Algebraic K-theory, $\mathbf{A}^1$-homotopy and Riemann-Roch theorems

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Abstract

In this article, we show that the combination of the constructions done in SGA 6 [1] and the $\mathbf{A}^1$-homotopy theory [32] naturally leads to results on higher algebraic K-theory. This applies to the operations on algebraic K-theory, Chern characters and Riemann-Roch theorems.

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The starting point of this article is the theorem which represents the algebraic K-theory of regular schemes in the $\mathbf{A}^1$-homotopy theory:

Theorem 0.1 (Morel-Voevodsky [32, Theorem 3.13, page 140]) *Let S be a regular scheme. Then, for any $n\in\mathbf{N}$ and $X\in\mathbf{Sm}/S$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(S^n\wedge X_+, \mathbf{Z}\times\mathbf{Gr}) \simeq K_n(X)\ .$$

Here, $\mathbf{Gr}$ is the colimit of the system $(\mathbf{Gr}_{d,r})_{(d,r)\in\mathbf{N}^2}$ in the category of presheaves over $\mathbf{Sm}/S_{\mathrm{Nis}}$ where $\mathbf{Gr}_{d,r}$ is the Grassmann scheme which parametrises subbundles of rank d in the trivial bundle of rank $d+r$. To make the definition of the transition morphisms unambiguous enough, we may say that they are of the form $\mathbf{Gr}_{d,r}\to\mathbf{Gr}_{1+d,r}$ and $\mathbf{Gr}_{d,r}\to\mathbf{Gr}_{d,r+1}$ and that the place where "1" appears tells us on which side a trivial bundle of rank 1 is added.

It should be pointed out that theorem 0.1 only applies to regular schemes because it can be true only over schemes where the algebraic K-theory is known to be $\mathbf{A}^1$-invariant. This is the reason why the assumption that the base scheme is regular will appear throughout the paper.

From theorem 0.1, it follows that the endomorphisms of $\mathbf{Z}\times\mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$ act on all the algebraic K-groups of schemes in $\mathbf{Sm}/S$. The basic result we obtain in section 1 is that these endomorphisms are completely characterised by their action on K_0:

Theorem 0.2 *Let S be a regular scheme. We let $K_0(-)$ be the presheaf of sets on $\mathbf{Sm}/S$ which maps X to $K_0(X)$. Then, the map induced by theorem 0.1 is a bijection:*

$$\mathrm{End}_{\mathcal{H}(S)}(\mathbf{Z}\times\mathbf{Gr}) \xrightarrow{\sim} \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-))\ ,$$

where $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$ is the category of presheaves of sets on $\mathbf{Sm}/S$.

It follows that the operations defined in [1] at the level of K_0 (*e.g.*, λ^n, Ψ^k) uniquely lift in $\mathcal{H}(S)$. From there, using theorem 0.1, we can make them act on higher algebraic K-theory. This principle also works for operations involving several operands (*e.g.*, products) and in a sense which will be made precise in section 2, we obtain a machinery which takes as an input the algebraic structures on K_0 and outputs such a structure on $\mathbf{Z}\times\mathbf{Gr}$ inside $\mathcal{H}(S)$. Thus, $\mathbf{Z}\times\mathbf{Gr}$ is equipped with a structure of special λ-ring with duality.

Structures of (special) λ-ring had already been obtained on higher K-theory, with different scales of generality. We may mention constructions of products, λ-operations or Adams operations by Loday [30], Waldhausen [47], Kratzer [26], Soulé [42], Grayson [18], Lecomte [27] and Levine [28]. We compare the structures on $K_\star(X)$ for X regular obtained by our method to these previous constructions in section 3. The comparison with Waldhausen's product (see proposition 3.2.1) may seen surprisingly straightforward, but it is a typical use of theorem 0.2 and its variants involving several operands (see theorem 1.1.4).

Section 4 relates our results to virtual categories, an insight of Deligne [10]. We show that, after inverting 2, constructions done at the level of K_0 refine to these virtual categories, which embodies both K_0 and K_1. This theory was used by Dennis Eriksson in his thesis [12] in order to refine Riemann-Roch theorems at the level of these virtual categories.

In section 5, we focus on operations $\tau\colon K_0(-) \to K_0(-)$ such that $\tau(x+y) = \tau(x)+\tau(y)$, *i.e.*, H-group endomorphisms of $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$. We compute them using the splitting principle. We show that the datum of τ is equivalent to the datum of an element in $K_0(S)[[U]]$. Then, we construct, up to a unique isomorphism in the stable homotopy category $\mathcal{SH}(S)$, the $\mathbf{P}^1$-spectrum $\mathbf{BGL}$ which represents algebraic K-theory and study its endomorphisms (it is somewhat related but quite different from the methods of [4, Chapter 6], [3] and [5]). After tensoring with $\mathbf{Q}$, we show that this spectrum decomposes in $\mathcal{SH}(S)$ as the direct sums of "eigenspaces" for the Adams operations. Alternate interesting descriptions of stable operations on algebraic K-theory (and more general oriented theory) have been obtained by very different methods by Naumann, Østvær and Spitzweck in [33].

We prove in section 6 that these ideas can be used to obtain an homotopical variant of some Riemann-Roch theorems in the case of a smooth and projective morphism $f\colon X \to S$. Basically, we prove that certain Riemann-Roch formulas are satisfied on zeroth K-groups if and only if they are satisfied on the whole higher algebraic K-theory. In that section, we give formulas for the group of morphisms $\mathbf{BGL} \to \mathbf{H}_A[n]$ in $\mathcal{SH}(k)$ where k is a perfect field and $\mathbf{H}_A$ the motivic Eilenberg-Mac Lane spectrum with coefficients in A. This computation gives a simple example of nonzero stably phantoms morphisms in the $\mathbf{P}^1$-stable homotopy category $\mathcal{SH}(k)$: all morphisms $\mathbf{BGL} \to \mathbf{H}_{\mathbf{Z}}[1]$ are stably phantoms. There is an homologous computation in the standard topological stable homotopy category: this gives a more concrete example than the one constructed in [8, Proposition 6.10].

If section 6 stands as a significant exception, most of these results appeared in my thesis [36] and were announced in [38] (however, when different proofs were available, my choices have tended to be different). Hence, I would like to thank Yves André, Joseph Ayoub, Denis-Charles Cisinski, Frédéric Déglise, Dennis Eriksson, Hinda Hamraoui, Bruno Kahn, Florence Lecomte, Georges Maltsiniotis, Fabien Morel, Christophe Soulé, Burt Totaro, Jörg Wildeshaus for their useful comments or discussions.

1 First unstable results

1.1 Statements

In this paper, we shall say that a scheme is regular if it is noetherian separated and that all its local rings are regular local rings (see [40, IV §D]). For any scheme S, the category of smooth and separated schemes of finite type over S is denoted $\mathbf{Sm}/S$.

For regular schemes, all the standard definitions of algebraic K-theory agree. Then, we may define some objects in the category $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$ of presheaves of sets over $\mathbf{Sm}/S$: for any natural number n, the presheaf that maps $X \in \mathbf{Sm}/S$ to its nth algebraic K-group $K_n(X)$ is denoted $K_n(-)$.

Theorem 1.1.1 *Let S be a regular scheme. For any natural transformation $\tau\colon K_0(-) \to K_0(-)$ of presheaves of sets on $\mathbf{Sm}/S$ such that $\tau(0) = 0$, there is a unique reasonable way to define an extension of $\tau\colon K_n(-) \to K_n(-)$ for all n.*

This theorem is a consequence of the following $\mathbf{A}^1$-homotopy theoretic statement:

Theorem 1.1.2 *Let S be a regular scheme. Then, the canonical map induced by the isomorphism of theorem 0.1 is a bijection:*

$$\mathrm{End}_{\mathcal{H}(S)}(\mathbf{Z} \times \mathbf{Gr}) \xrightarrow{\sim} \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-)) .$$

Indeed, if $\tau\colon K_0(-) \to K_0(-)$ is a natural transformation, the theorem says that there exists a unique morphism $\tilde{\tau}\colon \mathbf{Z} \times \mathbf{Gr} \to \mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ inducing τ on $K_0(-)$. As $\mathbf{Z} \times \mathbf{Gr}$ has a structure of H-group (see [32, page 139]), if we assume $\tau(0) = 0$, then we see that $\tilde{\tau}$ can be identified to an endomorphism of $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$. Such endomorphisms not only induce natural transformations on $K_0(-)$ but also on $K_n(-)$ for all n as one may evaluate them on higher homotopy groups.

This theorem applies to operations like the λ-operations λ^n for all $n \in \mathbf{N}$ [1, V 2.2 b], γ-operations γ^n for all $n \in \mathbf{N} - \{0\}$ [1, V 3.2] and Adams operations Ψ^k [1, V 7.1] for all $k \in \mathbf{Z}$. Then, to construct these operations on higher K-groups, the only specific information we need to know is how to define them on K_0, which is usually easy using the presentation of these groups by generators and relations.

Remark 1.1.3 *One can prove similar results for $\mathbf{Gr}$ instead of $\mathbf{Z} \times \mathbf{Gr}$: endomorphisms of $\mathbf{Gr}$ in $\mathcal{H}(S)$ identify to endomorphisms of $\tilde{K}_0(-)$ in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$ where $\tilde{K}_0(X)$ is the kernel of the rank map $K_0(X) \to \mathbf{Z}^{\pi_0(X)}$. Moreover, in the situation of theorem 1.1.1, if we use the fact that the loop space $\mathbf{R}\Omega(\mathbf{Z})$ of $\mathbf{Z}$ is $\bullet$, we see that $\tau\colon K_n(-) \to K_n(-)$ for $n \geq 1$ only depends on the natural transformation $\tilde{K}_0(-) \to \tilde{K}_0(-)$ induced by $\tau\colon K_0(-) \to K_0(-)$.*

The operations considered above are unary operations on algebraic K-theory. One may also consider operations involving several operands (*e.g.*, the product law $K_0(X) \times K_0(X) \to K_0(X)$):

Theorem 1.1.4 *Let S be a regular scheme. Let n be a natural number. Then, the canonical map is a bijection:*

$$\mathrm{Hom}_{\mathcal{H}(S)}((\mathbf{Z} \times \mathbf{Gr})^n, \mathbf{Z} \times \mathbf{Gr}) \to \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-)^n, K_0(-)) .$$

As we shall see, the method of the proof allows to consider not only operations on algebraic K-theory but also maps from algebraic K-theory to other cohomology theories. However, we need to know that the cohomology theory is represented by an object in $\mathcal{H}_\bullet(S)$, which means that it can be expressed as homotopy presheaves of an object in $\mathcal{H}_\bullet(S)$:

Definition 1.1.5 *Let S be a noetherian scheme. Let E be an object in $\mathcal{H}(S)$. We let $\pi_0 E$ be the presheaf of sets on $\mathbf{Sm}/S$ defined by $\pi_0 E(X) = \mathrm{Hom}_{\mathcal{H}(S)}(X, E)$. If E belongs to $\mathcal{H}_\bullet(S)$ and n is any natural number, we define a presheaf $\pi_n E$ by the formula $\pi_n E(X) = \pi_0 \mathbf{R}\Omega^n E$, where $\mathbf{R}\Omega\colon \mathcal{H}_\bullet(S) \to \mathcal{H}_\bullet(S)$ is the loop space functor.*

Theorem 0.1 states that for any natural number n and S a regular scheme, we have a canonical isomorphism $\pi_n(\mathbf{Z} \times \mathbf{Gr}) \simeq K_n(-)$ in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$.

Theorem 1.1.6 *Let S be a regular scheme. Let E be an object in $\mathcal{H}_\bullet(S)$. If we assume that E satisfies property (K) (a mild technical assumption, see definition 1.2.2), then the canonical map is a bijection:*

$$\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{Z}\times\mathbf{Gr}, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-), \pi_0 E)\ .$$

This set of morphisms can also be identified to an infinite product indexed by $\mathbf{Z}$ of copies of the projective limit $\lim_{(d,r)\in\mathbf{N}^2}\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{Gr}_{d,r}, E)$.

As above, there is a similar homotopical description of natural transformations $K_0(-)^n \to \pi_0 E$ involving n operands.

We may focus on the 1-operand case. If a natural transformation $\tau\colon K_0(-)\to\pi_0 E$ verifies $\tau(0)=0$, it corresponds to a unique morphism $\mathbf{Z}\times\mathbf{Gr}\to E$ in $\mathcal{H}_\bullet(S)$. Then, in the same way we mentioned it for operations on algebraic K-theory, τ will induce natural transformations $\tau\colon K_n(-)\to\pi_n E$ for all n.

The proof of theorems 1.1.2, 1.1.4 and 1.1.6 will also supply a concrete computation of the set of all operations on algebraic K-theory. In the 1-operand case, it gives:

Theorem 1.1.7 *Let S be a regular scheme. The sets of endomorphisms $\mathrm{End}_{\mathcal{H}(S)}(\mathbf{Z}\times\mathbf{Gr}) \simeq \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-))$ can be identified to the product $R^{\mathbf{Z}}$ of an infinite number of copies of a ring $R = K_0(S)[[\tilde{\gamma}^1, \tilde{\gamma}^2, \dots]]$ of formal power series with an infinite number of variables and coefficient ring $K_0(S)$. The elements $\tilde{\gamma}^n$ are related to the usual γ-operations on algebraic K-theory.*

The computation of the set of morphisms $\mathbf{Z}\times\mathbf{Gr}\to\mathbf{R}\Omega^i(\mathbf{Z}\times\mathbf{Gr})$ in $\mathcal{H}(S)$ is given by a similar formula, where $K_0(S)$ is replaced by $K_i(S)$.

1.2 Proofs

Lemma 1.2.1 *Let S be a noetherian scheme. Let E be a group object in $\mathcal{H}_\bullet(S)$ (*i.e.*, E is an H-group). Let $(X_i)_{i\in\mathcal{I}}$ be a direct system indexed by a directed ordered set $\mathcal{I}$. The colimit of this system in the category of presheaves over $\mathbf{Sm}/S$ is denoted $\mathcal{X}$. We assume that $\mathcal{I}$ has a cofinal sequence (*i.e.*, there exists a functor $x\colon\mathbf{N}\to\mathcal{I}$ such that for any $i\in I$, there exists $n\in\mathbf{N}$ such that $i\leq x_n$). Then, there is an exact sequence of groups.*

$$1\to\mathbf{R}^1\lim_{i\in\mathcal{I}}\pi_1 E(X_i)\to\mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{X},E)\to\lim_{i\in\mathcal{I}}\pi_0 E(X_i)\to 1\ .$$

Using a cofinal sequence $\mathbf{N}\to\mathcal{I}$, one may assume that $\mathcal{I}=\mathbf{N}$. In that case, it follows from the usual Milnor exact sequence [17, Proposition VI.2.15].

Definition 1.2.2 *With the notations of lemma 1.2.1, we say that the direct system $(X_i)_{i\in\mathcal{I}}$ does not unveil phantoms in E if the group $\mathbf{R}^1\lim_{i\in\mathcal{I}}\pi_1 E(X_i)$ vanishes. We say that E satisfies property (K) if the direct system $(\mathbf{Gr}_{d,r})_{(d,r)\in\mathbf{N}^2}$ does not unveil phantoms in E. More generally, for any natural number n, we say that E satisfies property (K) with n operands if the direct system $(\prod_{i=1}^n\mathbf{Gr}_{d_i,r_i})_{(d_1,r_1,\dots,d_n,r_n)\in\mathbf{N}^{2n}}$ does not unveil phantoms in E.*

Thus, whenever an inductive system $(X_i)_{i\in\mathcal{I}}$ does not unveil phantoms in E, the datum of a morphism $\operatorname{colim}_{i\in\mathcal{I}} X_i\to E$ in $\mathcal{H}(S)$ is equivalent to the datum of a compatible family of morphisms $X_i\to E$ in $\mathcal{H}(S)$.

Definition 1.2.3 *We let $\mathcal{T}$ be the family of morphisms in $\mathbf{Sm}/S$ of the form $T \to X$ where T is a torsor under a vector bundle over X.*

Locally on the base, morphisms in $\mathcal{T}$ are of the form $\mathbf{A}^n \times X \to X$. This implies that they induce $\mathbf{A}^1$-weak equivalences. The important fact we need about this family of maps is:

Theorem 1.2.4 (Jouanolou [23, Lemme 1.5], Thomason [49, Proposition 4.4])
Let S be a regular scheme. For any $X \in \mathbf{Sm}/S$, there exists a morphism $T \to X$ in $\mathcal{T}$ such that T is an affine scheme.

We require that the scheme T is affine; as S is separated, it implies that $T \to S$ in an affine morphism, but the converse implication is not true. In the sequel, the word "affine" will be used in that absolute sense only.

Definition 1.2.5 *Let S be a regular scheme. Let $\mathcal{X}$ be a presheaf of sets on $\mathbf{Sm}/S$. Then $\mathcal{X}$ defined an object in $\mathcal{H}(S)$ and a presheaf of sets $\pi_0\mathcal{X}$ is attached to it. We say that $\pi_0\mathcal{X}$ is generated by $\mathcal{X}$ up to $\mathcal{T}$ if for any affine scheme $U \in \mathbf{Sm}/S$, the map $\mathcal{X}(U) \to \pi_0\mathcal{X}(U)$ is onto.*

We will give an explanation for this terminological choice in remark 1.2.7. First, we see how one may apply this definition to algebraic K-theory:

Lemma 1.2.6 *Let S be a regular scheme. If $\mathcal{X} = \mathbf{Z} \times \mathbf{Gr}$, then $\pi_0\mathcal{X}$ is generated by $\mathcal{X}$ up to $\mathcal{T}$. The same conclusion applies to $(\mathbf{Z} \times \mathbf{Gr})^n$ for any natural number n and also to $(\mathbf{P}^\infty)^n$.*

Obviously, the condition we have to check is stable under finite products. Then, we shall first focus on the case $\mathcal{X} = \mathbf{Z} \times \mathbf{Gr}$. It is implicit in the proof of theorem 0.1 that for any $n \in \mathbf{Z}$ and $(d, r) \in \mathbf{N}^2$, if we consider the canonical inclusion $\iota_{d,r,n} \colon \mathbf{Gr}_{d,r} = \{n\} \times \mathbf{Gr}_{d,r} \to \mathbf{Z} \times \mathbf{Gr}$ as an element in $\mathcal{X}(\mathbf{Gr}_{d,r})$, its image in $\pi_0\mathcal{X}(\mathbf{Gr}_{d,r})$ corresponds to the class $[\mathcal{M}'_{d,r}] - d + n$ in $K_0(\mathbf{Gr}_{d,r})$ under the isomorphism of theorem 0.1, where $\mathcal{M}'_{d,r}$ is the universal vector bundle of rank d on $\mathbf{Gr}_{d,r}$. Then, the lemma follows from the obvious fact that if U is a connected affine scheme in $\mathbf{Sm}/S$, any class $x \in K_0(U)$ is of the form $x = [\mathcal{M}] - d + n$ for some integers d, n, and $\mathcal{M}$ a vector bundle of rank d on U. Indeed, as U is affine, $\mathcal{M}$ is isomorphic to a direct factor of $\mathcal{O}_U^{d+r}$ for a big enough r. Then, by definition of Grassmann varieties, there exists an S-morphism $f \colon U \to \mathbf{Gr}_{d,r}$ such that $f^\star\mathcal{M}'_{d,r} \simeq \mathcal{M}$. It follows that the element in $\mathcal{X}(U)$ corresponding to the composition $\iota_{d,r,n} \circ f \colon U \to \mathcal{X}$ maps to $x = f^\star([\mathcal{M}'_{d,r} - d + n])$ in $\pi_0\mathcal{X}(U) \simeq K_0(U)$.

The case $\mathcal{X} = \mathbf{P}^\infty$ is similar: it uses the identification $\pi_0\mathbf{P}^\infty = \mathrm{Pic}(-)$, see [32, Proposition 3.8, page 138].

