GROTHENDIECK–SERRE IN THE QUASI-SPLIT UNRAMIFIED CASE

KĘSTUTIS ČESNAVIČIUS

ABSTRACT. The Grothendieck–Serre conjecture predicts that every generically trivial torsor under a reductive group scheme G over a regular local ring R is trivial. We settle it in the case when G is quasi-split and R is unramified. Some of the techniques that allow us to overcome obstacles that have so far kept the mixed characteristic case out of reach include a version of Noether normalization over discrete valuation rings, as well as a suitable presentation lemma for smooth relative curves in mixed characteristic that facilitates passage to the relative affine line via excision and patching.

1. THE GROTHENDIECK–SERRE CONJECTURE

The subject of this article is the following conjecture, due to Serre [Ser58, page 31, Remarque] and Grothendieck [Gro58, pages 26–27, Remarques 3°], [Gro68, Remarques 1.11 a)], on triviality of torsors under reductive groups.

Conjecture 1.1 (Grothendieck–Serre). *For a regular local ring R and a reductive R-group scheme G, no nontrivial G-torsor trivializes over the fraction field of R, in other words,*

$$\mathrm{Ker}\left(H^1(R, G) \to H^1(\mathrm{Frac}(R), G)\right) = \{*\}.$$

The conjecture is settled when R contains a field but its remaining mixed characteristic case has so far been widely open: see the recent survey [Pan18] for a detailed review of the state of the art, as well as §1.4 below for a summary. The goal of this article is to resolve the mixed characteristic case under the assumption that R is unramified and G is quasi-split. We recall that a regular local ring

$(R, \mathfrak{m})$ with $p := \operatorname{char}(R/\mathfrak{m})$ is *unramified* if the ring R/pR is also regular, in other words, if either R contains a field or $p \notin \mathfrak{m}^2$.

Theorem 1.2. *For an unramified regular local ring R and a quasi-split reductive R-group scheme G,*

$$\operatorname{Ker}\left(H^1(R, G) \to H^1(\operatorname{Frac}(R), G)\right) = \{*\};$$

moreover, a reductive R-group scheme H is split if and only if its generic fiber $H_{\operatorname{Frac}(R)}$ is split.

In fact, our result is stronger: we allow the regular ring R to be semilocal, flat, and geometrically regular over some Dedekind ring $\mathcal{O}$, so that the case $\mathcal{O} = \mathbb{Z}$ with R local recovers the above, see Theorems 9.1 and 9.3 and Example 9.2. The semilocal version is worth the extra effort because in many ways it is a more natural starting point. In equal characteristic, we strengthen the last aspect of Theorem 1.2: for an equicharacteristic regular semilocal R, we show that a reductive R-group scheme H is quasi-split if and only if its generic fiber $H_{\operatorname{Frac}(R)}$ is quasi-split, see Theorem 9.5.

The Grothendieck–Serre conjecture is known for its numerous concrete consequences. We illustrate them with the following product formula that seems to resist any direct attack. For a further consequence that concerns quadratic forms over regular semilocal rings, see Corollary 9.6.

Corollary 1.3. *For an unramified regular local ring R, an $r \in R\backslash\{0\}$, and the r-adic completion $\widehat{R}$,*

$$G(\widehat{R}[\tfrac{1}{r}]) = G(\widehat{R})G(R[\tfrac{1}{r}]) \quad \text{for every quasi-split reductive } R\text{-group } G.$$

Indeed, if the double coset on the right side did not exhaust the left side, then one could use patching (for instance, Lemma 7.1) to build a nontrivial G-torsor that trivialized over $R[\tfrac{1}{r}]$ (and also over $\widehat{R}$).

1.4. Known cases. Previous results on Conjecture 1.1 fall into the following categories.

(1) The case when G is a torus was settled by Colliot-Thélène and Sansuc in [CTS87].

(2) The case when R is 1-dimensional, that is, a discrete valuation ring, was settled by Nisnevich in [Nis82], [Nis84], with corrections and a generalization to semilocal Dedekind rings by Guo in [Guo20]. Subcases of the 1-dimensional case (resp., of its semilocal generalization) appeared in [Har67], [BB70], [BT87] (resp., [PS16], [BVG14], [BFF17], [BFFH19]).

(3) The case when R is Henselian was settled in [BB70] and [CTS79, Assertion 6.6.1]. For such R, one may test the triviality of a G-torsor after base change to the residue field, so one may choose a height 1 prime $\mathfrak{p} \subset R$ for which $R/\mathfrak{p}$ is regular, apply the Nisnevich result, and induct on $\dim R$.

(4) The case when R contains a field, that is, when R is of equicharacteristic, was settled by Fedorov–Panin in [FP15] when the field is infinite (with significant inputs from [PSV15], [Pan20b]), and by Panin [Pan20a] when the field is finite, with substantial simplifications due to Fedorov [Fed21a]. Various subcases of the equicharacteristic case appeared in [Oja80], [CTO92], [Rag94], [PS97], [Zai00], [OP01], [OPZ04], [Pan05], [Zai05], [Che10], [PSV15].

(5) Sporadic cases, in which either R or G is of specific form but with R possibly of mixed characteristic, appeared in [Gro68, Remarques 1.11 a)], [Oja82], [Nis89], [Fed21b], [Fir22], [BFFP20], [Pan19b].

In the cases (1)–(4), the results also include the variant when the regular ring R is only semilocal.

For arguing Theorems 1.2 and 9.1, we use the toral case [CTS87] and the semilocal Dedekind case [Guo20] but no other known case of Conjecture 1.1. In fact, our argument simultaneously streamlines the case when R contains a field, although we do not pursue this here beyond the case of quasi-split

G contained in Theorem 1.2 because we see no point in repeating the same additional reductions that Fedorov–Panin, Panin, and Fedorov used for handling general G over such R.

1.5. The point of departure. A key feature of the Grothendieck–Serre conjecture and, in fact, of problems of its flavor, for example, of the Bass–Quillen conjecture, is that one cannot easily "enlarge" the ring R, essentially, because this may trivialize torsors, one can only "shrink" it. The key to progress therefore lies in better understanding the geometry of R, and our point of departure is precisely in this for unramified R of mixed characteristic $(0, p)$: we apply Popescu approximation to assume that R is essentially smooth over $\mathbb{Z}_{(p)}$ and then use a presentation theorem in the style of Gabber–Gros–Suwa [CTHK97, Theorem 3.1.1] to spread R out to a fibration $U \to S$ into smooth affine curves over an open

$$S \subset \mathbb{A}_{\mathbb{Z}_{(p)}}^{\dim(R)-1}$$

in such a way that a given small closed subscheme $Y \subset \mathrm{Spec}(R)$ spreads out to be *finite* over S, see Proposition 4.1 for a precise statement. This structural result may be viewed as a version of the Noether normalization in mixed characteristic and is reminiscent of presentation lemmas of Quillen and Gabber from [Qui73, Lemma 5.12] and [Gab94, Lemma 3.1].

For us, Y is such that a generically trivial G-torsor E that we wish to trivialize reduces to a B-torsor over $\mathrm{Spec}(R) \backslash Y$ for a Borel $B \subset G$. The valuative criterion of properness applied to E/B allows us to make this Y be of codimension $\geqslant 2$ in $\mathrm{Spec}(R)$, and this codimension requirement appears difficult to relax while arguing our mixed characteristic "Noether normalization." In equal characteristic, Y being of codimension $\geqslant 1$ suffices and is immediate to arrange from the generic triviality of E without using any Borel, and this distinction is one of the main sources of complications in comparison to works of Panin and Fedorov. Although in mixed characteristic virtually every step seems to require either new ideas or new techniques, the works of Panin and Fedorov in equicharacteristic have provided us with invaluable guidance for what the structure of the overall argument might be.

1.6. The stages of the proof of Theorem 1.2. In Theorem 1.2, the key assertion is the triviality of every generically trivial G-torsor E. For this, our argument proceeds as follows.

(1) In §2, we follow a suggestion by one of the referees and present a quick reduction of the quasi-split case of the Grothendieck–Serre conjecture to the setting when our quasi-split G is also semisimple and simply connected. This additional assumption simplifies the steps (4) and (6) below, but it could also be avoided at the cost of more refined techniques in these steps. Upon insistence of a referee, however, we take full advantage of this simplification.

(2) In §3, we use the Bertini theorem and some version of the Gabber–Gros–Suwa presentation theorem in the context of semilocal Dedekind bases $\mathcal{O}$ to build the aforementioned fibration $U \to S$ starting from a projective, flat compactification of R over $\mathcal{O}$, see Variant 3.7. We do not separate into cases according to whether the residue fields of $\mathcal{O}$ are all infinite or not, but finite residue fields lead to complications that concern Bertini theorems (as do imperfect residue fields: in the case of perfect residue fields, we could replace our version of the Gabber–Gros–Suwa presentation theorem with ideas from Artin's technique of "good neighborhoods" from [SGA 4$_{\mathrm{III}}$, Exposé XI][1]). We resolve these complications with Gabber's approach [Gab01] to Bertini theorems in positive characteristic. *Op. cit.* is more convenient for us than the generally finer approach of Poonen [Poo04] because it can guarantee that a suitable hypersurface exists for every large enough degree divisible by the characteristic, which helps in making sure that this degree is uniform across all the residue fields of $\mathcal{O}$ at maximal ideals.

---

[1]See the previous arXiv versions of this article for detailed arguments along such lines.

In this step, a major simplification in comparison to the strategy in equicharacteristic is that we do not seek a fibration into *projective* curves (nor even the complement of a relatively finite subscheme in a projective relative curve) but are nevertheless able to ensure that Y spreads out to a *finite* S-scheme in the notation of §1.5. Even in equicharacteristic, this allows us to dispose of much effort usually spent in analyzing the "boundary" in subsequent steps.

(3) In §4, we deduce the mixed characteristic "Noether normalization" mentioned in §1.5 and then use it to lift our generically trivial G-torsor E to a torsor $\mathscr{E}$ over a smooth affine R-curve C equipped with a section $s \in C(R)$ such that $\mathscr{E}$ pulls back to E via s and reduces to a torsor under the unipotent radical of a Borel over $C\backslash Z$ for some R-finite $Z \subset C$. The R-finiteness (as opposed to mere R-quasi-finiteness) of Z is critical for later steps and comes from the finiteness of the spreading out of Y. The appearance of the unipotent radical is a new phenomenon: in equicharacteristic, $C\backslash Z$ is affine and $\mathscr{E}|_{C\backslash Z}$ is a trivial torsor.

Our (C, s, Z) is a simplification of what Panin and Fedorov keep track of with the notion of a "nice triple." The latter is a variant of a "standard triple" of Voevodsky [MVW06, Definition 11.5] used in his construction of the triangulated category of motives. In general, it is convenient to work in terms of the relative R-curve C instead of directly with the fibration $U \to S$ because this gives the flexibility of changing C. In this, we reap the benefits of our C being affine: we need to work less in subsequent reductions than "nice triples" would require.

(4) Our $\mathscr{E}$ is not a G_C-torsor but a $\mathscr{G}$-torsor for some reductive C-group scheme $\mathscr{G}$ equipped with a Borel $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, so, in order to proceed, in §5 we modify C to reduce to $\mathscr{B} \subset \mathscr{G}$ being $B_C \subset G_C$. For this, we use the locally constant nature of the scheme parametrizing isomorphisms between two quasi-pinned reductive group schemes to show, at the cost of shrinking C, that $\mathscr{G}$ and G_C become isomorphic (compatibly with the Borels) over some *finite* étale $\widetilde{C} \to C$ to which s lifts.

(5) After simplifying $\mathscr{G}$ in §5, we turn to simplifying C in §§6–7, namely, to replacing C by $\mathbb{A}^1_R$. In §6, we construct an affine open $U \subset C$ containing Z and s as well as a quasi-finite R-morphism $\pi\colon C \to \mathbb{A}^1_R$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_R$ whose preimage in U is precisely Z. The R-finiteness of Z is critical for this, and the argument is simpler than its versions in the literature because C is affine (as opposed to projective). It uses Panin's tricks with finite fields to first prepare C and Z for building π: when some residue fields of R are finite, the initial Z could have too many rational points to fit inside $\mathbb{A}^1_R$.

