_{(k-1)})$ (if k is odd). In conclusion to this discussion and to this article, we would like to say that such an identity (11.7) is one of the key fact which explains the mystery of the CR regularity properties in the minimal case, because *this identity exhibits a natural relation between the reflection identities and the flows generating the Segre chains, some two a priori different objects which reveal therefore to be in fact intimately related with each other.*

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