Boundary value problems in domains with boundaries of high or mixed codimension

Problèmes aux limites dans des domaines à bords de codimension plus grande que 1 ou de codimension mixe

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THESE D'HABILITATION

Title : Boundary value problems in domains with boundaries of high or mixed codimension

Keywords : Boundary value problems, Uniform rectifiability, Elliptic measures, Green functions, Carleson measures.

Abstract : In this memoir, I introduce the reader to my postdoctoral work, which focuses on characterizing the uniform rectifiability (UR) of sets of codimension higher than one using estimates for solutions of elliptic PDEs. To this end, we developed an elliptic theory in the complement of a given set $S \subset \mathbb{R}^n$ based on operators that are uniformly elliptic with respect to a degenerate weight adapted to the dimension $d < n-1$ of S. Within this framework, we sought a suitable analogue of the Laplacian that could serve to characterize UR sets.

I outline the state of the art prior to our contribution, the difficulties we encountered, and the solution we ultimately propose.

Titre : Problèmes aux limites dans des domaines à bords de codimension plus grande que 1 ou de codimension mixe

Mots clés : Problèmes aux limites, Rectifiabilité uniforme, Mesures elliptiques, Functions de Green, Mesures de Carleson.

Résumé : Dans ce mémoire, j'introduis le lecteur à mon travail postdoctoral, qui porte sur la caractérisation de la rectifiabilité uniforme (UR) de sous-ensembles de codimension supérieure à un, au moyen d'estimations sur les solutions d'EDP elliptiques. À cette fin, nous avons développé une théorie elliptique dans le complément d'un ensemble donné $S \subset \mathbb{R}^n$, fondée sur des opérateurs uniformément elliptiques par rapport à un poids dégénéré adapté à la dimension de S. Dans ce cadre, nous avons recherché un analogue approprié du laplacien, susceptible de permettre la caractérisation des ensembles UR.

Dans ce qui suit, je présente l'état de l'art avant notre contribution, les difficultés rencontrées au cours de ce travail, ainsi que la solution que nous proposons.

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Acknowledgements

This memoir supports my application for the HDR (Habilitation à Diriger des Recherches). It outlines my research accomplishments from 2015 to 2024, which include 16 articles and 2 surveys. My collaborative work resulted in eleven articles and two surveys with the following colleagues : Svitlana Mayboroda (7 articles + 2 surveys), Guy David (4 articles + 2 surveys), Linhan Li (3 articles), Zanbing Dai (2 articles), Bruno Poggi (1 article) and Zihui Zhao (1 article). Additionally, I authored five articles independently. I am grateful to all my collaborators for their valuable contributions and the engaging research experiences we shared.

The present memoir is loosely based on a book project I am working on with Svitlana Mayboroda. The book is intended for the CBMS series, following the conference *Analysis, Geometry, and Partial Differential Equations in a Lower-Dimensional World,* held in Florida in May 2022. While this memoir focuses on my personal contributions and perspectives, some overlap with Svitlana Mayboroda's conference presentation is inevitable. However, I have made an effort to minimize similarities. For example : all pictures are my original designs ; the organization of the content is different ; all the material has been revised to the point that only Section 1.1 is a bit close to the original material ; the list of open problems is compiled based on my own discussions with various individuals.

I would like to express my warmest thanks to Pascal Auscher, Jill Pipher, and Xavier Tolsa for generously agreeing to serve as reviewers for my HDR. I am also grateful to the other members of the jury - Jean-François Babadjian, Hajer Bahouri, Patrick Gérard, Michael Goldman, and Séverine Rigot - for their time and their endorsement.

I extend my sincere thanks to Svitlana Mayboroda, Antoine Julia, Linhan Li, Cole Jeznach, and my mother, all of whom read parts of this memoir and provided insightful comments.

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Introduction

The behavior of solutions to a partial differential equation (PDE) is inherently linked to both the geometry of the domain's boundary and the regularity of the associated operator. For instance, it has been established that, if u satisfies $Lu = 0$ in a domain Ω, where $L := -\operatorname{div} A\nabla$ is a uniformly elliptic operator, then u belongs to the Hölder space $C^{k+1}_\alpha(\overline{\Omega})$ for any $k \in \mathbb{N}$ provided that Ω is a C^{k+1}_α domain and the coefficients A are in $C^k_\alpha(\overline{\Omega})$. However, when we assume less regularity on the domain (e.g. Lipschitz domains) or the coefficients (e.g. L^∞), the behavior of solutions becomes more complicated to study.

The intricate relationship between PDEs and the geometry of the domain remains a topic of significant mathematical interest. One seeks to understand how the geometry of the boundary influences the qualitative and quantitative behavior of solutions. For instance, in practical applications, this includes determining which boundary structures optimize noise dampening, or identifying the lung surface geometry that maximizes oxygen transfer to the bloodstream. Additionally, understanding the behavior of solutions in the worst case scenario (bad boundary or initial data) for a given geometry of a domain is crucial in areas such as fluid dynamics (e.g., analyzing river flow) and climatology (e.g., predicting climate patterns).

In this memoir, we will mainly focus on one particular geometry of the boundary : uniform rectifiability, and one specific PDE property : the absolute continuity of the harmonic or elliptic measure with respect to the surface measure. Let us quickly remind the reader that the harmonic measure $\omega^X_{\Omega,-\Delta}(E)$ represents the probability that a Brownian particle starting at X exits the domain Ω through the set E ; the elliptic measure is a generalization of the harmonic measure corresponding to other stochastic processes. Furthermore, we consider a **quantitative and scale-invariant**[1] formulation of absolute continuity, which is known to be equivalent to the solvability of the Dirichlet problem for L^p boundary data, where $p \in (1, \infty)$ is sufficiently large.

The characterization of the (quantitative) absolute continuity of the harmonic measure in terms of the geometry of the domain is the culmination of decades of research by numerous mathematicians. This line of inquiry began with the work of the bothers Frigyes and Marcel Riesz in the 1910s for planar domains. For domains in $\mathbb{R}^n$, $n \geq 3$, the first breakthrough contribution on the topic is from Björg Dahlberg in the 1970s, and shows that the Dirichlet problem for the Laplacian is solvable in any Lipschitz domain and any L^2 boundary data, which implies that, in Lipschitz domains, the harmonic measure is (qualitatively) absolutely continuous with respect to the surface measure. In the 1980s and the 1990s, research in this area soared with the involvement of Carlos Kenig, David Jerison, and Jill Pipher, Charles Fefferman, and Tatiano Toro, to name a few. In that time, the research was mainly on extended the class of operators in Lipschitz domains for which the L^p Dirichlet problem and the L^p Neumann problem was solvable, and introduce the world to the notion of t-independent operators, Carleson perturbations, and DKP (for Dahlberg, Kenig, Pipher) operators. After a comparably calmer decade, Steve Hofmann and José Maria Martell reignited the area by taking a turn towards geometric measure theory and linking the quantitative absolute continuity of

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[1] Although non-quantitative results exist, they are not the focus of this work and will not be discussed here.

the harmonic measure to domains with uniformly rectifiable boundaries, which are a bit beyond Lipschitz domains (but that do not go up to domains with Hölder regular boundaries) and where introduced by Guy David and Stephen Semmes in the early 1990s. They were quickly joined by Ignacio Uriarte-Tuero, Svitlana Mayboroda, Jonas Azzam, Mihalis Mourgoglou, Xavier Tolsa, Alexander Volberg, Kaj Nyström, John Garnett, Tatiano Toro, Le Phi, Matthew Badger, Simon Bortz, Zihui Zhao, Murat Akman. In parallel, progress has been made in the 2010s in the study of elliptic operators with complex or elliptic systems by Pascal Auscher, Andreas Rosén, Mihalis Mourgoglou, Moritz Egert, and Olli Saari (for t-independent operators) and Martin Dindoš, Jill Pipher, Sukjung Hwang, David Rule, and Marius Mitrea (for operators verifying a Carleson condition). Of course, they are probably many more that deserve to be named here, and the contributions are not equally distributed across people listed here, but it gives an idea of the scale and the length of the research undertaken to just study the solvability of boundary value problem with L^p data. Moreover, we will not just drop a list of articles here to show the contributions of each, as the reader will find more precise results and the corresponding bibliography in Chapter 1.

Before proceeding, we introduce the notion of “smallness" or “proximity." The concept of “smallness" most suitable for the study of the quantitative absolute continuity of the elliptic measure is typically expressed in terms of Carleson measures or functions in the space of bounded mean oscillation (BMO), depending on the context. If $E \subset \mathbb{R}^n$ is the support of a doubling measure σ, and f is a function on $E \times (0,\infty)$, we say that f is **often almost zero** if there is a $C > 0$ such that, for any ball $\Delta(x,r) \subset E$, we have

$$\int_{\Delta(x,r)} \int_0^r |f(y,t)|^2 \frac{dt}{t}\, d\sigma \leq C\sigma(\Delta(x,r)).$$

Alternatively, if f is a function on $\mathbb{R}^n \setminus E$ and $E \subset \mathbb{R}^n$ has dimension d, and if we choose σ to be the Hausdorff measure $\mathcal{H}^d|_E$ then we say that f is **often almost zero** if there is a $C > 0$ such that, for any ball $B(x,r)$ centered on E, we have

$$\int_{B(x,r)\setminus E} |f(X)|^2 \operatorname{dist}(X,E)^{d-n}\, dX \leq C\sigma(B(x,r) \cap E). \tag{0.1}$$

Other ways to express smallness and proximity will have a similar spirit.

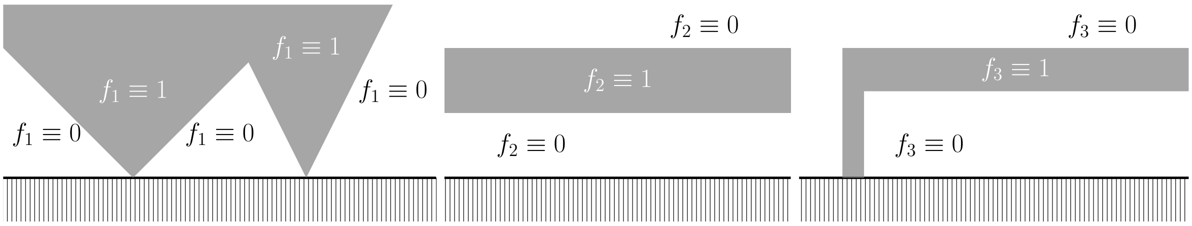

Figure 1 – f_1 and f_2 are *often almost zero* in $\mathbb{R}^2_+$, while f_3 is not.

We begin by presenting alternative PDE characterizations of the quantitative absolute continuity of the elliptic measure, utilizing our notion of smallness. In this introduction, we will state the results in two forms : (1) their exact formulation, omitting technical definitions that will be provided in the main body of the manuscript, and (2) their “informal” counterparts, which offer a more intuitive understanding but lack full mathematical precision.

Theorem 0.1 (Informal). *Let $\Omega \subset \mathbb{R}^n$ be a domain with quantitative non-tangential access to its entire boundary, the boundary being of dimension $n-1$ in a scale invariant way. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator. TFAE :*

(i) The elliptic measure of L is absolutely continuous with respect to the surface measure in a quantitative and scale invariant way*.*

(ii) The elliptic measure of L is often almost equivalent to *a multiple of the surface measure.*

(iii) Bounded solutions to $Lu = 0$ are often almost constant.

Theorem 0.1 (Formal). *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator whose elliptic measure is ω^X. TFAE :*

(i) The elliptic measure ω_L^X is A_∞-absolutely continuous with respect to $\sigma := \mathcal{H}^{n-1}_{|\partial\Omega}$.

(ii) The elliptic measure ω_L^X is absolutely continuous with respect to $\sigma := \mathcal{H}^{n-1}_{|\partial\Omega}$ and the Poisson kernel $k^X := \frac{d\omega^X}{d\sigma}$ is such that

$$\log(k^X) \in BMO(\partial\Omega, \sigma).$$

(iii) There exists $C > 0$ such that, for any bounded solution to $Lu = 0$ in Ω and any ball $B(x,r)$ centered on $\partial\Omega$, we have

$$\int_{B(x,r)\setminus E} \operatorname{dist}(X,E)|\nabla u(X)|^2\, dX \leq C\|u\|_{L^\infty(\Omega)}\sigma(B(x,r)\cap E).$$

By the 2010s, a comprehensive understanding of the relationship between the geometry of the domain and the absolute continuity of harmonic measure was finally achieved, revealing that :

Theorem 0.2 (Informal). *Let $\Omega \subset \mathbb{R}^n$ be a domain with quantitative non-tangential access to its entire boundary, the boundary being of dimension $n-1$ in a scale invariant way. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator whose coefficients are* often almost constant. *TFAE :*

(i) The elliptic measure of L is often almost equivalent to *a multiple of the surface measure.*

(ii) The boundary $\partial\Omega$ is often almost flat.

Theorem 0.2 (Formal)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator whose coefficients satisfy the DKP condition. TFAE :*

(i) The elliptic measure of L is A_∞-absolutely continuous with respect to $\sigma := \mathcal{H}^{n-1}_{|\partial\Omega}$.

(ii) The boundary $\partial\Omega$ is uniformly rectifiable.

There exist multiple versions of the above theorem, each depending on the specific assumptions imposed *a priori* on the operator and the domain. Note that a particular instance of the elliptic operator L in Theorem 0.2 is the Laplacian. In this case, the theorem asserts that the boundary of Ω is uniformly rectifiable if and only if bounded harmonic functions in Ω are "often almost constant."

The concept of uniform rectifiability extends to any integer dimension $d < n$. Intuitively, uniformly rectifiable sets of dimension d are subsets of $\mathbb{R}^n$ that are *often almost flat*, i.e. *often close to* a d-dimensional plane. However, a prerequisite for the characterization provided in Theorem 0.2 is that the boundary has dimension $n-1$, hence limiting the characterization to codimension 1 sets.

Question 1 : Do we have a notion of harmonic/elliptic measure on sets E of codimension bigger than 1?

Answer 1 : The harmonic measure or elliptic measure of uniformly elliptic operators can only be defined for sets of dimension $d \in (n-2, n)$; for further details, see Section 1.5. To extend the definition of elliptic measures on sets E of any dimension $d \in [0, n)$, my coauthors, Guy David and Svitlana Mayboroda, and I developed an elliptic theory based on operators in the form $L := -\operatorname{div} A\nabla$ where A satisfies an ellipticity and boundedness condition with respect to a carefully chosen weight.

More precisely, if $E \subset \mathbb{R}^n$ is an Ahlfors regular set of dimension $d \in [0, n)$ - that is if E looks d-dimensional at all scales - then we define uniformly elliptic operators on $\mathbb{R}^n \setminus E$ as the operators L in the form $-\operatorname{div} A\nabla$, where the coefficients A verify

$$A(X)\xi \cdot \xi \geq C^{-1} \underbrace{\operatorname{dist}(X,E)^{d+1-n}}_{=:w_E(X)} |\xi|^2, \quad \text{for } X \in \mathbb{R}^n \setminus E, \, \xi \in \mathbb{R}^n \tag{0.2}$$

and

$$|A(X)\xi \cdot \zeta| \leq C^{-1} \underbrace{\operatorname{dist}(X,E)^{d+1-n}}_{=w_E(X)} |\xi||\zeta|, \quad \text{for } X \in \mathbb{R}^n \setminus E, \, \xi, \zeta \in \mathbb{R}^n. \tag{0.3}$$

The appropriate space for solutions is then the weighted homogeneous Sobolev space

$$W := \{f \in L^1_{loc}(\mathbb{R}^n), \, \nabla f \in L^2(\mathbb{R}^n, w_E \, dX)\}.$$

More precisely, any functions in W have traces in $L^2_{loc}(E, \mathcal{H}^d|_E)$, and the Lax-Milgram theorem guarantees the existence of solutions to $Lu = 0$ in W with a given trace $g \in C^\infty_0(E)$. From there, many results established for classical uniformly elliptic operators can be adapted to our setting. In particular, the following holds :

- A maximum principle : for any solution $u \in W$ to $Lu = 0$, we have

$$\sup_{\mathbb{R}^n \setminus E} u \leq \sup_E u.$$

- De Giorgi-Nash-Moser estimates : any solution $u \in W$ with traces in $C^{0,\alpha}(E)$ lies in $C^{0,\alpha}(\mathbb{R}^n)$.
- A Green function associated with L can be constructed.
- An elliptic measure associated with L can be constructed.

The elliptic theory was developed to provide the necessary tools to prove PDE characterizations of the A_∞-absolute continuity of the elliptic measure similar to Theorem 0.1.

References 1 : This elliptic theory was established by my coauthors and myself in [DFM21] and further extended in [DFM20]. An even more general version is presented in Chapter 2 of this memoir. With Svitlana Mayboroda and Zihui Zhao, we also proved a Moser estimate for operators with complex coefficients in [FMZ21], by adapting and improving the Moser estimate shown by Dindoš and Pipher in [DP19] (operators with complex coefficients lie outside of the scope of this memoir).

PDE characterizations of the quantitative absolute continuity of the elliptic measure with respect to $\mathcal{H}^d|_E$ are given by Mayboroda and Zhao in [MZ19] in the setting of [DFM21]. In the setting of [DFM20], a few characterizations were provided by Bruno Poggi and myself in [FP22], and additional ones were proven by Cao and Yabuta in [CY25]. The results, along with new proofs, are presented in Section 3.2.

Question 2 : When $E \subset \mathbb{R}^n$ is of dimension $d < n-1$, it is immediately clear that the coefficients of the uniformly elliptic operators defined in (0.2)–(0.3) are never constant - nor even *often almost constant* - as it is the case in Theorem 0.2. Therefore, when $d < n-1$, which operator will be the analogue of the Laplacian in the statement of Theorem 0.2?

Answer 2 : Our initial choice was to take $L = -\operatorname{div}[w_E \nabla]$. However, this operator turned out to be insufficiently smooth for our purpose, because the coefficients w_E were not suited to the Carleson-type condition (0.1). Instead, we opted for the following "distance"

$$D_\alpha(X) := \left(\int_E |X-y|^{-d-\alpha} d\mathcal{H}^d|_E(y) \right)^{-\frac{1}{\alpha}},$$

which is equivalent to $\operatorname{dist}(X,E)$ but smoother, and then define

$$L_\alpha := -\operatorname{div}[D_\alpha^{d+1-n} \nabla]. \tag{0.4}$$

We generalized one implication of Theorem 0.2, namely that the elliptic measure ω_α^X of L_α is A_∞-absolutely continuous with respect to $\mathcal{H}^d|_E$ whenever E is uniformly rectifiable. The converse is false when $\alpha_0 := n-d-2 > 0$ and the operator is L_{α_0} : in this case, it fails spectacularly, as we have the equivalence of measures

$$C^{-1}\mathcal{H}^d|_E \leq \omega_{\alpha_0}^X \leq C\mathcal{H}^d|_E.$$

We conjecture that, excluding this special value of α, the converse will indeed hold.

References 2 : D_α was introduced for the first time by Guy David, Svitlana Mayboroda and myself in [DFM19a]. The proof of the fact that uniformly rectifiable boundaries of low dimension imply A_∞-absolute continuity of the elliptic measure was established by David and Mayboroda in [DM23] and by myself in [Fen22a], with intermediate results provided in [DFM19a]. The counterexample to the converse is given by David, Engelstein and Mayboroda in [DEM21].

The choice of D_α and L_α is explained in Section 3.5, while the adaptation of Theorem 0.2 to domains with boundaries of codimension higher than 1 is discussed in Section 3.6.

Question 3 : Even though a characterization of uniformly rectifiable sets via the absolute continuity of the elliptic measure of the L_α operator can be achieved, having to exclude L_{α_0} - which is very similar to the other L_α - seems like a major setback, does it not ?

Answer 3 : We prefer to view this counterexample as an opportunity rather than a setback. While it is true that it makes the theory appear less stable, it also highlights the fact that there is still much to be understood. In fact, this apparent setback has led to most of the novel ideas of the project.

When we take $\alpha_0 := n - d - 2 > 0$ and the specific operator L_{α_0} on $\mathbb{R}^n \setminus E$, the function D_{α_0} is a positive solution of L_{α_0} with zero boundary data, i.e. D_{α_0} acts as a Green function "with pole at infinity". Prior to this discovery, explicit expressions for the Green functions were extremely rare - typically only in the cases where $L = -\Delta$ and Ω is either the ball, the half space, the quarter space, and a few other cases.

Amidst all the L_α, the operator L_{α_0} is also the only one where we can compute explicit solutions. When the boundary E is irregular and $\alpha \neq \alpha_0$, explicit solutions to $L_\alpha u = 0$ in $\mathbb{R}^n \setminus E$ is far from achievable. From this, we learned several important facts :

1. First, L_{α_0} is a "lucky" guess that works exceptionally well and computations for this case differ significantly from those for $\alpha \neq \alpha_0$. Thus, we strongly believe the converse statement - that the A_∞-absolute continuity of the elliptic measure with respect to the Hausdorff measure implies E is uniformly rectifiable - remains true by excluding the value α_0. However, it is highly likely that proving this will require distinguishing between the cases $\alpha > \alpha_0$ and $\alpha < \alpha_0$.
2. Second, as the question suggests, L_α is indeed close to L_{α_0}. Since we know an operator close to L_α for which the elliptic measure is A_∞-absolutely continuous with respect to the Hausdorff measure, we can use a perturbation technique to obtain the A_∞-absolute continuity of the elliptic measure of L_α for uniformly rectifiable boundaries with significantly less effort. This approach was completed by myself in [Fen22a], providing a shorter and alternative proof compared to the original one found in [DM23].

   It does not end here. Following the ideas of [Fen22a], we obtained an entirely new estimate on the Green function when the domain has uniformly rectifiable boundaries. The estimate was first established in the case $d < n - 1$ in the article [DFM23c] written by Guy David, Svitlana Maydoroda and myself, and its proof relies on the existence of the special operator L_{α_0} [2]. This was later extended to $d = n - 1$ with a more complex method by Linhan Li, Svitlana Maydoroda and myself in [FLM24]. We will discuss this further when answering the next question.
3. Finally, we now understand that we can (probably) always construct an operator **adapted to** the boundary such that the elliptic measure and the Hausdorff measure become equivalent. So far, we only know of such an operator for some specific boundaries : David and Mayboroda constructed such an operator in the

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[2] A careful reader will notice that we set $d < n-1$, and $\alpha_0 = n-d-2 > 0$ exists only when $d < n-2$. The proof was actually done originally when $d < n - 2$, and then adapted to $d = n - 2$; see Subsection 3.6.2 for details.

complement of the 4 corners Cantor sets in [DM21], and Perstneva did the same in the complement of a Wolff snowflake in [Per23a].

In [Vol22], Volberg made the following conjecture :

Conjecture 0.3 (Informal). *Let $\Omega \subset \mathbb{R}^n$ be a domain with quantitative non-tangential access to its entire boundary, the boundary being of dimension d in a scale invariant way. If moreover, on the boundary $\partial\Omega$*

the harmonic measure is equivalent to the d-dimensional Hausdorff measure,

then $d = n-1$ and $\partial\Omega$ is uniformly rectifiable.

Conjecture 0.3 (Formal). *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with d-Ahlfors regular boundaries, $d \in (n-2, n)$. Assume that there exists $C > 0$, such that for any $X \in \Omega$ and any Borel $E \subset 8B_X \cap \partial\Omega$, the harmonic measure satisfies*

$$C^{-1}\mathcal{H}^d(E) \leq \omega^X_{-\Delta}(E) \leq C\mathcal{H}^d(E),$$

where $B_X := B(X, \operatorname{dist}(X, \partial\Omega)/2)$. Then

$$d = n-1 \text{ and } \partial\Omega \text{ is uniformly rectifiable.}$$

However, the conjecture fails, as demonstrated by David, Jeznach and Julia in [DJJ23], where they constructed a Cantor set of dimension $d < 1$ in $\mathbb{R}^2$ on which the harmonic measure and the Hausdorff measure are equivalent. This result illustrates that, in Theorem 0.2, one cannot simply replace the assumption that the boundary is $(n-1)$-Ahlfors regular by the fact that $\partial\Omega$ is d-Ahlfors regular for a $d \in (n-2, n)$. Consequently, this reinforces the idea that a control on the harmonic/elliptic measure does not automatically translate to any flatness of the boundary.

Question 4 : Is there any possibility of salvaging a characterization of uniformly rectifiable sets of codimension greater than 1 via PDE in the complement?

Answer 4 : After encountering challenges in obtaining a characterization of the rectifiability through the absolute continuity of the harmonic measure, we pursued an alternative approach by studying the Green function[3]. The Green function of an operator and the elliptic measure are deeply connected to each other : if the domain $\Omega \subset \mathbb{R}^n$ is sufficiently connected and its boundary has dimension d (both quantitatively), and if L is a uniformly elliptic operator with Green function $G(X, Y)$ and elliptic measure ω^Y, then the following well known equivalence holds

$$C^{-1} \operatorname{dist}(X, \partial\Omega)^{n-2} G(Y, X) \leq \omega^Y(4B_X) \leq C \operatorname{dist}(X, \partial\Omega)^{n-2} G(Y, X)$$

for $Y \in \Omega$, $8B_X \subset 4B_Y$, and $B_X := B(X, \operatorname{dist}(X, \partial\Omega)/2)$ as before. Long story short : if ω^Y behaves like a $(n-1)$-Ahlfors regular measure, then

$$\omega^Y(4B_X) \text{ is comparable to } \left(\frac{\operatorname{dist}(X, \partial\Omega)}{\operatorname{dist}(Y, \partial\Omega)}\right)^{n-1}$$

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[3] Morally, the Green function $G_L(., y)$ is the solution to $Lu = \delta_y$ in Ω with zero trace ; see Subsection 2.3.3.

and hence

$$G(Y,X) \text{ is comparable to } \frac{\operatorname{dist}(X,\partial\Omega)}{\operatorname{dist}(Y,\partial\Omega)^{n-1}}.$$

Thus we seek to compare the function $G(Y,.)$ with a distance to the boundary.

If $G^Y := G(Y,.)$ is the Green function of an operator with smooth coefficients, it will also be smooth far from the pole. Moreover, let us remind the reader that D_{α_0} is a "Green function with pole at infinity" associated with L_{α_0}. So it makes more sense to compare G^Y to the smooth distance D_α than to compare G^Y to $\operatorname{dist}(.,\partial\Omega)$. Our first result is :

Theorem 0.4 (Informal). *Let E be a d-dimensional* often almost flat *set, $\Omega := \mathbb{R}^n \setminus E$, and L_α be as in* (0.4). *Then the "Green function with pole at infinity" is* often almost equal *to a multiple of D_α.*

Theorem 0.4 (Formal). *Let $\Omega := \mathbb{R}^n \setminus E$ be a domain, where E is a d-uniformly rectifiable set, $d < n-1$. Let $\alpha > 0$ and L_α be as in* (0.4). *There exists $C > 0$ such that, if $x \in E$ and $r > 0$, then any positive solution u to $Lu = 0$ in $B(x,2r) \setminus E$ with zero trace on $E \cap B(x,2r)$ satisfies*

$$\int_{B(x,r)} \left|\nabla \ln\left(\frac{u}{D_\alpha}\right)\right|^2 D_\alpha^{d+2-n}\, dX \leq Cr^d.$$

Since D_{α_0} is the Green function with pole at infinity of L_{α_0} for all sets of dimension $d < n-2$, it is still not possible to achieve a characterization of uniformly rectifiable sets of low dimension using this approach. Nevertheless, this bound was previously unknown, even for domains with boundaries of codimension 1. Following this, we were able to obtain the result in the case of codimension 1 :

Theorem 0.5 (Informal). *With the assumptions of Theorem 0.2 and $\alpha > 0$, TFAE :*

(i) The "Green function with pole at infinity" is often almost equal *to a multiple of D_α.*

(ii) $\partial\Omega$ is often almost flat*.*

Theorem 0.5 (Formal). *With the assumptions of Theorem 0.2 and $\alpha > 0$, TFAE :*

(i) There exists $C > 0$ such that, if $x \in \partial\Omega$ and $r > 0$, then any positive solution u to $Lu = 0$ in $B(x,2r) \cap \Omega$ with zero trace on $B(x,2r) \cap \partial\Omega$ satisfies

$$\int_{B(x,r)\cap\Omega} \left|\nabla \ln\left(\frac{u}{D_\alpha}\right)\right|^2 D_\alpha\, dX \leq Cr^d.$$

(ii) $\partial\Omega$ is uniformly rectifiable.

Ultimately, we managed to obtain a characterization of uniform rectifiability using PDE in all dimensions and codimensions.

Theorem 0.6 (Informal). *With the same assumptions as Theorem 0.4, TFAE :*

(i) The length of the gradient of the "Green function with pole at infinity" is often almost constant*.*

(ii) $d \in \mathbb{N}$ and $\partial\Omega$ is often almost flat*.*

Theorem 0.6 (Formal)**.** *With the same assumptions as Theorem 0.4, TFAE :*

(i) There exists $C > 0$ such that, if $x \in E$ and $r > 0$, then any positive solution u to $Lu = 0$ in $B(x, 2r) \cap \Omega$ with zero trace on $B(x, 2r) \cap E$ satisfies

$$\int_{B(x,r)} \frac{|\nabla|\nabla u||^2}{u^2} D_\alpha^{d+4-n}\, dX \leq C r^d.$$

(ii) $d \in \mathbb{N}$ and $\partial\Omega$ is uniformly rectifiable.

x

Theorem 0.7 (Informal)**.** *With the same assumptions as Theorem 0.5, TFAE :*

(i) The gradient of the "Green function with pole at infinity" is often almost constant*.*

(ii) The length of the gradient of the "Green function with pole at infinity" is often almost constant*.*

(iii) $d \in \mathbb{N}$ and $\partial\Omega$ is often almost flat*.*

Theorem 0.7 (Formal)**.** *With the same assumptions as Theorem 0.5, TFAE :*

(i) There exists $C > 0$ such that, if $x \in E$ and $r > 0$, then any positive solution u to $Lu = 0$ in $B(x, 2r) \cap \Omega$ with zero trace on $B(x, 2r) \cap \partial\Omega$ satisfies

$$\int_{B(x,r)\cap\Omega} \frac{|\nabla^2 u||^2}{u^2} D_\alpha^3\, dX \leq C r^d.$$

(ii) Same as (i) but with the bound

$$\int_{B(x,r)\cap\Omega} \frac{|\nabla|\nabla u||^2}{u^2} D_\alpha^3\, dX \leq C r^d.$$

(iii) $d \in \mathbb{N}$ and $\partial\Omega$ is uniformly rectifiable.

References 4 : Theorem 0.4 is a result from [DFM23c], co-authored by Guy David, Svitlana Mayboroda and myself . The adaptation to codimension 1, presented in Theorem 0.5, was established by Linhan Li, Svitlana Mayboroda and myself in [FLM24]. The characterizations provided in Theorems 0.6 and 0.7 appear in the article [FL23] written by Linhan Li and myself, and rely on an estimate on D_α proved by David, Engelstein, and Mayboroda in [DEM21].

Question 5 : What comes next ?

Answer 5 : In this introduction, I only highlighted a few key results from this memoir. A vast literature is dedicated to exploring all possible operators for which the elliptic measure is absolutely continuous with respect to the surface or the Hausdorff measure. I barely address these results in this introduction, although I have many results in this direction, particularly for boundaries of mixed or high codimension (see [FMZ21, Fen22c, FP22, Fen22b, Fen24]). The absolute continuity of the elliptic measure is also closely related to the solvability of the L^p Dirichlet problem, and I have studied other boundary value problems, such as the regularity problem or the Neumann problem, in works like [DFM23a, DFM23b, Fen23, FL24].

Finally, to emphasize the vastness of what remains unknown and yet to be explored, I have compiled a list of open questions in Chapter 5. Enjoy !

Organization of the memoir

The memoir is organized as follows.

The first chapter provides a thorough introduction to the geometric and analytic concepts used throughout the work. We will introduce uniformly rectifiable sets and explain what we mean by the solvability of the Dirichlet problem with L^p data. This chapter will primarily serve as an overview of the state of the art before I began working on the topics presented here, with minimal contributions from my own work. In contrast, the subsequent chapters will focus on my contributions, either original results or developments stemming from my work. I will conclude the chapter by outlining the direction in which we want to extend the results, and by explaining why we need to have a new elliptic theory is necessary to even makes sense to the generalization.

The second chapter builds upon the elliptic theory developed in [DFM21] and [DFM20] for studying sets with high codimension of mixed dimension, and extends the elliptic theory by relaxing the connectedness condition on the domain. While the results presented here are new, the huge majority of the proofs are either identical or only slightly different those found in [DFM21] and [DFM20]. For completeness, I gave in Subsection 2.2.3 a proof of the equivalence between boundary Poincaré inequalities and capacity. As such, the elliptic theory presented here - unlike in [DFM21] and [DFM20] - incorporates the elliptic theory based on capacity, as developed in [HKM93] [4].

The third chapter presents my contributions to the Dirichlet boundary problem when the boundaries are of high or mixed codimension. We also include our new proof of the equivalence between A_∞-absolute continuity of the elliptic measure and solvability of the Dirichlet problem in BMO, which works in the elliptic setting given in Chapter 2.

The fourth chapter presents my work on Green functions. In particular, we show how a certain Carleson-type bound on the Green function imply uniform rectifiability of the boundary, and we provide a unified approach to all the known cases.

The last chapter presents a list of open questions that naturally follow from my work, suggesting avenues for future research.

Where to find my results in the memoir?

Below is a table listing my research articles in chronological order, with references to the corresponding sections of the memoir. Let me point out that none of the articles mentioned here are covered in my Ph.D. thesis.

[DFM21] The entire Chapter 2, together with [DFM20].
[DFM19a] Section 3.4, an early version of Theorem 3.26;
    the entire Section 3.5, whose main result is Theorem 3.32.
[DFM19b] Section 1.5, Proposition 1.55.
[DFM20] The entire Chapter 2, together with [DFM21].
[Fen22a] The entire Subsection 3.6.2, whose main result is 3.35.
[Fen22c] Section 3.4, an intermediate version of Theorem 3.26;
    Section 4.1, Example 4.1.
[FP22] Section 3.1, Theorem 3.11, (ii);
    Section 3.2, part of Theorem 3.19;

---

[4] Actually, [HKM93] develop an elliptic theory around the p-Laplacian and p-capacity, while our elliptic theory only incorporate $p = 2$. Our theory is superior in the sense that it includes unbounded domains, and the study of Green functions, elliptic measure and comparison principle.

Section 3.3, Theorem 3.23, first part.
[DFM23c] Subsection 4.4.2, Theorem 4.31.
[Fen22b] Subsection 1.2.2, Proposition 1.38 ;
Section 3.4, Theorem 3.26 ;
[DFM23a] Section 3.7, Theorem 3.40, (1).
[DFM23b] Section 3.7, Theorem 3.41.
[FLM24] Section 4.3, some cases of Theorems 4.20 and 4.23 ;
Subsection 4.4.2, Theorem 4.32.
[FL23] Section 4.2, some parts of Proposition 4.7 and Corollary 4.9 ;
Subsection 4.4.2, Theorems 4.33 and 4.34.
[Fen23] Section 1.3, Theorem 1.45, (1') ;
Section 3.7, Theorem 3.40, (2).
[Fen24] Subsection 1.2.2, Theorem 1.36 (1') ;
Section 3.3, Theorem 3.23, second part.
[FL24] Section 1.4, Proposition 1.50.

1 - Definitions and history

In this chapter, we aim to introduce the reader to the state of the art prior to the author's involvement with the topic. We do not intend to be exhaustive, as that would be an immense task ; for example, we will not cover most qualitative results.

1.1 . Geometry and topology

1.1.1 . Qualitative geometry

Let us begin by discussing the concept of the dimension of a set. We will provide an overview of the key definitions in this section, and for more in-depth discussions, the reader is referred to Chapter 2 of [Fal86] or Chapter 4 of [Mat95].

It is well understood that a line is one-dimensional, a square is two-dimensional, and a cube is three-dimensional or higher. But what about sets $E \subset \mathbb{R}^n$ with more complicated geometries ? For these, we turn to the Hausdorff dimension, the definition of which we briefly recall here.

Definition 1.1 (Hausdorff Measure, Hausdorff Dimension)**.** For any set $U \subset \mathbb{R}^n$, we define the diameter of U as $\operatorname{diam}(U) := \sup |x-y|, , x, y \in U$.

Now, let $E \subset \mathbb{R}^n$ and $0 \leq s < \infty$. For $\delta > 0$, we define the δ-Hausdorff measure of E as

$$\mathcal{H}^s_\delta(E) := \inf \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s,$$

where the infimum is taken over all countable collections of sets $U_{i_{i=1}}^{\infty}$ such that

$$E \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \sup_{i \geq 1} \operatorname{diam}(U_i) \leq \delta.$$

The Hausdorff measure of E is then

$$\mathcal{H}^s(E) := \sup_{\delta \to 0} \mathcal{H}^s_\delta(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

The Hausdorff dimension of E is defined as

$$\dim_{\mathcal{H}} E := \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}$$

with the convention that $\dim_{\mathcal{H}} \emptyset = -\infty$.

Alternatively, the Hausdorff dimension can also be defined using coverings by spherical balls instead of arbitrary sets.

A set can have a fractional dimension. For example, the Koch snowflake, shown in Figure 1.1, is a fractal with fractional dimension. However, the reverse is not true : a fractal set can have an integer dimension, such as the 4-corners Cantor set (shown in Figure 1.2), which has Hausdorff dimension 1.

Calculating the Hausdorff measure of a set is often quite challenging, even for fractals. However, calculating the dimension is often more intuitive.

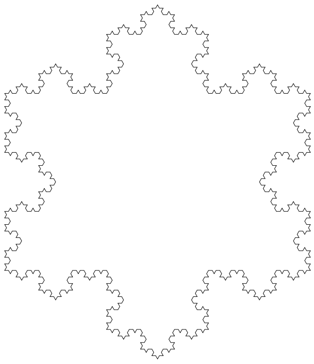

Figure 1.1 – Koch snowflake

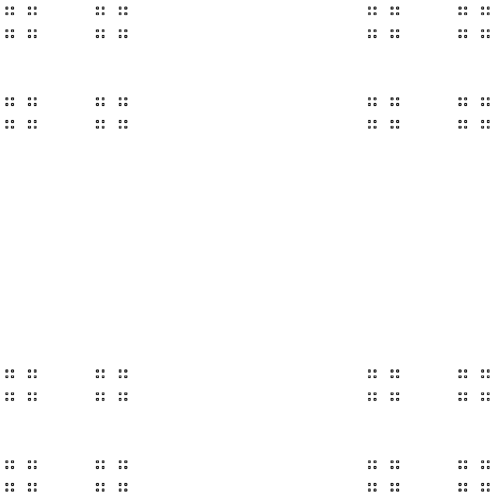

Figure 1.2 – 4-corners Cantor set

Even though both the 4-corners Cantor set and a smooth curve in $\mathbb{R}^2$ have Hausdorff dimension 1, they exhibit very different properties. The smooth curve has tangents at every point and is connected, whereas the Cantor set is neither connected nor has tangents at any point. To distinguish these two behaviors, we introduce the notion of rectifiability.

Definition 1.2. A set $E \subset \mathbb{R}^n$ is said to be d-rectifiable if $\dim_{\mathcal{H}}(E) = d$ and there exists a countable collection of continuously differentiable functions $f_i : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathcal{H}^d\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^d)\right) = 0. \tag{1.1}$$

A set $E \subset \mathbb{R}^n$ is said to be purely unrectifiable if $0 < \mathcal{H}^d(E) < \infty$ and $\mathcal{H}^d(E \cap R) = 0$ for any rectifiable set R.

In this context, rectifiability and pure unrectifiability represent two extreme cases. Any set E can be decomposed as $E = A \cup B$, where A is rectifiable and B is purely unrectifiable.

Remarks 1.3.
- Rectifiable sets have approximate tangents at almost every point.
- In Definition 1.2, the functions f_i can be taken to be Lipschitz functions rather than continuously differentiable.

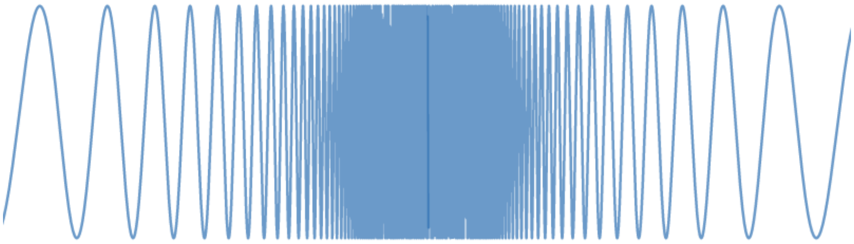

Figure 1.3 – The function $x \mapsto \sin(1/x)$

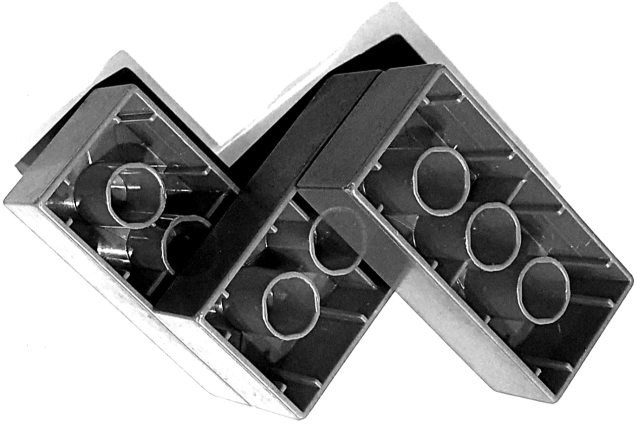

Figure 1.4 – Some Lego structures viewed as 2-dimensional sets in $\mathbb{R}^3$

- Rectifiability is essentially the simplest concept that contains the images of Lipschitz functions and is stable under countable unions.
- The 4-corners Cantor set (Figure 1.2) is purely unrectifiable.
- The sets shown in Figures 1.3 and 1.4 are rectifiable.

We now highlight a few theorems that are fundamental in geometric measure theory and further illuminate the concept of rectifiability.

Theorem 1.4 (Besicovitch-Federer Projection Theorem, see [Bes39] in $\mathbb{R}^2$ and [Fed47] in $\mathbb{R}^n$)**.** *A set E is d-purely unrectifiable if, for almost every d-dimensional affine plane P, the projection of E onto P has zero $\mathcal{H}^d$-measure.*

According to the above theorem, purely unrectifiable sets are "almost invisible", since they cast no significant shadow in almost every direction.

Let us end the paragraph with a nice result, that can be found in for instance [Fal86, Theorem 3.14].

Theorem 1.5. *If $E \subset \mathbb{R}^n$ is a compact connected set with $\mathcal{H}^1(E) < \infty$, then E is rectifiable (and arcwise connected).*

1.1.2 . Quantitative geometry

Take a look at Figures 1.5 and 1.6.

In the first image, the Hausdorff dimension of the set is 1. However, if we look at it from a distance, it appears to be a flat surface, which would be more appropriately

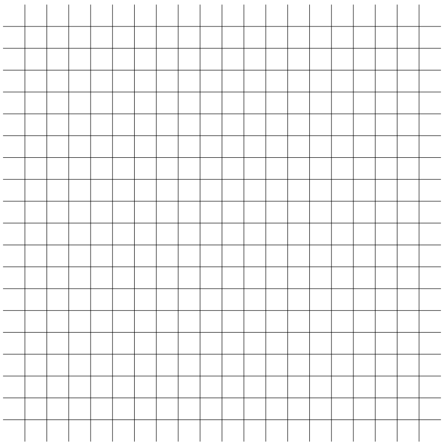

Figure 1.5 – A planar mesh (S_1)

Figure 1.6 – An infinite cylinder (S_2)

modeled as a 2-dimensional plane. In the second image, we see the opposite : an infinite cylinder with Hausdorff dimension 2, yet from a sufficient distance, it resembles a 1-dimensional string. This illustrates that the Hausdorff dimension only captures the dimension at the smallest scale.

One might naively assume there are only two dimensions : one at the small scale and another at the large scale, as seen, for example, in Lie groups. However, this is not the case. The next image combines the two previous examples : a $\mathbb{Z}^2$ mesh rolled up to form a cylinder.

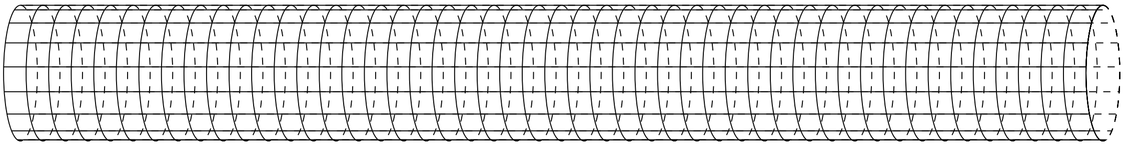

Figure 1.7 – Rolled mesh (S_3)

In this third case, the Hausdorff dimension is 1 at a very small scale, becomes 2 at an intermediate scale, and returns to 1 when viewed from afar. But the complexity doesn't stop there. The cylinder could loop back, forming a part of an object that appears as a ring from a distance, and as a dot—effectively a set with zero dimension—from even further away. These varying dimensions are crucial because the properties of the solution we want to study in the complement, around a point $X \in \mathbb{R}^n \setminus E$ will reflect the dimension of E seen from the viewpoint of X.

For this reason, it is useful to focus on properties of E that are uniform—i.e., independent of the position in $\mathbb{R}^n \setminus E$ from which we observe E. A set is "uniformly d-dimensional" if it satisfies the following property.

Definition 1.6 (Ahlfors regular)**.** A closed set E is d-Ahlfors regular if there exists $C > 0$

$$C^{-1} r^d \leq \mathcal{H}^d(B(x,r) \cap E) \leq C r^d \qquad \text{for } x \in E,\ r \in (0, \operatorname{diam} E). \tag{1.2}$$

More generally, a measure μ is d-Ahlfors regular if,

$$C^{-1} r^d \leq \sigma(B(x,r)) \leq C r^d \qquad \text{for } x \in \operatorname{supp} \sigma,\ r \in (0, \operatorname{diam} \operatorname{supp} \sigma). \tag{1.3}$$

Remarks 1.7.

- It is important to note that the Ahlfors regularity is quite distinct from regularity in the sense of smoothness!
- The 4 corners Cantor set (Figure 1.2) is 1-Ahlfors regular, as it is scale-invariant (up to harmless rotations).
- The sets S_1 and S_2 (Figure 1.6) above are not Ahlfors regular. The set S_3 (Figure 1.7) is actually Ahlfors regular, but with a very large constant. This reflects the fact that the dimension 1 is a bad dimension for the set at the intermediate scale.
- The graph of $x \mapsto \sin(1/x)$ (Figure 1.3) is not 1-Ahlfors regular, as visually, the graph of the function looks like a 2-dimensional object near $\{0\} \times [-1, 1]$.

Lemma 1.8. *If a measure σ is d-Ahlfors regular, then $\operatorname{supp}(\sigma)$ is Ahlfors regular.*

Proof. The result is well known, but let us give a proof for practice. Write E for $\operatorname{supp}(\sigma)$. We want to prove that, whenever σ is Ahlfors regular, there exists a constant C' such that for $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} E)$, we have

$$C'^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C' r^d.$$

In one hand, by definition of the Hausdorff measure, for any $\epsilon > 0$, we have a cover $\{B_i\}_{i \in I}$ of $E \cap B(x,r)$ by balls of radius r_i such that $\mathcal{H}^d(E \cap B(x,r)) \geq \sum_{i \in I} (r_i)^d - \epsilon$. Consequently, using the Ahlfors regularity of σ,

$$C^{-1} r^d \ \leq\ \sigma(E \cap B(x,r)) \ \leq\ \sum_{i \in I} \sigma(E \cap B_i) \ \leq\ C \sum_{i \in I} (r_i)^d \ \leq\ C H^d(E \cap B(x,r)) + \epsilon.$$

Since ϵ is as small as we want, we obtain $C^{-2} r^d \leq H^d(E \cap B(x,r))$.

In the other hand, take $\delta > 0$. By Vitali's covering lemma, we can take a non-overlapping subcollection $\{B_i\}_{i \in I}$ of $\{B(y, \delta/5)\}_{y \in E \cap B(x,r)}$ such that $\{5B_i\}_{i \in I}$ covers $E \cap B(x,r)$ and thus

$$\mathcal{H}^d_\delta(E \cap B(x,r)) \leq \sum_{i \in I} \delta^d \leq C \sum_{i \in I} \sigma(E \cap B_i) \leq C \sigma(E \cap B(x, 2r)) \leq 2^d C^2 r^d.$$

Since the result is true for all $\delta > 0$, we have $\mathcal{H}^d(E \cap B(x,r)) \leq C' r^d$ as desired. □

The quantitative and scale invariant version of rectifiability is simply called uniform rectifiability and is introduced by David and Semmes in [DS91, DS93].

Definition 1.9 (uniform rectifiability)**.** Let $d \in \mathbb{N}$. A set E is d-uniformly rectifiable if E is d-Ahlfors regular and there exists $\epsilon > 0$ and $M > 0$ such that for any $x \in E$ and $r \in (0, \operatorname{diam} E)$, there exists a M-Lipschitz function $f_{x,r} : \mathbb{R}^d \mapsto \mathbb{R}^n$ such that

$$\mathcal{H}^d(E \cap B(x,r) \cap f_{x,r}(\mathbb{R}^d)) \geq \epsilon r^d. \tag{1.4}$$

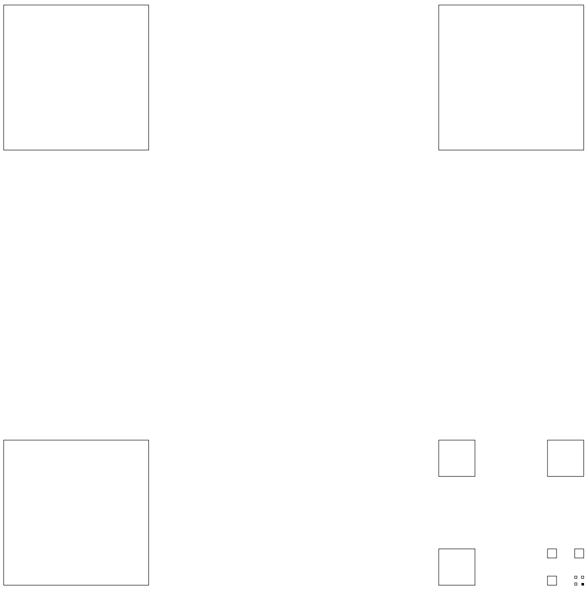

Figure 1.8 – The bottom right corner is a rescaling of the full picture.

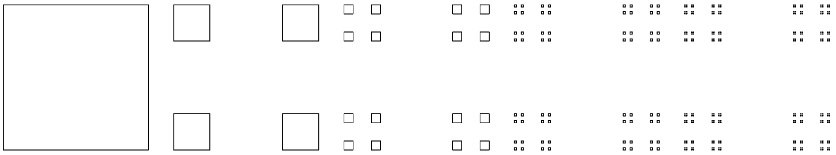

Figure 1.9 – The unbounded set consisting of the succession of the boundary of E_k, where E_k is the k$^{\text{th}}$ step of the construction of the 4 corners Cantor set.

Remarks 1.10.

- The sets despicted in Figures 1.4 and 1.7 are uniformly rectifiable. In constrast, the ones in Figures 1.1, 1.2, 1.3, 1.5, 1.6 are not uniformly rectifiable.
- The set in Figure 1.8 is uniformly rectifiable because, for any ball centered at the boundary, there is a cube within the ball whose side length is comparable to the radius of the ball.
- The set shown in Figure 1.9 is rectifiable and 1-Ahlfors regular, but not uniformly rectifiable. A bounded, rectifiable and 1-Ahlfors regular set can be obtained using the same strategy. The key is to have, for any $k \in \mathbb{N}$, a rescaled version of E_k within the set.
- The definition of uniform rectifiability has to be constrasted with that of purely unrectifiability : there is a Lipschitz image (special case of rectifiable set) in which the set has a fair intersection.
- Although the function $f_{x,r}$ might seem to capture only a small portion of E, suggesting that the rest is uncontrolled, this is not the case. Indeed, if $y \in E \cap B(x,r) \setminus f_{x,r}(\mathbb{R}^d)$, we can use the definition of uniform rectifiability at a smaller scale to say that there is a Lipschitz image closer to y than $f_{x,r}(\mathbb{R}^d)$. Roughly speaking, the property (1.4) captures 1% of the set $E \cap B(x,r)$, but using the property at a smaller scale will capture an extra 1% of the remaining of $E \cap B(x,r)$. Iterating the process will eventually capture all $E \cap B(x,r)$.
- The images of Lipschitz functions are, by definitions, uniformly rectifiable. Uniform rectifiability is the notion which encompasses images of Lipschitz functions and which is stable by locally finite unions.

Uniformly rectifiable sets are those that can be well approximated by planes at almost every point across most scales. Before we provide a precise definition, let's first introduce some tools to measure the distance to planes. The β-numbers, introduced by Peter Jones in his work [Jon89], serve as a key tool for this purpose.

Definition 1.11 (Peter Jones' β number)**.** We let $\mathcal{P}$ be the collection of all affine d-planes in $\mathbb{R}^n$. If E is a d-Ahlfors regular set and $q \in (0, \infty)$, then we define β_q on $E \times (0, \operatorname{diam} E)$ as

$$\beta_q(x,r) := \inf_{P\in\mathcal{P}} \left(r^{-d} \int_{E\cap B(x,r)} \Big(\frac{\operatorname{dist}(y,P)}{r} \Big)^q d\mathcal{H}^d(y) \right)^{\frac{1}{q}}.$$

Instead of using the Hausdorff measure $\mathcal{H}^d$ in the definition of β_q above, we could use any d-Ahlfors regular measure σ whose support is E. We also define

$$b\beta_\infty(x,r) := \inf_{P\in\mathcal{P}} \left(\sup_{y\in E\cap B(x,r)} \frac{\operatorname{dist}(y,P)}{r} + \sup_{z\in P\cap B(x,r)} \frac{\operatorname{dist}(z,E)}{r} \right).$$

An analogue of this tool for Ahlfors regular measures σ - in the sense that it also penalizes the oscillations of $d\sigma/d\mathcal{H}^d$ - has been introduced by Tolsa in [Tol09]..

Definition 1.12 (Tolsa's α number)**.** We let $\mathcal{F}$ be the collection of all flat measures, that is

$$\mathcal{F} := \{c\mathcal{H}^d_P,\ c > 0 \text{ and } P \in \mathcal{P}\}.$$

If σ is a d-Ahlfors regular measure, then we define α on $E \times (0, \operatorname{diam} E)$ as

$$\alpha(x,r) := \frac{1}{r} \inf_{\mu\in\mathcal{F}} \sup_{f\in \operatorname{Lip}(x,r)} \left| \int f\, d\sigma - \int f\, d\mu \right|,$$

where $\operatorname{Lip}(x,r)$ is the space of 1-Lipschitz functions on $\mathbb{R}^n$ supported in $B(x,r)$.

Using these tools, we can establish several characterizations of uniform rectifiability. These characterizations help capture the geometric structure of a set, revealing how closely it can be approximated by planes at various scales.

Theorem 1.13 (David-Semmes, Tolsa)**.** *Let E be a d-Ahlfors regular set and let σ be any Ahlfors regular measure whose support equals E. We say that ν is a Carleson measure on $E \times (0, \operatorname{diam} E)$ if there exists $C > 0$ such that*

$$\sup_{x\in E} \sup_{r\in(0,\operatorname{diam} E)} r^{-d} \int_0^r \int_{B(x,r)} d\nu \le C.$$

The following statements are equivalents :

(i) E is uniformly rectifiable,

(ii) given any $q < \frac{2d}{d-2}$ ($q \le \infty$ if $d = 1$),

$$\beta_q^2(x,t) \frac{d\sigma(x)\, dt}{t} \quad \textit{is a Carleson measure on } E \times (0, \operatorname{diam} E),$$

(iii) for all $\epsilon > 0$,

$$\mathbb{1}_{b\beta_\infty(x,t)>\epsilon} \frac{d\sigma(x)\, dt}{t} \quad \textit{is a Carleson measure on } E \times (0, \operatorname{diam} E),$$

(iv) $\alpha^2(x,t)\dfrac{d\sigma(x)\,dt}{t}$ *is a Carleson measure on* $E \times (0, \operatorname{diam} E)$.

Theorem 1.13 is a consequence of [DS91] and [Tol09], and gives a small taste of all the known characterizations of uniform rectifiability. Many additional characterizations can be found in the pioneer books [DS91] and [DS93]; and let us give a special mention to the David-Semmes conjecture :

Conjecture 1.14 (David-Semmes). *Let* $n \in \mathbb{N}$ *and* $0 < d < n$. *Let* E *be a* d*-Ahlfors regular set, and write* σ *for a Ahlfors regular measure on* E. *Then the following are equivalent :*

(i) $d \in \mathbb{N}$ *and* E *is uniformly rectifiable;*

(ii) the Riesz transform $\mathcal{R}_\sigma$ *defined as*

$$\mathcal{R}_\sigma(f)(x) := p.v. \int_E \frac{x-y}{|x-y|^{d+1}} f(y)\, d\sigma(y) \tag{1.5}$$

is bounded on $L^2(\sigma)$.

The implication $(i) \implies (ii)$ is true and is a special case of the theory developed by David and Semmes in [DS91]. The conjecture pertains to the converse. Twenty-five years after it was made, Nazarov, Tolsa, and Volberg proved in [NTV14] the David-Semmes conjecture in the case when $d = n-1$. The general case is still open.

1.1.3 . Quantitative topology

Let us now turn our focus on the domain of "observation", denoted by Ω, where the solution to our boundary value problem will exist. This domain will be either $\mathbb{R}^n \setminus E$ or one of its connected component. To simplify the notation, we use

$$\delta_{\partial\Omega}(X) := \operatorname{dist}(X, \partial\Omega) \tag{1.6}$$

and

$$B_X := B(X, \operatorname{dist}_{\partial\Omega}(X))/4. \tag{1.7}$$

Our next assumption we introduce is the quantitative openness. The idea behind this is as follows : we want to ensure that for any given point x and scale r, we can find a suitable location in the domain where we can "observe" the ball $B_E(x,r) := B_{\mathbb{R}^n}(x,r) \cap E$. If such a location exists, it is called a corkscrew point for x at scale r.

Definition 1.15 (corkscrew point). Let $\Omega \subset \mathbb{R}^n$. We say that Ω satisfies the corkscrew point condition if there exists $\epsilon \in (0,1)$ such that, for any $x \in \partial\Omega$ and any $r \in (0, \operatorname{diam}\Omega)$, there exists

$$X \in \Omega \cap B(x,r) \text{ satisfying } B(X,\epsilon) \subset \Omega. \tag{1.8}$$

We call ϵ**-corkscrew point** for x at scale r any point X satisfying (1.8); the ball $B(X, \epsilon r)$ is called a corkscrew ball. If Ω satisfies the corkscrew point condition (with the constant ϵ_Ω), we call corkscrew point (for x at scale r) any ϵ_Ω-corkscrew point (for x at scale r). Alternatively, a corkscrew point for a boundary ball $\Delta = B(x,r) \cap \partial\Omega$ is a corkscrew point for x at scale r.

The next concept we need to introduce is quantitative connectedness. This notion allows us to connect corkscrew points within Ω using paths that are both relatively short and not too narrow.

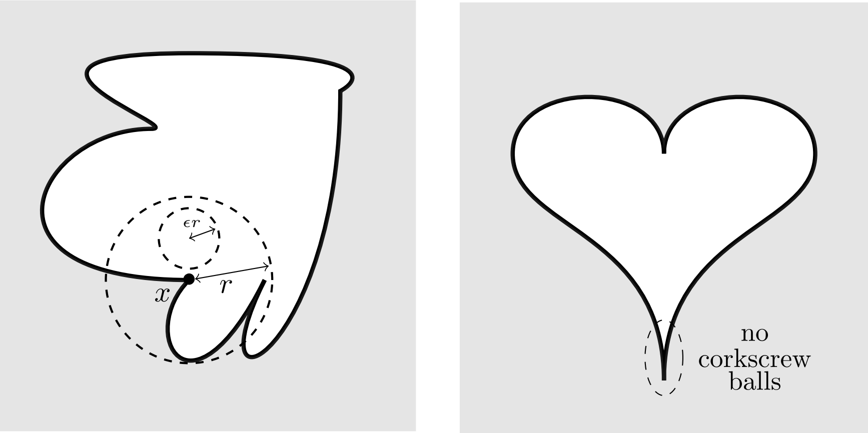

Figure 1.10 – On the left, the small ball is radius ϵr of corkscrew ball for x at scale r.
On the right, a domain that does not satisfy the corkscrew point condition.