Since C is Cohen–Macaulay, our quasi-finite π is necessarily flat, so the idea is to descend $\mathscr{E}$ to a $G_{\mathbb{A}^1_R}$-torsor via patching. We carry this out in §7: the main complication is the *a priori* nontriviality of $\mathscr{E}_{C\backslash Z}$, which we overcome by bootstrapping enough excision for $H^1(-, \mathscr{R}_u(B))$ from excision for quasi-coherent cohomology, see Lemma 7.2. Relatedly, since Z need not be principal, the patching is slightly more delicate than usual and uses [MB96].

(6) The final step is the analysis of a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z' \subset \mathbb{A}^1_R$. This is a problem of Bass–Quillen type that is significantly simpler when G is semisimple, simply connected, and quasi-split (the general case could be approached using the geometry of the affine Grassmannian, similarly to the techniques of Fedorov from [Fed21a]). The key inputs to this analysis are the study of the case when R is a field carried out in [Gil02] (or in the earlier [RR84]) and the unramified nature of the Whitehead group [Gil09, Fait 4.3, Lemme 4.5] that builds on earlier work of Tits and uses the assumptions on G.

Globally, our method may be viewed as a geometric reduction of the Grothendieck–Serre conjecture for G over R to its case for the torus $T := B/\mathscr{R}_u(B)$ over R. It is tempting to expect that if G is no

longer quasi-split but has a parabolic subgroup $P \subset G$ with a Levi M, then one could find a way to reduce from G to M. As it stands, the sticking point in achieving this generalization is in the proof of Proposition 4.2: there we extend a T-torsor across a closed subscheme of codimension $\geqslant 2$ (across Y in the notation of §1.5) and such extendibility fails beyond tori (although knowing how to resolve the Colliot-Thélène–Sansuc purity question [CTS79, Question 6.4] would help).

1.7. Conventions and notation. As in [SP], our algebraic spaces need not be quasi-separated. For a scheme S (resp., a ring R), we let k_s (resp., $k_{\mathfrak{p}}$) denote the residue field of a point $s \in S$ (resp., a prime ideal $\mathfrak{p} \subset R$). Intersections $Y \cap Y'$ of closed subschemes $Y, Y' \subset S$ are always scheme-theoretic, and we recall from [EGA IV$_1$, Chapitre 0, Définition 14.1.2] that $\dim(\varnothing) = -\infty$. We denote the (always open) S-smooth locus of an S-scheme X by X^{sm}. A scheme is *Cohen–Macaulay* if it is locally Noetherian and its local rings are Cohen–Macaulay. We use the definition [EGA IV$_1$, Chapitre 0, Définition 15.1.7, Section 15.2.2] of a regular sequence (so there is no condition on quotients being nonzero). A ring $\mathcal{O}$ is *Dedekind* if it is Noetherian, normal, and of dimension $\leqslant 1$; by [SP, Lemma 034X], any such $\mathcal{O}$ is a finite product of Dedekind domains.

We always consider *right* torsors, for instance, we want sections of G/H to give rise to H-torsors. As already seen in §1.6 (4)–(6), we use scheme-theoretic notation when talking about torsors, that is, we base change the group in order to be unambiguous about what the base is; in the rare exceptions when this would make notation too cumbersome, we make sure that no confusion is possible. For a reductive group scheme G, we let $Z(G)$, $\mathrm{rad}(G)$, G^{der}, and G^{ad} denote its center, maximal central torus, derived subgroup, and adjoint quotient (see [SGA 3$_{\mathrm{III \ new}}$, Exposé XXII, Corollaire 4.1.7, Définition 4.3.6, Théorème 6.2.1 (iv)]); for a semisimple group scheme G, we let G^{sc} denote its simply-connected cover (see [Con14, Exercise 6.5.2 (i)] or [Čes22, Proposition A.3.4]). For a parabolic group scheme P, we let $\mathscr{R}_u(P)$ denote its unipotent radical constructed in [SGA 3$_{\mathrm{III \ new}}$, Exposé XXVI, Proposition 1.6 (i)] (already in [SGA 3$_{\mathrm{III \ new}}$, Exposé XXII, Proposition 5.6.9 (ii)] for a Borel).

2. THE QUASI-SPLIT CASE REDUCES TO SIMPLY-CONNECTED GROUPS

Following suggestions of one of the referees, we show that for quasi-split reductive groups the Grothendieck–Serre conjecture 1.1 reduces to the case when the group is also semisimple and simply connected, see Proposition 2.2. This reduction is very short and simple, and its key input is the following result from [SGA 3$_{\mathrm{III \ new}}$].

Lemma 2.1. *For a scheme S, a semisimple S-group scheme G that is either simply connected or adjoint, and a maximal S-torus with a Borel S-subgroup $T \subset B$,*

$$T \cong \mathrm{Res}_{S'/S}(\mathbb{G}_{m, S'}) \quad \textit{for some finite étale } S\textit{-scheme } S';$$

in particular, if S is affine and semilocal, then $H^1(S, T) \cong H^1(S', \mathbb{G}_m) \cong 0$.

Proof. The claim is a special case of [SGA 3$_{\text{III new}}$, Exposé XXIV, Proposition 3.13, Corollaire 3.14, Proposition 8.4]. The relevant S' is the scheme of Dynkin diagrams of G defined in [SGA 3$_{\text{III new}}$, Exposé XXIV, Section 3.2 and below]. $\qquad\square$

Proposition 2.2. *For a Noetherian semilocal ring R whose strict Henselization at any prime ideal is a unique factorization domain (for example, R could be a regular semilocal ring), the total ring of fractions $K := \text{Frac}(R)$, and a quasi-split reductive R-group scheme G, if*

$$\text{Ker}\left(H^1(R, (G^{\text{der}})^{\text{sc}}) \to H^1(K, (G^{\text{der}})^{\text{sc}})\right) = \{*\}, \quad then\ also \quad \text{Ker}\left(H^1(R, G) \to H^1(K, G)\right) = \{*\}.$$

Proof. Let $T \subset B \subset G$ be a maximal R-torus and an R-Borel subgroup of G (see [SGA 3$_{\text{III new}}$, Exposé XXIV, Section 3.9]), and let $T' \subset B' \subset (G^{\text{der}})^{\text{sc}}$ be the corresponding maximal R-torus and Borel R-subgroup of $(G^{\text{der}})^{\text{sc}}$ (see [SGA 3$_{\text{III new}}$, Exposé XXVI, Proposition 1.19]). We have compatible isogenies

$$\text{rad}(G) \times (G^{\text{der}})^{\text{sc}} \to G \quad and \quad \text{rad}(G) \times T' \to T$$

that have a common finite kernel Z of multiplicative type, and Lemma 2.1 ensures that $H^1(R, T') = 0$. We will use the resulting commutative diagram of long exact cohomology sequences

$$
\begin{array}{ccccccc}
H^1(R, Z) & \longrightarrow & H^1(R, \text{rad}(G)) & \longrightarrow & H^1(R, T) & \longrightarrow & H^2(R, Z) \\
\| & & \downarrow{\scriptstyle \text{id} \times \{*\}} & & \downarrow & & \| \\
H^1(R, Z) & \longrightarrow & H^1(R, \text{rad}(G)) \times H^1(R, (G^{\text{der}})^{\text{sc}}) & \longrightarrow & H^1(R, G) & \longrightarrow & H^2(R, Z),
\end{array}
$$

as well as its analogue over K. Thus, to proceed, we fix an $\alpha \in \text{Ker}(H^1(R, G) \to H^1(K, G))$.

By [CTS87, Theorem 4.3] (which is where we use the assumption on the strict Henselizations of R), the map $H^2(R, Z) \to H^2(K, Z)$ is injective. Chasing the diagram above and its analogue over K, we therefore find that α comes from some pair

$$(\beta, \gamma) \in H^1(R, \text{rad}(G)) \times H^1(R, (G^{\text{der}})^{\text{sc}})$$

and that, by the nature of the second vertical arrow there, $\gamma|_K$ is trivial. The assumption on $(G^{\text{der}})^{\text{sc}}$ then implies that γ is trivial, so that α lifts to a $\beta \in H^1(R, \text{rad}(G))$ for which $\beta|_K$ lifts to $H^1(K, Z)$. The image of this β in $H^1(R, T)$ lifts α and is trivial over K. By [CTS87, Theorem 4.1] (the Grothendieck–Serre conjecture 1.1 for T), this image of β is trivial, so α is also trivial. $\qquad\square$

3. Fibrations into smooth relative curves

We begin with the geometric part of our approach to the Grothendieck–Serre conjecture for quasi-split G: for a discrete valuation ring $\mathcal{O}$ and a smooth, affine $\mathcal{O}$-scheme U of relative dimension $d > 0$ equipped with an $\mathcal{O}$-fiberwise nowhere dense closed subscheme $Y \subset U$, we wish to construct a smooth morphism

$$\pi \colon U \to S \subset \mathbb{A}_{\mathcal{O}}^{d-1}$$

whose fibers over the affine open S are of dimension 1 such that $Y \cap U$ is *finite* over S. Roughly, the idea is to cut U by $d-1$ suitably transversal hypersurfaces supplied by Bertini theorem and then let their defining equations be the images of the standard coordinates of $\mathbb{A}_{\mathcal{O}}^{d-1}$. The actual argument given in Proposition 3.6 (vii) and Variant 3.7 (vii) is more subtle because ensuring the S-finiteness of $Y \cap U$ is slightly more delicate. To achieve this finiteness, we start from a projective compactification of U and use a geometric presentation theorem of Gabber–Gros–Suwa [CTHK97, Theorem 3.1.1] (if the residue fields of the maximal ideals of $\mathcal{O}$ are perfect, ideas from Artin's construction of "good neighborhoods" in [SGA 4$_{\text{III}}$, Exposé XI, Proposition 3.3] suffice, too).

Before turning to Bertini, we review the following avoidance lemma that we will use repeatedly.

Lemma 3.1. *For a ring R, a quasi-projective, finitely presented R-scheme X, a very R-ample line bundle $\mathscr{L}$ on X, a finitely presented closed subscheme $Z \subset X$ not containing any positive-dimensional irreducible component of any R-fiber of X, and points $y_1, \ldots, y_n \in X \backslash Z$, there is an $N_0 > 0$ such that for every $N \geqslant N_0$ there is an $h \in \Gamma(X, \mathscr{L}^{\otimes N})$ whose vanishing scheme is a hypersurface $H \subset X$ containing Z but not any y_i or any positive-dimensional irreducible component of any R-fiber of X.*

Proof. The claim is a special case of [GLL15, Theorem 5.1] (with definitions reviewed in [GLL15, page 1207]). $\qquad\square$

In the case when R is a field, the following Bertini lemma allows us to impose a smoothness requirement on $X^{\mathrm{sm}} \cap H$. Its most delicate case is when the base field is finite, in which it amounts to a mild extension of [Gab01, Corollaries 1.6 and 1.7], whose argument is actually our key technique.

Lemma 3.2. *Let k be a field, let X be a projective k-scheme of pure dimension, let $Y_1, \ldots, Y_m, Z \subset X$ be closed subschemes with $Z = Z_1 \sqcup Z_0$ for some reduced 0-dimensional $Z_0 \subset X^{\mathrm{sm}}$ all of whose residue fields are separable extensions of k, and fix a*

$$t \leqslant \min(\dim(X), \dim(X) - \dim(Z)) \quad (\textit{recall from §1.7 that } \dim(\varnothing) = -\infty).$$

For an ample line bundle $\mathscr{O}_X(1)$ on X, there are hypersurfaces $H_1, \ldots, H_t \subset X$ such that

(i) *$H_1 \cap \ldots \cap H_t$ is of pure dimension $\dim(X) - t$ and contains Z;*

(ii) *$(X^{\mathrm{sm}} \backslash Z_1) \cap H_1 \cap \ldots \cap H_t$ is k-smooth;*

(iii) *$\dim((Y_j \backslash Z) \cap \bigcap_{\ell \in L} H_\ell) \leqslant \dim(Y_j \backslash Z) - \#L$ for $1 \leqslant j \leqslant m$ and $L \subset \{1, \ldots, t\}$;*

moreover, we may iteratively choose $H_1, \ldots, H_t$ so that, for each i, with $H_1, \ldots, H_{i-1}$ already fixed, H_i may be chosen to have any sufficiently large degree divisible by the characteristic exponent of k.