Remark 1.2.7 *The category of presheaves on $\mathbf{Sm}/S$ contains the full subcategory of the category of presheaves $\mathcal{X}$ such that for any $f \colon T \to X$ in $\mathcal{T}$, the map $f^\star \colon \mathcal{X}(X) \to \mathcal{X}(T)$ is a bijection. This subcategory can be identified to the category of presheaves on the localised category $\mathbf{Sm}/S[\mathcal{T}^{-1}]$ (see [15, Lemma I.1.2]). For any presheaf $\mathcal{X}$ on $\mathbf{Sm}/S$, there exists a universal presheaf $\mathcal{X}[\mathcal{T}^{-1}]$ on $\mathbf{Sm}/S[\mathcal{T}^{-1}]$ equipped with a morphism $\mathcal{X} \to \mathcal{X}[\mathcal{T}^{-1}]$ (see [2, I 5.1]). As $\pi_0\mathcal{X}$ factors through $\mathbf{Sm}/S[\mathcal{T}^{-1}]$, the canonical morphism $\mathcal{X} \to \pi_0\mathcal{X}$ induces a morphism $\mathcal{X}[\mathcal{T}^{-1}] \to \pi_0\mathcal{X}$. Using theorem 1.2.4, it is easy to check that the condition stated in definition 1.2.5 implies that $\mathcal{X}[\mathcal{T}^{-1}] \to \pi_0\mathcal{X}$ is an epimorphism. The converse implication* *is also true, but we will not need it in the sequel. This is the reason why we chose to refer to "generation up to $\mathcal{T}$" in the terminology.*

Moreover, the proof of lemma 1.2.6 actually shows that as a presheaf F on $\mathbf{Sm}/S[\mathcal{T}^{-1}]$ satisfying $F(X \sqcup Y) \xrightarrow{\sim} F(X) \times F(Y)$ for all X and Y in $\mathbf{Sm}/S$, $K_0(-) \simeq \pi_0(\mathbf{Z} \times \mathbf{Gr})$ is generated by the elements $u_{d,r} + n$ for all $(d,r) \in \mathbf{N}^2$ and $n \in \mathbf{Z}$.

Remark 1.2.8 *If is easy to deduce from theorem 1.2.4 that the localised category $\mathbf{Sm}/S[\mathcal{T}^{-1}]$ is equivalent to $\mathbf{SmAff}_S[\mathcal{H}_{\mathbf{A}^1}^{-1}]$ where $\mathbf{SmAff}_S$ is the fullsubcategory of $\mathbf{Sm}/S$ consisting of affine schemes and $\mathcal{H}_{\mathbf{A}^1}$ is the family of projections $X \times \mathbf{A}^1 \to X$ for $X \in \mathbf{SmAff}_S$ (see [25, §7.4]). Hence, the category of $\mathcal{T}$-invariant presheaves on $\mathbf{Sm}/S$ is equivalent to the category of $\mathbf{A}^1$-invariant presheaves on $\mathbf{SmAff}_S$.*

Proposition 1.2.9 *Let S be a regular scheme. Let $E \in \mathcal{H}_\bullet(S)$ be an H-group. Let $(X_i)_{i\in\mathcal{I}}$ be a direct system in $\mathbf{Sm}/S$ that does not unveil phantoms in E. We let $\mathcal{X}$ be the colimit of this system in the category of presheaves over $\mathbf{Sm}/S$. We assume that $\pi_0\mathcal{X}$ is generated by $\mathcal{X}$ up to $\mathcal{T}$. Then, the following obvious maps are bijections:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{X}, E) \arrow[r, "\sim", "\alpha"'] \arrow[dr, "\sim", "\gamma"'] & \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(\pi_0\mathcal{X}, \pi_0 E) \arrow[d, "\beta"', "\sim"] \\ & \lim\limits_{i\in\mathcal{I}} \pi_0 E(X_i) \end{tikzcd}$$

Using lemma 1.2.1, we see that the assumption that $(X_i)_{i\in\mathcal{I}}$ does not unveil phantoms on E precisely says that γ is a bijection. To finish the proof, we only have to prove that β is an injection. To do this, we may observe that $\lim\limits_{i\in\mathcal{I}} \pi_0 E(X_i)$ identifies to $\mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(\mathcal{X}, \pi_0 E) \simeq \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(\mathcal{X}[\mathcal{T}^{-1}], \pi_0 E)$. Then, β identifies to the map obtained by applying the functor $\mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(-, \pi_0 E)$ to the canonical map $\mathcal{X}[\mathcal{T}^{-1}] \to \pi_0\mathcal{X}$, which is an epimorphism as $\mathcal{X} \to \pi_0\mathcal{X}$ is an epimorphism up to $\mathcal{T}$ (see remark 1.2.7). Thus, β is injective.

At this stage, theorem 1.1.6 is proved as lemma 1.2.6 implies that it is a special case of proposition 1.2.9. To finish the proof of theorems 1.1.1, 1.1.2 and 1.1.4, the remaining step is the following lemma:

Lemma 1.2.10 *Let S be regular scheme. Let n be a natural number. The object $\mathbf{Z} \times \mathbf{Gr}$ satisfies property (K) with n operands. This conclusion also applies to the loop spaces $\mathbf{R}\Omega^j(\mathbf{Z} \times \mathbf{Gr})$ for any $j \in \mathbf{N}$.*

On the one hand we have to notice the technical fact that $\mathbf{Z}\times\mathbf{Gr}$ has a structure of H-group (see [32, page 139]). On the other hand, we have to prove the vanishing of the $\mathbf{R}^1 \lim$ of some projective systems. To do this, one may use the Mittag-Leffler condition, which is obviously satisfied when all transition maps are onto. Then, we need to know that the canonical map $K_{j+1}(\prod_{i=1}^n \mathbf{Gr}_{d'_i,r'_i}) \to K_{j+1}(\prod_{i=1}^n \mathbf{Gr}_{d_i,r_i}))$ is onto whenever $d_i \leq d'_i$ and $r_i \leq r'_i$.

An S-scheme X is cellular if there exists a sequence of closed subschemes $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_k = X$ of S such that $Z_i - Z_{i-1}$ is isomorphic to an affine space $\mathbf{A}^d$ over S for $1 \leq i \leq k$.

It is well known that Grassmann varieties are cellular (see [11]) and it is easy to prove the following formulas:

- if X is a smooth cellular S-scheme, then for any $j \in \mathbf{N}$, $K_j(S) \otimes_{K_0(S)} K_0(X) \xrightarrow{\sim} K_j(X)$;

- if X is a smooth cellular S-scheme, T a regular scheme and $T \to S$ a morphism, then $K_0(T) \otimes_{K_0(S)} K_0(X) \xrightarrow{\sim} K_0(T \times_S X)$;

- if X and Y are smooth cellular S-schemes, then $K_0(X) \otimes_{K_0(S)} K_0(Y) \xrightarrow{\sim} K_0(X \times_S Y)$.

We see that we only have to prove that $K_0(\mathbf{Gr}_{d',r'}) \to K_0(\mathbf{Gr}_{d,r})$ is onto whenever $d \leq d'$ and $r \leq r'$. One may also assume that $S = \mathrm{Spec}(\mathbf{Z})$. Then, for any tuple $(d,r) \in \mathbf{N}^2$, $K_0(\mathbf{Gr}_{d,r})$ is generated as a λ-ring by the class $u_{d,r} = [\mathcal{M}'_{d,r}] - d$ (see [1, VI 4.6]). With the notations above, the lemma follows from the obvious fact that the inverse image of $u_{d',r'}$ by the inclusion $\mathbf{Gr}_{d,r} \to \mathbf{Gr}_{d',r'}$ is $u_{d,r}$.

Remark 1.2.11 *The particular case $d = d' = 1$ in the proof shows that the direct system $(\mathbf{P}^n)_{n\in\mathbf{N}}$ does not unveil phantoms in the objects $\mathbf{R}\Omega^j(\mathbf{Z} \times \mathbf{Gr})$. This gives an interpretation of morphisms $\mathbf{P}^\infty \to \mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ as natural transformation $\mathrm{Pic}(-) \to K_0(-)$ in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$.*

To finish the proof of theorem 1.1.7, we have to determine the structure of the ring $R = \lim\limits_{(d,r)\in\mathbf{N}^2} K_0(\mathbf{Gr}_{d,r})$. If we fix d, we know from [1, VI 4.10] that $\lim\limits_{r\in\mathbf{N}} K_0(\mathbf{Gr}_{d,r}) \simeq K_0(S)[[\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d]]$ where $\tilde{\gamma}^i$ is given by the compatible family $\gamma^i(u_{d,r})$. Then, R identifies to $\lim\limits_{d\in\mathbf{N}} K_0(S)[[\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d]]$. One can easily see that the induced transition maps

$$K_0(S)[[\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d, \tilde{\gamma}^{d+1}]] \to K_0(S)[[\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d]]$$

are obtained by making $\tilde{\gamma}^{d+1}$ vanish. It proves that R identifies to the ring of formal power series with an infinite number of variables $\tilde{\gamma}^1, \tilde{\gamma}^2, \ldots$ and coefficient ring $K_0(S)$.

2 Algebraic structures

We shall see that the previous results show that the algebraic structures on the sets $K_0(X)$, $X \in \mathbf{Sm}/S$ uniquely refine to structures of the same type on $\mathbf{Z} \times \mathbf{Gr}$ in the category $\mathcal{H}(S)$. Thus, $\mathbf{Z} \times \mathbf{Gr}$ shall be endowed with the structure of a special λ-ring with duality in $\mathcal{H}(S)$. In this section, we shall use similar notions to those appearing in [9].

2.1 Abstract operators, formulas, algebraic structures

Definition 2.1.1 *We define a language $\mathcal{L}$ as the datum of a family of elements $(l_i)_{i\in I}$ called abstract operators, where each of these operators is equipped with its arity $n_i \in \mathbf{N}$.*

Definition 2.1.2 *A formula of the language $\mathcal{L} = (l_i, n_i)_{i\in I}$ involving variables $(x_v)_{v\in V}$ (V is assumed to be finite) is the set of expressions inductively built from the following rules:*

- *for any $v \in V$, x_v is a formula;*

- *for any $i \in I$, if $F_1, \ldots, f_{n_i}$ are formulas, then $l_i(F_1, \ldots, F_{n_i})$ is a formula.*

Definition 2.1.3 *An abstract algebraic structure is the datum of a language $\mathcal{L}$ and of a family of pairs $(A_r, B_r)_{r\in R}$ of formulas of $\mathcal{L}$ involving variables in some finite set V_r. These pairs are called "relations" and shall be denoted $A_r = B_r$.*

Example 2.1.4 *The abstract algebraic structure of group is defined as follows. The language $\mathcal{L}$ is made of a 0-ary operator e (we may say that e is a constant), a binary operator μ and an unary operator i. The relations are:*

- $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z))$;
- $\mu(e, x) = x$;
- $\mu(x, e) = x$;
- $\mu(x, i(x)) = e$;
- $\mu(i(x), x)) = e$.

Each of these relations involves a subset of $\{x, y, z\}$ as set of variables.

2.2 Algebraic structures on objects

Definition 2.2.1 *Let $\mathcal{L} = (l_i, n_i)_{i\in I}$ be a language. An $\mathcal{L}$-object consists of an object X of a category $\mathcal{C}$ such that all finite products X^n exist and of a family of morphisms $X^{n_i} \to X$ denoted l_i, for all $i \in I$.*

A morphism of $\mathcal{L}$-objects $X \to Y$ in a category $\mathcal{C}$ is a morphism $F \colon X \to Y$ in $\mathcal{C}$ such that for any $i \in I$, the obvious diagram commutes:

$$\begin{tikzcd} X^{n_i} \arrow[r, "l_i"] \arrow[d, "{(F,\dots,F)}"'] & X \arrow[d, "F"] \\ Y^{n_i} \arrow[r, "l_i"] & Y \end{tikzcd}$$

If X is an $\mathcal{L}$-object, then one can inductively define a morphism $F \colon X^V \to X$ for any formula F of $\mathcal{L}$ involving a finite set of variables V.

Definition 2.2.2 *Let $\mathcal{S} = (\mathcal{L}, (A_r = B_r)_{r\in R})$ be an abstract algebraic structure. An object equipped with an $\mathcal{S}$-structure is an $\mathcal{L}$-object X in some category $\mathcal{C}$ such that for any $r \in R$, the morphisms $X^{V_r} \to X$ defined by A_r and B_r are equal. We may also say that X is an $\mathcal{S}$-object or that X is a model of $\mathcal{S}$ in the category $\mathcal{C}$.*

We may define the category of $\mathcal{S}$-objects as a full subcategory of the category of $\mathcal{L}$-objects.

Proposition 2.2.3 *Let $\mathcal{S}$ be an abstract algebraic structure. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We assume that finite products exist in $\mathcal{C}$ and that F commutes with these products. If X is an $\mathcal{S}$-object in $\mathcal{C}$, then FX has a natural structure of an $\mathcal{S}$-object in $\mathcal{D}$.*

Conversely, if the canonical map $\mathrm{Hom}_{\mathcal{C}}(X^n, X) \to \mathrm{Hom}_{\mathcal{D}}(F(X^n), FX)$ is a bijection for any $n \in \mathbf{N}$ and some object X of $\mathcal{C}$, then an $\mathcal{S}$-structure on FX uniquely arises from an $\mathcal{S}$-structure on X.

Furthermore, let X and Y be two $\mathcal{S}$-objects. We assume that for any $n \in \mathbf{N}$, the map $\mathrm{Hom}_{\mathcal{C}}(X^n, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X^n), FY)$ is a bijection. Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. Then, f is a morphism of $\mathcal{S}$-objects in $\mathcal{C}$ if and only if $Ff \colon FX \to FY$ is a morphism of $\mathcal{S}$-objects in $\mathcal{D}$.

This is a pliantly true.

2.3 Structures on $\mathbf{Z} \times \mathbf{Gr}$

The example 2.1.4 shows that there is an obvious abstract algebraic structure whose models in the category of sets are groups. The same applies to commutative rings (with unit): the underlying language of the corresponding abstract algebraic structure involves the 0-ary operators 0 and 1, the unary operator $-$ and the binary operators $+$ and $\times$. Following [1, RRR I 1], if we add a family of unary operators $(\lambda^n)_{n\in\mathbf{N}}$, we can define the abstract algebraic structures of λ-rings and of special λ-rings. One may also introduce the abstract algebraic structure of special λ-rings with duality: we add an unary duality operator that should be an involution commuting with the other operators.

Theorem 2.3.1 *Let S be a regular scheme. In the category $\mathcal{H}(S)$, there exists a unique structure of a special λ-ring with duality on the object $\mathbf{Z} \times \mathbf{Gr}$ such that the corresponding induced structures of λ-rings with duality on $K_0(X)$ for all $X \in \mathbf{Sm}/S$ are the usual ones.*

For any $X \in \mathbf{Sm}/S$, the set $K_0(X)$ is endowed with the structure of a special λ-ring with duality [1, VI 3.2]. All these structures are compatible with inverse image maps $f^\star\colon K_0(X) \to K_0(Y)$ for morphisms $f\colon Y \to X$. This shows that, as a presheaf of sets on $\mathbf{Sm}/S$, $K_0(-) = \pi_0(\mathbf{Z} \times \mathbf{Gr})$ is endowed with the structure of a special λ-ring with duality. Proposition 2.2.3 and theorem 1.1.4 shows that it lifts to a unique structure of a special λ-ring with duality on $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$.

Proposition 2.3.2 *Let $f\colon Y \to X$ be a morphism of regular schemes. Let $\mathbf{Z} \times \mathbf{Gr}_X \in \mathcal{H}(X)$ (resp. $\mathbf{Z} \times \mathbf{Gr}_Y \in \mathcal{H}(Y)$) be the special λ-rings with duality defined in theorem 2.3.1. The structures on $\mathbf{Z} \times \mathbf{Gr}_X$ induce a structure of special λ-rings with duality on $f^\star(\mathbf{Z} \times \mathbf{Gr}_X)$. Then, the obvious isomorphism $f^\star(\mathbf{Z}\times\mathbf{Gr}_X) \simeq \mathbf{Z}\times\mathbf{Gr}_Y$ in $\mathcal{H}(Y)$ is an isomorphism of special λ-rings with duality.*

We can use the construction of proposition 2.2.3 because the functor $f^\star\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ (see [32, page 108]) commutes with finite products. Using theorem 1.1.4, it suffices to compare the two induced special λ-rings with duality structures on the presheaf $K_0(-)$ on $\mathbf{Sm}/Y$. If f is smooth, one may argue by saying that the structures on $\pi_0 f^\star(\mathbf{Z} \times \mathbf{Gr}_X)$ are obtained from those on $\pi_0(\mathbf{Z} \times \mathbf{Gr}_X)$ by applying the "restriction" functor $\mathbf{Sm}/X^{\mathrm{opp}}\mathbf{Sets} \to \mathbf{Sm}/Y^{\mathrm{opp}}\mathbf{Sets}$ obtained by composition with the "forgetful" functor $\mathbf{Sm}/Y \to \mathbf{Sm}/X$. In the general case, we may observe that it suffices to check that the two special λ-rings structures considered on $K_0(-)$ in $\mathbf{Sm}/Y^{\mathrm{opp}}\mathbf{Sets}$ agree on the "universal" elements $u_{d,r} + n \in K_0(\mathbf{Gr}_{d,r,Y})$ (see remark 1.2.7) and this follows from the fact that the presheaves $K_0(-)$ on $\mathbf{Sm}/X$ or $\mathbf{Sm}/Y$ come from a presheaf of special λ-rings with duality on the category of all regular schemes.

Remark 2.3.3 *Similar arguments can be used to prove that, through the interpretation of operations as formal power series (see theorem 1.1.7), the map $f^\star\colon \mathrm{End}_{\mathcal{H}(X)}(\mathbf{Z} \times \mathbf{Gr}_X) \to \mathrm{End}_{\mathcal{H}(Y)}(\mathbf{Z} \times \mathbf{Gr}_Y)$ corresponds to the extension of scalars of formal power series along the morphism $f^\star\colon K_0(X) \to K_0(Y)$.*

2.4 Structures on higher K-groups

Let S be a regular scheme. We have constructed structures on $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$. For any $\mathcal{X} \in \mathcal{H}(S)$, they induce structures on the set $\mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{X}, \mathbf{Z} \times \mathbf{Gr})$, which we denote $K_0(\mathcal{X})$.

As a result, these sets $K_0(\mathcal{X})$ are special λ-rings with duality. To extend some structures to the higher K-groups $K_n(\mathcal{X}) = \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(S^n \wedge \mathcal{X}_+, \mathbf{Z} \times \mathbf{Gr})$, one has to refine some morphisms in $\mathcal{H}(S)$ to morphisms in $\mathcal{H}_\bullet(S)$.

Theorem 1.1.2 and the subsequent comments shows that the families of operations $(\Psi^k)_{k\in\mathbf{Z}}$, $(\lambda^n)_{n\in\mathbf{N}}$ and $(\gamma^n)_{n\in\mathbf{N}-\{0\}}$ and more generally all operations $\tau\colon K_0(-) \to K_0(-)$ such that $\tau(0) = 0$ naturally act on these sets $K_n(\mathcal{X})$. Moreover, relations known at the level of K_0 implies similar relations on all the K-groups: for instance, the formula $\Psi^k \circ \Psi^{k'} = \Psi^{kk'}$ is satisfied by the corresponding operations on $K_\star(\mathcal{X})$.

This also applies to operations involving several operands like $+$ and $\times$. The commutative group structure on $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ comes from a commutative group structure on $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$. Using this H-group structure, we obtain abelian group structures on the sets $K_n(\mathcal{X})$ for all $n \in \mathbf{N}$. Using the argument of [31, page 74], the product law $\times\colon (\mathbf{Z} \times \mathbf{Gr})^2 \to \mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ can easily be refined to a pairing $\mu\colon (\mathbf{Z} \times \mathbf{Gr}) \wedge (\mathbf{Z} \times \mathbf{Gr}) \to \mathbf{Z} \times \mathbf{Gr}$, which induce pairings $K_i(\mathcal{X}) \times K_j(\mathcal{Y}) \to K_{i+j}(\mathcal{X} \times \mathcal{Y})$ for $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{H}(S)$. Using this construction in the case $\mathcal{Y} = \mathcal{X}$ and the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, we get a product law on the graded abelian group $K_\star(\mathcal{X})$. It formally follows from the commutative ring structure on $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ that with these definitions, $K_\star(\mathcal{X})$ is a graded commutative ring. One can easily check compatibilities between the λ-operations and the product. For instance, if $k \in \mathbf{Z}$, the fact that Ψ^k is an endomorphism of the ring $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}(S)$ shows that the operation $\Psi^k\colon K_\star(\mathcal{X}) \to K_\star(\mathcal{X})$ is an endomorphism of graded rings.