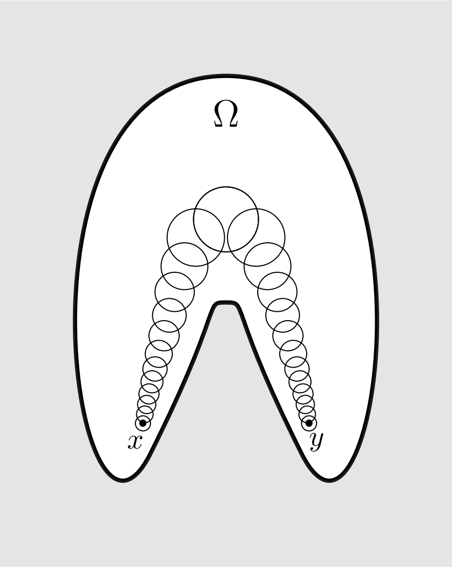

Figure 1.11 – An example of Harnack chain between 2 points x and y

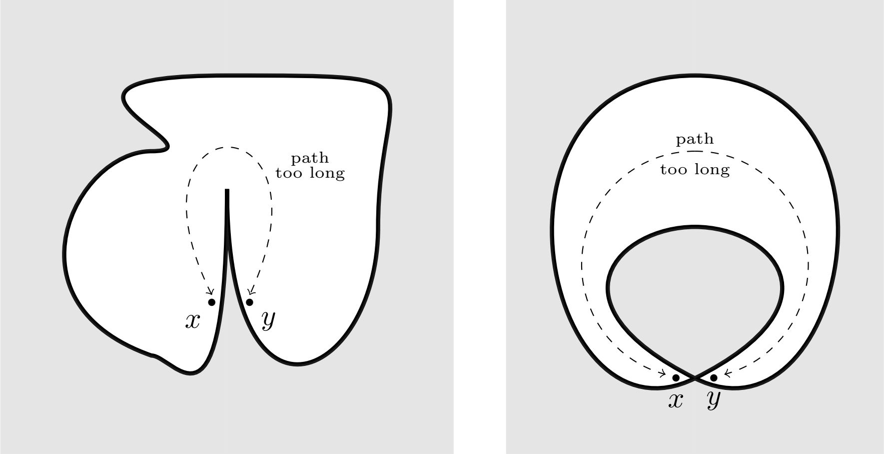

Figure 1.12 – Two domains which do not satisfy the Harnack chain condition

Definition 1.16 (Harnack chain)**.** Let $\Omega \subset \mathbb{R}^n$. We say that Ω satisfies the Harnack chain condition if, for any $\Lambda > 0$, there exists $C_\Lambda > 0$ such that, for any $X, Y \in \Omega$ satisfying $|X - Y| \leq \Lambda \min\{\delta_{\partial\Omega}(X), \delta_{\partial\Omega}(Y)\}$, there exists $N \leq C_\Lambda$ and a collection of points $\{Z_i\}_{0 \leq i \leq N}$ such that $Z_0 = X$, $Z_n = Y$, and $|Z_i - Z_{i+1}| \leq \delta_{\partial\Omega}(Z_i)$ for all $0 \leq i \leq N - 1$.

The collection $\{B(Z_i, \delta_{\partial\Omega}(Z_i)/2)\}_{0 \leq i \leq N}$ is called a Harnack chain of balls linking X to Y.

Definition 1.17 (uniform)**.** A domain Ω is uniform if it satisfies both the corkscrew point condition and the Harnack chain condition.

It is important to note that any Lipschitz domain, or any domain that lies above a Lipschitz graph, is uniform. In fact, the understanding of uniform domains may be clearer when approached from their original definition.

Proposition 1.18. *A domain Ω is uniform if and only if there exists $C > 0$ such that, for any $X, Y \in \overline{\Omega}$, there exists a smooth "cigar with bounded turning" curve linking X to Y, that is to say a rectifiable curve $\gamma : [0,1] \mapsto \Omega$ such that $\gamma(0) = X$, $\gamma(1) = Y$, $\min\{\ell_\gamma(X, \gamma(t)), \ell_\gamma(Y, \gamma(t))\} \leq C\delta_{\partial\Omega}(\gamma(t))$ for all $t \in (0,1)$ and $\ell_\gamma(X, Y) \leq C|X - Y|$. Here $\ell_\gamma(A, B)$ denotes the length of the curve γ between the two points A and B.*

Remark 1.19. As an immediate corollary of the proposition, one can observe that, if the domain Ω is uniform, then we can actually choose $C_\Lambda = C\log(\Lambda)$ in the Harnack chain condition.

Proof. Proposition 1.18 is well known, although the authors cannot pinpoint the first time it was proved. One direction is a consequence of [DFM20, Proposition 2.18], and the other one is fairly immediate. Several other characterizations of uniform domains are provided in [V̈88]. ☐

Note that the notion of uniform domain allows us to construct, for any boundary point $x \in \partial\Omega$, connected "cones" with vertex at x. The cones will give us non-tangentially access to the boundary, which means - with our extended metaphor - that any points in the boundary are "observable". Those cones can be constructed as follows. Given

$x \in \partial\Omega$, we let the set of corkscrew points be

$$CP(x) := \{X \in \Omega,\ \text{there exists } r \in (0, \operatorname{diam}\partial\Omega) \text{ such that } X \text{ if a corkscrew point for } x \text{ at scale } r\}.$$

Then for any couple $X, Y \in CP(x)$, we take a smooth "cigar with bounded turning" curve $\gamma_{X,Y}$ between X and Y as given by Proposition 1.18, the "cigar" set $\mathcal{C}(X,Y)$ is $\{Z \in \Omega,\ \exists t \in [0,1] :\ |Z - \gamma(t)| \leq \delta_{\partial\Omega}(\gamma(t))/2\}$. And then the cones are

$$\Gamma_{unif}(x) := \bigcup_{X,Y \in CP(x)} \mathcal{C}(X,Y). \tag{1.9}$$

This notion of cones is coherent with the one that we will introduce later, as we have :

Proposition 1.20. *For $a > 0$ and $x \in \partial\Omega$, we define*

$$\Gamma_a(x) := \{X \in \Omega,\ |X - x| < (1+a)\delta_{\partial\Omega}(X)\}. \tag{1.10}$$

If Ω is a uniform domain, then there exists $0 < \alpha < \beta$ such that, for all $x \in \partial\Omega$,

$$\Gamma_\alpha(x) \subset \Gamma_{unif}(x) \subset \Gamma_\beta(x).$$

Proof. Let $\epsilon_{CP} \in (0,1)$ be the constant in the definition of corkscrew points, and $C_{unif} \geq 1$ be the constant in Proposition 1.18. We choose $\alpha > 0$ such that $(1+\alpha)^{-1} < \epsilon_{CP}$ and we check that any point in Γ_α is a corkscrew point for x at scale $|X - x|$, which means that $\Gamma_\alpha \subset CP(x) \subset \Gamma_{unif}$. To get the reverse, let $Z \in \Gamma_{unif}(x)$. So there exists $X \in CP(x)$ and $Y = \gamma(t) \in \Omega$ such that

$$|Z - Y| \leq \delta_{\partial\Omega}(Y)/2,\ |Y - X| \leq \ell_\gamma(X,Y) \leq C_{unif}\delta_{\partial\Omega}(Y), \text{ and } \delta_{\partial\Omega}(X) \geq \epsilon_{CP}|X - x|.$$

Observe that these three estimates imply

$$\delta_{\partial\Omega}(X) \leq \delta_{\partial\Omega}(Y) + \ell_\gamma(X,Y) \leq (C_{unif}+1)\delta_{\partial\Omega}(Y) \leq 2(C_{unif}+1)\delta_{\partial\Omega}(Z)$$

and

$$|Z - x| \leq |Z - Y| + |Y - X| + |X - x| \leq \frac{1}{2}\delta_{\partial\Omega}(Y) + C_{unif}\delta_{\partial\Omega}(Y) + (\epsilon_{CP})^{-1}\delta_{\partial\Omega}(X).$$

The combination of these two inequalities easily gives $|Z - x| < (1+\beta)\delta_{\partial\Omega}(Z)$ for $1 + \beta = 2(1 + (\epsilon_{CP})^{-1})(1 + C_{unif})$. The proposition follows. □

For our next definition and results, we combine the quantitative geometry and the quantitative topology.

Proposition 1.21. *If $d \in [0, n-1)$ and $E \subset \mathbb{R}^n$ is d-Ahlfors regular, then $\Omega := \mathbb{R}^n \setminus E$ is uniform.*

Proof. Proposition 1.21 is the combination of Lemmas 2.2 and 11.6 in [DFM21]. □

Note that in the above proposition, d is not necessarily an integer.

Theorem 1.22 ([AHM+17])**.** *If $\Omega \subset \mathbb{R}^n$ is uniform and $\partial\Omega$ is $(n-1)$-Ahlfors regular, then the following are equivalent :*

1. *$\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew point condition.*
2. *$\partial\Omega$ is uniformly rectifiable.*

We call Chord-Arc Domain (CAD for short) any domain satisfying the assumptions of the Theorem and either (1) or (2).

Remarks 1.23.
- If $\partial\Omega$ is d-Ahlfors regular with $d < n-1$, then $\mathbb{R}^n \setminus \overline{\Omega}$ is empty. Consequently, the above characterization of uniform rectifiability cannot be extended to these cases.
- In the case where Ω is the Koch snowflake, Ω is uniform, $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew point condition, and $\partial\Omega$ is Ahlfors regular (with some fractional dimension). However, Ω is not uniformly rectifiable.
- The two previous remarks highlight that the theorem cannot be extended to Ahlfors regular boundaries $\partial\Omega$ with dimension $d \neq n-1$.

Our final definition is the weak local John condition, which will be instrumental in characterizing the solvability of the Dirichlet problem. Intuitively, this condition ensures that each point of the domain is well-connected to a significant portion of its nearby boundary.

Definition 1.24 (weal local John)**.** We say that a set $\Omega \subset \mathbb{R}^n$ with $(n-1)$-Ahlfors regular boundaries is weak local John if there exists $C \geq 1$, $\Lambda \geq 8$, and $\epsilon > 0$ such that, for any point $X \in \Omega$,

$$\frac{\sigma(\{y \in \partial\Omega \cap \Lambda B_X, \ \ X \text{ and } y \text{ are linked by a “cigar with bounded turning”}\})}{\sigma(\Lambda B_X)} \geq \epsilon,$$

i.e., if $E_{\Lambda,C,X}$ is the set of points in $\partial\Omega \cap \Lambda B_X$ such that there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = y$, $\gamma(1) = X$, $\ell_\gamma(y, \gamma(t)) \leq C\delta_{\partial\Omega}(\gamma(t))$ for all $t \in (0,1)$, and $\ell_\gamma(y, X) \leq C|X-y|$, then $\sigma(E_{\Lambda,C,X})/\sigma(\Lambda B_X) \geq \epsilon$.

1.2 . The Dirichlet problem with L^p boundary data

The Dirichlet problem - or more specifically the Dirichlet boundary value problem - involves finding a solution to an equation with prescribed data on the boundary. In this memoir, we focus on uniformly elliptic operators, which are operators in the form $L = -\operatorname{div} A\nabla$ on $\Omega \subset \mathbb{R}^n$/ Here $A = A(X)$ is a (Lebesgue) measurable $n \times n$ matrix function on Ω that satisfies the conditions

$$|A(X)\xi \cdot \zeta| \leq C|\xi||\zeta| \qquad \text{for } X \in \Omega, \ \xi, \zeta \in \mathbb{R}^n \tag{1.11}$$

and

$$A(X)\xi \cdot \xi \geq C^{-1}|\xi|^2 \qquad \text{for } X \in \Omega, \ \xi \in \mathbb{R}^n, \tag{1.12}$$

where of course $C > 0$ is independent of X, ξ and ζ.

We say that u is solution to $Lu = f$ in Ω, where $f \in L^2_{loc}(\Omega)$, when $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int_\Omega A\nabla u \cdot \nabla\varphi \, dX = \int_\Omega f\varphi \, dX. \tag{1.13}$$

One of the initial boundary value problems to examine is the Dirichlet problem with continuous data. It is widely known (see [Wie24] in $\mathbb{R}^2$, [LSW63] in $\mathbb{R}^n$) that :

Theorem 1.25 (Wiener criterion)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator. Then the following are equivalent :*

(i) The continuous Dirichlet is solvable, that is, for any $g \in C^0_0(\partial\Omega)$, there is a unique $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ such that u is a weak solution to $Lu = 0$ satisfying $u|_{\partial\Omega} = g$ and $\lim_{X\in\Omega, X\to\infty} u(X) = 0$.

(ii) Ω is Wiener regular, i.e., for all $x \in \partial\Omega$,

$$\int_0^1 \frac{\operatorname{Cap}(B(x,r) \setminus \Omega, B(x,2r))}{r^{n-1}}\, dr < \infty.$$

The concept of capacity will not be introduced here, as it is not pertinent to our current discussion and will be covered in the next chapter (Definition 2.38). However, it is worth noting that domains with $(n-1)$-Ahlfors regular boundaries are Wiener regular.

The next step is to move beyond continuous boundary data and explore the solvability of the Dirichlet problem with data in spaces such as L^p. Since the extension of discontinuous boundary data cannot be continuous on $\overline{\Omega}$ anymore, it is more challenging to define $u = g$ on $\partial\Omega$. To address this, we need the notion of non-tangential convergence, which involves (generalized) cones in Ω with vertex at $x \in \partial\Omega$ as previously defined in (1.10), i.e.

$$\Gamma_\alpha(x) := \{X \in \Omega, |X - x| < (1+\alpha)\delta_{\partial\Omega}(X)\},$$

where $\delta_{\partial\Omega} := \operatorname{dist}(X, \partial\Omega)$. The non-tangential maximal function N_a is

$$N_\alpha(u)(x) := \sup_{\Gamma_\alpha(x)} |u|, \qquad \text{for } u \in L^\infty_{loc}(\Omega) \text{ and } x \in \partial\Omega,$$

and the square function is

$$S_\alpha(u)(x) := \int_{\Gamma_\alpha(x)} \delta_{\partial\Omega} |\nabla u|^2 \, dX, \qquad \text{for } u \in W^{1,2}_{loc}(\Omega) \text{ and } x \in \partial\Omega.$$

The parameter α will play little role in our theory, as we have the equivalence

$$\|N_\alpha(u)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\alpha,\beta} \|N_\beta(u)\|_{L^p(\partial\Omega,\sigma)} \quad \text{and} \quad \|S_\alpha(u)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\alpha,\beta} \|S_\beta(u)\|_{L^p(\partial\Omega,\sigma)}$$

whenever $\alpha, \beta > 0$, with a constant independent of the function u (see for instance [SM93]). Consequently, we write N and S for N_1 and S_1 respectively, and the parameter α is needed solely for intermediate computations. It will be also useful to introduce the averaged non-tangential maximal function $\widetilde{N}_{\alpha,c}$ as

$$\widetilde{N}_{\alpha,c}(u) := \sup_{X\in\Gamma_\alpha(x)} \Big(\fint_{B(X, c\delta_{\partial\Omega})} |u|^2 \, dX \Big)^{\frac{1}{2}},$$

Note that we also have

$$\|\widetilde{N}_{\alpha,c}(u)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\alpha,\beta,c,c'} |\widetilde{N}_{\beta,c'}(u)\|_{L^p(\partial\Omega,\sigma)}$$

whenever $\alpha, \beta > 0$ and $c, c' \in (0,1)$; see [MPT23]. As such, we write $\widetilde{N}$ for $\widetilde{N}_{1,1/4}$ and the parameter c will also only be used in proofs.

Definition 1.26 (L^p Dirichlet problem)**.** Let $\Omega \subset \mathbb{R}^n$ be an open (possibly unbounded) domain with $(n-1)$-Ahlfors regular boundaries and $L = -\operatorname{div} A\nabla$ is a uniformly elliptic operator on Ω. We say that the L^p Dirichlet problem is solvable if there exists $C > 0$ such that, for any $g \in C_0^\infty(\mathbb{R}^n)$, the solution $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ to the continuous Dirichlet problem satisfies

$$\|N(u)\|_{L^p(\partial\Omega,\sigma)} \leq C\|g\|_{L^p(\partial\Omega,\sigma)},$$

where we remind the reader that σ is any Ahlfors regular measure on $\partial\Omega$.

1.2.1 . Equivalent characterizations of the solvability of the L^p Dirichlet problem.

Elliptic measures and Green functions are fundamental tools for solving the Dirichlet problem. The precise definitions and properties of these tools are provided in the more general framework developed in Chapter 2. Therefore, we will only give a brief overview here. Assuming that our domain $\Omega \subset \mathbb{R}^n$ has $(n-1)$-Ahlfors regular boundaries and $L = -\operatorname{div} A\nabla$ is a uniformly elliptic operator on Ω, the elliptic measure[1] is the tool used to solve the Dirichlet problem. Specifically, the elliptic measure $\omega_L := \{\omega_L^X\}_{X\in\Omega}$ is a collection of probability measures such that, for any $g \in C_0^0(\partial\Omega)$, the function defined for all $X \in \Omega$ as

$$u_{g,0}(X) := \int_{\partial\Omega} g(y)d\omega_L^X(y)$$

is the solution to the continuous Dirichlet problem $Lu = 0$ with boundary data g. The Green function $G_L : \Omega \times \Omega \to \mathbb{R}$ is used to solve inhomegenous equations. In fact, for any $f \in C_0^\infty(\Omega)$, the function defined for all $X \in \Omega$ as

$$u_{0,f}(X) := \int_\Omega G_L(X,Y)f(Y)\,dX \tag{1.14}$$

is the solution to the continuous Dirichlet problem $Lu = f$ with boundary data 0.

The solvability of the L^p Dirichlet problem can be characterized using the elliptic measure and the Green function. Here is an example of the existing equivalences.

Theorem 1.27. *Let $\Omega \subset \mathbb{R}^n$ with $(n-1)$-Ahlfors regular boundaries and satisfying the corkscrew point condition, let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω, and let $p \in (1,\infty)$. Then the following are equivalent.*

(i) The L^p-Dirichlet problem is solvable (for the operator L in the domain Ω).

(ii) The elliptic measure ω_L is absolutely continuous with respect to σ and there are $\Lambda, C \geq 2$ such that the Poisson kernel $k_L^X := d\omega_L^X/d\sigma$ satisfies the reverse Hölder bound

$$\left(\fint_{B(x,\Lambda r)} |k_L^X|^{p'} d\sigma\right)^{\frac{1}{p'}} \leq \frac{C}{\sigma(B(x,r))}$$

for all $x \in \partial\Omega$, $r \in (0, \operatorname{diam}\partial\Omega/2)$, $X \in \Omega \cap \{\delta_{\partial\Omega} \geq r/\Lambda\}$.

(iii) The Green function satisfies the bound

$$\|\widetilde{N}(\nabla G_L(X,.)\mathbb{1}_{10B_X\setminus B_X})\|_{L^p(\partial\Omega,\sigma)} \leq C\delta_{\partial\Omega}^{(1-n)/p'}(X),$$

where $B_X := B(X, \delta_{\partial\Omega}(X)/4)$ and C is independent of X.

---

[1] The elliptic measure is called harmonic measure in the case where L is the Laplacian.

(iv) For any $f \in C_0^\infty(\Omega)$, the solution $u_{0,f}$ constructed as in (1.14) *verifies*

$$\|\widetilde{N}(u_{0,f})\|_{L^p(\partial\Omega,\sigma)} \leq C\|\widetilde{A}^1(\delta_{\partial\Omega}^2 f)\|_{L^p(\partial\Omega,\sigma)},$$

where $\widetilde{A}^1(h)(x) := \int_{\gamma(x)}(\sup_{B_X} h)\, dX/\delta_{\partial\Omega}^n$, and C is of course independent of f.

Proof. The equivalence $(i) \iff (ii)$ is found as Theorem 9.2 in [MT24], while $(i) \iff (iii) \iff (iv)$ is a slight variant of Theorem 1.22 in [MPT23] (partial results are found in [KP95]).

The idea here is to see that the solvability of the L^p-Dirichlet problem is equivalent to a condition on the elliptic measure [condition (ii)]. At the same time, the L^p solvability of the Dirichlet problem is equivalent to the L^p solvability of the inhomogeneous "Poisson-Dirichlet" problem [condition (iv)], which can be characterized by a bound on the Green function [condition (iii)]. Many similar characterizations exist, and we refer to [MPT23] for the extended literature on the topic. ☐

Corollary 1.28. *Let $\Omega \subset \mathbb{R}^n$ with $(n-1)$-Ahlfors regular boundaries and satisfying the corkscrew point condition, let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω. If the L^p Dirichlet problem is solvable for some $p \in (1,\infty)$, then there exists $\epsilon > 0$ such that the L^q Dirichlet problem is solvable for all $q \in (p-\epsilon,\infty)$.*

Proof. The characterization (ii) in Theorem 1.27 is a reverse-Hölder estimate on the Poisson kernel, so solvability of the L^p Dirichlet problem immediately implies the one of the L^q Dirichlet problem for $q \in (p,\infty)$. The L^q solvability in the range $q \in (p-\epsilon,p)$ is a consequence of the well known fact that reverse Hölder estimates self-improve ([Gia83, Proposition 1.1]). ☐

In many instances, the question is not whether the L^p Dirichlet problem is solvable for a specific p, but rather whether it is solvable for some (large) $p \in (1,\infty)$. Even in the simplest case where $p = 2$, a necessary and sufficient condition on the boundary for the solvability of the L^2 Dirichlet problem remains unknown. While a sufficient condition is known - specifically, that Ω is Lipschitz and $L = -\Delta$ - extending beyond Lipschitz boundaries to uniformly rectifiable sets only ensures the solvability of the L^p Dirichlet problem for some $p \in (1,\infty)$.

Theorem 1.29. *Let $\Omega \subset \mathbb{R}^n$ with $(n-1)$-Ahlfors regular boundaries and satisfying the corkscrew point condition, and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω. The following are equivalent :*

(i) The L^p Dirichlet problem is solvable for some $p \in (1,\infty)$.

(ii) The elliptic measure is weak A_∞-absolutely continuous with respect to the Ahlfors regular measure, meaning that there are positive constants C and s such that, for every ball B centered on $\partial\Omega$ with associated boundary ball $\Delta := B \cap \partial\Omega$,

$$\omega^X(E) \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^s \omega^X(2\Delta) \qquad \text{for all } X \in \Omega\backslash 4B \text{ and for all Borel } E \subset \Delta. \quad (1.15)$$

If, in addition, we assume that Ω satisfies the Harnack chain condition - hence is uniform - then the above statements are equivalent to :

(ii') The elliptic measure is A_∞-absolutely continuous with respect to the Ahlfors regular measure, meaning that there are positive constants C, s such that, for every ball B centered on $\partial\Omega$ with associated boundary ball $\Delta := B \cap \partial\Omega$,

$$\omega^X(E) \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^s \omega^X(\Delta) \qquad \text{for all } X \in \Omega \setminus 4B \text{ and for all Borel } E \subset \Delta. \tag{1.16}$$

(iii) The Dirichlet problem is solvable in BMO, i.e. there exists $C > 0$ such that, for any $g \in C_0^0(\partial\Omega)$, the solution $u_{g,0}$ to the continuous Dirichlet problem satisfies, for any ball B centered on $\partial\Omega$ with associated boundary ball Δ,

$$\int_{\Omega\cap B} \delta_{\partial\Omega}|\nabla u|^2\, dX \leq C\|g\|_{BMO(\sigma)}\sigma(\Delta). \tag{1.17}$$

(iv) The solutions to $Lu = 0$ satisfy a Carleson measure estimate, i.e. there exists $C > 0$ such that, for any bounded solution to $Lu = 0$ in Ω, any ball B centered on $\partial\Omega$ with associated boundary ball Δ,

$$\int_{\Omega\cap B} \delta_{\partial\Omega}|\nabla u|^2\, dX \leq C\|u\|_{L^\infty(\Omega)}\sigma(\Delta). \tag{1.18}$$

(v) For any $\epsilon \in (0,1)$, bounded solutions to $Lu = 0$ in Ω are ϵ-approximable, i.e. there exists a constant $C_\epsilon > 0$ and a function Φ^ϵ such that

(a) $\|u - \Phi^\epsilon\|_{L^\infty(\Omega)} \leq \epsilon\|u\|_{L^\infty(\Omega)}$,

(b) Φ^ϵ satisfies the L^1-type Carleson measure estimate,

$$\int_{\Omega\cap B} |\nabla\Phi^\epsilon|\, dX \leq C_\epsilon\|u\|_{L^\infty(\Omega)}\sigma(\Delta),$$

where B is any ball centered on $\partial\Omega$ and $\Delta := B \cap \partial\Omega$ as usual,

(c) $\|\delta_{\partial\Omega}\nabla\Phi\|_{L^\infty(\Omega)} \leq C_\epsilon\|u\|_{L^\infty(\Omega)}$,

(d) there exists $\phi \in L^\infty(\partial\Omega)$ such that

$$\lim_{\substack{X\to x\\ X\in\gamma(x)}} \Phi^\epsilon(X) = \varphi(x) \qquad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

Proof. The proof of $(iii) \implies (ii) \implies (i)$ can be found in [HL18], while $(ii') \iff (iii)$ is in [DKP11] and $(i) \implies (ii)$ is [Hof19, Proposition 2]. The equivalence between (ii) and (ii') is a consequence of the doubling property of the elliptic measure when Ω is uniform, which can be found in [CFMS81, Theorem 2.3] (for Lipschitz domains) and [Ken94, Corollary 1.3.6] . The implication $(ii') \implies (iv)$ is easier than $(ii') \implies (iii)$, so we can refer to [HL18] and [DKP11], or to the [FP22] which deals with the situation in a more general setting. The inverse $(iv) \implies (ii')$ requires that Ω is uniform and is proved in [KKPT16] (when Ω is Lipschitz) and [CHMT20, Theorem 1.1]. Finally, $(ii') \iff (v)$ is established in [BPTT24], see also [Dah80, KKPT00, HKMP15b, HMM16, GMT18, BT19, HT20, Gar22] for previous works on ϵ-approximability.

Let us also mention [CHPM24] for some nice equivalences involving weaker versions of (ii) and (iv) when we do not assume that Ω is uniform. □

With these characterizations established, let us proceed to examine explicit scenarios where the Dirichlet problem is solvable in L^p.

1.2.2 . Positive results on the solvability of the Dirichlet problem.

We will present here a non-exhaustive collection of cases where the L^p-Dirichlet problem is solvable.

Theorem 1.30. *Let* $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$, *and let* $L := -\operatorname{div} A\nabla$ *be a uniformly elliptic operator on* Ω_0 *with coefficients independent of* t, *i.e.* $A(x,t) = A(x)$. *Then there exists a* $p \in (1,\infty)$ *such that the* L^p *Dirichlet problem is solvable for the operator* L.

If A *is symmetric, then the* L^2 *Dirichlet problem is solvable.*

Proof. The symmetric case was proved in [JK81a], the non-symmetric case in $\mathbb{R}^2$ is in [KKPT00], and the non-symmetric case in $\mathbb{R}^n$ was left open for a long time before being solved in [HKMP15b].

Many other important results were proved in the case of complex coefficients, or for systems (e.g. [AAH08, AAM10, AAA$^+$11]), but we will not present them here since it is not the direction that interests us. ☐

As a corollary, we can solve the Dirichlet problem in Lipschitz domains.

Corollary 1.31. *Let* Ω *be a bounded Lipschitz domain, or a special Lipschitz domain - i.e.* Ω *is above the graph of a Lipschitz function. Then the Dirichlet problem is solvable in* L^2 *for the Laplacian.*

Proof. This result, which is actually anterior to the theorem, is due to Dahlberg in [Dah77], and is often considered as the pioneer work in the area . ☐

If the operator is the Laplacian, how much can we weaken the assumptions on the domain ? After decades of research and successive improvements, an optimal condition was finally identified in the late 2010s.

Theorem 1.32. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with* $(n-1)$*-Ahlfors regular boundaries and which satisfies the corkscrew point condition. Then the following are equivalent :*

(i) *There is a* $p \in (1,\infty)$ *such that* L^p *Dirichlet problem is solvable for the Laplacian.*

(ii) *The harmonic measure is weak* A_∞*-absolutely continuous with respect to the Ahlfors regular measure.*

(iii) $\partial\Omega$ *is uniformly rectifiable and* Ω *is weak local John.*

Proof. $(i) \iff (ii)$ is Theorem 1.29, and the equivalence of $(i)-(ii)$ to (iii) is the main result of [AHM$^+$20]. The proof of the fact that the weak-A_∞ absolute continuity of the harmonic measure implies uniformly rectifiable is in [HLMN17]. Similar results assuming more *a priori* conditions include [DJ90, HM14, HMUT14, Azz21]. ☐

Remark 1.33. Corollary 1.31 deals with the solvability for the Laplacian of the Dirichlet problem in L^2, while Theorem 1.32 deals with the solvability of the Dirichlet problem in L^p for some $p \in (1,\infty)$. Currently, we do not know what are the optimal conditions for the solvability of the Dirichlet problem in L^2, but people knew early on that, for any $p \in (1,\infty)$, there is a uniform domain with uniformly rectifiable boundaries such that the L^p Dirichlet problem for the Laplacian is not solvable, see [Jer83].

If we are interested in the other characterizations in Theorem 1.29, we have :

Theorem 1.34. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with* $(n-1)$*-Ahlfors regular boundaries and which satisfies the corkscrew point condition. Then the following are equivalent :*

(i) $\partial\Omega$ is uniformly rectifiable.

(ii) The bounded harmonic functions satisfy the Carleson estimate (1.18).

(iii) Bounded harmonic functions are ϵ-approximable.

Proof. (i) $\implies$ (ii), (iii) is [HMM16, Theorems 1.1 and 1.3] and (ii), (iii) $\implies$ (i) is [GMT18, Theorem 1.1] ☐

Theorem 1.35. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $(n-1)$-Ahlfors regular boundaries and that satisfies the corkscrew point condition. Then the following are equivalent :*

(i) The harmonic measure is A_∞-absolutely continuous with respect to the surface measure σ.

(ii) $\partial\Omega$ is uniformly rectifiable and Ω is semi-uniform, meaning that there exists $C \geq 1$ such that, for any point $X \in \Omega$ and $y \in \partial\Omega$, X and y are linked by a "cigar with bounded turning", more precisely there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = y$, $\gamma(1) = X$, $\ell_\gamma(y, \gamma(t)) \leq C\delta_{\partial\Omega}(\gamma(t))$ for all $t \in (0,1)$, and $\ell_\gamma(y, X) \leq C|X - y|$.

Proof. A more general version of this result is found in [Azz21]. In particular, [Azz21] shows that the harmonic measure is doubling if and only if Ω is semi-uniform. ☐

Researchers have also explored the flexibility of the coefficients of the operator L, specifically investigating which modifications to these coefficients preserve the solvability of the L^p Dirichlet problem. It is important to note that if we consider all uniformly elliptic operators, even in the half-space $\mathbb{R}^n_+$, there exist counterexamples. Indeed, we can find uniformly elliptic operators $L := -\operatorname{div} A\nabla$ on $\mathbb{R}^n_+$ for which the L^p Dirichlet problem is not solvable for any of the $p \in (1,\infty)$; see for instance [MM81, CFK81].

Therefore, the goal is to identify conditions on the coefficients of our operators that ensure the solvability of the L^p Dirichlet problem when the Laplacian is solvable. For domains without a privileged direction, these conditions involve Carleson measures. For convenience, we will denote that $f \in CM_\sigma$ - or $CM_\sigma(M)$ when highlighting the constant - when the function f on Ω satisfies the smallness condition

$$\int_{\Omega\cap B} |f|^2 \frac{dX}{\delta_{\partial\Omega}} \leq M\sigma(\Delta) \qquad \text{for all ball } B \text{ centered on } \partial\Omega,\ \Delta := B \cap \partial\Omega.$$

This condition is not new for the reader, as it appears in (1.17) as

$$\delta_{\partial\Omega}|\nabla u_{g,0}| \in CM_\sigma(C\|g\|_{BMO(\sigma)}),$$

and in (1.18) as

$$\delta_{\partial\Omega}|\nabla u| \in CM_\sigma(C\|u\|_{L^\infty(\Omega)}).$$

The L^p Dirichlet problem is stable under Carleson perturbations.

Theorem 1.36. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries. Assume that $L_0 := -\operatorname{div} A_0\nabla$ and $L_1 := -\operatorname{div} A_1\nabla$ are two uniformly elliptic operators, and there is a $p \in (1,\infty)$ such that the L^p Dirichlet problem is solvable for L_0.*

(1) If $X \to \sup_{B_X} |A_0 - A_1| \in CM_\sigma$, then there is a $r \in (1,\infty)$ such that the L^r Dirichlet problem is solvable for L_1.

(1') If $|A_0 - A_1| \in CM_\sigma$ and $\delta_{\partial\Omega}|\nabla A_i| \in L^\infty(\Omega)$ for either $i = 0$ or $i = 1$, then there is a $r \in (1,\infty)$ such that the L^r Dirichlet problem is solvable for L_1.

(2) There is a $\epsilon = \epsilon(\Omega, L_0, p) > 0$ such that if $X \to \sup_{B_X} |A_0 - A_1| \in CM_\sigma(\epsilon)$, then the L^p Dirichlet problem is also solvable for L_1.

Here we remind the reader that $B_X := B(X, \delta_{\partial\Omega}(X)/4)$.

Proof. On smooth domains, (1) and (2) are proved in [FKP91], and weaker notions of Carleson perturbations can be found in the earlier works [FJK84, Dah86, Fef89]. It was then generalized for weaker geometry in [MT10, MPT, CHM19, CHMT20]. The variant (1') is showed by the author in [Fen24].

We could have a (2') in the same spirit as (1') but for small perturbations. Moreover, (2) would stay true if we do not assume that Ω is uniform (but only that Ω satisfies the corkscrew point condition). However, as far as the author knows, none of those two facts are written anywhere. ☐

In Theorem 1.32, we can replace the Laplacian by a more general class of operators called DKP operators (for Dahlberg-Kenig-Pipher).

Theorem 1.37. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries, and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator whose coefficients satisfy $\delta_{\partial\Omega}|\nabla A| \in L^\infty(\Omega) \cap CM_\sigma$. Then the following are equivalent :*

(i) There is a $p \in (1, \infty)$ such that the L^p Dirichlet problem for L is solvable.

(ii) The elliptic measure ω_L is A_∞-absolutely continuous with respect to the Ahlfors regular measure.

(iii) $\partial\Omega$ is uniformly rectifiable.

Note that this equivalence is compatible with Theorem 1.36, and we can thus enlarge the class of operators for which the equivalence holds.

Proof. $(i) \iff (ii)$ is again proved by Theorem 1.29. $(iii) \implies (ii)$ is a combination of [KP01], which treats the case of Lipschitz domains, that then either [DJ90] or [HMM24], which says that the bound (1.18) is transferred to a CAD as long as it holds for every Lipschitz subdomain. The last implication $(ii) \implies (iii)$ is the purpose of [HMM+21]. ☐

Note that Theorem 1.37 requires Ω to *a priori* be uniform, unlike Theorem 1.32. It is conjectured that a more refined version of Theorem 1.37 exists, where we only assume that Ω satisfies the corkscrew point condition and where we deduce that Ω is weak local John from the weak-A_∞-absolute continuity of the elliptic measure. Many researchers have attempted to prove this, but to the author's knowledge, no one has succeeded yet.

To conclude this section, we highlight a recent result by the author that, under certain conditions, allows us to view DKP operators as Carleson perturbations (see [Fen22b]).

Proposition 1.38. *Let $\Omega = \mathbb{R}^n_+$ and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator that satisfies the DKP condition $\delta|\nabla A| \in CM_\sigma(M)$. Then there exists a bi-Lipschitz change of variable ρ that fixes $\mathbb{R}^{n-1}$ such that the conjugate $L_\rho := -\operatorname{div} A_\rho\nabla$ of L by ρ satisfies*

$$A_\rho = \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline \mathbf{0} & 1 \end{array}\right] + C_\rho \tag{1.19}$$

with $\delta|\nabla B_1| + \delta|\nabla B_1| + |C_\rho| \in CM_\sigma(CM)$. The bi-Lipschitz constants of ρ depend only on the elliptic and boundedness constants of L.

Assume that A is written

$$A := \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3 & a_4 \end{array}\right].$$

Then we can choose

$$B_1 = a_4 A_1 - A_3 A_2 \text{ and } B_2 = A_2 - A_3^T. \tag{1.20}$$

Note that $L_0^*(t) = 0$ whenever $L_0 = -\operatorname{div} A_0 \nabla$ and A_0 has the form

$$A_0 = \left[\begin{array}{c:c} A_{0,1} & A_{0,2} \\ \hdashline \mathbf{0} & 1 \end{array}\right]$$

By showing that t is a "Green function with pole at infinity" and by comparing Green functions and elliptic measure, it is not too hard to demonstrate that the L^p Dirichlet problem for L_0 is solvable for **any** $p \in (1, \infty)$. One can readily verify that any statement in Theorem 1.29 remains stable under a bi-Lipschitz change of variables. Consequently, by combining Proposition 1.38 and Theorem 1.36 (1), we can establish that the L^p Dirichlet problem for DKP operators on $\Omega = \mathbb{R}^n_+$ is always solvable for a sufficiently large $p \in (1, \infty)$.

Furthermore, if we consider small Carleson perturbations instead of large ones, we can combine Proposition 1.38 and Theorem 1.36 (2) to obtain the following result for operators with a "small DKP constant", originally proved in [DPP07] :

Theorem 1.39. *Let $\Omega_0 := \{(x, t) \in \mathbb{R}^{n-1} \times (0, \infty)\}$, let $L := -\operatorname{div} A \nabla$ be a uniformly elliptic operator, and let $p \in (1, \infty)$. There exists ϵ depending on p and the elliptic constants of L such that if $t|\nabla A| \in CM_\sigma(\epsilon)$, then the L^p Dirichlet problem for L is solvable.*

1.3 . The L^p regularity problem

The regularity problem is the natural follow up to the Dirichlet problem : instead of solving the Dirichlet problem with L^p boundary data, we aim to solve it with $W^{1,p}$ boundary data. To study the regularity problem beyond Lipschitz domains, we need a definition of $W^{1,p}(\partial\Omega)$ that applies to potentially very rough boundaries, ideally up to Ω-Ahlfors regular boundaries.

For this purpose, we will employ the concept of the Hajłasz gradient, which can be defined on any metric measure space since it relies on the notion of Lipschitz functions. The Hajłasz gradient was introduced in [Haj96] and was first applied to the study of the regularity problem in [MT24].

Definition 1.40. (Hajłasz upper gradient) If g is a Borel function on a metric measure space (Σ, d, m), then h is a Hajlasz upper gradient of g if h is a positive Borel measurable function and

$$|g(x) - g(y)| \leq d(x, y) \frac{h(x) + h(y)}{2} \qquad \text{for } m\text{-a.e. } x, y \in \Sigma.$$

Let $D(g)$ be the collection of Hajłasz upper gradient of g, and we define the norm

$$\|g\|_{\dot{W}^{1,p}(\Sigma,m)} := \inf_{h \in D(g)} \|h\|_{L^p(\Sigma,m)}.$$

When $p \in (1, \infty)$, due to the uniform convexity of $L^p(\Sigma, m)$, there exists a unique function h_g that reaches the infimum $\inf_{h \in D(g)} \|h\|_{L^p(\Sigma,m)}$, and we write $\nabla_{H,p} g$ for h_g.

We always have $\nabla_{H,p} g \geq |\nabla g|$, indicating that the Hajłasz gradient is always larger than the classical gradient, if the later exists. If Σ is uniformly rectifiable, and $\nabla_t g$ denotes the "classical" notion of gradient (obtained by approximating the sets by tangent planes - which exist σ-almost everywhere, and taking the gradient on those tangent planes), then $\|\nabla_{H,p} g\|_p$ and $\|\nabla_t g\|_p$ are equivalent whenever Σ supports a $(1,p)$-Poincaré inequality on balls; see for instance Lemma 1.3 in [MT24]. We will not delve further into this question, but it is important to note that the two notions of gradients on the boundary are equivalent when $\Omega = \mathbb{R}^n_+$ or Ω is Lipschitz. Therefore, in such scenarios, we will simply write ∇ instead of $\nabla_{H,p}$ or ∇_t.

The Hajłasz gradient is indeed the natural gradient to use for boundary values problems, especially when the domain is uniform, as showed by the next proposition.

Proposition 1.41. *Let Ω be a uniform domain with $(n-1)$-Ahlfors regular boundaries, and let L be a uniformly elliptic operator. The solution u_g denotes the solution to the continuous Dirichlet problem ($Lu = 0$) with data $g \in C^0_0(\partial\Omega)$.*

Then there exist $K \geq 1$ and $\alpha > 0$ such that, for any $g \in C^0_0(\partial\Omega)$, the quantity $K\widetilde{N}_\alpha(\nabla u_g)$ is a Hajłasz upper gradient of g. Consequently, for $p \in (1,\infty)$, we necessarily have

$$\|\widetilde{N}(\nabla u_g)(x)\|_{L^p(\partial\Omega,\sigma)} \geq C^{-1}\|\nabla_{H,p} g\|_{L^p(\partial\Omega,\sigma)}$$

with a constant C independent of g.

Proof. Let $x, y \in \partial\Omega$. Since Ω is uniform, it satisfies the corkscrew point condition with the constant ϵ_Ω. So we can find $Z \in \Omega$ which is a ϵ_Ω-corkscrew point to both x and y at scale $2|x-y|$.

Let us write $v_g(X)$ for $\fint_{B_X} u\, dX$, where we remind the reader that $B_X := B(X, \delta_{\partial\Omega}(X)/4)$. We have

$$|u_g(x) - u_g(y)| = |u_g(x) - v_g(Z)| + |u_g(y) - v_g(Z)|$$

so the proposition will be proven once we show that

$$|u_g(x) - v_g(Z)| \leq K|x-Z|\widetilde{N}_\alpha(\nabla u_g)(x) \quad \text{and} \quad |u_g(y) - v_g(Z)| \leq K|y-Z|\widetilde{N}_\alpha(\nabla u_g)(y) \tag{1.21}$$

for some $K \geq 1$, $\alpha > 0$.

By symmetry of the roles of x and y, we just need to prove the first part of (1.21). By the corkscrew point condition, we can take an infinite collection $\{A_i\}_{i\in\mathbb{N}}$ such that $A_0 = Z$ and A_i is a corkscrew point for x at scale $2^{1-i}|x-y|$. By the Harnack chain condition, we can find N such that two successive corkscrew points are linked by a Harnack chain of length at most N. Altogether, we can find α, K' (that both depend only on the corkscrew point and Harnack chain constants of Ω) and a sequence of points $\{Z_i\}_{i\in\mathbb{N}}$ in Ω such that

i. $Z_0 = Z$ and $\lim_{i\to\infty} Z_i = x$;

ii. for each $i \in \mathbb{N}$, $|Z_i - Z_{i+1}| \leq \frac{1}{4}\min\{\delta_{\partial\Omega}(Z_i), \delta_{\partial\Omega}(Z_{i+1})\}$, in particular $\delta_{\partial\Omega}(Z_i) \approx \delta_{\partial\Omega}(Z_{i+1})$;

iii. for each $j \in \mathbb{N}$, $\sum_{i\geq j} \delta_{\partial\Omega}(Z_i) \leq K'\delta_{\partial\Omega}(Z_j)$;

iv. $Z_i \in \Gamma_\alpha(x)$ for all $i \in \mathbb{N}$.

The proof of this fact is just a variant of Proposition 1.18.

With those intermediate points in hand, and since $\lim_{X\to x} v(X) = u(x)$, we have

$$|u_g(x) - v_g(Z)| \leq \sum_{i\in\mathbb{N}} |v_g(Z_i) - v_g(Z_{i+1})|.$$

But note that, by construction, $|B_{Z_i}|$, $|B_{Z_{i+1}}|$ and $|B_{Z_i} \cap B_{Z_{i+1}}|$ have a comparable size, so the Poincaré inequality entails that

$$\begin{aligned} |v_g(Z_i) - v_g(Z_{i+1})| &\leq \left| \fint_{B_{Z_i}\cap B_{Z_{i+1}}} u\, dY - v(Z_i)\right| + \left| \fint_{B_{Z_i}\cap B_{Z_{i+1}}} u\, dY - v(Z_{i+1})\right| \\ &\leq C\left(\fint_{B_{Z_i}} |u - v_{Z_i}|\, dY + \fint_{B_{Z_{i+1}}} |u - v_{Z_{i+1}}|\, dY\right) \\ &\leq C\left(\delta_{\partial\Omega}(Z_i)\fint_{B_{Z_i}} |\nabla u|\, dY + \delta_{\partial\Omega}(Z_{i+1})\fint_{B_{Z_{i+1}}} |\nabla u|\, dY\right) \\ &\leq C\delta_{\partial\Omega}(Z_i)\widetilde{N}_\alpha(\nabla u)(x). \end{aligned}$$

by the properties ii. and iv. of the collection $\{Z_i\}_{i\in\mathbb{N}}$. Summing over $i \in \mathbb{N}$ entails

$$|u_g(x) - v_g(Z)| \leq C\widetilde{N}_\alpha(\nabla u)(x)\sum_{i\in\mathbb{N}} \delta_{\partial\Omega}(Z_i) \leq C\delta_{\partial\Omega}(Z)\widetilde{N}_\alpha(\nabla u)(x) \leq C|x - Z|\widetilde{N}_\alpha(\nabla u)(x),$$

where we successively use the property iii. of the $\{Z_i\}_{i\in\mathbb{N}}$ and the fact that Z is a corkscrew point. The proposition follows. □

We are ready for the definition of the L^p regularity problem.

Definition 1.42 (L^p regularity problem)**.** Let Ω be a domain with $(n-1)$-Ahlfors regular boundaries and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator. Then we say that the L^p-regularity problem is solvable if there exists $C > 0$ such that, for any $g \in C_0^\infty(\mathbb{R}^n)$, the solution $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ to the continuous Dirichlet problem satisfies

$$\|\widetilde{N}(\nabla u)\|_{Lp(\partial\Omega,\sigma)} \leq C\|\nabla_{H,p} g\|_{L^p(\partial\Omega,\sigma)}.$$

In the sequel, we will write (D$_p$)$_L$ and (R$_p$)$_L$ to say that the L^p- Dirichlet and regularity (respectively) problems are solvable.

Proposition 1.43. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $(n-1)$-Ahlfors regular boundaries and satisfying the corkscrew point condition. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator, and write L^* for its adjoint. Let $p, q \in (1,\infty)$ and write p', q' for their Hölder conjugate.*

(1) (R$_p$)$_L$ $\Longrightarrow$ (D$_{q'}$)$_{L^*}$ for all $q \in (p' - \epsilon, \infty)$ and some $\epsilon > 0$;

(2) (R$_p$)$_L$ $\Longrightarrow$ (R$_q$)$_L$ for all $q \in (1, p)$;

(3) if Ω is uniform, (R$_p$)$_L$ + (D$_q$)$_{L^*}$ $\Longrightarrow$ (R$_q$)$_L$.

Proof. Conclusion 1 is [MT24, Theorem A.2, Remark 9.3], see [KP93] for an earlier version in smooth domains. Conclusion 2 is [GMT25, Theorem 1.3], with weaker results in [MT24], [DK12] and [Ken94]. The main result from [She07] shows conclusion 3 in Lipschitz domains, but the proof immediately extends to uniform domains. One can probably remove the condition that Ω is uniform by combining the proofs of [She07] and [GMT25, Theorem 1.6], but I don't think that the result is written anywhere. Finally, the cited articles limit themselves to $\Omega \subset \mathbb{R}^n$, $n \geq 3$, but the arguments extend easily to $\Omega \subset \mathbb{R}^2$. □

The best results on the L^p-regularity problem to date are :

Theorem 1.44. *Let $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$, and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω_0 with coefficients independent of t, i.e. $A(x,t) = A(x)$. Then there exists a $p \in (1,\infty)$ such that the L^p regularity problem is solvable for the operator L.*

If A is symmetric, then the L^2 regularity problem is solvable.

Proof. The general case is in [HKMP15a], while the symmetric case is a consequence of the Rellich identity ([JK81a]). ☐

Theorem 1.45. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries. Assume that $L_0 := -\operatorname{div} A_0\nabla$ and $L_1 := -\operatorname{div} A_1\nabla$ are two uniformly elliptic operators, and there is a $p \in (1,\infty)$ such that the L^p regularity problem is solvable for L_0.*

(1) If $X \to \sup_{B_X} |A_0 - A_1| \in CM_\sigma$, then there is a $r \in (1,\infty)$ such that the L^r regularity problem is solvable for L_1.

(1') If $|A_0 - A_1| \in CM_\sigma$ and $\delta_{\partial\Omega}|\nabla A_i| \in L^\infty(\Omega)$ for either $i = 0$ or $i = 1$, then there is a $r \in (1,\infty)$ such that the L^r regularity problem is solvable for L_1.

(2) There is a $\epsilon = \epsilon(\Omega, L_0, p) > 0$ such that, if $X \to \sup_{B_X} |A_0 - A_1| \in CM_\sigma(\epsilon)$, then the L^p regularity problem is also solvable for L_1.

Proof. (1) is [KP95] in smooth domains, and [DFM23a] in uniform domains. The variation (1') is Theorem 2.11 in [Fen23]. (2) is an easy consequence of Theorem 1.36 (2) and Proposition 1.43. ☐

Theorem 1.46. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $(n-1)$-uniformly rectifiable boundaries and satisfying the corkscrew point condition, let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator satisfying $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma$, and let $p \in (1,\infty)$. Then*

$$(D_{p'})_{L^*} \implies (R_p)_L$$

Proof. The result is [MPT23, Theorem 1.33], see also [MT24], [DHP23] and [Fen23]. Note that, when $L = -\Delta$, we can assume that Ω has $(n-1)$-Ahlfors regular boundaries instead of assuming that $\partial\Omega$ is uniformly rectifiable because Theorem 1.32 gives that (D$_p$) $\implies$ UR. However, if we only assume that $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma$, Theorem 1.37 does not allow us to say that (D$_p$) $\implies$ UR. ☐

1.4 . The L^p Neumann problem

The elliptic measure and the Green function are essential tools for studying Dirichlet and regularity boundary value problems. The analogous tool for the Neumann problem is the Neumann function, which possesses the following properties :

Theorem 1.47. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with $(n-1)$-Ahlfors regular boundaries, and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator. There exists a unique function N - called Neumann function - defined on $\overline{\Omega}\times\overline{\Omega}$ such that N is continuous except on the diagonal $\{(X,X),\ X \in \overline{\Omega}\}$, $N(X,.)$ is locally integrable in Ω, $\int_{\partial\Omega} N(X,y)\, d\sigma(y) = 0$, and such that the following representation formula holds : for any $Y \in \overline{\Omega}$, any $\Phi \in W^{1,2}(\Omega) \cap C^0(\Omega)$,*

$$\int_\Omega A(X)\nabla_X N(X,Y) \cdot \nabla\Phi(X) dX = \Phi(Y) - \fint_{\partial\Omega} \Phi\, d\sigma. \tag{1.22}$$

Moreover,

(i) for any $X, Y \in \Omega$, $N^T(Y,X) = N(X,Y)$, where $N^T(X,Y)$ is the Neumann function for the adjoint operator $L^* = -\operatorname{div}(A^T\nabla)$;

(ii) for any ball $B = B(x,r)$ centered on the boundary and for $Y \in \Omega \setminus 2B$, we have

$$\left(\fint_{B\cap\Omega} |\nabla_X N(X,Y)|^2\,dX\right)^{\frac{1}{2}} \leq Cr^{1-n} \tag{1.23}$$

or equivalently

$$\operatorname*{osc}_{X\in B\cap\Omega} N(X,Y) \leq Cr^{2-n} \tag{1.24}$$

with a constant that depends only on Ω and L.

Proof. Most of the proof follows the path from [KP93] but will diverge when we need to prove (ii) because [KP93] assume the boundary to be smooth. We also assume that basic estimates on solution with zero Neumann data are known, see [DDE$^+$24]; and that the reader is familiar with the notion of trace of function in $W^{1,2}(\Omega)$, which can be otherwise found - although in a more general setting - in Chapter 2 below.

We define the space $\widehat{W}^{1,2}(\Omega)$ to be

$$\widehat{W}^{1,2}(\Omega) := \left\{u \in W^{1,2}(\Omega),\ \int_{\partial\Omega} \operatorname{Tr} u\,d\sigma = 0\right\},$$

which is complete with the norm $\|.\|_{\widehat{W}^{1,2}(\Omega)} = \|\nabla .\|_{L^2(\Omega)}$.

The definition makes sense because traces exist when the boundaries are $(n-1)$-Ahlfors regular regular, such a fact is proved in Theorem 2.34 below in a general setting that includes our scenario.

For each $Y \in \Omega$, the Lax-Milgram theorem allows us to construct a function $v_Y \in \widehat{W}^{1,2}(\Omega)$ such that

$$\int_\Omega A\nabla v_Y \cdot \nabla\phi\,dX = \int_{\partial\Omega} (\operatorname{Tr}\phi)\,d\omega^Y_{L^*} \qquad \forall \phi \in \widehat{W}^{1,2}(\Omega), \tag{1.25}$$

meaning that

$$\int_\Omega A\nabla v_Y \cdot \nabla\phi\,dX = \int_{\partial\Omega} (\operatorname{Tr}\phi)\left[d\omega^Y_{L^*} - \frac{d\sigma}{\sigma(\partial\Omega)}\right] \qquad \forall \phi \in W^{1,2}(\Omega). \tag{1.26}$$

The fact that $\phi \in \int_{\partial\Omega}(\operatorname{Tr}\phi)\,d\omega^Y_{L^*}$ belongs to the dual $[\widehat{W}^{1,2}(\Omega)]^*$ is proved like [KP93, Lemma 2.3] and requires an interior Moser estimate on the solutions and a Poincaré inequality on the boundary. We define $N(X,Y) = G(X,Y) + v_Y(X)$. Since

$$\int_\Omega A(X)\nabla_X G(X,Y) \cdot \nabla\Phi(X) dX = \Phi(Y) - \fint_{\partial\Omega} \operatorname{Tr}\phi\,d\omega^Y_{L^*} \qquad \forall \phi \in W^{1,2}(\Omega), \tag{1.27}$$

we immediately get (1.22). The identity $N(X,Y) = N^T(Y,X)$ is then proved as in [KP93, Lemma 2.5].

So there is only the proof of (ii) left, and more precisely the proof of (1.23) since (1.24) follows from (1.23), a Moser estimate at the boundary for solution with zero Neumann data (found in [DDE$^+$24, Lemma 3.2]) and a Poincaré inequality at the boundary. Moreover, observe that the estimate

$$\left(\fint_{B\cap\Omega} |\nabla_X G(X,Y)|^2\,dX\right)^{\frac{1}{2}} \leq Cr^{1-n},$$

where $B := B(x_B, r)$ is centered on the boundary and $Y \in \Omega \backslash 2B$, is a classical estimate on the Green function; see (2.64) below for the estimate in a general setting or [GW82] for the first article where this estimate was established. So we just need to prove the estimate

$$\Big(\fint_{B\cap\Omega} |\nabla v_Y|^2 \, dX \Big)^{\frac12} \leq C r^{1-n} \tag{1.28}$$

for B is radius r centered on the boundary and $Y \in \Omega \setminus 2B$. The proof is similar to [DFM20, Lemma 14.6], so we only sketch the proof. We define w_Y as $v_Y - \fint_{B\cap\Omega} v_Y$ and then $\Omega_t := \{Z \in \Omega, |w_Y| > t\}$. We let the reader check that

$$\int_\Omega A\nabla |w_Y| \cdot \nabla \phi \, dX \leq \int_{\partial\Omega} (\operatorname{Tr} \phi) \operatorname{sgn}(w_Y) \Big[d\omega^Y_{L^*} - \frac{d\sigma}{\sigma(\partial\Omega)} \Big] \qquad \forall \phi \in W^{1,2}(\Omega). \tag{1.29}$$

and, by using it with

$$\phi := \max\Big\{0, \frac{2}{t} - \frac{1}{|w_Y|}\Big\},$$

we obtain that

$$\int_{\Omega_{t/2}} \frac{|\nabla w_Y|^2}{|w_Y|^2} \, dX \leq \frac{C}{t},$$

where C depends only on the ellipticity constant of L. The Sobolev inequality entails that

$$\Big(\int_{\Omega_{t/2}\cap 2B} \Big| \ln(|w_Y|/2t)_+ \Big|^{2^*} dX \Big)^{\frac{1}{2^*}} \leq C \Big(\int_{\Omega_{t/2}\cap 2B} \frac{|\nabla w_Y|^2}{|w_Y|^2} dX \Big)^{\frac12} \leq C t^{-\frac12},$$

where r is the radius of B, and then

$$\frac{|\Omega_t \cap 2B|}{|2B|} \leq C r^n t^{-n/(n-2)}.$$

Using Cavalieri's formula, we get

$$\fint_{\Omega\cap 2B} |w_Y| \, dX \leq C \int_0^\infty \frac{|\Omega_t \cap 2B|}{|2B|} \, dt \leq C \int_0^\infty \min\{1, r^n t^{-n/(n-2)}\} \, dt \leq C r^{2-n}. \tag{1.30}$$

We just need to conclude. Recall that $v_Y = N(.,Y) - G(.,Y)$, and both the Neumann and the Green functions satisfy Caccioppoli and Moser estimates at the boundary, so

$$\Big(\fint_{\Omega\cap B} |\nabla v_Y|^2 \, dX \Big)^{\frac12} \leq C r \fint_{\Omega\cap\frac32 B} |w_Y| \, dX + \sup_{X\in\Omega\cap\frac32 B} G(X,Y) \leq C r^{2-n}$$

by (1.30) and the pointwise bound on the Green function (see Theorem 2.68 below). The theorem follows. □

Proposition 1.48. *Let Ω be a bounded uniform domain with $(n-1)$-Ahlfors regular boundaries and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator. We define the conormal derivative of a solution $u \in W^{1,2}(\Omega)$ to $Lu = 0$ as the distribution*

$$\langle \partial^A_\nu u, h \rangle := \int_\Omega A\nabla u \cdot \nabla H \, dX \qquad \text{for } H \in W^{1,2}(\Omega) \text{ and } h = \operatorname{Tr} H.$$

We have the following properties :

(1) *When $u \in W^{1,2}(\Omega)$ is a solution to $Lu = 0$, the quantity $\langle \partial_\nu^A u, h \rangle$ depends on h and not on the choice of the extension H.*

(2) *For any $g \in C^\infty(\mathbb{R}^n)$ verifying $\int_{\partial\Omega} g \, d\sigma = 0$, the function $u^g \in \widehat{W}^{1,2}(\Omega)$ defined as*

$$u^g(X) := \int_\Omega N(X,y) g(y) \, d\sigma(y)$$

is a weak solution to $Lu = 0$ that satisfies $\partial_\nu^A u = g|_{\partial\Omega}$ (in the sense of distribution).

Proof. (1) is a simple consequence of the fact that u is a solution to $Lu = 0$. (2) is obtained from (1.22) after a careful interchange of integrals. ☐

Definition 1.49 (Solvability of the L^p Neumann problem)**.** Let Ω be a bounded uniform domain with $(n-1)$-Ahlfors regular boundaries and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator. We say that the L^p Neumann problem is solvable - (N$_p$)$_L$ for short - if there exists $C > 0$ such that, for all $g \in C^\infty(\mathbb{R}^n)$, the solution u^g defined as

$$u^g(X) := \int_\Omega N(X,y) g(y) \, d\sigma(y)$$

satisfies

$$\|\widetilde{N}(\nabla u^g)\|_{L^p(\partial\Omega,\sigma)} \leq C\|g\|_{L^p(\partial\Omega,\sigma)}.$$

The Neumann problem is much less understood than the Dirichlet and even the regularity problem. Even the basic properties of the Neumann problem requires to invoke the Dirichlet or regularity problems.