We do not know how to ensure that the hypersurfaces H_i in Lemma 3.2 all have the same degree.

Proof. The pure dimension hypothesis means that all the irreducible components of X have the same dimension, so [EGA IV$_2$, Proposition 5.2.1] ensures that X is biequidimensional in the sense that the saturated chains of specializations of its points all have the same length. Similarly to [Čes21, Section 4.1], part (i) then ensures that each $X \cap H_1 \cap \ldots \cap H_i$ inherits biequidimensionality, so is also of pure dimension. This reduces us to $t = 1$: by applying this case iteratively and at each step adjoining to the Y_j's all their possible intersections with some of the already chosen H_i's (to ensure (iii)), we will obtain the general case. In the case $t = 1$, we fix closed points $y_1, \ldots, y_n \in X \backslash Z$ that jointly meet every irreducible component of X and of every $Y_j \backslash Z$. Both (iii) and the dimension aspect of (i) will hold as soon as H_1 contains no y_j, so at the cost of requiring this we may forget about the Y_j.

For the rest of the argument, we begin with the case when char $k = 0$, in which we will use the "classical" Bertini theorem (one could also choose to skip this case because we will only use Lemma 3.2 for finite k). For this, we first claim that for every large $N > 0$ there are global sections h_i of $\mathscr{O}_X(N)$ whose common zero locus contains Z and set-theoretically equals to it. Indeed, by repeatedly applying [EGA III$_1$, Corollaire 2.2.4] to shrink the base locus, we first build such $h'_{i'}$ (resp., $h''_{i''}$) for some N' (resp., N'') that is a power of 2 (resp., of 3), then express every large N as $aN' + bN''$ with $a, b > 0$, and, finally, let h_i be the collection of all the $h'^a_{i'} h''^b_{i''}$. By [EGA III$_1$, Corollaire 2.2.4] and [EGA IV$_4$, Corollaire 17.15.9] (which uses the separable residue field assumption), granted that N is

7

sufficiently large, we may build another global section h_0 of $\mathscr{O}_X(N)$ whose associated hypersurface contains Z and is smooth at every point of Z_0. We adjoin this h_0 to the h_i and then discard linear dependencies to assume that the h_i are k-linearly independent. By [EGA II, Proposition 4.2.3], the h_i determine a morphism

$$X \backslash Z \to \mathbb{P}_k^{N'}$$

such that the pullback of $\mathscr{O}_{\mathbb{P}_k^{N'}}(1)$ is our $\mathscr{O}_{X \backslash Z}(N)$. The hyperplanes in $\mathbb{P}_k^{N'}$ and, compatibly, the nonzero k-linear combinations of the h_i up to scaling are parametrized by the dual projective space. Due to the existence of a k-linear combination of the h_i whose associated hypersurface does not contain a fixed y_j, a generic such hypersurface contains no y_j. Likewise, due to the openness of the smooth locus, the existence of a k-linear combination of the h_i whose associated hypersurface is smooth at all the points in Z_0, and [EGA IV$_3$, Théorème 11.3.8 b')⇔c)], a generic such hypersurface is smooth at all the points in Z_0. Finally, by the Bertini theorem [Jou83, Corollaire 6.11 2)], the hypersurface H associated to a generic k-linear combination of the h_i is such that $(X^{\mathrm{sm}} \backslash Z) \cap H$ is k-smooth. In conclusion, since k is infinite, we may choose our desired H_1 to be a generic such H.

The remaining case when $\mathrm{char}\, k = p > 0$ is a very minor sharpening of [Gab01, Corollary 1.6] that is proved as there. Namely, we use the pure dimension hypothesis to apply [Gab01, Theorem 1.1][2] with

- U there being our $X^{\mathrm{sm}} \backslash (Z \cup \{y_1, \ldots, y_n\})$ and $\mathcal{E}$ there being Ω_U^1;

- Z there being our $Z_1 \cup \{y_1, \ldots, y_n\} \cup \bigcup_{z \in Z_0} \underline{\mathrm{Spec}}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}_z^2)$;

- Σ there being our $Z_0 \cup \{y_1, \ldots, y_n\}$;

- m_0 there being 0; and

- σ_0 there being 0 on our Z_1, a unit on each of our $y_1, \ldots, y_n$, and a nonzero cotangent vector at $z \in Z_0$ on each of our $\underline{\mathrm{Spec}}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}_z^2)$;

to find a finite set of closed points

$$F \subset X^{\mathrm{sm}} \backslash (Z \cup \{y_1, \ldots, y_n\})$$

and, for every large N divisible by p, a global section h of $\mathscr{O}_X(N)$ whose associated hypersurface contains Z, has a k-smooth intersection with $X^{\mathrm{sm}} \backslash (F \cup Z \cup \{y_1, \ldots, y_n\})$, passes through every $z \in Z_0$ and is k-smooth there (for this we use [EGA IV$_4$, Corollaire 17.15.9] and the separable residue field assumption as in the characteristic 0 case), and does not pass through any y_j. By [EGA III$_1$, Corollaire 2.2.4], if this N is sufficiently large, then there is a global section h' of $\mathscr{O}_X(N/p)$ that vanishes on $Z \cup \{y_1, \ldots, y_n\}$ and is such that $h + h'^p$ does not vanish at any point of F. We may then let H_1 be the hypersurface associated to $h + h'^p$. $\qquad\square$

Remark 3.3. In (iii), if $(Y_j \backslash Z) \cap \bigcap_{\ell \in L} H_\ell \neq \varnothing$, then the inequality is actually an equality because, unless the intersection is empty, cutting by $\#L$ hypersurfaces decreases dimension by at most $\#L$.

For our purposes, the main drawback of Lemma 3.2 is its requirement that the residue fields of the points in Z_0 be separable over k: this is automatic if k is perfect but cannot be removed if k is imperfect because then (i) and (ii) cannot hold simultaneously (consider the case $t = \dim(X)$). To accommodate for imperfect k, we will relax (i) by no longer requiring that $Z \subset H_i$, see Proposition 3.6 for the precise statement, which uses the following review of weighted blowups.

---

[2]*Loc. cit.* is stated in the case when the base field k is finite but its proof continues to work whenever k is any field of positive characteristic p.

3.4. Weighted projective spaces. For $w_0, \ldots, w_d \in \mathbb{Z}_{>0}$, we consider the polynomial algebra $\mathbb{Z}[t_0, \ldots, t_d]$ that is $\mathbb{Z}_{\geqslant 0}$-graded by declaring each t_i to be of weight w_i (and the constants $\mathbb{Z}$ to be of weight 0), and we let the resulting *weighted projective space* be

$$\mathbb{P}_{\mathbb{Z}}(w_0, \ldots, w_d) := \mathrm{Proj}(\mathbb{Z}[t_0, \ldots, t_d]).$$

We repeat the same construction over any scheme S to build $\mathbb{P}_S(w_0, \ldots, w_d)$, although the latter is simply $\mathbb{P}_{\mathbb{Z}}(w_0, \ldots, w_d) \times_{\mathbb{Z}} S$ because the formation of Proj commutes with base change [EGA II, Proposition 3.5.3]. We will only use weighted projective spaces when $w_0 = 1$, in which case the open subscheme of $\mathbb{P}_S(1, w_1, \ldots, w_d)$ given by $\{t_0 \neq 0\}$ is the affine space $\mathbb{A}_S^d$ with coordinates

$$t_1/t_0^{w_1}, \quad \ldots, \quad t_d/t_0^{w_d}.$$

3.5. Weighted blowups. For a scheme X, a line bundle $\mathscr{L}$ on X, and sections

$$h_0 \in \Gamma(X, \mathscr{L}^{\otimes w_0}), \quad \ldots, \quad h_d \in \Gamma(X, \mathscr{L}^{\otimes w_d}) \quad \text{with} \quad w_0, \ldots, w_d > 0,$$

we define the *weighted blowup* of X with respect to $h_0, \ldots, h_d$ as

$$\mathrm{Bl}_X(h_0, \ldots, h_d) := \underline{\mathrm{Proj}}_{\mathscr{O}_X}(\mathscr{O}_X[h_0, \ldots, h_d]), \quad \text{where} \quad \mathscr{O}_X[h_0, \ldots, h_d] \subset \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$

is the quasi-coherent, graded $\mathscr{O}_X$-subalgebra generated by the h_i. The *center* of this weighted blowup is the closed subscheme of X cut out by the h_i. By [EGA II, Proposition 3.1.8 (i)], the map

$$\mathrm{Bl}_X(h_0, \ldots, h_d) \to X \quad \text{is an isomorphism away from the center.}$$

By §3.4 and the functoriality of Proj, the weighted blowup $\mathrm{Bl}_X(h_0, \ldots, h_d)$ admits a morphism

$$\mathrm{Bl}_X(h_0, \ldots, h_d) \to \mathbb{P}_{\mathbb{Z}}(w_0, \ldots, w_d) \quad \text{determined by} \quad t_i \mapsto h_i. \tag{3.5.1}$$

In the case when $w_0 = \ldots = w_d$, our $\mathrm{Bl}_X(h_0, \ldots, h_d)$ is identified with the usual blowup of X along the closed subscheme cut out by the h_i: this is evident when also $\mathscr{L} = \mathscr{O}_X$, and the general case reduces to this one because Proj is insensitive to twisting by line bundles [EGA II, Proposition 3.1.8 (iii)].

Proposition 3.6. *Let k be a field, let X be a projective k-scheme of pure dimension d, let $\mathscr{O}_X(1)$ be an ample line bundle on X, let $W \subset X^{\mathrm{sm}}$ be an open, let $x_1, \ldots, x_n \in W$, and let $Y \subset X$ be a closed subscheme of codimension > 0. Upon replacing $\mathscr{O}_X(1)$ by any large power, there exist nonzero*

$$h_0 \in \Gamma(X, \mathscr{O}_X(1)), \quad h_1 \in \Gamma(X, \mathscr{O}_X(w_1)), \quad \ldots, \quad h_{d-1} \in \Gamma(X, \mathscr{O}_X(w_{d-1})) \quad \text{with} \quad w_1, \ldots, w_{d-1} > 0,$$

such that

   (i) *the hypersurface $H_0 := V(h_0) \subset X$ does not contain any x_i;*

   (ii) *the hypersurfaces $H_i := V(h_i) \subset X$ satisfy $Y \cap H_0 \cap \ldots \cap H_{d-1} = \varnothing$;*

   (iii) *in the following commutative diagram with vertical maps determined by the $h_0, \ldots, h_{d-1}$:*

$$
\begin{array}{ccccc}
X \backslash H_0 & \hookrightarrow & X \backslash (H_0 \cap \ldots \cap H_{d-1}) & \hookrightarrow & \overline{X} := \mathrm{Bl}_X(h_0, \ldots, h_{d-1}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \overline{\pi}} \; {\scriptstyle (3.5.1)} \\
\mathbb{A}_k^{d-1} & \xrightarrow{\S 3.4} & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) & = & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}),
\end{array}
$$

   for every i, the map π is smooth (of relative dimension 1) at x_i;

   (iv) *for every i, we have $Y \cap H_0 \cap \overline{\pi}^{-1}(\pi(x_i)) = \varnothing$;*

   (v) *if $Y \backslash X^{\mathrm{sm}}$ is of codimension > 1 in X, then π is smooth at each $Y \cap \overline{\pi}^{-1}(\pi(x_i))$;*

   (vi) *if even $Y \backslash W$ is of codimension > 1 in X, then, for every i, we have $(Y \backslash W) \cap \overline{\pi}^{-1}(\pi(x_i)) = \varnothing$;*

(vii) *if even $Y \backslash W$ is of codimension > 1 in X, then there are affine opens*

$$S \subset \mathbb{A}_k^{d-1} \quad and \quad x_1, \ldots, x_n \in U \subset W \cap \pi^{-1}(S) \subset X \backslash H_0$$

such that $\pi \colon U \to S$ is smooth of relative dimension 1 and $Y \cap U = Y \cap \pi^{-1}(S)$ is S-finite;

moreover, we may iteratively choose $h_0, \ldots, h_{d-1}$ so that, for each i, with $h_0, \ldots, h_{i-1}$ already fixed, the degree w_i may be any sufficiently large integer divisible by the characteristic exponent of k.