Example 2.4.1 *The following confusing example should warn the reader against misinterpretations of the previous results. Let $\tau\colon K_0(-) \to K_0(-)$ be the operation defined by $\tau(x) = x^2$ for any $x \in K_0(X)$ and $X \in \mathbf{Sm}/S$. This operation satisfies $\tau(0) = 0$; then it induces maps $\tau\colon K_n(X) \to K_n(X)$ for all $n \in \mathbf{N}$ and $X \in \mathbf{Sm}/S$. However, this operation on higher K-groups is unrelated to the squaring map $K_n(X) \to K_{2n}(X)$ unless $n = 0$. Indeed, a simple computation using the splitting principle shows that $\tau = \Psi^2 + 2\lambda^2$. To the latter, we associated maps $K_n(X) \to K_n(X)$ rather than maps $K_n(X) \to K_{2n}(X)$.*

3 Comparison with previous constructions

3.1 Models of algebraic K-theory

Definition 3.1.1 *Let S be a regular scheme. A candidate model of algebraic K-theory (over S) is an object $\mathcal{K} \in \mathcal{H}_\bullet(S)$ equipped with a morphism $\alpha_\mathcal{K}\colon K_0(-) \to \pi_0\mathcal{K}$ of presheaves of pointed sets on $\mathbf{Sm}/S$. We say that $(\mathcal{K}, \alpha_\mathcal{K})$ is strict if $\alpha_\mathcal{K}$ is an isomorphism.*

For such an object $\mathcal{K}$, $\mathcal{X} \in \mathcal{H}(S)$ and $n \in \mathbf{N}$, we define $K_n^\mathcal{K}(\mathcal{X})$ to be the set of morphisms $\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(S^n \wedge \mathcal{X}_+, \mathcal{K})$.

A morphism of candidate models $(\mathcal{K}, \alpha_\mathcal{K}) \to (\mathcal{K}', \alpha_{\mathcal{K}'})$ is the datum of a morphism $f\colon \mathcal{K} \to \mathcal{K}'$ in $\mathcal{H}_\bullet(S)$ such that $\alpha_{\mathcal{K}'} = \pi_0(f) \circ \alpha_\mathcal{K}$.

Proposition 3.1.2 *Candidate models of algebraic K-theory can be associated to the following definitions of algebraic K-theory :*

- *Quillen's Q-construction [34, 7.1];*
- *Waldhausen's [47, §1.9];*

- *Thomason-Trobaugh's [43, 3.5.3].*

For each of these constructions, there is a well-defined presheaf $\mathcal{K}$ of pointed simplicial sets of $\mathbf{Sm}/S$ such that the corresponding K-groups are the homotopy groups of the spaces $\mathcal{K}(X)$ for all $X \in \mathbf{Sm}/S$. This presheaf $\mathcal{K}$ defines an object in $\mathcal{H}_\bullet(S)$ and there are canonical maps for all $X \in \mathbf{Sm}/S$ (see definition 1.1.5) :

$$\pi_0(\mathcal{K}(X)) \to \operatorname{Hom}_{\mathcal{H}_\bullet(S)}(X, K) = (\pi_0\mathcal{K})(X) .$$

For any of these definitions of algebraic K-theory, in degree zero, $\pi_0(\mathcal{K}(X))$ is identified to the Grothendieck group $K_0(X)$ of the exact category of vector bundles on X. Then, we get the expected map $\alpha_\mathcal{K} \colon K_0(-) \to \pi_0\mathcal{K}$ in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}_\bullet$.

Thanks to theorem 0.1, the object $\mathbf{Z}\times\mathbf{Gr}$ is endowed with a structure of a (strict) candidate model of algebraic K-theory. The map $\alpha_{\mathbf{Z}\times\mathbf{Gr}} \colon K_0(-) \to \pi_0(\mathbf{Z}\times\mathbf{Gr})$ has the (characteristic) property that the class $u_{d,r} + n \in K_0(\mathbf{Gr}_{d,r})$ is mapped to the homotopy class of the inclusion $\mathbf{Gr}_{d,r} \subset \{n\} \times \mathbf{Gr} \subset \mathbf{Z} \times \mathbf{Gr}$.

The following proposition shows that this model $(\mathbf{Z}\times\mathbf{Gr}, \alpha_{\mathbf{Z}\times\mathbf{Gr}})$ plays an almost universal role:

Proposition 3.1.3 *Let S be a regular scheme. Let $(\mathcal{K}, \alpha_\mathcal{K})$ be a candidate model of algebraic K-theory over S. Then, there exists a morphism $(\mathbf{Z}\times\mathbf{Gr}, \alpha_{\mathbf{Z}\times\mathbf{Gr}}) \to (\mathcal{K}, \alpha_\mathcal{K})$ of candidate models of algebraic K-theory. If this morphism is an isomorphism, then it is unique and we shall say that $(\mathcal{K}, \alpha_\mathcal{K})$ is a genuine model of algebraic K-theory.*

May $\mathcal{K}$ not be an H-group, the surjectivity part of the Milnor exact sequence stated in lemma 1.2.1 is still true. Then, there exists a morphism $f \colon \mathbf{Z}\times\mathbf{Gr} \to \mathcal{H}$ in $\mathcal{H}_\bullet(S)$ such that the morphism of presheaves α_K and $\pi_0(f) \circ \alpha_{\mathbf{Z}\times\mathbf{Gr}}$ in $\operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-), \pi_0\mathcal{K})$ coincide on the universal classes $u_{d,r} + n \in K_0(\mathbf{Gr}_{d,r})$. Then remark 1.2.7 implies that that they are equal which proves that f is a morphism of candidate models of algebraic K-theory.

If f is an isomorphism, then we may replace $(\mathcal{K}, \alpha_\mathcal{K})$ by $(\mathbf{Z}\times\mathbf{Gr}, \alpha_{\mathbf{Z}\times\mathbf{Gr}})$ and the uniqueness of f means that there exists a unique endomorphism of $\mathbf{Z}\times\mathbf{Gr}$ which induces the identity on $\pi_0(\mathbf{Z}\times\mathbf{Gr}) = K_0(-)$, which is known thanks to theorem 1.1.2.

Corollary 3.1.4 *Let S be a regular scheme. If $(\mathcal{K}, \alpha_\mathcal{K})$ and $(\mathcal{K}', \alpha_{\mathcal{K}'})$ are two genuine models of algebraic K-theory, they are canonically isomorphic and the associated K-groups are also canonically isomorphic for all $\mathcal{X} \in \mathcal{H}(S)$ and $n \in \mathbf{N}$:*

$$K_n^{\mathcal{K}}(\mathcal{X}) \simeq K_n^{\mathcal{K}'}(\mathcal{X}) .$$

It follows from the fact that both genuine models are canonically isomorphic to $(\mathbf{Z}\times\mathbf{Gr}, \alpha_{\mathbf{Z}\times\mathbf{Gr}})$.

Proposition 3.1.5 *Let S be a regular scheme. The candidate models defined in proposition 3.1.2 are genuine models of algebraic K-theory.*

The proofs of the comparison theorems between Quillen's, Waldhausen's and Thomason-Trobaugh's constructions ([47, 1.9] and [43, proposition 3.10]) are functorial enough to imply that the three corresponding presheaves of pointed simplicial sets induce isomorphic objects in the pointed homotopy category of the site $\mathbf{Sm}/S_{\mathrm{Nis}}$. Moreover, these objects satisfy the Nisnevich descent property [43, theorem 10.8] and the homotopy invariance of algebraic K-theory for regular schemes [34, §6] shows that they are $\mathbf{A}^1$-local. As a result, if $\mathcal{K}$ is one of these presheaves of pointed simplicial sets, the obvious maps $\pi_n(\mathcal{K}(X)) \to \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(S^n \wedge X_+, \mathcal{K})$ are bijections for all $X \in \mathbf{Sm}/S$. In particular, the map $\alpha_{\mathcal{K}} \colon K_0(-) \to \pi_0\mathcal{K}$, which is part of the datum of a candidate model, is an isomorphism. These candidate models are strict ones. Then, the proposition follows from the fact that the object $\mathcal{K}$ associated to Quillen's Q-construction is isomorphic to $\mathbf{Z} \times \mathbf{Gr}$, which is implicit in the proof of theorem 0.1.

3.2 Products

Proposition 3.2.1 *Let S be a regular scheme. For all $X \in \mathbf{Sm}/S$, $(i,j) \in \mathbf{N}^2$, the pairing $K_i(X) \times K_j(X) \to K_{i+j}(X)$ defined in subsection 2.4 is the same as the one defined by Waldhausen [47].*

First, thanks to the results of subsection 3.1, it truly makes sense to say that these pairings coincide as the different flavours of models of algebraic K-theory give canonically isomorphic groups. Then, as Waldhausen's product on $K_\star(X)$ obviously extends the standard one on $K_0(X)$, theorem 1.1.4 shows that we only need to observe that Waldhausen's pairing is functorial enough to be defined at the level of presheaves of pointed simplicial sets on $\mathbf{Sm}/S$ and thus induces a morphism $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$ in $\mathcal{H}_\bullet(S)$ where $\mathcal{K}$ is the model of algebraic K-theory associated to Waldhausen's definition.

Remark 3.2.2 *Using similar arguments, one may prove that the pairing $K_i(X) \times K_j(X) \to K_{i+j}(X)$ coincides with the one defined by Quillen (only for $i = 0$ or $j = 0$). For $i \neq 0$, $j \neq 0$ and X affine, one may also compare them with the product defined by Loday using the $+$-construction [30]; the arguments would be similar to the arguments in subsection 3.3 below. In particular, Waldhausen's pairing coincide with those defined by Quillen and Loday. This comparison was already known (see [48]).*

3.3 Operations involving one operand

In his article [42], Soulé defined an action of $\mathrm{R}_{\mathbf{Z}}\mathbf{GL} = \lim_{d \in \mathbf{N}} \mathrm{R}_{\mathbf{Z}}\mathbf{GL}_d$ on the higher algebraic K-theory of schemes, where $\mathrm{R}_{\mathbf{Z}}\mathbf{GL}_d$ is the Grothendieck group defined by Serre [39]. If we fix a regular base scheme S, theorem 1.1.1 introduces such an action on K-theory of smooth S-schemes for elements $\tau \in \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}_\bullet}(K_0(-))$. As we would like to state a compatibility between these two constructions, we shall introduce a common input for both of them.

Definition 3.3.1 *Let $d \in \mathbf{N}$. We let Univ_d be the universal special λ-ring equipped with an element $\underline{\mathrm{id}}_d$ satisfying the following conditions:*

(i) $\lambda^d(\underline{\mathrm{id}}_d)$ *is invertible;*

(ii) $\lambda^k(\underline{\mathrm{id}}_d)$ *vanishes for* $k \geq d+1$.

The special λ-ring Univ_∞ *is the projective limit of the system* $(\mathrm{Univ}_d)_{d\in\mathbf{N}}$ *where the transition map* $\mathrm{Univ}_{d+1}\to\mathrm{Univ}_d$ *maps* $\underline{\mathrm{id}}_{d+1}$ *to* $\underline{\mathrm{id}}_d+1$*.*

Obviously, for any d, there is a canonical morphism of special λ-rings $\mathrm{Univ}_d\to\mathrm{R}_{\mathbf{Z}}\mathbf{GL}_d$ that maps $\underline{\mathrm{id}}_d$ to the class of the tautological representation $\mathrm{id}\colon\mathbf{GL}_d\to\mathbf{GL}_d$ of rank d of the group scheme $\mathbf{GL}_d$. Serre's computation [39, §3.8] shows that this sequence of morphisms consists of isomorphisms. Then, the canonical morphism $\mathrm{Univ}_\infty\to\mathrm{R}_{\mathbf{Z}}\mathbf{GL}$ is an isomorphism.

We may also use the universal properties of the special λ-rings Univ_d to define a morphism of special λ-rings $\mathrm{Univ}_\infty\to K_0(\mathbf{Gr})=\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{Gr},\mathbf{Z}\times\mathbf{Gr})$. It is induced by the morphisms

$$\mathrm{Univ}_d\to\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{Gr}_{d,\infty},\mathbf{Z}\times\mathbf{Gr})\simeq\lim_{r\in\mathbf{N}}K_0(\mathbf{Gr}_{d,r})$$

sending $\underline{\mathrm{id}}_d$ to the compatible family of classes $([\mathcal{M}'_{d,r}])_{r\in\mathbf{N}}$ (see the proof of lemma 1.2.6 for this notation).

We let $(\mathrm{Univ}_\infty)_0$ and $(\mathrm{R}_{\mathbf{Z}}\mathbf{GL})_0$ be the kernel of the rank morphism from these groups to $\mathbf{Z}$. Similarly, we denote $\tilde{K}_0(\mathbf{Gr})$ the kernel of the restriction to the base-point $K_0(\mathbf{Gr})\to K_0(S)$. The comparison theorem announced above is the following:

Theorem 3.3.2 *Let S be a regular scheme. For any $n\geq 1$, the following diagram commutes:*

$$\begin{tikzcd}
 & (\mathrm{Univ}_\infty)_0 \arrow[dl, "\sim"'] \arrow[dr] & \\
(\mathrm{R}_{\mathbf{Z}}\mathbf{GL})_0 \arrow[dr] & & \tilde{K}_0(\mathbf{Gr}) \arrow[dl] \\
 & \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_n(-))\ , &
\end{tikzcd}$$

where the two upper maps are the ones mentioned above, the lower-left one is the one defined by Soulé and the lower-right one arises from theorem 0.1. (Thanks to previous results, this lower-right map can be interpreted as the canonical map $\mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}_\bullet}(\tilde{K}_0(-))\to\mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_n(-))$*.)*

The strategy of the proof consists in the construction of an horizontal map $(\mathrm{R}_{\mathbf{Z}}\mathbf{GL})_0\to\tilde{K}_0(\mathbf{Gr})$ which makes both upper and lower triangles commute. This map is induced by a morphism $\mathrm{R}_{\mathbf{Z}}\mathbf{GL}\to K_0(\mathbf{Gr})$ and is a particular case of a more general construction:

Proposition 3.3.3 *Let G be a smooth group scheme over* $\mathrm{Spec}(\mathbf{Z})$*. We let $\mathrm{R}_{\mathbf{Z}}G$ be the Grothendieck group of finitely generated free $\mathbf{Z}$-modules endowed with a linear action of G (see [39, §2.3]). We let $\mathbf{B}G\in\mathcal{H}_\bullet(S)$ be the classifying space of G (where G is considered as a sheaf of groups on $\mathbf{Sm}/S_{\mathrm{Nis}}$). Let $\rho\colon G\to\mathbf{GL}(M)$ be a free finitely generated $\mathbf{Z}$-module endowed with a linear action of G (we shall say that M is a representation of G). The choice of a $\mathbf{Z}$-basis of M identifies ρ with a morphism of group schemes $G\to\mathbf{GL}_d$ over* $\mathrm{Spec}(\mathbf{Z})$ *where $d=\mathrm{rk}\,M$. We let $[\rho]\in\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{B}G,\mathbf{Z}\times\mathbf{Gr})$ be the morphism obtained from $\mathbf{B}\rho\colon\mathbf{B}G\to\mathbf{BGL}_d$ by composing with the canonical morphism $\mathbf{BGL}_d\simeq\mathbf{Gr}_{d,\infty}\simeq\{d\}\times\mathbf{Gr}_{d,\infty}\subset\mathbf{Z}\times\mathbf{Gr}$. Then, this assignment $\rho\longmapsto[\rho]$ does not depend on the the choice of $\mathbf{Z}$-bases and induces a morphism of special λ-rings with duality $\mathrm{R}_{\mathbf{Z}}G\to\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{B}G,\mathbf{Z}\times\mathbf{Gr})=K_0(\mathbf{B}G)$.*

The choice of two different $\mathbf{Z}$-bases of a representation M of G would lead to morphisms $G \to \mathbf{GL}_d$ which would differ by an inner automorphism of $\mathbf{GL}_d$ (induced by an element of $\mathbf{GL}_d(\mathbf{Z})$): the associated morphisms $\mathbf{B}G \to \mathbf{BGL}_d$ are equal in $\mathcal{H}(S)$ (and also in $\mathcal{H}_\bullet(S)$ after composition with $\mathbf{BGL}_d \to \mathbf{BGL}_\infty$ because $\mathbf{BGL}_\infty$ is an H-group).

To prove that $\rho \longmapsto [\rho]$ induces a morphism at the level of the Grothendieck group of representations of G, we use the following two lemmas:

Lemma 3.3.4 *Let* $+\colon \mathbf{BGL}_\infty \times \mathbf{BGL}_\infty \to \mathbf{BGL}_\infty$ *be the* H*-group structure coming from the usual group structure on* $\tilde{K}_0(-)$ *(see remark 1.1.3). For any* $d, d' \geq 0$*, the following diagram commutes in* $\mathcal{H}_\bullet(S)$*:*

$$\begin{tikzcd} \mathbf{BGL}_d \times \mathbf{BGL}_{d'} \arrow[r, "\mathbf{B}\oplus"] \arrow[d] & \mathbf{BGL}_{d+d'} \arrow[d] \\ \mathbf{BGL}_\infty \times \mathbf{BGL}_\infty \arrow[r, "+"] & \mathbf{BGL}_\infty \end{tikzcd},$$

where the vertical morphisms are the obvious ones and the upper one is the morphism $\mathbf{B}\oplus$ *deduced from the "direct sum" morphism* $\oplus\colon \mathbf{GL}_d \times \mathbf{GL}_{d'} \to \mathbf{GL}_{d+d'}$*.*

The correspondence between $\mathbf{GL}_d$-torsors on schemes and rank-d vector bundles provides a functorial map $H^1(X, \mathbf{GL}_d) \to \tilde{K}_0(X)$ (we substract the rank in $K_0(X)$ so as to get elements in $\tilde{K}_0(X)$). An obvious verification leads to the following commutative square which states a compatibility between this correspondence, the sum in $\tilde{K}_0(X)$ and the map induced on cohomology by the morphism $\oplus\colon \mathbf{GL}_d \times \mathbf{GL}_{d'} \to \mathbf{GL}_{d+d'}$:

$$\begin{tikzcd} H^1(X, \mathbf{GL}_d) \times H^1(X, \mathbf{GL}_{d'}) \arrow[r, "\oplus_\star"] \arrow[d] & H^1(X, \mathbf{GL}_{d+d'}) \arrow[d] \\ \tilde{K}_0(X) \times \tilde{K}_0(X) \arrow[r, "+"] & \tilde{K}_0(X) \end{tikzcd}$$

The two morphisms we want to compare are in $\operatorname{Hom}_{\mathcal{H}(S)}(\mathbf{BGL}_d \times \mathbf{BGL}_{d'}, \mathbf{BGL}_\infty) \simeq \lim_{(r,r')} \tilde{K}_0(\mathbf{Gr}_{d,r} \times \mathbf{Gr}_{d',r'})$. Then, the lemma follows from the commutativity mentioned above in the case where X is a product of Grassmann varieties and where the torsors corresponds to the universal vector bundles on these varieties.

Lemma 3.3.5 *Let* $0 \to \rho' \to \rho \to \rho'' \to 0$ *be an exact sequence of representations of* G*. Then,* $[\rho] = [\rho' \oplus \rho'']$ *in* $\operatorname{Hom}_{\mathcal{H}(S)}(\mathbf{B}G, \mathbf{Z} \times \mathbf{Gr})$.

Let $d' = \operatorname{rk} \rho'$, $d = \operatorname{rk} \rho$ and $d'' = \operatorname{rk} \rho''$. Using the obvious functoriality of the constructions with respect to the group G, we may assume that we are in the universal situation where $\rho\colon G \to \mathbf{GL}_d$ is the inclusion of the subgroup of matrices of the form $g = \begin{pmatrix} g' & h \\ 0 & g'' \end{pmatrix}$ where $g' \in \mathbf{GL}_{d'}$, $g'' \in \mathbf{GL}_{d''}$ and h is an d'-by-d'' matrix and where the representations ρ' and ρ'' correspond to the obvious morphisms $G \to \mathbf{GL}_{d'}$ and $G \to \mathbf{GL}_{d''}$.

Let $D = \mathbf{GL}_{d'} \times \mathbf{GL}_{d''}$ be the subgroup of G consisting of matrices of the previous form such that $h = 0$. Obviously, the restriction of the representations ρ and $\rho' \oplus \rho''$ from G to D are isomorphic. Then, to finish the proof, it suffices to know that the restriction map $K_0(\mathbf{B}G) \to K_0(\mathbf{B}D)$ is an injection. Indeed, this map is a bijection because $D \to G$

is an $\mathbf{A}^1$-weak equivalence and thus $\mathbf{B}D \to \mathbf{B}G$ is also an $\mathbf{A}^1$-weak equivalence (see [32, Proposition 2.14, page 74]).

We have constructed a morphism of abelian groups $\mathrm{R}_{\mathbf{Z}}G \to K_0(\mathbf{B}G)$. To finish the proof of the proposition, it remains to show that this is a morphism of special λ-rings with duality. The compatibility of the construction with external powers and duality can be checked in the same way as we did it for direct sums (see lemma 3.3.4).