Proposition 1.50. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with $(n-1)$-Ahlfors regular boundaries, let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator, and let $p \in (1,\infty)$.*

$$(\mathrm{N}_p)_L + (\mathrm{D}_{p'})_{L^*} \implies (\mathrm{N}_q)_L \text{ for all } q \in (1, p+\epsilon) \text{ and some } \epsilon > 0.$$

Proof. This result is proved in [FL24]. We actually proved that the solvability of the so-called L^p Poisson-Neumann problem - (PN$_p$)$_L$ for short - extrapolates and is equivalent to the solvability of Neumann problem, assuming the solvability of the Dirichlet problem. More precisely, we proved that

$$(\mathrm{N}_p)_L + (\mathrm{D}_{p'})_{L^*} \implies (\mathrm{PN}_p)_L \implies (\mathrm{N}_p)_L$$

and

$$(\mathrm{PN}_p)_L \implies (\mathrm{PN}_q)_L \text{ for all } q \in (1, p+\epsilon) \text{ and some } \epsilon > 0.$$

Anterior results that require (R$_p$)$_L$ instead of (D$_{p'}$)$_{L^*}$ have been shown in [KP93] (when the domain is smooth) and [HS24]. ☐

As for actual cases where we know the solvability of the L^p Neumann problem, we have :

Theorem 1.51. *Take $n \geq 3$. Let $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$ and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω_0 with coefficients independent of t, i.e. $A(x,t) = A(x)$.*

(1) *If A is symmetric, then the L^2 Neumann problem is solvable.*

(2) *Let $A = A_s + A_a$ be the decomposition of A into its symmetric and antisymmetric parts. There exists $\epsilon > 0$ (depending on A_s) such that the L^2 Neumann problem (for L) is solvable whenever $\|A_a\|_{L^\infty} < \epsilon$.*

Proof. The symmetric case is a consequence of the Rellich identity (see [JK81a]); see also [JK81b] for a result on Lipschitz domain. The small perturbation result is a consequence of a much stronger result on elliptic operators with complex coefficients; see [AAA+11, Theorem 1.14]. ☐

Theorem 1.52. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with $(n-1)$-Ahlfors regular boundaries. Assume that $L_0 := -\operatorname{div} A_0 \nabla$ and $L_1 := -\operatorname{div} A_1 \nabla$ are two uniformly elliptic operators and that there is a $p \in (1,\infty)$ such that the L^p regularity problem is solvable for L_0.*

(1) *There is a $\epsilon > 0$ (that depends on Ω, L_0, and p) such that, if $X \to \sup_{B_X} |A_0 - A_1| \in CM_\sigma(\epsilon)$, then the L^p Neumann problem is also solvable for L_1.*

(2) *If $\delta_{\partial\Omega}|\nabla A_i| \in L^\infty(\Omega)$, there is a $\epsilon > 0$ (that depends on Ω, L_0, and p) such that, if $|A_0 - A_1| \in CM_\sigma(\epsilon)$, then the L^p Neumann problem is also solvable for L_1.*

Proof. (1) is [KP95, Theorem 2.2] in smooth domains, and easily extend to our seeting. To prove (2), we just need to follow the proof of [KP95, Theorem 2.2], and use the Moser estimate $|\nabla u(X)| \leq C \fint_{B_X} |\nabla u|\, dY$ - which is true when $\delta_{\partial\Omega}|\nabla A_i| \in L^\infty(\Omega)$ and established for instance at (2.15) in [Fen23] - at the appropriate moment. ☐

Theorem 1.53. *Let $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$ and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator, and let $p \in (1,\infty)$.*

(1) *If $n = 2$ and $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma$, then there exists $p \in (1,\infty)$ such that the L^p Neumann problem is solvable.*

(2) *Then there exists $\epsilon > 0$ depending on n, p, and the elliptic constants of L such that, if $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma(\epsilon)$, then the L^p Neumann problem is solvable.*

Proof. (1) is in [DR10] and uses a trick from [KR09] that works only in $\mathbb{R}^2_+$ and transforms a Neumann problem into a regularity problem for a different operator, which allows us to get the result from Theorem 1.46. (2) is [DPR17, Theorem 7.1]. ☐

1.5 . Why do we need a new elliptic theory to study sets of higher codimension ?

The purpose of this memoir is to study the geometry of a set E by examining the properties of harmonic or elliptic extensions in $\mathbb{R}^n \setminus E$ corresponding to L^p data on E.

Consider a d-Ahlfors regular set $E \subset \mathbb{R}^n$ and investigate how the regularity or flatness of E interacts with solutions of an elliptic operator L in $\mathbb{R}^n \setminus E$. When $d \in (n-2, n)$, the Ahlfors regular set E has non-zero capacity (with quantifiable, scale invariant properties), meaning that a rich elliptic theory based on classical uniformly elliptic operators can be developed. However, when $d \leq n-2$, studying E through harmonic solutions in $\mathbb{R}^n \setminus E$ becomes pointless because the Laplacian does not "detect" the boundary. To discuss the solvability of the (continuous, L^p, etc.) Dirichlet problem, we need to consider weak solutions $u \in W(\mathbb{R}^n \setminus E, \mathcal{L}^n)$ to $-\Delta u = 0$ in $\mathbb{R}^n \setminus E$, i.e.

$$\int_{\mathbb{R}^n \setminus E} \nabla u \cdot \nabla \varphi \, dX = 0 \qquad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n \setminus E). \tag{1.31}$$

However, sets of codimension $d \leq n-2$ have zero capacity, which means that we have the following density result :

Proposition 1.54. *Let $E \subset \mathbb{R}^n$ be a Ahlfors regular set of dimension $d \leq n-2$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, there exists a sequence $\varphi_k \in C_0^\infty(\mathbb{R}^n \setminus E)$ such that*

$$\lim_{k\to\infty} \|\nabla(\varphi_k - \varphi)\|_{L^2(\mathbb{R}^n,\mathcal{L}^n)} = 0.$$

So $W(\mathbb{R}^n \setminus E, \mathcal{L}^n;$ locally in $\mathbb{R}^n) = W_{loc}(\mathbb{R}^n, \mathcal{L}^n)$ and (1.31) immediately self-improves to

$$\int_{\mathbb{R}^n\setminus E} \nabla u \cdot \nabla\varphi \, dX = 0 \qquad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1.32}$$

In conclusion, the only bounded weak solutions to $-\Delta u = 0$ in $\mathbb{R}^n \setminus E$ are the constants, rendering the notion of harmonic measure void. From a probabilistic perspective, it means that, for any finite t_0, the probability that a particle in $\mathbb{R}^n \setminus E$ subject to a Brownian motion will hit E in less than t_0 units of time is zero.

Proof of Proposition 1.54. We simply need to prove that $\|\nabla(\varphi\psi_\epsilon)\|_{L^2}$ converges to 0 as ϵ tends to 0, where ψ_ϵ is a cut-off function that is 1 on a neighborhood of E. The case $d < n-2$ only requires the use of the "classical" cut off function $0 \leq \psi_\epsilon \leq 1$ that takes the value 1 on $\{\delta_E \leq \epsilon\}$, 0 when $\{\delta_E \geq 2\epsilon\}$ and that satisfies $|\nabla\psi_\epsilon| \leq 2/\epsilon$.

In order to include the case $d = n-2$, we will use a better cut-off function. Take $\Psi \in C^\infty(\mathbb{R})$ to be such that $0 \leq \Psi \leq 1$, $\Psi \equiv 0$ on $(-\infty, 1]$, $\Psi \equiv 1$ on $[2,\infty)$, and $|\Psi'| \leq 3$. Then, for $\epsilon \in (0, \frac{1}{2})$, we construct

$$\psi_\epsilon(X) := \Psi\Big(\frac{\ln \delta_E(X)}{\ln \epsilon}\Big)$$

and we observe that $\psi_\epsilon(X) = 0$ when $\delta_E(X) \geq \epsilon$, $\psi_\epsilon(X) = 1$ when $\delta_E(X) \leq \epsilon^2$, and $|\nabla\psi_\epsilon(X)| \leq C/(\delta_E(X)|\ln\epsilon|)$. The function $\varphi(1-\psi_\epsilon) \in C_0^\infty$ and we need to prove that $\|\nabla(\varphi\psi_\epsilon)\|_{L^2} \to 0$. If K_φ denotes $\{X \in \Omega, \operatorname{dist}(X, \operatorname{supp}\varphi \leq 1\}$ and S_ϵ is $\{X \in K_\varphi,\, \delta_E(X) \leq \epsilon\}$, observe that

$$\int_\Omega |\nabla[\varphi\psi_\epsilon]|^2 dX \leq 2\int_{S_\epsilon} |\nabla\varphi|^2\psi_\epsilon^2 dX + \frac{C}{|\ln\epsilon|^2}\|\varphi\|_\infty^2 \int_{S_\epsilon\setminus S_{\epsilon^2}} \frac{1}{\delta_E(X)^2} dX =: I_1 + I_2.$$

The term I_1 converges to 0, as $\nabla\varphi \in L^2(\mathbb{R}^n)$ and $|S_\epsilon| \to 0$. As for the term I_2, we use the Ahlfors regularity of $\partial\Omega$ to get that

$$|S_\eta \setminus S_{\eta/2}| \leq |S_\epsilon| \leq C\eta^{n-d}\sigma(E \cap K_\varphi)$$

where σ is the Ahlfors regular measure of E. Consequently,

$$I_2 \leq C\|\varphi\|_\infty^2 \frac{1}{|\ln\epsilon|^2}\sigma(E\cap K_\varphi) \sum_{2\ln_2\epsilon \leq k \leq \ln_2\epsilon} 2^{k(n-d-2)} \leq \frac{C}{|\ln\epsilon|}\|\varphi\|_\infty^2 \sigma(E\cap K_\varphi) \to 0$$

when ϵ tends to 0. The proposition follows. ☐

Let us define an appropriate notion of harmonic and elliptic measure on sets of high codimension. To begin, consider the set $\mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\, t = 0\}$, where $d < n-1$, which has a flat boundary. For this domain, we can readily construct acceptable solutions that share properties with those in $\mathbb{R}^{d+1}_+$. For instance, the Laplacian in

$\mathbb{R}^{d+1}_+$ will become $-\operatorname{div}|t|^{d-n}\nabla$ in $\mathbb{R}^n \setminus \mathbb{R}^d$. More generally, for a given uniformly elliptic operator L on $\mathbb{R}^{d+1}_+$, by leveraging the radial symmetry of the domain, we can find $\mathcal{L}$ such that our solutions $v(x,t)$ to $\mathcal{L}v = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^d$ to be such that $v(x,t) = u(x,|t|)$ and $Lu = 0$ in $\mathbb{R}^{d+1}_+$. The construction of such operators $\mathcal{L}$ is given by the following proposition.

Proposition 1.55 (Subsection 4.1 in [DFM19b]). *Let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in $\mathbb{R}^{d+1}_+$ ($m = \mathcal{L}^n$). Write*

$$A = \left[\begin{array}{c:c} A_{||} & B \\ \hdashline C & a_\perp \end{array}\right] = (a_{i,j})_{1\leq i,j\leq d+1}$$

for the coefficients of A. Assume that v is such that $v(x,t) = u(x,|t|)$ with $u \in W_{loc}(\mathbb{R}^{d+1}_+, \mathcal{L}^n)$ being a weak solution to $Lu = 0$ in $\mathbb{R}^{d+1}_+$. Then $v \in W_{loc}(\mathbb{R}^n \setminus \mathbb{R}^d, |t|^{d-n}dxdt)$ and is a weak solution to $\mathcal{L}v = -\operatorname{div}\mathcal{A}\nabla v = 0$, where

$$\mathcal{A} = |t|^{d+1-n}\left[\begin{array}{c:c} A_{||} & B\dfrac{t}{|t|} \\ \hdashline \dfrac{t^T}{|t|}C & a_\perp Id_{d-n} \end{array}\right] = |t|^{d+1-n}\left[\begin{array}{c:c} a_{i,j} & a_{i,d+1}\dfrac{t_j}{|t|} \\ \hdashline a_{d+1,j}\dfrac{t_i}{|t|} & \delta_{i,j}a_{d+1,d+1} \end{array}\right]. \tag{1.33}$$

In the above matrix expression, $t = (t_{d+1}, t_n)$ is an horizontal vector, $\delta_{i,j}$ is the Kronecker symbol, and the sizes of the bloc matrices are $d \times d$, $d \times (n-d)$, $(n-d) \times d$, $(n-d) \times (n-d)$ respectively. Moreover, in the definition of $\mathcal{A} = \mathcal{A}(x,t)$, we use the lightened notation $a_{i,j}$ for $a_{i,j}(x,|t|)$.

Let us now consider a domain $\Omega = \mathbb{R}^n \setminus \partial\Omega$, where $\partial\Omega$ is an Ahlfors regular set of dimension $d < n-1$ (with d is not necessarily an integer). The class of "uniformly elliptic" operators in Ω is relatively simple. Observing that the coefficent $|t|^{d-n} = \delta_{\partial\Omega}(X)^{n-d}$ naturally appears in (1.33), we define a uniformly elliptic operator $L = -\operatorname{div} A\nabla$ on Ω by the following conditions :

$$\begin{aligned} |A(X)\xi\cdot\zeta| &\leq C\delta_{\partial\Omega}(X)^{d+1-n}|\xi||\zeta| \quad \text{for } X \in \Omega \text{ and } \xi,\zeta \in \mathbb{R}^n; \\ A(X)\xi\cdot\xi &\geq C^{-1}\delta_{\partial\Omega}(X)^{d+1-n}|\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{1.34}$$

Those operators fit within the framework of elliptic theory given in the next chapter (see Subsection 2.1.2), enabling us to construct an elliptic measure and analyze the solvability of the Dirichlet problem in L^p.

We immediately have from the results of this chapter :

- Examples of elliptic operators in $\mathbb{R}^n \setminus \mathbb{R}^d$ for which the elliptic measure is A_∞-absolutely continuous with respect to the measure $\mathcal{L}^d = \mathcal{H}^d$ on $\mathbb{R}^d$, as shown by combining Proposition 1.55 and Theorems 1.30 and 1.37).
- Examples of elliptic operators in $\mathbb{R}^n \setminus \mathbb{R}^d$ for which the elliptic measure is singular, derived from Proposition 1.55 and the counterexamples in [MM81, CFK81]).

Our goals for the incoming chapters are :

1. To develop an elliptic theory that allows us to study boundary value problems when the boundary is too thin to be detected by the Laplacian.
2. To identify a "good operator" that will serve as the alternative of the Laplacian when studying the boundary value problems in $\Omega \setminus E$, where E is Ahlfors regular with dimension $d \leq n-2$.

2 - Elliptic theory for domains with mixed dimensional boundaries

2.1 . Elliptic theory : guiding examples

Let us present some examples where our elliptic theory, that will be introduced later, can be applied.

2.1.1 . Classical elliptic operators

Let $L := -\operatorname{div} A\nabla$ be a classical uniformly elliptic operator in the divergence form, that is A is a matrix valued function on a domain Ω that satisfies the elliptic and boundedness conditions

$$\begin{aligned} |A(X)\xi\cdot\zeta| \leq C|\xi||\zeta| \quad & \text{for } X\in\Omega \text{ and } \xi,\zeta\in\mathbb{R}^n; \\ A(X)\xi\cdot\xi \geq C^{-1}|\xi|^2 \quad & \text{for } X\in\Omega \text{ and } \xi\in\mathbb{R}^n. \end{aligned} \tag{2.1}$$

To have the rich elliptic theory provided by this manuscript, it is sufficient for $\partial\Omega$ to be d-Ahlfors regular with $d\in(n-2,n)$. Indeed, we just need to prove that, in such context, we have a boundary Poincaré inequality. The result is classical, but let us give a quick proof. Take $u\in C_0^\infty(\Omega)$ and a ball $B=B(x_B,r_B)$ centered on $\partial\Omega$. We write $\mathcal{B}(y,k)$ for the ball $B(2^{1-k}x_B+(1-2^{1-k})y, 2^{-k}r_B)$, $\mathcal{B}^*(y,k)$ for $B(y,2^{1-k}r_B)$, which contains both $\mathcal{B}(y,k)$ and $\mathcal{B}(y,k+1)$. We also write u_D for $\fint_D u\,dX$. For any $y\in\partial\Omega$, we have

$$|u_{B/2}| = |u_{B/2}-u(y)| \leq \sum_{k\geq 1}|u_{\mathcal{B}(y,k)}-u_{\mathcal{B}(y,k+1)}| \leq C\sum_{k\geq 1}(2^{-k}r_B)^{1-n}\int_{\mathcal{B}^*(y,k)}|\nabla u(Z)|\,dZ$$

thanks to the (classical) Poincaré inequality on balls. If we average the last estimate over $y\in B/2\cap\partial\Omega$, and if we write S_k for the set $\bigcup_{y\in\partial\Omega\cap B/2}B^*_{y,k}$ - whose measure is bounded by $C(r_B)^n 2^{-k(n-d)}$ due to the Ahlfors regularity of $\partial\Omega$ - we obtain

$$\begin{aligned} |u_{B/2}| &\leq C\sum_{k\geq 1}(2^{-k}r_B)^{1-n}\fint_{y\in B/2\cap\partial\Omega}\int_{\mathcal{B}^*(y,k)}|\nabla u(Z)|\,dZ\,d\sigma(y) \\ &\leq C(r_B)^{-d}\sum_{k\geq 1}(2^{-k}r_B)^{1+d-n}\int_{S(y,k)}|\nabla u(Z)|\,dZ \\ &\leq C(r_B)^{1-n/2}\sum_{k\geq 1}(2^{-k})^{1+(d-n)/2}\left(\int_{S(y,k)}|\nabla u(Z)|^2\,dZ\right)^{\frac{1}{2}} \\ &\leq C(r_B)^{1-n/2}\left(\int_B|\nabla u(Z)|^2\,dZ\right)^{\frac{1}{2}}\sum_{k\geq 1}(2^{-k})^{1+(d-n)/2} \\ &\leq Cr_B\left(\fint_B|\nabla u(Z)|^2\,dZ\right)^{\frac{1}{2}} \end{aligned}$$

as long as $1+(d-n)/2>0$, which is true if $d>n-2$. We conclude by saying

$$\fint_B|u(Z)|dZ \leq u_{B/2}+\fint_B|u(Z)-u_{B/2}|dZ \leq Cr_B\left(\fint_B|\nabla u(Z)|\,dZ\right)^{\frac{1}{2}}$$

due to our previous computations and the Poincaré inequality. The boundary Poincaré inequality requested for our elliptic theory to work follows.

2.1.2 . Complement of low dimensional Ahlfors regular set

In this paragraph, we take Ω to be $\mathbb{R}^n \setminus E$, where E is a Ahlfors regular set of dimension $d < n-1$ (d is not necessarily an integer). Note that, in this case, the domain Ω is well connected and thus automatically uniform. When $d \in (n-2, n-1)$, an elliptic theory can be obtained by using the classical elliptic operators defined above, but we also want to be able to treat $0 \leq d \leq n-2$. The simplest choice of "uniformly elliptic operators" would be in the form $L := -\operatorname{div} A\nabla$ where the matrix A is elliptic and uniform with respect to the weight $w(X) = \operatorname{dist}(X,E)^{d+1-n}$, that is

$$\begin{aligned} |A(X)\xi \cdot \zeta| \leq C \operatorname{dist}(X,E)^{d+1-n}|\xi||\zeta| & \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n; \\ A(X)\xi \cdot \xi \geq C^{-1} \operatorname{dist}(X,E)^{d+1-n}|\xi|^2 & \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.2}$$

Instead of living in the homogeneous space $\dot{W}^{1,2}(\Omega)$, the solutions exist in some weighted Sobolev spaces, with the weight as in (2.2), that is $w(X) = \operatorname{dist}(X,E)^{d+1-n}$. Note that, in the limit case $d = n-1$, we recover the classical uniformly elliptic operators. The elliptic theory in this context, and the fact that Ω is automatically uniform, is proved in [DFM21].

2.1.3 . Caffarelli and Silvestre fractional operators

We can get more freedom on the degeneracy of the operator. Indeed, assume that $L := -\operatorname{div} A\nabla$, where there exists γ such that

$$\begin{aligned} |A(X)\xi \cdot \zeta| \leq C \operatorname{dist}(X,E)^{\gamma}|\xi||\zeta| & \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n; \\ A(X)\xi \cdot \xi \geq C^{-1} \operatorname{dist}(X,E)^{\gamma}|\xi|^2 & \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.3}$$

Indeed, as long as the dimension d of the Ahlfors regular boundary $\partial\Omega \subset \mathbb{R}^n$ and the degeneracy γ of the weight $w(X) := \operatorname{dist}(X,E)^{\gamma}$ satisfy the estimate $|\gamma - (d+1-n)| < 1$, then our elliptic theory applies

Those operators can be seen as the generalization of the ones considered by Caffarelli and Silvestre in [CS07]. Let us talk a bit about it. Take $\Omega := \mathbb{R}^n_+ = \{(x,t) \in \mathbb{R}^n,\ t > 0\}$ (the upper plane), which means $\partial\Omega = \mathbb{R}^d$ for $d = n-1$. Caffarelli and Silvestre have shown that the effect of the fractional operator $(-\Delta)^s$ - for $s \in (0,1)$ and $\gamma := 2s-1 \in (-1,1)$ - on a function f defined on $\mathbb{R}^d$ can be seen as the trace of a Dirichlet to Neuman boundary value problem. That is, up to a harmless constant,

$$(-\Delta)^s f(x) = -\lim_{t\to 0} t^{\gamma} \frac{\partial u}{\partial t}, \tag{2.4}$$

where u is the solution to $L_\gamma := -\operatorname{div} t^{\gamma}\nabla u = 0$ in $\mathbb{R}^n_+$ with boundary data f. The operators L_γ, $\gamma \in (-1,1)$, are a special case of operators verifying (2.3) and will satisfy the assumptions for the elliptic theory that we are going to provide.

We can generalize some of this to the context of the complement Ω of a d-Ahlfors regular set $E \subset \mathbb{R}^n$, where the operator is for instance $L = -\operatorname{div} \operatorname{dist}(X,E)^{\gamma}\nabla$ with $|\gamma - (d+1-n)| < 1$. When $F \in C_0^\infty(\mathbb{R}^n)$, the Lax-Milgram theorem (Theorem 2.47) allows us to solve the Dirichlet problem with boundary data $f = F|_E$, i.e. find a function $u \in W^{1,2}(\Omega, \operatorname{dist}(X,\Gamma)^{-\gamma}dX)$ such that $Lu = 0$ and $u = F$ on E. Then we can define an operator T that generalizes the fractional Laplacian on the plane, by saying that Tf is a distribution on E such that

$$\langle Tf, \phi \rangle = \int_\Omega \operatorname{dist}(X,E)^{-\gamma}\, \nabla u(X) \cdot \nabla\Phi(X) dX, \tag{2.5}$$

where $\Phi \in C_0^\infty(\mathbb{R}^n)$ is a smooth extension of ϕ. A simple integration by parts and (2.4) show that, when E is an hyperplane, the distribution Tf is the fractional Laplacian on E.

2.1.4 . Boundary of mixed dimensions

There are two situations that easily appear when working with low dimensional Ahlfors regular boundaries. We take d-Ahlfors regular sets $E \subset \mathbb{R}^n$ with $d < n-1$ and we want to study solutions of degenerate elliptic - i.e. satisfying (2.2) - operators $L = -\operatorname{div} A\nabla$, but not in the full domain $\Omega := \mathbb{R}^n \setminus E$.

In the first scenario, instead of working in Ω, we want to work in $B \cap \Omega = B \setminus E$, where B is a ball that intersects E. In this case, the boundary of the new domain will be composed of the boundary of B (which is of dimension $n-1$) and the boundary of Ω (which is of a different dimension d).

In the second situation, we consider the so-called "saw-tooth" domains. An example of such domains is

$$\Omega_s := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ |t| > \varphi(x)\},$$

where $\varphi : \mathbb{R}^d \mapsto [0,\infty)$ is a Lipschitz function for which the set $\{x \in \mathbb{R}^d,\ \varphi(x) = 0\}$ has non-zero Lebesgue measure.

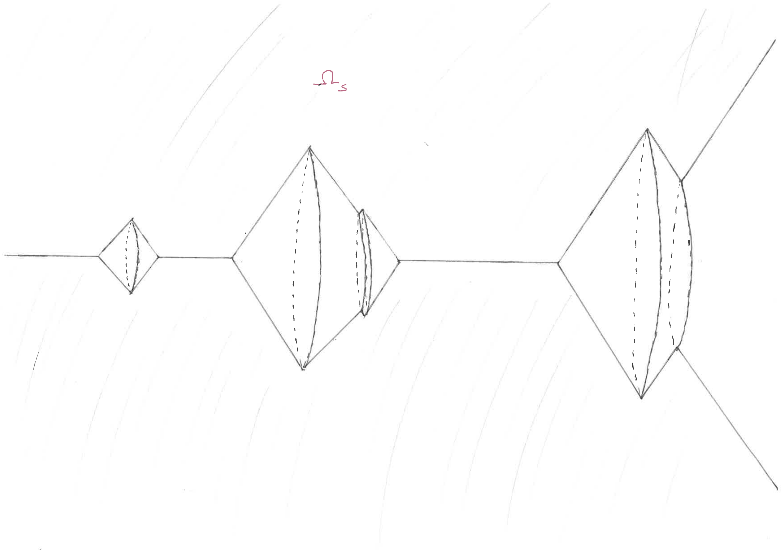

Figure 2.1 – A saw-tooth domain in $\mathbb{R}^3$ whose boundary is partially of dimension 1 and partially of dimension 2.

Our elliptic theory can be used in the two above cases, and the verification of our assumptions is done in [DFM20, Section 3].

2.1.5 . Nearly t-independent A_2-weights

The t-independent elliptic operators have a special status among divergence form operators, in particular because some control on the behavior of the coefficients in the direction transversal to the boundary is necessary for absolute continuity of harmonic measure with respect to the Lebesgue measure – see [CFK81, MM81].

Let $\omega : \mathbb{R}^{n-1} \to \mathbb{R}_+$ be any A_2-weight on $\mathbb{R}^d$ (see [Jou83, GCRdF85] for details). We use it to define a weight w on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times \mathbb{R}$ by $w(x,t) = \omega(x)$ (and then $dm(x,t) = w(x,t)dxdt$). We consider t-independent elliptic operators $L := -\operatorname{div} A\nabla$ where, as usual, A satisfies the ellipticity and boundedness condition relatively to $w(x,t)$, that is

$$\begin{aligned} |A(x,t)\xi\cdot\zeta| \leq C\omega(x)|\xi||\zeta| \quad & \text{for } (x,t)\in\mathbb{R}^n_+ \text{ and } \xi,\zeta\in\mathbb{R}^n; \\ A(x,t)\xi\cdot\xi \geq C^{-1}\omega(x)|\xi|^2 \quad & \text{for } (x,t)\in\mathbb{R}^n_+ \text{ and } \xi\in\mathbb{R}^n. \end{aligned} \tag{2.6}$$

We can construct solutions in the weighted Sobolev space $W^{1,2}(\mathbb{R}^n_+, w)$, and we refer to [DFM20, Subsection 3.6] for the proof of the fact that this situation satisfies the elliptic theory given in [DFM20], which requires stronger assumptions than the one given here.

2.2 . Our most general assumptions for the elliptic theory

In this section, we present the elliptic theory that will be used later to study boundary value problems. This theory is a contribution of the author, building upon and improving the elliptic framework previously developed in collaboration with Guy David and Svitlana Mayboroda, as outlined in [DFM21] and [DFM20]. The main innovation here is the integration of our earlier work with the elliptic theory developed under a capacity condition in [HMTon]. Our approach is largely inspired by the methods presented in [HKM93].

2.2.1 . Poincaré inequalities, traces, and weighted Sobolev spaces

The cornerstone in our theory is the careful selection of a weight w. From the weight w, we define operator $L = -\operatorname{div} A\nabla$ that are uniformly elliptic and bounded with respect to w, and a measure m whose Radon-Nikodym derivative with respect to the Lebesgue measure is the weight w.

Our goal is for the measure mm to be well adapted to both the domain Ω and its boundary $\partial\Omega$. By "adapted", we mean that m satisfies certain carefully chosen Poincaré inequalities on balls, ensuring regularity properties for solutions of L. Specifically, we require :

- A Poincaré inequality inside Ω, which guarantees Hölder continuity of solutions within the domain.
- A Poincaré inequality on the boundary, which ensures Hölder continuity up to $\partial\Omega$. This, in turn, plays a key role in constructing an associated elliptic measure and Green function.

We first focus on defining m. Since we require a Radon-Nikodym derivative, we will assume throughout this book that mm is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. This assumption, while convenient, is somewhat restrictive. In principle, we could (and eventually will) define a differentiation structure around m, allowing us to extend our theory to Riemannian manifolds, fractals, or graphs. However, for now, we choose to remain within the familiar setting of $\mathbb{R}^n$, postponing such generalizations for future work.

Although m is initially only required to be defined on Ω, we will assume it is also defined on all of $\mathbb{R}^n$. This assumption is both practical and natural in many cases - particularly when Ω is the complement of a set with empty interior. Extending m to all of $\mathbb{R}^n$ simplifies localization arguments : if m satisfies the desired properties in Ω, it will also retain these properties in any subdomain, making the analysis significantly more manageable.

We now outline the precise conditions we impose on m.

Definition 2.1 (derivative compatible). We say that a measure m is derivative compatible in Ω if the following property holds. Whenever $K \Subset \Omega$ is compact, $u_i \in C^\infty(K)$ is a sequence of functions such that $\int_K |u_i|\,dm \to 0$, and $\int_K |\nabla u_i - v|^2\,dm \to 0$ as $i \to \infty$ where v is a vector valued function in $L^2(K,m)$, we necessary have $v \equiv 0$.

Definition 2.2 (interior Poincaré inequality). We say that a measure m satisfies

- the interior doubling property if there exists $C > 0$ such that, for any ball B satisfying $2B \subset \Omega$, there holds
$$m(2B) \leq Cm(B)$$
- the interior Poincaré inequality if there exists $C > 0$ such that, for any ball B satisfying $2B \subset \Omega$ and any function $u \in C^\infty(\mathbb{R}^n)$, one has the weak Poincaré inequality
$$\fint_B |u - u_B|\,dm \leq Cr\left(\fint_{2B} |\nabla u|^2\,dm\right)^{\frac{1}{2}},$$
where u_B stands for $\fint_B u\,dm$ and r is the radius of B.

Definition 2.3 (suitable for PDE inside the domain). At last, we say that a measure m is suitable for PDE inside the domain Ω if m is absolutely continuous with respect to the Leguesgue measure, doubling inside Ω, derivative compatible in Ω, and satisfies the interior Poincaré inequality.

We are not particularly interested in rough weights w or measures m that are not comparable to the Lebesgue measure inside Ω. Instead, our focus lies in the opposite direction : identifying the optimal weight w and the most regular measure m that allow us to formulate and study boundary value problems effectively.

With this in mind, we will primarily work within the following, much simpler setting.

Proposition 2.4. *If m is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and if there exists $C > 0$ such that the Radon-Nykodym derivative satisfies*

$$\sup_B \frac{dm}{d\mathcal{L}^n} \leq C \inf_B \frac{dm}{d\mathcal{L}^n} \quad \textit{for any ball } B \textit{ such that } 2B \subset \Omega, \tag{2.7}$$

then m is suitable for PDE inside the domain.

Proof. The Lebesgue measure $\mathcal{L}^n$ is suitable for PDE inside the domain $\mathbb{R}^n$, and thanks to (2.7), the property immediately transfers to m. ☐

Our final condition ensures that m properly detects the boundary. A common approach relies on capacity, but we instead prefer to use Poincaré inequalities. In Subsection 2.2.3, we will demonstrate that, in the classical setting, these two approaches are equivalent.

Definition 2.5 (boundary Poincaré inequality). We say that a measure m satisfies

- the boundary doubling condition if there exists $C > 0$ such that for any $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$, we have $m(B(x,2r)) \leq Cm(B(x,r))$;
- the boundary Poincaré inequality if there exists $C > 0$ such that, for any $x \in \partial\Omega$, any $r \in (0, \operatorname{diam}\Omega)$, and any $u \in C_0^\infty(\Omega)$, we have
$$\fint_{B(x,r)} |u|\,dm \leq Cr\left(\fint_{B(x,2r)} |\nabla u|^2\,dm\right)^{\frac{1}{2}}.$$

Definition 2.6 (suitable for PDE)**.** We say that a measure m is suitable for PDE in Ω if m is suitable for PDE inside the domain Ω, and satisfies both the boundary doubling property and the boundary Poincaré inequality.

Remarks 2.7.

- It is unknown to the author whether the doubling property of m and the interior Poincaré inequality alone imply that m is absolutely continuous with respect to the Lebesgue measure or that it is derivative compatible in Ω.
- The assumption derivative compatible is essential for constructing a weighted Sobolev space. Without it, there is no *a priori* guarantee that a well-defined notion of derivative - acting as a linear and local operator - exists when completing $C^\infty(\overline{\Omega})$.
- The interior Poincaré inequality ensures that m behaves locally in a manner sufficiently similar to the Lebesgue measure. For example, this holds if w is in a A_2 Muckenhoupt weight.
- The boundary Poincaré inequality is the key condition that ensures m is adapted to the boundary $\partial\Omega$. In particular, if $\partial\Omega$ has a low dimension in $\mathbb{R}^n$, this condition forces the weight w to blow up to $+\infty$ near the boundary. This serves as our substitute for capacity considerations (see Subsection 2.2.3)
- The multiplicative constant 2 in the balls appearing on the right-hand sides of the interior Poincaré inequality and the boundary Poincaré inequality isarbitrary. Any constant $\kappa \geq 1$ could be used instead, yielding an equivalent definition of measures suitable for PDE. See in particular Theorem 2.10 below, which shows that replacing 2 by 1 leads to an equivalent formulation.

One of the fundamental properties of this elliptic theory is its stability under restriction to subdomains, as demonstrated in the next proposition.

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be open and m be suitable for PDE in Ω. Take $D \subset \mathbb{R}^n$ be an open set such that both D and $\mathbb{R}^n \setminus \overline{D}$ satisfy the corkscrew point condition. Then m is suitable for PDE in D.*

Remark 2.9. From the proposition, if $B(x,r)$ is a ball centered on $\partial\Omega$, then m is suitable for PDE in $\Omega \cap B(x,r)$. Hence we are able to use all the elliptic theory developed in the next subsection, in particular we can construct Green functions and elliptic measures on $\Omega \cap B(x,r)$.

Proof. The measure m is trivially derivative compatible in D and satisfies the interior Poincaré inequality (both come directly from the same properties on m). The existence of corkscrew points implies the porosity of ∂D, hence its zero Lebesgue measure. As a consequence, since m is absolutely continuous with respect to the Lebesgue measure, $m(\partial D \cap \Omega) = 0$. The (interior and boundaries) doubling properties of m are true for D since they are true for Ω.

It only remains to show that m satisfies the boundary Poincaré inequality. Take $x' \in \partial D$, $r > 0$, and $u \in C_0^\infty(D) \subset C_0^\infty(\Omega)$. If $\operatorname{dist}(x', \partial\Omega) < 2r$, then take $x \in \partial\Omega$ such that $|x - x'| < 2r$, and we have

$$\begin{aligned}\fint_{B(x',r)} |u| \, dm = \fint_{B(x',r)} |u| \, dm \leq \fint_{B(x,3r)} |u| \, dm \\ \leq Cr \left(\fint_{B(x,6r)} |\nabla u|^2 \, dm \right)^{\frac{1}{2}} \leq Cr \left(\fint_{B(x',8r)} |\nabla u|^2 \, dm' \right)^{\frac{1}{2}}\end{aligned}$$

because m is doubling and satisfies the boundary Poincaré inequality. We established a boundary Poincaré inequality with a scaling constant 8 instead of 2 for the balls in the integral in right-hand side, but it does not matter, because we can always self-improve the Poincaré inequalities and ultimately get 1 once we run the argument used to prove Theorem 2.10 below.

When $\operatorname{dist}(x', \partial\Omega) \geq 2r$, let $B' = B(X, \epsilon r)$ where X is an exterior corkscrew point for x' at scale r, and ϵ is the corkscrew constant of $\mathbb{R}^n \setminus D$. We have $u_{B'} = 0$ and thus

$$\begin{aligned}\fint_{B(x',r)} |u|\, dm = \fint_{B(x',r)} |u - u_{B'}|\, dm \\ \leq \fint_{B(x',r)} |u - u_{B(x',r)}|\, dm + \fint_{B'} |u - u_{B(x',r)}|\, dm \\ \leq C \fint_{B(x',r)} |u - u_{B(x',r)}|\, dm \leq Cr \left(\fint_{B(x',2r)} |\nabla u|^2\, dm' \right)^{\frac{1}{2}}\end{aligned}$$

by the doubling property of m and then the interior Poincaré inequality. ☐

Theorem 2.10 (Sobolev-Poincaré). *Let $\Omega \subset \mathbb{R}^n$. Let m satisfy the interior doubling property and the interior Poincaré inequality.*

(i) There exist $p_0 \in (1,2)$, $k > 1$, and $C > 0$ such that, for any $p \in [p_0, 2]$, any open ball B satisfying $2B \subset \Omega$, and any function $u \in C^\infty(\mathbb{R}^n)$, we have

$$\left(\fint_B |u - u_B|^{kp}\, dm \right)^{\frac{1}{kp}} \leq Cr \left(\fint_B |\nabla u|^p\, dm \right)^{\frac{1}{p}}, \tag{2.8}$$

where u_B stands for $\fint_B u\, dm$ and r is the radius of B.

(ii) If, in addition, m satisfies the boundary doubling property and the boundary Poincaré inequality, then there exist $p_0 \in (1,2)$, $k > 1$, and $C > 0$ such that, for any $x \in \partial\Omega$, any $r \in (0, \operatorname{diam}\Omega)$ and any function $u \in C_0^\infty(\Omega)$, we have

$$\left(\fint_{B(x,r)\cap\Omega} |u|^{kp}\, dm \right)^{\frac{1}{kp}} \leq Cr \left(\fint_{B(x,r)\cap\Omega} |\nabla u|^p\, dm \right)^{\frac{1}{p}}. \tag{2.9}$$

Proof. The improvement (i) is a combination of [KZ08, Theorem 1.0.1] and [HK00, Corollary 9.8]. See also [HK95, Theorem 1].

The second part (ii) is a variant. When Ω is uniform, the result is morally [DFM20, Corollary 7.9]. The author intended to write the proof without assuming that Ω is uniform, because the strategy is similar to [KZ08]. However, if we admit that boundary Poincaré inequalities are equivalent to a notion of capacity (see Subsection 2.2.3 for such equivalence when $m = \mathcal{L}^{n-1}$), then the result is [BMS01, Theorem 1.2]. ☐

With this powerful theorem in hand, we want to push the limits on the functions u that can satisfy (2.8) and (2.9).

Definition 2.11. Let $D \subseteq \Omega$ be an open set. A function u belongs to $W(D,m)$ if $u \in L^1_{loc}(D,m)$ and there exists a vector valued function $v \in L^2(D,m)$ such that, for some sequence $\{\varphi_i\}_{i\in\mathbb{N}} \in C^\infty(D)$, we have

$$\lim_{i\to\infty} \int_K |\varphi_i - u|\, dm = 0 \quad \text{for any compact } K \Subset D \tag{2.10}$$

and

$$\lim_{i\to\infty}\int_D |\nabla\varphi_i - v|^2\,dm = 0. \tag{2.11}$$

We write $u \in W(D,m;\ \text{locally in } E)$ if $u \in W(D\cap K, m)$ for any compact $K \Subset E$.

Remark 2.12. A sample of Sobolev spaces that we will use are :

- $W(\Omega,m)$ for global solutions, for instance given via the Lax-Milgram theorem ;
- when $B \subset \Omega$ is a ball, $W(B,m)$ or $W_{loc}(B,m) := W(B,m;\ \text{locally in } B)$ for local solutions inside the domain ;
- when B is a ball centered on the boundary, $W(B\cap\Omega,m)$ or $W(B\cap\Omega,m;\ \text{locally in } B)$ for local solutions with a trace ;
- $W(\Omega,m;\ \text{locally in } \mathbb{R}^n)$ for global solutions whose behavior at ∞ is not controlled - like Green functions with pole at infinity - but still have a trace ;
- $W_{loc}(\Omega,m) := W(\Omega,m;\ \text{locally in } \Omega)$ for global solutions without control at the boundary.

The notion of derivative is formally established in the following proposition.

Proposition 2.13. *If m is derivative compatible, then, for an open set $D \subsetneq \Omega$ and a given $u \in W(D,m)$, the function v in Definition 2.11 is unique and locally defined. We write ∇u for such v and we equip $W(\Omega,m)$ with the seminorm*

$$\|u\|_{W(\Omega,m)} = \|u\|_W := \|\nabla u\|_{L^2(\Omega,m)}. \tag{2.12}$$

Proof. Obvious. □

We emphasize that the mapping $u \mapsto \nabla u$ defines a linear operator, though it does not necessarily coincide with the gradient in the sense of distributions. An example illustrating this distinction can be found on page 13 of [HKM93]. However, as one might expect, in many cases, these two notions of differentiation do, in fact, coincide.

Proposition 2.14. *Let m be absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$.*

- *If the Radon-Nykodym derivative $w := dm/d\mathcal{L}$ belongs to the Muckenhoupt class $\mathcal{A}_2$ - that is if*

$$\sup_B \left(\fint_{B\cap\Omega} w\,dX\right)\left(\fint_{B\cap\Omega} w^{-1}\,dX\right) < +\infty, \tag{2.13}$$

  where the supremum is taken over the balls such that $2B \subsetneq D$ - then m is derivative compatible and, for any $u \in W(D,m)$, ∇u is the distribution gradient of u in D.
- *Let m be derivative compatible and $W^{1,\infty}_{loc}(D)$ be the space of locally Lipschitz functions. If $u \in W(D,m) \cap W^{1,\infty}_{loc}(D)$, then the two notions of gradient of u carried over from $W(D,m)$ and $W^{1,\infty}_{loc}(D)$ coincide.*

Proof. See page 14 and Lemma 1.11 (both) in [HKM93]. □

The next result addresses the fact that the approximating sequence in (2.10)–(2.11) can be chosen to preserve the same bounds as u.

Proposition 2.15. *Let $D \subsetneq \Omega$ be open and m be derivative compatible.*

Take $u \in W(D,m)$ such that $u \geq 0$ a.e. on D. Then there exists $\{\varphi_i\}_{i\in\mathbb{N}} \in C^\infty(D)$ that satisfies $\varphi_i \geq 0$ for all $i \in \mathbb{N}$, and (2.10)–(2.11).

Analogously, if $u \in W(D,m)$ is such that $L \leq u \leq M$ a.e. on D, then there exists $\{\varphi_i\}_{i\in\mathbb{N}} \in C^\infty(D)$ that satisfies $L \leq \varphi_i \leq M$ for all $i \in \mathbb{N}$, and (2.10)–(2.11).

Proof. We take $u \geq 0$, and let $\{\varphi_i\}_i$ be a function that approximates $u \in W$ in the sense of (2.10)–(2.11).

In the first case, we take $f_i \in C^\infty(\mathbb{R})$ such that $f_i \geq -1/i$, $f_i(t) = t$ when $t \geq 0$, and $|f_i'| \leq 1$. With similar computations as the ones used to prove Lemma 6.1 (a) in [DFM21], check that $\{f_i \circ \varphi_i + \frac{1}{i}\}_i$ - which is non-negative - also approximates u in the sense (2.10)–(2.11).

If u is essentially bounded, without loss of generality, we can take $L = 0$ and $M = 1$. We choose $f_i \in C^\infty(\mathbb{R})$ such that $-1/i \leq f_i \leq 1 + 1/i$, $f_i(t) = t$ when $t \in [0, 1]$, and $|f_i'| \leq 1$. We then check that $\{\frac{1}{1+2/i}(f_i \circ \varphi_i + \frac{1}{i})\}_i$ is a C^∞ approximating sequence of u that stays between 0 and 1. ☐

We also want to define the functions with zero trace.

Definition 2.16. We define $W(D, m;\ 0 \text{ on } F)$ as the subspace of $W(D, m)$ composed of the functions u for which we can find an approximating sequence $\{\varphi_i\}_{i \in \mathbb{N}} \in C^\infty(D)$ - that is the sequence satisfying (2.10)–(2.11) for some $v \in L^2(D, m)$ - such that $\operatorname{supp} \varphi_i \cap F = \emptyset$.

To lighten the notion, we write $W_0(\Omega, m)$ for $W(\Omega, m;\ 0 \text{ on } \partial\Omega)$.

Furthermore, the space $W(D, m;\ 0 \text{ on } F;\ \text{locally in } E)$ contains the functions that belong to $W(D \cap K, m;\ 0 \text{ on } F)$ for every compact set $K \subset E$.

With those definitions, we easily have :

Theorem 2.17 (Sobolev-Poincaré). *If m is suitable for PDE inside the domain, the estimate* (2.8) *is true for any $u \in W(B, m)$. Furthermore, if m is suitable for PDE, the bound* (2.9) *is true for any $u \in W(B \cap \Omega, m;\ 0 \text{ on } \partial\Omega \cap B)$. In any case, the parameter p_0 and the constants $k > 1$ and $C > 0$ do not change by extending to such a class of functions.*

Proof. Immediate consequence of Theorem 2.10 and our definitions of $W(D, m)$ and $W_0(D, m)$. ☐

Let us now state some basic properties of the Sobolev spaces that we constructed.

Proposition 2.18. *Let m be derivative compatible and satisfy the interior Poincaré inequality, and let $u \in W(\Omega, m)$. Then $\|u\|_W = 0$ if and only if, on each connected component of Ω, u is m-almost everywhere equal to a constant function.*

Proof. Use Lemma 4.10 in [DFM20] on each connected component of Ω. Note that, if Ω' is connected (and open), it means that we can always find a chain of ball linking any two points of Ω'. ☐

The spaces $W(D, m)$ and $W(D, m;\ 0 \text{ on } F)$ are compatible with the notion of cut-off functions. For instance, we have the following equality of spaces :

Proposition 2.19.
- *Assume that m is suitable for PDE inside the domain. Let B be a ball such that $2B \subset \Omega$. Then*

$$W_{loc}(B, m) = \{u \in L^1_{loc}(B) : \forall \phi \in C_0^\infty(B),\ u\phi \in W_0(\Omega, m)\}.$$

- *Assume that m is suitable for PDE. Let B be a ball centered on $\partial\Omega$. Then*

$$\begin{aligned} &W(B \cap \Omega, m;\ 0 \text{ on } \partial\Omega \cap B;\ \text{locally in } B) \\ &\qquad = \{u \in L^1_{loc}(B \cap \overline{\Omega}) : \forall \phi \in C_0^\infty(B),\ u\phi \in W_0(\Omega, m)\}. \end{aligned}$$

Proof. Let us prove only the first point, since the second one is similar.

($\subset$) Let $u \in W_{loc}(B,m)$. Let $\{\varphi_i\}_{i\in\mathbb{N}} \in C^\infty(B)$ that satisfies (2.10)–(2.11). Then the approximating sequence of $u\phi$ is $\varphi_i\phi \in C_0^\infty(\Omega)$ and $\nabla(\varphi_i\phi)$ converges to $\phi\nabla u + u\nabla\phi$ in $L^2(\Omega)$. Indeed, (2.10) implies the convergence of $\phi\nabla\varphi_i$ to $\phi\nabla u$ in L^2, while the convergence of $\varphi_i\nabla\phi$ to $u\nabla\phi$ is due to a combination of (2.10), (2.11), and Theorem 2.17.

($\supset$) Take $u \in \{u \in L^1_{loc}(B) : \forall\phi \in C_0^\infty(B),\, u\phi \in W_0(\Omega,m)\}$. Choose $\phi_j \in C_0^\infty(B)$ such that $\phi_j \equiv 1$ on $B_j := (1-1/j)B$. From the definition of $W_0(\Omega,m)$, we can construct $\varphi_{j,k} \in C^\infty(\Omega)$ such that

$$\int_{B_j} |u-\varphi_{j,k}|\, dm = \int_{B_j} |u\phi_j - \varphi_{j,k}|\, dm \leq \frac{1}{k}$$

and

$$\int_{B_j} |\nabla\varphi_{j,k} - \nabla u|^2\, dm \leq \int_\Omega |\nabla\varphi_{j,k} - \nabla(u\phi_j)|^2\, dm \leq \frac{1}{k}.$$

For each $i \leq j$, $\{\varphi_{j,k}\}_{k\geq j}$ is an approximating sequence for u in B_i, so the fact that m is derivative compatible gives that the $\{\nabla(u\phi_j)\}_{j>i}$ are all equal on B_i. So we can construct $v \in L^2_{loc}(B)$ such that, for $j \geq i$, $v := \nabla(u\phi_j)$ on B_i. The sequence $\varphi_{j,j} \in C^\infty(B)$ is then an approximating sequence of u in any compact $K \subset B$, which shows that $u \in W_{loc}(B,m)$. □

The space $W(\Omega,m)$ is complete 'up to constants', in the following sense.

Proposition 2.20. *Let $\Omega \subset \mathbb{R}^n$ be a connected open domain. Let m be suitable for PDE inside the domain. The quotient space $\dot{W}(\Omega,m) := W(\Omega,m)/\mathbb{R}$, equipped with the quotient norm $\|\cdot\|_W$, is complete. Also, if a sequence $\{u_k\}_{k=1}^\infty$ in $W(\Omega,m)$ and a function $u_\infty \in W(\Omega,m)$ are such that $\|u_k - u_\infty\|_W \to 0$, then there exist constants $c_k \in \mathbb{R}$ such that $u_k - c_k \to u$ in $L^1_{loc}(\Omega,m)$.*

In particular, if $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $W(\Omega,m)$ - in the sense that $\|u_i - u_j\|_W \to 0$ as $i,j \to \infty$ - and if $u_k \to u$ in $L^1_{loc}(\Omega,m)$, then $u \in W(\Omega,m)$ and $\|u_k - u\|_W \to 0$.

Proof. See Lemma 9.1 and 9.10 in [DFM20]. Instead of using T_{2Q^j} and $U^*_{Q^0}$ like in [DFM20], use a collection $\{\Omega_j\}_{j\in\mathbb{N}}$ of C^∞ precompact domains such that $\Omega_j \uparrow \Omega$. Since the domains Ω_j are C^∞ and m satisfies the interior Poincaré inequality, we still have a Poincaré inequality on Ω_j (possibly with large constants, but it does not matter), see for instance [HK95, Theorem 1]. □

As for the space $W_0(D,m)$, it is complete in the classical sense.

Proposition 2.21. *Let $\Omega \subset \mathbb{R}^n$. Let m be suitable for PDE. The space $W_0(\Omega,m)$ equipped with the inner product $\langle u,v\rangle_W := \int_\Omega \nabla u \cdot \nabla v\, dm$ is a Hilbert space.*

Proof. Without loss of generality, we can assume that Ω is connected. Then Theorem 2.17 shows that the constants are not in any $W_0(\Omega,m)$, so $\|.\|_W$ is a norm on $W_0(\Omega,m)$. The completeness under $\|.\|_W$ is now a simple consequence of the construction of $W_0(\Omega,m)$. □

Note the interesting following result.

Proposition 2.22. *Let $\Omega \subset \mathbb{R}^n$ and m be suitable for PDE. The space $C_0^\infty(\Omega)$ is dense in $W_0(\Omega,m)$.*

Proof. By definition of $W_0(\Omega,m)$ (and knowing now that $\|.\|_W$ is a norm), $W_0(\Omega,m)$ is the completion of the functions that lie in $W(\Omega,m)\cap C^\infty(\Omega)$ and have a support that does not intersect $\partial\Omega$. If Ω is bounded, it means that $W_0(\Omega,m)$ is the completion of $C_0^\infty(\partial\Omega)$. If Ω is unbounded, we need to make sure that we can choose the approximate functions so that their support is 0 at infinity. The proof of this part is the same as Part (ii) in the proof of [DFM21, Lemma 5.5]. □

We want now to give a notion of trace on $\partial\Omega$. We will stay vague at first on the space of traces, and a trace will simply be a map from $W(\Omega,m)$ to a space containing the functions on $\partial\Omega$ that satisfy the following properties.

Definition 2.23 (Trace). A trace $\mathrm{Tr}: W(\Omega,m)\to S\supset\mathcal{F}(\partial\Omega)$ is such that :

(i) if $\varphi\in C^\infty(\Omega)\cap W(\Omega,m)$, then $\mathrm{Tr}(\varphi)=\varphi|_{\partial\Omega}$;

(ii) if $u\in W_0(\Omega,m)$, then $\mathrm{Tr}\,u\equiv 0$;

(iii) Tr is a linear operator.

Moreover, if a trace exists on $W(\Omega,m)$, we can extend it to functions in $W(B\cap\Omega,m;\text{ locally in } B)$ - where B is a ball centered on $\partial\Omega$ - in the following manner. We say that $\mathrm{Tr}\,u=g$ on $B\cap\partial\Omega$ if for any $\varphi\in C_0^\infty(B)$, $\mathrm{Tr}(u\varphi)=g\varphi|_{\partial\Omega}$.

Remark 2.24. A few comments on our definition of traces.

- We will not prove here that our definition is well-posed. For example, we will not show that if $u\in W(\Omega,m)$, then $u|_{B\cap\Omega}\in W(\Omega\cap B)$ and $\mathrm{Tr}\,u|_{B\cap\Omega}=(\mathrm{Tr}\,u)|_{B\cap\partial\Omega}$ on $B\cap\partial\Omega$. These verifications are left to the reader. Computations similar to those are carried out in the proofs of [DFM21, Lemma 5.4, Lemma 6.1].
- The existence of an operator satisfying conditions (i) and (ii) is guaranteed only by the boundary Poincaré inequality inequality. To illustrate this, consider $\Omega=B_{\mathbb{R}^4}(0,1)\setminus D$, where D is a diameter of the ball $B_{\mathbb{R}^4}(0,1)$. When $m=\mathcal{L}^4$ is the Lebesgue measure, the function $1-|x|^2$ can be approximated by a sequence of functions in $C_0^\infty(\Omega)$ in the sense of (2.10)–(2.11) (this fact is left as an exercise). Thus $1-|x|^2\in W_0(\Omega,\mathcal{L}^4)$ and by (ii), we have $\mathrm{Tr}(1-|x|^2)=0$ on D. However, since $1-|x|^2\in C^\infty(\Omega)$, by (i), we should have $\mathrm{Tr}(1-|x|^2)=1-|x|^2\neq 0$ on D.
- Note that $\mathrm{Tr}\,u$ is not necessarily a function on $\partial\Omega$. For example, nothing in our assumptions prevents the space from being disconnected, as in $\mathbb{R}^n\setminus\mathbb{R}^{n-1}$, or not well connected, as in $\mathbb{R}^2\setminus\{(x,0)\in\mathbb{R}^2,\ x\leq 0\}$. In such cases, $\mathrm{Tr}\,u$ would conceptually represent the combination of the functions $\mathrm{Tr}_+\,u$ and $\mathrm{Tr}_-\,u$ on $\partial\Omega$ corresponding to the values of u approaching the boundary from above or below, respectively.
- The trace will be a function in $L^1_{loc}(\partial\Omega,\sigma)$ when (Ω,m,σ) is suitable for PDE and traces ; see Definition 2.32 below. In this context, we will be able to express $\mathrm{Tr}\,u$ as a function in the fractional Sobolev space $H(\Omega,m,\sigma)$ constructed in Definition 2.33.

Let us study the stability of $W(\Omega,m)$ and $W_0(\Omega,m)$.

Proposition 2.25. *Let $\Omega\subset\mathbb{R}^n$ and let m be derivative compatible. The following properties hold :*

(i) Let $f\in C^1(\mathbb{R})$ be such that f' is bounded, and let $u\in W(\Omega,m)$. Then $f\circ u\in W(\Omega,m)$, $\nabla(f\circ u)=f'(u)\nabla u$, and $\mathrm{Tr}(f\circ u)=\mathrm{Tr}(f\circ\bar{u})$ whenever $\mathrm{Tr}\,u=\mathrm{Tr}\,\bar{u}$.

(ii) Let $u, v \in W(\Omega, m)$*. Then* $\max\{u, v\}$ *and* $\min\{u, v\}$ *lie in* $W(\Omega, m)$*,*

$$\nabla \max\{u,v\}(x) = \begin{cases} \nabla u(x) & \textit{if } u(x) \geq v(x) \\ \nabla v(x) & \textit{if } v(x) \geq u(x), \end{cases}$$

$$\nabla \min\{u,v\}(x) = \begin{cases} \nabla u(x) & \textit{if } u(x) \leq v(x) \\ \nabla v(x) & \textit{if } v(x) \leq u(x). \end{cases}$$

Moreover, $\operatorname{Tr}(\max\{u, v\}) = \operatorname{Tr}(\max\{\bar{u}, \bar{v}\})$ *and* $\operatorname{Tr}(\min\{u, v\}) = \operatorname{Tr}(\min\{\bar{u}, \bar{v}\})$ *whenever* $\operatorname{Tr} u = \operatorname{Tr} \bar{u}$ *and* $\operatorname{Tr} v = \operatorname{Tr} \bar{v}$*.*

(iii) Let $u, v \in W(\Omega, m) \cap L^\infty(\Omega)$*. Then* $uv \in W(\Omega, m) \cap L^\infty(\Omega)$*, with* $\nabla[uv] = v\nabla u + u\nabla v$*, and* $\operatorname{Tr}(uv) = \operatorname{Tr}(\bar{u}\bar{v})$ *whenever* $\operatorname{Tr} u = \operatorname{Tr} \bar{u}$ *and* $\operatorname{Tr} v = \operatorname{Tr} \bar{v}$*.*

Each time, the identities on the gradients hold by seeing the two quantities as the same element in $L^2(\Omega, m)$ *(so they are equal almost everywhere).*

Remark 2.26. Again, by virtue of Proposition 2.19, the properties discussed above can be extended to the spaces $W_{loc}(B, m)$ and $W(B \cap \Omega, m;\ 0 \text{ on } B \cap \partial\Omega;\ \text{locally in } B)$, where B is either a ball satisfying $2B \subset \Omega$ or a ball centered on $\partial\Omega$.

Remark 2.27. Here is an example to clarify how we should interpret the statement on traces. If $\operatorname{Tr} u = 0$, then $\operatorname{Tr}(f \circ u) = f(0)$. Similarly, if $\operatorname{Tr}(u) = \varphi$ for some non-positive $\varphi \in C^\infty(\mathbb{R}^n)$, then $\operatorname{Tr}(\max\{u, 0\}) = 0$.

Proof. See paragraphs 1.18 to 1.23 in [HKM93], or Lemmas 6.1 and 6.3 in [DFM21]. The proof relies on the fact that every $u \in W(\Omega, m)$ can be approached by smooth functions. The statement on the traces is fairly easy : let us do (i) as an example. If $\operatorname{Tr} u = \operatorname{Tr} \bar{u}$, then we can find two smooth sequences φ_k and ψ_k that approximate - in the sense of Definition 2.11 - u and $\bar{u}$ respectively and such that $\varphi_k = \psi_k$ on a neighborhood of $\partial\Omega$. We approach f by a sequence of C^∞ functions such that $\|f' - f_k'\|_\infty \to 0$ and we observe that $f_k \circ \varphi_k$ and $f_k \circ \psi_k$ are approximating sequences of $f \circ u$ and $f \circ \bar{u}$ such that $f_k \circ \varphi_k = f_k \circ \psi_k$ on a neighborhood of Ω. That is $\operatorname{Tr}(f \circ u) = \operatorname{Tr}(f \circ \bar{u})$. ☐

In view of Remark 2.27, we can define a partial order on the trace as follows :

Definition 2.28. We say that $\operatorname{Tr} u \leq \operatorname{Tr} v$ if $\operatorname{Tr}(\max\{u, v\}) = \operatorname{Tr} v$ or equivalently if $\max\{u, v\} - v \in W_0(\Omega, m)$.