Proof. Closed points are dense in a finite type scheme over a field [SP, Lemma 02J6], so each x_i specializes to a closed point of X^{sm}. By replacing the x_i by such specializations, we assume for the rest of the proof that the points $x_1, \ldots, x_n$ are closed in X.

In (iii), the parenthetical relative dimension aspect follows from a dimension count. The intersections in (iv) make sense because (ii) ensures that

$$Y \subset X \backslash (H_0 \cap \ldots \cap H_{d-1}),$$

that is, that Y does not meet the center of the weighted blowup $\overline{X} \to X$ (see §3.5). Moreover, (iv) implies that over the open neighborhood of $\bigcup_{i=1}^n \pi(x_i)$ given by the complement of $\overline{\pi}(Y \cap H_0)$, our closed Y even lies in $X \backslash H_0$. Since H_0 is a hyperplane section, intersecting with it cuts the dimension of any positive-dimensional closed subvariety of X by at most 1, so (iv) implies that $Y \cap \overline{\pi}^{-1}(\pi(x_i))$ is finite. The openness of the quasi-finite locus [SP, Lemma 01TI] and the finiteness of proper, quasi-finite morphisms [SP, Lemma 02OG] then imply that $Y \cap \pi^{-1}(S)$ is S-finite for every sufficiently small affine open $S \subset \mathbb{A}_k^{d-1}$ containing $\pi(x_1), \ldots, \pi(x_n)$. Consequently, by spreading out from the semilocalization at $\pi(x_1), \ldots, \pi(x_n)$, for any affine open $U \subset \pi^{-1}(S)$ containing all the x_i and all the points of $Y \cap \overline{\pi}(\pi^{-1}(x_i))$, at the cost shrinking S further we may also arrange that

$$Y \cap U = Y \cap \pi^{-1}(S).$$

In conclusion, (vii) follows from (i)–(v) (we stated (vii) explicitly because we will use it below).

For the rest, we may assume that $d > 0$ and begin with the case when k is perfect, in which we will deduce the claim from Lemma 3.2, even with the additional requirement that each H_i passes through $x_1, \ldots, x_n$: this will ensure that $\pi(x_i)$ is the origin of $\mathbb{A}_k^{d-1}$ for every i, so that

$$\overline{\pi}^{-1}(\pi(x_i)) = (H_1 \cap \ldots \cap H_{d-1}) \backslash (H_0 \cap \ldots \cap H_{d-1}) \quad in \quad X \backslash (H_0 \cap \ldots \cap H_{d-1}), \tag{3.6.2}$$

to the effect that (iv) will follow from (ii). We begin by using Lemma 3.1 to choose a hypersurface $H_0 \subset X$ of any sufficiently large degree w_0 such that H_0 does not contain any x_i nor any irreducible component of Y. We then replace $\mathscr{O}_X(1)$ by its w_0-th power and, since the extensions $k(x_i)/k$ are separable, use Lemma 3.2 to choose hypersurfaces $H_1, \ldots, H_{d-1}$ containing every x_i such that

- $X^{\mathrm{sm}} \cap H_1 \cap \ldots \cap H_{d-1}$ is k-smooth of pure dimension 1;

- $Y \cap H_0 \cap \ldots \cap H_{d-1} = \varnothing$;

- if $Y \backslash X^{\mathrm{sm}}$ is of codimension > 1 in X, then $(Y \backslash X^{\mathrm{sm}}) \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$; and

- if $Y \backslash W$ is of codimension > 1 in X, then $(Y \backslash W) \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

These properties ensure the desired (i)–(vi): indeed, by (3.6.2), the x_i and, if $Y \backslash X^{\mathrm{sm}}$ is of codimension > 1 in X, then also the $Y \cap \overline{\pi}^{-1}(\pi(x_i))$ all lie in the smooth locus of the zero fiber of π and, by the dimensional flatness criterion [EGA IV$_2$, Proposition 6.1.5], also in the smooth locus of π. The aspect about arranging the degrees w_i to be large follows from its counterpart in Lemma 3.2.

In the remaining case when $d > 0$ and k is infinite (or even imperfect), we will deduce the claim from a geometric presentation theorem of Gabber–Gros–Suwa [CTHK97, Theorem 3.1.1], according to which:

> for a smooth, affine, irreducible k-scheme V of finite type and dimension d, points $v_1, \ldots, v_m \in V$, and a closed subscheme $Z \subset V$ of codimension > 0, there is a k-map $V \to \mathbb{A}_k^{d-1}$ that is smooth at every v_i and such that Z is finite over $\mathbb{A}_k^{d-1}$; without the smoothness assumption on V, the same holds granted that $m = 0$.　($\star$)

Loc. cit. is much more general but ($\star$) will suffice for us. For convenience, we added the aspect about nonsmooth V, which follows from Noether normalization [SP, Lemma 00OY]. To apply ($\star$), we first use Lemma 3.1 to choose a hypersurface $H_0 = V(h_0) \subset X$ of any sufficiently large degree w_0 such that H_0 does not contain any x_i, does not contain any irreducible component of Y of codimension 1 in X, contains the intersections of any two distinct irreducible components of X, contains the nonsmooth locus of any generically smooth irreducible component of X, and contains $Y \backslash X^{\mathrm{sm}}$ if the latter is of codimension > 1 in X. Then $X \backslash H_0$ is affine and a nonempty disjoint union of irreducible components V of dimension d, each of which is either smooth or contains no x_i. By applying ($\star$) to each such V with the v_j being those x_i that lie on V and Z being $Y \cap V$, we obtain a k-morphism

$$\pi' : X \backslash H_0 \to \mathbb{A}_k^{d-1}$$

that is smooth at every x_i and such that $Y \backslash H_0$ is finite over $\mathbb{A}_k^{d-1}$.

We replace $\mathscr{O}_X(1)$ by its w_0-th power, and consequently view h_0 as a global section of $\mathscr{O}_X(1)$, in other words, as a morphism $\mathscr{O}_X \to \mathscr{O}_X(1)$ determined by $1 \mapsto h_0$, which for any $w > 0$ gives rise to a morphism $\mathscr{O}_X \to \mathscr{O}_X(w)$ determined by $1 \mapsto h_0 \otimes \cdots \otimes h_0$. Via the latter, any section of $\mathscr{O}_X$ over $X \backslash H_0$ extends to a section of $\mathscr{O}_X(w)$ over X granted that w is large enough, and two fixed such extensions become equal granted that we enlarge w further (the uniqueness allows one to work on a finite cover of X by affines that trivialize $\mathscr{O}_X(1)$, and the claim then reduces to the observation that for any ring R, any $r \in R$, and any $h \in R[\frac{1}{r}]$, the element $r^w h \in R[\frac{1}{r}]$ lifts to R granted that $w > 0$ is large, and any two such lifts agree granted that we multiply them by $r^{w'}$ for a large $w' > 0$). In particular, the images under π' of the standard coordinates of $\mathbb{A}_k^{d-1}$ extend to nonzero sections

$$h_1' \in \Gamma(X, \mathscr{O}_X(w_1)), \quad \ldots, \quad h_{d-1}' \in \Gamma(X, \mathscr{O}_X(w_{d-1})) \tag{3.6.3}$$

for all large enough $w_1, \ldots, w_{d-1} > 0$. By construction, with $H_i' := V(h_i')$, the map

$$\overline{\pi}' : X \backslash (H_0 \cap H_1' \cap \cdots \cap H_{d-1}') \to \mathbb{P}_k(1, w_1, \ldots, w_{d-1})$$

that results from h_0 and these h_i' as in (iii) restricts to π', satisfies (i), and is smooth at every x_i.

To proceed further, for h_0 and $w_1, \ldots, w_{d-1}$ fixed above, we consider the k-scheme $\mathcal{M}$ that parametrizes all $(d-1)$-tuples of global sections like in (3.6.3), so that $\mathcal{M}$ is noncanonically isomorphic to some affine space $\mathbb{A}_k^N$. The base change $X_{\mathcal{M}}$ comes equipped with the universal hypersurfaces

$$\mathcal{H}_1, \ldots, \mathcal{H}_{d-1} \subset X_{\mathcal{M}}$$

and, of course, also with the base changes of $x_1, \ldots, x_n$, Y, and H_0. The defining equations of H_0 and of the $\mathcal{H}_1, \ldots, \mathcal{H}_{d-1}$ give the universal diagram of maps

$$
\begin{array}{ccc}
X_{\mathcal{M}} \backslash (H_0)_{\mathcal{M}} & \hookrightarrow & X_{\mathcal{M}} \backslash ((H_0)_{\mathcal{M}} \cap \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_{d-1}) \\
\downarrow & & \downarrow \\
\mathbb{A}_{\mathcal{M}}^{d-1} & \xrightarrow{\ \S 3.4\ } & \mathbb{P}_{\mathcal{M}}(1, w_1, \ldots, w_{d-1})
\end{array}
$$

whose base changes to k-points of $\mathcal{M}$ are left squares of diagrams as in (iii). To finish the proof, we will argue that for all sufficiently large $w_1, \ldots, w_{d-1}$ that are divisible by the characteristic exponent of k and that may be chosen iteratively as in the statement, the conditions (i)–(vi) define a nonempty open of $\mathcal{M}$: since $\mathcal{M} \simeq \mathbb{A}_k^N$ and k is infinite, this open will have a k-point that, by construction, will correspond to the desired $h_1, \ldots, h_{d-1}$ satisfying (i)–(vi).

First of all, since h_0 is fixed, (i) holds by construction. Since the image of a proper morphism is closed, the nonvanishing requirement on the h_i and the condition (ii) define an open of $\mathcal{M}$ that, granted that the $w_1, \ldots, w_{d-1}$ are large enough, is nonempty by Lemma 3.1. Due to the existence of the h_i' as in (3.6.3) and the openness of the smooth locus, the smoothness requirement (iii) likewise defines a nonempty open. Granted (ii), since the image of a proper morphism is closed, the requirements

$$Y \cap H_0 \cap \overline{\pi}^{-1}(\pi(x_i)) = \varnothing$$

of (vii) define an open. Granted that the $w_1, \ldots, w_{d-1}$ are iteratively chosen to be large enough and divisible by the characteristic exponent of k, this open is nonempty thanks to Lemma 3.2 (applied after base change to $\overline{k}$ and similarly to the case of a perfect k treated above), and even remains a nonempty open after imposing the further requirements (v) and (vi). In summary, the conditions (i)–(vi) indeed define a nonempty open of $\mathcal{M}$, as desired. □

For our mixed characteristic purposes, we need to extend Proposition 3.6 by allowing k be a semilocal Dedekind ring $\mathcal{O}$. We deduce such an extension in the following variant: this is straight-forward by specialization if the x_i all lie above the maximal ideals of $\mathcal{O}$, and we reduce to this case via a trick that adds an auxiliary maximal ideal to $\mathcal{O}$.

Variant 3.7. *Let $\mathcal{O}$ be a semilocal Dedekind ring, let X be a projective, flat $\mathcal{O}$-scheme with fibers of pure dimension d, let $\mathcal{O}_X(1)$ be an $\mathcal{O}$-ample line bundle on X, let $W \subset X^{\mathrm{sm}}$ be an open, let $x_1, \ldots, x_n \in W$, and let $Y \subset X$ be a closed subscheme that is $\mathcal{O}$-fiberwise of codimension > 0. Upon replacing $\mathcal{O}_X(1)$ by any large power, there exist nonzero*

$$h_0 \in \Gamma(X, \mathcal{O}_X(1)), \quad h_1 \in \Gamma(X, \mathcal{O}_X(w_1)), \quad \ldots, \quad h_{d-1} \in \Gamma(X, \mathcal{O}_X(w_{d-1})) \quad \text{with} \quad w_1, \ldots, w_{d-1} > 0,$$

such that

(i) *the hypersurface $H_0 := V(h_0) \subset X$ does not contain any x_i;*

(ii) *the hypersurfaces $H_i := V(h_i) \subset X$ satisfy $Y \cap H_0 \cap \ldots \cap H_{d-1} = \varnothing$;*

(iii) *in the following commutative diagram with vertical maps determined by the $h_0, \ldots, h_{d-1}$:*

$$
\begin{array}{ccccc}
X \backslash H_0 & \hookrightarrow & X \backslash (H_0 \cap \ldots \cap H_{d-1}) & \hookrightarrow & \overline{X} := \mathrm{Bl}_X(h_0, \ldots, h_{d-1}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \overline{\pi}} \quad {\scriptstyle (3.5.1)} \\
\mathbb{A}_{\mathcal{O}}^{d-1} & \xhookrightarrow{\S 3.4} & \mathbb{P}_{\mathcal{O}}(1, w_1, \ldots, w_{d-1}) & = & \mathbb{P}_{\mathcal{O}}(1, w_1, \ldots, w_{d-1}),
\end{array}
$$

for every i, the map π is smooth (of relative dimension 1) at x_i;

(iv) *for every i, we have $Y \cap H_0 \cap \overline{\pi}^{-1}(\pi(x_i)) = \varnothing$;*

(v) *if $Y \backslash X^{\mathrm{sm}}$ is $\mathcal{O}$-fiberwise of codimension > 1 in X, then π is smooth at each $Y \cap \overline{\pi}^{-1}(\pi(x_i))$;*

(vi) *if even $Y \backslash W$ is $\mathcal{O}$-fiberwise of codimension > 1 in X, then $(Y \backslash W) \cap \overline{\pi}^{-1}(\pi(x_i)) = \varnothing$ for all i;*

(vii) *if even $Y \backslash W$ is $\mathcal{O}$-fiberwise of codimension > 1 in X, then there are affine opens*

$$S \subset \mathbb{A}_{\mathcal{O}}^{d-1} \quad \text{and} \quad x_1, \ldots, x_n \in U \subset W \cap \pi^{-1}(S) \subset X \backslash H_0$$

such that $\pi\colon U \to S$ is smooth of relative dimension 1 and $Y \cap U = Y \cap \pi^{-1}(S)$ is S-finite.