To prove theorem 3.3.2, we apply proposition 3.3.3 to the cases $G = \mathbf{GL}_d$ for all d. It provides a morphism of special λ-rings $\mathrm{R}_{\mathbf{Z}}\mathbf{GL}_d \to K_0(\mathbf{Gr}_{d,\infty})$. Taking the projective limit over all d and considering the rank-0 part leads to the expected morphism $(\mathrm{R}_{\mathbf{Z}}\mathbf{GL})_0 \to \tilde{K}_0(\mathbf{Gr})$. The universal property of Univ_d and the fact that the morphisms $\mathrm{R}_{\mathbf{Z}}\mathbf{GL}_d \to K_0(\mathbf{Gr}_{d,\infty})$ are morphisms of special λ-rings shows that the upper triangle commutes. The fact that the lower triangle commutes follows easily from the very definition in Soulé's paper [42].

4 Virtual categories

Virtual categories were introduced by Deligne in [10]. They are refinements of K_0-groups. More precisely, if $X \in \mathbf{Sm}/S$ (S regular), the category $\mathcal{V}(X)$ is identified to the fundamental groupoid of $\mathcal{K}(X)$ where $\mathcal{K}$ is some $\mathbf{A}^1$-fibrant genuine model of algebraic K-theory. Any vector bundle $\mathcal{E}$ on X defines an object $\mathcal{E}$ of the category $\mathcal{V}(X)$ whose isomorphism class corresponds to $[\mathcal{E}]$ in $K_0(X)$. When we have a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we not only have an equality of classes $[\mathcal{E}] = [\mathcal{E}' \oplus \mathcal{E}'']$, which means that $\mathcal{E}$ and $\mathcal{E}' \oplus \mathcal{E}''$ become isomorphic in $\mathcal{V}(X)$ but we have a specific isomorphism $\mathcal{E} \simeq \mathcal{E}' \oplus \mathcal{E}''$ in this category $\mathcal{V}(X)$.

4.1 The Thom spectrum of a virtual bundle

The construction of this paragraph will be used only in §6.1.3. It appears here because it favours the understanding of virtual categories.

Proposition 4.1.1 *Let X be a scheme. The construction of the Thom spectrum $\mathrm{Th}_X \mathcal{E}$ of a vector bundle $\mathcal{E}$ on X (see [32, Definition 2.16, page 111]) extends to a functor $\mathrm{Th}_X \colon \mathcal{V}(X) \to \mathcal{SH}(X)$.*

(See also [6, Théorème 1.5.18].) One may first check that the Thom spectrum of a vector bundle is invertible for the $\wedge$-product in $\mathcal{SH}(X)$; one is reduced to the case of a trivial bundle because the invertibility can be checked locally for the Zariski topology on X. Then, using the universal property of $\mathcal{V}(X)$ as a Picard category, one has to define an isomorphism $\mathrm{Th}_X \mathcal{E}' \wedge \mathrm{Th}_X \mathcal{E}'' \simeq \mathrm{Th}_X \mathcal{E}$ for any short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles. If the sequence splits, a splitting of it gives such an isomorphism (see [32, Proposition 2.17, page 112]) and explicit $\mathbf{A}^1$-homotopies show that it is independant of the splitting. The general case reduces to this because we can use a torsor $T \to X$ under a vector bundle such that the inverse image of the sequence splits over T. If Jouanolou's trick is available (see theorem 1.2.4), we may use it; otherwise, as I learned from Dennis Eriksson, we can always use the scheme which parametrises the sections of $\mathcal{E} \to \mathcal{E}''$: it is a torsor under the vector bundle $\mathbf{Hom}(\mathcal{E}'', \mathcal{E}')$. To check the needed coherence properties, we may split a finite number of short exact sequences of vector bundles as above; then, it becomes straightforward.

Definition 4.1.2 *Let $f\colon X \to S$ be a smooth morphism between noetherian schemes. Proposition 4.1.1 defines a functor $\mathrm{Th}_X\colon \mathcal{V}(X) \to \mathcal{SH}(X)$. We also denote $\mathrm{Th}_X\colon \mathcal{V}(X) \to \mathcal{SH}(S)$ the functor obtained by composition with $f_\sharp\colon \mathcal{SH}(X) \to \mathcal{SH}(S)$ (see [37, Proposition 4.4]).*

4.2 Inverting primes on $\mathbf{Z} \times \mathbf{Gr}$

Definition 4.2.1 *Let S be a regular scheme. Let $a \in \mathbf{N} - \{0\}$. For any $(d, r) \in \mathbf{N}^2$, we define a morphism $\mathbf{Gr}_{d,r} \to \mathbf{Gr}_{ad,ar}$ which sends an admissible subbundle $\mathcal{M}' \subset \mathcal{O}^n$ ($n = d + r$) of rank d to $\delta_{a,n}(\mathcal{M}'^{\oplus a})$ where $\delta_{a,n}\colon (\mathcal{O}^n)^{\oplus a} \to \mathcal{O}^{an}$ is the isomorphism that sends $(s_1^1, \dots, s_n^1), \dots, (s_1^a, \dots, s_n^a)$ to $(s_1^1, \dots, s_1^a, \dots, s_n^1, \dots, s_n^a)$. This compatible family of morphisms induces a morphism $m_a\colon \mathbf{Gr} \to \mathbf{Gr}$ of presheaves of pointed sets. We also denote $m_a\colon \mathbf{Z} \times \mathbf{Gr} \to \mathbf{Z} \times \mathbf{Gr}$ the morphism which is the multiplication by a on $\mathbf{Z}$ and m_a on $\mathbf{Gr}$.*

Lemma 4.2.2 *Let a and b be two positive natural numbers. Then, the endomorphisms m_{ab} and $m_a \circ m_b$ of $\mathbf{Z} \times \mathbf{Gr}$ are equal in the category of presheaves of pointed sets.*

Definition 4.2.3 *For any $x \in \mathbf{N} - \{0\}$, we set $\frac{1}{x}(\mathbf{Z} \times \mathbf{Gr}) = \mathbf{Z} \times \mathbf{Gr}$. If $y \in \mathbf{N} - \{0\}$ is a multiple of x, the endomorphism $m_{y/x}$ of $\mathbf{Z} \times \mathbf{Gr}$ defines a canonical morphism $\frac{1}{x}(\mathbf{Z} \times \mathbf{Gr}) \to \frac{1}{y}(\mathbf{Z} \times \mathbf{Gr})$.*

Lemma 4.2.2 says that this defines a direct system $(\frac{1}{x}(\mathbf{Z} \times \mathbf{Gr}))_{x \in \mathbf{N}-\{0\}}$ of sheaves of pointed sets of $\mathbf{Sm}/S$. It is indexed by $\mathbf{N} - \{0\}$, which is ordered by divisibility.

Definition 4.2.4 *Let n be a supernatural number (see [41, §I.1.3]). We denote $(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$ the colimit of the system $\frac{1}{x}(\mathbf{Z} \times \mathbf{Gr})$ where x varies in the set of positive natural numbers dividing n^∞.*

Proposition 4.2.5 *Let S be a regular scheme. Let $i \in \mathbf{N}$. Let n be a supernatural number. Then, the canonical maps are bijections:*

$$\mathrm{Hom}_{\mathcal{H}(S)}(\mathbf{Z} \times \mathbf{Gr}, \mathbf{R}\Omega^i(\mathbf{Z} \times \mathbf{Gr})[\tfrac{1}{n}]) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-), K_i(-)[\tfrac{1}{n}])\ ,$$

$$\mathrm{Hom}_{\mathcal{H}(S)}((\mathbf{Z} \times \mathbf{Gr})[\tfrac{1}{n}], \mathbf{R}\Omega^i(\mathbf{Z} \times \mathbf{Gr})[\tfrac{1}{n}]) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-)[\tfrac{1}{n}], K_i(-)[\tfrac{1}{n}])\ .$$

A variant of lemma 1.2.10 shows that $\mathbf{R}\Omega^i(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$ satisfies property (K). Hence, theorem 1.1.6 gives the first bijection. The second bijection needs additional arguments. From lemma 1.2.6, it is easy to show that $(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$ generates $\pi_0((\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}])$ up to $\mathcal{T}$. The definition also gives an expression of $(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$ as the colimit of some direct system $(X_i)_{i \in \mathcal{I}}$ of representable sheaves, where $\mathcal{I}$ is an ordered set which has a cofinal sequence. Then, using proposition 1.2.9, we have to show that $(X_i)_{i \in \mathcal{I}}$ does not unveil phantoms in $\mathbf{R}\Omega^i(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$. Reasoning like in the proof of lemma 1.2.10, it suffices to check that for a natural number a dividing a power of n, the morphisms $m_{a,d,r}\colon \mathbf{Gr}_{d,r} \to \mathbf{Gr}_{ad,ar}$ induce surjections $m_{a,d,r}^\star\colon K_0(\mathbf{Gr}_{ad,ar})[\frac{1}{n}] \to K_0(\mathbf{Gr}_{d,r})[\frac{1}{n}]$. This is true because $K_0(\mathbf{Gr}_{d,r})[\frac{1}{n}]$ is generated by $u_{d,r}$ as a $K_0(S)[\frac{1}{n}]$-λ-algebra and $m_{a,d,r}^\star(u_{ad,ar}) = a u_{d,r}$.

We leave the variants involving several operands to the reader.

4.3 Operations on virtual categories

Definition 4.3.1 *Let S be a regular scheme. Let n be a supernatural number. We let $\mathcal{V}(-)[\frac{1}{n}]$ be the presheaf of groupoids that sends $X \in \mathbf{Sm}/S$ to the fundamental groupoid of $\mathcal{K}[\frac{1}{n}](X)$ where $\mathcal{K}[\frac{1}{n}]$ is an $\mathbf{A}^1$-fibrant replacement of $(\mathbf{Z} \times \mathbf{Gr})[\frac{1}{n}]$.*

Theorem 4.3.2 *Let n be an even supernatural number. Let $\tau\colon K_0(-)[\frac{1}{n}] \to K_0(-)[\frac{1}{n}]$ be a morphism in $\mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})^{\mathrm{opp}}\mathbf{Sets}$. Then, up to a unique isomorphism, we can define a family of functors $\tilde{\tau}_X\colon \mathcal{V}(X)[\frac{1}{n}] \to \mathcal{V}(X)[\frac{1}{n}]$ for $X \in \mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})$ which induces τ on sets of isomorphisms classes in $\mathcal{V}(-)[\frac{1}{n}]$ and such that for any morphism $f\colon Y \to X$ in $\mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})$, it satisfies the equality $f^\star \circ \tilde{\tau}_X = \tilde{\tau}_Y \circ f^\star$. (Variants involving several operands are also true.)*

From proposition 4.2.5, we know that τ corresponds to an endomorphism of $\mathcal{K}[\frac{1}{n}]$ in $\mathcal{H}(\operatorname{Spec}(\mathbf{Z}))$. As $\mathcal{K}[\frac{1}{n}]$ is $\mathbf{A}^1$-fibrant, τ lifts to a morphism $\tilde{\tau}\colon \mathcal{K}[\frac{1}{n}] \to \mathcal{K}[\frac{1}{n}]$. Passing to fundamental groupoids, we get a family of functors $\tilde{\tau}_X\colon \mathcal{V}(X)[\frac{1}{n}] \to \mathcal{V}(X)[\frac{1}{n}]$ for $X \in \mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})$.

We let $E = \mathbf{hom}(\mathcal{K}[\frac{1}{n}], \mathcal{K}[\frac{1}{n}])$ be the simplicial set of endomorphisms of $\mathcal{K}[\frac{1}{n}]$ (it is given by the simplicial structure). The morphism $\tilde{\tau}$ corresponds to a 0-simplex in E. If $\tilde{\tau}'\colon \mathcal{K}[\frac{1}{n}] \to \mathcal{K}[\frac{1}{n}]$ is in the same homotopy class as $\tilde{\tau}$, the choice of an homotopy (*i.e.*, a path between $\tilde{\tau}$ and $\tilde{\tau}'$ in E gives an isomorphism between the associated families of functors $(\tilde{\tau}_X)$ and $(\tilde{\tau}'_X)$. The question is whether this isomorphism is uniquely determined or not. It will be so if there exists a unique homotopy class of paths $\tilde{\tau} \to \tilde{\tau}'$. As E is an H-group, it means that the connected components of E are simply connected, *i.e.*, $\pi_1 E = 0$. This group identifies to $\operatorname{Hom}_{\mathcal{H}(\operatorname{Spec}(\mathbf{Z}))}(\mathcal{K}[\frac{1}{n}], \Omega\mathcal{K}[\frac{1}{n}])$, which identifies to $\operatorname{Hom}_{\mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})^{\mathrm{opp}}\mathbf{Sets}}(K_0(-)[\frac{1}{n}], K_1(-)[\frac{1}{n}])$. To prove that this group vanishes, we can use proposition 1.2.9 which expresses it as a projective limit of some groups $K_1(\mathbf{Gr}_{d,r})[\frac{1}{n}]$. The result then follows from the fact that $K_1(\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$.

Remark 4.3.3 *In theorem 4.3.2, we may replace $\mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})$ by any small full subcategory $\mathbf{Reg}$ of the category of regular schemes. Indeed, we may assume that $\operatorname{Spec}(\mathbf{Z}) \in \mathbf{Reg}$ and that for any $S \in \mathbf{Reg}$, objects in $\mathbf{Sm}/S$ belong to $\mathbf{Reg}$. Then, we may work in the $\mathbf{A}^1$-homotopy category $\mathcal{H}(\mathbf{Reg})$ of the site $\mathbf{Reg}_{\mathrm{Nis}}$ equipped with the interval $\mathbf{A}^1$. Arguments leading to theorem 4.3.2 can be made with the category $\mathcal{H}(\mathbf{Reg})$ instead of $\mathcal{H}(\operatorname{Spec}(\mathbf{Z}))$. We may also deduce results for $\mathbf{Reg}$ from the case of $\mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})$ by using the fully faithful functor $p^\star\colon \mathcal{H}(\operatorname{Spec}(\mathbf{Z})) \to \mathcal{H}(\mathbf{Reg})$ associated to the obvious reasonable continuous map of sites $p\colon \mathbf{Reg}_{\mathrm{Nis}} \to \mathbf{Sm}/\operatorname{Spec}(\mathbf{Z})_{\mathrm{Nis}}$.*

5 Additive and stable results

5.1 The splitting principle

Now, we shall focus on natural transformations $K_0(-) \to K_0(-)$ which are compatible with the abelian group structures on K-groups, *i.e.*, morphisms in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}$ rather than in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$. From theorem 1.1.4 and proposition 2.2.3, we know that these additive operations precisely correspond to endomorphisms of $\mathbf{Z} \times \mathbf{Gr}$ as an H-group (*i.e.*, a group object in $\mathcal{H}_\bullet(S)$).

To compute these additive transformations, we shall use the "splitting principle". We let $\operatorname{Pic}(-)$ be the presheaf of sets on $\mathbf{Sm}/S$ (for a regular scheme S) that maps $U \in \mathbf{Sm}/S$ to the Picard group $\operatorname{Pic}(U)$, considered as a set. We denote $c\colon \operatorname{Pic}(-) \to K_0(-)$ the morphism in $\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}$ that maps the isomorphism class of a line bundle $\mathcal{L}$ to the class $[\mathcal{L}]$ in the Grothendieck group of vector bundles.

Proposition 5.1.1 *Let S be a regular scheme. For any integer i, the map induced by c*

$$c^\star\colon \operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-),K_i(-)) \to \operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(\operatorname{Pic}(-),K_i(-))$$

is a bijection. Moreover, the latter group identifies to

$$\operatorname{Hom}_{\mathcal{H}(S)}(\mathbf{P}^\infty, \mathbf{R}\Omega^i(\mathbf{Z}\times\mathbf{Gr})) \simeq \lim_n K_i(\mathbf{P}^n) \simeq K_i(S)[[U]]\ ,$$

where $U=[\mathcal{O}(1)]-1$ is the obvious compatible family in $\lim_n K_0(\mathbf{P}^n)$.

The injectivity of $c^\star$ follows easily from the "splitting principle": if $\mathcal{M}$ is a vector bundle of rank r on a scheme $X\in\mathbf{Sm}/S$, the complete flag scheme $\mathbf{D}(\mathcal{M})\overset{\pi}{\to} X$ is such that $[\pi^\star\mathcal{M}]$ decomposes in $K_0(\mathbf{D}(\mathcal{M}))$ as a sum of the classes of r line bundles and $\pi^\star\colon K_i(X)\to K_i(\mathbf{D}(\mathcal{M}))$ is injective.

Proposition 1.2.9, lemma 1.2.6, lemma 1.2.10 and remark 1.2.11 show that we have bijections:

$$\operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(\operatorname{Pic}(-),K_i(-)) \simeq \operatorname{Hom}_{\mathcal{H}(S)}(\mathbf{P}^\infty, \mathbf{R}\Omega^i(\mathbf{Z}\times\mathbf{Gr})) \simeq \lim_n K_i(\mathbf{P}^n)\ .$$

The identification of this group with $K_i(S)[[U]]$ follows from the computation of the algebraic K-theory of projective spaces: $K_i(\mathbf{P}^n)\simeq K_0(\mathbf{P}^n)\otimes_{K_0(S)} K_i(S)$ and $K_0(\mathbf{P}^n)\simeq K_0(S)[U]/(U^{n+1})$.

It remains to show that $c^\star$ is surjective. Using the previous identifications, we rewrite it as a map $c^\star\colon \operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-),K_i(-))\to K_i(S)[[U]]$. First, we observe that for any $k\in\mathbf{N}$ and $x\in K_i(S)$, we may denote $x\Psi^k$ the natural transformation $K_0(-)\to K_i(-)$ that maps y to $x\cdot\Psi^k(y)$ and see that it satisfies $c^\star(x\Psi^k)=x(1+U)^k$. This proves that the image of $c^\star$ contains $K_i(S)[U]$. To finish the proof, we use the following lemma:

Lemma 5.1.2 *Let $(\tau_n)_{n\in\mathbf{N}}$ be a sequence of additive natural transformations $K_0(-)\to K_i(-)$ such that $c^\star(\tau_n)$ converges to zero in $K_i(S)[[U]]$ for the infinite product topology, where $K_i(S)$ is endowed with the discrete topology; in other words, we assume that for each $k\in\mathbf{N}$, the coefficient of U^k in $c^\star(\tau_n)$ eventually vanishes. Then, for any $X\in\mathbf{Sm}/S$ and $x\in K_0(X)$, there exists $N\in\mathbf{N}$ such that for all $n\geq N$, $\tau_n(x)=0$ and it makes sense to define a natural transformation $\tau\colon K_0(-)\to K_i(-)$ by the formula $\tau(x)=\sum_{n=0}^\infty \tau_n(x)$ and we have the equality $c^\star(\tau)=\sum_{n\in\mathbf{N}} c^\star(\tau_n)$ in $K_i(S)[[U]]$.*

We have to prove that given $X\in\mathbf{Sm}/S$ and $x\in K_0(X)$, $\tau_n(x)$ eventually vanishes. The assumption says that it is true for the class $x=[\mathcal{O}(1)]$ on $\mathbf{P}^n$ for all n. Taking inverse images of these classes by morphisms $f\colon X\to\mathbf{P}^n$ enables to obtain the more general case of classes of line bundles generated by their global sections, *e.g.*, line bundles on affine schemes. Using $\mathcal{T}$ (see theorem 1.2.4), we get the case of line bundles on any $X\in\mathbf{Sm}/S$. Then, the general case follows from the splitting principle.

Remark 5.1.3 *We may define a topology on $\operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-),K_i(-))$ by considering the weakest topology for which the evaluation maps at all elements $x\in K_0(X)$ for all $X\in\mathbf{Sm}/S$ are continuous, where all groups $K_i(X)$ are endowed with the discrete topology. The argument of the lemma shows that the bijection $c^\star\colon \operatorname{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-),K_i(-))\to K_i(S)[[U]]$ is an homeomorphism.*

Remark 5.1.4 *The composition of endomorphisms endows*

$$\mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-)) \simeq K_0(S)[[U]]$$

with a structure of a (possibly non-commutative) ring. If this law on $K_0(S)[[U]]$ is denoted $\bigstar$, one may characterise it by the fact that it is continuous and that for all $(x,y) \in K_0(S)^2$, $(k,k') \in \mathbf{N}^2$, $(x(1+U)^k)\bigstar(y(1+U)^{k'}) = (x\Psi^k(y)(1+U)^{kk'}$. More generally, we have a graded ring structure on $\oplus_{i\in\mathbf{N}}K_i(S)[[U]]$ which comes from the fact that $K_i(S)[[U]]$ identifies to the group of homomorphisms $\mathbf{Z}\times\mathbf{Gr} \to \mathbf{R}\Omega^i(\mathbf{Z}\times\mathbf{Gr})$ of abelian groups inside $\mathcal{H}(S)$; the multiplication can be described similarly as it has been described in degree 0.