For $L, M \in \mathbb{R}$, we write $\operatorname{Tr} u \leq M$ if $\operatorname{Tr}(\max\{u, M\}) = M$ and $\operatorname{Tr} u \geq L$ if $\operatorname{Tr}(\min\{u, L\}) = L$.

If B is a ball centered at the boundary and if $u \geq 0$ a.e. in a neighborhood of $\partial\Omega \cap B$ (seen as a subset of $B \cap \overline{\Omega}$), then $\operatorname{Tr} u \geq 0$ on $\partial\Omega \cap B$.

Proposition 2.29. *Let* m *be derivative compatible. If* $u, v \in W(\Omega, m)$ *and* $u \leq v$ *a.e. on a neighborhood of* $\partial\Omega$*, then* $\operatorname{Tr} u \leq \operatorname{Tr} v$*.*

Proof. By linearity, we only need to prove that $u \leq 0$ on a neighborhood of $\partial\Omega$ implies $\operatorname{Tr}(u) \leq 0$. But $u \leq 0$ on a neighborhood of $\partial\Omega$ yields that $u - \min\{0, u\} \in W_0$, so $\operatorname{Tr}(u) = \operatorname{Tr}(\min\{0, u\})$, and thus we have $\operatorname{Tr}(\max\{0, u\}) = \operatorname{Tr}(\max\{0, \min\{0, u\}\}) = 0$ by Proposition 2.25. The latter means, by definition, that $\operatorname{Tr}(u) \leq 0$. The proposition follows. ☐

Proposition 2.30. *Let* m *be suitable for PDE. For* $u \in W(\Omega, m)$*, we define* $\max(\operatorname{Tr} u)$ *and* $\min(\operatorname{Tr} u)$ *as*

$$\max(\operatorname{Tr} u) := \min\{M \in \mathbb{R},\ \operatorname{Tr} u \leq M\} \textit{ and } \min(\operatorname{Tr} u) := \max\{L \in \mathbb{R},\ \operatorname{Tr} u \geq L\}.$$

Proof. We need to prove that the minimum in $M_\infty := \min\{M \in \mathbb{R},\, \operatorname{Tr} u \leq M\}$ exists. Let $M_k \geq M_\infty$ be such that $\operatorname{Tr} u \leq M_k$ and $M_k \to M_\infty$. Define $u_k := \max\{u, M_k\} - M_k$ and $u_\infty := \max\{u, M_\infty\} - M_\infty$. We have $\|u_\infty - u_k\|_W \to 0$ by the Lebesgue domination convergence (and (ii) of Proposition 2.25) and $u_k \in W_0$ by choice of M_k. Therefore $u_\infty \in W_0$ since W_0 is complete, that is $\operatorname{Tr} u \leq M_\infty$. The existence of the maximum $\max\{m \in \mathbb{R},\, \operatorname{Tr} u \geq m\}$ follows from the same arguments. ☐

2.2.2 . The measure σ on the boundary

The previous subsection provides the appropriate framework for the elliptic theory we want to develop. However, in many of our key examples, we will consider a measure on the boundary $\partial\Omega$. This is not surprising - in fact, it is essential - because the study of the Dirichlet problem in L^p requires extending functions from $L^p(\partial\Omega, \sigma)$ to the domain, ... and σ is a measure on $\partial\Omega$.

Proposition 2.31. *Let Ω be a uniform domain in $\mathbb{R}^n$. Let m be a doubling and derivative compatible measure on Ω that satisfies the interior Poincaré inequality. Let σ be a doubling measure on $\partial\Omega$. If there exist a small $\epsilon > 0$ and large $C > 0$ such that*

$$\frac{m(B(x,r)\cap\Omega)}{m(B(x,s)\cap\Omega)}\frac{\sigma(B(x,s))}{\sigma(B(x,r))} \leq C\left(\frac{r}{s}\right)^{2-\epsilon} \quad \text{for all } x \in \Omega,\ r \in (0, \operatorname{diam}\partial\Omega), \tag{2.14}$$

then m satisfies the boundary Poincaré inequality and hence is suitable for PDE.

Proof. The proof is based on the fact that, when Ω is uniform, we can find K and $T(x,r)$ such that $B(x,r)\cap\Omega \subset T(x,r) \subset B(x,Kr)$ and $T(x,r)$ is itself uniform (with constants independent of x and r). The latter is equivalent to the fact that $T(x,r)$ satisfies a $C(\kappa, M)$ chain condition and is proved in [DFM20, Lemma 5.23]. Now, since all the $T(x,r)$ are uniform (with the same constant), they satisfy a Poincaré inequality (with averages) with a uniform constant, see [DFM20, Theorem 5.24]. The proof of the boundary Poincaré inequality is then done as in the proof of [DFM20, Theorem 7.1]. ☐

In view of the above proposition, we define :

Definition 2.32 (suitable for PDE and traces)**.** We say that (Ω, m, σ) is suitable for PDE and traces if Ω, m, and σ satisfy the assumptions of Proposition 2.31, including (2.14).

We write $\rho(x,r)$ for the dimensionless quantity

$$\rho(x,r) := \frac{m(B(x,r)\cap\Omega)}{r\mu(B(x,r))}, \tag{2.15}$$

and we observe that (2.14) is equivalent to saying that

$$\frac{\rho(x,r)}{\rho(x,s)} \leq C\left(\frac{r}{s}\right)^{1-\epsilon} \quad \text{for all } x \in \partial\Omega,\ r \in (0, \operatorname{diam}\Omega). \tag{2.16}$$

Ideally, we have $C^{-1} \leq \rho(x,r) \leq C$, and so $\rho(x,r)$ is morally some error term whose growth needs to be controlled.

This quantity is a corrective term that will appear in the definition of the space of traces $H(\partial\Omega, \sigma, m)$.

Definition 2.33. The function $g \in H(\partial\Omega, \sigma, m)$ if g is σ-measurable on $\partial\Omega$ and $\|g\|_H < +\infty$, where

$$\|g\|_H = \|g\|_{H(\partial\Omega,\sigma,m)} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{\rho(x,|x-y|)^2 |g(x)-g(y)|^2}{m(B(x,|x-y|)\cap\Omega)} \, d\sigma(x)\, d\sigma(y) \right)^{\frac{1}{2}}.$$

If (Ω, m, σ) is suitable for PDE and traces, then we can identify the traces of functions in $W(\Omega, m)$ as functions in $H(\partial\Omega, \sigma, m)$. All this is given by the following nice theorem.

Theorem 2.34 (Trace Theorem). *Assume that (Ω, m, σ) is suitable for PDE and traces. Then Tr is a bounded linear operator from $W(\Omega, m)$ to $H(\partial\Omega, \sigma, m)$. That is, when $u \in W(\Omega, m)$, we can define a σ-measurable function $\mathrm{Tr}\, u$ on $\partial\Omega$ which satisfies the properties given in Definition 2.23 and for which we have $\|\mathrm{Tr}\, u\|_H \leq C\|u\|_W$ for some $C > 0$ independent of u.*

Moreover, the trace of $u \in W(\Omega, m)$ is such that

$$M_\epsilon(u)(x) := \fint_{B(x,\epsilon)\cap\Omega} |u - \mathrm{Tr}\, u(x)|\, dm \tag{2.17}$$

converges to 0 in $L^2_{loc}(\partial\Omega, \sigma)$. In particular, we have the Lebesgue density property

$$\lim_{\epsilon\to 0} \fint_{B(x,\epsilon)\cap\Omega} |u - \mathrm{Tr}\, u(x)|\, dm = 0 \qquad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{2.18}$$

and, if $\Gamma(y) := \Gamma_{unif}(y)$ are the cones constructed in (1.9) *with vertex at y, we have*

$$\mathrm{Tr}\, u(x) = \lim_{\epsilon\to 0} \fint_{B(x,\epsilon)\cap\Omega} u\, dm = \lim_{\substack{X\in\Gamma(x)\\ \delta(X)\to 0}} \fint_{B(X,\delta(X)/2)} u\, dm \qquad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega. \tag{2.19}$$

Proof. Let (Ω, m, σ) be suitable for PDE and traces. We first pick $X_{x,\epsilon}$ to be a corkscrew point for x at scale $\epsilon/2$. We can replace $X_{x,\epsilon}$ by any other point in $\Gamma_{unif}(x) \cap B(x,\epsilon)$ that is ϵ/C away from x. Let $B_{x,\epsilon} := B(X_{x,\epsilon}, \delta_{\partial\Omega}(X_{x,\epsilon})/2) \subset B(x,\epsilon)\cap\Omega$. We define

$$\mathrm{Tr}'_\epsilon(u)(x) := \fint_{B_{x,\epsilon}} u\, dm, \tag{2.20}$$

$$\mathrm{Tr}'(u)(x) := \lim_{\epsilon\to 0} \mathrm{Tr}'_\epsilon(u)(x),$$

and

$$M'_\epsilon(u)(x) := \fint_{B_{x,\epsilon}} |u - \mathrm{Tr}'(u)(x)|\, dm. \tag{2.21}$$

The bound (6.14) in [DFM20] shows that

$$M'_\epsilon(u) \text{ converges to 0 in } L^2_{loc}(\partial\Omega, \sigma) \tag{2.22}$$

and [DFM20, Theorem 6.6] shows that

$$\mathrm{Tr}' \text{ is a linear map from } W(\Omega, m) \text{ to } H(\partial\Omega, \sigma, m). \tag{2.23}$$

We proved in [DFM20, Lemma 9.18] that $W_0(\Omega, m)$ is the subspace of $W(\Omega, m)$ made of functions u verifying $\mathrm{Tr}'\, u = 0$[1]. This means that Tr' satisfies the condition (ii) of

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[1] In [DFM20], we defined W_0 as the subspace of $W = W(\Omega, m)$ with $\mathrm{Tr}'\, u = 0$, and steps (i) and (iii) of the proof of [DFM20, Lemma 9.18] show that W_0 is the completion of the smooth functions which are zero on a neighborhood of $\partial\Omega$, hence $W_0(\Omega, m)$

Definition 2.23; but since Tr' easily satisfies the conditions (i) and (iii), it entails that Tr' is a trace as given in Definition 2.23. Thus, for the sequel, we set $\mathrm{Tr} := \mathrm{Tr}'$ and $\mathrm{Tr}_\epsilon := \mathrm{Tr}'_\epsilon$.

It only remains to show that $M_\epsilon(u)$ converges to 0 in $L^2_{loc}(\partial\Omega, \sigma)$, since this convergence immediately implies (2.18) and (2.19), the two other statements that we have not demonstrated yet. To this end, we write

$$M_\epsilon(u) \leq \fint_{B(x,\epsilon)\cap\Omega} |u - \mathrm{Tr}_\epsilon(u)(x)|\, dm + |\mathrm{Tr}_\epsilon(u)(x) - \mathrm{Tr}(u)(x)|$$

$$\leq C\epsilon \left(\fint_{B(x,K\epsilon)\cap\Omega} |\nabla u|^2 dm \right)^{\frac{1}{2}} + M'_\epsilon(u)(x)$$

for some K independent of ϵ and x, thanks to Theorem 5.24, Lemma 5.23, and (5.15) in [DFM20]. Since both terms in the right-hand side above converge to 0 in $L^2_{loc}(\partial\Omega)$ - the first convergence is just by Fubini and the second one is (2.22) - then the convergence of $M_\epsilon(u)$ to 0 in $L^2_{loc}(\partial\Omega, \sigma)$ follows. □

We also prove that Tr is a surjective map.

Theorem 2.35 (Extension Theorem). *Let (Ω, m, σ) be suitable for PDE and traces. There exists a map Ext such that, for any $g \in L^1_{loc}(\partial\Omega, \sigma)$, we have*

$$\mathrm{Tr} \circ \mathrm{Ext}\, g = g \qquad \sigma\text{-a.e. in } \partial\Omega. \tag{2.24}$$

Moreover, Ext is a bounded linear operator from $H(\partial\Omega, \sigma, m)$ to $W(\Omega, m)$, i.e. there exists $C > 0$ such that, for any $g \in H(\partial\Omega, \sigma, m)$,

$$\|\mathrm{Ext}(g)\|_W \leq C\|g\|_H.$$

Finally, there exists $\mathrm{Ext}_k\, g \in W \cap C^\infty(\overline{\Omega})$ such that $\mathrm{Ext}_k\, g \to \mathrm{Ext}\, g$ in $L^1_{loc}(\Omega, m)$ and $\|\mathrm{Ext}_k\, g - \mathrm{Ext}\, g\|_W \to 0$.

Proof. Theorem 2.35 is mainly [DFM20, Theorem 8.5]. The only missing thing is the last part, that is the fact that $\mathrm{Ext}\, g$ can be approached by functions in C^∞ **up to the boundary**. With the notation from [DFM20, Section 8], we define $\mathcal{W}_k$ as the non-overlapping covering of Ω composed of all the dyadic cubes in $\mathcal{W}$ of side length $\ell(Q) \geq 2^{-k}$ completed with dyadic cubes of side length 2^{-k}. For a dyadic cube in $I \in \mathcal{W}_k \setminus \mathcal{W}$, we take similarly $\varphi_I \in C^\infty_0(I)$ such that $0 \leq \varphi_I \leq 1$, $|\nabla\varphi_I| \leq C2^{-k}$ and $\sum_{I\in\mathcal{W}_k} \varphi_I$, and we construct B_I, y_I like in [DFM20, Section 8]. We set $\mathrm{Ext}_k\, g(X) := \sum_{I\in\mathcal{W}_k} \varphi_I(X) y_I$, which is obviously a function in $C^\infty(\overline{\Omega})$. By construction, we have $\mathrm{Ext}_k\, g(X) = \mathrm{Ext}\, g(X)$ if $\delta_{\partial\Omega}(X) \geq K2^{-k}$ for K large enough, so in particular $\mathrm{Ext}_k\, g$ converges to $\mathrm{Ext}\, g$ in $L^1_{loc}(\Omega, m)$. In order to prove that $\|\mathrm{Ext}_k\, g - \mathrm{Ext}\, g\|_W \to 0$, we mimic the proof of [DFM20, Theorem 8.5] and we obtain

$$\|\mathrm{Ext}_k\, g - \mathrm{Ext}\, g\|^2_W \leq C \int_\Gamma \int_\Gamma \frac{\rho(x, |x-y|)^2 |g(x) - g(y)|^2}{m(B(x, |x-y|))} \mathbb{1}_{|x-y|\leq 100K2^{-k}} d\sigma(x)\, d\sigma(y).$$

The right-hand side above converges to 0 by the Lebesgue domination theorem when $k \to \infty$, as desired. □

The last part of the above theorem allows us to deduce that the restrictions to Ω of functions in $C^\infty(\mathbb{R}^n)$ are dense in $W(\Omega, m)$, as shown in the next proposition.

Proposition 2.36. *Let (Ω, m, σ) be suitable for PDE and traces. Then, $C^\infty(\mathbb{R}^n) \cap W(\Omega, m)$ is dense in $W(\Omega, m)$, that is, for any $u \in W(\Omega, m)$, we can choose $\{\varphi_i\}_{i\in\mathbb{N}}$ in* (2.10)–(2.11) *to lie in $C^\infty(\mathbb{R}^n)$ instead of $C^\infty(\Omega)$.*

As a consequence, functions in $H(\partial\Omega, \sigma, m)$ are well approximated by smooth functions. That is, for every $g \in H(\partial\Omega, \sigma, m)$, we can find a sequence of functions $(g_k)_{k\in\mathbb{N}} \in C^\infty(\mathbb{R}^n)$ whose restrictions to $\partial\Omega$ (we still call them g_k) belong to $H(\partial\Omega, \sigma, m)$ and such that g_k converges to g in $L^2_{loc}(\partial\Omega, \sigma)$ and σ-a.e., and such that $\|g_k - g\|_H \to 0$ as $k \to \infty$.

Proof. Proposition 2.36 is proved like [DFM20, Lemma 9.19], but we use the last statement of Theorem 2.35 to approach $\mathrm{Ext} \circ \mathrm{Tr}\, u$. The second part of the proposition is an easy consequence of the first part and Theorem 2.34, it is also [DFM20, Lemma 8.12]. ☐

At last, we finish by completing Proposition 2.25.

Proposition 2.37. *Let (Ω, m, σ) be suitable for PDE and traces. The following properties hold :*

(i) Let $f \in C^1(\mathbb{R})$ be such that f' is bounded, and let $u \in W(\Omega, m)$. Then $\mathrm{Tr}(f \circ u) = f \circ (\mathrm{Tr}\, u)$.

(ii) Let $u, v \in W(\Omega, m)$. Then $\mathrm{Tr}(\max\{u, v\}) = \max\{\mathrm{Tr}\, u, \mathrm{Tr}\, v\}$ and $\mathrm{Tr}(\min\{u, v\}) = \min\{\mathrm{Tr}\, u, \mathrm{Tr}\, v\}$.

(iii) Let $u, v \in W(\Omega, m) \cap L^\infty(\Omega)$. Then $\mathrm{Tr}(uv) = \mathrm{Tr}\, u \cdot \mathrm{Tr}\, v$ and lies in the space $H(\Omega, \sigma, m) \cap L^\infty(\Omega)$.

The identity of traces holds σ-a.e.

Proof. The proof is a simple variant of Lemma 6.1 in [DFM21] and Lemma 9.20 in [DFM20]. ☐

2.2.3 . Link to capacity

A classical setting for studying elliptic operators is when the domain satisfies the capacity condition. In this section, we will present this condition and explain how it connects to our theory. It would not be surprising if the results discussed here are already established in the literature, though we cannot pinpoint the exact references, and our results are relatively straightforward to prove.

There are many equivalent notions of capacity, and we have chosen the definition that the author has encountered most frequently in the literature.

Definition 2.38 (Capacity)**.** Given an open set $D \subset \mathbb{R}^n$, $n \geq 2$, and a compact set $K \Subset D$, we define the capacity of K relatively to D as the quantity

$$\mathrm{Cap}(K, D) := \inf\Big\{ \int_D |\nabla u(X)|^2 \, dX, \ u \in C_0^\infty(D), \ u \geq 1 \text{ on } K \Big\}. \tag{2.25}$$

We say that the domain $\Omega \subset \mathbb{R}^n$ satisfies the **capacity density condition** if there exists $c > 0$ such that

$$\frac{\mathrm{Cap}(\overline{B(x,r)} \setminus \Omega, B(x, 2r))}{\mathrm{Cap}(\overline{B(x,r)}, B(x, 2r))} \geq c \qquad \text{for } x \in \partial\Omega, \ 0 < r < \mathrm{diam}(\Omega). \tag{2.26}$$

Lemma 2.39. *Let $B \subset \mathbb{R}^n$ be an open ball of radius r_B. There exists c_n that depends only on n such that if, $K \Subset B$ is a compact set of (Lebesgue) measure $|K|$, then*

$$\mathrm{Cap}(K, B) \geq c\frac{|K|}{r_B^2}. \tag{2.27}$$

Moreover, there exists $c_n' > 0$ that depends only on n such that

$$\operatorname{Cap}(\overline{B}, 2B) = c_n' r_B^{n-2}. \tag{2.28}$$

Proof. Let $u \in C_0^\infty(B)$ be such that $u \geq 1$ on K. The Poincaré inequality entails that

$$\int_B |u|^2 \, dX \leq C r_B^2 \int_B |\nabla u|^2 \, dX,$$

where r_B is the radius of B and C depends only on n. Using the fact that $u \geq 1$ on K, we have

$$|K| \leq C r_B^2 \int_B |\nabla u|^2 \, dX$$

as desired.

The first part shows that $\operatorname{Cap}(\overline{B_0}, 2B_0) =: c_n' > 0$, where $B_0 = B(0,1)$. The general case follows from homogeneity. □

Lemma 2.40. *If $D \subset \mathbb{R}^n$ is open and $K \Subset D$ is compact,*

$$\operatorname{Cap}(K, D) = \inf \Big\{ \int_D |\nabla u(X)|^2 \, dX, \ u \in C^\infty(\mathbb{R}^n), \\ u = 0 \text{ on a neighborhood of } K, \ u \geq 1 \text{ on } \mathbb{R}^n \setminus D \Big\}. \tag{2.29}$$

Proof. We write $\widetilde{\operatorname{Cap}}(K, D)$ for the right-hand side of (2.29). We will only prove that $\widetilde{\operatorname{Cap}}(K, D) \leq \operatorname{Cap}(K, D)$, since the other inequality is entirely similar.

Let $(\eta_\delta)_{\delta > 0}$ be a mollifier, that is $\eta \in C_0^\infty(B(0,1))$, $\eta \geq 0$, $\int_{\mathbb{R}^n} \eta \, dX = 1$, and $\eta_\delta(X) = \delta^{-n} \eta(X/\delta)$. Given $u \in C_0^\infty(D)$, $u \geq 1$ on K, we construct

$$u_{\epsilon,\delta}(X) = \max\{1, (1+\epsilon)u\} * \eta_\delta \in C^\infty.$$

Check that for $\delta = \delta(\epsilon)$ small enough depending on ϵ and u, we have $u_{\epsilon,\delta} = 1$ on a neighborhood of K. We construct then $v_\epsilon := 1 - u_{\epsilon,\delta(\epsilon)}$, which belongs to $u \in C_0^\infty(\mathbb{R}^n \setminus K)$ and satisfies $u = 1$ on $\mathbb{R}^n \setminus D$. By the properties of mollifiers,

$$\limsup_{\epsilon \to 0} \int_D |\nabla v_\epsilon(X)|^2 \, dX \leq \iint_D |\nabla u(X)|^2 \, dX.$$

Taking the infimum on all the possible u, we obtain $\widetilde{\operatorname{Cap}}(K, D) \leq \operatorname{Cap}(K, D)$, as desired. □

A third lemma will provide us with a bit more flexibility to work with the capacity.

Lemma 2.41. *Let $B \subset$ be a ball, $K \subset \overline{B}$, $E \subset \overline{2B}$ be two compact sets such that $K \cap E = \emptyset$ and $|E| \geq \left|\frac{3}{2}B\right|$. Then there exists $C > 0$ independent of B, K and E such that*

$$\operatorname{Cap}(K, 2B) \leq C \operatorname{Cap}(K, \mathbb{R}^n \setminus E).$$

Proof. **Step 1 : Construction of the minimizer.** Set $\alpha := \operatorname{Cap}(K, \mathbb{R}^n \setminus E)$. Let u_j be a sequence of functions that reaches the infimum in the definition of $\widetilde{\operatorname{Cap}}(K, \mathbb{R}^n \setminus E)$, that is $u_j \in C^\infty(\mathbb{R}^n)$, $u = 0$ on a neighborhood of K, $u \geq 1$ on E, and

$$\alpha \leq \int_{\mathbb{R}^n} |\nabla u_j|^2 \, dX \leq \alpha + \frac{1}{j}.$$

Without loss of generality, we can even take u_j such that $0 \leq u_j \leq 1$. Up to a subsequence, for any compact $J \Subset \mathbb{R}^n$, u_j converges in $L^2(J)$ to a function u_∞, and ∇u_j converges to ∇u_∞ weakly in $L^2(\mathbb{R}^n \setminus E)$. The limit verifies $0 \leq u_\infty \leq 1$, $u_\infty = 1$ a.e. on E and

$$\int_{\mathbb{R}^n} |\nabla u_\infty|^2 \, dX \leq \inf \int_{\mathbb{R}^n} |\nabla u_j|^2 \, dX = \alpha.$$

Step 2 : The minimizer is superharmonic in $\mathbb{R}^n \setminus K$ and harmonic in $\mathbb{R}^n \setminus (E \cup K)$. Let φ be either in $C_0^\infty(\mathbb{R}^n \setminus (K \cup E))$ or a non-negative function in $C_0^\infty(\mathbb{R}^n \setminus K)$. Take any $\epsilon > 0$. We have

$$\begin{aligned} 2 \int_{\mathbb{R}^n} \nabla u_\infty \cdot \nabla \varphi \, dX &= \lim_{j \to \infty} 2 \int_{\mathbb{R}^n} \nabla u_j \cdot \nabla \varphi \, dX \\ &= \frac{1}{\epsilon} \left(\lim_{j \to \infty} \int_{\mathbb{R}^n} \nabla (u_j + \epsilon \varphi) \cdot \nabla (u_j + \epsilon \varphi) \, dX - \int_{\mathbb{R}^n} \nabla u_j \cdot \nabla u_j \, dX \right) - \epsilon \int_{\mathbb{R}^n} \nabla \varphi \cdot \nabla \varphi \, dX. \end{aligned}$$

Since $u_j + \varphi$ lies in $C^\infty(\mathbb{R}^n)$ and satisfies $u = 0$ on K and $u \geq 1$ on E, we have

$$\int_{\mathbb{R}^n} \nabla (u_j + \epsilon \varphi) \cdot \nabla (u_j + \epsilon \varphi) \, dX \geq \alpha \geq \int_{\mathbb{R}^n} \nabla u_j \cdot \nabla u_j \, dX - \frac{1}{j}.$$

As a consequence,

$$2 \int_{\mathbb{R}^n} \nabla u_\infty \cdot \nabla \varphi \, dX \geq \frac{1}{\epsilon} \liminf_{j \to \infty} \Big(- \frac{1}{j} \Big) - \epsilon \int_{\mathbb{R}^n} \nabla \varphi \cdot \nabla \varphi \, dX.$$

But since $\epsilon > 0$ is arbitrary, the right-hand side above can be as close to 0 as we want, so we conclude that

$$\int_{\mathbb{R}^n} \nabla u_\infty \cdot \nabla \varphi \, dX \geq 0$$

whenever φ is a nonnegative function in $C_0^\infty(\mathbb{R}^n \setminus K)$ or $\varphi \in C_0^\infty(\mathbb{R}^n \setminus (K \cup E))$. The first set of test functions gives that u_∞ is superharmonic in $\mathbb{R}^n \setminus K$, the second set of test functions implies that u_∞ is harmonic in $\mathbb{R}^n \setminus (K \cup E)$.

Step 3 : Conclusion We proved that u_∞ is a non-negative superharmonic function in $4B \setminus B$. Moreover,

$$|\{X \in 4B \setminus B, \, u(X) \geq 1\}| \geq |E \setminus B| \geq c|4B \setminus B|$$

by assumption. So a density lemma (variant of Lemma 2.65 in our much simpler context, see [HL11] for instance) gives

$$\inf_{3B \setminus 2B} u_\infty \geq M^{-1}$$

for a constant $M > 0$ independent of B, E and K.

We define $u_j' = (1 - \eta) u_\infty + \eta u_j$, where $\eta \in C^\infty(2B)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on B, and $|\nabla u| \leq 2/r_B$, we then can notice that

$$\int_{\mathbb{R}^n} |\nabla u_j'|^2 \, dX \leq 2 \int_{\mathbb{R}^n} |\nabla (u_\infty - u_j)|^2 + 8 r_B^{-2} \int_{4B} |u_\infty - u_j|^2 \, dX \leq C \Big(\alpha + \frac{1}{j} \Big)$$

where $C > 0$ is independent of j, B, K and E, and where we use the Poincaré inequality for the function $u_\infty - u_j$ (which is 0 on E).

Now, observe that $Mu'_j \in C^\infty(\mathbb{R}^n)$, $Mu'_j = 0$ on a neighborhood of K, $Mu'_j \geq 1$ on $4B \setminus 2B$. So, by definition of $\mathrm{Cap}(\overline{B} \setminus E, 2B)$, we have

$$\mathrm{Cap}(K, 2B) \leq M^2 \int_{2B} |\nabla u'_j|^2 \, dx \leq CM^2 \Big(\mathrm{Cap}(K, \mathbb{R}^n \setminus E) + \frac{1}{j} \Big).$$

Since j is arbitrary, the lemma follows. □

Let us turn to our objective of the paragraph.

Theorem 2.42. *Let $\Omega \subset \mathbb{R}^n$ be an open domain equipped with the Lebesgue measure. Then Ω satisfies the capacity density condition if and only if the Lebesgue measure on Ω is suitable for PDE.*

Proof. The Lebesgue measure on Ω is suitable for PDE in Ω if it satisfies the boundary Poincaré inequality. So we have to link the notion of capacity to the boundary Poincaré inequality.

The fact that $\mathcal{L}|_\Omega$ satisfies the boundary Poincaré inequality implies the capacity density condition. By translation and dilatation invariance, it suffices to prove (2.26) when $B(x, r) = B(0, 1) =: B_0$. Take $u \in C^\infty(\mathbb{R}^n)$, $u = 0$ on a neighborhood of $B_0 \setminus \Omega$, and $u \geq 1$ on $\mathbb{R}^n \setminus 2B_0$.

We assume that $\mathcal{L}|_\Omega$ satisfies the boundary Poincaré inequality, which means by (ii) of Theorem 2.10 below that

$$\int_{B_0} |u|^2 \, dX \leq C \int_{B_0} |\nabla u|^2 \, dX$$

for a constant $C > 0$ independent of u'. However, by using the classical Poincaré inequality on balls, we have

$$\int_{4B_0} |u - u_{B_0}|^2 \, dX \leq C' \int_{4B_0} |u - u_{4B_0}|^2 \, dX \leq C'' \int_{4B_0} |\nabla u|^2 \, dX.$$

Combining the two estimates gives

$$\int_{4B_0} |u|^2 \, dX \leq C''' \int_{4B_0} |\nabla u|^2 \, dX.$$

But since $u \equiv 1$ on $4B_0 \setminus 2B_0$, we get

$$\int_{2B_0} |\nabla u|^2 \, dX \geq c$$

for some $c > 0$ that depends only on n and the constant in the boundary Poincaré inequality. Taking the infimum on all the possible u gives

$$\mathrm{Cap}(B_0 \setminus \Omega, 2B_0) \geq c'$$

as desired.

The capacity density condition implies the fact that $\mathcal{L}|_\Omega$ satisfies the boundary Poincaré inequality. Let B be a ball centered on $\partial\Omega$ and of radius r, $u \in C_0^\infty(\overline{B} \cap \Omega)$. We want to prove that

$$\fint_{B \cap \Omega} |u| \, dX \leq Cr \left(\fint_{B \cap \Omega} |\nabla u|^2 \, dX \right)^{\frac{1}{2}}$$

with a constant $C > 0$ independent of B and u. By translation and dilatation invariance, we can assume that $B = B_0 := B(0,1)$. Moreover, we can assume that $u \geq 0$ and $\fint_B u\, dX = 1$. Under those assumptions, we want to prove that

$$\int_{B_0 \cap \Omega} |\nabla u|^2\, dX \geq c$$

where c depends only on n and a lower bound on $\operatorname{Cap}(\frac{1}{2}\overline{B_0} \setminus \Omega, B_0)$.

Using the classical Poincaré inequality (using average), we have

$$\left|\left\{X \in 2B_0,\, u < \frac{1}{2}\right\}\right| \leq 2 \int_{B_0 \cap \Omega} |u-1| dX \leq C \left(\int_{B_0 \cap \Omega} |\nabla u|^2\, dX \right)^{\frac{1}{2}},$$

so

$$\frac{\left|\left\{X \in B_0,\, u > \frac{1}{2}\right\}\right|}{|B_0|} \geq 1 - M \left(\int_{B_0} |\nabla u|^2\, dX \right)^{\frac{1}{2}}$$

for some $M > 0$ depending only on n.

Either

$$1 - M \left(\int_{B_0} |\nabla u|^2\, dX \right)^{\frac{1}{2}} \leq \frac{|3B_0/4|}{|B_0|}$$

and we have

$$\int_{B_0} |\nabla u|^2\, dX \geq \frac{1}{M^2}\Big(1 - \frac{|3B_0/4|}{|B_0|}\Big)^2;$$

or

$$1 - M \left(\int_{B_0} |\nabla u|^2\, dX \right)^{\frac{1}{2}} \geq \frac{|3B_0/4|}{|B_0|}$$

and then the ball $\frac{1}{2}B_0$, and the sets $K := \frac{1}{2}B_0 \setminus \Omega$ and $E := \left\{X \in B_0,\, u > \frac{1}{2}\right\}$ satisfy the assumption of Lemma 2.41, so we have

$$\operatorname{Cap}(K, B_0) \leq C \operatorname{Cap}(K, \mathbb{R}^n \setminus E) \leq 4C \int_{B_0 \cap \Omega} |\nabla u|^2\, dX$$

where the last inequality comes from the definition of the capacity for the function $2u$. In any case, we have

$$\int_{B_0 \cap \Omega} |\nabla u|^2\, dX \geq \min \left\{ \frac{1}{M^2}\Big(1 - \frac{|3B_0/4|}{|B_0|}\Big)^2, \frac{1}{4C} \operatorname{Cap}(\frac{1}{2}\overline{B_0} \setminus \Omega, B_0) \right\} > 0.$$

The theorem follows. □

2.3 . Basic elliptic theory

In this section, we take an open domain $\Omega \subset \mathbb{R}^n$ and a measure m on Ω which is - as always in our report - absolutely continuous with respect to the Lebesgue measure. We write w for the weight $dm/d\mathcal{L}^n$, i.e. $dm(X) = w(X) dX$ or

$$m(E) = \int_E w(X) dX \qquad \text{for any Borel } E \subseteq \Omega. \tag{2.30}$$

In this section, we study the solutions of $Lu = 0$, where $L = -\operatorname{div} A\nabla$ is an operator in divergence form.

Definition 2.43 (uniformly elliptic)**.** We say that $L = -\operatorname{div} A\nabla$ is a uniformly elliptic operator if the matrix A satisfies the elliptic and boundary conditions with respect to the weight w, that is there exists $C > 0$ such that

$$\begin{aligned} |A(X)\xi \cdot \zeta| &\leq C|\xi||\zeta|w(X) \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n; \\ A(X)\xi \cdot \xi &\geq C^{-1}w(X)|\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.31}$$

That is, the reduced matrix $\mathcal{A}(X) := A(X)/w(X)$ satisfies the classical elliptic conditions

$$\begin{aligned} |\mathcal{A}(X)\xi \cdot \zeta| &\leq C|\xi||\zeta| \quad \text{for } X \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n; \\ \mathcal{A}(X)\xi \cdot \xi &\geq C^{-1}|\xi|^2 \quad \text{for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.32}$$

The notion of weak solutions, subsolutions, and supersolutions is given in the weak sense as follows. But first, we need to introduce the space $W^{-1}(\Omega, m) = (W_0(\Omega, m))^*$ as the set of linear forms f on W_0 such that

$$\|f\|_{W^{-1}} := \sup_{\substack{\varphi \in W_0 \\ \|\varphi\|_W = 1}} \langle f, \varphi \rangle_{W^{-1}, W_0} < +\infty.$$

Definition 2.44 (solution)**.** Consider an open set $D \subseteq \Omega$ and a uniformly elliptic operator L on Ω. Let $f \in W^{-1}(\Omega, m)$. We say that $u \in W(D, m)$ is a weak solution to $Lu = f$ in D if

$$\int_D A\nabla u \cdot \nabla\varphi \, dX = \int_D \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = \langle f, \varphi \rangle \qquad \text{for } \varphi \in C_0^\infty(D). \tag{2.33}$$

We say that $u \in W(D, m)$ is a subsolution [resp. supersolution] (to $Lu = f$) if

$$\int_D A\nabla u \cdot \nabla\varphi \, dX = \int_D \mathcal{A}\nabla u \cdot \nabla\varphi \, dm \leq \text{ [resp. } \geq \text{] } \langle f, \varphi \rangle \qquad \text{for } \varphi \in C_0^\infty(D). \tag{2.34}$$

If we say that u is a weak solution (resp. subsolution, supersolution) without mentioning $Lu = f$, it means that $f \equiv 0$ and L is clear from context.

Remark 2.45. If $f \in L^\infty(\Omega)$ and $F \in L^\infty(\Omega, \mathbb{R}^n)$ are compactly supported in $\mathbb{R}^n$, we define the form

$$T_{f,F}(u) := \int_D [fu + F \cdot \nabla u] \, dm.$$

The boundary Poincaré inequality proves that $T_f \in W^{-1}(\Omega, m)$, so, from now on, for $\mathbb{R}^n$-compactly supported $f \in L^\infty(\Omega)$, we are allowed to talk about weak solutions to $Lu = fw + \operatorname{div}[wF]$ in Ω, which are such that

$$\int_\Omega A\nabla u \cdot \nabla\varphi \, dX = \int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = \int_D [fu + F \cdot \nabla u] \, dm \qquad \text{for } \varphi \in C_0^\infty(\Omega).$$

Notice that the assumption that f and F are bounded is far from optimal. For instance, Theorem 2.17 shows that we can take $f \in L^{2-\epsilon}(\Omega, m)$ for some $\epsilon > 0$.

Remark 2.46. Note that, since $W_0(\Omega, m)$ is the completion of smooth functions (when m is suitable for PDE), we can enlarge for free the set of functions φ that satisfies (2.33), and (2.34). Moreover, we only need to assume u to be in a local versions of the Sobolev spaces $W(D, m)$. So, in particular, we have (2.33)–(2.34) when

(i) $u \in W(\Omega, m)$ and $\varphi \in W_0(\Omega, m)$;

(ii) [if B is a ball verifying $2B \subset \Omega$:] $u \in W_{loc}(B, m)$ and $\varphi \in W(B, m)$ is such that $\operatorname{supp} \varphi \subset B$;

(iii) [if B is a ball centered at the boundary :] $u \in W(B \cap \Omega, m;$ locally in $B)$ and $\varphi \in W(B \cap \overline{\Omega}, m; 0$ on $B \cap \partial\Omega)$ is such that $\operatorname{supp} \varphi \subset B$.

The proof of this fact is done for instance in [DFM21, Lemma 8.3]. When subsolutions and supersolutions are involved, we need to invoke Proposition 2.15 for the proof.

2.3.1 . Existence of solutions

Our initial goal is to verify the existence of weak solutions. Indeed, such solutions exist and can be derived using the Lax-Milgram theorem.

Theorem 2.47 (Lax-Milgram Theorem). *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, and L be a uniformly elliptic operator.*

(a) For any $f \in W^{-1}(\Omega, m)$ and any $v \in W(\Omega, m)$, there exists a unique weak solution $u \in W(\Omega, m)$ to $Lu = f$ such that $\operatorname{Tr} u = \operatorname{Tr} v$. Moreover

$$\|u\|_W \leq C(\|f\|_{W^{-1}} + \|v\|_W).$$

(b) If (Ω, m, σ) is suitable for PDE and traces, then for any $f \in W^{-1}(\Omega, m)$ and any $g \in H(\partial\Omega, \sigma, m)$, there exists a unique weak solution $u \in W(\Omega, m)$ to $Lu = f$ such that $\operatorname{Tr} u = g$. Moreover,

$$\|u\|_W \leq C(\|f\|_{W^{-1}} + \|g\|_H).$$

In both cases, the constant $C > 0$ depends only on the constant in (2.31).

Proof. The Lax-Milgram theorem does not classically have the v or g boundary terms. We can include the boundary terms by using the extension theorem, see for instance [DFM21, Lemma 9.1]. ☐

Another straightforward approach to obtaining solutions is through the use of elliptic measures. These measures can be viewed as a consequence of the maximum principle. One key advantage of elliptic measures is that they do not require the space of traces $H(\partial\Omega, \sigma, m)$, enabling the construction of solutions without assuming the connectedness of the domain. Additionally, elliptic measures provide a simple method to handle solutions with highly irregular boundary data, as long as they are Borel measurable and bounded.

However, to establish desirable properties for these solutions derived from elliptic measures, we rely on the De Giorgi-Nash-Moser estimates, also known as the Hölder continuity of solutions. These estimates necessitate a well-developed elliptic theory, which will be presented in the following subsection.

Before we proceed with the maximum principle, let us recall a stability result from Stampacchia.

Lemma 2.48. *Let $D \subseteq \Omega$ be an open set, m be suitable for PDE, L be uniformly elliptic, and $f \in W^{-1}(\Omega, m)$. If $u, v \in W(D, m)$ are subsolutions [resp. supersolutions] to $Lu = f$ in D, then $\max\{u, v\}$ [resp. $\min\{u, v\}$] is also a subsolution [resp. supersolution] to $Lu = f$ on D.*

Remark 2.49. The above lemma will be used in the following situation : if u is a weak solution, then $\max\{u, 0\}$ and $|u|$ are subsolutions to $Lu = 0$, while $\min\{u, 0\}$ is a supersolution.

Proof. Lemma 2.48 is proved like [Sta65, Theorem 3.5] (see also [DFM21, Lemma 8.23]), since it only uses convex analysis. ☐

This stability is used to prove the following maximum principles :

Proposition 2.50. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, and L be uniformly elliptic. If $u \in W(\Omega, m)$ is a weak solution to $Lu = 0$, then*

$$\min(\operatorname{Tr} u) \leq u(X) \leq \max(\operatorname{Tr} u) \qquad \textit{for a.e. } X \in \Omega.$$

In addition, if u is a supersolution such that $\min(\operatorname{Tr} u) \geq 0$, then $u \geq 0$ almost everywhere.

Proof. See the proof of [DFM21, Lemma 9.2]. ☐

Corollary 2.51. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, and L be uniformly elliptic. If a closed set F and an open set E are such that $\operatorname{dist}(F, \mathbb{R}^n \setminus E) > 0$, then for any $u \in W(E \cap \Omega, m)$ satisfying*

(i) $\operatorname{Tr} u \geq 0$ *on* $\partial\Omega \cap E$ *and*

(ii) $u \geq 0$ *a.e. in* $(E \setminus F) \cap \Omega$,

we have $u \geq 0$ a.e. in $E \cap \Omega$.

Proof. See the proof of [DFM21, Lemma 11.3]. ☐

Remark 2.52. Corollary 2.51 is just a localized version of Proposition 2.50. Sometimes, it will be easier to use Corollary 2.51 instead of using Proposition 2.50 on a subdomain $E \cap \Omega \subsetneq \Omega$, because the later would require to check that $m|_{E \cap \Omega}$ is suitable for PDE.

Using Theorem 2.47 (Lax Milgram) and the continuity of solutions (Propositions 2.62 and 2.63), we can construct a bounded map U which sends $\varphi \in C_0^\infty(\mathbb{R}^n)$ - or more precisely its restriction to $\partial\Omega$ - to a solution to $Lu = 0$ in $W \cap C_b^0(\overline{\Omega})$ - the space of bounded functions that are continuous up to the boundary. Proposition 2.50 (Maximum Principle) shows that the map U is bounded from $(C_0^\infty(\mathbb{R}^n), \|.\|_\infty)$ to $(C_b^0(\overline{\Omega}), \|.\|_\infty)$. The density of $C_0^\infty(\mathbb{R}^n)$ in $C_0^0(\mathbb{R}^n)$ - the space of continuous and compactly supported functions on $\mathbb{R}^n$ - shows that we can extend by continuity the map U into one from $C_0^0(\mathbb{R}^n)$ to $C^0(\overline{\Omega})$. Since each function in $C_0^0(\partial\Omega)$ is the restriction to $\partial\Omega$ of a function in $C_0^0(\mathbb{R}^n)$, we can define a continuous map $U : (C_0^0(\partial\Omega), \|.\|_\infty) \mapsto (C_b^0(\Omega), \|.\|_\infty)$. As such, the Riesz representation lemma gives the existence of a (unique) positive Borel regular measure ω_L^X on $\partial\Omega$ such that

$$Ug(X) = \int_{\partial\Omega} g(y) d\omega^X(y)$$

for any $X \in \Omega$ and any $g \in C_0^0(\partial\Omega)$. We gather the properties of the elliptic measure ω_L^X in the next theorem.

Theorem 2.53. *Let $\Omega \subset \mathbb{R}^n$, m be suitable for PDE, and L be uniformly elliptic. There exists a family of positive Borel regular measures $\{\omega_L^X\}_{X \in \Omega}$ on $\partial\Omega$ - we write ω^X or ω_L^X for short - such that, for any $g \in C_0^\infty(\mathbb{R}^n)$, the function u_g constructed as*

$$u_g(X) := \int_{\partial\Omega} g(y) d\omega^X(y), \quad X \in \Omega, \tag{2.35}$$

is the unique weak solution in $W(\Omega, m)$ to $Lu = 0$ in Ω with $\operatorname{Tr} u = g|_{\partial\Omega}$ given by Theorem 2.47. Moreover, ω^X is called the **elliptic measure** *of L on $\partial\Omega$ and enjoys the following properties :*

(i) The elliptic measure is Borel regular, that is, for any Borel set $E \subseteq \partial\Omega$,

$$\omega^X(E) = \sup\{\omega^X(K) : E \supset K,\ K \textit{compact}\} = \inf\{\omega^X(V) : E \subseteq V,\ V \textit{ open}\}.$$

(ii) ω^X is a probability measure, i.e. $\omega^X(\partial\Omega) = 1$ and, for any Borel set $E \subseteq \partial\Omega$, we have $\omega^X(E) = 1 - \omega^X(\partial\Omega \setminus E)$.

(iii) If Ω is connected, the measures ω^X are all absolutely continuous with respect to each other, i.e. if $\omega^X(E) = 0$ for a Borel set E and a pole $X \in \Omega$, then $\omega^Y(E) = 0$ for all $Y \in \Omega$.

Let $E \subseteq \partial\Omega$ be a Borel set and let us write u_E for the function defined on Ω by $u_E(X) = \omega^X(E)$.

(iv) The function u_E lies in $W_{loc}(\Omega, m)$ and is a weak solution to $Lu = 0$ in Ω.

(v) If $B \subset \mathbb{R}^n$ is a ball centered at the boundary such that $B \cap E = \emptyset$, then $u_E \in W(B \cap \Omega, m;\ 0 \text{ on } B \cap \partial\Omega;\ \text{locally in } B)$. In particular $\operatorname{Tr} u_E = 0$ on $\partial\Omega \cap B$.

(vi) Similarly, if $B \subset \mathbb{R}^n$ is a ball centered at the boundary such that $E \Subset B$, then $u_E \in W(\Omega \setminus \overline{B}, m;\ 0 \text{ on } \partial\Omega \setminus \overline{B})$. In particular $\operatorname{Tr} u_E = 0$ on $\partial\Omega \setminus \overline{B}$.

Proof. See Lemmas 12.15 and 12.19 in [DFM20] (see also Lemmas 9.6 and 9.7 in [DFM21]) except for (vi), but the proof of the latter is very similar to (v). ☐

Given an elliptic measure $\omega^X_{L,\Omega}$, one can construct numerous weak solutions using the formula (2.35), as demonstrated in the following corollary.

Corollary 2.54. *Let $\Omega \subset \mathbb{R}^n$, m be suitable for PDE and L be uniformly elliptic. If $g \in C^0_b(\partial\Omega)$ - the space of continuous and bounded functions on $\partial\Omega$ - then the function u_g constructed as in* (2.35) *lies in $W_{loc}(\Omega, m)$, is a weak solution to $Lu = 0$ in Ω and can be extended by continuity to $\overline{\Omega}$ and the extension $\overline{u_g}$ satisfies $\overline{u_g}|_{\partial\Omega} = g$.*

Moreover, if B is a ball centered at the boundary such that $\operatorname{supp} g \Subset B$, then $u_g \in W(\Omega \setminus B, m;\ 0 \text{ on } \partial\Omega \setminus B)$. Similarly, if B is a ball centered at the boundary such that $\operatorname{supp} g \subset \partial\Omega \setminus \overline{B}$, then $u_g \in W(B \cap \Omega, m;\ 0 \text{ on } B \cap \partial\Omega)$.

Proof. Corollary 2.53 is Lemma 12.13 in [DFM20] (see also Lemma 9.4 [DFM21]) when g is compactly supported. For the general case, we approach $g \in C^0_b(\partial\Omega)$ uniformly by step functions and the proof is similar and easier than (iv)-(vi) of Theorem 2.53. ☐

2.3.2 . Interior and boundary estimates for solutions

The elliptic theory for degenerate elliptic equations has been explored in various works, including [Anc90, FKS82, FJK82]. Our presentation below builds upon and slightly enhances the content from [DFM17], [DFM21], and [DFM20], drawing inspiration from the proofs in [GT01], [HL11], and [GW82].

We begin with the Caccioppoli inequality, which essentially states that if u is a solution, we have an inverse for the interior Poincaré inequalityand the boundary Poincaré inequalityinequalities.

Proposition 2.55 (interior Caccioppoli inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE inside the domain and L be uniformly elliptic.*

If $E \subset \Omega$ is an open set and $u \in W(E, m)$ is a non-negative subsolution in E, then, for any $\alpha \in C^\infty_0(E)$,

$$\int_\Omega \alpha^2 |\nabla u|^2 dm \le C \int_\Omega |\nabla\alpha|^2 u^2 dm, \tag{2.36}$$

where $C > 0$ depends only on the constant in (2.31).

In particular, if B is a ball of radius r such that $2B \subset \Omega$ and $u \in W(2B, m)$ is a non-negative subsolution in $2B$, then

$$\int_B |\nabla u|^2 dm \leq Cr^{-2} \int_{2B} u^2 dm. \tag{2.37}$$

Proposition 2.56 (Caccioppoli inequality on the boundary)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE and L be uniformly elliptic.*

If $E \subset \mathbb{R}^n$ is an open set and $u \in W(E \cap \Omega, m,\ 0 \text{ on } E \cap \partial\Omega)$ is a non-negative subsolution in E, then, for any $\alpha \in C_0^\infty(E)$,

$$\int_\Omega \alpha^2 |\nabla u|^2 dm \leq C \int_\Omega |\nabla \alpha|^2 u^2 dm, \tag{2.38}$$

where $C > 0$ depends only on the constant in (2.31).

In particular, if B is a ball of radius r centered at the boundary and $u \in W(2B \cap \Omega, m;\ 0 \text{ on } 2B \cap \partial\Omega)$ is a non-negative subsolution in $2B \cap \Omega$, then

$$\int_{B \cap \Omega} |\nabla u|^2 dm \leq Cr^{-2} \int_{2B \cap \Omega} u^2 dm. \tag{2.39}$$

Remarks 2.57.
- Instead of taking $2B$ in the left hand side of (2.37), we can take λB for any $\lambda > 1$. In this case, the constant C depends also on λ and will explode as $\lambda \to 1$.
- Note that, if $u \in W(E, m)$ is a weak solution, then $|u|$ is a subsolution by Lemma 2.48, and $|\nabla |u|| = |\nabla u|$ a.e. on E by Proposition 2.25. So the statements in Propositions 2.55 and 2.56 remain true when u is a weak solution, even when they are not nonnegative.

Proofs of Proposition 2.55 and Proposition 2.56 are simple and proved in for instance [DFM21, Lemmas 8.6 and 8.11]. ☐

One consequence of the Caccioppoli inequality is the Moser estimates, which essentially allow for the equivalence of L^p-norms of solutions across different values of p. Notably, solutions that are initially assumed to be in $L^{2+\epsilon}$ (as per Theorem 2.17) are actually locally bounded.

Proposition 2.58 (interior Moser estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE inside the domain, L be uniformly elliptic, $p > 0$. There exists $C = C_p > 0$ such that, if B is a ball of radius r such that $2B \subset \Omega$, $f \in L^\infty(\Omega)$, $F \in L^\infty(\Omega, \mathbb{R}^n)$ and $u \in W(2B, m)$ is a non-negative subsolution to $Lu = fw + \operatorname{div}[Fw]$ in $2B$, then*

$$\sup_B u \leq C \left(\fint_{2B} u^p \, dm \right)^{\frac{1}{p}} + Cr^2 \|f\|_{L^\infty(2B)} + Cr \|F\|_{L^\infty(2B)}. \tag{2.40}$$

Proposition 2.59 (Moser estimates on the boundary)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, L be uniformly elliptic, and $p > 0$. There exists $C = C_p > 0$ such that, if B is a ball centered on $\partial\Omega$ and $u \in W(2B \cap \Omega, m;\ 0 \text{ on } 2B \cap \partial\Omega)$ is a non-negative subsolution to $Lu = 0$ in $2B \cap \Omega$, then*

$$\sup_{B \cap \Omega} u \leq C_p \left(\fint_{2B \cap \Omega} |u|^p dm \right)^{\frac{1}{p}}. \tag{2.41}$$

Remarks 2.60.

- As for the Caccioppoli inequalities, in the above propositions, we can replace $2B$ by λB for $\lambda > 1$. The constant C will then depend on λ and explode when $\lambda \to 1$.
- In Proposition 2.58, we can actually allow $f \in L^q(\Omega, m)$ for $q > k/(k-1)$, where k is the parameter in Theorem 2.10. In this case, $\|f\|_{L^\infty(2B)}$ needs to be replaced by $\fint_{2B} f^q \, dm$.

Proofs of Proposition 2.59 and Proposition 2.58 when $f = F = 0$ can be found in [KS00, Theorems C.4 and C.5], [FKS82, Lemmas 2.3.1 and 2.4.1], [DFM20, Lemmas 11.18 and 11.20], [DFM21, Lemmas 8.7 and 8.12]. □

Proof of the inhomogeneous version of Proposition 2.58. Our proof will be similar to the one of Theorem 8.17 in [GT01], so we only give the outline of the proof. Pick $\epsilon > 0$ and set $\mathfrak{f} := r^2\|f\|_{L^\infty(2B)} + r\|F\|_{L^\infty(2B)} + \epsilon$. We write $\bar{u}$ for $u + \mathfrak{f}$, and then v for $\eta^2 \bar{u}^\beta$ for some $\eta \in C_0^\infty(2B)$ and $\beta \in (0, \infty)$. We have

$$\nabla v = 2\eta \bar{u}^\beta \nabla \eta + \beta \eta^2 \bar{u}^{\beta-1} \nabla u$$

and, assuming that $\bar{u}^\beta \in W_{loc}(2B, m)$, one gets $v \in W_0(\Omega, m)$ and, using v as a test function against the weak solution u, we have

$$\beta \int_\Omega \mathcal{A}\nabla u \cdot \nabla u \, \bar{u}^{\beta-1} \eta^2 \, dm + 2 \int_\Omega \mathcal{A}\nabla u \cdot \nabla \eta \, \bar{u}^\beta \eta \, dm \leq \int_\Omega f \eta^2 \bar{u}^\beta \, dm. \tag{2.42}$$

Using the ellipticity of $\mathcal{A}$, the Cauchy-Schwarz inequality, the boundedness of $\mathcal{A}$, and the fact that $|f| \leq |\bar{u}|/r^2$ on $\operatorname{supp} \eta$, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^2 \bar{u}^{\beta-1} \eta^2 \, dm &\leq C \int_\Omega |\nabla u| \bar{u}^\beta \Big(\eta \nabla \eta + \frac{\eta^2}{r}\Big) + C \int_\Omega \Big(\frac{\eta^2}{r^2} + \frac{\eta \nabla \eta}{r}\Big) \bar{u}^{\beta+1} \, dm \\
&\leq C \left(\int_\Omega |\nabla u|^2 \bar{u}^{\beta-1} \eta^2 \, dm\right)^{\frac12} \left(\int_\Omega \bar{u}^{\beta+1} \Big(\frac{\eta^2}{r^2} + |\nabla \eta|^2\Big) \, dm\right)^{\frac12} \\
&\qquad + C \int_\Omega \Big(\frac{\eta^2}{r^2} + |\nabla \eta|^2\Big) \bar{u}^{\beta+1} \, dm,
\end{aligned}$$

that is

$$\int_\Omega |\nabla u|^2 \bar{u}^{\beta-1} \eta^2 \, dm \leq C \int_\Omega \bar{u}^{\beta+1} \Big(\frac{\eta^2}{r^2} + |\nabla \eta|^2\Big) \, dm. \tag{2.43}$$

If $\mathfrak{u}(x) := \bar{u}^{(\beta+1)/2}$, we can rewrite (2.43) as

$$\int_\Omega |\nabla \mathfrak{u}|^2 \eta^2 \, dm \leq C \int_\Omega \mathfrak{u}^2 \Big(\frac{\eta^2}{r^2} + |\nabla \eta|^2\Big) \, dm.$$

If $k > 1$ is the parameter from Theorem 2.10, we can write

$$\left(\fint_{2B} |\mathfrak{u}|^{2k} \eta^{2k} \, dm\right)^{1/2k} \leq C \left(\fint_{2B} \mathfrak{u}^2 (\eta^2 + r^2 |\nabla \eta|^2) \, dm\right)^{\frac12}.$$

Now, we pick $1 \leq \kappa_1 < \kappa_2 \leq 2$ and we construct η such that $\eta \in C^\infty(\kappa_2 B)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $\kappa_1 B$ and $|\nabla \eta| \leq Cr/(\kappa_2 - \kappa_1)$. We deduce

$$\left(\fint_{\kappa_1 B} |\mathfrak{u}|^{2k} \, dm\right)^{1/2k} \leq \frac{C}{\kappa_2 - \kappa_1} \left(\fint_{\kappa_2 B} \mathfrak{u}^2 \, dm\right)^{\frac12}.$$

We write $\Phi(p,\kappa) := \left(\fint_{\kappa B} |\bar u|^p \, dm\right)^{1/p}$ and so we proved that, for $1 \leq \kappa_1 < \kappa_2 \leq 2$ and $p \in (1,+\infty) \setminus \{1\}$,

$$\Phi(kp,\kappa_1) \leq \left(\frac{C}{\kappa_2 - \kappa_1}\right)^{\frac{2}{p}} \Phi(p,\kappa_2) \tag{2.44}$$

provided that $(\bar u)^{p-1} \in W_{loc}(2B,m)$. The later is automatically true for $p \leq 2$, since $\bar u > \mathfrak{f} > 0$. If $p > 2$, then we actually need to rerun the argument with $\bar u_N := \min\{\bar u, N\}$ and $v_N = \eta^2 \bar u (\bar u_N)^{p-2}$, using the p-ellipticity in order to prove that

$$\int_\Omega |\nabla u|^2 (\bar u_N)^{p-2} \eta^2 \, dm \leq C \int_\Omega \mathcal{A} \nabla u \cdot \nabla[\bar u (\bar u_N)^{p-2}] \eta^2;$$

see for instance (3.4) in [FMZ21]. Ultimately, we get the bound

$$\left(\fint_{\kappa_1 B} \bar u^2 (\bar u_N)^{kp-2} \, dm\right)^{1/pk} \leq \left(\frac{C}{\kappa_2 - \kappa_1}\right)^{2/p} \left(\fint_{\kappa_2 B} \bar u^2 (\bar u_N)^{p-2} \, dm\right)^{\frac{1}{2}}$$

with a constant C independent of N; see (3.17) in [FMZ21]. Taking $N \to \infty$ gives (2.44).

We take $\kappa_i = \frac{3}{2} - 2^{-i}$ (note that $\kappa_1 = 1$) and, by iterating (2.47)–(2.44), we have, for all $m \in \mathbb{N}$ and all $p > 0$[2],

$$\Phi\Big(k^m p, \frac{1}{2}\Big) \leq \left\{ \prod_{i=0}^{m-1} (C 2^i)^{2\kappa^{-i}/p} \right\} \Phi(p,\kappa_m).$$

Taking the limit as $m \to \infty$, we deduce

$$\sup_B \bar u = \lim_{m\to\infty} \Phi(k^m p, 1) \leq C_p \Phi\Big(p, \frac{3}{2}\Big). \tag{2.45}$$

The proposition follows by taking $\epsilon \to 0$. $\square$

These properties are valid when u is a subsolution. For weak solutions, one can anticipate even stronger properties.

Proposition 2.61 (Harnack Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE inside the domain, and L be uniformly elliptic.*

There exists $C > 0$ such that, if B is a ball of radius r such that $2B \subset \Omega$, $f \in L^\infty(\Omega)$, $F \in L^\infty(\Omega,\mathbb{R}^n)$ and $u \in W(2B,m)$ is a non-negative weak solution to $Lu = fw + \operatorname{div}[wF]$ in $2B$, we have

$$\sup_B u \leq C\Big(\inf_B u + r^2 \|f\|_{L^\infty(2B)} + r \|F\|_{L^\infty(2B)}\Big). \tag{2.46}$$

Proof. The Harnack inequality for solutions to $Lu = 0$ can be found as [FKS82, Lemma 2.3.5] and [DFM20, Lemma 11.35]. In pur general setting, the proof is similar to the one of Theorem 8.18 in [GT01]. We keep the notations used for the proof of Proposition 2.58.