Proof. If $\mathcal{O}$ is a field, then the claim follows from Proposition 3.6, so we may assume that $\mathcal{O}$ is a domain that is not a field and set $K := \mathrm{Frac}(\mathcal{O})$. As in the proof Proposition 3.6, we may replace each x_i by its suitable specialization to assume without losing generality that the points $x_1, \ldots, x_n$ are closed in their $\mathcal{O}$-fibers of X. Moreover, (vii) again follows from the rest by an argument as there.

For the rest, we let $C \subset \mathrm{Spec}(\mathcal{O})$ be the union of the closed points of $\mathrm{Spec}(\mathcal{O})$, and we begin with the case when each x_i lies over C. By Proposition 3.6, the desired h_i exist after base change to C. Moreover, the last aspect of Proposition 3.6 ensures that these h_i may be chosen to have constant degrees w_i on C and, by [EGA III$_1$, Corollaire 2.2.4], to be such that they lift to sections of $\mathscr{O}_X(w_i)$. Since the image of a proper morphism is closed, these lifts still satisfy the desired (i)–(vi) (the formation of the weighted blowup $\mathrm{Bl}_X(h_0, \ldots, h_{d-1})$ in (iii) need not commute with base change to C, but the formation of the map

$$\overline{\pi}\colon X\backslash(H_0 \cap \ldots \cap H_{d-1}) \to \mathbb{P}_{\mathcal{O}}(1, w_1, \ldots, w_{d-1})$$

does and this suffices).

More generally, consider the case when each x_i that lies in the generic $\mathcal{O}$-fiber of X specializes to some point in $W \cap X_C$. By replacing each such x_i by this specialization, we reduce to $x_1, \ldots, x_n$ all lying over C, that is, to the previous paragraph. In general, such specializations need not exist, but we will reduce to the case when they do by enlarging $\mathrm{Spec}(\mathcal{O})$ and suitably extending X and Y. For this, we begin by letting $\mathfrak{m} \subset \mathcal{O}$ range over the maximal ideals and noting that, since $\bigcup_{\mathfrak{m}} \mathrm{Spec}(\mathcal{O}_{\mathfrak{m}})$ is an open cover of $\mathrm{Spec}(\mathcal{O})$, the map

$$\mathcal{O} \xrightarrow{\sim} \bigcap_{\mathfrak{m}} \mathcal{O}_{\mathfrak{m}}$$

is an isomorphism. For every subfield $K' \subset K$, we consider the discrete valuation rings $\mathcal{O}'_{\mathfrak{m}} := \mathcal{O}_{\mathfrak{m}} \cap K'$ and we note that, by [Mat89, Theorem 12.2], the intersection

$$\mathcal{O}' := \bigcap_{\mathfrak{m}} \mathcal{O}'_{\mathfrak{m}}$$

is a semilocal Dedekind subdomain of $\mathcal{O}$ whose localizations at maximal ideals are precisely the $\mathcal{O}'_{\mathfrak{m}}$. As K' ranges over the finitely generated subfields of K, the subrings $\mathcal{O}'$ exhaust $\mathcal{O}$. Thus, we may use a limit argument to replace $\mathcal{O}$ by some such $\mathcal{O}'$, and hence to reduce to the case when K is finitely generated (and automatically separable) over its prime subfield. By [EGA IV$_4$, Corollaire 17.15.9] and spreading out, such a K is the fraction field of a regular domain A that is smooth either over this prime subfield or even over $\mathbb{Z}$. Moreover, this A cannot be 0-dimensional: else, K would be a finite field, which would contradict our assumption that $\mathcal{O}$ is not a field.

By localizing A if needed, we may assume that

- X_K spreads out to a projective, flat A-scheme $\mathcal{X}$ with A-fibers of pure dimension d (see [EGA IV$_3$, Théorème 12.2.1 (ii) and (v)]);

- $\mathscr{O}_{X_K}(1)$ spreads out to an A-ample line bundle $\mathscr{L}$ on $\mathcal{X}$ (see [EGA IV$_3$, Corollaire 9.6.4]);

- W_K and Y_K spread out to A-flat open (resp., closed) subschemes

$$\mathcal{W} \subset \mathcal{X}^{\mathrm{sm}} \quad \text{and} \quad \mathcal{Y} \subset \mathcal{X}$$

  such that $\mathcal{Y}$ is A-fiberwise of codimension > 0 and, if $Y_K\backslash W_K$ (resp., $Y_K\backslash X_K^{\mathrm{sm}}$) is of codimension > 1 in X, then $\mathcal{Y}\backslash\mathcal{W}$ (resp., $\mathcal{Y}\backslash\mathcal{X}^{\mathrm{sm}}$) is A-fiberwise of codimension > 1 in $\mathcal{X}$ (see [EGA IV$_3$, Théorème 11.3.1, Corollaire 12.2.2 (i)]); and

- each x_i that lies over K spreads out to an A-finite closed subscheme of $\mathcal{W}$.

Since A is positive-dimensional, it has infinitely many primes $\mathfrak{p}$ of height 1. In particular, for some such $\mathfrak{p}$, the discrete valuation subring $R := A_{\mathfrak{p}}$ of K is distinct from each $\mathcal{O}_{\mathfrak{m}}$. Since these discrete valuation subrings share a common fraction field, there can be no inclusion relations between them. Thus, [Mat89, Theorems 12.2 and 12.6] ensure that

$$\widetilde{\mathcal{O}} := R \cap \textstyle\bigcap_{\mathfrak{m}} \mathcal{O}_{\mathfrak{m}}$$

is a semilocal Dedekind domain whose spectrum is obtained by glueing $\mathrm{Spec}(\mathcal{O})$ and $\mathrm{Spec}(R)$ along their common open $\mathrm{Spec}(K)$. Consequently, we may glue X with $\mathcal{X}_R$ along X_K to extend X to a projective, flat $\widetilde{\mathcal{O}}$-scheme $\widetilde{X}$ with fibers of pure dimension d that comes equipped with an ample line bundle $\mathcal{O}_{\widetilde{X}}(1)$ obtained by glueing $\mathcal{O}_X(1)$ with $\mathscr{L}_R$ along $\mathcal{O}_{X_K}(1)$, with an open subscheme $\widetilde{W} \subset \widetilde{X}^{\mathrm{sm}}$ obtained by glueing W with $\mathcal{W}_R$ along W_K, as well as with a closed subscheme $\widetilde{Y} \subset \widetilde{X}$ obtained by glueing Y with $\mathcal{Y}_R$ along Y_K. We may now replace X and $\mathcal{O}$ by $\widetilde{X}$ and $\widetilde{\mathcal{O}}$ without losing generality. This has the advantage that, by construction, we are left with the already treated case in which each x_i that lies in X_K specializes to some point in $W \cap X_C$. $\qquad\square$

4. Lifting the torsor to a smooth relative curve

In practice, we start from a smooth affine scheme W, not from its projective compactification X as in Variant 3.7. The following proposition recasts the results of the previous section from this vantage point. We thank Panin for extracting its formulation from the initial version of this article.

Proposition 4.1. *For*

- *a semilocal Dedekind ring $\mathcal{O}$;*
- *a smooth $\mathcal{O}$-algebra A that is everywhere of positive relative dimension over $\mathcal{O}$;*
- *$x_1, \ldots, x_n \in \mathrm{Spec}(A)$;*
- *a closed subscheme $Y \subset \mathrm{Spec}(A)$ that is of codimension $\geqslant 2$;*

there are

(i) *an affine open $U \subset \mathrm{Spec}(A)$ containing all the x_i;*

(ii) *an affine open $S \subset \bigsqcup_{d \geqslant 0} \mathbb{A}^d_{\mathcal{O}}$; and*

(iii) *a smooth $\mathcal{O}$-morphism $\pi\colon U \to S$ of pure relative dimension 1 such that $Y \cap U$ is S-finite.*

Proof. We decompose $\mathrm{Spec}(\mathcal{O})$ and $W := \mathrm{Spec}(A)$ into connected components to assume that $\mathcal{O}$ and A are domains, so that A is of pure relative dimension $d > 0$ over $\mathcal{O}$. We embed W into some affine space over $\mathcal{O}$ and then take the schematic image in the corresponding projective space to build an open immersion $W \hookrightarrow X$ into a projective, flat $\mathcal{O}$-scheme X. On the K-fiber with $K := \mathrm{Frac}(\mathcal{O})$ this immersion has dense image, so X_K is of pure dimension d. It then follows from [SP, Lemmas 0D4J and 02FZ] that X is of pure relative dimension d over $\mathcal{O}$. We will obtain our U and π from X via Variant 3.7 (vii). To apply the latter, all we need to do is to check that the schematic image $\overline{Y}$ of Y in X is such that $Y' := \overline{Y} \backslash Y$ is $\mathcal{O}$-fiberwise of codimension > 1 in X.

By [SP, Lemma 01R8], set-theoretically we have $\overline{Y} = \bigcup_y \overline{\{y\}}$ where y ranges over the generic points of Y and the closures are in X. Each y is of height $\geqslant 2$ in X, so each $\overline{\{y\}}$ intersects the $\mathcal{O}$-fiber of X containing y in a closed subscheme of dimension $\leqslant d - 1$ (even $\leqslant d - 2$ if the fiber is generic). Thus, since $\overline{\{y\}}$ has a nonempty open $\overline{\{y\}} \cap W$, the contribution of y to its $\mathcal{O}$-fiber of Y' is of dimension $\leqslant d - 2$. The only situation in which $\overline{\{y\}}$ may contribute to other $\mathcal{O}$-fibers of Y' is when y lies in

14

the generic $\mathcal{O}$-fiber of X and $\mathcal{O}$ is not a field. However, since the local rings of X are of dimension $\leqslant d+1$, then the intersection of $\overline{\{y\}}$ with any closed $\mathcal{O}$-fiber of X is of dimension $\leqslant d-2$. In conclusion, Y' is $\mathcal{O}$-fiberwise of dimension $\leqslant d-2$, that is, $\mathcal{O}$-fiberwise of codimension > 1 in X. $\quad\square$

The following consequence of Proposition 4.1 starts a string of reductions that will eventually lead to Theorem 1.2. In comparison to versions in the literature, for instance, to [Fed21b, Proposition 4.4], the main new phenomena are that the group $\mathscr{G}$ is only defined over a small affine C and that $C \backslash Z$ need not be affine, which will cause additional subtleties in §7. It is not necessary to assume that our quasi-split group is simply connected but, upon insistence of a referee, we assume this anyway.