Remark 5.1.5 *The surjectivity of the map*

$$c^\star\colon \mathrm{Hom}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-), K_i(-)) \to K_i(S)[[U]]$$

may be proved using a different argument. First, we may assume that the formal power series $f = \sum_{k\geq 0} a_k U^k$ is such that $a_0 = 0$, so that we actually have to construct a natural transformation $\tau\colon \tilde{K}_0(-) \to K_i(-)$. After the application of Jouanolou's trick and the splitting principle, an element $x \in \tilde{K}_0(X)$ can be expressed as $x = \sum_{i=1}^n u_i$ where $u_i = [\mathcal{L}_i] - 1$ for a family of vector bundles $L_1, \ldots, L_n$ for a big enough n. Then, $\tau(x)$ should be $f(u_1) + \cdots + f(u_n) = \sum_{k\geq 1} a_k(u_1^k + \cdots + u_n^k)$. Using the theory of symmetric polynomials and once we have noticed that the elementary symmetric functions of the $u_1, \ldots, u_n$ are the elements $\gamma^1(x), \ldots, \gamma^n(x)$, we get the existence of an element in $K_i(S)[[\tilde{\gamma}^1, \tilde{\gamma}^2, \ldots]]$ whose associated natural transformation $\tau\colon \tilde{K}_0(-) \to K_i(-)$ (see theorem 1.1.7) is additive and such that $\tau(u) = f(u)$ whenever $u = [\mathcal{L}] - 1$ and $\mathcal{L}$ is a line bundle.

Remark 5.1.6 *The method of remark 5.1.5 can be used for the study of natural transformation $\tau\colon K_0(-) \to K_0(-)$ which induces group morphisms $(K_0(X), +) \to (K_0(X)^\times, \cdot)$ for all $X \in \mathbf{Sm}/S$,* i.e., *classes which are multiplicative on short exact sequences (* e.g., *the Todd class, which is defined after tensoring with $\mathbf{Q}$). The result is that for any series $f = \sum_{k\geq 0} a_k U^k$ such that a_0 is invertible in $K_0(S)$, there exists a unique $\tau\colon K_0(-) \to K_0(-)$ as above such that for any line bundle $\mathcal{L}$, $\tau([\mathcal{L}]) = f(u)$ where $u = [\mathcal{L}] - 1$. The proof follows the same pattern: reduce to the case $a_0 = 1$ and then consider $f(u_1)f(u_2)\ldots f(u_n)$ instead of $f(u_1) + f(u_2) + \cdots + f(u_n)$.*

Exercise 5.1.7 (Optional) *Assume that S is a regular scheme such that $K_0(S) \simeq \mathbf{Z}$. Prove that any ring endomorphism φ of $\mathbf{Z}\times\mathbf{Gr}$ in $\mathcal{H}(S)$ is of the form Ψ^k for some $k \in \mathbf{Z}$. (Hint: φ corresponds to a series $f \in \mathbf{Z}[[U]]$ which satisfies $f(0) = 1$ and $f(U)\cdot f(V) = f(U + V + UV)$. Then, ratiocinate in $\mathbf{Q}[[U]]$ to prove that f is of the form $(1+U)^\alpha$ for $\alpha \in \mathbf{Q}$.)*

5.2 The $\mathbf{P}^1$-spectrum BGL

Let S be a regular scheme. We define a morphism $\sigma\colon \mathbf{P}^1 \wedge (\mathbf{Z}\times\mathbf{Gr}) \to \mathbf{Z}\times\mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$ (where ∞ is the base-point of $\mathbf{P}^1$) as the composition

$$\mathbf{P}^1 \wedge (\mathbf{Z}\times\mathbf{Gr}) \xrightarrow{u\wedge \mathrm{id}} (\mathbf{Z}\times\mathbf{Gr}) \wedge (\mathbf{Z}\times\mathbf{Gr}) \xrightarrow{\mu} \mathbf{Z}\times\mathbf{Gr}$$

where $u\colon \mathbf{P}^1 \to \mathbf{Z}\times\mathbf{Gr}$ corresponds to the class $u = [\mathcal{O}(1)] - 1 \in \ker(\infty^\star\colon K_0(\mathbf{P}^1) \to K_0(S))$ and μ is the pairing defined in subsection 2.4. We denote $\tilde{\sigma}\colon \mathbf{Z}\times\mathbf{Gr} \to \mathbf{R}\,\mathbf{Hom}_\bullet(\mathbf{P}^1, \mathbf{Z}\times\mathbf{Gr})$

the morphism in $\mathcal{H}_\bullet(S)$ corresponding to σ by adjunction. It follows from the projective bundle theorem that $\tilde{\sigma}$ is an isomorphism.

We can use this to define an object of the naive variant $\mathcal{SH}_{\text{naïve}}(S)$ (see [37, §6]) of the stable homotopy category $\mathcal{SH}(S)$, *i.e.*, an (Ω)-$\mathbf{P}^1$-spectrum up to homotopy. More precisely, an object of $\mathcal{SH}_{\text{naïve}}(S)$ consists in the datum of a sequence $(\mathbf{E}_n)_{n\in\mathbf{N}}$ of objects of $\mathcal{H}_\bullet(S)$ and of bonding morphisms $\sigma\colon \mathbf{P}^1 \wedge \mathbf{E}_n \to \mathbf{E}_{n+1}$ in $\mathcal{H}_\bullet(S)$ which are supposed to be such that the adjoint morphisms $\mathbf{E}_n \to \mathbf{R}\,\mathbf{Hom}_\bullet(\mathbf{P}^1, \mathbf{E}_{n+1})$ are isomorphisms for all $n \in \mathbf{N}$. The object $\mathbf{BGL}_{naive} \in \mathcal{SH}_{\text{naïve}}(S)$ is defined by the fact that $(\mathbf{BGL}_{naive})_n = \mathbf{Z} \times \mathbf{Gr}$ and that all bonding morphisms identifies to the morphism σ defined above.

We shall see that we may define an object $\mathbf{BGL} \in \mathcal{SH}(S)$ up to a unique isomorphism and that it lifts $\mathbf{BGL}_{naive}$. The obstruction we may encounter to do this lies in the notion of stably phantom morphisms. More precisely, if $\mathbf{E}$ and $\mathbf{F}$ are objects of $\mathcal{SH}(S)$, represented by Ω-spectra, for any $i \in \mathbf{N}$, the sequence of groups $(\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbf{E}_n, \mathbf{R}\Omega^i\mathbf{F}_n))_{n\in\mathbf{N}}$ is equipped with the structure of a projective system, and it follows from the Milnor exact sequence that we have a short exact sequence (see [37, Lemme 6.5]):

$$0 \to \mathbf{R}^1 \lim_n \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbf{E}_n, \mathbf{R}\Omega\mathbf{F}_n) \to \mathrm{Hom}_{\mathcal{SH}(S)}(\mathbf{E}, \mathbf{F}) \\ \to \mathrm{Hom}_{\mathcal{SH}_{\text{naïve}}(S)}(\mathrm{oub}\,\mathbf{E}, \mathrm{oub}\,\mathbf{F}) \to 0\ ,$$

where $\mathrm{oub}\colon \mathcal{SH}(S) \to \mathcal{SH}_{\text{naïve}}(S)$ is the forgetful functor. The group on the right identifies to $\lim_n \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbf{E}_n, \mathbf{F}_n)$ and the group on the left is the subgroup of stably phantom morphisms $\mathbf{E} \to \mathbf{F}$.

An object of $\mathcal{SH}_{\text{naïve}}(S)$ always lifts to an object of $\mathcal{SH}(S)$, unique up to isomorphism; however, this lifting is unique up to a *unique* isomorphism if and only if a given lifting has no nonzero stably phantom endomorphisms [37, Proposition 6.3]. We will see that it is the case for $\mathbf{BGL}_{naive}$ if $K_1(S)$ is finite (*e.g.*, $S = \mathrm{Spec}(\mathbf{Z})$), which is sufficient to construct a canonical $\mathbf{BGL} \in \mathcal{SH}(S)$ for all regular schemes S as we may take the inverse image by $S \to \mathrm{Spec}(\mathbf{Z})$ (see [37, Proposition 4.4]) of the unique one in $\mathcal{SH}(\mathrm{Spec}(\mathbf{Z}))$. This appeared in my thesis [36] and in [21] similar arguments reappeared. This being said, until the end of this subsection, we choose a lifting $\mathbf{BGL}$ of $\mathbf{BGL}_{naive}$ in $\mathcal{SH}(S)$.

Remark 5.2.1 *In the study of projective systems $(\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbf{E}_n, \mathbf{R}^1\Omega^i\mathbf{F}_n))_{n\in\mathbf{N}}$, for some $i \in \mathbf{N}$, we may focus on the subsystem made of H-group morphisms, which may be denoted $(\mathrm{Hom}^+_{\mathcal{H}_\bullet(S)}(\mathbf{E}_n, \mathbf{R}^1\Omega^i\mathbf{F}_n))_{n\in\mathbf{N}}$. Indeed, the cokernel of this inclusion is a projective system with zero transition maps, which implies that the inclusion induce isomorphisms on* $\lim$ *and* $\mathbf{R}^1\lim$.

Definition 5.2.2 *Let A be an abelian group. We set A^Ω to be the following projective system indexed by $\mathbf{N}$:*

$$\cdots \to A[[U]] \overset{\Omega_{\mathbf{P}^1}}{\to} A[[U]] \overset{\Omega_{\mathbf{P}^1}}{\to} A[[U]] \overset{\Omega_{\mathbf{P}^1}}{\to} A[[U]]\ ,$$

where the map $\Omega_{\mathbf{P}^1}\colon A[[U]] \to A[[U]]$ is defined by $\Omega_{\mathbf{P}^1}(f) = (1+U)\frac{df}{dU}$.

Proposition 5.2.3 *Let S be a regular scheme. The projective system*

$$(\mathrm{Hom}^+_{\mathcal{H}_\bullet(S)}((\mathbf{BGL})_n, \mathbf{R}^i\Omega(\mathbf{BGL})_n))_{n\in\mathbf{N}}$$

canonically identifies to $K_i(S)^\Omega$.

From proposition 5.1.1, we already know that $\mathrm{Hom}^+_{\mathcal{H}_\bullet(S)}((\mathbf{BGL})_n, \mathbf{R}\Omega^i(\mathbf{BGL})_n)$ identifies degreewise to the group $K_i(S)[[U]]$. We let $\omega\colon K_i(S)[[U]] \to K_i(S)[[U]]$ be the morphism corresponding to the transition maps on the projective system

$$(\mathrm{Hom}^+_{\mathcal{H}_\bullet(S)}((\mathbf{BGL})_n, \mathbf{R}\Omega^i(\mathbf{BGL})_n))_{n\in\mathbf{N}}$$

under this identification. We have to prove that $\omega = \Omega_{\mathbf{P}^1}$.

Let $\tau = \sum_{n\geq 0} a_n U^n \in K_i(S)[[U]]$. It corresponds to an additive natural transformation $(\tau_X\colon K_0(X) \to K_i(X))_{X\in\mathbf{Sm}/S}$ which is such that $\tau_X([\mathcal{L}]) = \sum_{n\geq 0} a_n([\mathcal{L}]-1)^n$ for all line bundles $\mathcal{L}$. The natural transformation $K_0(X) \to K_i(X)$ associated to $\omega(\tau)$ is characterised by the formula:

$$\omega(\tau)_X(x) \boxtimes v = \tau_{X\times\mathbf{P}^1}(x \boxtimes v)\ ,$$

where $v = [\mathcal{O}(1)]-1 \in K_0(\mathbf{P}^1)$ and $\boxtimes$ is the external product $K_\star(X)\times K_0(\mathbf{P}^1) \to K_\star(X\times\mathbf{P}^1)$. Assume that $x = [\mathcal{L}]$ is the class of a line bundle $\mathcal{L}$ on a scheme $X \in \mathbf{Sm}/S$. Then, $x \boxtimes v = [\mathcal{L}\boxtimes\mathcal{O}(1)] - [\mathcal{L}\boxtimes\mathcal{O}_{\mathbf{P}^1}]$. We may apply $\tau_{X\times\mathbf{P}^1}$ to this difference; if we set $u = x-1$ and use that $K_\star(X\times\mathbf{P}^1) \simeq K_\star(X)[v]/(v^2)$, we get:

$$\begin{aligned}
\tau_{X\times\mathbf{P}^1}(x\boxtimes v) &= \sum_{n\geq 0} a_n\,[(1+u)(1+v)-1]^n - \sum_{n\geq 0} a_n u^n \\
&= \sum_{n\geq 0} a_n\,[(u+v(1+u))^n - u^n] \\
&= \sum_{n\geq 1} n a_n (1+u)u^{n-1} v\ .
\end{aligned}$$

Then, $\omega(\tau)_X(x) = \sum_{n\geq 1} n a_n(1+u)u^{n-1}$ which proves that $\omega(\tau) = \sum_{n\geq 1} n a_n (1+U)U^{n-1} = (1+U)\frac{d\tau}{dU} = \Omega_{\mathbf{P}^1}(\tau)$.

Corollary 5.2.4 *Let S be a regular scheme. For all $i\in\mathbf{Z}$, we have a canonical short exact sequence:*

$$0 \to \mathbf{R}^1 \lim K_{i+1}(S)^\Omega \to \mathrm{Hom}_{\mathcal{SH}(S)}(\mathbf{BGL}, \mathbf{BGL}[-i]) \to \lim K_i(S)^\Omega \to 0\ .$$

Proposition 5.2.5 *Let A be an abelian group. If A is either finite or divisible, then*

$$\mathbf{R}^1 \lim A^\Omega = 0\ .$$

If A is divisible, the map $\Omega_{\mathbf{P}^1}\colon A[[U]] \to A[[U]]$ is surjective. Hence, the result is obvious in this case.

As a sequence of abelian groups $0 \to A' \to A \to A'' \to 0$ leads to a short exact sequence of projective systems $0 \to A'^\Omega \to A^\Omega \to A''^\Omega \to 0$, a simple *dévissage* reduces the case of a finite abelian group A to the special case of $A = \mathbf{F}_p$ for a prime number p. Then, we are reduced to the following lemma, which was suggested by Yves André :

Lemma 5.2.6 *Let p be a prime number. We define $L_{\mathbf{F}_p} \subset \mathbf{F}_p[[U]]$ as the subgroup of series $f = \sum_{n\geq 0} a_n U^n$ such that for all $k\in\mathbf{N}$, $\sum_{i=0}^{p-1} a_{kp+i} = 0$. Then,*

(i) The image of $\Omega_{\mathbf{P}^1}\colon \mathbf{F}_p[[U]] \to \mathbf{F}_p[[U]]$ is $L_{\mathbf{F}_p}$;

(ii) *If $f \in L_{\mathbf{F}_p}$, there exists a unique $g \in L_{\mathbf{F}_p}$ such that $\Omega_{\mathbf{P}^1}(g) = f$;*

(iii) *The canonical map $\lim \mathbf{F}_p^\Omega \to (\mathbf{F}_p^\Omega)_0$ induces a bijection $\lim \mathbf{F}_p^\Omega \simeq L_{\mathbf{F}_p}$;*

(iv) *The projective system $L_{\mathbf{F}_p}$ satisfies Mittag-Leffler condition. In particular, $\mathbf{R}^1 \lim \mathbf{F}_p^\Omega = 0$.*

Let $f = \sum_{n\geq 0} a_n U^n$ and $g = \sum_{b\geq 0} b_n U^n$ be two elements of $\mathbf{F}_p[[U]]$. The relation $\Omega_{\mathbf{P}^1}(g) = f$ is equivalent to the equalities $nb_n + (n+1)b_{n+1} = a_n$ for all $n \geq 0$. They can be restated as $nb_n = (-1)^{n-1}\sum_{k=0}^{n-1} a_k$ for all $n \in \mathbf{N}$. It follows that f is in the image of $\Omega_{\mathbf{P}^1}$ if and only if $\sum_{k=0}^{n-1} a_k = 0$ whenever p divides k, *i.e.*, $f \in L_{\mathbf{F}_p}$. Then, the relation $\Omega_{\mathbf{P}^1}(g) = f$ determines the coefficients b_n for p not dividing n but says nothing about the coefficients b_{kp} for all $k \in \mathbf{N}$. There is a unique possible choice for those so as to obtain $g \in L_{\mathbf{F}_p}$. We have proved (i) and (ii). (iii) and (iv) immediately follow.

Corollary 5.2.7 *Let S be a regular scheme. Let $i \in \mathbf{Z}$. If $K_{i+1}(S)$ is finite or divisible, then*

$$\mathrm{Hom}_{\mathcal{SH}(S)}(\mathbf{BGL}, \mathbf{BGL}[-i]) \simeq \lim_i K_i(S)^\Omega \,.$$

*In particular, if $K_1(S)$ is finite (*e.g.*, $S = \mathrm{Spec}(\mathbf{Z})$), $\mathrm{End}_{\mathcal{SH}(S)}(\mathbf{BGL}) \simeq \lim K_0(S)^\Omega$, $\mathbf{BGL}$ has no nonzero stably phantom endomorphism in $\mathcal{SH}(S)$ and thus $\mathbf{BGL}_{naive} \in \mathcal{SH}_{\text{naïve}}(S)$ lifts to an object $\mathbf{BGL} \in \mathcal{SH}(S)$ which is defined up to a unique isomorphism.*

Proposition 5.2.8 *Let A be a torsionfree abelian group such that $\mathrm{Hom}(\mathbf{Q}, A) = 0$ (*e.g.*, $A = \mathbf{Z}$). Then, the map $\lim A^\Omega \to (A^\Omega)_0 = A[[U]]$ is injective.*

To prove this, it suffices to check that if $f \in A[[U]]$ is such than $\Omega_{\mathbf{P}^1}(\Omega_{\mathbf{P}^1}(f)) = 0$, then $\Omega_{\mathbf{P}^1}(f) = 0$. Indeed, let $g = \Omega_{\mathbf{P}^1}(f)$. The equality $\Omega_{\mathbf{P}^1}(g) = 0$ implies that g is constant, *i.e.*, $g \in A$. Then, we have $\frac{df}{dU} = \frac{g}{1+U}$ so that there exists $h \in A$ such that $f = g \log(1+U) + h$. This series, which makes sense in $(A \otimes_{\mathbf{Z}} \mathbf{Q})[[U]]$ does not lie in $A[[U]]$ unless g is in the image of a morphism $\mathbf{Q} \to A$. It follows that $\Omega_{\mathbf{P}^1}(f) = g = 0$.

Remark 5.2.9 *Thanks to corollary 5.2.7, endomorphisms of $\mathbf{BGL}$ in $\mathcal{SH}(\mathrm{Spec}(\mathbf{Z}))$ can be described as compatible families of series in $\mathbf{Z}[[U]]$. Proposition 5.2.8 shows that this information can be reduced to a single element in $\mathbf{Z}[[U]]$. However, I do not know to which subgroup of $\mathbf{Z}[[U]]$ these endomorphisms correspond. It obviously contains $1+U$ and $1/(1+U)$, which corresponds to the identity $\mathbf{\Psi}^1$ and the duality $\mathbf{\Psi}^{-1}$ (see subsection 5.3). According to [3], this group is strictly bigger and even uncountable!*

5.3 Adams operations on $\mathbf{BGL}_{\mathbf{Q}}$

The triangulated category $\mathcal{SH}(S)$ may be localised so as to invert certain or all primes. For instance, we may define $\mathcal{SH}(S)_{\mathbf{Q}}$ as the full subcategory of $\mathcal{SH}(S)$ consisting of objects A such that for any prime p, the multiplication by p on A is an isomorphism. The left adjoint $-_{\mathbf{Q}} \colon \mathcal{SH}(S) \to \mathcal{SH}(S)_{\mathbf{Q}}$ to this inclusion is called the $\mathbf{Q}$-localisation functor. We let $\mathbf{BGL}_{\mathbf{Q}}$ be the image of $\mathbf{BGL}$ by this functor. Then, for any finitely presented object X of $\mathcal{SH}(S)$ [1],

[1] An object X in a triangulated category $\mathcal{T}$ where coproducts exist is finitely presented if the functor $\mathrm{Hom}_{\mathcal{T}}(X, -)$ from $\mathcal{T}$ to the category of abelian groups commutes with (infinite) coproducts. They constitute a triangulated subcategory $\mathcal{T}^{\mathrm{pf}}$ of $\mathcal{T}$. In the case $\mathcal{T} = \mathcal{SH}(S)$, $\mathcal{SH}(S)^{\mathrm{pf}}$ is the pseudo-abelian hull of the triangulated subcategory generated by objects of the form $(\mathbf{P}^1)^{-n} \wedge U_+$ for $U \in \mathbf{Sm}/S$ (see [35, Proposition 1.2]).

the canonical map $\mathrm{Hom}_{\mathcal{SH}(S)}(X,\mathbf{BGL})\otimes_{\mathbf{Z}}\mathbf{Q}\to\mathrm{Hom}_{\mathcal{SH}(S)}(X,\mathbf{BGL_Q})$ is a bijection and the methods used to obtain corollaries 5.2.4 and 5.2.7 give the following result:

Corollary 5.3.1 *Let S be a regular scheme. For all $i\in\mathbf{Z}$, we have a canonical isomorphism:*

$$\mathrm{Hom}_{\mathcal{SH}(S)_{\mathbf{Q}}}(\mathbf{BGL_Q},\mathbf{BGL_Q}[-i])\simeq\lim(K_i(S)\otimes_{\mathbf{Z}}\mathbf{Q})^{\Omega}\ .$$

Definition 5.3.2 *For all $k\in\mathbf{Z}-\{0\}$, we let $\mathbf{\Psi}^k\in\mathrm{End}_{\mathcal{SH}(S)_{\mathbf{Q}}}(\mathbf{BGL_Q})$ be the endomorphism corresponding to the family $(k^{-n}(1+U)^k)_{n\geq 0}\in\lim\mathbf{Q}^{\Omega}$ (this family will also be denoted $\mathbf{\Psi}^k$).*

We obviously have the relations $\mathbf{\Psi}^k\circ\mathbf{\Psi}^{k'}=\mathbf{\Psi}^{kk'}$. These Adams operations are constructed here with $\mathbf{Q}$-coefficients, but it suffices to invert k to define $\mathbf{\Psi}^k$ (there might exist an obstruction to uniqueness in $\mathbf{R}^1\lim K_1(S)[\frac{1}{k}]^{\Omega}$, in which case we may, as above, construct it first on $\mathrm{Spec}(\mathbf{Z})$ and change the base).