Note that the same argument can be run when $\beta < 0$ and u is a supersolution to $Lu = fw + \operatorname{div}[Fw]$ in $2B$, and we still have (2.42) and then (2.43). When $p \in (-\infty, 0)$, instead of (2.44), we have

$$\Phi(kp,\kappa_1) \geq \left(\frac{C}{\kappa_2 - \kappa_1}\right)^{\frac{2}{p}} \Phi(p,\kappa_2) \tag{2.47}$$

---

[2] We have a problem if $k^m p = 1$ for some m, but we can always take at any step $k' \in (0,k]$ instead k, which allows us to avoid any discrete set of values.

for $1 \leq \kappa_1 < \kappa_2 \leq 2$, and therefore

$$\Phi\Big(p, \frac{3}{2}\Big) \leq C_p \inf_B \bar{u} = C_p \lim_{m \to \infty} \Phi\Big(k^m p, 1\Big). \tag{2.48}$$

If u is a weak solution to $Lu = fw + \mathrm{div}[Fw]$, u is a supersolution to $Lu = fw + \mathrm{div}[Fw]$ so (2.48) is true for all $p < 0$. In addition, Proposition 2.58 gives (2.45) for all $p > 0$. Now, we want to prove that there exists $p_0 > 0$ small enough such that

$$\Phi\Big(p_0, \frac{3}{2}\Big) \leq C\Phi\Big(-p_0, \frac{3}{2}\Big). \tag{2.49}$$

By taking $\beta = -1$ in (2.43), B' of radius r' such that $2B' \subset 2B$, and $\eta \in C_0^\infty(2B')$ such that $\eta \equiv 1$ on B', we have

$$\fint_{B'} |\nabla(\ln \bar{u})|^2 \, dm \leq C(r')^2,$$

or, together with the interior Poincaré inequality (or more precisely Theorem 2.17),

$$\fint_{B'} |\ln \bar{u} - (\ln \bar{u})_{B'}| \, dm \leq C.$$

If means that $\ln \bar{u}$ is a function in $BMO_{loc}(2B, m)$, so the John-Nirenberg lemma (See [HL11, Chapter 3, Theorem 1.5], ;the context is different but the proof only rely on the Calderón-Zygmund decomposition, which is true since we assume that m is doubling) entails the existence of $p_0 > 0$ and $C > 0$ such that

$$\fint_{\frac{3}{2}B} e^{p_0 |\ln \bar{u} - (\ln \bar{u})_{3B/2}|} \, dm \leq C$$

and thus

$$\Phi\Big(p_0, \frac{3}{2}\Big)^{p_0} \Phi\Big(-p_0, \frac{3}{2}\Big)^{-p_0} \leq C e^{p_0 (\ln \bar{u})_{3B/2}} e^{-p_0 (\ln \bar{u})_{3B/2}} \leq C.$$

The claim (2.49) follows. The combination of (2.45), (2.48) and (2.49) implies (2.46), as desired. □

Building upon the previous proposition, we can establish the Hölder continuity of solutions.

Proposition 2.62 (interior Hölder continuity for inhomogeneous solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE inside the domain, and L be uniformly elliptic.*

For every $\epsilon > 0$, there exists $\theta > 0$ such that, for any $f \in L^\infty(\Omega)$, any $F \in L^\infty(\Omega, \mathbb{R}^n)$, any ball $B \subset \Omega$ of radius r and any weak solution $u \in W(B, m)$ to $Lu = fw + \mathrm{div}[Fw]$, we have

$$\operatorname*{osc}_{B/2} u \leq \theta \operatorname*{osc}_B u + \epsilon(r^2 \|f\|_\infty + r\|F\|_\infty). \tag{2.50}$$

As a consequence, there exist $C > 0$ and $\alpha \in (0, 1]$ such that, for any $\tau \in (0, 1/2]$, any $f \in L^\infty(\Omega)$, any ball $B \subset \Omega$ of radius r, and any weak solution $u \in W(B, m)$ to $Lu = fw + \mathrm{div}[Fw]$ in B, we have

$$\operatorname*{osc}_{\tau B} u \leq C\tau^\alpha \left(\fint_B |u| \, dm + r^2 \|f\|_\infty + r\|F\|_\infty \right). \tag{2.51}$$

Proof. For homogeneous solutions, the result can be found as [FKS82, Lemmas 2.3.11, Theorems 2.3.12], [HL11, Section 4.3, Theorem 2.4]. Our proof with inhomogeneous terms is similar to the one of Theorem 8.22 in [GT01]. We keep the notations introduced in the proofs of Propositions 2.58 and 2.61. We invoke Proposition 2.61 for $u - (\inf_B u)$ and $(\sup_B u) - u$, and we obtain

$$\sup_{B/2} u - \inf_B u + r^2\|f\|_\infty \leq C_\epsilon(\inf_{B/2} u - \inf_B u + \epsilon r^2\|f\|_\infty)$$

and

$$-\inf_{B/2} u + \sup_B u + r^2\|f\|_\infty \leq C_\epsilon(-\sup_{B/2} u + \sup_B u + \epsilon r^2\|f\|_\infty).$$

By summing, we obtain

$$\operatorname*{osc}_{B/2} u + \operatorname*{osc}_B u + 2r^2\|f\|_\infty \leq C_\epsilon(-\operatorname*{osc}_{B/2} u + \operatorname*{osc}_B u + 2\epsilon r^2\|f\|_\infty),$$

that is

$$\operatorname*{osc}_{B/2} u \leq \frac{C_\epsilon - 1}{C_\epsilon + 1}(\operatorname*{osc}_B u + 2\epsilon r^2\|f\|_\infty).$$

The bound (2.50) follows. Thanks to Lemma 8.23 in [GT01], we obtain for all $\tau \in (0, 1/2)$ the estimate

$$\operatorname*{osc}_{\tau B} u \leq C\tau^\alpha\Big(\operatorname*{osc}_{B/2} u + Cr^2\|f\|_\infty\Big).$$

Check that $|u|$ is a subsolution to $Lu = |f|$, so Proposition 2.58 entails that $\operatorname{osc}_{B/2} u \leq \sup_{B/2}|u| \leq C \fint_B |u|\, dm$. The proposition follows. ☐

Proposition 2.63 (Hölder continuity at the boundary). *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, and L be uniformly elliptic. Set*

$$\operatorname*{osc}_{B\cap\partial\Omega} \operatorname{Tr} u := \max_{\substack{\eta\in C_0^\infty(B)\\ \|\eta\|_\infty\leq 1}} (\max \operatorname{Tr}(\eta u) - \min \operatorname{Tr}(\eta u)), \tag{2.52}$$

where the quantities $\max \operatorname{Tr}(v)$ and $\min \operatorname{Tr}(v)$ have been defined in Proposition 2.30.

There exists $\theta \in (0, 1)$ such that, for any ball $B \subset \mathbb{R}^n$ centered on $\partial\Omega$ and any weak solution $u \in W(2B \cap \Omega, m)$, we have

$$\operatorname*{osc}_{B\cap\Omega} u \leq \theta \operatorname*{osc}_{2B\cap\Omega} u + (1-\theta) \operatorname*{osc}_{2B\cap\partial\Omega} \operatorname{Tr} u. \tag{2.53}$$

As a consequence, there exist $C > 0$ and $\alpha \in (0, 1]$ such that, for any $\lambda \geq 2$, any ball B centered on $\partial\Omega$ and any weak solution $u \in W(\lambda B \cap \Omega, m)$, we have

$$\operatorname*{osc}_{B\cap\Omega} u \leq C\lambda^{-\alpha} \operatorname*{osc}_{\lambda B\cap\Omega} u + C \operatorname*{osc}_{\sqrt{\lambda}B\cap\Omega} \operatorname{Tr} u. \tag{2.54}$$

When $\operatorname{Tr} u = 0$ on $\lambda B \cap \partial\Omega$, we can further obtain

$$\operatorname*{osc}_{B\cap\Omega} u \leq C\lambda^{-\alpha} \fint_{B\cap\Omega} |u| dm. \tag{2.55}$$

Remark 2.64. The reader might be a bit scared by the definition (2.52), but there is nothing to be worried about. The cut-off function is needed to make sense to the trace in our general setting. When (Ω, m, σ) is suitable for PDE and traces, then, the trace is given by (2.19) and $\operatorname{osc}_{B\cap\partial\Omega} u$ is simply the difference between the σ-essential supremum and the σ-essential infimum of u on $B \cap \partial\Omega$.

Proof. See [FKS82, Lemma 2.4.5, Theorem 2.4.6] and [DFM21, Lemma 8.16]. ☐

Interesting variants of the Harnack inequality and Hölder continuity include the following density lemmas.

Lemma 2.65 (Density lemmas)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE inside the domain and L be uniformly elliptic.*

Let $\epsilon > 0$. There exists $c_\epsilon > 0$ such that if B is a ball satisfying $4B \subset \Omega$ and $u \in W(4B, m)$ is a non-negative supersolution in $4B$ such that

$$m(\{X \in 2B,\, u(X) \geq 1\}) \geq \epsilon m(2B),$$

then

$$\inf_B u \geq c_\epsilon. \tag{2.56}$$

Let m be suitable for PDE. There exists $c > 0$ such that, if B is a ball centered on $\partial\Omega$ and $u \in W(2B, m)$ is a non-negative supersolution in $2B$ such that $\operatorname{Tr} u \geq 1$ on $2B \cap \partial\Omega$, we have

$$\inf_{B\cap\Omega} u \geq c. \tag{2.57}$$

Proof. The interior result can be copied from the one of Density Theorem (Section 4.3, Theorem 4.9) in [HL11]. The boundary result is similar to [DFM21, Lemma 8.14]. ☐

Theorem 2.66 (boundary Harnack inequality)**.** *Let Ω be a uniform domain, m be suitable for PDE, and L be uniformly elliptic.*

There exists $C > 0$ such that, for any ball B centered on $\partial\Omega$, any corkscrew point A_B for the boundary ball $B \cap \partial\Omega$ ...

(i) ... and any non-negative weak solution $u \in W(2B \cap \Omega, m;\, 0 \text{ on } 2B \cap \partial\Omega)$, we have

$$\sup_{B\cap\Omega} u \leq Cu(A_B);$$

(ii) ... and any positive weak solutions $u, v \in W(2B \cap \Omega, m;\, 0 \text{ on } 2B \cap \partial\Omega)$, we have

$$\sup_{B\cap\Omega} \frac{u(X)}{v(X)} \leq C\frac{u(A_B)}{v(A_B)},$$

which is equivalent to

$$C^{-1}\frac{u(Y)}{v(Y)} \leq \frac{u(X)}{v(X)} \leq C\frac{u(Y)}{v(Y)} \qquad \text{for } X, Y \in B \cap \Omega;$$

(iii) ... and any positive weak solutions $u, v \in W(\Omega \setminus B, m;\, 0 \text{ on } \partial\Omega \setminus B)$, we have

$$C^{-1}\frac{u(Y)}{v(Y)} \leq \frac{u(X)}{v(X)} \leq C\frac{u(Y)}{v(Y)} \qquad \text{for } X, Y \in \Omega \setminus 2B.$$

Proof. Theorem 2.66. Conclusion (i) was proved as [DFM20, Lemma 15.14]; earlier results include : [JK82] for CAD domains, [Aik04, Theorem 1.2] and [Aik08, Corollary 2] for a characterization of uniform domains. Conclusion (ii) is [DFM20, Lemma 15.64], generalizing the breakthrough result [CFMS81, Theorem 1.4]; see also [JK82] for CAD domains and [Aik01, Aik06] for a characterization of uniform domains. Conclusion (iii) is proved similarly to (ii). ☐

The subsequent result enhances the Moser inequality for non-negative weak solutions, as opposed to general solutions or non-negative subsolutions. The boundary analogue of this interior result is also known as the“Carleson estimat” or“comparison principle”.

Combining this with Hölder continuity, we obtain an improved version of Theorem 2.66.

Corollary 2.67. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, m be suitable for PDE, and L be uniformly elliptic.*

There exist $C > 0$ and $\alpha \in (0,1]$ such that, for any $\lambda \geq 1$, any ball B centered on $\partial\Omega$, any corkscrew point $A_{x,r}$ for the boundary ball $B \cap \partial\Omega$ ...

(i) ... and any non-negative weak solution $u \in W(2B \cap \Omega, m;\ 0$ on $2B \cap \partial\Omega)$, we have

$$\sup_{\lambda^{-1}B\cap\Omega} u \leq C\lambda^{-\alpha} u(A_{x,r});$$

(ii) ... and any positive weak solutions $u, v \in W(2B \cap \Omega, m;\ 0$ on $2B \cap \partial\Omega)$, we have

$$\sup_{X,Y\in\lambda^{-1}B\cap\Omega} \left| \frac{u(X)v(Y)}{v(X)u(Y)} - 1 \right| \leq C\lambda^{-\alpha};$$

(iii) ... and any positive weak solutions $u, v \in W(\Omega \setminus B, m;\ 0$ on $\partial\Omega \setminus B)$, we have

$$\sup_{X,Y\in\Omega\setminus 2\lambda B} \left| \frac{u(X)v(Y)}{v(X)u(Y)} - 1 \right| \leq C\lambda^{-\alpha}.$$

Proof. (i) is trivial from Theorem 2.66 (i) and Proposition 2.63. (ii) and (iii) are proved like [DEM21, Corollary 6.4]. □

2.3.3 . Green functions

Another important tool is the Green functions associated to a uniformly elliptic operator. For $Y \in \Omega$ and $r \in (0, \delta_{\partial\Omega}(Y)/10]$, let us introduce the quantity

$$\gamma_Y(r) := \int_r^{\delta_{\partial\Omega}(Y)} \frac{s^2}{m(B(Y,s))} \frac{ds}{s} \tag{2.58}$$

and note that

$$\frac{d\gamma_Y}{dr}(r) = -\frac{r}{m(B(Y,s))}$$

and

$$\gamma_Y(r) \geq c\frac{r^2}{m(B(Y,r))}.$$

We have the following properties.

Theorem 2.68. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE and L be uniformly elliptic. There exists a unique function $G : \Omega \times \Omega \mapsto \mathbb{R} \cup \{+\infty\}$ such that $G(X,.)$ is continuous on $\Omega \setminus \{X\}$ and locally integrable in Ω for every $X \in \Omega$, and such that, for every $f \in C_0^\infty(\Omega)$, the function u given by*

$$u(X) := \int_\Omega G(X,Y)f(Y)dm(Y) \tag{2.59}$$

belongs to $W_0(\Omega, m)$ *and is a solution of* $Lu = f$ *in the sense that*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = \int_\Omega f\varphi \, dm \qquad \text{for every } \varphi \in W_0(\Omega, m). \tag{2.60}$$

Such a function G *is called Green function and satisfies the following properties.*

(i) *If* $\eta \in C_0^\infty(\Omega)$ *is such that* $\eta \equiv 1$ *on a neighborhood of* $\{Y\}$*, then* $(1-\eta)g(.,Y) \in W_0(\Omega, m)$*.*

(ii) *For* $Y \in \Omega$ *and* $\varphi \in C_0^\infty(\Omega)$*,*

$$\int_\Omega A\nabla_X G(X,Y) \cdot \nabla\varphi(X) dX = \varphi(Y). \tag{2.61}$$

In particular, $G(.,Y)$ *is a solution of* $Lu = 0$ *in* $\Omega \setminus \{Y\}$*.*

(iii) *There exists* $\alpha \in (0,1]$ *such that, for* $X, Y \in \Omega$ *satisfying* $|X-Y| \geq \delta_{\partial\Omega}(Y)/10$*,*

$$0 \leq G(X,Y) \leq C\delta_{\partial\Omega}(X)^\alpha \frac{|X-Y|^{2-\alpha}}{m(B(Y,|X-Y|)\cap\Omega)}. \tag{2.62}$$

In particular, $G(.,X)$ *can be extended by continuity to* $\partial\Omega$*, and* $G(.,X) \equiv 0$ *on* $\partial\Omega$*.*

(iv) *For* $X, Y \in \Omega$ *such that* $|X-Y| \leq \delta_{\partial\Omega}(Y)/2$*,*

$$c\gamma_Y(|X-Y|) \leq G(X,Y) \leq C\gamma_Y(|X-Y|). \tag{2.63}$$

The constant $c > 0$ *above does not depend on the constant in the boundary Poincaré inequality.*

(v) *For* $r \in (0, \delta_{\partial\Omega}(Y)/2)$ *and* $y \in \Omega$*,*

$$\int_{\Omega\setminus B(Y,r)} |\nabla_x G(X,Y)|^2 dm(x) \leq C\gamma_Y(r). \tag{2.64}$$

(vi) *There exists* $q > 1$ *(depending only on the doubling constant of* m*) such that, for* $Y \in \Omega$*,*

$$\left(\fint_{B(Y,\delta_{\partial\Omega}(Y))} |\nabla_x G(X,Y)|^q dm(X)\right)^{\frac{1}{q}} \leq C\frac{\delta_{\partial\Omega}(Y)}{m(B(Y,\delta_{\partial\Omega}(Y)))}. \tag{2.65}$$

In particular, $\nabla G(.,Y) \in L^q_{loc}(\Omega)$*.*

(vii) *Finally, for any* $X, Y \in \Omega$*, we have*

$$G(X,Y) = G_T(Y,X), \tag{2.66}$$

where g_T *is the Green function for the adjoint operator* $L^T := -\operatorname{div} A^T\nabla$*. In particular, all the above estimates stay true if* x *is replaced by* y*.*

Proof. The statements and the proofs can be found in [DFM20]. Note that our conditions are slightly more general but the proofs are exactly the same as in [DFM20]. The existence and uniqueness in Theorem 2.68 are given in [DFM20, Lemmas 14.87 and 14.91] once you know that $\mathfrak{g}^\rho$ is continuous (this is given by Proposition 2.62). The bounds (i), (iii), (iv) and (v) are given by [DFM20, Theorem 14.60], (ii) by Lemma [DFM20, Lemma 14.83], and (vi) is [DFM20, Lemma 14.78]. Some of the bounds are found in [FJK82], and the proofs in the codimension 1 case are in [Sta65] and [GW82]. ☐

Corollary 2.69. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE and L be uniformly elliptic.*

For $Y \in \Omega$ and $\rho \leq \delta_{\partial\Omega}(Y)/100$, take a non-negative function $f_{Y,\rho} \in L^\infty(\Omega)$ such that $\operatorname{supp} f_{Y,\rho} \subset B(Y,\rho)$ and $\int_\Omega f_{Y,\rho}\, dm = 1$. The map $v \in W_0(\Omega,m) \to \int_\Omega f_{Y,\rho}\varphi\, dm$ belongs to $W^{-1}(\Omega,m)$ and so we can construct $G^\rho(X,Y)$ as the unique weak solution in $W_0(\Omega,m)$ to $Lu = f_{Y,\rho}$ given by Theorem 2.47. Then, as $\rho \to 0$, we have the following convergences :

(a) $G^\rho(.,Y) \to G(.,Y)$ uniformly on compact of $\overline{\Omega} \setminus \{Y\}$,

(b) $G^\rho(.,Y) \to G(.,Y)$ strongly in $W(\Omega \setminus B_Y, m;\ 0 \text{ on } \partial\Omega)$, where $B_Y := B(Y, \delta_{\partial\Omega}(Y)/4)$,

(c) $\nabla_X G^\rho(.,Y) \rightharpoonup \nabla_X G^\rho(.,Y)$ weakly in $L^q(2B_Y, m)$.

Moreover, G^ρ satisfies the following estimates with constants independent of ρ.

(i) For $X, Y \in \Omega$ satisfying $|X-Y| \geq \delta_{\partial\Omega}(Y)/10$,

$$0 \leq G^\rho(X,Y) \leq C \frac{|X-Y|^2}{m(B(Y,|X-Y|) \cap \Omega)}.$$

(ii) For $X, Y \in \Omega$ such that $10\rho \leq |X-Y| \leq \delta_{\partial\Omega}(Y)/2$,

$$c\gamma_Y(|X-Y|) \leq G^\rho(X,Y) \leq C\gamma_Y(|X-Y|),$$

where $c > 0$ above does not depend on (ρ, X, Y, and) the constant in the boundary Poincaré inequality.

(iii) For $Y \in \Omega$ and $r \in (0, \delta_{\partial\Omega}(Y)/2]$,

$$\int_{\Omega \setminus B(Y,r)} |\nabla_X G^\rho(X,Y)|^2 dm(X) \leq C\gamma_Y(r).$$

(iv) There exists $q > 1$ (depending only on the doubling constant of m) such that

$$\left(\fint_{2B_Y} |\nabla_X G^\rho(X,Y)|^q dm(X) \right)^{\frac{1}{q}} \leq C \frac{\delta_{\partial\Omega}(Y)}{m(B_Y)}.$$

Proof. The uniform bounds in Corollary 2.69 are a consequences of the proofs in [DFM20], the convergence is due to the fact that a limit $G'(x,y)$ of any convergent subsequence of $G^\rho(x,y)$ satisfies (2.59) and (2.60), so $G'(x,y)$ has to be $G(x,y)$ by uniqueness. ☐

Proposition 2.70. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain, m be suitable for PDE and L be uniformly elliptic. There exists a non-zero non-negative solution $G \in W(\Omega, m;\ 0 \text{ on } \partial\Omega;\ \text{locally in } \mathbb{R}^n)$ to $Lu = 0$ in Ω. By Propositions 2.62 and 2.63, such a function G is Hölder continous on $\overline{\Omega}$ and $G \equiv 0$ on $\partial\Omega$.*

By Proposition 2.61, if Ω is connected, then G is positive. By (ii) of Corollary 2.67, if Ω is uniform, then G is unique up to a constant, that is, if $G_1 \in W(\Omega, m;\ 0 \text{ on } \partial\Omega;\ \text{locally in } \mathbb{R}^n$ and $G_2 \in W(\Omega, m;\ 0 \text{ on } \partial\Omega;\ \text{locally in } \mathbb{R}^n)$ are two non-zero non-negative solutions to $Lu = 0$ in Ω, then G_1/G_2 is a positive constant. We call such a function the **Green function with pole at infinity.**

Proof. It can be proved like Lemma 3.7 in [KT99], Lemma 6.5 in [DEM21] or Lemma 3.2 in [DFM23a]. ☐

2.3.4 . Properties of the elliptic measures

The interaction between the Green function and harmonic measure is crucial, as they are primary tools in the study of solutions. Specifically, the comparison between these two quantities is a key element in the proof of Theorem 2.66 parts (ii) and (iii), which pertain to the comparison principle and boundary Harnack principle. These results were initially established for classical uniformly elliptic operators in [CFMS81], and we have extended them to more general contexts in [DFM21, DFM20].

In this subsection, $\omega^X := \omega^X_{L,\Omega}$ denotes the elliptic measure constructed in Theorem 2.53, and $G(X,Y) = G_L(X,Y)$ represents the Green function constructed in Theorem 2.68.

Our first result is the non-degeneracy of the harmonic measure, which is a specific case of the second part of Lemma 2.65.

Lemma 2.71. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, L be uniformly elliptic. For any $\alpha \in (0,1)$, there exists $C > 0$ such that, for any ball B centered at the boundary,*

$$\inf_{X \in \alpha B} \omega^X(B \cap \partial\Omega) \geq C^{-1}.$$

As a consequence, if B is a ball centered on $\partial\Omega$ and X is a ϵ-corkscrew point for $B \cap \partial\Omega$, then

$$\omega^X(B \cap \partial\Omega) \geq C_\epsilon^{-1},$$

where C_ϵ depends on X only via ϵ.

Proof. The first part of the lemma is given by Lemma 2.65 when $\alpha = 1/2$, but the proof is the same for a general $\alpha \in (0,1)$. For the second part of the lemma, let ϵ be the one in the definition of corkscrew points. Observe that a point in $B(X,\epsilon/2) \cap (1-\epsilon/4)B$ is non-empty, so contains a point Y. By the first part, $\omega^Y(B \cap \partial\Omega) \geq C$, and we conclude by using the Harnack inequality (Proposition 2.61) in $B(X,\epsilon/2)$. □

The Green function and the elliptic measure are connected through the following formula.

Proposition 2.72. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE, and L be uniformly elliptic. Let $G(X,Y) = G_L(X,Y)$ be the Green function from Theorem 2.68 and $\omega^X = \omega^X_L$ be the measure from Theorem 2.53. For any $X \in \Omega$ and any $\varphi \in C^\infty(\mathbb{R}^n)$ which is 0 on a neighborhood of $\{X\}$, we have*

$$\int_\Omega \mathcal{A}\nabla\varphi(Y) \cdot \nabla_y G(X,Y)\, dm(Y) = -\int_{\partial\Omega} \varphi(y) d\omega^X(y).$$

Proof. See for instance Lemma 2.18 in [DFM23a]. □

This relationship allows us to establish a comparison between Green functions and elliptic measures, serving as an intermediate step in the proof of Theorem 2.66 part (iii).

Proposition 2.73. *Let $\Omega \subset \mathbb{R}^n$ be open, m be suitable for PDE and L be uniformly elliptic. We write $G(X,Y) = G_L(X,Y)$ be the Green function from Theorem 2.68 and $\omega^X = \omega^X_L$ be the measure from Theorem 2.53.*

There exists $C > 0$ such that, for any $x \in \partial\Omega$, any $r > 0$, and any corkscrew point $X_{x,r}$ for x at scale r, we have

$$\frac{m(B(x,r) \cap \Omega)}{r^2} G(Y, X_{x,r}) \leq C\omega^Y(B(x,r) \cap \partial\Omega) \quad \text{for } Y \in \Omega \setminus B(X_{x,r}, \delta_{\partial\Omega(X_{x,r})}/4).$$

Moreover, if Ω is uniform, then

$$\omega^Y(B \cap \partial\Omega) \leq C\frac{m(B(x,r)\cap\Omega)}{r^2}G(Y, X_{x,r}) \quad \text{for } Y \in \Omega \setminus 2B.$$

Remark 2.74. Note that the first inequality in the above proposition is a direct consequence of the non-degeneracy of the harmonic measure (Lemma 2.71), the pointwise bound (2.63) on the Green function, and the maximum principle (Corollary 2.51).

Proof. See Lemma 15.28 in [DFM20], generalizing [CFMS81, Lemma 2.2] (see also [DFM21, Lemma 11.11]). ☐

Applying Theorem 2.66 part (iii) to two elliptic measures yields the following result :

Proposition 2.75 (Change of poles for the elliptic measure). *Let Ω be a uniform domain, m be suitable for PDE, L be uniformly elliptic, and $\omega^X = \omega^X_L$ be the measure from Theorem 2.53.*

There exists $C > 0$ such that, for any $x \in \partial\Omega$, any $r > 0$, any corkscrew point $X_{x,r}$ for x at scale r, and any Borel sets $E, F \subset B(x,r) \cap \partial\Omega$, we have

$$C^{-1}\frac{\omega^X(E)}{\omega^X(F)} \leq \frac{\omega^Y(E)}{\omega^Y(F)} \leq C\frac{\omega^X(E)}{\omega^X(F)} \qquad \text{for } X, Y \in \Omega \setminus B(x, 2r)$$

and

$$C^{-1}\omega^{X_{x,r}}(E) \leq \frac{\omega^Y(E)}{\omega^Y(B(x,r)\cap\partial\Omega)} \leq C\omega^{X_{x,r}}(E) \qquad \text{for } Y \in \Omega \setminus B(x, 2r).$$

Proof. The change of poles is Lemma 15.61 in [DFM20] (see also [DFM21, Lemma 11.16]), but it would be fair to say that it is only 2.66 applied to the elliptic measure. ☐

By combining Proposition 2.73 with the Harnack inequality (Proposition 2.61), we can easily derive the doubling property of the harmonic measure for uniform domains. This doubling property is expected to hold in the weaker "semi-uniform" domains. Such a proof has been obtained in [Azz21] for classical uniformly elliptic operators, but has yet to be verified in our more general context.

Proposition 2.76. *Let Ω be a uniform domain, m be suitable for PDE, L be uniformly elliptic, and $\omega^X = \omega^X_L$ be the measure from Theorem 2.53. Then, there exists $C > 0$ such that, for any $x \in \partial\Omega$, any $r > 0$ and any $X \in \Omega \setminus B(x, 3r)$, we have*

$$\omega^X(B(x,2r)\cap\partial\Omega) \leq C\omega^X(B(x,r)\cap\partial\Omega).$$

Proof. See for instance [DFM20, Lemma 15.43]. ☐

To conclude this section, let us construct the elliptic measure "with pole at infinity".

Proposition 2.77. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded uniform domain, m be suitable for PDE, and $L := \operatorname{div} A\nabla$ be uniformly elliptic. Let G_{L^*} be a positive solution in $W(\Omega, m;\ 0 \text{ on } \partial\Omega;\ \text{locally in } \mathbb{R}^n)$ to $L^*u = \operatorname{div} A^T\nabla u = 0$ in Ω (as given by Proposition 2.70).*

Then there exists a Borel measure $\omega := \omega^\infty_L$ on $\partial\Omega$ such that

$$\int_\Omega A^T\nabla G_{L^*} \cdot \nabla\varphi \, dm = -\int_{\partial\Omega} \varphi \, d\omega \qquad \text{for } \varphi \in C^\infty_0(\mathbb{R}^n).$$

Since G_* is unique up to a positive constant, ω is also unique to L up to a positive constant. We call ω **the elliptic measure with pole at ∞**.

Proof. Proposition 2.77 can be proved like Lemma 6.5 in [DEM21] or Lemma 3.2 in [DFM23a]. □

In most cases, the pole is not particularly relevant. To avoid discussing the pole, it is convenient to use the following quantity instead.

Definition 2.78. Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, m be suitable for PDE and $L := \operatorname{div} A\nabla$ be uniformly elliptic. We define the function G_L and the measure ω_L as follows

- if Ω is bounded, we take $G_L = G_L(., X_0)$ and $\omega_L = \omega_L^{X_0}$, where X_0 satisfies $B(X_0, \epsilon_\Omega \operatorname{diam} \Omega) \subset \Omega$ and ϵ_Ω is the corkscrew point constant of Ω;
- if Ω is unbounded, we take G_L to be the function (defined up to a constant) given by Proposition 2.70 and ω_L to be the measure given by Proposition 2.77.

Proposition 2.79. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain (ϵ_Ω is its corkscrew constant), m be suitable for PDE and $L := \operatorname{div} A\nabla$ be uniformly elliptic. Let $G_{L^*}(X,Y)$, ω_L^X be respectively the Green function associated to L^* and the elliptic measure associated to L, and let G_{L^*}, ω_L be the objects defined in Definition 2.78. There exists a constant $C > 0$ such that, for any boundary ball $x \in \partial\Omega$, any $r \in (0, \epsilon_\Omega \operatorname{diam} \Omega/2)$ and any corkscrew point $X_{x,r}$ for x at scale r, we have*

$$C^{-1} \frac{m(B(x,r) \cap \Omega)}{r^2} G_{L^*}(X_{x,r}) \leq \omega_L(B(x,r) \cap \partial\Omega) \leq C \frac{m(B(x,r) \cap \Omega)}{r^2} G_{L^*}(X_{x,r}).$$

If, moreover, E is a Borel subset of $B(x,r) \cap \partial\Omega$, then

$$C^{-1} \omega_L^X(E) \leq \frac{\omega_L(E)}{\omega_L(B(x,r) \cap \partial\Omega)} \leq C \omega_L^X(E).$$

At last, if $Y \in \Omega \setminus B(x,2r)$[3]*, we have*

$$C^{-1} G_{L^*}(X_{x,r}, Y) \leq \frac{G_{L^*}(Y)}{\omega_L(B(x,r) \cap \partial\Omega)} \leq C G_{L^*}(X_{x,r}, Y).$$

Proof. This is simply Propositions 2.73 and 2.75. This is immediate when Ω is bounded. It works also when Ω is unbounded because G_{L^*} and ω_L are constructed by taking a limit of (appropriately rescaled) functions $G_{L^*}(., X_i)$ and measures $\omega_L^{X_i}$ when $|X_i| \to \infty$ (thanks to (iii) of Corollary 2.67, the convergence to G_{L^*} will be on compacts sets of $\overline{\Omega}$ and the convergence to ω_L will be for all Borel sets). □

2.3.5 . Absolute continuity of the elliptic measure

In our manuscript, we will not delve into the dimension of the support of the harmonic measure, which is an entire theory by itself. Instead, we will take a step further by investigating when the harmonic measure (or the elliptic measure) is mutually absolutely continuous with respect to the Hausdorff measure, as well as its quantitative version, known as A_∞-absolute continuity.

Definition 2.80 (absolute continuity)**.** Let σ, μ be two Borel measures on E.

We say that μ is absolutely continuous with respect to σ if, for any Borel set $F \subset E$, $\sigma(F) = 0$ implies $\mu(F) = 0$.

---

[3] If Ω is bounded, we also assume that Y is not close to the pole X_0 used to define G_{L^*}

We say that μ is A_∞-absolutely continuous with respect to σ - and we write $\mu \in A_\infty(\sigma)$ - if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any "boundary" ball $Q \subset E$, and any Borel set $F \subset Q$,

$$\frac{\mu(F)}{\mu(Q)} < \delta \Rightarrow \frac{\sigma(F)}{\sigma(Q)} < \epsilon.$$

We say that the collection of elliptic measures $\{\omega_L^X\}_{X\in\Omega}$ is A_∞-absolutely continuous with respect to σ - and we still write $\omega_L^X \in A_\infty(\sigma)$ - if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any ball B centered on $\partial\Omega$, any $F \subset B \cap \partial\Omega$ and any $X \in \Omega \setminus 2B$,

$$\frac{\omega_L^X(F)}{\omega_L^X(Q)} < \delta \Rightarrow \frac{\sigma(F)}{\sigma(Q)} < \epsilon. \tag{2.67}$$

Of course, we can also say that $\omega_{L_1}^X$ is A_∞-absolutely continuous with respect to another elliptic measure $\omega_{L_0}^X$ if (2.67) is replaced by

$$\frac{\omega_{L_1}^X(F)}{\omega_{L_1}^X(Q)} < \delta \Rightarrow \frac{\omega_{L_0}^X(F)}{\omega_{L_0}^X(Q)} < \epsilon.$$

It is not practical to discuss a 'collection' of measures. In fact, we can consolidate all the information of the elliptic measure - which is essentially a collection of measures - into a single measure.

Proposition 2.81. *Let Ω be a uniform domain, m be suitable for PDE, and L be uniformly elliptic, and let σ be a Borel measure on $\partial\Omega$. Let ω_L^X be the collection of elliptic measures constructed in Theorem 2.53 and ω_L be the (single) elliptic measure from Definition 2.78. Then $\omega_L^X \in A_\infty(\sigma)$ if and only if $\omega_L \in A_\infty(\sigma)$.*

Proof. The proposition is an immediate consequence of the second equivalence in Proposition 2.79. ☐

Several fundamental properties and characterizations of A_∞-absolute continuity are presented in the following results, which can be found as Theorem 1.4.13 in [Ken94], and draw from the works of [Muc74, CF74].

Proposition 2.82. *(i) If $\mu \in A_\infty(\sigma)$, then μ is absolutely continuous with respect to σ.*

(ii) A_∞ is an equivalence relationship, meaning that $\mu \in A_\infty(\sigma)$ if and only if $\sigma \in A_\infty(\mu)$.

Theorem 2.83. *If σ and μ are two measures on E, then the following are equivalent :*

(i) $\mu \in A_\infty(\sigma)$;

(ii) there exist $\epsilon, \delta \in (0,1)^2$ such that, for any ball $Q \subset E$ and any Borel set $F \subset Q$,

$$\frac{\sigma(F)}{\sigma(Q)} < \delta \Rightarrow \frac{\mu(F)}{\mu(Q)} < \delta;$$

(iii) there exist $C > 0$, $\eta > 0$ such that, for any ball $Q \subset E$ and any Borel set $F \subset Q$,

$$\frac{\mu(F)}{\mu(Q)} \le C \left(\frac{\sigma(F)}{\sigma(Q)}\right)^\eta;$$

(iv) $\mu \in \bigcup_{q>1} B_q(\sigma)$*, where* $\mu \in B_q(\sigma)$ *if* μ *is absolutely continuous with respect to* σ *and there exists* $C > 0$ *such that, for any ball* $Q \subset E$*, the Radon derivative* $k = \frac{d\mu}{d\sigma}$ *verifies*

$$\left(\fint_Q k^q \, d\sigma \right)^{\frac{1}{q}} \leq C \fint_Q k \, d\sigma;$$

Moreover, $\mu \in B_q(\sigma)$ *implies that there exists* $\epsilon > 0$ *such that* $\mu \in B_r(\sigma)$ *for* $r \in [1, q + \epsilon)$*; and* $\mu \in B_q(\sigma)$ *is equivalent to the* $L^{q'}(\sigma)$*-boundedness of the maximal operator* $\mathcal{M}_\mu$*, where* $\frac{1}{q} + \frac{1}{q'} = 1$ *and*

$$\mathcal{M}_\mu(f)(x) := \sup_{\substack{Q\textit{ball} \\ Q \ni x}} \fint_Q |f| d\mu.$$

3 - The Dirichlet problem on domains with higher or mixed codimensional boundaries

The purpose of this chapter is to adapt the theorem presented in Sections 1.2 and 1.3 to the framework developed in Chapter 2. We will start from scratch and reintroduce all the necessary definitions.

3.1 . The non-tangential maximal function N and the square function S

Our goal is to work with a measure on the boundary. Therefore, we will present the literature for the case where (Ω, m, σ) is suitable for PDE and traces. Given that this memoir focuses on domains that are complements of low-dimensional sets - which are inherently uniform by Proposition 1.21 - this restriction is reasonable. However, it is important to note that in the classical setting, where the measure m is the Lebesgue measure, many recent works have aimed to weaken the uniform and Ahlfors regular assumptions as much as possible. Therefore, the result on the equivalent characterization of the solvability of the L^p Dirichlet problem are inherently not optimal.

The key functionals in this theory are the non-tangential maximal function and the square function, defined below.

Definition 3.1. We define the cone with vertex on $x \in \partial\Omega$ and aperture $\alpha > 0$ by

$$\Gamma_\alpha(x) := \{Y \in \Omega,\, |Y - x| < (1+\alpha)\delta_{\partial\Omega}(Y)\},$$

which is used to defined some functionals on $\partial\Omega$: the non-tangential maximal function

$$N_\alpha u(x) := \sup_{Y \in \Gamma_\alpha(x)} |u|.$$

If we write B_Y for $B(Y, \delta_{\partial\Omega}(Y)/2)$, we also define and the square functions

$$\mathfrak{A}_\alpha u(x) := \left(\int_{\Gamma_\alpha(x)} |u(Y)|^2 \, \frac{dm(Y)}{m(B_Y)} \right)^{\frac{1}{2}},$$

and

$$S_\alpha u(x) := \left(\int_{\Gamma_\alpha(x)} |\delta_{\partial\Omega}(Y) \nabla u(Y)|^2 \, \frac{dm(Y)}{m(B_Y)} \right)^{\frac{1}{2}},$$

that is $S_\alpha(u) := \mathfrak{A}_\alpha(\delta_{\partial\Omega} \nabla u)$. The functionals are defined when u makes sense, that is $u \in L^\infty_{loc}(\Omega)$ for N_α, $u \in L^2_{loc}(\Omega, m)$ for $\mathfrak{A}_\alpha$, and $u \in W_{loc}(\Omega, m)$ for S_α.

Remark 3.2. Let us mention that in the classical setting when m is the n-dimensional Lebesgue measure, then

$$m(B(Y, \delta_{\partial\Omega}(Y)) = c_n \delta_{\partial\Omega}(Y)^n$$

where c_n is the volume of the unit balls. Since $\delta_{\partial\Omega}(Y) \leq |Y - x| \leq (1+\alpha)\delta_{\partial\Omega}(Y)$ when $Y \in \Gamma_\alpha(x)$, in the classical setting, the square functionals can we rewritten as

$$\mathfrak{A}_\alpha u(x) := \left(\int_{\Gamma_\alpha(x)} |u(Y)|^2 \, \frac{dY}{|X - y|^n} \right)^{\frac{1}{2}},$$

and $S_\alpha u := \mathfrak{A}_\alpha(\delta_{\partial\Omega} \nabla u)$.

Let us study the dependence in the aperture α. The first result is quite straightforward.

Proposition 3.3. *Let m and σ be a doubling measures on Ω and $\partial\Omega$ respectively, and let $\alpha > 0$. Then, for any $u \in L^2_{loc}(\Omega, m)$,*

$$C_\alpha^{-1} \int_\Omega u^2(Y) \frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y) \leq \int_{\partial\Omega} |\mathfrak{A}_\alpha(u)|^2 \, d\sigma \leq C_\alpha \int_\Omega u^2(Y) \frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y),$$

where $B_Y := B(Y, \delta_{\partial\Omega}(Y)/2)$ and C_α depends only in α and the doubling constants of m and σ.

Proof. Simple use of Fubini's lemma. See the proof of [FP22, Proposition 3.2] for a similar computation. ☐

Remark 3.4. Keep in mind that when m is the n-dimensional Lebesgue measure and σ is a $(n-1)$-Ahlfors regular measure,

$$\frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y) \approx \frac{dY}{\delta_{\partial\Omega}(Y)}. \tag{3.1}$$

If $dm = \delta_{\partial\Omega}^{d+1-n} dY$ and σ is a d-Ahlfors regular measure, we have instead

$$\frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y) \approx \frac{dm(Y)}{\delta_{\partial\Omega}(Y)} = \frac{dY}{\delta_{\partial\Omega}^{n-d}(Y)}. \tag{3.2}$$

If (Ω, m, σ) is suitable for PDE and traces, then

$$\frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y) \approx \rho(y, \delta_{\partial\Omega}(Y))^{-1} \frac{dm(Y)}{\delta_{\partial\Omega}(Y)}, \tag{3.3}$$

where ρ is the dimensionless quantity introduced in Definition 2.32 and y is such that $|y - Y| = \delta_{\partial\Omega}(Y)$.

Finally, let Ω be uniform (ϵ_Ω is the corkscrew constant), m be suitable for PDE, and L be uniformly elliptic. Set ω_L and G_{L^*} to be the elliptic measure (associated to L) and the Green function (associated to L^*) from Definition 2.78. If $\sigma = \omega_L$ and $B(Y, \epsilon_\Omega \operatorname{diam} \Omega/2)$ intersects $\partial\Omega$, then we have

$$\frac{\sigma(4B_Y)}{m(B_Y)}\, dm(Y) \approx \frac{G_{L^*}(Y)}{\delta_{\partial\Omega}(Y)^2}\, dm(Y). \tag{3.4}$$

This is the first step in demonstrating independence from α. However, we can achieve even more, as shown by the following :

Proposition 3.5. *Let m and σ be doubling measures on Ω and $\partial\Omega$ respectively, and let $\alpha, \beta > 0$. Then, for any function $u \in L^\infty_{loc}(\Omega)$ and any $t > 0$,*

$$\sigma(\{x \in \partial\Omega,\ N_\alpha(u) > t\} \leq C_{\alpha,\beta} \sigma(\{x \in \partial\Omega,\ N_\beta(u) > t\}$$

Moreover, if $r > 0$, for any $\gamma \in (0,1)$, any $u \in L^2_{loc}(\Omega, m)$, and any $t > 0$,

$$\sigma(\{x \in \partial\Omega,\ \mathfrak{A}_\alpha(u) > t,\ \mathcal{M}_{\sigma,r}(\mathfrak{A}_\beta(u)) \leq \gamma r\} \leq C_{\alpha,\beta,r} \gamma^2 \sigma(\{x \in \partial\Omega,\ \mathfrak{A}_\alpha(u) > t/2\},$$

where $\mathcal{M}_{\sigma,r}$ is the uncentered maximal function

$$\mathcal{M}_{\sigma,r}(g)(x) := \sup_{B \ni x} \left(\fint_B |g|^r d\sigma \right)^{\frac{1}{r}}.$$

Here $C_{\alpha,\beta}$ depends on α, β, and the doubling constants of m and σ, while $C_{\alpha,\beta,r}$ depends also on r.

Proof. The first part of Proposition 3.5 (about the non-tangential maximal function) is due to Stein in the half-plane (see [SM93, Section 2.5.1]) but the proof extends immediately to spaces with doubling measures. The second part (about the square function) is a well known result that can be easily proven using a good-λ argument, although the authors cannot pinpoint where it is proven. ☐

Together with the Cavalieri formula, the L^p norm of the functional N_α (resp. $\mathfrak{A}_\alpha$) are comparable to the one of N_β (resp. $\mathfrak{A}_\beta$). So in view of the next result, we will drop the parameter α for the functionals N_α, $\mathfrak{A}_\alpha$, and S_α, and only write it in the proofs when it becomes relevant.

Corollary 3.6. *Let m and σ be doubling measures on Ω and $\partial\Omega$ respectively. Let $\alpha, \beta > 0$ and $p > 0$. Then there exists $C_{\alpha,\beta,p} > 0$ such that*

$$\|N_\alpha(u)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\alpha,\beta,p} \|N_\beta(u)\|_{L^p(\partial\Omega,\sigma)}$$

whenever $u \in L^\infty_{loc}(\Omega)$ and

$$\|\mathfrak{A}_\alpha(u)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\alpha,\beta,p} \|\mathfrak{A}_\beta(u)\|_{L^p(\partial\Omega,\sigma)}$$

whenever $u \in L^2_{loc}(\Omega, m)$.

The square function and the non-tangential maximal functions are interrelated through the Carleson inequality, which also requires a notion of Carleson measure.

Definition 3.7. We define the functional $\mathcal{C}_\sigma$ for $u \in L^2_{loc}(\Omega, m)$ as

$$\mathcal{C}_\sigma(u)(x) := \sup_{B \ni x} \left(\fint_{\partial\Omega} |\mathfrak{A}_\alpha(u \mathbb{1}_B)|^2 d\sigma \right)^{\frac{1}{2}} \qquad \text{for } x \in \partial\Omega.$$

We say that u satisfies the Carleson measure condition with respect to σ - or $u \in CM_\sigma$ for short - if $\mathcal{C}_\sigma(u) \in L^\infty(\partial\Omega, \sigma)$. We write $u \in CM_\sigma(M)$ when we want to refer to the norm $M := \|\mathcal{C}_\sigma(u)\|_{L^\infty(\partial\Omega,\sigma)}$.

Remark 3.8. We can define equivalently $\mathcal{C}_\sigma$ as

$$\mathcal{C}_\sigma(u)(x) := \sup_{B \ni x} \left(\frac{1}{\sigma(B)} \int_B |u(Y)|^2 \frac{\sigma(4B_Y)}{m(B_Y)} \, dm(Y) \right)^{\frac{1}{2}}.$$

By equivalently, we mean that when σ and m are doubling, the Carleson norm $\|\mathcal{C}_\sigma(u)\|_{L^\infty(\partial\Omega,\sigma)}$ is equivalent with both definition of $\mathcal{C}_\sigma$, and the proof of this fact is similar to the one of Proposition 3.3.

One of our key tool is the following Carleson inequality.

Theorem 3.9 (Carleson inequality). *Let m and σ be doubling measures on Ω and $\partial\Omega$ respectively. Then, for any $u \in L^\infty_{loc}(\Omega)$ and any $v \in L^2_{loc}(\Omega, m)$, we have*

$$\int_{\partial\Omega} |\mathfrak{A}(uv)|^2 \, d\sigma \leq C^2 \int_{\partial\Omega} |N(u)|^2 |\mathcal{C}_\sigma(v)|^2 \, d\sigma$$

for some $C > 0$ that depends only on the doubling constants of m and σ. In particular, if $v \in CM_\sigma(M)$, then

$$\|\mathfrak{A}(uv)\|_{L^2(\partial\Omega,\sigma)} \leq CM \|N(u)\|_{L^2(\partial\Omega,\sigma)}.$$

Proof. Theorem 3.9 is the generalization of Theorem 2 in [SM93, Section 2.2] in our context (to which the proof of [SM93] extends). ☐

Remark 3.10. Let us restate Theorem 3.9. By Proposition 3.3, we have

$$\int_\Omega u^2(Y)v^2(Y)\frac{\sigma(4B_Y)}{m(B_Y)}\,dm(Y) \leq C\|\mathcal{C}_\sigma(v)\|^2_{L^\infty(\partial\Omega,\sigma)}\int_{\partial\Omega}|N(u)|^2\,d\sigma.$$

Then Remark 3.4 allows us to replace $\frac{\sigma(4B_Y)}{m(B_Y)}\,dm(Y)$ depending on the setting.

Let us complete Theorem 2.83 with the following property on A_∞-absolute continuity.

Theorem 3.11. *If σ and μ are two doubling measures on $E \subset \mathbb{R}^n$ such that $\sigma \in A_\infty(\mu)$, then we have :*

(i) the spaces $BMO(E,\sigma)$ and $BMO(E,\mu)$ are equal as Banach spaces, that is there exists $C > 0$ such that

$$C^{-1}\sup_{\text{ball }\Delta\subset E}\fint_\Delta|f-f_{\Delta,\sigma}|\,d\sigma \leq \sup_{\text{ball }\Delta\subset E}\fint_\Delta|f-f_{\Delta,\mu}|\,d\mu \leq C\sup_{\text{ball }\Delta\subset E}\fint_\Delta|f-f_{\Delta,\sigma}|\,d\sigma,$$

where Δ are boundary balls and $f_{\Delta,\nu} := \fint_\Delta f\,d\nu$;

(ii) There exists $C > 0$ such that, for any measure m on Ω and any $u \in L^2_{loc}(\mathbb{R}^n \setminus E, m)$,

$$C^{-1}\|\mathcal{C}_\sigma(u)\|_{L^\infty(E,\sigma)} \leq \|\mathcal{C}_\mu(u)\|_{L^\infty(E,\mu)} \leq C\|\mathcal{C}_\sigma(u)\|_{L^\infty(E,\sigma)}.$$

In particular, $u \in CM_\sigma(M) \implies u \in CM_\mu(CM)$.

Proof. The first part of Theorem 3.11 is a simple consequence of the John-Nirenberg inequality and the fact that $k = d\sigma/d\mu$ belongs to the reverse Hölder class B_q for some $q > 1$ (see Theorem 2.83). The second part is similar, and its proof is also simple with an analogue of the John-Nirenberg lemma for the Carleson measures. The result was hinted in [DKP11] and the proof in our setting can be found in [FP22, Lemma 3.30] . ☐

Remark 3.12. We emphasize that the choice of the measure m in part (ii) of the above theorem is inconsequential. Thus, (ii) remains a property of the A_∞-absolute continuity of measures. It would not be surprising if the converse - that (i) or (ii) implies $\sigma \in A_\infty(\mu)$ - were true. However, the author is unaware of any proof of such a result.

3.2 . Equivalent characterizations of the solvability of the Dirichlet problem

For the next definitions, $\Omega \subset \mathbb{R}^n$ is open, m is suitable for PDE, σ is a Borel measure supported on $\partial\Omega$, and $L := -\operatorname{div} A\nabla$ is a uniformly elliptic operator (with respect to the weight $w := dm/dx$). In particular, Theorem 2.53 gives the existence of an elliptic measure ω^X_L on $\partial\Omega$.

Definition 3.13. Given $p \in (1,\infty)$ and σ a doubling measure on $\partial\Omega$, we say that the Dirichlet problem for L is solvable in $L^p(\partial\Omega,\sigma)$ - or that $(\mathrm{D}_{p,\sigma})_L$ holds - if there exists $C > 0$ such that for every $g \in C^\infty_0(\partial\Omega)$, the solution u_g given as in (2.35) by the elliptic measure, that is

$$u_g(X) := \int_{\partial\Omega} g(y)\,d\omega^X_L(y) \tag{3.5}$$

satisfies

$$\|Nu_g\|_{L^p(\partial\Omega,\sigma)} \leq C\|g\|_{L^p(\partial\Omega,\sigma)}. \tag{3.6}$$

Proposition 3.14. *If $(D_{p,\sigma})_L$ holds, then for any $g \in L^p(\partial\Omega, \sigma)$, the construction* (3.5) *makes sense and gives a solution $u_g \in W_{loc}(\Omega, m)$ to $Lu = 0$ in Ω that satisfies* (3.6) *and*

$$\lim_{Y \in \Gamma(x)} u_g(Y) = g(x) \qquad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{3.7}$$

Proof. By density, we can find $g_n \in C_0(\partial\Omega)$ such that $\|g - g_n\|_{L^p(\partial\Omega,\sigma)} \leq 1/n$. The bound (3.6) implies that u_{g_n} is a Cauchy sequence for the uniform convergence on compact sets of Ω; in particular $u_g(X)$ is well defined - i.e. is independent on the the approximating sequence g_n - and is the limit of the Cauchy sequence $u_{g_n}(X)$. Then the Caccioppoli inequality (Proposition 2.55) implies that u_{g_n} converges in $W_{loc}(\Omega, m)$ to u_g and, since the set of weak-solution is closed under the weak-topology of $W_{loc}(\Omega, m)$, we obtain that u_g is a solution to $Lu = 0$.

It remains to show (3.7). Since $N(u_{g_n} - u_g)$ and $g_n - g$ converge to 0 in L^p, up to a subsequence, they also converge σ-almost everywhere. Take a point x where they both converge and $\epsilon > 0$. If $Y \in \Gamma(x)$, we have

$$|u_g(Y) - g(x)| \leq N(u_g - u_{g_n})(x) + |u_{g_n}(Y) - g_n(x)| + |g_n(x) - g(x)|.$$

Choose first n (independent of Y) such that $N(u_g - u_{g_n})(x) + |g_n(x) - g(x)| < \epsilon/2$. But since g_n is continuous, Proposition 2.63 shows that $u_{g_n}(Y)$ converges to $\operatorname{Tr} u_{g_n}(x) = g_n(x)$ and we can take Y close enough to x such that $|u_{g_n}(Y) - g_n(x)| < \epsilon/2$. The proposition follows. ☐

Proposition 3.15. *Let (Ω, m, σ) be suitable for PDE and traces, $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω, and $p \in (1, \infty)$. Are equivalent :*

(i) *$(D_{p,\sigma})_L$;*

(ii) *the elliptic measure ω_L defined in* (2.78) *is absolutely continuous with respect to σ and, there exists $C > 0$ such that the Poisson kernel $k_L := \frac{d\omega_L}{d\sigma}$ verifies*

$$\left(\fint_{B(x,r)\cap\partial\Omega} |k_L|^{p'} d\sigma \right)^{\frac{1}{p'}} \leq C \qquad \text{for } x \in \partial\Omega,\ r > 0.$$

(iii) *the elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ are absolutely continuous with respect to σ and, there exists $C > 0$ such that the Poisson kernels $k_L^X := \frac{d\omega_L^X}{d\sigma}$ verify*

$$\left(\fint_{B(x,r)\cap\partial\Omega} |k_L^X|^{p'} d\sigma \right)^{\frac{1}{p'}} \leq C \qquad \text{for } x \in \partial\Omega,\ r > 0,\ X \in \Omega \setminus B(x, 2r).$$

Proof. $(\mathbf{ii}) \iff (\mathbf{iii})$. This is the change of pole (Proposition 2.75).

$(\mathbf{ii}) \implies (\mathbf{i})$. If $g \in C_0(\partial\Omega)$ is the trace of a solution u, then we can construct a solution to the Dirichlet problem with boundary data g as in see (2.35) and get

$$u(X) := \int_{\partial\Omega} g(y)\, d\omega_L^X(y),$$

where Ω_L^X is the elliptic measure. Let $x \in \partial\Omega$ and $X \in \Gamma(x)$. Define Δ_k as $B(x, 2^k|X - x|) \cap \partial\Omega$, $u_0(X) := \int_{\Delta_0} |g|\, d\omega_L^X$. For $k \geq 1$, we set X_k to be a corkscrew point for Δ_k

and $u_k(X) := \int_{\Delta_k \setminus \Delta_{k-1}} |g| \, d\omega_L^X$. Using the Hölder continuity at the boundary (Proposition 2.63, the Carleson principle ((i) of Theorem 2.66), and the the change of pole in Proposition 2.79, we obtain that

$$u_k(X) \leq C 2^{-k\alpha} u_k(X_k) \leq C 2^{-k\alpha} \fint_{\Delta_k} |g| \, d\omega_L \leq C 2^{-k\alpha} \mathcal{M}_{\omega_L}(u)(x)$$

As a consequence

$$|u(X)| \leq \sum_{k \geq 0} u_k(X) \leq C \sum_{k \geq 0} 2^{-k\alpha} \mathcal{M}_{\omega_L}(g)(x) \leq C \mathcal{M}_{\omega_L}(g)(x)$$

and hence

$$N(u)(x) \leq C \mathcal{M}_{\omega_L}(g)(x). \tag{3.8}$$

The Poisson kernel $k_L := d\omega_L / d\sigma$ is reverse class $B_{p'}$, so by Theorem 2.83, the maximal Hardy-Littlewood functional $\mathcal{M}_{\omega_L}$ is bounded from $L^p(\sigma)$ to $L^p(\sigma)$. Together with (3.8), we have $\|N(u)\|_{L^p(\sigma)} \leq \|g\|_{L^p(\sigma)}$ as desired for $(ii) \Rightarrow (i)$.

$(\mathbf{i}) \implies (\mathbf{ii})$. First take $g \in C^0(\partial\Omega) \cap L^p(\partial\Omega, \sigma)$, and u_g defined as

$$u_g(X) := \int_{\partial\Omega} g(y) \, d\omega_L^X(y).$$

$X \in \Omega$. By setting $r := \delta(X)$ and then $x \in \partial\Omega$ such that $|X - x| = r$, we have that $X \in \Gamma(z)$ for all $z \in \Delta_X := B(x, r) \cap \partial\Omega$. As a consequence,

$$u_g(X) \leq \fint_{\Delta_X} N(u_g)(z) \, d\sigma(z) \leq \left(\fint_{\Delta_X} |N(u_g)|^p \, d\sigma \right)^{\frac{1}{p}} \leq C \sigma(\Delta_X)^{-1/p} \|g\|_{L^p(\partial\Omega, \sigma)} \tag{3.9}$$

by (D$_{p,\sigma}$)$_L$, with a constant independent of X.

Let now B be a ball centered on $\partial\Omega$, $\Delta := B \cap \partial\Omega$, X be a corkscrew point associated to Δ. Define $k_L := \frac{d\omega_L}{d\sigma}$ and $k_L^X := \frac{d\omega_L^X}{d\sigma}$ the Poisson kernels, where ω_L is the measure from Definition 2.78. Proposition 2.79 yields that

$$C^{-1} k_L^X(y) \leq \frac{k_L(y)}{\omega_L(\Delta)} \leq C k_L^X(y) \qquad \text{for } y \in \Delta,$$

so, by the duality of $L^p(\partial\Omega, \sigma)$ and $L^{p'}(\Omega, \sigma)$ and by the density of continuous functions in L^p, we obtain

$$\begin{aligned} \frac{1}{\omega_L(\Delta)} \left(\fint_{\Delta} |k_L|^{p'} \, d\sigma \right)^{\frac{1}{p'}} \leq C \left(\fint_{\Delta} |k_L^X|^{p'} \, d\sigma \right)^{\frac{1}{p'}} &= \sup_{\substack{g \in C_0^0(\Delta) \\ \|g\|_p \leq 1}} \int g(y) k^X(y) \, d\sigma \\ &= \sup_{\substack{g \in C_0^0(\Delta) \\ \|g\|_p \leq 1}} u_g(X) \leq C \sigma(\Delta)^{-1/p} \end{aligned}$$

by (3.9) and the fact that $\sigma(\Delta) \approx \sigma(\Delta_X)$, since σ is doubling. ☐

Corollary 3.16. *Let (Ω, m, σ) be suitable for PDE and traces, $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω, and $p \in (1, \infty)$. Then (D$_{p,\sigma}$)$_L$ implies (D$_{q,\sigma}$) for $q \in (p - \epsilon, \infty)$ and some $\epsilon > 0$.*

Proof. The characterization (ii) in Theorem 3.15 is a reverse-Hölder estimate on the Poisson kernel, so the solvability of the L^q Dirichlet problem for $q \in (p, \infty)$. The L^q solvability in the range $q \in (p-\epsilon, p)$ is a consequence of the well known fact that reverse Hölder estimates self-improve; see [Gia83, Proposition 1.1, p. 122]. $\square$

Our goal is to establish an analogue of Theorem 1.29, providing equivalent characterizations of the solvability of the L^p Dirichlet problem for some $p \in (1, \infty)$. We will begin by properly introducing the necessary concepts.