Proposition 4.2. *For a semilocal Dedekind ring $\mathcal{O}$, the semilocalization R of a smooth $\mathcal{O}$-algebra A at finitely many primes $\mathfrak{p}$, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

(i) *a smooth, affine R-scheme C of pure relative dimension 1;*

(ii) *a section $s \in C(R)$;*

(iii) *an R-finite closed subscheme $Z \subset C$;*

(iv) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$;*

(v) *a $\mathscr{G}$-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(\mathscr{B})$-torsor over $C \backslash Z$;*

in fact, for any extension of $B \subset G$ to a quasi-pinning[3] of G, one may find the objects as above such that $\mathscr{B} \subset \mathscr{G}$ extends to a quasi-pinning of $\mathscr{G}$ that is compatible via s with the quasi-pinning of G.

Proof. We decompose $\mathrm{Spec}(\mathcal{O})$ and $\mathrm{Spec}(R)$ into connected components to assume that $\mathcal{O}$ and R are domains, and then likewise assume that A is a domain. If A is of relative dimension 0, then R is a Dedekind domain, so, by [Guo20, Theorem 1], our torsor E is trivial and we may choose $C = \mathbb{A}^1_R$ and $\mathscr{E} := E_{\mathbb{A}^1_R}$, the closed subscheme Z being empty and s being the zero section. Thus, we may assume that A is $\mathcal{O}$-fiberwise of pure dimension $d > 0$. Moreover, we localize A and spread out to assume (abusively, from a notational standpoint) that G, B, and E begin life over A and, for the last aspect of the claim, that a fixed extension of $B \subset G$ to a quasi-pinning of G likewise exists over A.

By [SGA 3$_{\mathrm{III\ new}}$, Exposé XXVI, Corollaire 3.6, Lemme 3.20], the quotient E/B is representable by a projective A-scheme. Thus, due to the generic triviality of E and the valuative criterion of properness, there is a closed subscheme $Y \subset \mathrm{Spec}(A)$ of codimension $\geqslant 2$ such that $(E/B)_{\mathrm{Spec}(A) \backslash Y}$ has a section that generically lifts to E, in other words, such that $E_{\mathrm{Spec}(A) \backslash Y}$ reduces to a generically trivial $B_{\mathrm{Spec}(A) \backslash Y}$-torsor E^B. Consider the torus

$$T := B/\mathscr{R}_u(B) \quad \text{and the induced } T_{\mathrm{Spec}(A) \backslash Y}\text{-torsor} \quad E^T := E^B/\mathscr{R}_u(B).$$

Since Y is of codimension $\geqslant 2$ in the regular scheme $\mathrm{Spec}(A)$, by [CTS79, Corollaire 6.9], there is a unique T-torsor $\widetilde{E^T}$ that extends E^T to all of $\mathrm{Spec}(A)$. Since the Grothendieck–Serre conjecture is known for tori [CTS87, Theorem 4.1 (i)] and $\widetilde{E^T}$ is generically trivial, the base change of $\widetilde{E^T}$ to $\mathrm{Spec}(R)$ is trivial. Thus, we may localize A further around the maximal ideals of R to assume that $\widetilde{E^T}$ is trivial, so that E^T is also trivial and $E_{\mathrm{Spec}(A) \backslash Y}$ reduces to an $\mathscr{R}_u(B)$-torsor.

---

[3]We recall from [SGA 3$_{\mathrm{III\ new}}$, Exposé XXIV, Sections 3.8–3.9] that a *quasi-pinning* of a reductive group scheme G is the datum of a Borel subgroup $B \subset G$, a maximal torus $T \subset B$, and, on the scheme $\underline{\mathrm{Dyn}}(G)$ of those T-roots that are simple with respect to B, a trivialization of the line bundle given by the universal root space that is simple with respect to B. A reductive group scheme is *quasi-split* if it admits a quasi-pinning—over a semilocal base this amounts to having a Borel subgroup, but in general it is a more stringent condition.

15

We now apply Proposition 4.1 to obtain

- an affine open $U \subset \mathrm{Spec}(A)$ containing $\mathrm{Spec}(R)$;

- an affine open $S \subset \mathbb{A}_{\mathcal{O}}^{d-1}$; and

- a smooth $\mathcal{O}$-morphism $U \to S$ of pure relative dimension 1 such that $Y \cap U$ is S-finite.

Since R is a localization of the coordinate ring of U, we then set

$$C := U \times_S \mathrm{Spec}(R) \quad \text{and} \quad Z := (Y \cap U) \times_S \mathrm{Spec}(R).$$

The R-scheme C comes equipped with an R-point $s \in C(R)$ induced by the diagonal of $\mathrm{Spec}(R)$ over S, and, by base change, (i)–(iii) hold. Finally, we let $\mathscr{G}$, $\mathscr{B}$, and $\mathscr{E}$ be the base changes to C of the restrictions of G, B, and E to U, so that, by construction, their s-pullbacks are G, B, and E, respectively. Since $U \subset \mathrm{Spec}(A)$ and, by construction, $E_{\mathrm{Spec}(A)\backslash Y}$ reduces to an $\mathscr{R}_u(B)$-torsor, the restriction of $\mathscr{E}$ to $C\backslash Z$ reduces to an $\mathscr{R}_u(\mathscr{B})$-torsor. In particular, (iv) and (v) hold. To likewise arrange the final claim about quasi-pinnings, simply take the fixed extension of $B \subset G$ to a quasi-pinning of G and base change it to C. $\qquad\square$

Remark 4.3. Proposition 4.2 is significantly simpler in the case when $\mathcal{O}$ is a field, in which the assumption that G be quasi-split could be removed with $\mathscr{E}$ in (v) being trivial over $C\backslash Z$. The point is that in this case, even if Y is only of codimension > 0, as is immediate to arrange from the generic triviality of E, one nevertheless gets that Y' in the proof of Proposition 4.1 is of codimension > 1 and the argument goes through without worrying about B and with $\mathscr{E}|_{C\backslash Z}$ even being trivial.

5. Changing the relative curve to equate $\mathscr{G}$ and G_C

To reach our main result on the Grothendieck–Serre conjecture we will gradually simplify the structure exhibited in Proposition 4.2 and eventually reduce to studying G-torsors over $\mathbb{A}_R^1$. As the first step towards this, in Proposition 5.2 we reduce to the case when the C-group scheme $\mathscr{G}$ appearing there is constant, that is, when $\mathscr{G}$ is simply G_C. Although this step could be pursued more generally, here we simplify it by taking advantage of the quasi-splitness assumption. The following lemma is close in spirit to [OP01, Proposition 7.1], [PSV15, Proposition 5.1], or [Pan20b, Theorem 3.4].

Lemma 5.1. *For a Noetherian semilocal ring A whose localizations at prime ideals are geometrically unibranch, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibers have the same type, fixed quasi-pinnings of G and G' extending Borel A-subgroups $B \subset G$ and $B' \subset G'$, and an A/I-group isomorphism*

$$\iota \colon G_{A/I} \xrightarrow{\sim} G'_{A/I} \quad \text{respecting the quasi-pinnings, so that, in particular,} \quad \iota(B_{A/I}) = B'_{A/I},$$

there are

(i) *a faithfully flat, finite, étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a \colon \widetilde{A} \twoheadrightarrow A/I$; and*

(ii) *an $\widetilde{A}$-group isomorphism $\widetilde{\iota} \colon G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ respecting the quasi-pinnings such that $a^*(\widetilde{\iota}) = \iota$.*

Proof. We consider the functor X that parametrizes those isomorphisms between the base changes of G and G' that preserve the (corresponding base changes of the) fixed quasi-pinnings. By [SGA 3$_{\mathrm{III\ new}}$, Exposé XXIV, Corollaire 1.10, Section 3.10], this X is representable by a scheme that becomes constant étale locally on A. Thus, by [SGA 3$_{\mathrm{II}}$, Exposé X, Corollaire 5.14] (with [EGA I, Corollaire 6.1.9]), the geometrically unibranch assumption ensures that the connected components of X are open subschemes that are finite étale over A. The A/I-point ι of X meets finitely many such components, whose union is then the spectrum of a finite étale A-algebra $\widetilde{A}$. By

adjoining further components if needed, we may ensure that the closed A-fibers of $\widetilde{A}$ are nonzero, so that $\widetilde{A}$ is faithfully flat over A, as desired. $\qquad\square$

Proposition 5.2. *For a semilocal Dedekind ring $\mathcal{O}$, the semilocalization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

(i) *a smooth, affine R-scheme C of pure relative dimension 1;*

(ii) *a section $s \in C(R)$;*

(iii) *an R-finite closed subscheme $Z \subset C$;*

(iv) *a G_C-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(B)$-torsor over $C \backslash Z$.*

Proof. We decompose $\mathrm{Spec}(R)$ into connected components to assume that R is a domain. By Proposition 4.2, there are such C, s, Z, and $\mathscr{E}$, except that $\mathscr{E}$ there is a torsor under a quasi-split reductive C-group scheme $\mathscr{G}$ that may not be G_C but that comes equipped with a Borel C-subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$ and that extends to a quasi-pinning of $\mathscr{G}$ whose s-pullback is a fixed quasi-pinning of G extending B. We replace C by its connected component containing the image of s to arrange that C be connected. Thus, the geometric C-fibers of $\mathscr{G}$ and G_C are of constant types, so that, by the condition on the s-pullback, these types are the same.

To replace $\mathscr{B} \subset \mathscr{G}$ by $B_C \subset G_C$, we first use prime avoidance [SP, Lemma 00DS] to construct the semilocalization $\mathrm{Spec}(A)$ of C at the union of the closed points of Z and of those of the image of s. Since C is R-smooth, the ring A is regular. The image of s gives rise to a closed subscheme $\mathrm{Spec}(R) \subset \mathrm{Spec}(A)$ cut out by an ideal $I \subset A$ and, by assumption, $\mathscr{B}_{A/I} \subset \mathscr{G}_{A/I}$ agrees with $B \subset G$, and even the quasi-pinnings of $\mathscr{G}$ and G are likewise compatible. Thus, by Lemma 5.1, there is a finite étale $\mathrm{Spec}(A)$-scheme $\mathrm{Spec}(\widetilde{A})$ equipped with an R-point $\widetilde{s}$ lifting s such that $\mathscr{B}_{\widetilde{A}} \subset \mathscr{G}_{\widetilde{A}}$ is isomorphic to $B_{\widetilde{A}} \subset G_{\widetilde{A}}$ compatibly with the fixed identification of $\widetilde{s}$-pullbacks. We may spread out

$$\mathrm{Spec}(\widetilde{A}) \to \mathrm{Spec}(A) \quad \text{to a finite étale morphism} \quad \widetilde{C} \to C'$$

for some affine open $C' \subset C$ that contains Z and the image of s, while preserving an $\widetilde{s} \in \widetilde{C}(R)$ and an isomorphism between $\mathscr{B}_{\widetilde{C}} \subset \mathscr{G}_{\widetilde{C}}$ and $B_{\widetilde{C}} \subset G_{\widetilde{C}}$. To arrive at the desired conclusion, it then remains to replace C, s, Z, and $\mathscr{E}$ by $\widetilde{C}$, $\widetilde{s}$, $Z \times_C \widetilde{C}$, and $\mathscr{E} \times_C \widetilde{C}$, respectively. $\qquad\square$

6. A Lindel trick in the setting of Cohen–Macaulay relative curves

Having corrected $\mathscr{G}$, our next goal is to reduce to the case when the affine relative curve C is $\mathbb{A}^1_R$. For this, we need a suitable flat morphism $C \to \mathbb{A}^1_R$, whose construction is the goal of this section. We summarize the resulting relevant for us refinement of Proposition 5.2 in Proposition 6.5.

To be able to later use patching to pass from C to $\mathbb{A}^1_R$, it is key to arrange that on some open subscheme $C' \subset C$ containing Z our desired flat map $C \to \mathbb{A}^1_R$ does not change Z in the sense that the latter is precisely the scheme-theoretic preimage in C' of some closed subscheme $Z' \subset \mathbb{A}^1_R$ to which Z maps isomorphically. This is reminiscent of Lindel's insight [Lin81, page 321, Lemma] that led to the resolution of the Bass–Quillen conjecture in the "geometric" case and says that an étale map $B \to A$ of local rings with the same residue field is an isomorphism modulo powers of a well-chosen element in the maximal ideal of B (compare also with [CTO92, Lemme 1.2] or [CT95, Section 3.7 and proof of Theorem 5.1.1]). In our situation, however, there is a basic obstruction to the existence of Z': if some residue fields of R are finite, then Z could have too many rational points to fit into $\mathbb{A}^1_R$.