To obtain a better understanding of the ring of endomorphisms of $\mathbf{BGL_Q}$, we focus on projective systems A^{Ω} in the case where A is a $\mathbf{Q}$-vector space:

Definition 5.3.3 *Let $n\geq 0$. We define $p_n=\frac{1}{n!}\log^n(1+U)\in\mathbf{Q}[[U]]$. For any $\mathbf{Q}$-vector space A, we define an application $\sigma\colon A^{\mathbf{N}}\to A[[U]]$ by the formula*

$$\sigma((a_n)_{n\in\mathbf{N}})=\sum_{n=0}^{\infty}a_np_n\ .$$

The infinite sum makes sense because the U-valuation of p_n equals n and thus tends to $+\infty$.

Lemma 5.3.4 *For any $\mathbf{Q}$-vector space A, the morphism $\sigma\colon A^{\mathbf{N}}\to A[[U]]$ is an isomorphism of topological groups. If we let $s\colon A^{\mathbf{N}}\to A^{\mathbf{N}}$ be the shift operator $s((a_n)_{n\geq 0})=(a_{n+1})_{n\geq 0}$, we have the equality $\sigma\circ s=\Omega_{\mathbf{P}^1}\circ\sigma$.*

The topologies considered on $A^{\mathbf{N}}$ and $A[[U]]$ are the infinite product topologies of the discrete topology on A. Then, the first statement obviously follows from the fact that the U-valuation of p_n is n. The second follows from the equalities $\Omega_{\mathbf{P}^1}(p_n)=p_{n-1}$ for all $n\geq 1$ and $\Omega_{\mathbf{P}^1}(p_0)=0$.

Proposition 5.3.5 *For any $\mathbf{Q}$-vector space, we may define $\Sigma\colon A^{\mathbf{Z}}\to\lim A^{\Omega}$ by the formula*

$$\Sigma((a_n)_{n\in\mathbf{Z}})=(\sigma(a_n,a_{n+1},a_{n+2},\dots))_{n\geq 0}\ ,$$

i.e., $\Sigma((a_n)_{n\in\mathbf{Z}})=\sum_{n\in\mathbf{Z}}a_n\pi_n$ *where $\pi_n=(p_{n+k})_{k\geq 0}\in\lim\mathbf{Q}^{\Omega}$ (with p_i set to zero for $i<0$).*

It immediately follows from lemma 5.3.4 which identifies the projective system A^{Ω} to the projective system

$$\dots\xrightarrow{s}A^{\mathbf{N}}\xrightarrow{s}A^{\mathbf{N}}\xrightarrow{s}A^{\mathbf{N}}\ ,$$

whose projective limit is $A^{\mathbf{Z}}$.

Remark 5.3.6 *If $A = K_0(S) \otimes_{\mathbf{Z}} \mathbf{Q}$, a variant of proposition 5.1.1 identifies $A[[U]]$ to*

$$\mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-) \otimes_{\mathbf{Z}} \mathbf{Q}) \ ,$$

so that the composition law induces a law $\bigstar$ on $A[[U]]$ (see also remark 5.1.4). The operator $\Omega_{\mathbf{P}^1}$ defines an endomorphism of the ring $(A[[U]], +, \bigstar)$ so that $\lim A^{\Omega}$ inherits a structure of a topological ring, which is, as a ring, isomorphic to $\mathrm{End}_{\mathcal{SH}(S)_{\mathbf{Q}}}(\mathbf{BGL_Q})$.

Proposition 5.3.7 *If $\mathbf{Q}^{\mathbf{N}}$ is endowed with its obvious ring structure and $\mathbf{Q}[[U]]$ with the law $\bigstar$, then $\sigma\colon \mathbf{Q}^{\mathbf{N}} \to \mathbf{Q}[[U]]$ is an isomorphism of topological rings. The same conclusion applies to the isomorphism $\Sigma\colon \mathbf{Q}^{\mathbf{Z}} \xrightarrow{\sim} \lim \mathbf{Q}^{\Omega}$ whose target identifies to a subring of $\mathrm{End}_{\mathcal{SH}(S)_{\mathbf{Q}}}(\mathbf{BGL_Q})$ for any nonempty regular scheme S.*

We know that the $\mathbf{Q}$-vector space of $\mathbf{Q}[[U]]$ spanned by elements $\Psi^k = (1+U)^k$, $k \geq 0$, is dense in $\mathbf{Q}[[U]]$. Hence, it remains to prove the consistency of the formulas $\Psi^{kk'} = \Psi^k \bigstar \Psi^{k'}$ with respect to the application of $\sigma^{-1}\colon A[[U]] \xrightarrow{\sim} \mathbf{A}^{\mathbf{N}}$. This springs from the following lemma:

Lemma 5.3.8 *Let $k \in \mathbf{Z} - \{0\}$. Then,*

$$(1+U)^k = \sigma((k^n)_{n \geq 0}) \ , \quad \mathbf{\Psi}^k = \Sigma((k^n)_{n \in \mathbf{Z}}) \ .$$

Let $(\lambda_n)_{n \geq 0} = \sigma^{-1}(\Psi^k)$, where Ψ^k is identified to $(1+U)^k$. We know that $\Omega_{\mathbf{P}^1}(\Psi^k) = k\Psi^k$. Then, lemma 5.3.4 implies that for all $n \geq 0$, $\lambda_{n+1} = k\lambda_n$, so that $\lambda_n = k^n \lambda_0$. It remains to compute λ_0. But, as it is the constant term of the series $(1+U)^k$, we finally get $\lambda_0 = 1$.

Definition 5.3.9 *For any $n \in \mathbf{Z}$, the element $\pi_n \in \lim \mathbf{Q}^{\Omega}$ was introduced in proposition 5.3.5 and it is also the image by $\Sigma\colon \mathbf{Q}^{\mathbf{Z}} \to \lim \mathbf{Q}^{\Omega}$ of the characteristic function of $\{n\} \subset \mathbf{Z}$. Thanks to proposition 5.3.7, for any regular scheme S, π_n identifies to an idempotent of $\mathrm{End}_{\mathcal{SH}(S)_{\mathbf{Q}}}(\mathbf{BGL_Q})$. As $\mathcal{SH}(S)$ has infinite sums, it is pseudo-abelian (see [44, Proposition II.1.2.9]) and we may denote $\mathbf{BGL}_{\mathbf{Q}}^{(n)} \subset \mathbf{BGL_Q}$ the image of the projector π_n.*

Theorem 5.3.10 *Let S be a regular scheme. The obvious morphism*

$$\bigoplus_{n \in \mathbf{Z}} \mathbf{BGL}_{\mathbf{Q}}^{(n)} \to \mathbf{BGL_Q}$$

is an isomorphism in $\mathcal{SH}(S)$.

Let $n \geq 0$. We let $\chi_{[-n,n]}$ be the characteristic function of $\{-n, \ldots, n\} \subset \mathbf{Z}$. The corresponding element of $\lim \mathbf{Q}^{\Omega}$ *via* Σ and the associated endomorphism of $\mathbf{BGL_Q}$ are also denoted $\chi_{[-n,n]}$. It is the sum of the orthogonal idempotents π_i for $-n \leq i \leq n$. Then, the image of $\chi_{[-n,n]}$ identifies to $\bigoplus_{-n \leq k \leq n} \mathbf{BGL}_{\mathbf{Q}}^{(k)}$.

To prove that the morphism above is an isomorphism, it suffices to prove that for any finitely presented object $X \in \mathcal{SH}(S)$, the induced map

$$\mathrm{Hom}_{\mathcal{SH}(S)}(X, \bigoplus_{n \in \mathbf{Z}} \mathbf{BGL}_{\mathbf{Q}}^{(n)}) \to \mathrm{Hom}_{\mathcal{SH}(S)}(X, \mathbf{BGL_Q})$$

is a bijection. Due to previous observations, this map is injective and its image is made of elements $x \in \mathrm{Hom}_{\mathcal{SH}(S)}(X, \mathbf{BGL_Q})$ such that for a big enough n, $\chi_{[-n,n]}(x) = x$. As the sequence $(\chi_{[-n,n]})_{n \in \mathbf{N}}$ of elements of $\mathbf{Q}^{\mathbf{Z}}$ tends pointwise to the constant function $\mathbf{1}$, the theorem shall be a consequence of the following general lemma:

Lemma 5.3.11 *Let S be a regular scheme. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of elements in the group* $\lim(K_0(S)\otimes_{\mathbf{Z}}\mathbf{Q})^\Omega$ *which converges to an element f. Then, for any finitely presented object X in $\mathcal{SH}(S)$ and $x\in\mathrm{Hom}_{\mathcal{SH}(S)}(X,\mathbf{BGL_Q})$, there exists an integer N such that for all $n\geq N$, $f_n(x)=f(x)$, where f_n and f are identified to endomorphisms of $\mathbf{BGL_Q}$.*

Using the fact that the triangulated category $\mathcal{SH}(S)^{\mathrm{pf}}$ identifies to the pseudo-abelian hull of the category $SW(S)^{\mathrm{ft}}$ [45, page 591], we may assume that $X=(\mathbf{P}^1)^{\wedge-k}\wedge Y$ where Y is a space of finite type (*e.g.*, $S^i\wedge U_+$ where $i\geq 0$ and $U\in\mathbf{Sm}/S$). Then, we are reduced to an unstable lemma:

Lemma 5.3.12 *Let S be a regular scheme. Let $(\tau_n)_{n\in\mathbf{N}}$ be a sequence of elements in the group $K_0(S)_{\mathbf{Q}}[[U]]$ which converges to an element τ. Then, for any space of finite type $X\in\mathcal{H}_\bullet(S)$ and $y\in\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(X,\mathbf{Z}\times\mathbf{Gr})$, there exists $N\geq 0$ such that for all $n\geq N$, $\tau_n(y)=\tau(y)$.*

Variants of lemma 5.1.2 and remark 5.1.3 show that the lemma is true if $Y=U_+$ with $U\in\mathbf{Sm}/S$. It holds more generally if X is a pointed smooth S-scheme, for the obvious map $\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(X,\mathbf{Z}\times\mathbf{Gr})\to\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(X_+,\mathbf{Z}\times\mathbf{Gr})$ is a split monomorphism, which, after tensoring with $\mathbf{Q}$, commutes to τ and the τ_n.

The general case follows. As X is of finite type, any $x\in\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(Y,\mathbf{Z}\times\mathbf{Gr})$ will factor through a disjoint union of finite Grassmann varieties. Then, there exists a pointed smooth S-scheme U, $u\in\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(U,\mathbf{Z}\times\mathbf{Gr})$ and $f\colon Y\to U$ in $\mathcal{H}_\bullet(S)$ such that $y=f^\star(u)$. By the previous case, there exists an integer N such that $\tau_n(u)=\tau(u)$ for $n\geq N$. Hence, $\tau_n(y)=\tau_n(f^\star u)=f^\star\tau_n(u)=f^\star\tau(u)=\tau(y)$ for $n\geq N$.

Remark 5.3.13 *One may find some inspiration from lemma 5.3.11 so as to define a topology on groups of morphisms $\mathrm{Hom}_{\mathcal{T}}(\mathbf{E},\mathbf{F})$ in a triangulated category $\mathcal{T}$ (where coproducts exist): the weakest one such that for any morphism $x\colon X\to\mathbf{E}$ with $X\in\mathcal{T}^{\mathrm{pf}}$, the composition with x induces a continuous map $\mathrm{Hom}_{\mathcal{T}}(\mathbf{E},\mathbf{F})\to\mathrm{Hom}_{\mathcal{T}}(X,\mathbf{F})$ where the target is endowed with the discrete topology. Then, the lemma would say that in the case of $\mathrm{End}_{\mathcal{SH}(S)}(\mathbf{BGL_Q})$, this topology is the same as the one introduced in remark 5.3.6.*

Proposition 5.3.14 *For any $n\in\mathbf{Z}$, the direct factor $\mathbf{BGL}_{\mathbf{Q}}^{(n)}$ of $\mathbf{BGL_Q}$ is preserved by $\mathbf{\Psi}^k$ for all $k\in\mathbf{Z}-\{0\}$ and $\mathbf{\Psi}^k$ acts on it by multiplication by k^n.*

It follows from the following equalities in $\mathrm{End}_{\mathcal{SH}(S)}(\mathbf{BGL_Q})$:

$$\mathbf{\Psi}^k\circ\pi_n=\pi_n\circ\mathbf{\Psi}^k=k^n\pi_n\ ,$$

which can be proved using their interpretations in the commutative subring $\mathbf{Q}^{\mathbf{Z}}$ (see proposition 5.3.7).

Corollary 5.3.15 *For all $k\in\mathbf{Z}-\{0,\pm1\}$ and $n\in\mathbf{Z}$, the endomorphism $\mathbf{\Psi}^k-k^n\mathrm{id}$ of $\mathbf{BGL_Q}$ has a kernel which is $\mathbf{BGL}_{\mathbf{Q}}^{(n)}$.*

Using easy computations in $\mathbf{Q}^{\mathbf{Z}}$, we get the existence of an automorphism $\phi_{n,k}$ of $\mathbf{BGL_Q}$ such that $\phi_{n,k}\circ(\mathbf{\Psi}^k-k^n\mathrm{id})=\mathrm{id}-\pi_n$. Hence, the kernel of $\mathbf{\Psi}^k-k^n\mathrm{id}$ is the same as the kernel of $\mathrm{id}-\pi_n$, which is $\mathbf{BGL}_{\mathbf{Q}}^{(n)}$ by definition.

In other words, the decomposition of theorem 5.3.10 can be thought as a decomposition of $\mathbf{BGL_Q}$ into a sum of eigenspaces $\mathbf{BGL}_{\mathbf{Q}}^{(n)}$ for the Adams operations.

Remark 5.3.16 *For any $a \in K_i(S)_{\mathbf{Q}}$, the constant family $a(1+U)$ belongs to $\lim K_i(S)^{\Omega}_{\mathbf{Q}}$ (it can be interpreted as the natural transformation $K_0(-)_{\mathbf{Q}} \to K_i(-)_{\mathbf{Q}}$ given by the multiplication by a). It induces a morphism $\mu_a \colon \mathbf{BGL_Q} \to \mathbf{BGL_Q}[-i]$. If $a \in K_i(S)^{(r)}$, one easily sees that μ_a maps $\mathbf{BGL}^{(n)}_{\mathbf{Q}}$ to $\mathbf{BGL}^{(n+r)}_{\mathbf{Q}}[-i]$ for all $n \in \mathbf{Z}$. Hence, we get a map*

$$K_i(S)^{(r)} \to \mathrm{Hom}_{\mathcal{SH}(S)}(\mathbf{BGL}^{(n)}_{\mathbf{Q}}, \mathbf{BGL}^{(n+r)}_{\mathbf{Q}}[-i])$$

which is easily shown to be a bijection for all $n \in \mathbf{Z}$ and $r \in \mathbf{Z}$.

It S is a regular scheme of finite Krull dimension, the γ-filtration on $K_i(X)$ has finitely many steps for all $X \in \mathbf{Sm}/S$ (see [42, §2]); it can be used to prove that $\mathbf{BGL_Q}$ is not only the direct sum of the $\mathbf{BGL}^{(n)}_{\mathbf{Q}}$ but also their infinite product in $\mathcal{SH}(S)$. This allows to give a description of morphisms $\mathbf{BGL_Q} \to \mathbf{BGL_Q}[-i]$ as infinite matrices $(a_{m,n})_{(m,n)\in \mathbf{Z}^2}$ where $a_{m,n} \in K_i(S)^{(m-n)}$ corresponds to $\mu_{a_{m,n}} \colon \mathbf{BGL}^{(n)}_{\mathbf{Q}} \to \mathbf{BGL}^{(m)}_{\mathbf{Q}}[-i]$.

Definition 5.3.17 *Let S be a regular scheme. We set $\mathbf{H}_{\mathrm{B}} = \mathbf{BGL}^{(0)}_{\mathbf{Q}} \in \mathcal{SH}(S)_{\mathbf{Q}}$.*

Using the periodicity isomorphism $\mathbf{Hom}_{\bullet}(\mathbf{P}^1, \mathbf{BGL_Q}) \simeq \mathbf{BGL_Q}$, we get canonical isomorphisms $\mathbf{BGL}^{(n)}_{\mathbf{Q}} \simeq \mathbf{H}_B \wedge (\mathbf{P}^1)^{\wedge n}$ for all $n \in \mathbf{Z}$.

Remark 5.3.18 *By its "definition" as an eigenspace of Adams operations on the object $\mathbf{BGL_Q}$ which represents rationalized algebraic K-theory, this object $\mathbf{H}_{\mathrm{B}}$ represents motivic cohomology as it was first introduced by Beilinson (see [7]).*

6 Riemann-Roch theorems

6.1 Adams-Riemann-Roch

The Adams-Riemann-Roch theorem [14, Theorem 7.6] says that if $f \colon X \to S$ is a projective morphism between regular schemes, then for all $k \in \mathbf{Z} - \{0\}$ and $x \in K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q}$:

$$\Psi^k(f_\star x) = f_\star(\Psi^k x \cdot (\theta_k \Omega_f)^{-1}) ,$$

where $f_\star \colon K_0(X) \to K_0(S)$ is the direct image in K-theory and $\theta_k \Omega_f$ is Bott's cannibalistic class associated to the virtual cotangent bundle. It can be stated as a commutative square:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\Psi^k(-)\cdot(\theta_k\Omega_f)^{-1}} & K_0(X) \\ \downarrow{\scriptstyle f_\star} & & \downarrow{\scriptstyle f_\star} \\ K_0(S) & \xrightarrow{\Psi^k} & K_0(S) \end{array}$$

We shall obtain that for a projective and smooth morphism between regular schemes, this diagram can be refined as a commutative diagram in $\mathcal{SH}(S)$, where $K_0(S)$ is replaced by $\mathbf{BGL}_{\mathbf{Q},S}$ (we add the subscript S as a remainder of the base scheme) and $K_0(X)$ by $\mathbf{R}f_\star \mathbf{BGL}_{\mathbf{Q},X}$ where $\mathbf{R}f_\star \colon \mathcal{SH}(X) \to \mathcal{SH}(S)$ is the functor constructed in [37, Proposition 4.4]. The proof will proceed by showing that the diagram in $\mathcal{SH}(S)$ commutes if and only if the relation stated at the level of K_0 in the standard Adams-Riemann-Roch theorem is true not only for $f\colon X \to S$ but for all morphisms $f_T\colon X_T \to T$ deduced from f by base change along smooth morphisms $T \to S$.

One may expect that the homotopic version of Adams-Riemann-Roch we state below (see theorem 6.1.2.1) has both sense and truthfulness for more general projective morphisms between regular schemes. However, the assumption that f is projective and smooth shall be used at several steps and thus should be considered as important in this method.