Definition 3.17. We say that (D$_{BMO,\sigma}$)$_L$ holds - or alternatively that the Dirichlet problem is solvable in BMO[1] - if there exists $C > 0$ such that for any $g \in C_0^\infty(\partial\Omega)$ the solution u_g constructed as in (3.5) satisfies

$$\delta_{\partial\Omega}|\nabla u| \in CM_\sigma(C\|g\|_{BMO(\partial\Omega,\sigma)}).$$

In practice, we can utilize a weaker version of BMO-solvability.

Definition 3.18. We say that the solutions to $Lu = 0$ satisfies the σ-Carleson Measure Estimate - or L satisfies (CME$_\sigma$) - if there exists $C > 0$ such that any bounded weak solution u to $Lu = 0$ in Ω satisfies

$$\delta_{\partial\Omega}|\nabla u| \in CM_\sigma(C\|u\|_{L^\infty(\Omega)}).$$

Theorem 3.19. *Let (Ω, m, σ) be suitable for PDE and traces and let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator (with respect to the weight $w := dm/dx$). We write ω_L^X for the elliptic measure constructed in Theorem 2.53. Then the following are equivalent :*

(a) The elliptic measure ω_L^X is A_∞-absolutely continuous with respect to σ - or equivalently $\omega_L \in A_\infty(\sigma)$ for the measure ω_L from Definition 2.78.

(b) There exists $p \in (1, \infty)$ such that (D$_{p,\sigma}$)$_L$ is solvable in $L^p(\partial\Omega, \sigma)$.

(c) (D$_{BMO,\sigma}$)$_L$ is solvable.

(d) L satisfies (CME$_\sigma$).

(e) There exists $q \in (1, \infty)$ and $C > 0$ such that for any weak solution u to $Lu = 0$, we have

$$\|Su\|_{L^q(\partial\Omega,\sigma)} \leq C\|Nu\|_{L^q(\partial\Omega,\sigma)}. \tag{3.10}$$

(f) For any $q \in (0, \infty)$, there exists $C = C(q)$ such that we have (3.10) *for any weak solution u to $Lu = 0$.*

Remark 3.20. The implications $(a) \implies (c)$ and $(a) \implies (d)$ do not require Ω to be uniform, as shown by the proofs below.

Throughout this section, we will rigorously prove $(a) \iff (b) \implies (c), (d)$, and provide intuition for the other equivalences. For further details, all the given equivalences are proven in [CY25], although our approach will differ slightly from theirs.

Our strategy will involve proving $(a) \implies (c)$ by refining the method used by the author in [FP22] to establish $(a) \implies (d)$.

In simpler settings, many of the estimates can be found in [Ken94], for instance $(a) \iff (b)$ is [Ken94, Theorem 1.7.3], while (3.18) is [Ken94, Theorem 1.5.18], $(a) \Rightarrow$

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[1] While we could theoretically state that the Dirichlet problem is solvable in VMO, we will adhere to the common terminology here.

$(e), (f)$ is [Ken94, Theorem 1.5.10], the good-λ argument is used to prove [Ken94, Theorems 1.5.11 and 1.5.12]; note also that the results from [Ken94, Section 1.5] is based in [JK82]. For domains satisfying some capacity condition, the theorem is proved in [CDMT22]. The equivalence $(c) \iff (a)$ was first established in [DKP11], see also [HL18], [Zha18], [MZ19], [GH21] for more complicated settings or weaker assumptions. The implication $(d) \Rightarrow (a)$ was first proved in [KKPT16] improving an earlier result from [KKPT00], see also [DFM19a], [CHMT20], [CHM19].

$(\mathbf{a}) \iff (\mathbf{b})$. This is Proposition 3.15 and Theorem 2.83 $(i) \iff (iv)$.

$(\mathbf{a}) \implies (\mathbf{d})$. Let ω_L, G_{L^*} be the elliptic measure and the Green function from Definition 2.78. We have $\omega_L \in A_\infty(\sigma)$ so thanks to (ii) of Theorem 3.11, we only need to prove the following lemma

Lemma 3.21. *Let $\Omega \subset \mathbb{R}^n$, m be suitable for PDE, and $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator (with respect to the weight $w := dm/dx$). Then there exists $C > 0$ such that for any weak solution u to $Lu = 0$, we have*

$$\delta_{\partial\Omega}|\nabla u| \in CM_{\omega_L}(C\|u\|_\infty). \tag{3.11}$$

More precisely, for any $x \in \partial\Omega$ and $r > 0$, we have

$$\int_{\partial\Omega} |\mathfrak{A}(\delta_{\partial\Omega}|\nabla u|\mathbb{1}_{B(x,r)})|^2 \, d\omega_L \le C \int_{\partial\Omega} |N(u\mathbb{1}_{B(x,2r)})|^2 \, d\omega_L. \tag{3.12}$$

Proof. Observe that (3.11) is a direct consequence of (3.12), since the function $N(u\mathbb{1}_{B(x,2r)})$ is bounded by $\|u\|_\infty$ and supported in $B(x, 100r) \cap \partial\Omega$. With Remark 3.8, we need to establish that

$$\int_{B(x,r)\cap\Omega} |\delta_{\partial\Omega}(Y)\nabla u(Y)|^2 \frac{\omega_L(4B_Y)}{m(B_Y)} dm(Y) \le C \int_{\partial\Omega} |N(u\mathbb{1}_{B(x,2r)})|^2 \, d\omega_L.$$

We can assume that r is sufficiently smaller than $\operatorname{diam}\Omega$, as the estimate on $|\nabla u|(Y)$ when $\delta_{\partial\Omega}(Y) \approx \operatorname{diam}(\Omega)$ can be treated with the Caccioppoli inequality (Proposition 2.55). When $x \in \partial\Omega$ and r is sufficiently small, we have in $B(x, 2r) \cap \Omega$ that the function G_{L^*} is a solution to $L^*u = 0$ and that $\delta_{\partial\Omega}(Y)^2\omega_L(4B_Y)/m(B_Y) \approx G_{L^*}(Y)$. So we can write

$$\begin{aligned} I = \int_{B(x,r)\cap\Omega} |\delta_{\partial\Omega}(Y)\nabla u(Y)|^2 \frac{\omega_L(4B_Y)}{m(B_Y)} dm(Y) &\le \int_{B(x,r)\cap\Omega} |\nabla u|^2 \, G_{L^*} dm \\ &\le C \int_{B(x,r)\cap\Omega} \mathcal{A}\nabla u \cdot \nabla u \, G_{L^*} dm \end{aligned}$$

by the ellipticity of $A =: w\mathcal{A}$. Let $\varphi \in C_0^\infty((B(x, 2r))$ be a cut-off function for $B(x, r)$, that is $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B(x, r)$, and $|\nabla\varphi| \le 2/r$. One can check that

$$\mathfrak{A}(\delta_{\partial\Omega}|\nabla\varphi|)(y) \le \frac{2}{r}\mathfrak{A}(\delta_{\partial\Omega}\mathbb{1}_{B(x,r)}) \le C\mathbb{1}_{y\in B(x,10r)},$$

similarly

$$\mathfrak{A}(\delta_{\partial\Omega}^{1/2}|\nabla\varphi|^{1/2})(y) \le \sqrt{\frac{2}{r}}\mathfrak{A}(\delta_{\partial\Omega}^{1/2}\mathbb{1}_{B(x,r)}) \le C\mathbb{1}_{y\in B(x,10r)},$$

and even

$$\mathfrak{A}\Big(\delta_{\partial\Omega}^{1/2}|\nabla\varphi|^{1/2}\frac{\delta|\nabla G_{L^*}|}{G_{L^*}}\Big)(y) \leq C\mathbb{1}_{y\in B(x,10r)},$$

because of the Caccioppoli and the Harnack inequalities. That is, there is a $M > 0$ such that

$$\delta_{\partial\Omega}|\nabla\varphi| + \delta_{\partial\Omega}^{1/2}|\nabla\varphi|^{1/2}\Big(1 + \frac{\delta|\nabla G_{L^*}|}{G_{L^*}}\Big) \in CM_\sigma(M). \tag{3.13}$$

for any measure σ on $\partial\Omega$, in particular for $\sigma = \omega_L$.

Let us return to our integral. We have $I \leq CI_0$, where

$$\begin{aligned} I_0 &:= \int_\Omega \mathcal{A}\nabla u \cdot \nabla u\,(G_{L^*}\varphi^2)\,dm \\ &= \int_\Omega \mathcal{A}\nabla u \cdot \nabla[uG_{L^*}\varphi^2]\,dm - 2\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi\,(G_{L^*}\varphi u)\,dm \\ &\quad - \int_\Omega \mathcal{A}\nabla u \cdot \nabla G_{L^*}\,(u\varphi^2)\,dm =: I_1 + I_2 + I_3. \end{aligned} \tag{3.14}$$

The term I_1 is 0 because u is weak solution to $Lu = 0$. The term I_2 is bounded by

$$\begin{aligned} (I_0)^{1/2}\left(\int_\Omega |\nabla\varphi|^2 uG_{L^*}dm\right)^{\frac{1}{2}} &\leq C(I_0)^{1/2}\left(\int_{\partial\Omega} |\mathfrak{A}(\delta_{\partial\Omega}|\nabla\varphi|u)|^2 d\omega_L\right)^{\frac{1}{2}} \\ &\leq CM(I_0)^{1/2}\|N(u\mathbb{1}_{B(x,2r)})\|_{L^2(\partial\Omega,\omega_L)}, \end{aligned}$$

by the Carleson inequality (Theorem 3.9) and the fact that $\delta_{\partial\Omega}|\nabla\varphi| \in CM_\sigma(M)$. As for the term I_3, we want to use the fact that G_{L^*} is a solution to $L^*u = 0$ so we write

$$I_3 = -\frac{1}{2}\int_\Omega \mathcal{A}\nabla[u^2\varphi^2] \cdot \nabla G_{L^*}\,dm + \int_\Omega \mathcal{A}\nabla\varphi \cdot \nabla G_{L^*}\,(u^2\varphi)\,dm := I_4 + I_5.$$

The integral I_4 is zero, because G_{L^*} is a weak solution on $\operatorname{supp}\varphi$, and I_5 is bounded by

$$C\int_\Omega \delta_{\partial\Omega}|\nabla\varphi|\Big(1 + \frac{\delta_{\partial\Omega}^2|\nabla G_{L^*}|^2}{G_{L^*}^2}\Big)u^2\,G_{L^*}\,dm \leq C'M^2\|N(u\mathbb{1}_{B(x,2r)})\|_{L^2(\partial\Omega,\omega_L)}^2$$

again by the Carleson inequality and (3.13). We conclude that

$$I_0 \leq C(I_0)^{1/2}M\|N(u\mathbb{1}_{B(x,2r)})\|_{L^2(\partial\Omega,\omega_L)} + CM^2\|N(u\mathbb{1}_{B(x,2r)})\|_{L^2(\partial\Omega,\omega_L)}^2,$$

which self-improves[2] into

$$I \leq CI_0 \leq C\|N(u\mathbb{1}_{B(x,2r)})\|_{L^2(\partial\Omega,\omega_L)}^2 \tag{3.15}$$

as desired □

$(\mathbf{a}) \implies (\mathbf{c})$. As for $(a) \implies (d)$, Theorem 3.11 can be used to reduce the proof to the following lemma.

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[2] We should check that I_0 is *a priori* finite. It is done by taking the cut-off $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on $B(x,r)\cap\{Y \geq \epsilon\}$, $\varphi \equiv 0$ on $\Omega\cap\{Y \leq \epsilon/2\}$, $|\nabla\varphi| \leq 2/\epsilon$ on $\{\epsilon/2 < Y < \epsilon\}$. Under those extra condition, I_0 is finite, so we can do the self-improvement with the same upper bound which is independent on ϵ, and we conclude by taking $\epsilon \to 0$.

Lemma 3.22. *Let $\Omega \subset \mathbb{R}^n$, m be suitable for PDE, and $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator (with respect to the weight $w := dm/dx$). Then, there exists $C > 0$ such that for any $g \in C_0(\partial\Omega)$, $x \in \partial\Omega$, and $r > 0$, the weak solution u_g constructed in* (2.35) *satisfies*

$$\delta_{\partial\Omega}|\nabla u_g| \in CM_{\omega_L}(C\|g\|_{BMO(\partial\Omega,\omega_L)}). \tag{3.16}$$

Proof. Write B for $B(x,r)$ and Δ for $B \cap \partial\Omega$. We decompose

$$\begin{aligned} g = c_0 + (g-c_0)\mathbb{1}_{4\Delta} + \sum_{k\geq 1}(g-c_k)\mathbb{1}_{2^{k+2}\Delta\setminus 2^{k+1}\Delta} + \sum_{k\geq 1}(c_k - c_{k-1})\mathbb{1}_{\partial\Omega\setminus 2^{k+1}\Delta} \\ =: c_0 + \sum_{k\geq 0} g_k + \sum_{k\geq 1} h_k, \end{aligned}$$

where $c_k := \fint_{2^{k+2}\Delta} g\, d\omega_L$. and we define $u_k := u_{g_k}$ and $v_k := u_{h_k}$ as in (2.35). The function g_k is not continuous but that is not a problem. We have $u := u_g = \sum_{k\geq 0} u_k$ and thus

$$\begin{aligned} \|\mathfrak{A}(\delta_{\partial\Omega}|\nabla u|\mathbb{1}_B)\|_{L^2(\omega_L)} \\ \leq \sum_{k\geq 0}\|\mathfrak{A}(\delta_{\partial\Omega}|\nabla u_k|\mathbb{1}_B)\|_{L^2(\omega_L)} + \sum_{k\geq 1}\|\mathfrak{A}(\delta_{\partial\Omega}|\nabla v_k|\mathbb{1}_B)\|_{L^2(\omega_L)} \\ \leq C\sum_{k\geq 0}\|N(u_k\mathbb{1}_{2B})\|_{L^2(\omega_L)} + C\sum_{k\geq 1}\|N(v_k\mathbb{1}_{2B})\|_{L^2(\omega_L)} \end{aligned} \tag{3.17}$$

by (3.12). Arguing like in $(a) \iff (b)$, we obtain

$$N(u_0\mathbb{1}_{2B}) \leq C\mathcal{M}_{\omega_L}(|g-c_0|\mathbb{1}_{4\Delta}),$$

that is, with the $L^2(\omega_L)$ boundedness of $\mathcal{M}_{\omega_L}$,

$$\|N(u_0\mathbb{1}_{2B})\|_{L^2(\omega_L)} \leq C\|g-c_0\|_{L^2(4\Delta,\omega_L)} \leq C\|g\|_{BMO(\omega_L)}\omega_L(\Delta)^{\frac{1}{2}}.$$

When $k \geq 1$, we have instead

$$N(u_k\mathbb{1}_{2B})(y) \leq C2^{-k\alpha}\mathcal{M}_{\omega_L}(|g-c_k|\mathbb{1}_{2^{k+2}\Delta\setminus 2^{k+1}\Delta})(y)$$

and thus,

$$\|N(u_k\mathbb{1}_{2B})\|_{L^2(\omega_L)} \leq C2^{-k\alpha}\|g-c_k\|_{L^2(\omega_L,2^{k+2}\Delta)} \leq C2^{-k\alpha}\|g\|_{BMO(\omega_L)}\omega_L(\Delta)^{\frac{1}{2}}.$$

Similarly, since $|c_k - c_{k-1}| \leq C\|g\|_{BMO(\omega_L)}$, we have

$$N(v_k\mathbb{1}_{2B})(y) \leq C2^{-k\alpha}|c_k - c_{k-1}| \leq C\|g\|_{BMO(\omega_L)},$$

hence

$$\|N(v_k\mathbb{1}_{2B})\|_{L^2(\omega_L)} \leq C2^{-k\alpha}\|g\|_{BMO(\omega_L)}\omega_L(\Delta)^{\frac{1}{2}}.$$

We gather the estimates so that (3.17) becomes

$$\omega_L(\Delta)^{-1/2}\|\mathfrak{A}(\delta_{\partial\Omega}|\nabla u|\mathbb{1}_B)\|_{L^2(\omega_L)} \leq C\|g\|_{BMO(\omega_L)}. \tag{3.18}$$

Since the bound is independent of Δ, we deduce that $\delta_{\partial\Omega}|\nabla u| \in CM_{\omega_L}(C\|g\|_{BMO(\omega_L)})$ as desired. ☐

$(\mathbf{a}) \implies (\mathbf{e}) \iff (\mathbf{f})$. A good-$\lambda$ argument is a type of method that will give the second part of Proposition 3.5 from Proposition 3.3, and that relies on the fact that the functionals N and $\mathfrak{A}$ are defined on cones.

With a good-λ argument, we can show that (3.12) self-improves into for any boundary ball

$$\Delta \subset S_{t/2} := \Big\{ \mathfrak{A}(\delta_{\partial\Omega} |\nabla u| \mathbb{1}_{B(x,r)}) > \frac{t}{2} \Big\}$$

satisfying $10\Delta \not\subset S_{t/2}$, any $\gamma \in (0,1)$, and any $r > 0$ we have

$$\frac{\omega_L(\Delta \cap S_t \cap \{\mathcal{M}_r[N(u\mathbb{1}_{B(x,2r)})] \leq \gamma t\})}{\omega_L(\Delta)} \leq C_r \gamma^2,$$

where γ is independent of x, r, u, and Δ. Since $\omega_L \in A_\infty(\sigma)$, we can apply (iii) of Theorem 2.83, which gives that

$$\frac{\sigma(\Delta \cap S_t \cap \{\mathcal{M}_r[N(u\mathbb{1}_{B(x,2r)})] \leq \gamma t\})}{\sigma(\Delta)} \leq C_r \gamma^p,$$

for some $p \in (1,\infty)$. By doing a Whitney decomposition of $S_{t/2}$ and applying the previous line on every Whitney region, we have

$$\sigma(S_t \cap \{\mathcal{M}_r[N(u\mathbb{1}_{B(x,2r)})] \leq \gamma t\}) \leq C_r \gamma^p \sigma(S_{t/2}),$$

and then, thanks to the Cavalieri formula,

$$\|\mathfrak{A}(\delta_{\partial\Omega} |\nabla u| \mathbb{1}_{B(x,r)})\|_{L^q(\sigma)} \leq C_q \|N(u\mathbb{1}_{B(x,2r)})\|_{L^q(\sigma)} \tag{3.19}$$

for any $q > 0$. (e) and (f) are then obtained by taking $r \to \infty$ in (3.19). Note that a similar "good-λ" argument gives $(e) \Rightarrow (f)$.

$(\mathbf{f}) \implies (\mathbf{d})$. We obtain (3.12) - which is stronger than (d) - by localizing the estimate $\|S(u)\|_{L^2(\sigma)} \leq C\|N(u)\|_{L^2(\sigma)}$.

$(\mathbf{c}), (\mathbf{d}) \implies (\mathbf{a})$. We want to prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that for any boundary ball $\Delta := B \cap \partial\Omega$ and any Borel set $E \subset \Delta$, we have

$$\frac{\omega_L(E)}{\omega_L(\Delta)} < \delta \implies \frac{\sigma(E)}{\sigma(\Delta)} < \epsilon. \tag{3.20}$$

The proof is similar for both cases (c) or (d). We need to construct a "ϵ_0-good cover" of E, namely we have

$$E = \mathcal{O}_0 \subset \mathcal{O}_1 \subset \cdots \subset \mathcal{O}_{2N} = \Delta,$$

where $\mathcal{O}_k = \bigcup_j Q_{k,j}$ is a union of dyadic cubes in $\partial\Omega$, and $\omega_L(\mathcal{O}_k \cap Q_{k+1,j}) \leq \epsilon_0 \omega_L(Q_{k+1,j})$ for some relevant dyadic cubes Q. The key property is that we can do this construction with number N depends only on $\omega_L(E)/\omega_L(\Delta)$, ϵ_0, and the doubling constant of ω_L, and such that $N \to \infty$ as $\omega_L(E)/\omega_L(\Delta) \to 0$. We construct f to be morally $\mathbb{1}_E + \sum_{k=1}^N \mathbb{1}_{\mathcal{O}_{2k} \setminus \mathcal{O}_{2k-1}}$ or a smooth approximation of it, and u to be the solution with boundary data f.

The solution u exhibits large oscillations on Whitney cubes, which in turn yield a large square function $\mathfrak{A}(|\nabla u| \mathbb{1}_B)(x)$ for any point $x \in E$, which is bounded from below by $c \ln(\omega_L(E)/\omega_L(\Delta))$. Since u is bounded by 1, having (c) or (d) implies then that

$$c \Big| \ln \Big(\frac{\omega_L(\Delta)}{\omega_L(E)} \Big) \Big|^2 \sigma(E) \leq \|\mathfrak{A}(|\nabla u| \mathbb{1}_{10B})\|_{L^2(\sigma)} \leq C\sigma(\Delta),$$

which gives (3.20).

3.3 . Carleson perturbations

We have identified a collection of operators for which the elliptic measure is A_∞-absolutely continuous with respect to the surface measure. An intriguing question is whether this A_∞-absolute continuity remains stable under perturbations of the coefficients. We will bypass the theory of t-independent perturbations, as t-independent operators are not well-suited for studying domains with rough boundaries, in particular of high or mixed codimension.

Given two uniformly elliptic operators $L_0 := -\operatorname{div} A_0 \nabla$ and $L_1 = \operatorname{div} A_0 \nabla$ in Ω, we ask : Is there a condition on $|A_0 - A_1|$ such that the elliptic measure of L^1 is A_∞-absolutely continuous with respect to the surface measure if and only if the elliptic measure of L_0 is? The answer is affirmative, and this condition is commonly referred to as "Carleson perturbation", due to its relation to Carleson measures and the Carleson measure condition (see Definition 3.7). This is not surprising, as the appropriate notion of perturbation was studied and established between 1984 and 1991, culminating in Theorem 1.36 for the codimension 1 setting.

The concept of Carleson perturbation extends naturally to our framework. Although many earlier proofs relied on certain geometric conditions on the boundaries, the proof provided by Bruno Poggi and the author in [FP22] leverages the rich elliptic theory available.

Theorem 3.23. *Let Ω be uniform and m be suitable for PDE. Take two operators $L_0 := -\operatorname{div} A_0 \nabla$ and $L_1 := -\operatorname{div} A_1 \nabla$ that are uniformly elliptic with respect to the weight $w := dm/dX$. Let ω_0 and ω_1 be the measures introduced in Definition 2.78 for L_0 and L_1 respectively. If the disagreement $|A_0 - A_1|$ satisfies*

$$X \to \sup_{B_X} \{w^{-1}|A_0 - A_1|\} \in CM_{\omega_0}(M) \tag{3.21}$$

for some constant M (where $B_X := B(X, \delta_{\partial\Omega}(X)/2)$ as before), then $\omega_1 \in A_\infty(\omega_0)$. By Theorem 3.11, if we a priori *had $\omega_0 \in A_\infty(\sigma)$, then we also have $\omega_1 \in A_\infty(\sigma)$.*

Moreover, if the function $G_0 = G_{L_0^*}$ constructed in Definition 2.78 satisfies the pointwise bound

$$\sup_\Omega \frac{\delta_{\partial\Omega}|\nabla G_0|}{G_0} < \infty^{3}, \tag{3.22}$$

then the condition (3.21) *on the disagreement can be replaced by*

$$w^{-1}|A_0 - A_1| \in CM_{\omega_0}(M). \tag{3.23}$$

Proof. The following proof is largely taken from [FP22] and [Fen24]. By Theorem 3.19, we only need to prove that the weak solutions u to $L_1 u = 0$ satisfies (CME$_{\omega_0}$). The proof is a variant of the one of Lemma 3.21, in particular, the preliminary computations are the same. We write $\mathcal{A}_i$ for $w^{-1}A_i$, $i \in \{0,1\}$. We replace then (3.14) by

$$\begin{aligned} I_0 &:= \int_\Omega \mathcal{A}_1 \nabla u \cdot \nabla u \, (G_0 \varphi^2) \, dm \\ &= \int_\Omega \mathcal{A}_1 \nabla u \cdot \nabla [u G_0 \varphi^2] \, dm - 2 \int_\Omega \mathcal{A}_1 \nabla u \cdot \nabla \varphi \, (G_0 \varphi u) \, dm \\ &\qquad - \int_\Omega \mathcal{A}_1 \nabla u \cdot \nabla G_0 \, (u \varphi^2) \, dm =: I_1 + I_2 + I_3. \end{aligned}$$

---

[3] This is the case when $w \equiv 1$ and $\delta_{\partial\Omega}|\nabla A_0| \in L^\infty$ for instance, see Lemma 3.1 in [GW82]

The term I_1 is 0, since u is a weak solution, and the term I_2 is treated as in the proof of Lemma 3.21. As for I_3, we write

$$\begin{aligned} I_3 = -\int_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G_0 \, (u\varphi^2) \, dm &+ \int_\Omega (\mathcal{A}_0 - \mathcal{A}_1) \nabla u \cdot \nabla G_0 \, (u\varphi^2) \, dm \\ = -\frac{1}{2} \int_\Omega \mathcal{A} \nabla [u^2\varphi^2] \cdot \nabla G_{L^*} \, dm &+ \int_\Omega \mathcal{A} \nabla \varphi \cdot \nabla G_{L^*} \, (u^2 \varphi) \, dm \\ &+ \int_\Omega (\mathcal{A}_0 - \mathcal{A}_1) \nabla u \cdot \nabla G_0 \, (u\varphi^2) \, dm =: I_4 + I_5 + I_6. \end{aligned}$$

The integral I_4 is 0 since G_0 is a solution to L_0^*, I_5 is treated as in the proof of Lemma 3.21, and the extra term I_6 is bounded as follows.

$$\begin{aligned} I_6 &\leq I_0^{1/2} \left(\int_\Omega |\mathcal{A}_0 - \mathcal{A}_1|^2 u^2 \varphi^2 \frac{\delta_{\partial\Omega}^2 |\nabla G_0|^2}{G_0^2} \, G_0 \, dm \right)^{\frac{1}{2}} \\ &\leq C I_0^{1/2} \left\| \mathcal{C}_{\omega_0} \left(|\mathcal{A}_0 - \mathcal{A}_1| \frac{\delta_{\partial\Omega} |\nabla G_0|}{G_0} \right) \right\|_{L^\infty(\omega_0)} \|N(u\varphi)\|_{L^2(\omega_0)}. \end{aligned}$$

by the Cauchy-Schwarz and the Carleson inequalities (Theorem 3.9). In both scenarios (3.21)–(3.23), we have

$$\left\| \mathcal{C}_{\omega_0} \left(|\mathcal{A}_0 - \mathcal{A}_1| \frac{\delta_{\partial\Omega} |\nabla G_0|}{G_0} \right) \right\|_{L^\infty(\omega_0)} \leq CM, \tag{3.24}$$

which allow us to say that

$$I_6 \leq C I_0^{1/2} M \|N(u\varphi)\|_{L^2}$$

and conclude as in the proof of Lemma 3.21.

Indeed, if $\delta_{\partial\Omega} |\nabla G_0| / G_0$ is uniformly bounded, then the bound (3.24) is simply the definition of the Carleson measure condition (3.23). In the other case, a simple use of Fubini's lemma[4] gives that, for any $f \in L^2_{loc}(\Omega, m)$, we have

$$\|C_{\omega_0}(f)\|_{L^\infty} \leq C \|C_{\omega_0}(\tilde{f})\|_{L^\infty}$$

with $\tilde{f}(X) = \left(\fint_{B_X} f^2 \, dm \right)^2$. Thus with our choice $f = |\mathcal{A}_0 - \mathcal{A}_1| \frac{\delta_{\partial\Omega} |\nabla G_0|}{G_0}$, we have

$$\tilde{f}(X) \leq C \Big(\sup_{B_X} |\mathcal{A}_0 - \mathcal{A}_1| \Big) \left(\fint_{B_X} \frac{\delta_{\partial\Omega}^2 |\nabla G_0|^2}{G_0^2} \, dm \right)^{\frac{1}{2}} \leq C \sup_{B_X} |\mathcal{A}_0 - \mathcal{A}_1| \tag{3.25}$$

by applying the Caccioppoli inequality (Proposition 2.55) and the Harnack inequality (Proposition 2.61) to G_0. The bound (3.24) is then a consequence of our assumption (3.21). □

Remark 3.24. As mentioned in [FKP91, Proposition 2.22] and proved in [DSU22]. We can weaken the assumption (3.21) to

$$X \to \left(\fint_{B_X} |w^{-1}(A_0 - A_1)|^r \, dm \right)^{\frac{1}{r}} \in CM_{\omega_0}(M) \tag{3.26}$$

---

[4] and Corollary 3.6, which allow us to use $\mathfrak{A}_\alpha$ for various values of α

for some r large enough. Indeed, the combination of the Cacciopoli inequality, the Harnack inequality and the Poincaré inequality gives that for any ball B such that $2B \subset \Omega$,

$$\left(\fint_B \left| \frac{\delta_{\partial\Omega} \nabla G_0}{G_0} \right|^2 dm \right)^{\frac{1}{2}} \leq C_p \left(\fint_{2B} \left| \frac{\delta_{\partial\Omega} \nabla G_0}{G_0} \right|^p dm \right)^{\frac{1}{p}}$$

for some $p < 2$ and C independent of B and that depends on L only via the constants in Definition 2.43, and as such, we can improve the bound to

$$\left(\fint_B \left| \frac{\delta_{\partial\Omega} \nabla G_0}{G_0} \right|^q dm \right)^{\frac{1}{q}} \leq C' \left(\fint_{2B} \left| \frac{\delta_{\partial\Omega} \nabla G_0}{G_0} \right|^2 dm \right)^{\frac{1}{2}} \leq C''$$

for some $q > 2$ that depends only on p and C_p, see for instance [Gia83, Proposition 1.1]. As a consequence, we can change (3.25) to

$$\begin{aligned} \tilde{f}(X) \leq C \left(\fint_{B_X} |w^{-1}(A_0 - A_1)|^r \, dm \right)^{\frac{1}{r}} \left(\fint_{B_X} \left| \frac{\delta_{\partial\Omega} \nabla G_0|}{G_0} \right|^q dm \right)^{\frac{1}{q}} \\ \leq C \left(\fint_{B_X} |w^{-1}(A_0 - A_1)|^r \, dm \right)^{\frac{1}{r}}, \quad (3.27) \end{aligned}$$

where $r = q/(q-2)$, which means that assuming (3.26) for this choice of r is sufficient.

Current results for small Carleson perturbations have not been proven in as general a setting as in Theorem 3.23.

Theorem 3.25. *Let $d \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be an open set such that $\partial\Omega$ is d-Ahlfors regular(σ is the Ahlfors regular measure). If $d = n-1$, we also assume that Ω is uniform. Take two operators $L_0 := -\operatorname{div} A_0 \nabla$ and $L_1 := -\operatorname{div} A_1 \nabla$ that are uniformly elliptic with respect to the weight $w := \delta_{\partial\Omega}^{d+1-n}$.*

Assume that ($D_{p,\sigma}$)$_{L_0}$ is solvable for some $p \in (1, \infty)$. Then there exists $\epsilon > 0$ such that, if the the disagreement $|A_0 - A_1|$ satisfies

$$X \to \sup_{B_X} \{w^{-1}|A_0 - A_1|\} \in CM_\sigma(\epsilon), \quad (3.28)$$

then ($D_{p,\sigma}$)$_{L_1}$ is solvable for the same p.

Proof. Theorem 3.25 is proved in [CHM19] in the classical case and [MP21] in higher codimension. The author claims that the theorem can be extended to the general setting assumed here - i.e. (Ω, m, σ) is suitable for PDE and traces - but the details, although not terribly complicated, are too long to place here. ☐

3.4 . Solvability of the L^p Dirichlet problem in domains with flat boundaries

In Section 1.5, we saw that when $d < n-1$ is an integer, the natural equivalent of the Laplacian in $\mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times (\mathbb{R}^{n-d} \setminus \{0\})\}$ is $L_0 := -\operatorname{div}[|t|^{d+1-n}\nabla]$. However, this operator does not naturally extend to sets of the form $\mathbb{R}^n \setminus E$, where E is a d-Ahlfors regular set. A naive choice for a "Laplacian" on these sets would be

$$L := -\operatorname{div}[\delta_E^{d+1-n}\nabla],$$

but the lack of regularity in its coefficients makes it unsuitable for our methods.

The plan for this section is as follows :

1. Identify the largest class of operators in $\mathbb{R}^n \setminus \mathbb{R}^d$ whose elliptic measure is A_∞-absolutely continuous with respect to $\mathcal{L}^d$, or equivalently, where the L^p Dirichlet problem is solvable for some $p \in (1, \infty)$.
2. Address domains that are complements of a Lipschitz graph E_φ. The usual strategy involves a change of variables that maps $\mathbb{R}^n \setminus E_\varphi$ to $\mathbb{R}^n \setminus \mathbb{R}^d$, transforming "simple" operators into those we can analyze from (1).
3. Determine a suitable choice of "simple" operators that can serve as the Laplacian in higher codimensions.

Thus, we will seek the largest class of operators in $\mathbb{R}^n \setminus \mathbb{R}^d = \{(x,t) \in \mathbb{R}^d \times (\mathbb{R}^{n-d} \setminus \{0\})\}$ for which the elliptic measure is A_∞-absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, or equivalently for which the L^p Dirichlet problem is solvable for a $p \in (1, \infty)$.

We will disregard operators $L := -\operatorname{div} A\nabla$ where $|t|^{d+1-n}A$ is independent of t, as these are ill-suited for studying boundaries without a uniform non-tangential direction. Instead, we will focus on operators satisfying a Carleson condition.

When the boundary is a plane, we have :

Theorem 3.26. *Let $L := -\operatorname{div} A\nabla$ to be a uniformly elliptic operator (with respect to $dm := |t|^{d+1-n}dt\,dx$) on $\mathbb{R}^n \setminus \mathbb{R}^d$ (the measure on $\mathbb{R}^d$ is $\sigma = \mathcal{L}^d$). Assume that there exists $\lambda \geq 1$, $M > 0$ such that the reduced matrix $\mathcal{A} = |t|^{n-d-1}A$ can be written as*

$$\mathcal{A} = \left[\begin{array}{c:c} \mathcal{A}_1 & \mathcal{A}_2 \\ \hdashline \mathcal{B}_3 + \mathcal{C}_3 & a_4 Id_{n-d} + \mathcal{C}_4 \end{array}\right]$$

where

(i) $\lambda^{-1} \leq a_4 \leq \lambda$,

(ii) $|\mathcal{C}_3| + |\mathcal{C}_4| + |t||\nabla a_4| \in CM_\sigma(M)$,

(iii) either there is a vector b_3 such that $\mathcal{B}_3 = \frac{t^T}{t}\mathbf{b}_3$ and $|t||\nabla \mathbf{b}_3| \in CM_{\mathcal{L}^d}(M)$, or $|t||\nabla \mathcal{B}_3| \in CM_{\mathcal{L}^d}(M)$.

Then the elliptic measure ω_L is A_∞-absolutely continuous with respect to $\mathcal{L}^d$ and (D$_{p,\sigma}$)$_L$ is solvable for a $p \in (1, \infty)$ large enough.

Furthermore, for any $p \in (1, \infty)$, there exists $\epsilon_p > 0$ that depends only on d, n, the elliptic constants of A, λ, and p, such that if $M < \epsilon_p$ in the above condition (i) and (iii), then (D$_{p,\sigma}$)$_L$ holds.

When the boundary is a plane, we observe that no assumptions are needed on the tangential directions of the matrix (i.e., the first d rows). The terms $\mathcal{C}_3$ and $\mathcal{C}_4$ are Carleson perturbations, which is expected if we seek the largest class. We want the bottom right corner of $\mathcal{A}$ to be, up to a Carleson perturbation, a scalar multiple of the identity. The intuition behind this is that we seek an operator $L_0 = -\operatorname{div} A_0\nabla$ close to L such that L_0^* has an explicit Green function with a pole at infinity, equivalent to the distance to the boundary. This allows us to perform integrations by parts, as in the proof of Lemma 3.21. To cancel the weight $|t|^{d+1-n}$, our best choice is to take $|t|$ as the Green function with a pole at infinity and the identity in the bottom right corner of A_0.

Proof. When $\mathcal{B}_3 = 0$, this theorem is [DFM19a, Theorem 7.10], and it was shown in [Fen22b, Theorem 3.16] why we can go back to the situation where $\mathcal{B}_3 = 0$ when either $|t||\nabla \mathcal{B}_3| \in CM_{\mathcal{L}^d}$ or $\mathcal{B}_3 = t^T\mathbf{b}_3$ with $|t||\nabla \mathbf{b}_3| \in CM_{\mathcal{L}^d}$.

When $\mathcal{B}_3 = t^T \mathbf{b}_3$ with $|t||\nabla \mathbf{b}_3| \in CM_\sigma(M)$, the small constant case is proved as in [FMZ21]. When $|t||\nabla \mathcal{B}_3| \in CM_{\mathcal{L}^d}$, we use the trick of [Fen22b] that allow us to go back to $\mathcal{B}_3 = 0$ at a price of a small Carleson perturbation.

Interested readers can refer to the above references for the full demonstration. However, let us build some intuition and provide elements of the proof here (for large Carleson perturbations). To simplify the proof, we will not be rigorous; in particular, we will not justify why certain quantities are well-defined or *a priori* finite. Furthermore, we will not include the term $\mathcal{B}_3$ in the proof.

Our goal is to prove that $\|S(u)\|_{L^2(\mathbb{R}^d)} \leq C\|N(u)\|_{L^2(\mathbb{R}^d)}$, from which the L^p solvability of the Dirichlet problem will follow by Theorem 3.19. Using Fubini's theorem, the ellipticity of A, and the fact that $b_4 \approx 1$, we have

$$\|S(u)\|_2^2 \leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} b_4^{-1}\mathcal{A}\nabla u \cdot \nabla u \, |t|^{d+2-n} dt\, dx.$$

But since u is a solution, one has by integration by parts[5] that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} b_4^{-1}\mathcal{A}\nabla u \cdot \nabla u \, |t|^{d+2-n} dt\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} \mathcal{A}\nabla u \cdot \nabla[|t| u b_4^{-1}] \, |t|^{d+1-n} dt\, dx \\ - \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} b_4^{-1}\mathcal{A}\nabla u \cdot \nabla|t| \, u \, |t|^{d+1-n} dt\, dx &- \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} \mathcal{A}\nabla u \cdot \nabla b_4 \, \frac{u}{b_4^2} |t|^{d+1-n} dt\, dx \\ &=: 0 + I_1 + I_2.\end{aligned}$$

Using the expression of $\mathcal{A}$, we have that $\mathcal{A}\nabla u \cdot \nabla|t| = \mathcal{C}\nabla u \cdot \nabla_t|t| + b_4 \nabla_t u \cdot \nabla_t|t|$, where $\mathcal{C}$ is the matrix $[\mathcal{C}_3, \mathcal{C}_4]$, so

$$I_1 = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} b_4^{-1}\mathcal{C}\nabla u \cdot \nabla_t|t| \, u \, |t|^{d+1-n} dt\, dx - \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} \nabla u \cdot \nabla|t| \, u \, |t|^{d+1-n} dt\, dx =: I_3 + I_4.$$

Note that $\operatorname{div} |t|^{d+1-n}\nabla|t| = 0$, so another integration by parts gives that

$$I_4 = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} \nabla|t| \cdot \nabla[u^2] \, |t|^{d+1-n} dt\, dx = \int_{\mathbb{R}^d} |u|^2 \, dx \leq \|N(u)\|_{L^2(\mathbb{R}^d)}^2.$$

As for I_2 and I_3, they are bounded in a similar manner - using the equivalence $b_4 \approx 1$, the Carleson inequality (Theorem 3.9) and then the fact that $|\mathcal{C}| + |t||\nabla b_4| \in CM_\sigma$ - as follows

$$\begin{aligned}|I_2| + |I_3| &\leq C \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} [|\mathcal{C}| + |t||\nabla b_4|] \, |u|^2 \, |t|^{d-n} dt\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} |\nabla u|^2 \, |t|^{d+2-n} dt\, dx \right)^{\frac{1}{2}} \\ &\leq C \underbrace{\|\mathcal{C}_\sigma([|\mathcal{C}| + |t||\nabla b_4|])\|_{L^\infty(\mathbb{R}^d)}}_{\text{finite by assumption and indep. of } u} \|N(u)\|_{L^2(\mathbb{R}^d)} \|S(u)\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

To summarize, we have

$$\|S(u)\|_{L^2(\mathbb{R}^d)}^2 \leq C\|N(u)\|_{L^2(\mathbb{R}^d)}^2 + C'\|N(u)\|_{L^2(\mathbb{R}^d)}\|S(u)\|_{L^2(\mathbb{R}^d)}$$

such self-improves to

$$\|S(u)\|_{L^2(\mathbb{R}^d)} \leq C\|N(u)\|_{L^2(\mathbb{R}^d)}$$

as desired. □

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[5] a.k.a the divergence theorem

3.5 . Solvability of the L^p Dirichlet problem in domains with Lipschitz boundaries

Solving the Dirichlet problem in Lipschitz domains is an intermediate step toward addressing domains with uniformly rectifiable boundaries. The strategy for handling Lipschitz domains or domains above a Lipschitz graph has traditionally involved using changes of variables. In this section, we will recall previously employed changes of variables that, unfortunately, are not well-suited to higher codimensional settings. We will then introduce a new change of variables proposed by Guy David, Svitlana Mayboroda, and the author in [DFM19a].

3.5.1 . Changes of variables in codimension 1

The domain that we consider is above a Lipschitz graph, i.e.

$$\Omega = \Omega_\varphi := \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R},\, t > \varphi(x)\} \tag{3.29}$$

for some Lipschitz function φ. We leave it to the reader to verify that the L^p solvability of the Dirichlet problem is preserved under bi-Lipschitz changes of variable. This means that studying the Laplacian in Ω_φ is equivalent to studying a uniformly elliptic operator $L := -\operatorname{div} A_\rho \nabla$ on $\mathbb{R}^n_+$, where $\rho = \rho_\varphi$ is a bi-Lipschitz map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that sends Ω_φ to $\mathbb{R}^n_+$. Here Jac_ρ is the Jacobian matrix of ρ, and A_ρ is the conjugate matrix given by $|\det(\operatorname{Jac}_\rho)|^{-1} \operatorname{Jac}_\rho^T \operatorname{Jac}_\rho$.

If Ω is defined by a Lipschitz function like in (3.29), a simple initial choice for ρ is the one defined by

$$\rho_1(x,t) := (x, t - \varphi(x)) \tag{3.30}$$

which has the associated conjugate matrix

$$A_{\rho_1}(x,t) := \left[\begin{array}{c:c} I_{n-1} & \nabla\varphi(x) \\ \hdashline (\nabla\varphi(x))^T & 1 + |\nabla\varphi(x)|^2 \end{array}\right] \tag{3.31}$$

The change of varaible ρ_1 is one that results in conjugate matrix independent of t. This independence is one reason for the extensive literature on the solvability of boundary value problems for operators with t-independent coefficients.

The second change of variable for Ω_φn introduced by Kenig and Pipher in [KP01], is defined as follows :

$$\rho_2^{-1}(x,t) = (x, \kappa t + \varphi_t(x)), \tag{3.32}$$

where $\kappa > 0$ is chosen large enough and φ_t is the convolution of φ with a mollifier, that is $\varphi_t = \varphi * \eta_t$, where $\eta \in C_0^\infty(B(0,1), \mathbb{R}_+)$, $\int_{\mathbb{R}^{n-1}} \eta\, dx = 1$, and $\eta_t(x) = t^{1-n}\eta(x/t)$. The Jacobian matrix is

$$\operatorname{Jac}(\rho_2^{-1})(x,t) = \left[\begin{array}{c:c} I_{n-1} & \nabla_x\varphi_t(x) \\ \hdashline 0 & \kappa + \partial_t\varphi_t(x). \end{array}\right]$$

It is a simple exercise to verify that $\partial_t\varphi_t \leq C_\eta \|\nabla\varphi\|_\infty$. Thus, by choosing κ sufficiently large, ρ_2^{-1} is indeed a bi-Lipschitz change of variable that maps $\mathbb{R}^n_+$ to Ω_φ. The conjugate of the Laplacian by ρ_2^{-1} is given by $L_{\rho_2} = -\operatorname{div} A_{\rho_2}\nabla$, where

$$A_{\rho_2} = \left[\begin{array}{c:c} I_{n-1} & -\dfrac{\nabla_x\varphi_t(x)}{\kappa + \partial_t\varphi_t(x)} \\ \hdashline -\dfrac{\nabla_x\varphi_t(x)}{\kappa + \partial_t\varphi_t(x)} & \dfrac{1 + |\nabla_x\varphi|^2}{(\kappa + \partial_t\varphi_t(x))^2}. \end{array}\right]$$

A simple application of the Littlewood-Paley theory shows that

$$|\partial_t \varphi_t| + t|\nabla^2_{x,t}\varphi_t| \in CM_\sigma(M),$$

hence A_{ρ_2} satisfies $\delta_{\partial\Omega}\nabla A_{\rho_2} \in CM_\sigma$ and L_φ is a DKP operator. This change of variable is one of the reason why DKP operators are a popular study.

3.5.2 . The change of variable in higher codimension

Let $E_\varphi := \{((x,\varphi(x)),\, x \in \mathbb{R}^d\}$, where $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is a Lipschitz function. We aim for our "simple" operator on $\Omega = \mathbb{R}^n \setminus E_\varphi$ - which will serve as our substitute of the Laplacian in higher codimension - to exhibit isotropic behavior. Thus, we seek an operator of the form $L = -\operatorname{div} a\nabla$, where a is a scalar function equivalent to $\delta_{\partial\Omega}^{d+1-n}$. Why don't we take $a = \delta_{\partial\Omega}^{d+1-n}$? We allow ourselves some extra flexibility for now[6], as $\delta_{\partial\Omega}^{d+1-n}$ might not be the optimal choice (for instance, it is not smooth).

We want our change of variable ρ from $\Omega = \mathbb{R}^n \setminus E_\varphi$ to $\mathbb{R}^n \setminus \mathbb{R}^d$ to be so that the conjugate of L by ρ is an operator that satisfies the assumption of Theorem 3.26. The reader can verify that the two bi-Lipschitz changes of variables introduced in the previous paragraph will not be adequate. Specifically, ρ_1 is not adapted to the Carleson measure conditions CM_σ, and the conjugate of ρ_2 will not preserve the nearly scalar structure of the bottom right corner. So we want a new change of variable which is an isometry in t (up to errors controllable with Carleson measures).

For our third choice of change of variable, we still write $\varphi_t = \varphi * \eta_{|t|}$, where η_r is a smooth and compactly supported mollifier. We introduce $P(x,t)$ to be the d-plane tangent to the graph of φ_t at the point $(x,\varphi_t(x))$, and $R_{x,t}$ to be a linear isometry of $\mathbb{R}^n$ that maps $\mathbb{R}^d$ to the d-plane $P(x,t)$[7] We define the map from $\mathbb{R}^n \setminus \mathbb{R}^d$ to $\mathbb{R}^n \setminus E_\varphi$ as

$$(\rho_3)^{-1}(x,t) = (x,\varphi_t(x)) + R_{x,t}(0,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \setminus \mathbb{R}^d, \tag{3.33}$$

see Figure 3.1.

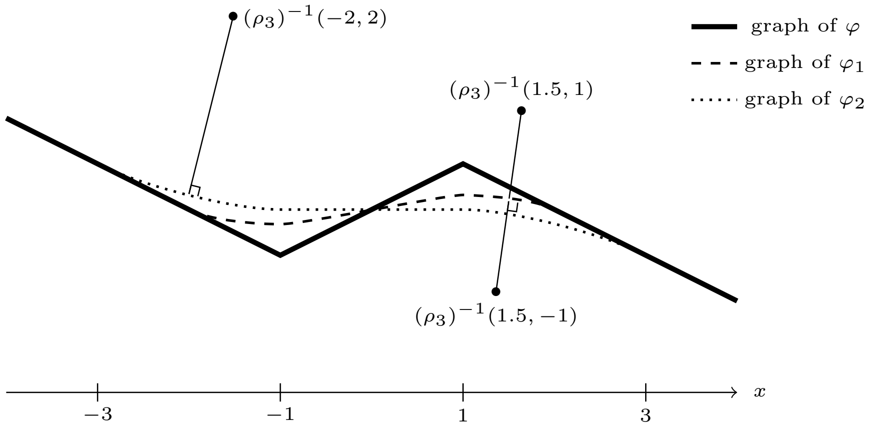

Figure 3.1 – Contruction of $(\rho_3)^{-1}$.

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[6] The authors initially attempted to use $L = -\operatorname{div} \delta_{\partial\Omega}^{d+1-n}\nabla$, but found it unsuitable for our proofs.

[7] Note that $\varphi_t(x)$, $P(x,t)$ and $R_{x,t}$ depends on t only via $|t|$. We avoided writing $\varphi_{|t|}(x)$, $P(x,|t|)$ and $R_{x,|t|}$ to lighten the notation.

Proposition 3.27 (Theorem 3.57 in [DFM19a]). *There exists $\epsilon_0 > 0$ (that depends on the mollifier η_r) such that if $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{n-d}$ is a ϵ_0-Lipschitz function, the map*

$$(x,t) \in \mathbb{R}^n \setminus \mathbb{R}^d \mapsto (x, \varphi_t(x)) + R_{x,t}(0,t) \tag{3.34}$$

is bi-Lipschitz from $\mathbb{R}^d \mapsto \mathbb{R}^{n-d}$ to $\mathbb{R}^d \setminus E_\varphi := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\, t \neq \varphi(x)\}$.

The reader can verify that the map in (3.34) may not remain injective if the Lipschitz constant of φ is not sufficiently small. Aside from this limitation to Lipschitz graph with small constants, the change of variable (3.34) meets our needs, as it turns $L = -\operatorname{div} a\nabla$ into an elliptic operator with coefficients displaying the correct block form.

Proposition 3.28 (Lemma 4.4 in [DFM19a]). *Let ϵ_0 as in Proposition 3.27 and take $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{n-d}$ is a ϵ_0-Lipschitz function. Set $E_\varphi := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\, t \neq \varphi(x)\}$ and let a be a scalar function satisfying*

$$C_a^{-1} \delta_{E_\varphi}^{d+1-n}(X) \leq a(X) \leq C_a \delta_{E_\varphi}^{d+1-n}(X)$$

for $X \in \mathbb{R}^n \setminus E_\varphi = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\, t \neq \varphi(x)\}$. Then the conjugate of $L := -\operatorname{div} a\nabla$ by ρ_3 - where ρ_3^{-1} is given in by (3.33) *- can be written as $L_{\rho_3} := -\operatorname{div}[|t|^{d+1-n} A_{\rho_3} \nabla]$ where*

$$\mathcal{A}_{\rho_3} = \left(\frac{a \circ \rho_3^{-1}}{|t|^{d+1-n}} \right) \left[\begin{array}{c:c} \mathcal{B}_1 & \mathbf{0} \\ \hdashline \mathbf{0} & b_4 \cdot I_{n-d} \end{array} \right] + \mathcal{C}$$

and

(i) $M^{-1} \leq b \leq M$,

(ii) $|\mathcal{C}| + |t||\nabla B_1| + |t||\nabla b_4| \in CM_{\mathcal{L}^d}(M)$.

for some $M > 0$ that depends only on C_a.

By combining this proposition with Theorem 3.26, we derive the following corollary, which provides insight into the desired properties of a.

Corollary 3.29. *Let φ, a, and L as in Proposition 3.28. Assume moreover that we have the decomposition*

$$\frac{a(\rho_3^{-1}(x,t))}{|t|^{d+1-n}} = b(x,t) + c(x,t), \qquad |\nabla b| + |c| \in CM_{\mathcal{L}^d}(C'_a) \tag{3.35}$$

Then the elliptic measure ω_L associated to L is A_∞-absolutely continuous with respect to $\sigma := \mathcal{H}^d_{E_\varphi}$ and (D$_{p,\sigma}$)$_L$ is solvable for a large enough $p \in (1, \infty)$.

3.5.3 . The choice of the Laplacian in higher codimension

We are seeking a substitute for the Laplacian in higher codimension. When the Lipschitz graph has a dimension $d < n-1$, we cannot use constant coefficient elliptic operators. Instead, the coefficients must depend on the distance to the boundary. It is also reasonable to consider operators of the form $L = -\operatorname{div}[a\nabla]$ with a being a scalar, representing isotropic diffusion.

A natural choice for the scalar function $a_\infty := \delta_{\partial\Omega}^{d+1-n}$. We need to verify if condition (3.35) is satisfied. Intuitively, a is the inverse of a minimum and is related to Peter Jones' β_∞ as defined in Definition 1.11. Verifying (3.35) involves checking whether the β_∞ are "Carleson packing", i.e., $\beta_\infty \in CM_\sigma$. According to Theorem 1.13, this is true when (and only when) the dimension d is 1. Thus, we have our first result :

Theorem 3.30 (Corollary 6.44 in [DFM19a]). *There exists $\epsilon_0 > 0$ such that if $\varphi : \mathbb{R} \mapsto \mathbb{R}^{n-1}$ is a ϵ_0-Lipschitz function, E_φ is the graph of φ, and L_∞ is the operator $-\operatorname{div}[\delta_{E_\varphi}^{2-n}\nabla]$, then the elliptic measure ω_{L_∞} associated to L_∞ is A_∞-absolutely continuous with respect to $\sigma := \mathcal{H}^1_{E_\varphi}$ and (D$_{p,\sigma}$)$_L$ is solvable for a large enough $p \in (1, \infty)$.*

For Lipschitz boundaries of other dimensions, the behavior of these operators is unknown. However, we suspect that the elliptic measure of $L_\infty := -\operatorname{div}[\delta_{E_\varphi}^{2-n}\nabla]$ may not always be $A_\infty(\mathcal{H}^d_{E_\varphi})$, making L_∞ an inappropriate substitute for the Laplacian.

What is our next best choice? Because we aim to characterize uniformly rectifiable, we want a construction of the operator L that is possible in the complement of any Ahlfors regular set E of any dimension. We seek a construction that is explicit and simple. Our choice is the coefficient :

$$a_\alpha := D_\alpha^{d+1-n}, \tag{3.36}$$

where $\alpha > 0$,

$$D_\alpha(X) = D_{\alpha,\sigma}(X) := \left(\int_E |X-y|^{-d-\alpha}\, d\sigma(y) \right)^{-\frac{1}{\alpha}}, \tag{3.37}$$

with σ being an Ahlfors regular measure on E. Note that the weight a_α (and thus the operator $L_\alpha := -\operatorname{div}[a_\alpha\nabla]$) satisfies the elliptic theory developed in the previous chapter. D_α acts as a “regularized distance," being smooth and satisfying :

Proposition 3.31 (Lemma 5.1 in [DFM19a]). *Let $E \subset \mathbb{R}^n$ be an unbounded d-Ahlfors regular set (with Ahlfors regular measure σ) and $\alpha > 0$. Then there exists C depending only on d, α, and the constant in* (1.3) *such that*

$$C^{-1}\delta_E(X) \leq D_\alpha(X) \leq \delta_E(X) \qquad \text{for } X \in \mathbb{R}^n \setminus E.$$

Moreover, if $k = (k_1, \dots, k_n) \in \widetilde{N}^n$ is a multi-indice, $|k| = \sum_i k_i$, and $\partial^k = \partial^{k_1}_{x_1} \dots \partial^{k_n}_{x_n}$, then there exists C_k depending only on d, α, the constant in (1.3)*, and k such that*

$$|\partial^k D_\alpha(X)| \leq C_k \delta_E(X)^{1-|k|} \qquad \text{for } X \in \mathbb{R}^n \setminus E.$$

If E is bounded instead, the above inequalities hold only when X lies in $B_E := B(e, 1000\operatorname{diam}(E))$, where $e \in E$.

Proof. Given $X \in \mathbb{R}^n \setminus E$, we take $x \in E$ such that $\delta_E(X) = |X - x|$. We define $B_X = B(x, 2|X-x|)$. We have then

$$\begin{aligned} D_\alpha^{-\alpha} &= \int_{B_X} |X-y|^{-d-\alpha} d\sigma(y) + \sum_{j\geq 1} \int_{2^j B_X \setminus 2^{j-1}B_X} |X-y|^{-d-\alpha} d\sigma(y) \\ &\approx \delta_E^{-d-\alpha}(X)\sigma(B_X) + \sum_{j \geq 1} 2^{-j(d+\alpha)}\sigma(2^j B_X \setminus 2^{j-1}B_X). \end{aligned} \tag{3.38}$$

But σ is Ahlfors regular, so $\sigma(B_X) \approx \delta_E(X)^d$ and

$$\sigma(2^j B_X \setminus 2^{j-1}B_X) \leq \sigma(2^j B_X) \approx 2^{jd}\delta_E(X)^d.$$

Using those identities in (3.38) gives $D_\alpha^{-\alpha} \approx \delta_E^{-\alpha}(X)$, as desired.

It is fairly easy to check that

$$|\partial^k[D_\alpha^{-\alpha}]| \leq C_k D_{\alpha+|k|^{-\alpha-|k|}},$$

so the bound on the derivatives can be showed by a simple induction. ☐

As a_∞ is linked to β_∞, a_α is related to Tolsa's α-number. This means $a_\alpha(X)$ and $D_{E,\alpha}(X)$ consider an average of points in E weighted by their distance to X, rather than just the nearest point. Since Tolsa's α-numbers are "Carleson packing" by Theorem 1.13, a_α satisfies (3.35), leading to our conclusion :

Theorem 3.32. *There exists $\epsilon_0 > 0$ such that if $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{n-d}$ is a ϵ_0-Lipschitz function, E_φ is the graph of φ, and L_α is the operator $-\operatorname{div}[a_\alpha \nabla]$, then the elliptic measure ω_{L_α} associated to L_α is A_∞-absolutely continuous with respect to $\sigma := \mathcal{H}^d_{E_\varphi}$ and (D$_{p,\sigma}$)$_L$ is solvable for a large enough $p \in (1,\infty)$.*

Proof. The condition (3.35) is a consequence of Lemmas 5.49 and 5.59 in [DFM19a]. ☐

3.6 . L^p Dirichlet problem in domains with rough boundaries.

3.6.1 . Magic α

Our ultimate goal is to characterize low-dimensional uniformly rectifiable sets using solutions in their complements, similar to how Theorems 1.32 and 1.37 characterize uniformly rectifiable boundaries of codimension 1. Along the way, we faced numerous challenges, but our primary obstacle was proving the converse : that the solvability of the L^p Dirichlet problem implies the uniform rectifiability of the boundary.

To be direct, at the time of writing this memoir, our best result regarding the converse is the stability result from [Per23b]. Additionally, we discovered early on that the converse can be false if not approached carefully. Specifically, there exists a choice of α in (3.37) such that the L^p Dirichlet problem for the operator $L_\alpha := -\operatorname{div} D_\alpha^{d+1-n}\nabla$ is solvable for all Ahlfors regular boundary and all $p \in (1,\infty)$.

Lemma 3.33 (Section 6 in [DEM21])**.** *Let $d < n-2$ and E be a d-Ahlfors regular set (σ is any Ahlfors regular measure). If $\alpha_0 := n-d-2 > 0$, then*

$$L_{\alpha_0} D_{\alpha_0}(X) = 0 \qquad \text{for } X \in \mathbb{R}^n \setminus E.$$

As a consequence, the Green function and elliptic measure with pole at infinity associated to L_{α_0} (Definition 2.78) are respectively $G^\infty = D_{\alpha_0}$ and $\omega^\infty = \sigma$; and so the L^p Dirichlet problem for L_{α_0} is solvable for all $p \in (1,\infty)$.

The value $\alpha_0 = n-d-2$ will be referred as "magic α".

Remark 3.34. This is surprising because there are very few instances where the Green function or the Green function with a pole at infinity is explicitly known (e.g., the disc, the half-plane, the quarter-plane, and only when the operator is the Laplacian). Remarkably, for this particular α, we have explicit expressions for the Green function and the elliptic measure with a pole at infinity for any Ahlfors regular set.