The purpose of the following crucial statement essentially taken from the literature is to circumvent this obstacle.

Lemma 6.1. *For a semilocal ring R, a quasi-projective, finitely presented R-scheme C, its R-finite closed subscheme Z, and an $s \in Z(R)$, there is a finite morphism $\widetilde{C} \to C$ that is étale at the points in $\widetilde{Z} := Z \times_C \widetilde{C}$ such that s lifts to $\widetilde{s} \in \widetilde{C}(R)$ and, for every maximal ideal $\mathfrak{m} \subset R$, we have*

$$\#\{z \in \widetilde{Z}_{k_\mathfrak{m}} \mid [k_z : k_\mathfrak{m}] = d\} < \#\{z \in \mathbb{A}^1_{k_\mathfrak{m}} \mid [k_z : k_\mathfrak{m}] = d\} \quad \text{for every} \quad d \geqslant 1$$

(a vacuous condition if the residue field $k_\mathfrak{m}$ is infinite).

Lemmas like 6.1 and 6.3 below go back to Panin's [Pan15, Lemma 14.1 and Theorem 11.7].

Proof. The lemma is a variant of, for instance, [Pan19a, Lemma 6.1] or [Fed21b, Lemma 4.5], and we will prove it by using similar arguments as there due to Panin. Since R is semilocal, the finite R-scheme Z has finitely many closed points, which all lie over maximal ideals of R. Thus, we begin by using Lemma 3.1 to construct the semilocalization S of C at the closed points of Z, so that Z is also a closed subscheme of S and $s \in S(R)$. It then suffices to construct a finite étale S-scheme $\widetilde{S}$ such that s lifts to an R-point $\widetilde{s} \in \widetilde{S}(R)$ and the preimage $\widetilde{Z} \subset \widetilde{S}$ of Z satisfies the displayed inequalities: indeed, once this is done, we may first spread $\widetilde{S}$ out to a finite étale scheme over an open neighborhood of S in C and then form its schematic image [SP, Lemma 01R8] in the factorization supplied by the Zariski Main Theorem [EGA IV$_4$, Corollaire 18.12.13] to further extend to a desired finite morphism $\widetilde{C} \to C$.

We view s as a closed subscheme $\operatorname{Spec}(R) \subset Z$ and we list the closed points of Z (that is, of S):

- the closed points $y_1, \ldots, y_m$ of Z not in s with an infinite residue field;
- the closed points $z_1, \ldots, z_n$ of Z not in s with a finite residue field;
- the closed points $y'_1, \ldots, y'_{m'}$ of s with an infinite residue field;
- the closed points $z'_1, \ldots, z'_{n'}$ of s with a finite residue field.

For any $N > 1$, we may choose monic polynomials

- $f_{y_i} \in k_{y_i}[t]$ that are products of N distinct linear factors; and
- $f_{z_j} \in k_{z_j}[t]$ that are irreducible of degree N.

Likewise, we may choose a monic polynomial $f_s \in tR[t]$ of degree N such that

- the image of f_s in each $k_{y'_i}[t]$ is a product of N distinct linear factors; and
- the image of f_s in each $k_{z'_j}[t]$ is a product of t and an irreducible polynomial not equal to t.

Finally, since $s \sqcup \bigsqcup_{i=1}^m y_i \sqcup \bigsqcup_{j=1}^n z_j$ is a closed subscheme of S, by lifting coefficients we may choose a monic polynomial $f \in \Gamma(S, \mathscr{O}_S)[t]$ of degree N that restricts to f_{y_i} on each y_i, to f_{z_j} on each z_j, and to f_s on s. This f defines a finite étale S-scheme $\widetilde{S}$, which, by construction, is equipped with an R-point $\widetilde{s} \in \widetilde{S}(R)$ lifting s (cut out by the factor t of f_s) and is such that the number of closed points with finite residue fields in the preimage $\widetilde{Z} \subset \widetilde{S}$ of Z stays bounded as N grows but, except for the points in $\widetilde{s}$, the cardinalities of these residue fields grow uniformly. Thus, since, for a finite field $\mathbb{F}$, the number of closed points of $\mathbb{A}^1_\mathbb{F}$ with a given residue field grows unboundedly together with the degree of that residue field over $\mathbb{F}$, for large N our $\widetilde{S}$ meets the requirements. $\square$

We turn to the Lindel trick in our setting, namely, to building the desired flat map $C \to \mathbb{A}_R^1$ in Lemma 6.3. Its numerous variants appeared in works of Panin, for instance, in [OP99, Section 5], [PSV15, Theorem 3.4], or [Pan19a, Theorem 3.8], but with the more stringent smoothness assumption on C, and preparation lemmas of similar flavor can be traced back at least to [Gab94, Lemma 3.1] (compare also with [CTHK97, Theorem 3.1.1]). As we show, Cohen–Macaulayness of C suffices. The argument uses the following simple lemma that characterizes residue fields of closed points on smooth curves.

Lemma 6.2. *For a field k, a smooth connected k-curve C, and a closed point $c \in C$, the extension k_c/k is generated by a single element, that is, k_c is the residue field of a closed point of $\mathbb{A}_k^1$.*

Proof. By [EGA IV$_4$, Corollaire 17.11.4], an open neighborhood $U \subset C$ of c has an étale k-morphism

$$U \to \mathbb{A}_k^1.$$

Thus, there is a subextension ℓ/k of k_c/k generated by a single element with k_c/ℓ separable. By the primitive element theorem, we need to check that this forces k_c/k to only have finitely many subextensions k'/k. Since there are finitely many possibilities for $k' \cap \ell$, we replace k by $k' \cap \ell$ to reduce to considering those k' for which $k' \cap \ell = k$. Like any finite separable extension, the separable closure of k in k_c has only finitely many subextensions. Thus, there are finitely many possibilities for the maximal separable subextension k''/k of k'/k. By replacing k by k'' and ℓ by $k''\ell$, we therefore reduce to the case when k'/k is purely inseparable. Then the subextension $k'\ell/\ell$ of the separable extension k_c/ℓ is also purely inseparable, to the effect that $k' \subset \ell$. However, ℓ/k is generated by a single element, so, by the primitive element theorem, it has only finitely many subextensions. $\square$

Lemma 6.3. *For*

- *a semilocal ring R;*

- *a flat, affine R-scheme C with Cohen–Macaulay fibers of pure dimension 1;*

- *R-finite closed subschemes $Y \subset C$ and $Z \subset C^{\mathrm{sm}}$ such that, for every maximal ideal $\mathfrak{m} \subset R$,*

$$\#\{z \in Z_{k_{\mathfrak{m}}} \mid [k_z : k_{\mathfrak{m}}] = d\} < \#\{z \in \mathbb{A}_{k_{\mathfrak{m}}}^1 \mid [k_z : k_{\mathfrak{m}}] = d\} \quad \text{for every} \quad d \geqslant 1$$

(a vacuous condition if the residue field $k_{\mathfrak{m}}$ is infinite)[4];

there are

(i) *an affine open $C' \subset C$ containing Y and Z;*

(ii) *a quasi-finite, flat R-map $C' \to \mathbb{A}_R^1$ that maps Z isomorphically onto a closed subscheme*

$$Z' \subset \mathbb{A}_R^1 \quad \text{such that} \quad Z \cong Z' \times_{\mathbb{A}_R^1} C';$$

so that, in particular, $C' \to \mathbb{A}_R^1$ is étale along Z and, for every $n \geqslant 0$, maps the n-th infinitesimal neighborhood of Z in C' isomorphically to the n-th infinitesimal neighborhood of Z' in $\mathbb{A}_R^1$.

Proof.[5] The étaleness follows from the flatness and the isomorphy over Z' of the map $C' \to \mathbb{A}_R^1$, and it implies the infinitesimal neighborhood aspect. For the rest, every closed point $z \in Z$ lies over

---

[4]If every closed point of Y lies on Z, then it suffices to require the same with nonstrict inequalities $\leqslant$ instead.

[5]We loosely follow [Pan19a, proof of Theorem 3.8], with several improvements and simplifications whose purpose is to avoid assuming that C be R-smooth or that R be the semilocal ring at finitely many closed points of a smooth variety over a field. Notably, in Remark 6.4 we give a more direct and more general argument for the final portion of *loc. cit.*

some maximal ideal $\mathfrak{m} \subset R$ and, since $z \in C_{k_\mathfrak{m}}^{\mathrm{sm}}$, the ideal sheaf $\mathscr{I}_z \subset \mathscr{O}_{C_{k_\mathfrak{m}}}$ is generated at z by a uniformizer $u_z \in \mathscr{O}_{C_{k_\mathfrak{m}}, z}$. Consequently, by [BouAC, Chapitre IX, Section 3, numéro 3, Théorème 1], the thickening[†]

$$\varepsilon_z := \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{k_\mathfrak{m}}}}(\mathscr{O}_{C_{k_\mathfrak{m}}}/\mathscr{I}_z^2) \quad \text{is isomorphic to} \quad \mathrm{Spec}(k_z[u_z]/(u_z^2)).$$

Letting y range over the closed points of Y not in Z and z range over the closed points of Z, we set

$$\varepsilon_Y := \bigsqcup\nolimits_y y \subset C, \quad \varepsilon_Z := \bigsqcup\nolimits_z \varepsilon_z \subset C, \quad \text{and} \quad \varepsilon := \varepsilon_Y \sqcup \varepsilon_Z = \bigsqcup\nolimits_y y \sqcup \bigsqcup\nolimits_z \varepsilon_z \subset C.$$

By Lemma 6.2 and the assumption on the numbers of points of $Z_{k_\mathfrak{m}}$, we may find an R-morphism

$$j \colon \varepsilon \to \bigsqcup\nolimits_\mathfrak{m} \mathbb{A}^1_{k_\mathfrak{m}} \subset \mathbb{A}^1_R \quad \text{that restricts to a closed immersion} \quad \varepsilon_Z \hookrightarrow \mathbb{A}^1_R$$

and, for each $\mathfrak{m}$, maps the points of ε_Y above $\mathfrak{m}$ to an $k_\mathfrak{m}$-point of $\mathbb{A}^1_{k_\mathfrak{m}} \backslash \varepsilon_Z$. We fix two disjoint sets of closed points

$$d_1, \ldots, d_n \in C \backslash (Y \cup Z) \quad \text{and} \quad d'_1, \ldots, d'_{n'} \in C \backslash (Y \cup Z)$$

lying over maximal ideals of R such that each set jointly meets every irreducible component of every closed R-fiber of C. Since $\varepsilon \cup \bigsqcup_{i=1}^n d_i \cup \bigsqcup_{i=1}^{n'} d'_i$ is a closed subscheme of C, we may find an $s \in \Gamma(C, \mathscr{O}_C)$ that

- vanishes on every d_i but does not vanish on any d'_i;
- on ε equals the j-pullback of the coordinate of $\mathbb{A}^1_R$.

By mapping the coordinate of $\mathbb{A}^1_R$ to s, we obtain an R-morphism

$$\pi \colon C \to \mathbb{A}^1_R.$$

The behavior of s at d_i and d'_i ensures that the locus where π is quasi-finite, which, by [SP, Lemma 01TI], is an open of C, contains every closed R-fiber of C. In particular, we may use prime avoidance [SP, Lemma 00DS] to replace C by some affine open subset containing Y and Z (equivalently, containing the closed points of Y and Z) to arrange that π is quasi-finite.

Since C is R-flat with Cohen–Macaulay fibers of pure dimension 1, the flatness criteria [EGA IV$_2$, Proposition 6.1.5], [EGA IV$_3$, Corollaire 11.3.11] ensure that π is flat. By construction $\pi|_\varepsilon = j$, so, by checking on the closed R-fibers, [EGA IV$_4$, Théorème 17.11.1] shows that π is étale around Z. Since $Z_{k_\mathfrak{m}}$ and ε_Z have the same underlying reduced subscheme $\bigsqcup_z z$, the agreement with j also shows that $\pi|_{Z_{k_\mathfrak{m}}}$ is a closed immersion. Since Z is R-finite, Nakayama lemma [SP, Lemma 00DV] then ensures that $\pi|_Z$ is also a closed immersion, so that π maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_R$.