6.1.1 Pushforwards on BGL

Proposition 6.1.1.1 *Let $f\colon X \to S$ be a projective and smooth morphism between regular schemes. There exists a morphism $\mathbf{R}f_\star\mathbf{BGL}_X \xrightarrow{f_\star} \mathbf{BGL}_S$ in $\mathcal{SH}(S)$ such that for any $n \in \mathbf{Z}$, $i \in \mathbf{N}$, $T \in \mathbf{Sm}/S$, the map induced after applying the functor $\mathrm{Hom}_{\mathcal{SH}(S)}((\mathbf{P}^1)^{\wedge n} \wedge S^i \wedge T_+, -)$ identifies to the usual pushforward in K-theory $f_\star\colon K_i(X_T) \to K_i(T)$ where $X_T = X \times_S T$.*

Lemma 6.1.1.2 *Let $f\colon X \to S$ be a projective and smooth morphism between regular schemes. There exists a morphism $\mathbf{R}f_\star(\mathbf{Z} \times \mathbf{Gr}_X) \xrightarrow{f_\star} \mathbf{Z} \times \mathbf{Gr}_S$ in $\mathcal{H}_\bullet(S)$ such that after the application of $\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(S^i \wedge T_+, -)$ for all $T \in \mathbf{Sm}/S$, we get the usual pushforward in K-theory $f_\star\colon K_i(X_T) \to K_i(T)$ where $X_T = X \times_S T$.*

We have to use an homotopical description of these pushforwards in a way which should be stricly functorial in $T \in \mathbf{Sm}/S$. We use Thomason's model [43, Lemma 3.5.3]: for any regular scheme X, we consider the complicial biWaldhausen category $\mathcal{C}(X)$ of perfect bounded above complexes of flat $\mathcal{O}_X$-modules [2]. For any (regular) base scheme S, it is easy to turn this construction into a presheaf $\mathcal{C}_S$ of complicial biWaldhausen categories over $\mathbf{Sm}/S$, with $\mathcal{C}_S(X)$ equivalent to $\mathcal{C}(X)$ for all $X \in \mathbf{Sm}/S$. Then, the associated presheaf of K-theory spaces $\mathbf{K}\mathcal{C}_S$ is a model of algebraic K-theory (*i.e.*, it is canonically isomorphic to $\mathbf{Z} \times \mathbf{Gr}$ in $\mathcal{H}_\bullet(S)$, see Proposition 3.1.5). At this stage, it is obvious that $\mathbf{K}\mathcal{C}_X$ is acyclic for the functor $f_\star$, *i.e.*, we have a canonical isomorphism $\mathbf{R}f_\star\mathbf{K}\mathcal{C}_X \simeq f_\star\mathbf{K}\mathcal{C}_X$ in $\mathcal{H}_\bullet(S)$.

We shall construct the expected morphism $f_\star\colon \mathbf{R}f_\star(\mathbf{Z} \times \mathbf{Gr}_X) \to \mathbf{Z} \times \mathbf{Gr}_S$ as a morphism $f_\star\mathbf{K}\mathcal{C}_X \to \mathbf{K}\mathcal{C}_S$. The details follow. We choose a finite open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of X such that all the induced morphisms $f_i\colon U_i \to S$ are affine (as we assumed S separated, any affine open cover of X has this property). For any nonempty subset I of $\{1, \dots, n\}$, we set $U_I = \cap_{i \in I} U_i$ and denote $f_I\colon U_I \to S$ the restriction of f to these subschemes.

For any $T \in \mathbf{Sm}/S$, we consider the base change $f_T\colon X_T \to T$ of f along $T \to S$ and introduce the morphisms $f_{I,T}\colon U_I \times_S T \to T$ deduced from f_I for all nonempty subsets I of $\{1, \dots, n\}$. These morphisms $f_{I,T}$ are affine and flat. For any $\mathcal{M} \in \mathcal{C}(X_T)$, we define $(f_{\bullet,T})_\star\mathcal{M}$ as the total complex of the Čech type bicomplex:

$$\cdots \to 0 \to \oplus_{1 \leq i \leq n}(f_{i,T})_\star\mathcal{M} \to \oplus_{1 \leq i < j \leq n}(f_{i,j,T})_\star\mathcal{M} \to \dots\ ,$$

where the first *a priori* non trivial object lies in cohomological degree 0. As f is flat, the object $(f_{\bullet,T})_\star\mathcal{M}$ is a bounded complex of flat $\mathcal{O}_T$-modules and from standard results in coherent cohomology (see [20, Théorème 3.2.1]), $(f_{\bullet,T})_\star\mathcal{M}$ represents $\mathbf{R}f_{T\star}\mathcal{M}$ in the derived category $D(T, \mathcal{O}_T)$ and is perfect. Hence, we have defined a functor $(f_{\bullet,T})_\star\colon \mathcal{C}(X_T) \to \mathcal{C}(T)$ for any $T \in \mathbf{Sm}/S$. This construction commutes up to canonical isomorphisms with the inverse image functors (*i.e.*, the presheaf structure on $\mathcal{C}_S$) associated to morphisms $T' \to T$ in $\mathbf{Sm}/S$. It

[2] Note that we have to fix suitable cardinality bounds so as to get (essentially) small categories.

is an easy game to modify the definitions so as to get strict compatibilities. Finally, we may apply the K-theory functor to obtain the expected morphism $f_\star \mathbf{K}\mathcal{C}_X \xrightarrow{f_\star} \mathbf{K}\mathcal{C}_S$ of presheaves of pointed sets on $\mathbf{Sm}/S$.

The compatibility between pushforwards and external products implies that we may use the morphism from lemma 6.1.1.2 to define a morphism $\mathbf{R}f_\star\mathbf{BGL}_X \to \mathbf{BGL}_S$ up to stably phantom maps (*i.e.*, in $\mathcal{SH}_{\text{naïve}}(S)$). In the statement of proposition 6.1.1.1, there is no uniqueness claim. However, we shall see in the sequel that it will be the case after tensoring with $\mathbf{Q}$.

6.1.2 Statement of the theorem

Theorem 6.1.2.1 *Let $f\colon X \to S$ be a projective and smooth morphism between regular schemes. Then, the following diagram in $\mathcal{SH}(S)$ commutes:*

$$\begin{tikzcd}
\mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \arrow[r, "\mathbf{R}f_\star(\theta_k(\Omega_f)^{-1}\cdot\mathbf{\Psi}^k(-))"] \arrow[d, "f_\star"] & \mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \arrow[d, "f_\star"] \\
\mathbf{BGL}_{\mathbf{Q},S} \arrow[r, "\mathbf{\Psi}^k"] & \mathbf{BGL}_{\mathbf{Q},S}
\end{tikzcd}$$

where both vertical maps are the pushforward morphism constructed in proposition 6.1.1.1 (tensored with $\mathbf{Q}$), the lower map is $\mathbf{\Psi}^k \in \mathrm{End}_{\mathcal{SH}(S)}(\mathbf{BGL}_{\mathbf{Q},S})$ (see definition 5.3.2) and the upper map is obtained by applying $\mathbf{R}f_\star$ to the endomorphism of $\mathbf{BGL}_{\mathbf{Q},X}$ corresponding to $\mathbf{\Psi}^k$ multiplied by the inverse of Bott's cannibalistic class [3].

Corollary 6.1.2.2 *Let $f\colon X \to S$ be a projective and smooth morphism between regular schemes. Then, the following diagram commutes for any $i \in \mathbf{Z}$:*

$$\begin{tikzcd}
K_i(X) \arrow[r, "\Psi^k(-)\cdot(\theta_k\Omega_f)^{-1}"] \arrow[d, "f_\star"] & K_i(X) \arrow[d, "f_\star"] \\
K_i(S) \arrow[r, "\Psi^k"] & K_i(S)
\end{tikzcd}$$

Corollary 6.1.2.2 is deduced from the statement of theorem 6.1.2.1 by applying functors $\mathrm{Hom}_{\mathcal{SH}(S)}(S^i \wedge T_+, -)$. Conversely, I claim that two morphisms $\mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \to \mathbf{BGL}_{\mathbf{Q},S}$ in $\mathcal{SH}(S)$ are equal as soon as they induce equal maps after the application of functors $\mathrm{Hom}_{\mathcal{SH}(S)}((\mathbf{P}^1)^{\wedge -n} \wedge T_+, -)$ for all $n \in \mathbf{N}$ and $T \in \mathbf{Sm}/S$. This will be the goal of theorem 6.1.3.2 in the paragraph which follows. Then, theorem 6.1.2.1 shall follow from the classical Adams-Riemann-Roch theorem (*i.e.*, the case $i = 0$ in corollary 6.1.2.2).

6.1.3 Morphisms $\mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \to \mathbf{BGL}_{\mathbf{Q},S}$

Definition 6.1.3.1 *For all $(i, j) \in \mathbf{Z}^2$, we define a functor $\pi_{i,j}\colon \mathcal{SH}(S) \to \mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}$ by*

$$(\pi_{i,j}\mathbf{E})(U) = \mathrm{Hom}_{\mathcal{SH}(S)}((\mathbf{P}^1)^{\wedge j} \wedge S^{i-2j} \wedge U_+, \mathbf{E})\ ;$$

they are the functors "presheaves of stable homotopy groups".

[3] It makes sense as previous results show that $\mathrm{End}_{\mathcal{SH}(X)}(\mathbf{BGL}_{\mathbf{Q},X})$ is a module over $K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Theorem 6.1.3.2 *Let $f\colon X \to S$ be a projective and smooth morphism between regular schemes. Let $\tau\colon \mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \to \mathbf{BGL}_{\mathbf{Q},S}$ be a morphism in $\mathcal{SH}(S)$ such that for all $n \in \mathbf{Z}$, $\pi_{2n,n}(\tau) = 0$* [4]*. Then, $\tau = 0$.*

We use the theory of stable homotopic functors (see [6] and also [37, Remarque 4.6]). Thus, we have a direct image functor with proper support $\mathbf{R}f_!\colon \mathcal{SH}(X) \to \mathcal{SH}(S)$ which has a right adjoint $f^!$. As f is projective, we have a canonical isomorphism $\mathbf{R}f_! \xrightarrow{\sim} \mathbf{R}f_\star$. Then, by adjunction, the morphism $\tau\colon \mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \to \mathbf{BGL}_{\mathbf{Q},S}$ corresponds to a morphism $\tilde{\tau}\colon \mathbf{BGL}_{\mathbf{Q},X} \to f^!\mathbf{BGL}_{\mathbf{Q},S}$.

Lemma 6.1.3.3 *We let $f\colon X \to S$ be a projective and smooth morphism between regular schemes.*

(i) There exists a canonical isomorphism $f^!\mathbf{BGL}_{\mathbf{Q},S} \simeq \mathbf{BGL}_{\mathbf{Q},X}$ in $\mathcal{SH}(X)$.

(ii) For any vector bundle $\mathcal{E}$ over X, we have a canonical isomorphism $\mathbf{BGL}_{\mathbf{Q},X} \wedge \mathrm{Th}\,\mathcal{E} \simeq \mathbf{BGL}_{\mathbf{Q},X}$ in $\mathcal{SH}(X)$.

By definition of $f^!$, for any $\mathbf{E} \in \mathcal{SH}(S)$, we have an isomorphism $f^!\mathbf{E} \simeq f^\star\mathbf{E} \wedge \mathrm{Th}\,T_f$ where T_f is the relative tangent bundle of f and $\mathrm{Th}\,T_f$ its Thom space. As $f^\star\mathbf{BGL}_{\mathbf{Q},S}$ identifies to $\mathbf{BGL}_{\mathbf{Q},X}$, (i) will follow from (ii).

To prove (ii), we consider the isomorphism $\mathrm{Th}\,\mathcal{E} \simeq \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)/\mathbf{P}(\mathcal{E})$ [5] and the class ξ of the fundamental sheaf $\mathcal{O}(1)$ in $K_0(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X))$. We may set $v = \xi^r - [\wedge^1\mathcal{E}]\xi^{r-1} + [\wedge^2\mathcal{E}]\xi^{r-2} + \cdots + (-1)^r[\wedge^r\mathcal{E}] \in K_0(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X))$ where r is the rank of $\mathcal{E}$. The class v vanishes when restricted to $\mathbf{P}(\mathcal{E})$. Hence, v actually defines an element in $\tilde{K}_0(\mathrm{Th}\,\mathcal{E})$. In this paragraph, $\tilde{K}_0(Y)$ is the reduced K-theory of a pointed space Y, *i.e.*, $\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(Y, \mathbf{Z} \times \mathbf{Gr})$, which identifies to the kernel of the map $K_0(Y) \to K_0(S)$ given by the base-point. Even if we use the same notation, it should not be confused with the kernel of $\mathrm{rk}\colon K_0(X) \to \mathbf{Z}^{\pi_0(X)}$, which makes sense for $X \in \mathbf{Sm}/S$.

Using the multiplicative structure on $\mathbf{Z} \times \mathbf{Gr}$, we may consider the external product with v in $\mathcal{H}_\bullet(S)$:

$$\mathbf{Z} \times \mathbf{Gr} \to \mathbf{R}\,\mathbf{Hom}_\bullet(\mathrm{Th}\,\mathcal{E}, \mathbf{Z} \times \mathbf{Gr})\ ,$$

which is seen to be an isomorphism thanks to computations using the projective bundle formula. Using this morphism termwise, we get the expected isomorphism

$$\mathbf{BGL}_X \xrightarrow{\sim} \mathbf{R}\,\mathbf{Hom}_\bullet(\mathrm{Th}\,\mathcal{E}, \mathbf{BGL}_X)$$

in $\mathcal{SH}_{\text{naïve}}(X)$ [6]. As $\mathrm{Th}\,\mathcal{E}$ is invertible for the $\wedge$-product on $\mathcal{SH}(X)$ (see proposition 4.1.1), property (ii) follows.

[4] One may notice that $\pi_{2n,n}(\tau) = 0$ implies $\pi_{2(n+1),n+1}(\tau) = 0$.

[5] We, reluctantly, do not follow Grothendieck's convention. Here, $\mathbf{P}(\mathcal{E})$ is the projectivisation of the symmetric algebra of the dual of $\mathcal{E}$.

[6] This construction may also be deduced from a more universal pairing $(\mathbf{Z} \times \mathbf{Gr}) \wedge \mathbf{BGL} \to \mathbf{BGL}$ which should be constructed first in $\mathcal{SH}(\mathrm{Spec}(\mathbf{Z}))$. However, when one want to tackle the trouble of stably phantoms morphisms, one has to use different arguments than those appearing in this article. To do this, we can use [36, Lemma A.6] which we used there to obtain another proof of the construction of $\mathbf{BGL}$, see corollary 5.2.7. This method can be continued in order to obtain an associative and commutative pairing $\mathbf{BGL} \wedge \mathbf{BGL} \to \mathbf{BGL}$ in $\mathcal{SH}(S)$ (see [21, Theorem 2.2.1]).

Lemma 6.1.3.4 *Let $\psi\colon \mathbf{E} \to \mathbf{F}$ be a morphism in $\mathcal{SH}(S)$. We assume that $\mathbf{F}$ is such that for any $U \in \mathbf{Sm}/S$, vector bundle $\mathcal{E}$ on U and $n \in \mathbf{Z}$, the canonical map $\tilde{\mathbf{F}}^{2n,n}(\mathrm{Th}_U\,\mathcal{E}) \to \mathbf{F}^{2n,n}(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_U))$ is injective* [7]. *We also assume that $\pi_{2n,n}(\psi) = 0$ for all $n \in \mathbf{Z}$. Then, for any vector bundle $\mathcal{E}$ on $U \in \mathbf{Sm}/S$ and any $n \in \mathbf{Z}$, the map $\mathrm{Hom}_{\mathcal{SH}(X)}((\mathbf{P}^1)^{\wedge n} \wedge \mathrm{Th}_U(-\mathcal{E}), \psi)$ vanishes (see definition 4.1.2).*

Using Jouanolou's trick, we may assume that U is affine. Then, the virtual bundle $-\mathcal{E}$ identifies to a difference $\mathcal{F} - \mathcal{O}_U^k$ where $\mathcal{F}$ is a genuine vector bundle and $k \in \mathbf{N}$. Then, we want to prove that for any $n \in \mathbf{Z}$, the morphism $\mathrm{Hom}_{\mathcal{SH}(S)}((\mathbf{P}^1)^{\wedge n} \wedge \mathrm{Th}_U\,\mathcal{F}, \psi)$ vanishes. As, $\mathrm{Th}_U\,\mathcal{F} = \mathbf{P}(\mathcal{F} \oplus \mathcal{O}_U)/\mathbf{P}(\mathcal{F})$, the result follows from the second assumption for $U = \mathbf{P}(\mathcal{F} \oplus \mathcal{O}_U)$ and the injectivity stated in the first assumption.

Now, we shall prove theorem 6.1.3.2. We may apply lemma 6.1.3.4 to $\tau\colon \mathbf{R}f_\star\mathbf{BGL}_{\mathbf{Q},X} \to \mathbf{BGL}_{\mathbf{Q},S}$. Then, as $f_!\mathbf{E} \simeq f_\sharp(\mathbf{E} \wedge \mathrm{Th}_X(-Tf))$ for any $\mathbf{E} \in \mathcal{SH}(X)$, we obtain the vanishing of the maps $\mathrm{Hom}_{\mathcal{SH}(S)}(f_!((\mathbf{P}^1)^{\wedge n} \wedge U_+), \tau)$ for all $n \in \mathbf{Z}$ and $U \in \mathbf{Sm}/X$. By adjunction, it implies that the maps $\mathrm{Hom}_{\mathcal{SH}(X)}((\mathbf{P}^1)^{\wedge n} \wedge U_+, \tilde{\tau})$ vanish. As we know that $\tilde{\tau}$ can be identified to an endomorphism of $\mathbf{BGL}_{\mathbf{Q},X}$ (see lemma 6.1.3.3), we can use the results of section 5 to assert that $\tilde{\tau} = 0$. Finally, by adjunction, $\tau = 0$.

6.2 Motivic Eilenberg-Mac Lane spectra

6.2.1 Morphisms $\mathbf{Z} \times \mathbf{Gr} \to K(\mathbf{Z}(n), 2n)$

Definition 6.2.1.1 *Let k be a perfect field. For any $n \geq 0$, we denote $K(\mathbf{Z}(n), 2n)$ the motivic Eilenberg-Mac Lane space defined in [45, §6.1]. For $i \geq 0$. we let $K(\mathbf{Z}(n), 2n - i)$ be its ith loop space.*

By definition, for any $n \geq 0$ and $i \geq 0$, the group $\mathrm{Hom}_{\mathcal{H}(k)}(X, K(\mathbf{Z}(n), 2n-i))$ identifies to the motivic cohomology group $H^{2n-i}(X, \mathbf{Z}(n))$. The comparison with (higher) Chow groups [46] implies that for any $n \geq 0$, there is a canonical isomorphism $\pi_0 K(\mathbf{Z}(n), 2n) \simeq CH^n(-)$ in $\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}$ where $CH^n(-)$ is the presheaf $X \longmapsto CH^n(X)$.

Theorem 6.2.1.2 *Let k be a perfect field. Let $n \geq 0$. Then, the functor π_0 induces a bijection:*

$$\mathrm{Hom}_{\mathcal{H}(k)}(\mathbf{Z} \times \mathbf{Gr}, K(\mathbf{Z}(n), 2n)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Sets}}(K_0(-), CH^n(-)) \ .$$

Moreover, the graded algebra $(\mathrm{Hom}_{\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Sets}}(\tilde{K}_0(-), CH^n(-))_{n\in\mathbf{N}}$ identifies to the polynomial algebra $\mathbf{Z}[c_1, c_2, \dots]$ where c_i lies in degree i and corresponds to the ith Chern class $c_i\colon \tilde{K}_0(-) \to CH^i(-)$.

The first statement follows from the fact that whenever $d \leq d'$ and $r \leq r'$, the inclusion $\mathbf{Gr}_{d,r} \subset \mathbf{Gr}_{d',r'}$ induces a split monomorphisms $M(\mathbf{Gr}_{d,r}) \subset M(\mathbf{Gr}_{d',r'})$ on motives. This fact follows from the cellularity of $\mathbf{Gr}_{d,r}$, $\mathbf{Gr}_{d',r'}$ and $\mathbf{Gr}_{d',r'} - \mathbf{Gr}_{d,r}$ (see [24, §3] for a similar statement). Then, any object representing a cohomology which factors through the category of motives will satisfy property (K) with any number of operands (see definition 1.2.2) and we may use theorem 1.1.6.