Proof. Let $H_\alpha = D_\alpha^{-\alpha}$. We simply compute

$$\nabla D_\alpha(X) = \nabla[H_\alpha^{-1/\alpha}] = -\frac{1}{\alpha} H_\alpha^{-\frac{1}{\alpha}-1} \int_E \nabla_X[|X-y|^{-d-\alpha}]\, d\sigma(y)$$
$$= -\frac{1}{\alpha} D_\alpha^{1+\alpha} \int_E \nabla_X[|X-y|^{-d-\alpha}]\, d\sigma(y)$$

So if $\alpha = n-d-2$, we have

$$\nabla D_\alpha(X) = -\frac{1}{n-d-2} D_\alpha^{n-d-1} \int_E \nabla_X[|X-y|^{2-n}]\, d\sigma(y),$$

and thus

$$L_\alpha D_\alpha = \operatorname{div} D_\alpha^{d+1-n} \nabla D_\alpha(X) = -\frac{1}{n-d-2} \int_E \Delta_X[|X-y|^{2-n}]\, d\sigma(y) = 0$$

because $|X-y|^{2-n}$ is the fundamental solution of the Laplacian. Hence $G^\infty = D_\alpha$.

Then, if $\alpha = n-d-2$ as before, for $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\int_E \varphi d\omega^\infty &:= -\int_{\mathbb{R}^n \setminus E} D_\alpha^{d+1-n} \nabla G^\infty \cdot \nabla \varphi \, dX = -\int_{\mathbb{R}^n \setminus E} D_\alpha^{d+1-n} \nabla D_\alpha \cdot \nabla \varphi \, dX \\
&= \frac{1}{n-d-2} \int_{\mathbb{R}^n \setminus E} \left(\int_E \nabla_X[|X-y|^{2-n}]\, d\sigma(y) \right) \cdot \nabla \varphi \, dX \\
&= \frac{1}{n-d-2} \int_E \left(\int_{\mathbb{R}^n \setminus E} \nabla_X[|X-y|^{2-n}] \cdot \nabla \varphi \, dX \right) d\sigma(y) \\
&= \frac{1}{n-d-2} \int_E \varphi(y)\, d\sigma(y)
\end{aligned}$$

because $|X-y|^{2-n}$ is the fundamental solution of the Laplacian and E has zero Lebesgue measure. Since both G^∞ and ω^∞ are defined up to a constant, $\omega^\infty = \sigma$. ☐

3.6.2 . Uniformly rectifiable boundaries

Despite the significant setback from discovering that the converse is false, we remain confident that for any other values of α, solutions to $L_\alpha = 0$ will capture the non-flatness of the boundary. Specifically, we aim to prove that for each L_α with $\alpha > 0$, the L^p Dirichlet problem is solvable for some $p \in (1, \infty)$ whenever the domain is the complement of a uniformly rectifiable set.

The existence of a "magic" α has significantly aided our understanding of the theory. We now recognize that certain operators are particularly suited to specific types of boundaries. For example, the Laplacian is the operator for which the L^p Dirichlet problem is solvable for all $p \in (1, \infty)$ in the half plane. Since uniformly rectifiable often almost flat in a quantitative and scale invariant way, the Laplacian will be adequate to characterize uniformly rectifiable sets. However, this comes at the cost of reducing the range of $p \in (1, \infty)$ for which L^p Dirichlet problem is solvable.

In [DM21], David and Mayboroda constructed an isotropic operator on the complement of the 4-corners Cantor set where the elliptic measure is equivalent to the Hausdorff measure, ensuring that the L^p Dirichlet problem is solvable for all $p \in (1, \infty)$. Similarly, Perstneva, in [Per23a], built an operator adapted to the complement of a Wolff snowflake.

For our current problem - the solvability of the L^p Dirichlet problem for L_α in complement of uniformly rectifiable sets - the existence of the magic α is fortuitous. It allows for a simple proof of the following result.

Theorem 3.35 ([DM23], [Fen22a])**.** *Let $\Omega = \mathbb{R}^n \setminus E$ for a uniformly rectifiable set $E = \partial\Omega$ of dimension $d < n-1$ (hence automatically uniform). Let $\alpha > 0$ and define L_α as*

$$L_\alpha := -\operatorname{div}[D_\alpha^{d+1-n}\nabla].$$

Then the elliptic measure ω_α associated to L_α is A_∞-absolutely continuous with respect to $\sigma = \mathcal{H}^d|_{\partial\Omega}$, or equivalently the Dirichlet problem $(D_{p,\sigma})_L$ is solvable for a large $p \in (1,\infty)$.

Idea of proof : The solvability of the Dirichlet problem (Theorem 3.35) was proved in [DM23] for all the codimensions. An alternative proof - simpler, using the magic α, but that works only in higher codimension - was proposed by the author in [Fen22a], and its rough outline is given below.

Let $\alpha_0 = n-d-2$. Then the function D_{α_0} in $\mathbb{R}^n \setminus E$ satisfies $L_{\alpha_0}D_{\alpha_0}$, so it works like "$|t|$" in the proof of Theorem 3.26. Indeed,

$$\begin{aligned}\|S(u)\|^2_{L^2(E,\sigma)} &\approx 2\int_{\mathbb{R}^n\setminus E} |\nabla u|^2\, D_{\alpha_0}^{d+2-n}\, dX \\ &= 2\int_{\mathbb{R}^n\setminus E} \nabla u\cdot\nabla[D_{\alpha_0}u]\, D_{\alpha_0}^{d+1-n}\, dX - 2\int_{\mathbb{R}^n\setminus E} (\nabla u\cdot\nabla D_{\alpha_0})u\, D_{\alpha_0}^{d+1-n}\, dX \\ &= -2\int_{\mathbb{R}^n\setminus E} (\nabla u\cdot\nabla D_{\alpha_0})u\, D_{\alpha_0}^{d+1-n}\, dX = -\int_{\mathbb{R}^n\setminus E} \nabla[u^2]\cdot[D_{\alpha_0}^{d+1-n}\nabla D_{\alpha_0}]\, dX \\ &= \int_E |u(y)|^2\, d\sigma(y), \qquad (3.39)\end{aligned}$$

where the last equality comes from link between the Green function with pole at infinity ($G^\infty = D_{\alpha_0}$) and the elliptic measure with pole at infinity ($\omega^\infty = \sigma$), see Proposition 2.77.

The second part is to say that, if α is any positive number, then D_α and D_{α_0} are not too far from each other in the Carleson sense, more precisely

Lemma 3.36 (Lemma 1.27 in [Fen22a])**.** *Let $E \subset \mathbb{R}^n$ be a uniformly rectifiable set of dimension d, $d < n$. Then for any couple $\alpha, \beta > 0$ and any couple σ, μ of Ahlfors regular measure on E we have*

$$\delta_E\nabla\Big(\frac{D_{\sigma,\alpha}}{D_{\mu,\beta}}\Big) \in CM_\sigma,$$

where the Carleson measure constant depends only on α, β and the Ahlfors regular constants of σ and μ.

Set $b = [D_\alpha/D_{\alpha_0}]^{d+1-n}$.The operator is L_α and L_{α_0} only differ by a multiplicative scalar function on the coefficients. So we can adapt the proof of Theorem 3.26 by having b play the role of b_4, D_{α_0} play the role of $|t|$, and L_{α_0} play the role of $-\operatorname{div}|t|^{d+1-n}\nabla$ in the proof of Theorem 3.26.

What actually works. An attentive reader will notice that the above idea works when the magic α exists, specifically when $n-d-2>0$. To address all the higher codimensions, we must also consider the scenario where $d = n-2$. The strategy is to examine the difference between $D_\alpha^{d+1-n}\nabla D_\alpha$ and the divergence free quantity $D_{\alpha_0}^{d+1-n}\nabla D_{\alpha_0}$, rather than the difference between D_α and D_{α_0}. When $\alpha_0 = n-d-2$, we formally have

$$D_{\alpha_0}^{d+1-n}\nabla D_{\alpha_0} = \int_E \frac{X-y}{|X-y|^n}\, d\sigma(y) := \mathbf{H},$$

but while D_{α_0} exists only when the dimension of E is smaller than $n-d-2$, the divergence free vector $\mathbf{H}$ exists whenever E is a Ahlfors regular set of dimension $d < n-1$. Thus, we gain one additional dimension !

Lemma 3.37 (Lemma 1.20 in [Fen22a]). *Let $E \subset \mathbb{R}^n$ be a uniformly rectifiable set of dimension d, $d < n-1$. Then for any $\alpha > 0$ and any d-Ahlfors regular measure σ on E, there exists a scalar b and a vector $\mathbf{V}$ on $\mathbb{R}^n \setminus E$ such that*

$$D_\alpha^{d+1-n}(b\nabla D_\alpha + \mathbf{V}) = \int_E \nabla\mathcal{E}(X-y)\, d\sigma(y) =: \mathbf{H}, \tag{3.40}$$

where $\mathcal{E}$ is the fundamental solution of the Laplacian. Moreover, there exists $M \geq 1$ depending only on α, d, n, and the Ahlfors regular and uniformly rectifiable constants of E such that we can choose b and $\mathbf{V}$ satisfying

(i) $M^{-1} \leq b \leq M$;

(ii) $|\mathbf{V}| \leq M$,

(iii) $\delta_E|\nabla b| \in CM_\sigma(M)$,

(iv) $\mathbf{V} \in CM_\sigma(M)$.

Assuming the lemma, why is it enough ? Take a ball $B = B(x_B, r_B)$ centered on the boundary $E = \partial\Omega$. We want to prove that solutions to $L_\alpha u = 0$ satisfies $\|S(u)\|_{L^2(E,\sigma)} \leq C\|N(u)\|_{L^2(E,\sigma)}$. Our proof will be similar to the one of Theorem 3.26, and similarly, we will not justify the existence and finiteness of the quantities that we are using, and we refer to [Fen22a] for the actual proof. Let b, $\mathbf{V}$, and $\mathbf{H}$ as in Lemma 3.37. Since $b \geq M^{-1}$ and $D_\alpha \approx \delta_E$, we have

$$\|S(u)\|^2_{L^2(E,\sigma)} \leq C\int_\Omega |\nabla u|^2 b D_\alpha^{d+2-n} dX =: I.$$

But but using an integration by parts (with no boundary terms because $D_\alpha = 0$ on $\partial\Omega$), we have

$$\begin{aligned} I = \int_\Omega D_\alpha^{d+1-n}\nabla u\cdot\nabla u\,(bD_\alpha)\,dX &= -\int_\Omega \operatorname{div}[D_\alpha^{d+1-n}\nabla u]\,ubD_\alpha\,dX \\ &- \int_\Omega D_\alpha^{d+2-n}\nabla u\cdot\nabla b\,u\,dX - \int_\Omega \nabla u\cdot(bD_\alpha^{d+1-n}\nabla D_\alpha)\,u\,dX =: I_1 + I_2 + I_3. \end{aligned}$$

The term I_1 is 0, since u is a weak solution to $L_\alpha u = 0$. The term I_2 is bounded with the help of the Cauchy inequality and then the Carleson inequality.

$$|I_2| \leq CI^{\frac12}\left(\int_{2B} u^2|\nabla b|^2\, D_\alpha^{d+2-n}\,dX\right)^{\frac12} \lesssim I^{\frac12}\|N(u)\|_{L^2(E,\sigma)}$$

since $b \leq M$ and $\Delta_\alpha|\nabla b| \in CM_\sigma$, see Lemma 3.37 (i) and (iii). As for I_3, we use (3.40) to get

$$I_3 = \int_\Omega D_\alpha^{d+1-n}\nabla u\cdot\mathbf{V}\,u\,dX - \int_\Omega \nabla u\cdot\mathbf{H}\,u\,dX := I_4 + I_5.$$

The integral I_4 is treated like I_2, using the fact that $\mathbf{V} \in CM_\sigma$.

$$I_4 \lesssim I^{\frac12}\left(\int_\Omega u^2|\mathbf{V}|^2\, D_\alpha^{d-n}\,dX\right) \lesssim I^{\frac12}\|N(u)\|_{L^2(E,\sigma)}.$$

To bound I_5, we integrate by part again

$$I_5 = -\frac{1}{2}\int_\Omega \nabla[u^2]\cdot H\,dX = \frac{1}{2}\int_\Omega u^2 \operatorname{div} H\,dX - \frac{1}{2}\int_{\partial\Omega} u^2 \mathbf{H}\cdot\vec{n}\,d\sigma =: I_6 + I_7,$$

where $\mathbf{H}\cdot\vec{n}(x)$ morally denotes $\lim_{X\in x} D_\alpha^{n-d-1}(X)\mathbf{H}(X)\cdot\nabla D_\alpha(X)$. Since $\mathbf{H}$ is divergence free, we have $I_6 = 0$, and since $D_\alpha^{n-d-1}(X)\mathbf{H}(X)\cdot\nabla D_\alpha(X)$ is bounded, we have

$$|I_7| \leq C\int_{\partial\Omega} u^2\,d\sigma \leq C\|N(u)\|^2_{L^2(E,\sigma)}.$$

Altogether, we have

$$I \leq CI^{\frac{1}{2}}\|N(u)\|_{L^2(E,\sigma)} + C\|N(u)\|^2_{L^2(E,\sigma)}$$

which self-improves in $\|S(u)\|_{L^2(E,\sigma)} \leq CI^{1/2} \leq C'\|N(u)\|_{L^2(E,\sigma)}$ as desired. □

3.7 . The regularity problem in domains with lower dimensional boundaries

In the general setting introduced in Chapter 2, it is unclear what constitutes a good statement for the L^p regularity problem and the L^p Neumann problem. To the best of the author's knowledge, the most general context in which the L^p regularity problem has been considered is as follows :

Definition 3.38. Let $\Omega \subset \mathbb{R}^n$ be uniform, $\partial\Omega$ be d-Ahlfors regular for some $d \in (0,n)$, and let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator with respect to the weight $w(X) = \delta_{\partial\Omega}(X)$.

For $p \in (1,\infty)$, we say that the L^p regularity problem is solvable - $(\mathrm{R}_{p,\sigma})_L$ for short - if there exists $C > 0$ such that, for $g \in C_0^\infty(\mathbb{R}^n)$, the solution $u_g \in W(\Omega,m)\cap C^0(\Omega)$ given by

$$u(X) := \int_{\partial\Omega} g(y)d\omega_L^X(y)$$

verifies

$$\|N(\nabla u)\|_{L^p(\partial\Omega,\sigma)} \leq C\|\nabla_{H,p}g\|_{L^p(\partial\Omega,\sigma)},$$

where $\nabla_{H,p}g$ is the Hajłasz gradient defined in Definition 1.40.

We recall that when $\partial\Omega$ is flat or is the graph of a Lipschitz function, then the Hajłasz gradient is equivalent to the classical local tangential gradient. Moreover, it would be reasonable to assume that this definition extends to the case where (Ω,m,σ) is suitable for PDE and traces and the ratio $\rho(x,r)$ defined in (2.15) is equivalent to 1. This means

$$C^{-1} \leq \rho(x,r) \leq C \qquad \text{for } x \in \partial\Omega,\ r \in (0,\operatorname{diam}\Omega). \tag{3.41}$$

We may even be able to remove the Harnack chain condition, but it becomes clear that the quantity ρ must be involved in some way in the formulation of the L^p regularity problem.

The main results in this context are obtained by the author in collaboration with Dai and Mayboroda. The first result addresses Carleson perturbations.

Proposition 3.39. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with d-Ahlfors regular boundaries. Let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator with respect to the weight $w(X) = \delta_{\partial\Omega}(X)$. For $p, q \in (1,\infty)$, we have*

$$(R_{p,\sigma})_L \implies (D_{p',\sigma})_{L^*}$$

and

$$(R_{p,\sigma})_L + (D_{q,\sigma})_{L^*} \implies (R_{q,\sigma})_L.$$

In particular

$$(R_{p,\sigma})_L \implies (R_{q,\sigma})_L \text{ for } q \in (1, p+\epsilon).$$

Proof. The first implication is [DFM23a, Theorem 1.5]. The second implication can be proven by generalizing the proof of [She07] in the current setting. The proof relies heavily on the elliptic theory, so it would not be surprising that if proposition could be extended to domains without the Harnack chain condition or without Ahlfors regular boundaries. ☐

Theorem 3.40. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domains with d-Ahlfors regular boundaries. Let $L_0 = -\operatorname{div} A_0\nabla$ and $L_1 = -\operatorname{div} A_1\nabla$ be two uniformly elliptic operators with respect to the weight $w(X) = \delta_{\partial\Omega}^{d+1-n}(X)$. Take $p \in (1,\infty)$ and suppose that $(R_{p,\sigma})_{L_0}$ holds, that is the L^p regularity problem for L_0 is solvable. If*

(1) *either $X \to \sup_{B_X}\{w^{-1}|A_0 - A_1|\} \in CM_\sigma(M)$,*

(2) *or $\delta_{\partial\Omega} w^{-1}|\nabla A_0|$ and $w^{-1}|A_0 - A_1| \in CM_\sigma(M)$,*

then there exists $q \in (1,\infty)$ such that $(R_{q,\sigma})_{L_1}$ holds.

Moreover, there exists $\epsilon > 0$ depending on the constants in the $(R_{p,\sigma})_{L_0}$ such that if $M < \epsilon$ in either (1) or (2), then we preserve the solvability of the L^p regularity problem, i.e. $(R_{p,\sigma})_{L_1}$ holds.

Proof. The theorem under (1) in both the large and small Carleson perturbation scenario is [DFM23a, Corollary 1.6], and relies on the analogue result on the Dirichlet problem (given in [FP22] and [MP21]). The theorem (2) in this generality is not written anywhere but the arguments given in [Fen24] and [Fen23, Theorem 2.11] - which treat the case where $\Omega = \mathbb{R}^n_+$ - easily extend to this setting. ☐

Our final theorem establishes the solvability of the L^p regularity problem in $\mathbb{R}^n \setminus \mathbb{R}^d$ for DKP-type operators.

Theorem 3.41. *Let $\Omega_0 = \mathbb{R}^n \backslash \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times (\mathbb{R}^{n-d} \backslash \{0\})\}$, and let $L := -\operatorname{div}[|t|^{d+1-n}\mathcal{A}\nabla]$ be a uniformly elliptic operator with respect to the weight $|t|^{d+1-n}$. Assume that $\mathcal{A}$ can be written as*

$$\mathcal{A} = \begin{bmatrix} B_1 & B_2 \\ B_3 & b_4 I_{n-d} \end{bmatrix}$$

where I_{n-d} is the $n-d$ identity matrix, and either $|t||\nabla A| \in CM_{\mathcal{L}^d}(M)$ or $|t||\nabla \widetilde{A}| \in CM_{\mathcal{L}^d}(M)$, where $B_2 = \mathbf{b}_2 \frac{t}{|t|}$, $B_3 = \frac{t^T}{|t|}\mathbf{b}_3$, and

$$\widetilde{A} := \begin{bmatrix} B_1 & \mathbf{b}_2 \\ \mathbf{b}_3 & b_4 \end{bmatrix}.$$

(1) *Then there exists $p \in (1,\infty)$ such that the L^p regularity problem is solvable.*

(2) *For all $p \in (1, \infty)$, there exists ϵ_p that depends only on d, n, p, and the elliptic constant of L such that if $M < \epsilon_p$, then the L^p regularity problem is solvable.*

Remark 3.42*.* For more generality, the above theorem needs to be paired with Theorem 3.40 - that allows more elliptic operators. Furthermore, by the change of variable (3.33), we will also be able to solve the L^p regularity problem for L_α in the complement of the graph of a Lipschitz function with small constant.

Proof. The large constant case (1) is [Fen23, Theorem 3.6]. The small constant case (2) is a consequence of the first part, the L^p solvability of the Dirichlet problem for DKP operators with small constant (second part of Theorem 3.26), and the second implication of Proposition 3.39; see also [DFM23b] for an earlier proof for $p = 2$. □

4 - The Green function as an alternative for the elliptic measure

4.1 . What one can expect from the Green function and early results

In domains with codimensional 1 boundaries, the A_∞-absolute continuity of the harmonic measure (and the elliptic measure of DKP operators) is well understood. However, we have not yet a fully satisfactory analogue in domains with higher codimensional boundaries. Specifically, we have not succeeded in finding a PDE characterization of uniform rectifiability that applies to uniformly rectifiable sets of *any* dimension and codimension.

So we have shifted our focus from the harmonic measure to the Green function, two closely related objects. Can we reformulate the A_∞-absolute continuity of the harmonic measure in terms of bounds on the Green function? Can we characterize uniform rectifiability with bounds on the Green function? Does this characterization on Green function extends to domains with boundaries of codimension higher than 1? These are the questions that we will explore in this chapter, and we will provide some answers.

First, what does the A_∞-absolute continuity of the elliptic measure ω_L with respect to the surface measure σ imply for the Green function? Let us consider the classical theory of domains with $(n-1)$-Ahlfors regular boundaries. We will reuse the unformal definition from the introduction. The A_∞-absolute continuity of the elliptic measure means that ω_L is "often almost equivalent" to a multiple of σ, where "often" refers to the fact that the dyadic cubes for which this equivalence fails are Carleson packing. The boundary Harnack inequality for the Green function and the elliptic measure (see Proposition 2.73) states that, if $4B_X$ stands for $B(X, 2\delta_{\partial\Omega}(X)))$.

$$g_L(Y,X) \approx \delta_{\partial\Omega}(X)^{2-n}\omega_L^Y(4B_X) \quad \text{for } Y, X \in \Omega,\ Y \notin 8B_X. \tag{4.1}$$

So if $\frac{\omega_L^Y(B_X)}{\sigma(4B_X)} \approx M$, then

$$g_L(Y,X) \approx M\delta_{\partial\Omega}(X)^{2-n}\sigma(4B_X) \approx M\delta_{\partial\Omega}(X) \quad \text{for } Y, X \in \Omega,\ Y \notin 8B_X.$$

Thus, if $Y \in \Omega$ is a point far away from the considered region, the fact that ω_L^Y is "often almost equivalent" to a multiple of σ morally means that

$$g_L(Y,.) \text{ is "often almost equivalent" to a multiple of } \delta_{\partial\Omega}. \tag{4.2}$$

We will discuss the proper statements in the following sections, but all will relate this informal statement (4.2).

Compared to the harmonic and elliptic measures, fewer results exist for the Green function. Nevertheless, we must be cautious when stating that the informal statement (4.2) is equivalent to the A_∞-absolute continuity of the elliptic measure, as it will be false without additional assumptions.

Example 4.1. Let $\Omega = \mathbb{R}^n_+ := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$ be our domain, which means that in our case $\delta_{\partial\Omega}(x,t) = t$. We set $b(t) := 1/(2+\cos(t))$, and then $L_b := -\operatorname{div}[b(t)\nabla]$. We can compute explicitely the Green function with pole at infinity (defined in Proposition 2.70), and we obtain

$$G(x,t) := \int_0^t \frac{dr}{b(r)} = 2t + \sin(t).$$

However, there exists $\epsilon_0 > 0$ such that for any $c > 0$ and any $R \geq 100$,

$$\int_R^{2R} |cG(x,t) - t|^2 \, dX \geq \epsilon R.$$

It means that the Green function with pole at ∞ is not close to a multiple of $\delta_{\partial\Omega}$ in a Carleson sense, or in any of the sense that we will introduce in the next sections.

On the opposite, $C^{-1}t \leq G(x,t) \leq Ct$, which means that ω_{L_b} is $\mathcal{A}_\infty$-absolutely continuous with respect to the surface measure.

Example 4.1 is nice because we can compute many aspects of it. However, many other examples would work, provided the coefficients oscillate sufficiently.

Before the authors began studying the Green function as an alternative to the elliptic measure, to the best of the author's knowledge, only the following result had been established :

Theorem 4.2 (Theorem VI in [Azz19]). *Let $\Omega \subset \mathbb{R}^n$ be a CAD, and let σ be the Ahlfors regular measure on $\partial\Omega$. Write $G = G_{-\Delta}$ for the Green function in Ω associated to the Laplacian, as defined in Definition 2.78.*

Then $\partial\Omega$ is uniformly rectifiable if and only if $\delta_{\partial\Omega}^2|\nabla^2 G|/G \in CM_\sigma$, i.e. if and only if there exists $C > 0$ such that, for any ball $B \subset \mathbb{R}^n$ centered on $\partial\Omega$,

$$\int_{B\cap\Omega} \frac{|\nabla^2 G|^2}{G^2} \delta_{\partial\Omega}^3 \, dX \leq C\sigma(B \cap \partial\Omega). \tag{4.3}$$

Remark 4.3. The result in [Azz19] are presented in a slightly different context (bounded domain and only lower Ahlfors regular constant in (1.2)) but the proof easily extends to our assumptions. The proof of $\delta_{\partial\Omega}\nabla^2 G/G \in CM_\sigma$ for uniformly rectifiable sets is based on an integration by part from [HMT17]. The arguments of [HMT17] would allow us to extend Theorem 4.2 to a larger class of operators, namely the L^1-DKP operators, which is strict subset of our target class of operators (the DKP operators).

Elements of proof. For the characterization of A_∞-absolute continuity of the elliptic measure via a corona decomposition, see [GMT18, Proposition 3.1] or [CHPM24].

Let us skip the converse, that is $\delta_{\partial\Omega}^2|\nabla^2 G|/G \in CM_\sigma$ implies uniform rectifiability, which will be discussed intensively in Section 4.3. We want to prove that a domain with reasonable flat boundary implies (4.3).

Since Ω is CAD, the harmonic measure is A_∞-absolutely continuous with respect to the surface measure. A characterization of this fact is given in terms of "corona decomposition", which will not be explained here. Eventually, it means that for any ball B centered on $\partial\Omega$, we can decompose $\Omega \cap B$ into Lipschitz regions $\{\Omega_j\}_{j\in J}$ such that

(i) $\displaystyle\bigcup_{j\in J} \overline{\Omega_j} = \Omega \cap B$,

(ii) $\displaystyle\sum_{j\in J} \mathcal{H}^{n-1}(\partial\Omega_j) \lesssim \sigma(B \cap \partial\Omega)$,

(iii) for each $j \in J$, there exists c_j such that

$$\omega_{-\Delta}(B_X) \approx c_j \sigma(B_X) \quad \text{for } X \in \Omega_j,$$

where $B_X = B(X, 2\delta_{\partial\Omega}(X))$ as before.

Think of the above as a precise formulation of

$$\omega_{-\Delta} \text{ is "often almost" equivalent to a multiple of } \sigma.$$

Thanks to Proposition 2.79, point (iii) above becomes

$$G(X) \approx c_j \delta_{\partial\Omega}(X) \quad \text{for } X \in \Omega_j. \tag{4.4}$$

Moreover, it is easy to check that

$$\delta_{\partial\Omega}|\nabla^2 G| \lesssim |\nabla G(X)| \lesssim \frac{G(X)}{\delta_{\partial\Omega}} \lesssim c_j \quad \text{for } X \in \overline{\Omega_j}. \tag{4.5}$$

Note that ∇G is a vector of harmonic functions, so we have $\Delta|\nabla G|^2 = 2|\nabla^2 G|^2$. With all this in mind, we have by (4.4) that

$$\begin{aligned}\int_{\Omega_j} \frac{|\nabla^2 G|^2}{G^2} \delta_{\partial\Omega}^3 \, dX \approx 2c_j^{-3} \int_{\Omega_j} |\nabla^2 G|^2 G \, dX = c_j^{-3} \int_{\Omega_j} \Delta[|\nabla G|^2] G \, dX \\ = c_j^{-3} \int_{\partial\Omega_j} \partial_n |\nabla G|^2 G \, dX - c_j^{-3} \int_{\partial\Omega_j} |\nabla G|^2 \partial_n G \, dX,\end{aligned}$$

where ∂_n is the normal derivative. So

$$\int_{\Omega_j} \frac{|\nabla^2 G|^2}{G^2} \delta_{\partial\Omega}^3 \, dX \lesssim \mathcal{H}^{n-1}(\partial\Omega)$$

by (4.5). Summing over $j \in J$ gives, thanks to (ii), the desired bound (4.3). □

Why is Theorem 4.2 related to (4.2)? If $\partial\Omega$ is uniformly rectifiable, then $\delta_{\partial\Omega}$ is "often almost" the distance to a plane, so $\nabla\delta_{\partial\Omega}$ is "often almost" constant and $\nabla^2\delta_{\partial\Omega}$ is "often almost" 0.

If we write $A \sim B$ for A is "often almost" equal to B, the fact that $\partial\Omega$ is uniformly rectifiable means $\nabla^2\delta_{\partial\Omega} \sim 0$, or $\delta_{\partial\Omega}\nabla^2\delta_{\partial\Omega} \sim 0$ for homogeneity purpose. Then (4.2) is $\nabla(G/\delta_{\partial\Omega}) \sim 0$, or $\frac{\delta_{\partial\Omega}}{G}\nabla(G/\delta_{\partial\Omega}) \sim 0$ for homogeneity purpose. We deduce that

$$\frac{\delta_{\partial\Omega}}{G}\nabla G = \frac{\delta_{\partial\Omega}}{G}\nabla\left(\frac{G}{\delta_{\partial\Omega}} \cdot \delta_{\partial\Omega}\right) \sim \nabla\delta_{\partial\Omega},$$

and then

$$\begin{aligned}\frac{\delta_{\partial\Omega}^2}{G}\nabla^2 G = \frac{\delta_{\partial\Omega}^2}{G}\nabla\left(\frac{\delta_{\partial\Omega}}{G}\nabla G \cot \frac{G}{\delta_{\partial\Omega}}\right) \\ = \delta_{\partial\Omega}\nabla\left(\frac{\delta_{\partial\Omega}}{G}\nabla G\right) + \delta_{\partial\Omega}\left(\frac{\delta_{\partial\Omega}}{G}\nabla G\right)\left(\frac{\delta_{\partial\Omega}}{G}\nabla\left[\frac{G}{\delta_{\partial\Omega}}\right]\right) \\ \sim \delta_{\partial\Omega}\nabla^2\delta_{\partial\Omega} + 0 \sim 0,\end{aligned}$$

which, translated to Carleson measure conditions, gives $\frac{\delta_{\partial\Omega}^2}{G}\nabla^2 G \in CM_\sigma$ as desired.

The reader might rightfully claim that the above explanation is far from a proof. While it is true that nothing above is rigorous, the formal computations will lead to the generalization of Theorem 4.2 to all DKP operators in the following sections. The main difference is that the computations use the regularized distances $D_{\sigma,\alpha}$, which appears to be a much better analogue of $\delta_{\partial\Omega}$.

4.2 . Uniform rectifiability and the regularized distance

In this section, we will delve deeper into the regularized distance D_α. Recall that when $E \subset \mathbb{R}^n$ is an d-Ahlfors regular set with Ahlfors regular measure σ, and when $\alpha > 0$, the regularized distance D_α is

$$D_\alpha(X) = D_{\sigma,\alpha}(X) := \left(\int_E |X-y|^{-d-\alpha} \, d\sigma(y) \right)^{-\frac{1}{\alpha}}.$$

We refer to D_α as *regularized distance* or *smooth distance* because it is equivalent to δ_E - the distance to E - see Proposition 3.31, but unlike δ_E, D_α is smooth everywhere in $\mathbb{R}^n \setminus E$.

Let us introduce the quantity

$$\mathcal{H}_{\sigma,\alpha} := -\frac{1}{d+\alpha} \nabla [D_{\sigma,\alpha}^{-\alpha}] = \int_E \frac{X-y}{|X-y|^{d+2+\alpha}} \, d\sigma(y).$$

Note that $\mathcal{H}_{\sigma,\alpha}$ is conceptually close to $\mathcal{R}_\sigma 1$, where $\mathcal{R}_\sigma$ is the Riesz transform on E

$$(\mathcal{R}_\sigma f)(x) := p.v. \int_E \frac{x-y}{|x-y|^{d+1}} f(y) d\sigma(y), \qquad x \in E$$

that we defined previously in (1.5). Morally, if the value $\alpha = -1$ were permissible, we would have $\mathcal{R}_\sigma 1 = \mathcal{H}_{-1}$.

The next question is whether $\mathcal{H}_{\sigma,\alpha}$ is *better* than the Riesz transform. Indeed, can we use $\mathcal{H}_{\sigma,\alpha}$ instead of $\mathcal{R}_\sigma$ in the David-Semmes conjecture (Conjecture 1.14)? First, we need to account for homogeneity in the expression of $\mathcal{H}_{\sigma,\alpha}$. The unitless quantity associated to $\mathcal{H}_{\sigma,\alpha}$ is

$$\mathcal{R}_{\sigma,\alpha} 1 := D_{\sigma,\alpha}^{\alpha+1} \mathcal{H}_{\sigma,\alpha} = \frac{\alpha}{d+\alpha} \nabla D_{\sigma,\alpha}. \tag{4.6}$$

Our first attempt is to prove the David-Semmes conjecture with the modified Riesz transform $\mathcal{R}_{\sigma,\alpha}$. If $\mathcal{R}_\sigma 1$ is analogous to $\mathcal{R}_{\sigma,\alpha} 1$, then the linear operator naturally arising from $\nabla D_{\alpha,\alpha} = c\mathcal{R}_{\sigma,\alpha} 1$ is

$$\mathcal{R}_{\sigma,\alpha} f(x) := \lim_{\substack{X \in \Gamma(x) \\ X \to x}} \left(\int_E |X-y|^{-d-\alpha} \, d\sigma(y) \right)^{-\frac{1}{\alpha}-1} \int_E \frac{X-y}{|X-y|^{d+\alpha+2}} f(y) \, d\sigma(y), \tag{4.7}$$

for $x \in E$, where $\Gamma(x)$ is the cone $\{X \in \mathbb{R}^n \setminus E, |X-x| \leq 2\delta_E(X)\}$ as usual. But trying to prove a characterization of uniform rectifiability from the boundedness of the operator $\mathcal{R}_{\sigma,\alpha}$ on $L^2(\sigma)$ is pointless, as we have

Lemma 4.4. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, $0 < d < n$, with Ahlfors regular measure σ. Then, there exists C depending only on α and the Ahlfors regular constant such that*

$$\mathcal{R}_{\sigma,\alpha} f(x) \leq C \mathcal{M}_\sigma f(x), \tag{4.8}$$

where $\mathcal{M}_\sigma$ is the Hardy-Littlewood maximal operator

Proof. Let $x \in E$ and $X \in \Gamma(x)$. Set $B_X := B(x, \delta(X))$, $C_{0,X} := B_X \cap E$, and $C_{k,X} := \Big(2^k B_k \setminus 2^{k-1} B_k\Big) \cap E$ when $j \geq 1$. We have that

$$\left| \left(\int_E |X-y|^{-d-\alpha} \, d\sigma(y) \right)^{-\frac{1}{\alpha}-1} \int_E \frac{X-y}{|X-y|^{d+\alpha+2}} f(y) \, d\sigma(y) \right|$$
$$\leq D_{\sigma,\alpha}(X)^{\alpha+1} \int_E |X-y|^{-d-\alpha-2} f(y) \, d\sigma(y)$$
$$\lesssim D_{\sigma,\alpha}(X)^{\alpha+1} \sum_{k \geq 0} (2^k \delta_E(X))^{-d-\alpha-2} \int_{C_{k,X}} f(y) \, d\sigma(y)$$
$$\lesssim D_{\sigma,\alpha}(X)^{\alpha+1} \delta_E(X)^{-\alpha-1} \mathcal{M}_\sigma f(x) \lesssim \mathcal{M}_\sigma f(x)$$

since $D_{\sigma,\alpha}(X) \approx \delta_E(X)$. Taking the supremum over the $X \in \Gamma(x)$ gives (4.8), as desired. □

After this simple result, we see that we need to take more information on $\mathcal{R}_{\sigma,\alpha} f$ than just a simple upper bound. If we come back to the discussion from the previous subsection, if E is uniformly rectifiable, it would mean that $\nabla D_\alpha \approx \delta_E$ is "often almost" close to the distance to a plane, and when uniformly rectifiable sets are involved, we know that "often almost" has be interpreted as a quantity q on $E \times (0, \infty)$ or $\mathbb{R}^n \setminus E$ satisfies the Carleson measure condition. One natural way to achieve this would be to define q as

$$q(X) := \inf_{\substack{P \ d\text{-plane} \\ c \geq 0}} \int_{B_X} |\nabla D_\alpha - c \nabla \delta_P| \, dY$$

and hope that

$$q \in CM_\sigma \iff E \text{ is uniformly rectifiable.} \tag{4.9}$$

Actually, we do not know whether (4.9) is true. While it sounds completely reasonable, to our knowledge, nobody has carefully proven it yet. The actual result involves taking one extra derivative : if E is a $(n-1)$-plane, then $\nabla^2 \delta_E = 0$, so we have :

Theorem 4.5. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, where $d \in (0, n)$ not necessarily an integer, and σ be a Ahlfors regular measure on E. Choose any $\alpha > 0$.*

Then the set E is $(n-1)$-uniformly rectifiable if and only if $\delta_E \nabla^2 D_{\sigma,\sigma} \in CM_\sigma$, i.e. there exists $M > 0$ such that for any ball B centered on E, we have

$$\int_B |\nabla^2 D_{\sigma,\alpha}(X)|^2 \delta_E(X) \, dX \leq M \sigma(B).$$

Proof. This theorem is a special case of the theorems given below : Theorem 4.6 for the converse and Corollary 4.9 for the direct implication. □

Can we generalize these results to higher codimensional sets, i.e. when $d < n-1$? Actually, yes, we just need to consider slightly weaker condition : when E is plane of dimension $d < n-1$, $\nabla D_{\sigma,\alpha} = \nabla \delta_E$ is not constant, but its length $|\nabla D_{\sigma,\alpha}|$ is. Replacing $D_{\sigma,\alpha}$ with its length is actually sufficient to characterize rectifiability and uniform rectifiability.

Theorem 4.6 (Theorem 1.4 in [DEM21]). *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, where $d \in (0, n)$ not necessarily an integer, and σ be a Ahlfors regular measure on E. Choose any $\alpha > 0$.*

Then the following are equivalent

(i) *E is d-uniformly rectifiable;*

(ii) *$\delta_E \nabla |\nabla D_{\sigma,\sigma}| \in CM_\sigma$, i.e. there exists $C > 0$ such that for any ball B centered on E, we have*

$$\int_B |\nabla |\nabla D_{\sigma,\alpha}(X)||^2 \delta_E(X)\, dX \leq C\sigma(B).$$

(iii) *$\delta_E \nabla [|\nabla D_{\sigma,\sigma}|^2] \in CM_\sigma$, i.e. there exists $C > 0$ such that for any ball B centered on E, we have*

$$\int_B |\nabla [|\nabla D_{\sigma,\alpha}(X)|^2]|^2 \delta_E(X)\, dX \leq C\sigma(B).$$

Let us explain how uniformly rectifiable sets affects $D_{\sigma,\alpha}$ and its derivative. Which quantities should satisfy the Carleson measure condition when the set E is uniformly rectifiable? They are actually fairy easy to find : morally if a function $F(x_0, x_1, x_2, \dots,)$ satisfies

$$F(D_{\sigma,\alpha}, \nabla D_{\sigma,\alpha}, \nabla^2 D_{\sigma,\alpha}, \dots) = 0$$

whenever σ is a flat measure - i.e. when $D_{\sigma,\alpha}$ is the distance to a d-plane, then

$$F(D_{\sigma,\alpha}, \nabla D_{\sigma,\alpha}, \nabla^2 D_{\sigma,\alpha}, \dots) \in CM_\sigma$$

whenever σ is a d-Ahlfors regular measure on a uniformly rectifiable set, provided that $F(D_{\sigma,\alpha}, \nabla D_{\sigma,\alpha}, \nabla^2 D_{\sigma,\alpha}, \dots)$ is a “dimensionless” quantity[1]. The precise result is given below.

Proposition 4.7. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, and let σ be Ahlfors regular measure on E. Take $\alpha > 0$ and define*

$$c_\alpha := \int_{\mathbb{R}^d} (1 + |y|^2)^{-\frac{d+\alpha}{2}}\, dy.$$

If E is uniformly rectifiable, then there exists $C > 0$ and a function $a_{\sigma,\alpha} \in CM_\Omega(M)$, such that for any $X \in \mathbb{R}^n \setminus \Omega$, there exist a constant $c_X > 0$ and an affine d-plane P_X that satisfy

$$C^{-1} \leq c_X \leq C, \tag{4.10}$$

$$C^{-1}\delta_E(X) \leq \operatorname{dist}(X, P_X) \leq C\delta_E(X), \tag{4.11}$$

and, if $k \in \mathbb{N}^n$ and $D_{E,\alpha,X}(Y) := (c_\alpha)^{-1/\alpha} \operatorname{dist}(Y, P_X)$,

$$\left| \partial^\kappa D_{\sigma,\alpha}(X) - (c_X)^{-\frac{1}{\alpha}} \partial^\kappa D_{E,\alpha,X}(X) \right| \leq C_\kappa \delta_E(X)^{1-|\kappa|} a_{\sigma,\alpha}(X). \tag{4.12}$$

One possible choice of c_X is

$$c_X := \frac{c_1}{c_{1/2}^2} \frac{D_{\sigma,1}(X)}{D_{\sigma,1/2}(X)}, \tag{4.13}$$

in particular $\delta_E \nabla c_X \in CM_\sigma$, see Lemma 3.36.

The constant C depends only on the Ahlfors-regular constant in (1.6)*, C_κ depends also on κ, M depends only on α and the uniformly rectifiable constants of $\partial\Omega$ (more precisely $a_{\sigma,\alpha}$ is a sum of the α-Tolsa numbers defined in Definition 1.12).*

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[1] if not there is a $j \in \mathbb{R}$ such that $D^j_{\sigma,\alpha} F$ is “dimensionless”, and this rescaled quantity satisfies the Carleson measure condition.

Remark 4.8. The above proposition focuses on D_α. However, similar estimates can be obtained for the quantity $\mathbf{H}$ in Lemma 3.37. Actually, Lemma 3.36 and Lemma 3.37 are just variants of the above proposition.

Proof. Proposition 4.7 comes from [FL23, Corollary 3.8], which gathers and generalizes the computations from [Fen22a, Section 3] and [DEM21, Theorem 2.1], see also [DFM19a] for a earlier similar result. ☐

How to read the above proposition? Take for instance $d = n - 1$, the quantities $D_{E,\alpha,X}$ are - up to a harmless constant - the distance to the $(n-1)$-plane P_X, so in particular, $\nabla^2 D_{E,\alpha,X} = 0$. This means that, in the codimension 1 case, $\delta_E \nabla^2 D_{\sigma,\alpha} \in CM_\sigma$, as in Theorem 4.5. Similarly, we have the following consequences of Proposition 4.7 :

Corollary 4.9 (Corollary 3.13 in [FL23])**.** *Let $E \subset \mathbb{R}^n$ be a d-dimensional uniformly rectifiable set, and take σ to be a Ahlfors regular measure on E. Let $\alpha > 0$. There exists M depending only on α and the uniformly rectifiable constants of σ such that*

(i) if $d = n-1$, then $\delta_{\partial\Omega} \nabla^2 D_{\sigma,\alpha} \in CM_\sigma$;

(ii) $\delta_{\partial\Omega} \nabla[|\nabla D_{\sigma,\alpha}|] \in CM_\sigma$, and similarly $\delta_{\partial\Omega} \nabla[|\nabla D_{\sigma,\alpha}|^2] \in CM_\sigma$;

(iii) $\delta_{\partial\Omega}^{n-d} \operatorname{div} D_{\sigma,\alpha}^{d+1-n} \nabla D_{\sigma,\alpha} \in CM_\sigma$.

Cases (3) and (4) of Theorem 4.20 state the converse to (i) and (ii) of Corollary 4.9, but the converse to (iii) of Corollary 4.9 is false (see "magic α", Lemma 3.33) at least without any extra assumption. The proof of the converse, i.e. $\delta_{\partial\Omega} \nabla[|\nabla D_{\sigma,\alpha}|] \in CM_\sigma$ implies that E is uniformly rectifiable, consists in a "blow-up" argument that reduce the problem to the limit case where $\nabla|\nabla D_{\sigma,\alpha}| = 0$: we can prove that if σ is a flat measure whenever $\nabla|\nabla D_{\sigma,\alpha}| = 0$, then E is uniformly rectifiable whenever $\delta_{\partial\Omega} \nabla[|\nabla D_{\sigma,\alpha}|] \in CM_\sigma$. Let us give more details.

4.3 . Weak-type Green function estimates

In this section, we will explore the relationship between estimates on the Green function and uniform rectifiability. Our primary focus will be on the "free boundary" direction, more precisely the fact that estimates on the Green function implies uniform rectifiability.

This subsection is heavily inspired from [DM22]. But since the presentation will differ from [DM22], and since we will give more results than [DM22], we will cite the bibliography only at the end of the section (Subsection 4.3.4).

4.3.1 . The limit case

In the terminology of Section 4.1, uniform rectifiable sets are those that are "often almost flat". We can often reduce the proof of the uniform rectifiability of a set under certain condition - expressed in the form $f \in CM_\sigma$ - to the proof of the "simpler" limit case where we prove that a set is a plane whenever the quantity f is zero.

The purpose of this section is to present these "limit cases", which are easier to prove. In the literature, the lemmas presented in this subsection are not always stated as results, but are often part of the proof of the larger theorems down the road.

Lemma 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain and m be a measure suitable for PDE. Let $G \in C^{0,\alpha}_{loc}(\overline{\Omega})$ be a positive function on Ω that satisfies $G = 0$ on $\partial\Omega$. Assume that one of the following case holds :*

(1) there exists a d-plane P such that $G(X) = \operatorname{dist}(X, P)$ for all $X \in \Omega$;

(2) there exists a d-plane P such that $\nabla G(X) = \nabla \operatorname{dist}(X, P)$ for a.e. $X \in \Omega$,

(3) $G \in C^{1,\alpha}_{loc}(\overline{\Omega})$ and $\nabla^2 G = 0$ a.e. in Ω,

(4) $\Omega = \mathbb{R}^n \setminus \partial\Omega$, $G \in C^2(\Omega)$, and $|\nabla G| = 1$ on Ω,

(5) $L = -\operatorname{div} A\nabla$ is a uniformly elliptic operator with constant coeffcients and G is a weak solution to $Lu = 0$ satisfying $|\nabla G| = 1$ a.e. on Ω,

Then $\partial\Omega$ is a d-plane. In Case (1), $\partial\Omega = P$; in Case (2), $\partial\Omega$ is parallel to P; in Case (3), d is $n-1$.

Remark 4.11. Note that in most cases, we do not need to know that G is a solution.

Proof. **Case (1).** Take $y \in \partial\Omega$ and $Y_n \in \Omega$ such that $Y_n \to y$. We have

$$0 = G(y) = \lim_{n\to\infty} G(Y_n) = \lim_{n\to\infty} \operatorname{dist}(Y_n, P) = \operatorname{dist}(y, P).$$

We deduce $y \in P$ and then $\partial\Omega \subset P$. If by contradiction, we have $\partial\Omega \subsetneq P$, then it means that $\Omega = \mathbb{R}^n \setminus \partial\Omega$ and thus any $P \setminus \partial\Omega \subset \Omega$. However, $\operatorname{dist}(Y, P) = 0$ when $Y \in P \setminus E$ and $G(Y) > 0$ when $Y \in \Omega$. This leads to a contradiction, hence that $P = \partial\Omega$.

Case (2). Take $X \in \Omega$. The property says that $G(Y) = G(X) > 0$ for all $Y \in P_X$, where P_X is the plane parallel to P going through X. It means that $P_X \subset \Omega$. By approaching boundary points $x \in \partial\Omega$ by points in $X \in \Omega$, we show that P_x is cannot be in Ω, but for any $\epsilon > 0$, a ϵ-translation of P_x is in Ω. It means that $P_x \subset \partial\Omega$, and thus that $\partial\Omega$ is the union of parallel d-planes. Then we let the reader check that the identity $\nabla G(X) = \nabla \operatorname{dist}(X, P)$ doesn't allow the boundary to contain 2 planes.

Case (3). If $\nabla^2 G = 0$, then ∇G is constant, so we can find a $(n-1)$-plane and $c > 0$ such that $\nabla G = c \operatorname{dist}(., P)$. Case (3) follows then from case (2).

Case (4). Given $X \in \Omega$, we construct the path $\varphi_X(t)$ as the solution to

$$\begin{cases} \varphi_X(0) = X \\ \varphi'(t) = -\nabla G(\varphi(t)) \end{cases}$$

We check that $(G \circ \varphi_X)' \equiv -1$, which means that $G \circ \varphi_X(t) = G(X) - t$. So φ_X exists if $t < G(X)$, and we can extend it by continuity to $t = G(X)$, where we have $G(\varphi_X(G(X))) = 0$, or $\varphi_X(G(X)) \in \partial\Omega$, or $\delta_{\partial\Omega}(\varphi_X(G(X))) = 0$. Then we have

$$\delta_{\partial\Omega}(X) = \delta(\varphi_X(0)) \leq \delta(\varphi_X(G(X))) + G(X) = G(X)$$

since $\delta_{\partial\Omega} \circ \varphi_X$ is 1-Lipschitz. Moreover, if $x \in \partial\Omega$ is such that $|X - x| = \delta_{\partial\Omega}(X)$, we have

$$G(X) \leq G(x) + \delta_{\partial\Omega}(X) = \delta_{\partial\Omega}(X)$$

since G is 1-Lipschitz. We deduce that $G = \delta_{\partial\Omega}$.

Let us sketch the rest. First, $\mathbb{R}^n \setminus \Omega$ has to be convex, otherwise there will be a point $X \in \Omega$ for which there exists two different boundary points $x_1, x_2 \in \partial\Omega$ such that $\delta_{\partial\Omega}(X) = |X - x_1| = |X - x_2|$; at such point X, $\nabla \delta_{\partial\Omega}$ is not defined and thus the $G = \delta_{\partial\Omega}$ cannot be C^1. Second, the convex hull of a set E has a integer dimension d and is included in a d-plane, so since $\partial\Omega = \mathbb{R}^n \setminus \Omega$ is convex, $\partial\Omega$ is included in a d-plane P, where d is the dimension of (the convex hull of) $\partial\Omega$. Finally, the $\partial\Omega$ - seen as a subset of P - cannot have boundary points in P, otherwise $\delta_{\partial\Omega}$ will only be $C^{1,1}(\Omega)$ above the edges of $\partial\Omega$. We conclude that $\partial\Omega$ is a d-plane.

Case (5). This case is a variant of (4). Since G is a weak solution of a constant coefficient operator, then $G \in C^\infty(\Omega)$. So as in Case (4), we deduce that $\mathbb{R}^n \setminus \Omega$ is convex and $G = \delta_{\partial\Omega}$. We also know that $d = n-1$, because any other integer choice of d would make the measure $m = \mathcal{L}^n$ not suitable for PDE.

Here is the idea to help us conclude. We locally parametrize $\partial\Omega$ by a function φ and brutally compute $\Delta\delta_{\partial\Omega}$, and we observe that $\Delta\delta_{\partial\Omega}(X) = 0$ if and only if $\nabla^2\varphi(x) = 0$, where $(x, \varphi(x))\partial\Omega$ is the (unique) projection of X on $\partial\Omega$. We deduce that φ is affine, hence $\partial\Omega$ is flat. ☐

Instead of comparing G to the distance to a plane, it would also be interesting to compare the Green function to the regularized distances $D_{\sigma,\alpha}$. To this end, we define

Definition 4.12. Take $\alpha > 0$ and $\mathcal{A}$ an elliptic matrix with constant coefficients. We say that the property $\mathcal{Y}_{flat}(d, \alpha, \mathcal{A})$ holds if for any domain $\Omega \subset \mathbb{R}^n$ satisfying the corkscrew point condition and whose boundary is d-Ahlfors regular (σ is an Ahlfors regular measure on $\partial\Omega$), the fact that

$$-\operatorname{div} D_{\sigma,\alpha}^{d+1-n} \mathcal{A} \nabla D_{\sigma,\alpha} = 0 \tag{4.14}$$

implies that d is an integer, $\partial\Omega$ is a d-plane, and σ is a flat measure (i.e. $\sigma = c\mathcal{L}^d|_{\partial\Omega}$ for some constant $c > 0$).

We do not know whether $\mathcal{Y}_{flat}(d, \alpha, A)$ is true, even when $d = n-1$ and A is the identity matrix. But we do know that it can fail, as shown in Lemma 3.33, where for $d < n-2$, the property $\mathcal{Y}_{flat}(d, n-d-2, I)$ is false. We invite the reader verify that Definition 4.12 is coherent, i.e. that (4.14) holds whenever $\partial\Omega$ is a d-place and σ a flat measure. With this definition, we can extend Lemma 4.10 as follows :

Lemma 4.13. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with d-Ahlfors regular boundaries, let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator with respect to the weight $w = \delta_{\partial\Omega}^{d+1-n}$, and let $\alpha > 0$. Assume*

(6) there exists a Ahlfors regular measure σ such that $\mathcal{A} := D_{\sigma,\alpha}^{n-d-1} A$ is constant, $D_{\sigma,\alpha}$ is a weak solution to L, and the property $\mathcal{Y}_{flat}(d, \alpha, A)$ holds.

Then $\partial\Omega$ is a d-plane.

You can see that the above lemma is trivial ; it is merely a reformulation of the property $\mathcal{Y}_{flat}(d, \alpha, A)$. Nevertheless, like Lemma 4.10, Lemma 4.13 will be the “limit” case for an apparently stronger result.

We can still discuss the comparison of the Green function and the regularized distances D_β, but we need to take another approach that only works in codimension 1. This approach requires the “free boundary” implication of Theorem 1.37. However, this is not a direction we want to follow : we aim to develop a theory to replace elliptic measures by Green functions, hoping to advance further in the wild land of new knowledge. Basing our proof on previously known results for the elliptic measures would probably be unproductive.

Lemma 4.14. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundary, and let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator with constant coefficients. Assume that there exists $\alpha > 0$ and a Ahlfors regular measure σ such that $D_{\sigma,\alpha}$ is a weak solution to $Lu = 0$ in Ω. Then Ω is a CAD, in particular $\mathbb{R}^n \setminus \Omega$ satisfies the corkscrew point condition.*

Proof. If $D_{\sigma,\alpha}$ is a weak solution in Ω, then $D_{\sigma,\alpha}$ is the Green function with pole at ∞ (See Proposition 2.70). Then the elliptic measure measure with pole at ∞ is comparable to σ (see Proposition 2.79), which in turn means that Ω is a CAD (see Theorem 1.37). $\square$

4.3.2 . The compactness argument.

We aim to extend the lemmas from the previous paragraph into theorems that provide sufficient conditions for uniform rectifiability. To achieve this, we need to replace conditions like $G = \mathrm{dist}(., P)$ to conditions like G is close to $\mathrm{dist}(., P)$. For this purpose, we start by introducing a slightly weaker notion of smallness ; while the Carleson measure condition would suffice, we can be more general.

Definition 4.15. Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, and write σ for a Ahlfors regular measure on E. We say that a subset $\mathcal{B}$ of $E \times (0, +\infty)$ satisfies the **Carleson packing condition** if there exists a constant $C > 0$ such that, for any $x \in E$ and any $r > 0$,

$$\int_{E\cap B(x,r)} \int_0^r \mathbb{1}_{\mathcal{B}}(y,t)\, \frac{dt}{t}\, d\sigma(y) \leq C\sigma(E \cap B(x,r)). \tag{4.15}$$

We say that $\mathcal{G} \subset E \times (0, +\infty)$ is a **Carleson prevalent set** if $E \times (0, +\infty) \setminus \mathcal{G}$ satisfies the Carleson packing condition.

We use this to introduce the notion of weak Carleson measure condition.

Definition 4.16. Let $\Omega \subset \mathbb{R}^n$ be a domain with d-Ahlfors regular boundary, and write σ for a Ahlfors regular measure on $\partial\Omega$. We say that $f \in L^2_{loc}(\Omega)$ satisfies the **weak Carleson measure condition** if the set

$$\mathcal{G}_f(\tau, K) := \Big\{(x,r) \in \partial\Omega \times (0,\infty),\ \int_{W_K(x,r)} |f|\, dX \leq \tau r^n\Big\},$$

where

$$W_K(x,r) := \{X \in B(x, Kr),\ \delta_{\partial\Omega}(X) \geq r/K\},$$

is Carleson prevalent for each couple of constant $\tau > 0$, $K \geq 2$. We write $f \in wCM_\sigma$.

It would be simple enough to extend Definition 4.16 to the same setting as Definition 3.7, but we refrain from doing so here to keep the presentation friendly. As expected, the Carleson measure condition implies the weak Carleson measure condition.

Lemma 4.17. *Let $\Omega \subset \mathbb{R}^n$ be a domain with d-Ahlfors regular boundary, and write σ for a Ahlfors regular measure on $\partial\Omega$. Then $f \in CM_\sigma$ implies $f \in wCM_\sigma$.*

Proof. Let $x \in \partial\Omega$ and $r > 0$. Set $\mathcal{B}_f(\tau, K) = \partial\Omega \times (0,\infty) \setminus \mathcal{G}_f(\tau, K)$.

$$\begin{aligned}
&\int_{E\cap B(x,r)} \int_0^r \mathbb{1}_{\mathcal{B}}(y,t)\, \frac{dt}{t}\, d\sigma(y) \\
&\qquad \leq \tau^{-2} \int_{E\cap B(x,r)} \int_0^r \Big(t^{-n} \int_{W_K(y,t)} |f|\, dX\Big)^2 \frac{dt}{t}\, d\sigma(y) \\
&\qquad \leq C_n K^n \tau^{-2} \int_{E\cap B(x,r)} \int_0^r \int_{W_K(y,t)} |f|^2\, dX\, \frac{dt}{t^{n+1}}\, d\sigma(y) \\
&\qquad \leq C_n K^n \tau^{-2} \int_{\Omega\cap B(x,(K+1)r} |f(X)|^2 \int_{E\cap B(x,r)} \int_0^r \mathbb{1}_{X\in W_K(y,t)} \frac{dt}{t^{n+1}}\, d\sigma(y)
\end{aligned}$$

But since $X \in W_K(y,t)$ implies that $t \geq \delta_{\partial\Omega}(X)/K$ and $y \in B(X, K^2\delta_{\partial\Omega}(X))$, we deduce

$$\begin{aligned}\int_{E\cap B(x,r)} \int_0^r \mathbb{1}_{\mathcal{B}}(y,t)\,\frac{dt}{t}\,d\sigma(y) \\ \leq C_n K^n \tau^{-2} \int_{\Omega\cap B(x,(K+1)r}|f(X)|^2 (K\delta_{\partial\Omega}(X))^{-n}\sigma(B(X,K^2\delta_{\partial\Omega}(X))\cap\partial\Omega) \\ \leq C_{\sigma,n}K^{2(n+d)}\int_{\Omega\cap B(x,(K+1)r}|f|^2\delta_{\partial\Omega}^{d-n}\,dX \\ \leq C_{K,\tau,\sigma,n} M\sigma(B(x,r)\cap\partial\Omega)\end{aligned}$$

if $f \in CM_\sigma(M)$. The lemma follows. □

Remark 4.18. Note that the Carleson measure condition and the weak Carleson measure condition are actually different. Indeed, the function

$$f:\,(x,t)\in\mathbb{R}^{n-1}\times(0,\infty)\to(1+\sqrt{|\ln(t)|})^{-1}$$

satisfies the weak Carleson measure condition in $\mathbb{R}^n_+$ but not the Carleson measure condition.

We define the weakest DKP operators with the following proposition.

Proposition 4.19. *Let Ω be a domain satisfying the corkscrew point condition whose boundary os $(n-1)$-Ahlfors regular, and let $L := -\operatorname{div}[\mathcal{A}\nabla]$ be a uniformly elliptic operator on Ω. Then the following are equivalent*

(i) for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_{w*DKP}(\tau,K)$ that contains the couples $(x,r) \in \partial\Omega\times(0,\infty)$ such that

$$\inf_{\mathcal{A}_0 \text{ constant}} \int_{W_K(x,r)} |\mathcal{A}(X)-\mathcal{A}_0|\,dX \leq \tau r^n$$

is Carleson prevalent.