The second part arises from the computation of Chow groups of Grassmann varieties $\mathbf{Gr}_{d,r}$ for $d, r \geq 0$ (see [19]) and the passage to the limit $r \to \infty$ and $d \to \infty$ as it was done for the algebraic K-theory.

[7] We use standard implicit convention. More precisely, this map is the result of the application of the functor $\mathrm{Hom}_{\mathcal{SH}(S)}((\mathbf{P}^1)^{\wedge n} \wedge -, \mathbf{F})$ to the canonical morphism $\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_U)_+ \to \mathrm{Th}_U\,\mathcal{E}$ in $\mathcal{H}_\bullet(S)$.

6.2.2 Additive morphisms

The proof of theorem 6.2.1.2 applies not only to natural transformations $K_0(-) \to CH^n(-)$ but also to natural transformations involving several operands, *e.g.*, $K_0(-) \times K_0(-) \to CH^n(-)$. Hence, H-group morphisms $\mathbf{Z} \times \mathbf{Gr} \to K(\mathbf{Z}(n), 2n)$ correspond to morphisms $K_0(-) \to CH^n(-)$ in $\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}$ (see proposition 2.2.3). The group of these morphisms is described in the following proposition:

Proposition 6.2.2.1 *Let k be a perfect field. For any $n \geq 0$, the map given by the evaluation at $[\mathcal{O}(1)]$ in $K_0(\mathbf{P}^n)$ induces an isomorphism:*

$$\mathrm{Hom}_{\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}}(K_0(-), CH^n(-)) \xrightarrow{\sim} \lim_{r \in \mathbf{N}} CH^n(\mathbf{P}^r) \simeq CH^n(\mathbf{P}^n) \simeq \mathbf{Z} \ .$$

We denote $\chi_n \colon K_0(-) \to CH^n(-)$ the canonical generator given by this isomorphism. It is characterised by the fact that $\chi_n([\mathcal{L}]) = [D]^n$ anytime $\mathcal{L}$ is a line bundle on $X \in \mathbf{Sm}/S$ and D is the divisor of a rational section of $\mathcal{L}$.

The proof of the injectivity of the map

$$\mathrm{Hom}_{\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}}(K_0(-), CH^n(-)) \xrightarrow{\sim} \lim_{r \in \mathbf{N}} CH^n(\mathbf{P}^r)$$

is similar to that of proposition 5.1.1. The group $\lim_{r \in \mathbf{N}} CH^n(\mathbf{P}^r)$ is easily identified to the group $\mathbf{Z}$, generated by the compatible family made of nth powers of classes in hyperplanes in $\mathbf{P}^r$ for all $r \in \mathbf{N}$. For the surjectivity, *i.e.*, the existence of χ_n, we shall use the following lemma, which is a consequence of the theory of symmetric polynomials (hint: use [1, VI 4.3-4.4]):

Lemma 6.2.2.2 *Let $n \geq 1$. There exists a unique functorial homomorphism*

$$\chi_n \colon (1 + A[[t]]^+, \times) \to (A, +)$$

for all commutative rings A such that for any $x \in A$,

$$\chi_n(1 + xt) = x^n \ ,$$

and χ_n vanishes on the subgroup $1 + t^{n+1}A[[t]]$.

Note that by looking at the universal situation, we know that $\chi_n(\sum_{i \geq 0} a_i t^i)$ is given by a polynomial in $a_1, \dots, a_n$ and it is homogeneous of total degree n if we set $\deg a_i = i$.

For any $X \in \mathbf{Sm}/k$, $u \in K_0(X)$, we consider the Chern polynomial $c_t(u) \in CH^\star(X)[[t]]$ and apply the construction of the lemma to this series : $\chi_n(c_t(u)) \in CH^n(X)$. This constructs a natural transformation $K_0(-) \to CH^n(-)$ to which we give the same name χ_n. This finishes the proof of the proposition in the case $n \geq 1$; the remaining case $n = 0$ is trivial.

Remark 6.2.2.3 *As we have seen it, the natural transformation $\chi_n \colon K_0(-) \to CH^n(-)$ is given by a polynomial involving Chern classes. It can be computed inductively using Newton relations:*

$$\chi_k - c_1\chi_{k-1} + \cdots + (-1)^{k-1}c_{k-1}\chi_1 + (-1)^k k c_k = 0 \ .$$

For instance, $\chi_1 = c_1$, $\chi_2 = c_1^2 - 2c_2$, $\chi_3 = c_1^3 - 3c_1c_2 + 3c_3$.

The following similar result gives a computation of the group of H-group morphisms $\mathbf{Z} \times \mathbf{Gr} \to K(\mathbf{Z}(n), 2n)$ in $\mathcal{H}_\bullet(k)$.

Corollary 6.2.2.4 *Let k be a perfect field, $n \geq 0$, $i \geq 0$. For any $0 \leq j \leq \min(i,n)$ and $x \in H^{2j-i}(k, \mathbf{Z}(j))$, we define a natural transformation $x \cdot \chi_{n-j} \colon K_0(-) \to H^{2n-i}(-, \mathbf{Z}(n))$ of presheaves of abelian groups on $\mathbf{Sm}/k$, obtained as the composition of χ_{n-j} and the multiplication by x on motivic cohomology. Then, the group of natural transformations $K_0(-) \to H^{2n-i}(-, \mathbf{Z}(n))$ identifies to the direct sum of the groups $H^{2j-i}(k, \mathbf{Z}(j))$ for $0 \leq j \leq \min(i,n)$, as follows:*

$$\mathrm{Hom}_{\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}}(K_0(-), H^{2n-i}(-, \mathbf{Z}(n))) \simeq \bigoplus_{j=0}^{\min(i,n)} H^{2j-i}(k, \mathbf{Z}(j)) \cdot \chi_{n-j} \; .$$

6.2.3 Stable morphisms

The motivic Eilenberg-Mac Lane spectrum $\mathbf{H}_{\mathbf{Z}}$ is obtained from the sequence of objects $K(\mathbf{Z}(n), 2n)$ (see [45, §6.1]). We may describe its image in $\mathcal{SH}_{\text{naïve}}(k)$ by saying that the different Eilenberg-Mac Lane spaces are related by the canonical isomorphism $K(\mathbf{Z}(n), 2n) \simeq \mathbf{R}\,\mathbf{Hom}_\bullet(\mathbf{P}^1, K(\mathbf{Z}(n+1), 2n+2)$ induced by the external product with the class of the 1-codimensional cycle $[\infty]$ in $CH^1(\mathbf{P}^1)$. This construction generalises to give a $\mathbf{P}^1$-spectrum $\mathbf{H}_A$ for any coefficient abelian group A.

In order to study morphisms $\mathbf{BGL} \to \mathbf{H}_{\mathbf{Z}}[-i]$ for $i \geq 0$, we use the following definition.

Definition 6.2.3.1 *Let $n \geq 1$ and $i \geq 0$. Let $\tau \colon K_0(-) \to H^{2n-i}(-, \mathbf{Z}(n))$ be an additive natural transformation,* i.e., *a morphism in $\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Ab}$. We define a natural transformation $\Omega_{\mathbf{P}^1}(\tau) \colon K_0(-) \to H^{2n-2-i}(-, \mathbf{Z}(n-1))$ which shall be characterised by the commutativity of the following diagram for all $X \in \mathbf{Sm}/k$:*

$$\begin{tikzcd} K_0(X) \arrow[r, "u \boxtimes -"] \arrow[d, "\Omega_{\mathbf{P}^1}(\tau)"] & K_0(\mathbf{P}^1 \times X) \arrow[d, "\tau"] \\ H^{2n-2-i}(X, \mathbf{Z}(n-1)) \arrow[r, "{[\infty] \boxtimes -}"] & H^{2n-i}(\mathbf{P}^1 \times X, \mathbf{Z}(n)) \end{tikzcd}$$

where $u = [\mathcal{O}(1)] - 1 \in K_0(\mathbf{P}^1)$ and $[\infty]$ is the class of a rational point in $CH^1(\mathbf{P}^1) = H^2(\mathbf{P}^1, \mathbf{Z}(1))$.

Lemma 6.2.3.2 *Let k be a perfect field. For any $n \geq 1$, we have $\Omega_{\mathbf{P}^1}(\chi_n) = n\chi_{n-1}$.*

By the splitting principle, it suffices to check that $\Omega_{\mathbf{P}^1}(\chi_n)$ and $n\chi_{n-1}$ coincide on elements of the form $[\mathcal{L}] \in K_0(X)$ where $\mathcal{L}$ is a line bundle on some $X \in \mathbf{Sm}/k$. Let D be the divisor of a rational section of $\mathcal{L}$. Considering $CH^\star(X \times \mathbf{P}^1)$ both as an algebra over $CH^\star(X)$ and $CH^\star(\mathbf{P}^1)$, we get:

$$\begin{aligned} [\infty] \boxtimes \Omega_{\mathbf{P}^1}(\chi_n)([\mathcal{L}]) &= \chi_n(u \boxtimes [\mathcal{L}]) \\ &= \chi_n([\mathcal{O}(1) \boxtimes \mathcal{L}]) - \chi_n(\mathcal{O}_{\mathbf{P}^1} \boxtimes \mathcal{L}) \\ &= ([\infty] + [D])^n - [D]^n = n[\infty][D]^{n-1} \\ &= [\infty] \boxtimes (n\chi_{n-1}([\mathcal{L}])) \; , \end{aligned}$$

which proves the expected result: $\Omega_{\mathbf{P}^1}(\chi_n)([\mathcal{L}]) = n\chi_{n-1}([\mathcal{L}])$.

This lemma leads to a description of the projective system

$$(\mathrm{Hom}^+_{\mathcal{H}_\bullet(S)}(\mathbf{Z}\times\mathbf{Gr}, K(\mathbf{Z}(n),2n)))_{n\in\mathbf{N}}$$

deduced from the bonding morphisms on $\mathbf{BGL}$ and $\mathbf{H}_\mathbf{Z}$; it identifies to a projective system which we shall denote $\mathbf{Z}!$:

$$\cdots \to \mathbf{Z} \xrightarrow{5} \mathbf{Z} \xrightarrow{4} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{1} \mathbf{Z}\ .$$

We generalise this definition:

Definition 6.2.3.3 *Let A be an abelian group. We define a projective system $A!$ of abelian groups indexed by $\mathbf{N}$ by saying that in degree $n\in\mathbf{N}$, $(A!)_n = A$ and the transition map $(A!)_n \to (A!)_{n-1}$ is the multiplication by n on A.*

Definition 6.2.3.4 *If $X_\bullet = (\cdots \to X_n \xrightarrow{f_{n-1}} X_{n-1} \to \cdots \to X_1 \xrightarrow{f_0} X_0)$ is a projective system of abelian groups indexed by $\mathbf{N}$, we define a new projective system $sX_\bullet = (\cdots \to X_n \xrightarrow{f_{n-1}} X_{n-1} \to \cdots \to X_1 \xrightarrow{f_0} X_0 \to 0)$.*

Proposition 6.2.3.5 *Let A be an abelian group. We let $\mathbf{H}_A$ be the motivic Eilenberg-Mac Lane spectrum with coefficients in A. Then, for any $i\in\mathbf{Z}$, the projective system*

$$(\mathrm{Hom}^+_{\mathcal{H}_\bullet(k)}(\mathbf{Z}\times\mathbf{Gr}, K(A(n),2n-i))_{n\in\mathbf{N}}$$

associated to the $\mathbf{P}^1$-spectra $\mathbf{BGL}$ and $\mathbf{H}_A[-i]$ identifies to

$$\prod_{j=0}^{i} s^j H^{2j-i}(k, A(j))!\ .$$

For $i\geq 0$, it follows from A-valued variants of corollary 6.2.2.4 and lemma 6.2.3.2. If $i<0$, $K(A(n),2n-i)$ identifies to $\mathbf{R}\,\mathbf{Hom}_\bullet(\mathbf{G}_{\mathrm{m}}^{\wedge -i}, K(A(n),2n))$ and both projective systems vanish.

Then, we observe that for any abelian group A, $\lim A! \simeq \mathrm{Hom}(\mathbf{Q},A)$ and $\mathbf{R}^1\lim A! \simeq \mathrm{Ext}(\mathbf{Q},A)$, and that the shift functor s does not change $\lim$ and $\mathbf{R}^1\lim$ of projective systems. Thus, we get the following theorem:

Theorem 6.2.3.6 *Let k be a perfect field. Let A be an abelian group. Let $i\in\mathbf{Z}$. There is a canonical short exact sequence:*

$$0\to\prod_{j=0}^{i+1}\mathrm{Ext}(\mathbf{Q},H^{2j-i-1}(k,A(j)))\to\mathrm{Hom}_{\mathcal{SH}(k)}(\mathbf{BGL},\mathbf{H}_A[-i])$$
$$\to\prod_{j=0}^{i}\mathrm{Hom}(\mathbf{Q},H^{2j-i}(k,A(j)))\to 0\ ,$$

where the group on the right side identifies to morphisms in $\mathcal{SH}_{\text{naïve}}(k)$ and the group on the left to stably phantom morphisms.

Corollary 6.2.3.7 (Existence of nonzero stably phantom morphisms) *Let k be a perfect field. There exists an isomorphism*

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\mathbf{BGL}, \mathbf{H_Z}[1]) \simeq \mathrm{Ext}(\mathbf{Q}, \mathbf{Z}) \simeq \widehat{\mathbf{Z}}/\mathbf{Z} \ ,$$

and all these morphisms $f\colon \mathbf{BGL} \to \mathbf{H_Z}[1]$ are stably phantom, i.e., *for any morphism in $\mathcal{SH}(k)$ of the form $g\colon (\mathbf{P}^1)^{\wedge -n} \wedge W \to \mathbf{BGL}$ where $n \in \mathbf{Z}$ and $W \in \mathcal{H}_\bullet(k)$, the composition $f \circ g$ vanishes (see [37, Définition 6.6]).*

Remark 6.2.3.8 *Most of the results appearing in this article have homologues in the classical homotopy theory and are coherent with "complex points functors" from $\mathbf{A}^1$-homotopy categories to usual (topological) homotopy categories. In particular, the spectrum $\mathbf{BGL}(\mathbf{C})$ obtained as the image of $\mathbf{BGL}$ by the "complex points functors" $\mathcal{SH}(\mathbf{C}) \to \mathcal{SH}^{\mathrm{top}}$ represents topological complex K-theory (see [37, Remarque 2.16]). Then, if $\mathbf{H}_\mathbf{Z}^{\mathrm{top}} \in \mathcal{SH}^{\mathrm{top}}$ is the classical Eilenberg-Mac Lane spectrum, we get the same computation of the group $\mathrm{Hom}_{\mathcal{SH}^{\mathrm{top}}}(\mathbf{BGL}(\mathbf{C}), \mathbf{H}_\mathbf{Z}^{\mathrm{top}}[1])$. The example of stably phantom morphisms in $\mathcal{SH}^{\mathrm{top}}$ which we hereby get may be considered as simpler than those constructed by Christensen [8, Proposition 6.10].*

Definition 6.2.3.9 *Let k be a perfect field. We let $\mathrm{ch}\colon \mathbf{BGL} \to \mathbf{H_Q}$ be the canonical generator of $\mathrm{Hom}_{\mathcal{SH}(k)}(\mathbf{BGL}, \mathbf{H_Q}) \simeq \mathbf{Q}$. This is the Chern character. Using Bott periodicity ($\mathbf{BGL} \simeq \mathbf{R\,Hom}_\bullet(\mathbf{P}^1, \mathbf{BGL})$), we deduce from it a sequence of morphisms $\mathrm{ch}_i\colon \mathbf{BGL_Q} \to \mathbf{H}_{\mathbf{Q}(i)}[2i]$ where $\mathbf{H}_{\mathbf{Q}(i)} = \mathbf{H_Q} \wedge (\mathbf{P}^1)^{\wedge i}[-2i]$. The total Chern character is $\prod_i \mathrm{ch}_i$:*

$$\mathrm{ch}_t\colon \mathbf{BGL_Q} \to \prod_{i\in\mathbf{Z}} \mathbf{H}_{\mathbf{Q}(i)}[2i]$$

(the infinite product on the right is also a direct sum).

Remark 6.2.3.10 *Remark 5.3.18 may be continued as follows. One easily sees that the Chern character $\mathrm{ch}\colon \mathbf{BGL_Q} \to \mathbf{H_Q}$ vanishes on $\mathbf{BGL}_\mathbf{Q}^{(i)}$ (see theorem 5.3.10) for $i \neq 0$ so that it factors through its direct factor $\mathbf{H}_{\mathrm{B}}$ (see definition 5.3.17) as $\mathbf{BGL_Q} \to \mathbf{H}_{\mathrm{B}} \xrightarrow{\mathrm{ch}^{(0)}} \mathbf{H_Q}$. It follows from known results (see [29]) that $\mathrm{ch}^{(0)}\colon \mathbf{H}_{\mathrm{B}} \to \mathbf{H_Q}$ is an isomorphism; equivalently, $\mathrm{ch}_t\colon \mathbf{BGL_Q} \to \oplus_{i\in\mathbf{Z}} \mathbf{H}_{\mathbf{Q}(i)}[2i]$ is an isomorphism.*

6.3 Grothendieck-Riemann-Roch

For simplicity, we only consider the case of a projective and smooth morphism $f\colon X \to S$ in $\mathbf{Sm}/k$ where k is a perfect field. We let d be the relative dimension of f. The "restriction" of $\mathbf{H_Q} \in \mathcal{SH}(k)$ to X and S provides objects in $\mathcal{SH}(X)$ and $\mathcal{SH}(S)$ which shall also be denoted $\mathbf{H_Q}$; they satisfy $f^\star\mathbf{H_Q} \simeq \mathbf{H_Q}$. To f is attached a morphism of motives $\mathbf{Z}(d)[2d] \to M(X)$ in $\mathrm{DM}(S)$ (see [22, §I.4.4]) which induces a morphism $f_\star\colon \mathbf{R\,Hom}_\bullet(X_+, \mathbf{H_Q}) \to \mathbf{H}_{\mathbf{Q}(-d)}[-2d]$ in $\mathcal{SH}(S)$. This morphism induces the pushforward maps

$$f_\star\colon H^p(X \times_S T, \mathbf{Z}(q)) \to H^{p-2d}(T, \mathbf{Z}(q-d)) \ ,$$

for all $T \in \mathbf{Sm}/S$.

Theorem 6.3.1 *Let k be a perfect field. Let $f\colon X \to S$ be a projective and smooth morphism in $\mathbf{Sm}/k$. Then, the following diagram commutes in $\mathcal{SH}(S)$:*

$$\begin{array}{ccc} \mathbf{R}f_\star\mathbf{BGL_Q} & \xrightarrow{\mathbf{R}f_\star(\mathrm{ch}\cdot\mathrm{Td}\,T_f)} & \prod_{i\in\mathbf{Z}}\mathbf{R}f_\star\mathbf{H}_{\mathbf{Q}(i)}[2i] \\ \downarrow{\scriptstyle f_\star} & & \downarrow{\scriptstyle f_\star} \\ \mathbf{BGL_Q} & \xrightarrow{\mathrm{ch}} & \prod_{i\in\mathbf{Z}}\mathbf{H}_{\mathbf{Q}(i)}[2i] \end{array}$$

The proof is similar to that of theorem 6.1.2.1. This statement is equivalent to the usual Grothendieck-Riemann-Roch theorem for morphisms $f_T\colon X\times_S T \to T$ for all $T \in \mathbf{Sm}/S$ (which is known to be true, see [13, Chapter 15]). The reason for this is the variant of theorem 6.1.3.2: a morphism $\tau\colon \mathbf{BGL_Q} \to \mathbf{H}_{\mathbf{Q}(i)}[2i]$ vanishes if and only if it vanishes after the application of functors $\pi_{2n,n}\colon \mathcal{SH}(S) \to \mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}$ for all $n \in \mathbf{Z}$.

Corollary 6.3.2 *Let k be a perfect field. Let $f\colon X \to S$ be a projective and smooth morphism in $\mathbf{Sm}/k$. For any $j \in \mathbf{N}$, the following diagram commutes:*

$$\begin{array}{ccc} K_j(X) & \xrightarrow{\mathrm{ch}_t\cdot\mathrm{Td}\,Tf} & \prod_{i\in\mathbf{Z}} H^{2i-j}(S,\mathbf{Q}(i)) \\ \downarrow{\scriptstyle f_\star} & & \downarrow{\scriptstyle f_\star} \\ K_j(S) & \xrightarrow{\mathrm{ch}_t} & \prod_{i\in\mathbf{Z}} H^{2i-j}(S,\mathbf{Q}(i)) \end{array}$$

This gives another proof of some results by Gillet [16] on higher Riemann-Roch theorems.

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