(ii) $\mathcal{A}$ can be decomposed as $\mathcal{A}=\mathcal{B}+\mathcal{C}$ where $\delta_{\partial\Omega}|\nabla\mathcal{B}|+|\mathcal{C}| \in wCM_\sigma$.

We call a uniformly elliptic operator satisfying either (i) or (ii) a **weakest DKP operator***. Alternatively, if $\mathcal{A}$ satisfies (i) or (ii), we say that $\mathcal{A}$ satisfies the* **weakest DKP condition***.*

Clearly, DKP operators - and more generally uniformly elliptic operators $L = -\operatorname{div}[B+C]\nabla$ with $\delta_{\partial\Omega}|\nabla B|+|C| \in CM_\sigma$ - are weakest DKP operators. Now, we are ready for our main theorem of the section.

Theorem 4.20. *Let $\Omega \subset \mathbb{R}^n$ be a domain that satisfies the corkscrew point condition and whose boundaries are d-Ahlfors regular. Let σ be a Ahlfors regular measure on $\partial\Omega$. Let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator (with respect to the weight $w=\delta_{\partial\Omega}^{d+1-n}=dm/dx$).*

Let $\Omega_{fat}\subset\Omega$ be a **Carleson prevalent domain** *in Ω, that is, for any $K\geq 1$, the set*

$$\mathcal{D}_\Omega(K) := \Big\{(x,r)\in\partial\Omega\times(0,\infty),\ B(x,2Kr)\cap\Omega\subset\Omega_{fat}\Big\} \tag{4.16}$$

is Carleson prevalent. Assume also that there exists $C,\eta>0$ such that, for any $(x,r)\in\mathcal{D}_\Omega(1)$, there exists a function $G_{x,r}\in W(B(x,2r)\cap\Omega,m)$ with the following property

(i) $G_{x,r}$ is a positive weak solution to $Lu = 0$ in $\Omega \cap B(x, 2r)$ satisfying $\mathrm{Tr}(G_{x,r}) = 0$ on $B(x, 2r) \cap \partial\Omega$,

(ii) $G_{x,r}(X) \geq C^{-1} \left(\dfrac{\delta(X)}{r}\right)^{\eta} \sup\limits_{B(x,r)\cap\Omega} G_{x,r}$ for any $X \in B(x, r) \cap \Omega$.

Assume moreover that one of the following case holds :

(1) $G_{x,r}$ is Carleson prevalently close to the distance to a d-plane, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_1(\tau, K)$ of couple $(x, r) \in \mathcal{D}_\Omega(K)$ such that

$$\inf_{\substack{P\,d\text{-plane} \\ c>0}} \int_{W_K(x,r)} |cG_{x,Kr}(X) - \mathrm{dist}(X, P)|\, dX \leq \tau r^{n+1}$$

is Carleson prevalent;

(2) $\nabla G_{x,r}$ is Carleson prevalently close to the gradient of the distance to a d-plane, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_2(\tau, K)$ of couple $(x, r) \in \mathcal{D}_\Omega(K)$ such that

$$\inf_{\substack{P\,d\text{-plane} \\ c>0}} \int_{W_K(x,r)} |c\nabla G_{x,Kr}(X) - \nabla\,\mathrm{dist}(X, P)|\, dX \leq \tau r^{n}$$

is Carleson prevalent;

(3) $d = n - 1$, $\delta_{\partial\Omega}\nabla\mathcal{A} \in L^\infty(\Omega)$, and $\nabla^2 G_{x,r,K}$ is Carleson prevalently close to 0, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_3(\tau, K)$ of couple $(x, r) \in \mathcal{D}_\Omega(K)$ such that

$$\int_{W_K(x,r)} \frac{\delta^2_{\partial\Omega}|\nabla^2 G_{x,Kr}(X)|}{G_{x,Kr}(X)}\, dX \leq \tau r^{n}$$

is Carleson prevalent;

(4) $\Omega = \mathbb{R}^n \setminus \partial\Omega$, $|\delta_{\partial\Omega}^{n-d}\nabla\mathcal{A}| + |\delta_{\partial\Omega}^{n+1-d}\nabla^2\mathcal{A}| \in L^\infty(\Omega)$ and $\delta^3_{\partial\Omega}\nabla[|\nabla G_{x,Kr}|^2]/G^2 \in wCM_\sigma$, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_4(\tau, K)$ of couple $(x, r) \in \mathcal{D}_\Omega(K)$ such that

$$\int_{W_K(x,r)} \frac{\delta^3_{\partial\Omega}\big|\nabla|\nabla G_{x,Kr}(X)|^2\big|}{G_{x,Kr}(X)^2}\, dX \leq \tau r^{n}$$

is Carleson prevalent;

(5) $d = n - 1$, L is a weakest DKP operator, and $|\nabla G|$ is Carleson prevalently close to a constant, that is for any $\tau > 0$ and $K \geq 2$, the set of couple $(x, r) \in \mathcal{D}_\Omega(K)$ such that

$$\inf_{c>0} \int_{W_K(x,r)} \big|c|\nabla G_{x,Kr}(X)| - 1\big|\, dX \leq \tau r^{n}$$

is Carleson prevalent;

(6) $d = n - 1$, L is a weakest DKP operator, and there exists $\alpha > 0$ such that (a) the property $\mathcal{Y}_{flat}(n - 1, \alpha, A_0)$ holds for all constant matrices A_0 and (b) $G_{x,r}$ is prevalently close to $D_{\sigma,\alpha}$, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_6(\tau, K)$ of couple $(x, r) \in D_\Omega(K)$ such that

$$\inf_{c>0} \int_{W_K(x,r)} |cG_{x,Kr}(X) - D_{\sigma,\alpha}(X)|\, dX \leq \tau r^{n+1}$$

is Carleson prevalent;

(6bis) there exists $\alpha > 0$ such that (a) $L = -\operatorname{div}[D_{\sigma,\alpha}^{d+1-n}\nabla]$, (b) the property $\mathcal{Y}_{flat}(d,\alpha,I)$ holds, and (c) $G_{x,r}$ is prevalently close to $D_{\sigma,\alpha}$.

Then $\partial\Omega$ is a uniformly rectifiable.

Some remarks are in order.

Remark 4.21. The property (ii) on $G_{x,r}$ is automatically satisfied when Ω is uniform. That is a simple consequence of the Carleson estimate (Point (i) of Theorem 2.66), the existence of Harnack chains, and the Harnack inequality (Proposition 2.61, with $f \equiv 0$). However, it permits us to use the theorem even when the Harnack chain condition fails, as long as we choose our positive function properly.

Remark 4.22. We consider $G_{x,r} = G(.,Y_{x,r})$, where G is the Green function and $Y_{x,r} \in \Omega\backslash B(x,2r)$. We can either make $Y_{x,r}$ depend on x and r, or fix Y and exclude the (x,r) for which $Y \in B(x,2r)$. When the domain is not uniform, we want to take an appropriate linear combination of $g(.,Y)$ to ensures that (ii) of the theorem holds.

Elements of proof for Theorem 4.20. As the reader can imagine, all the 6 (or 7) cases can be treated in a similar manner. We present the ideas of the proof for Case (1), and just point out the differences for the other cases.

Case (1). The proof is by contradiction. We assume that the set $\partial\Omega$ is not uniformly rectifiable, which is implied - thanks to Theorem 1.13 - by the fact that there exists $\epsilon_0 > 0$ such that, for each $\tau > 0$, $K \geq 2$, the set $\mathcal{G}_1(\tau,K) \cap \mathcal{D}_\Omega(K)$ is not a subset of

$$\mathcal{G}_{ur}(\epsilon_0) := \{(x,r) \in \partial\Omega \times (0,\infty),\ b\beta_\infty(x,r) < \epsilon\}.$$

If it is not the case, it means that for any $j \geq 1$, there exists $(x_j,r_j) \in \big(\mathcal{G}_1(1/j,j) \cap \mathcal{D}(j)\big) \setminus \mathcal{G}_{ur}(\epsilon_0)$. We can translate and dilate the ambient space $\mathbb{R}^n$ to make $(x_j,r_j) = (0,1)$. Let Ω_j, σ_j, L_j, and G_j be the domain Ω, the measure σ, the operator L, and the function G_{x_j,r_j} after this translation and dilatation. Write B_0 for $B(0,1)$ and $\Omega_{j,j}$ for $\Omega_j \cap W_j(0,1)$. The collections $\{\Omega_j\}_j$, $\{\sigma_j\}_j$ and $\{G_j\}_j$ satisfy

(a) $0 \in \partial\Omega_j$,

(b) $$\inf_{P \text{ is a } d\text{-plane}} \left(\sup_{y \in \partial\Omega_j \cap B_0} \operatorname{dist}(y,P) + \sup_{z \in P \cap B_0} \operatorname{dist}(z,\partial\Omega_j) \right) > \epsilon_0,$$

(c) Ω_j satisfies the corkscrew point condition (with constant uniform in j),

(d) $\partial\Omega_j$ is d-Ahlfors regular regular, and σ_j is a Ahlfors regular measure on $\partial\Omega_j$ (with constants uniform in j),

(e) L_j are uniformly elliptic (with constants uniform in j).

(f) G_j are weak solution to $L_j u = 0$ in $\Omega_{j,j}$, in particular they are Hölder continuous (with constants uniform in j).

(g) $G_j \geq C^{-1}\delta_{\Omega_j}(X)^\eta \sup_{B(x,r)\cap\Omega_j} G_j$

(h) there exists a constant $c_j > 0$ and a d-plane P_j such that

$$\int_{\Omega_{j,j}} |c_j G_j(X) - \operatorname{dist}(X,P_j)|\, dX \leq \frac{1}{j} r^{n+1} \tag{4.17}$$

We then extract a limit. Such extraction is not proven here, but they are fairly straightforward with just basic set theory, functional analysis, and elliptic theory.

Up to a subsequence $\mathbb{R}^n \setminus \Omega_j$ and $\partial\Omega_j$ converges in the local Hausdorff distance to $\mathbb{R}^n \setminus \Omega_\infty$ and E_∞ respectively, where a sequence of set $S_j \ni 0$ converges to $S_\infty \ni 0$ in the local Hausdorff distance if, for all $K \geq 0$

$$\lim_{j\to\infty} \left(\sup_{y \in S_j \cap KB_0} \operatorname{dist}(y, S_\infty) + \sup_{z \in S_\infty \cap KB_0} \operatorname{dist}(z, S_j) \right) = 0.$$

Because the Ω_j satisfy the corkscrew point condition, one can then check that Ω_∞ also satisfies the corkscrew point condition, that Ω_j converges to Ω_∞ in the local Hausdorff distance, and that $\partial\Omega_\infty = E_\infty$ and is d-Ahlfors regular. Moreover, up to an extra subsequence, σ_j converges weakly-$*$ to a d-Ahlfors regular measure σ_∞ on $\partial\Omega_\infty$, and $c_j G_j$ converges uniformly on bounded sets of $\mathbb{R}^n$ to Hölder continuous function G_∞, which is positive on Ω_∞ and satisfies $G_\infty = 0$ on $\partial\Omega_\infty$. The important point is that all the constants relative to Ω_∞, $\partial\Omega_\infty$, σ_∞, and G_∞ (the corkscrew point constant of Ω, the Ahlfors regular constant of $\partial\Omega_\infty$ and σ_∞, the Hölder constants of G_∞) depends only on those of Ω, $\partial\Omega$, σ, and $G_{x,r}$.

Up to a subsequence, we also have that the plane P_j from (h) converges to a plane P_∞. Together with the convergence $G_j/c_j \to G_\infty$ and (4.17), we have $G_\infty = \operatorname{dist}(., P_\infty)$. The domain Ω_∞ and the function G_∞ verifies the assumptions of Lemma 4.10, Case (1), which allow us to conclude that $\partial\Omega$ is a d-plane. This is a contradiction, because we have

$$0 = \lim_{j\to\infty} \left(\sup_{y \in \partial\Omega_j \cap B_0} \operatorname{dist}(y, \partial\Omega_\infty) + \sup_{z \in \partial\Omega_\infty \cap KB_0} \operatorname{dist}(z, \partial\Omega_j) \right) > \epsilon_0$$

where we used : the fact that $\partial\Omega_j$ converges to $\partial\Omega_\infty$ - which is a plane - in the local Hausdorff distance, and then (b).

Case (2). Same as Case (1) without much difference : we take P_j to be the plane going through 0 to ensure that a subsequence of the P_j converges; and we construct the subsequence so that ∇G_j converges weakly in $L^2_{loc}(\Omega)$, the limit being necessarily to both ∇G_∞ and $\nabla \operatorname{dist}(., P_\infty)$. We invoke Case (2) of Lemma 4.10 to say that $\partial\Omega_\infty$ is flat and we conclude as in Case (1).

Case (3). The condition $\delta_{\partial\Omega}\nabla\mathcal{A} \in L^\infty(\Omega)$ is only here to guarantee that $\nabla^2 G \in L^2_{loc}(D_G(K))$, so that the quantity makes sense *a priori*. We take G_∞ as a limit of $G_j/G_j(X_0)$, where $X_{0,j}$ is a corkscrew point for $B_0 \cap \partial\Omega_j$. We have $\nabla^2 G_j/G_j(X_{0,j})$ converges weakly in $L^2_{loc}(\Omega)$ to $\nabla^2 G_\infty$ and 0. We invoke Case (3) of Lemma 4.10 to say that $\partial\Omega_\infty$ is flat and we conclude as in Case (1).

Case (4). We extract the limit G_∞ as in Case (3), in particular $\nabla^2 G_j/G_j(X_{0,j})$ converges weakly to $\nabla^2 G_\infty$. Since $\nabla|\nabla[G_j/G_j(X_{0,j})]|^2$ converges to 0, we deduce that $\nabla|\nabla G_\infty| = 0$, that is - up to multiplication by an harmless constant - G_∞ satisfies $|\nabla G_\infty| = 1$.

The coefficients of $L_j = -\operatorname{div} A_j \nabla$ are $C^{1,1}_{loc}(\Omega)$, which means that the G_j are $C^{2,1}_{loc}(\Omega)$. The constants are uniform so G_∞ is also $C^{2,1}_{loc}(\Omega)$.

We invoke Case (4) of Lemma 4.10 to say that $\partial\Omega_\infty$ is flat and we conclude as in Case (1).

Case (5). With a method similar to the previous cases, we can extract a limit G_∞ that satisfies $|\nabla G_\infty| = 1$.

The only real difference is the presence of weakest DKP operators. We want to prove this time that, for any $\epsilon > 0$, there exists $\tau > 0$ and $K \geq 2$ such that $\mathcal{G}_5(\tau, K) \cap \mathcal{D}_\Omega(K) \cap \mathcal{G}_{w*DKP}(\tau, K) \subset \mathcal{G}_{ur}(\epsilon)$. By contradiction, we assume that there exists ϵ_0 such that, for any $j \geq 1$,

$$\Big(\mathcal{G}_5(j^{-1}, j) \cap \mathcal{D}_\Omega(j) \cap \mathcal{G}_{w*DKP}(j^{-1}, j)\Big) \setminus \mathcal{G}_{ur}(\epsilon_0) \supset \{(x_j, r_j)\} \neq \emptyset.$$

What it means is that, in addition to (a)–(h) above, we also have that the collection $\{L_j = -\operatorname{div} A_j \nabla\}_j$ satisfies

(i) there exists a constant matrix $A_{0,j}$ such that

$$\int_{\Omega_{j,j}} |A_j - A_{0,j}| \, dX \leq \frac{1}{j} r^{n.} \tag{4.18}$$

Up to a subsequence, $A_{0,j}$ converges in to a constant matrix A_∞. By invoking then (4.18), the sequence A_j converges to A_∞ in $L^1_{loc}(\Omega)$ and - since A_j is uniformly bounded - also in $L^2_{loc}(\Omega)$. By the Caccioppoli inequality, we can take another subsequence so that ∇G_j converges weakly in $L^2_{loc}(\Omega)$ to ∇G_∞. Using those two facts, it is fairly easy to see that G_∞ is a weak solution to $L_\infty := -\operatorname{div} A_\infty \nabla$.

We invoke Case (5) of Lemma 4.10 to say that $\partial\Omega_\infty$ is flat and we conclude as in Case (1).

Cases (6) and (6bis). Since σ_j weakly-$*$ converges to σ_∞, we have that $D_{\sigma_j,\alpha}$ converges to $D_{\sigma_\infty,\alpha}$ uniformly on compact sets of Ω_∞. Ultimately, we find that $G_\infty = D_{\sigma_\infty,\alpha}$ and we use Case (6) of Lemma 4.13 to conclude. ☐

If Ω is uniform, we can prove Case (6) with a different method.

Theorem 4.23. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundaries. Let σ be a Ahlfors regular measure on $\partial\Omega$. Let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic weakest DKP operator.*

Let $\Omega_{fat} \subset \Omega$ be a Carleson prevalent domain, and let $\mathcal{D}_\Omega(K)$ be like in (4.16)*. Assume that for any $(x,r) \in \mathcal{D}_\Omega(1)$, there exists a function $G_{x,r} \in W^{1,2}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ with the following property*

(i) $G_{x,r}$ is a positive weak solution to $Lu = 0$ in $\Omega \cap B(x, 2r)$ satisfying $G_{x,r} = 0$ on $B(x, 2r) \cap \partial\Omega$,

(7) $G_{x,r}$ is prevalently close to $D_{\sigma,\alpha}$, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_7(\tau, K)$ of couple $(x,r) \in D_\Omega(K)$ such that

$$\inf_{c>0} \int_{W_K(x,r)} |cG_{x,Kr}(X) - D_{\sigma,\alpha}(X)| \, dX \leq \tau r^{n+1}$$

is Carleson prevalent.

(7bis) $\delta_{\partial\Omega} \nabla \ln(G_{x,r}/D_{\sigma,\alpha}) \in wCM_\sigma$, that is, for any $\tau > 0$ and $K \geq 2$, the set $\mathcal{G}_{7b}(\tau, K)$ of couple $(x,r) \in D_\Omega(K)$ such that

$$\int_{W_K(x,r)} \delta_{\partial\Omega} \left| \frac{\nabla G_{x,Kr}(X)}{G_{x,Kr}} - \frac{\nabla D_{\sigma,\alpha}(X)}{D_{\sigma,\alpha}} \right| dX \leq \tau r^n$$

is Carleson prevalent.

Then $\partial\Omega$ is uniformly rectifiable.

Elements of proof. We want to prove the existence of corkscrew point in $\mathbb{R}^n\setminus\Omega$. For $\epsilon > 0$, let us introduce $\mathcal{G}_{extCP}(\epsilon)$ as the couple $(x,r) \in \partial\Omega \times (0,\infty)$ such that

$$\mathcal{G}_{extCP}(\epsilon) := \Big\{(x,r) \in \partial\Omega \times (0,\infty),\ B(x,r) \setminus \{X \in \mathbb{R}^n, \operatorname{dist}(X,\Omega) \leq \epsilon r\} \neq \emptyset\Big\}.$$

We can see that if $(x,r) \in \mathcal{G}_{extCP}(\epsilon)$, then there exists a ϵ-corkscrew point associated to (x,r) in $\mathbb{R}^n \setminus \Omega$.

Case (7). We want to prove that there exists $\epsilon > 0$, $\tau > 0$ and $K \geq 2$ such that

$$\mathcal{G}_7(\tau,K) \cap \mathcal{D}_\Omega(K) \cap \mathcal{G}_{w*DKP}(\tau,K) \subset \mathcal{G}_{extCP}(\epsilon). \tag{4.19}$$

Why is the claim (4.19) enough? Since $\mathcal{G}_7(\tau,K) \cap \mathcal{D}_\Omega(K) \cap \mathcal{G}_{w*DKP}(\tau,K)$ is Carleson prevalent, the set $\mathcal{G}_{extCP}(\epsilon)$ is then also Carleson prevalent, meaning that there exists M such that, for any $x \in \partial\Omega$ and any $r > 0$

$$\int_{B(x,r)\cap\partial\Omega} \int_0^r \big(1 - \mathbb{1}_{\mathcal{G}_{extCP}(\epsilon)}\big) \frac{dt}{t}\, d\sigma(y) \leq M\sigma(B(x,r)\cap\partial\Omega).$$

We can then find N large, depending only on M and the Ahlfors regular constant, such that

$$\ln(N) \geq M\frac{\sigma(B(x,r)\cap\partial\Omega)}{\sigma(B(x,r/2)\cap\partial\Omega)}.$$

The two last inequalities means that for any couple $(x,r) \in \partial\Omega \times (0,\infty)$, we can find $(y,s) \in \mathcal{G}_{extCP}(\epsilon) \cap \big(B(x,r/2)\times(r/2N,r/2)\big)$. Observe then that the exterior ϵ-corkscrew point for y at scale s is an exterior $\epsilon/2N$-corkscrew point for x at scale r; that is $\mathcal{G}_{extCP}(\epsilon/2N) = \partial\Omega \times (0,\infty)$.

We prove the claim (4.19) by contradiction, that is we assume that for all $j \geq 1$,

$$\Big(\mathcal{G}_7(1/j,j) \cap \mathcal{D}_\Omega(j) \cap \mathcal{G}_{w*DKP}(1/j,j)\Big) \setminus \mathcal{G}_{extCP}(1/j) \supset \{(x_j,r_j)\} \neq \emptyset.$$

Like in Theorem 4.20, we obtain some collections $\{\Omega_j\}_j$, $\{\sigma_j\}_j$, $\{L_j = -\operatorname{div} A_j\nabla\}_j$, and $\{G_j\}_j$ verifying

(a) $0 \in \partial\Omega_j$,

(b) $\mathbb{R}^n \setminus \Omega_j$ has no $1/j$-corkscrew point associated to $(0,1)$,

(c) Ω_j is uniform (with constants uniform in j),

(d) $\partial\Omega_j$ is d-Ahlfors regular regular, and σ_j is a Ahlfors regular measure on $\partial\Omega_j$ (with constants uniform in j),

(e) L_j are uniformly elliptic (with constants uniform in j).

(f) G_j are weak solution to $L_j u = 0$ in $\Omega_{j,j}$, in particular they are Hölder continuous (with constants uniform in j),

(g) there exists a constant $c_j > 0$ such that

$$\int_{\Omega_{j,j}} |c_j G_j(X) - D_{\sigma_j,\alpha}|\, dX \leq \frac{1}{j} r^{n+1}, \tag{4.20}$$

(h) there exists a constant matrix $A_{0,j}$ such that

$$\int_{\Omega_{j,j}} |A_j - A_{0,j}|\, dX \leq \frac{1}{j} r^{n.} \tag{4.21}$$

We extract limits. Up to a subsequence, Ω_j and $\partial\Omega_j$ converges to the uniform domain Ω_∞ and the Ahlfors regular set $\partial\Omega_\infty$ in the local Hausdorff distances, σ_j converges weakly-$*$ to the Ahlfors regular measure σ_∞, $D_{\sigma_j,\alpha}$ converges to $D_{\sigma_\infty,\alpha}$ on compacts of Ω_∞, A_j converges to a constant matrix A_∞, and $c_j G_j$ converges to G_∞ both on compact sets of $\mathbb{R}^n$ and weakly in $W^{1,2}_{loc}(\Omega)$, G_∞ is a positive weak solution $L_\infty = -\operatorname{div} A_\infty \nabla u = 0$ in Ω_∞ with zero trace. The estimate (4.20) gives $G_\infty = D_{\sigma_\infty,\alpha}$.

We can then apply Lemma 4.14 to Ω_∞, L_∞ and G_∞, and we obtain that $\mathbb{R}^n \setminus \Omega_\infty$ satisfies the corkscrew point condition, with a constant ϵ that depends only on the uniform constant of Ω_∞, the Ahlfors regular constant of $\partial\Omega_\infty$ and σ_∞, the elliptic constants of A_∞, and α. But then, since $\mathbb{R}^n \setminus \Omega_j$ converges to $\mathbb{R}^n \setminus \Omega_\infty$, it means that for j large enough, a ϵ-corkscrew point of $\mathbb{R}^n \setminus \Omega_\infty$ associated to $(0,1)$ is a $\epsilon/2$-corkscrew point of $\mathbb{R}^n \setminus \Omega_\infty$ associated to $(0,1)$. This contradicts (b). The theorem follows.

Case (7b). We replace (g) by

(g) there holds

$$\int_{\Omega_{j,j}} \left| \nabla \ln \Big(\frac{G_j(X)}{D_{\sigma_j,\alpha}(X)} \Big) \right| dX \leq \frac{1}{j} r^n. \tag{4.22}$$

Of course, $\nabla D_{\sigma_j,\alpha}$ also converges to $\nabla D_{\sigma_\infty,\alpha}$ on compact supports of Ω_∞, which means that $\nabla \ln(G_\infty / D_{\sigma_\infty,\alpha}) = 0$, or $G_\infty = c D_{\sigma_\infty,\alpha}$. We conclude as in Case (7), applying Lemma 4.14 at the appropriate time. ☐

4.3.3 . Uniform rectifiability implies weak estimates on the Green function

In this paragraph, we examine the converse of Theorem 4.20 and Theorem 4.23. While these results may not be the most impressive, as we will derive better "strong" estimate on the Green function later. But they are still worth mentioning. This is because they require fewer assumptions on the operator (weakest DKP instead of weak DKP).

Theorem 4.24. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain whose boundary $\partial\Omega$ is $(n-1)$-uniformly rectifiable. Let $L := -\operatorname{div} \mathcal{A}\nabla$ be a weakest DKP uniformly elliptic operator. For any $(x,r) \in \partial\Omega \times (0,\infty)$ $G_{x,r} \in W(B(x,2r) \cap \Omega)$ be a collection of positive weak solution to $Lu = 0$ in $B(x,2r) \cap \Omega$ satisfying $\operatorname{Tr} G_{x,r} = 0$ on $B(x,2r) \cap \partial\Omega$. Then the following holds*

(i) $\nabla G_{x,r}$ is (L^∞)-Carleson prevalently close to the distance to a plane, that is, for any $\tau > 0$ and $K \geq 2$, the set of couple $(x,r) \in \partial\Omega \times (0,\infty)$ such that

$$\inf_{\substack{P\, d\text{-plane} \\ c>0}} \sup_{B(x,Kr)\cap\Omega} |cG_{x,Kr}(X) - \operatorname{dist}(X,P)| \leq \tau r$$

is Carleson prevalent;

(ii) $\nabla G_{x,r}$ is (L^2)-Carleson prevalently close to the distance to a plane, that is, for any $\tau > 0$ and $K \geq 2$, the set of couple $(x,r) \in \partial\Omega \times (0,\infty)$ such that

$$\inf_{\substack{P\, d\text{-plane} \\ c>0}} \int_{W_K(x,r)} |c\nabla G_{x,Kr}(X) - \nabla \operatorname{dist}(X,P)|^2\, dX \leq \tau r^n$$

is Carleson prevalent.

Moreover, for any $\beta > 0$ and any Ahlfors regular measure μ on $\partial\Omega$,

(iii) $G_{x,r}$ *is (L^∞)-Carleson prevalently close to $D_{\mu,\beta}$, that is, for any $\tau > 0$ and $K \geq 2$, the set of couple $(x,r) \in \partial\Omega \times (0,\infty)$ such that*

$$\inf_{c>0} \sup_{B(x,Kr)\cap\Omega} |cG_{x,Kr}(X) - D_{\mu,\beta}(X)| \leq \tau r$$

is Carleson prevalent;

(iv) $\nabla G_{x,r}$ *is (L^2)-Carleson prevalently close to $\nabla D_{\mu,\beta}$, that is, for any $\tau > 0$ and $K \geq 2$, the set of couple $(x,r) \in \partial\Omega \times (0,\infty)$ such that*

$$\inf_{c>0} \int_{W_K(x,r)} |c\nabla G_{x,Kr}(X) - \nabla D_{\mu,\beta}(X)|^2 \leq \tau r^n$$

is Carleson prevalent.

Alternatively, in higher codimension, we have

Theorem 4.25. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain whose boundary $\partial\Omega$ is d-uniformly rectifiable, and write σ for a Ahlfors regular measure on $\partial\Omega$. Let $\alpha > 0$ and set $L := -\operatorname{div}[D_{\sigma,\alpha}^{d+1-n}\nabla]$. For any $(x,r) \in \partial\Omega \times (0,\infty)$ $G_{x,r} \in W(B(x,2r)\cap\Omega, \delta_{\partial\Omega}^{d+1-n}dX)$ be a collection of positive weak solution to $Lu = 0$ in $B(x,2r)\cap\Omega$ satisfying $\operatorname{Tr} G_{x,r} = 0$ on $B(x,2r)\cap\partial\Omega$. Then the same conclusions as Theorem 4.24 hold*

(i) $\nabla G_{x,r}$ *is (L^∞)-Carleson prevalently close to the distance to a plane;*

(ii) $\nabla G_{x,r}$ *is (L^2)-Carleson prevalently close to the distance to a plane.*

And if $\beta > 0$ and μ is another Ahlfors regular measure on $\partial\Omega$,

(iii) $G_{x,r}$ *is (L^∞)-Carleson prevalently close to $D_{\mu,\beta}$;*

(iv) $\nabla G_{x,r}$ *is (L^2)-Carleson prevalently close to $\nabla D_{\mu,\beta}$.*

Remark 4.26. In the above theorem, we could allow a bit more flexibility on the operator. Indeed, the proof permits the operator to be in the form

$$L := -\operatorname{div}[D_{\sigma,\alpha}^{d+1-n}\mathcal{A}\nabla]$$

where $\mathcal{A}$ is a matrix which is Carleson prevalently close to the identity, that is, for any $\tau > 0$ and $K \geq 2$, the set that contains the couples $(x,r) \in \partial\Omega \times (0,\infty)$ such that

$$\inf_{c>0} \int_{W_K(x,r)} |c\mathcal{A}(X) - I|\, dX \leq \tau r^n$$

is Carleson prevalent. We could consider even more matrices with a suitable definition of DKP operators in higher codimension, but this project will be left for future work.

4.3.4 . References and comments on the section

Lemmas 4.10, 4.13 and 4.14 are not always stated, see but are part of the proof of the bigger theorems down the road. Lemma 4.13, Lemma 4.14, and Case (1) of Lemma 4.10 are in [DM22]. Cases (4) and (5) of Lemma 4.10 are part of [FL23, Section 6], although the proofs are not optimal in [FL23]. Definition 4.12 comes from [DM22, Section 8].

Definition 4.16 is taken from [DM22]. Proposition 4.19 is proven by a smoothing argument similar to [FLM24, Lemma 2.1] or [Fen23, Proposition 2.8], and is inspired by a similar smoothing method in [DPP07]. The extraction of limits from Theorems 4.20 and 4.23 are variants of the ones found in [HMM$^+$21, Theorem 4.8], [DM22, Theorem 2.19] or [FL23, Proposition 7.1]. Various cases of Theorem 4.20 are found as [DM22, Theorems 6.1, 8.7, 8.8] and [FL23, Theorem 7.14], although we notice in the present manuscript that we do not need to assume that the domain is uniform as long as G satisfies the lower Hölder bound (ii), which allow us - for instance - to take the sum of several Green functions with different poles. Theorem 4.23 is [DM22, Theorem 7.1] and [FLM24, Theorem 9.1].

The direct cases (Theorem 4.24 and 4.25) are [DM22, Theorems 3.1, 3.43, 4.5].

4.4 . Strong-type Green function estimates

In Subsection 4.3.3, we saw that the uniform rectifiability of the boundary implies some weak-type Carleson estimate on the Green function. Why weak type? If q is a dimensionless quantity that measures the difference between the Green function estimate and a distance, then a weak type estimate takes the form "$\mathbb{1} q > \epsilon \in CM_\sigma$ for all ϵ", while a strong-type estimate will simply be "$q \in CM_\sigma$."

We first encountered weak and strong Carleson estimates in Theorem 1.13 : uniformly rectifiable sets can be characterized equivalently by a strong Carleson estimate (see (ii) of Theorem 1.13) or by a weak Carleson estimate (ee (iii) of Theorem 1.13). The fact that both conditions equivalently characterize uniform rectifiability is one of the most surprising aspects of the theory developed by David and Semmes in [DS91, DS93].

Returning to our topic, uniformly rectifiable sets do not differentiate between weak or strong Carleson estimates. Therefore, we should extend Subsection 4.3.3 further and aim for strong-type Carleson estimates on the Green function. Two approaches have been developed, each with its advantages and disadvantages.

4.4.1 . Approach by compacity

Let us separate the codimension 1 case to the higher codimension setting, which, while not surprising, is more complex to articulate.

Theorem 4.27 (Theorem 1.13 in [DLM22b]). *Let* $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$, *and* $\sigma := \mathcal{L}^{n-1}|_{\mathbb{R}^{n-1}}$. *Take* $L := -\operatorname{div} A\nabla$ *be a uniformly elliptic operator such that* A *can be decomposed as* $A = B + C$ *where* $t|\nabla B| + |C| \in CM_\sigma(M)$. *Then for any ball* B *centered on the boundary, any positive solution* $u \in W^{1,2}(B \cap \Omega_0)$ *to* $Lu = 0$ *in* $B \cap \Omega_0$ *with* $\operatorname{Tr} u = 0$ *on* $B \cap \partial\Omega_0$, *and any* $\tau \in (0, 1/2)$, *we have*

$$\frac{J_{u,\tau B}}{E_{u,\tau B}} \in CM_\sigma(C\tau^\alpha + CM),$$

where $C, \alpha > 0$ *depend only on* n *and the ellipticity constant of* L,

$$E_{u,\tau B}(x,r) := \iint_{B(x,r)\cap\tau B\cap\Omega_0} |\nabla u|^2 dy\, dt$$

and

$$J_{u,\tau B}(x,r) := \inf_{a\in\mathbb{R}} \iint_{B(x,r)\cap\tau B\cap\Omega_0} |\nabla[u - at]|^2 dy\, dt. \tag{4.23}$$

Note that instead of taking the infimum in $a \in \mathbb{R}$ *in* (4.23), *we can equivalently take*

$$a := \frac{1}{B(x,r)\cap\Omega_0} \iint_{B(x,r)\cap\tau B\cap\Omega} \partial_t u(y,t)\, dy\, dt.$$

The theorem allows us to estimate the second derivatives, in the spirit of Theorem 4.2.

Corollary 4.28 (Corollary 1.17 in [DLM22b]). *Let* $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$*, and* $\sigma := \mathcal{L}^{n-1}|_{\mathbb{R}^{n-1}}$*. Take* $L := -\operatorname{div} A\nabla$ *be a uniformly elliptic operator and assume that* $t\nabla A \in CM_\sigma(M)$*. Then for any ball* B *centered on the boundary and any positive solution* $u \in W^{1,2}(2B \cap \Omega_0)$ *to* $Lu = 0$ *in* $2B \cap \Omega_0$ *with* $\operatorname{Tr} u = 0$ *on* $2B \cap \partial\Omega_0$*, we have*

$$\frac{t^2|\nabla^2 u|}{u} \in CM_\sigma(M'),$$

where M' *depend only on* n*, the ellipticity constant of* L*, and* M*.*

The higher codimension analogue of Theorem 4.27 is

Theorem 4.29 (Theorem 1.18 in [DLM22a]). *Let* $d < n-1$*,* $\Omega_0 := \{(x,t) \in \mathbb{R}^d \times ((\mathbb{R}^{n-d} \setminus \{0\})\}$*, and* $\sigma := \mathcal{L}^d|_{\mathbb{R}^d}$*. Take* $L := -\operatorname{div} A\nabla$ *be a uniformly elliptic operator such that* A *can be decomposed as* $A = B + C$ *where*

(a) B *can be written as*

$$B = \begin{bmatrix} B_1 & \mathbf{b}_2 \frac{t}{|t|} \\ \frac{t^T}{|t|}\mathbf{b}_3 & b_4 I_{n-d} \end{bmatrix}$$

(b) $t|\nabla \widetilde{B}| + |C| \in CM_\sigma(M)$*, where*

$$\widetilde{B} := \begin{bmatrix} B_1 & \mathbf{b}_2 \\ \mathbf{b}_3 & b_4 \end{bmatrix}$$

Then for any ball B *centered on the boundary, any positive solution* $u \in W^{1,2}(B \cap \Omega_0)$ *to* $Lu = 0$ *in* $B \cap \Omega_0$ *with* $\operatorname{Tr} u = 0$ *on* $B \cap \partial\Omega_0$*, and any* $\tau \in (0,1/2)$*, we have*

$$\frac{J_{u,\tau B}}{E_{u,\tau B}} \in CM_\sigma(C\tau^\alpha + CM),$$

where $C, \alpha > 0$ *depend only on* n *and the ellipticity constant of* L*,*

$$E_{u,\tau B}(x,r) := \iint_{B(x,r)\cap\tau B\cap\Omega_0} |\nabla u|^2 dy\, |t|^{d+1-n} dt$$

and

$$J_{u,\tau B}(x,r) := \inf_{a\in\mathbb{R}} \iint_{B(x,r)\cap\tau B\cap\Omega_0} |\nabla[u - a|t|]|^2 dy\, |t|^{d+1-n} dt. \tag{4.24}$$

Let us discuss about those results. The proof relies on a compactness argument : we first verify that the theorems hold when the operator L has constant coefficients. We then extend this to “weak DKP” operators (the operators that we consider) by saying that weak DKP operators are sufficiently close to constants coefficients operators.

One of the main advantage of this approach is its independence from previously known results on the elliptic measure, allowing us to derive new insights on the elliptic measure. Indeed, Theorem 4.27 is a key ingredient of the “small constant” analogue of Theorem 1.37 (iii) $\implies$ (ii).

Note that if $\Omega \subset \mathbb{R}^n$ is a CAD - i.e. a uniform domain with $(n-1)$-uniformly rectifiable boundaries - and $L := -\operatorname{div} A\nabla$ is a “weak DKP” uniformly elliptic operator -

i.e. a uniformly elliptic operator for which A can be decomposed as $A = B + C$ with $\delta_{\partial\Omega}|\nabla B| + |C| \in CM_\sigma$ - then Theorem 1.37 (iii) $\implies$ (ii) implies $\omega_L \in A_\infty(\sigma)$[2]. Consequently, (iv) of Theorem 2.83 implies $\log(k_L) \in BMO(\sigma)$. The small constant variant of this result is :

Theorem 4.30 ([BTZ23, DLM23]). *Let* $\Omega_0 := \{(x,t) \in \mathbb{R}^{n-1} \times (0,\infty)\}$*, and* $\sigma := \mathcal{L}^{n-1}|_{\mathbb{R}^{n-1}}$*. Take* $L := -\operatorname{div} A\nabla$ *be a uniformly elliptic operator.*

For all $\epsilon > 0$*, there exists* $\delta > 0$ *such that if* A *can be decomposed as* $A = B + C$ *with* $t|\nabla B| + |C| \in CM_\sigma(\epsilon)$*, then*

$$\| \ln(k_L) \|_{BMO(\mathbb{R}^{n-1},\sigma)} \leq \epsilon.$$

The above result can be extended to "Chord-Arc Surfaces with Small Constants" (CASSC) that can be seen as the "small constant" version of uniform domains with uniformly rectifiable boundaries. However, we will not delve into the definition here and we let the reader check [DLM23] for the statements.

The disadvantage of Theorems 4.27 and 4.29 is that it is not clear what would be the statements when the boundaries are Lipschitz, let alone uniformly rectifiable. We can see that Theorem 4.2 and Corollary 4.28 are two sides of the same coin : they are saying that the Green function G_L defined in Definition 2.78 satisfies

$$\frac{\delta_{\partial\Omega}^2 \nabla^2 G}{G} \in CM_\sigma \tag{4.25}$$

whenever

(a) either $L = -\Delta$ and Ω is a CAD (Theorem 4.2),

(b) or L is a DKP operator and $\Omega = \mathbb{R}^n_+$ (Corollary 4.28).

Thus, it is reasonable to expect that the estimate (4.25) holds when both L is a DKP operator and Ω is a CAD. However, none of the two proofs (of Theorem 4.2 and Corollary 4.28), which relies on completely different arguments, seem adaptable to this general setting.

4.4.2 . Approach by integration by parts

An alternative approach was initiated by David, Mayboroda, and the author. This approach began even before the results in Subsection 4.4.1. We aimed to adapt the strategy of [Fen22a] using the magic α from Subsection 3.6.1. Indeed, if $\alpha_0 := n-d-2 > 0$, then the Green function with pole at ∞ $G_{L_{\alpha_0}}$ is D_{α_0}. When E is d-uniformly rectifiable, L_α is a "multiplicative" Carleson perturbation of L_{α_0}, so the ratio between G_{L_α} and D_α should be close to a constant.

Theorem 4.31 (Theorem 1.8 in [DFM23c]). *Let* $d < n-1$*,* $E \subset \mathbb{R}^n$ *a* d*-uniformly rectifiable set with Ahlfors regular measure* σ*,and* $\Omega := \mathbb{R}^n \setminus E$*. For* $\alpha > 0$*, define the operator* $L_\alpha := -\operatorname{div} D_{\alpha,\sigma}^{d+1-n}\nabla$*, where* $D_{\alpha,\sigma}$ *is as in* (3.37)*, Then for any ball* B *centered on* E *and any solution* u *to* $Lu = 0$ *in* $2B \setminus E$ *with* $\operatorname{Tr}(u) = 0$ *on* $2B \cap E$*, we have*

$$\delta_E \left| \nabla \ln\left(\frac{u}{D_{\alpha,\sigma}}\right) \right| \mathbb{1}_B \in CM_\sigma.$$

---

[2] ω_L is the elliptic measure with pole at infinity (see Definition 2.78) and $k_L = d\omega_L/d\sigma$.

Subsequently, the previous theorem was adapted to boundaries of codimension 1 through a much more technical argument. This marked the first time that a higher codimension result was proved beforehand and was used to determine the existence of an estimate in the "classical" settings of domains with codimension 1 boundaries.

Theorem 4.32 (Theorem 1.12 in [FLM24])**.** *Let $\Omega \subset \mathbb{R}^n$ be a that is a uniform domain with $(n-1)$-Ahlfors regular boundaries. Let $L := -\operatorname{div} A\nabla$ be a uniformly elliptic operator that satisfies the DKP condition $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma$. Then are equivalent :*

- *(i) $\partial\Omega$ is uniformly rectifiable, or equivalently that Ω is CAD.*
- *(ii) for all $\alpha > 0$ and any Ahlfors regular measure μ on $\partial\omega$, there exists $M > 0$ such that for any ball B centered on $\partial\Omega$ and any solution u to $Lu = 0$ in $2B \cap \Omega$ with $\operatorname{Tr}(u) = 0$ on $2B \cap \partial\Omega$, we have*
$$\delta_{\partial\Omega}\Big|\nabla \ln\Big(\frac{u}{D_{\alpha,\mu}}\Big)\Big|\mathbb{1}_B \in CM_\sigma(M).$$
- *(iii) there exists $\alpha > 0$ such that the Green function G_L defined in Definition 2.78 verifies*
$$\delta_{\partial\Omega}\Big|\nabla \ln\Big(\frac{G_L}{D_{\alpha,\sigma}}\Big)\Big| \in CM_\sigma.$$

Note that the proof of (iii) $\implies$ (i) in Theorem 4.32 is a consequence of Theorem 4.23 (7bis).

The above estimates are actually the missing key ingredient needed to extend Theorem 4.2 to DKP operators or to domain with high codimensional boundaries. To present more readable statements, we separate the cases for codimension 1 boundaries and higher codimensional boundaries. For domains with codimension 1 boundaries, we have :

Theorem 4.33. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with $(n-1)$-Ahlfors regular boundary, and let $L = -\operatorname{div} \mathcal{A}\nabla$ be a DKP-operator, i.e. $\delta_{\partial\Omega}|\nabla A| \in CM_\sigma$. Then the following are equivalent :*

- *(i) $\partial\Omega$ is uniformly rectifiable ;*
- *(ii) there exists $M > 0$ such that for any ball B centered on $\partial\Omega$ and any positive weak solution $u \in W(\Omega, \mathcal{L}^n;$ locally in $\mathbb{R}^n)$ to $Lu = 0$ in $\Omega \cap 2B$ satisfying $\operatorname{Tr} u = 0$ in $\partial\Omega \cap 2B$, we have*
$$\delta^2_{\partial\Omega}\frac{\nabla^2 u(X)}{u(X)}\mathbb{1}_B \in CM_\sigma(M); \tag{4.26}$$
- *(iii) if G_L is the Green function defined in Definition 2.78,*
$$\delta^3_{\partial\Omega}\frac{\nabla[|\nabla G_L|^2]}{G_L^2} \in CM_\sigma. \tag{4.27}$$

For domains with higher codimensional boundaries, we have :

Theorem 4.34. *Let $d < n-1$, $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, $\Omega := \mathbb{R}^n \setminus E$, and $\alpha > 0$. Take $L_\alpha := -\operatorname{div} D^{d+1-n}_{\alpha,\sigma}\nabla$. Then the following are equivalent :*

- *(i) d is an integer and $\partial\Omega$ is d-uniformly rectifiable ;*
- *(ii) there exists $M > 0$ such that for any ball B centered on $\partial\Omega$ and any positive weak solution $u \in W(\Omega, \delta^{d+1-n}_{\partial\Omega} d\mathcal{L}^n;$ locally in $\mathbb{R}^n)$ to $L_\alpha u = 0$ in $\Omega \cap 2B$ satisfying $\operatorname{Tr} u = 0$ in $\partial\Omega \cap 2B$, we have*
$$\delta^2_{\partial\Omega}\frac{\nabla|\nabla u|}{u}\mathbb{1}_B \in CM_\sigma(M); \tag{4.28}$$

(iii) if G_{L_α} is the Green function defined in Definition 2.78,

$$\delta_{\partial\Omega}^3 \frac{\nabla[|\nabla G_{L_\alpha}|^2]}{G_{L_\alpha}^2} \in CM_\sigma. \tag{4.29}$$

Ideas of the proof of Theorem 4.34 from Theorem 4.31 and Corollary 4.9 : First, let us mention that the ideas to prove Theorem 4.33 are similar, using Theorem 4.32 instead. The proof of Theorem 4.34 (iii) $\implies$ (i) is a simple consequence of Theorem 4.20 (5) and the Poincaré inequality, so we will not mention this direction anymore, and we will focus on (i) $\implies$ (iii).

We write G for G_{L_α} and D for $D_{\alpha,\sigma}$. First note that there exists C depending only on α and σ such that

$$|\nabla D| + \frac{D|\nabla G|}{G} \leq C. \tag{4.30}$$

Then we write G as $D.\frac{G}{D}$, which leads to

$$\nabla G = \frac{G}{D}\nabla D + D\nabla\Big(\frac{G}{D}\Big) = \frac{G}{D}\nabla D + G\nabla \ln\Big(\frac{G}{D}\Big).$$

We compute then $\frac{D^3\nabla[|\nabla G|^2]}{G^2}$ from the above expression of ∇G, and we observe that

$$\frac{D^3|\nabla[|\nabla G|^2]|}{G^2} \leq C\left(D|\nabla[|\nabla D|^2]| + D\Big|\nabla \ln\Big(\frac{G}{D}\Big)\Big| + D^2\Big|\nabla^2 \ln\Big(\frac{G}{D}\Big)\Big|\right)$$

where we use (4.30) as many times as necessary.

Theorem 4.31 and Corollary 4.9 gives respectively that

$$D\nabla \ln\Big(\frac{G}{D}\Big) \in CM_\sigma \quad \text{and} \quad D\nabla[|\nabla D|^2] \in CM_\sigma,$$

so our theorem will be proved as long as we show that $D^2\nabla^2\ln(G/D) \in CM_\sigma$ as well. However, by a Caccioppoli-type argument, we can show that for any ball $B \subset \Omega$, we have

$$\int_{B/4} \Big|\nabla^2 \ln\Big(\frac{G}{D}\Big)\Big|^2 D^{d+4-n} dX \leq C\Big[\int_{B/2} \Big|\nabla \ln\Big(\frac{G}{D}\Big)\Big|^2 D^{d+2-n}\, dX \\ + \int_{B/2} |D^{n-d}\operatorname{div}(D^{d+1-n}\nabla D)|^2\, D^{d-n}\, dX\Big].$$

So $D^2\nabla^2\ln(G/D) \in CM_\sigma$ as long as $D\nabla\ln(G/D) \in CM_\sigma$ and $D^{n-d}\operatorname{div}(D^{d+1-n}\nabla D) \in CM_\sigma$, and the latter are given again by Theorem 4.31 and Corollary 4.9 respectively. □

5 - Open and on-going problems

Here is a non-exhaustive and randomly ordered list of problems to solve in relation with the contents of this memoir. Many of the questions listed here are not originally from me, but from discussions with my peers.

(1) Show the stability of the weak-A_∞ of the elliptic measure under large Carleson perturbations without assuming that the domain has Harnack chain, that is drop the existence of Harnack chains from Theorem 1.36.

(2) Characterize (D$_2$)$_\Delta$ - the L^2 solvability of the Dirichlet problem for the Laplacian - in terms of geometry and topology on the domain. More generally, characterize (D$_p$)$_\Delta$.

(3) Characterize the weak A_∞ of the elliptic measure of DKP operators in terms of geometry and topology of Ω, without *a priori* assuming connectedness. That is, prove Theorem 1.37 without assuming that Ω has Harnack chains.

(4) For each uniform domain Ω with $(n-1)$-Ahlfors regular set, find an isotropic operator L such that ω_L is equivalent to the Ahlfors regular measure σ, that is, adapt the construction from [DM21] to all domains with $(n-1)$-Ahlfors regular boundaries. We can also aim for something weaker : to construct L such that $\omega_L \in A_\infty(\sigma)$.

(5) Same as before, but assuming instead a capacity condition on the boundary. A non $(n-1)$-Ahlfors regular boundary was treated in [Per23a].

(6) Find a proof of $\omega_L \in A_\infty(\sigma)$ when $\partial\Omega$ is uniformly rectifiable without using a Corona decomposition, but by using the smooth distance D_α.

(7) Find the range of $d \in (n-2, n-1)$ for which we can find $\Omega \subset \mathbb{R}^n$ with d-Ahlfors regular boundary and $p \in (1,\infty)$ such that (D$_p$)$_\Delta$ holds. Partial results are given in [DJJ23] (we can find such Ω when $n = 2$ and d is suffisament petit) and [Tol23] (we cannot find such Ω when $n = 2$, $d > \frac{1}{2}$, and $\mathbb{R}^2 \setminus \Omega := \partial\Omega$ is included in a plane).

(8) Study the Dirichlet problem with data in Besov or Triebel-Lizorkin spaces, that is with data in $\dot{B}^{p,q}_s$ and $\dot{T}^{p,q}_s$ for $s \in (0,1)$, $p \in (1,\infty)$ and $q \in [1,\infty]$.

(9) Prove or disprove that the solvability of (D$_p$)$_L$ implies weak local John for any uniformly elliptic operator.

(10) Study the L^p Dirichlet problem without assuming *a priori* the corkscrew point condition.

(11) The Laplacian on the paraboloid domain $\{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R},\ t > |x|^2\}$ is outside the elliptic theory that was developed in Chapter 2. Can we say anything about the L^p Dirichlet problem ?

(12) We know from Proposition 1.41 that the Hajłasz gradient is optimal for uniform domains. But is it optimal when Ω is not uniform ? That is, can we find a weaker notion of gradient ∇_w such that we still have (D$_p$) $\implies$ (R$_p$) for the Laplacian (or DKP operators), but such that $\|\nabla_w f\|_{L^p(\partial\Omega,\sigma)}$ is not equivalent to $\|\nabla_{H,p} f\|_{L^p(\partial\Omega,\sigma)}$ when the domain is not uniform ?

(13) Define the Neumann problem in an unbounded domain. Is it even possible to do so in $\mathbb{R}^2$?

(14) Show that (N$_p$)$_L$ $\implies$ (N$_q$)$_L$ for $q \in (1,p)$ without assuming (D$_{p'}$)$_{L^*}$.

(15) Find a limit problem (N$_1$)$_L$ such that (N$_1$)$_L$ $\implies$ (N$_q$)$_L$ for one $q > 1$.

(16) If L is an operator on $\mathbb{R}^n_+$ with non-symmetric t-independent coefficients, is there a $p \in (1, \infty)$ such that (N$_p$)$_L$ holds? Morally, do we have Theorem 1.51 (2) without assuming that A_a is small?

(17) Is the solvability of the L^p Neumann problem for some $p \in (1, \infty)$ stable under large Carleson perturbations? I.e. do we have Theorem 1.52 for large Carleson perturbations?

(18) Is the L^p Neumann problem solvable for DKP operators with large constants? I.e. do we have a large constant analogue of Theorem 1.53?

(19) Are quantities in CM_σ even adapted to the Neumann problem? The Dirichlet problem relies on a $S < N$ type estimate that has not been proved for the Neumann problem, and it might be possible that one needs to rely on a completely different estimate for the Neumann problem, one for which DKP operators and uniformly rectifiable sets are not adapted to.

(20) Instead of using the regularity problem as an intermediate step towards the Neumann problem, use the subregular Neumann problem or the Poisson Neumann problem.

(21) Can we have a comparison principle for the Neumann problem? Which form will it take? Ultimately, can we have a tool that proves that the Poisson-Neumann problem from [FL24] is equivalent to the Neumann problem?

(22) Show that an interior Poincaré inequality on m implies that m is derivative compatible.

(23) Link the boundary Poincaré inequality and the capacity in our general setting.

(24) Define a space of traces for $W(\Omega, m)$ without assuming the Harnack chain condition. Basically, we want to construct a space of traces, maybe using currents, in which the direction of non-tangential limit matters. Then construct an elliptic measure and study (D$_p$) on this space of traces.

(25) On domains that are not well connected, define a distance based on the length of Harnack chains, and prove a comparison principle with this notion of distance.

(26) Find the maximum dimension of the harmonic measure in domains of $\mathbb{R}^n$, when $n \geq 3$. The case $n = 2$ is discussed in [Mak85a, Mak85b, JW88]. For the case $n \geq 3$, see [Bou87] (dim. harmonic measure $< n$), [BG24] (explicit upper bound), [Azz20] (dim. harmonic measure always drops for sets of codimension <1) .

(27) Study the link between dimension drop of the elliptic measure and oscillations of the coefficients of the operator.

(28) Prove Theorem 1.27 - the equivalence between the Dirichlet problem and the Poisson-Dirichlet problem - in the setting of Chapter 2.

(29) Define (R$_p$) and (N$_p$), $p \in (1, \infty)$, in the general setting of Chapter 2. The difficulty comes from the dimensionless quantity ρ defined in (2.15). The definition is pretty straightforward when ρ is equivalent to 1, but when ρ is not equivalent to 1, one will quickly see that ρ needs to be involved in the definitions of (R$_p$) and (N$_p$) to make reasonable sense.

(30) Find examples when (N$_p$)$_\Delta$ when Ω is not uniformly rectifiable. Find various examples of couples Ω and L such that (N$_p$)$_L$ holds. Find a case where (N$_p$) holds and the ρ in (2.15) is not equivalent to 1.

(31) Define ϵ-approximability in the general setting of Chapter 2, and prove an equivalence with the solvability of the L^p Dirichlet problem.

(32) Study the solvability/non-solvability of the L^p Dirichlet problem in $\mathbb{R}^n \setminus \mathbb{R}^d$ when the bottom right corner of the coefficients of the operator are not close to a scalar multiplicative of the identity. More precisely, assuming $L := -\operatorname{div} |t|^{d+1-n} A\nabla$ is such that

$$A := \begin{bmatrix} I_d & 0 \\ 0 & A_4 \end{bmatrix},$$

is there a condition (C) on scalar functions[1] such that $\omega_L \in A_\infty(\sigma)$ whenever each coefficient of A_4 satisfies (C)?

(33) Alternatively, define a notion of DKP operators that works in in domains whose boundary is non-flat and of higher codimension.

(34) Show whether $\nabla\Big(\frac{D_{\beta,\sigma}}{D_{\alpha,\sigma}}\Big) = 0 \implies \sigma$ is a flat measure. Knowing this, we will be able to show that $\nabla\Big(\frac{D_{\beta,\sigma}}{D_{\alpha,\sigma}}\Big) \in CM_\sigma \implies \sigma$ is a Ahlfors regular measure on a uniformly rectifiable set[2]. We can assume at first that the property $\nabla(D_{\beta,\sigma}/D_{\alpha,\sigma}) = 0$ holds for all couple $(\alpha,\beta) \in (0,\infty)^2$, and then aim for the property to be true for a single couple where $\alpha \neq \beta$.

(35) If $d < n$ and $L_\alpha := -\operatorname{div} D_\alpha^{d+1-n}\nabla$, show a converse to Theorem 3.35, that is

$$\omega_{L_\alpha} \in A_\infty(\sigma) \text{ for all } \alpha > 0 \implies \sigma \text{ is uniformly rectifiable,}$$

$$\omega_{L_\alpha} \in A_\infty(\sigma) \text{ for one } 0 < \alpha \neq n-d-2 \implies \sigma \text{ is uniformly rectifiable.}$$

(36) Define a Laplacian L_μ in all doubling measures μ such that, if μ is often well approximated by a plane of any dimension (in a quantitative way, using α or β numbers), then $\omega_{L_\mu} \in A_\infty$. Basically, can we have a mixed codimension analogue of Theorem 3.35?

(37) What happens if $\Omega = \mathbb{R}^n_+$ and we take a measure μ on $\mathbb{R}^{n-1}$ which is not absolutely continuous with respect to the surface measure? What behavior will have ω_{L_μ}?

(38) Prove that, if $\Omega = \mathbb{R}^n \setminus E$ with E being a d-uniformly rectifiable set of low dimension, then $(\mathrm{R}_p)_{L_\alpha}$ holds for some $p \in (1,\infty)$. I.e. prove the higher codimension analogue of Theorem 1.46.

(39) Do we have

$$G_{L_\beta} = D_{\alpha,\sigma} \implies \sigma \text{ is a flat measure}$$

when α or $\beta \neq n-d-2$?

(40) If Ω is a domain with uniformly rectifiable boundaries, do we have $\nabla \ln(G_L/D_\alpha) \in CM_\sigma$ if we don't assume that Ω is uniform? What are the minimal topological conditions on Ω that still ensure the equivalence of Theorem 4.32?

(41) Can we find some implications between $\nabla \ln(G_L/D_\alpha) \in CM_\sigma$ and $\omega_L \in A_\infty(\sigma)$? What are the minimal conditions on L and Ω that guarantee the equivalence? Can we relate bounds on the Green function to the L^p regularity problem?

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[1] A condition like the DKP condition, that does not distinguish between coefficients

[2] From now on, we say that σ is uniformly rectifiable if σ is a Ahlfors regular measure supported on a uniformly rectifiable set

(42) Same two questions as above, but with $\nabla|\nabla G_L| \in CM_\sigma$ instead of $\nabla \ln(G_L/D_\alpha) \in CM_\sigma$.

(43) Can we use the approach of [DLM22b] to domains with Lipschitz or uniformly rectifiable boundaries? That consists first in finding what would even be the right formulation of Theorem 4.27 for such domains.

(44) Prove the David-Semmes conjecture (Conjecture 1.14) in codimension different from 1. The conjecture has been proved in codimension 1 in [NTV14]. The idea of characterizing uniformly rectifiable sets of higher codimension via solution of a PDEs in the complement partially comes from the fact that such characterization could be helpful to solve the David-Semmes conjecture.

Those general directions have also been considered :

- Study the L^p Dirichlet problem (for the sub-Laplacian) in the Heisenberg group. To the best of the author's knowledge, so far, people only know such solvability in flag domains (see Theorem 1.8 in [OV23]).
- Look at higher order elliptic operators. The L^p boundary value problems for t-independent operators has been studied in for instance [BM13], [BM14], [BHM17] or [BHM19], but we are not aware of articles that deal with DKP-type operators and uniformly rectifiable boundaries.
- Similarly, study systems of equations ; and I am not aware that it has been touched when the domain is not Lipschitz.
- The L^p Robin problem is another boundary value problem that recently gathered interest (see [DDE$^+$24]).
- Study the L^p solvability of boundary value problem for parabolic equations.
- Study the L^p solvability of boundary value problem for elliptic operator with lower terms (drifts, potentials).
- All the estimates given here are quantitative estimates, and their qualitative analogues have not always been established.

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