A section of a separated, étale morphism is an isomorphism onto a clopen subscheme, so the étaleness of π around Z gives a decomposition

$$\pi^{-1}(Z') = Z \sqcup Z''$$

for some R-quasi-finite closed subscheme $Z'' \subset C$. By the agreement with j, the image under π of every closed point of Y not in Z does not lie in Z', so that $Y \cap Z'' = \varnothing$. Thus, prime avoidance

---

[†]*Added after publication.* As Gabber pointed out, the isomorphism $\varepsilon_z \simeq \mathrm{Spec}(k_z[u_z]/(u_z^2))$ deduced here from the Cohen structure theorem is *a priori* not an isomorphism of $k_\mathfrak{m}$-schemes unless the field $k_\mathfrak{m}$ is perfect (in which case $k_z/k_\mathfrak{m}$ is a finite separable extension and the claimed $k_\mathfrak{m}$-scheme isomorphism also follows from [EGA IV$_4$, Proposition 17.5.3] combined with the insensitivity of the étale site to nilpotents). This does not affect the subsequent argument because, for imperfect $k_\mathfrak{m}$, we may nevertheless construct the closed immersions $\varepsilon_z \hookrightarrow \mathbb{A}^1_{k_\mathfrak{m}}$ comprising j by applying the geometric presentation theorem [CTHK97, Theorem 3.1.1] to each $\varepsilon_z \subset C_{k_\mathfrak{m}}^{\mathrm{sm}}$ and postcomposing with suitable changes of coordinates $t \mapsto t + \alpha$ of $\mathbb{A}^1_{k_\mathfrak{m}}$ for $\alpha \in k_\mathfrak{m}$ to make the resulting closed immersions $\varepsilon_z \hookrightarrow \mathbb{A}^1_{k_\mathfrak{m}}$ have disjoint images.

[SP, Lemma 00DS] supplies a global section of C that vanishes on Z'' but does not vanish at any closed point of Y or Z. By inverting this section, we obtain the desired affine open $C' \subset C$. $\qquad\square$

Remark 6.4. If $\mathrm{Spec}(R)$ is connected, then any R-(finite locally free) closed subscheme $Z' \subset \mathbb{A}^1_R$ is cut out by a monic polynomial. This holds for any ring R with a connected spectrum: the coordinate t of $\mathbb{A}^1_R$ acts by multiplication on the projective R-module $\Gamma(Z', \mathscr{O}_{Z'})$, the characteristic polynomial of this action is monic and cuts out an R-(finite locally free) closed subscheme $H \subset \mathbb{A}^1_R$, and Cayley–Hamilton implies that $Z' \subset H$ inside $\mathbb{A}^1_R$, so, by comparing ranks over R, even $Z' = H$.

We now refine Proposition 5.2 to the following statement adapted to passing to $\mathbb{A}^1_R$ via patching.

Proposition 6.5. *For a semilocal Dedekind ring $\mathcal{O}$, the localization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

    (i) *a smooth, affine R-scheme C of pure relative dimension 1;*

    (ii) *a section $s \in C(R)$;*

    (iii) *an R-finite closed subscheme $Z \subset C$;*

    (iv) *a G_C-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to a $\mathscr{R}_u(B)$-torsor over $C \backslash Z$;*

    (v) *a flat R-map $C \to \mathbb{A}^1_R$ that maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_R$ with*

$$Z \cong Z' \times_{\mathbb{A}^1_R} C.$$

Proof. Proposition 5.2 supplies C, s, Z, and $\mathscr{E}$ that satisfy the present (i)–(iv). We view s as a closed subscheme of C and we apply Lemma 6.1 to the R-finite closed subscheme $(Z \cup s)^{\mathrm{red}}$ of C to see that we may change C to assume that, in addition, for every maximal ideal $\mathfrak{m} \subset R$,

$$\#\{z \in Z_{k_\mathfrak{m}} \mid [k_z : k_\mathfrak{m}] = d\} < \#\{z \in \mathbb{A}^1_{k_\mathfrak{m}} \mid [k_z : k_\mathfrak{m}] = d\} \quad \text{for every} \quad d \geqslant 1.$$

This allows us to apply Lemma 6.3 with $Y = s$ to shrink C and to arrange (v). $\qquad\square$

7. Descending to $\mathbb{A}^1_R$ via patching

With the suitable flat map $C \to \mathbb{A}^1_R$ already built in Proposition 6.5, descending the G_C-torsor $\mathscr{E}_C$ to $\mathbb{A}^1_R$ concerns patching along the closed subscheme Z. Since our Z need not be cut out by a single equation (relatedly, $C \backslash Z$ need not be affine), this patching is slightly more delicate than its most frequently encountered instances. Its precise statement is captured by the following lemma, which follows from more general results of Moret-Bailly [MB96] (for our purposes, we could also get by with the more basic patching of Ferrand–Raynaud [FR70, Proposition 4.2]).

Lemma 7.1. *Let $S' \to S$ be an affine, flat scheme map whose base change to a closed subscheme $Z \subset S$ cut out by a quasi-coherent ideal sheaf of finite type is an isomorphism and let $U' \to U$ be the base change to $U := S \backslash Z$. For a quasi-affine, flat, finitely presented S-group scheme G, base change induces an equivalence from the category of G-torsors to the category of triples consisting of a $G_{S'}$-torsor, a G_U-torsor, and a $G_{U'}$-torsor isomorphism between the two base changes to U'.*

Of course, the isomorphism condition $Z \times_S S' \xrightarrow{\sim} Z$ ensures that S' and U jointly cover S.

Proof. By [SP, Theorem 06FI], the classifying S-stack $\mathbf{B}G$ is algebraic and, by descent, its diagonal inherits quasi-affineness from G. Thus, the claim is a special case of [MB96, Corollaire 6.5.1 (a)]. $\qquad\square$

To be able to apply Lemma 7.1 in our setting, we need to descend $\mathscr{E}_{U \setminus Z}$ to a G-torsor over $\mathbb{A}_R^1 \setminus Z'$. To achieve this, we will use the following excision result that is similar (but simpler) than its counterparts that recently appeared in [ČS21, Theorem 5.4.4] and in [BČ21, Section 2.3].

Lemma 7.2. *Let $S' \to S$ be a flat morphism of affine, Noetherian schemes whose base change to a closed subscheme $Z \subset S$ is an isomorphism, and let $U' \to U$ be the base change to $U := S \setminus Z$.*

(a) *For a quasi-coherent $\mathscr{O}_S$-module $\mathscr{F}$ (or even a complex of such $\mathscr{O}_S$-modules), we have*
$$R\Gamma_Z(S, \mathscr{F}) \xrightarrow{\sim} R\Gamma_Z(S', \mathscr{F}_{S'}).$$

(b) *For an affine, smooth S-group (resp., U-group) F with a filtration*
$$F = F_0 \supset F_1 \supset \ldots \supset F_n = 0$$
by normal, affine, smooth S-subgroups (resp., U-subgroups) such that, for all $i \geqslant 0$, the quotient F_i/F_{i+1} is a vector group associated to a vector bundle on S (resp., such that the vector group F_i/F_{i+1} is also central in F/F_{i+1}), the map
$$H^1(U, F) \to H^1(U', F) \quad \text{has trivial kernel (resp., is surjective).}$$

Proof.

(a) We let A and A' be the coordinate rings of S and S', respectively. By [SP, Lemmas 0ALZ and 0955],
$$R\Gamma_Z(S, \mathscr{F}) \otimes_A^{\mathbb{L}} A' \xrightarrow{\sim} R\Gamma_Z(S', \mathscr{F}_{S'}).$$
Thus, since A' is A-flat, to obtain (a) it remains to note that, by [SP, Lemma 05E9], we have
$$H_Z^i(S, \mathscr{F}) \xrightarrow{\sim} H_Z^i(S, \mathscr{F}) \otimes_A A' \quad \text{for all} \quad i \in \mathbb{Z}.$$

(b) In the case when F is an S-group, the vanishing of quasi-coherent cohomology of affine schemes and the assumed filtration show that both $H^1(S, F)$ and $H^1(S', F)$ vanish. Thus, the assertion about the kernel simply amounts to the claim that every F_U-torsor that trivializes over U' extends to an F-torsor. This, however, is immediate from Lemma 7.1.

For the surjectivity assertion, we will induct on n. We begin with the case $n = 1$, in which F itself is the vector group associated to some vector bundle $\mathscr{F}$ on U. By applying (a) to $j_*(\mathscr{F})$, where $j \colon U \hookrightarrow S$ is the indicated open immersion, and again using the vanishing of quasi-coherent cohomology of affine schemes, we find that, for all $i \geqslant 1$, even
$$H^i(U, F) \cong H^i(U, \mathscr{F}) \cong H_Z^{i+1}(S, j_*(\mathscr{F})) \xrightarrow[\sim]{(a)} H_Z^{i+1}(S', j_*(\mathscr{F})) \cong H^i(U', \mathscr{F}_{U'}) \cong H^i(U', F).$$

For the inductive step, we assume that $n > 1$ and combine the inductive hypothesis, the preceding display for F_{n-1}, and the nonabelian cohomology sequences [Gir71, Chapitre IV, Remarque 4.2.10] of a central extension to obtain the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
H^1(U, F_{n-1}) & \longrightarrow & H^1(U, F) & \longrightarrow & H^1(U, F/F_{n-1}) & \longrightarrow & H^2(U, F_{n-1}) \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} \\
H^1(U', F_{n-1}) & \longrightarrow & H^1(U', F) & \longrightarrow & H^1(U', F/F_{n-1}) & \longrightarrow & H^2(U', F_{n-1}).
\end{array}
$$

We fix an $\alpha' \in H^1(U', F)$ that we wish to lift to $H^1(U, F)$ and note that, by a diagram chase, there at least is an $\alpha \in H^1(U, F)$ whose image in $H^1(U', F/F_{n-1})$ agrees with that of α'. Every inner fpqc form of F inherits an analogous filtration, even with the same subquotients

22

F_i/F_{i+1}, so the change of origin bijections [Gir71, Chapitre III, Proposition 2.6.1 (i)] allow us to twist F and reduce to the case when the common image of α and α' in $H^1(U', F/F_{n-1})$ vanishes. In this case, however, the surjectivity of the left vertical arrow suffices. $\qquad\square$

Example 7.3. For example, F in Lemma 7.2 (b) could be the unipotent radical $\mathscr{R}_u(P)$ of a parabolic S-subgroup (resp., U-subgroup) P of a reductive S-group (resp., U-group) G: in this case, [SGA 3$_{\text{III new}}$, Exposé XXVI, Proposition 2.1] supplies the desired filtration.

We can now reduce to the case when the relative curve C in Proposition 6.5 is $\mathbb{A}^1_R$.

Proposition 7.4. *For a semilocal Dedekind ring $\mathcal{O}$, the semilocalization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$, a quasi-split reductive R-group G, and a generically trivial G-torsor E, there are*

   (i) *a closed subscheme $Z \subset \mathbb{A}^1_R$ that is finite over R;*

   (ii) *a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ whose pullback along the zero section is E such that $\mathscr{E}$ is trivial over $\mathbb{A}^1_R\backslash Z$.*

Proof. Let $B \subset G$ be a Borel R-subgroup. Proposition 6.5 supplies a quasi-finite, affine, flat R-morphism $\pi\colon C \to \mathbb{A}^1_R$ whose base change to an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ (called Z' there) is an isomorphism, as well as an $s \in C(R)$ and a G_C-torsor $\widetilde{\mathscr{E}}$ (called $\mathscr{E}$ there) with s-pullback E such that $\widetilde{\mathscr{E}}$ reduces to a $\mathscr{R}_u(B)$-torsor over $C\backslash\pi^{-1}(Z)$. By Lemma 7.2 (b) and Example 7.3, this $\mathscr{R}_u(B)$-torsor descends to a $\mathscr{R}_u(B)$-torsor over $\mathbb{A}^1_R\backslash Z$, so $\widetilde{\mathscr{E}}_{C\backslash\pi^{-1}(Z)}$ descends to a $G_{\mathbb{A}^1_R\backslash Z}$-torsor. The patching lemma 7.1 then ensures that $\widetilde{\mathscr{E}}$ itself descends to a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that reduces to a $\mathscr{R}_u(B)$-torsor over $\mathbb{A}^1_R\backslash Z$. By postcomposing with a change of coordinate automorphism of $\mathbb{A}^1_R$ to ensure that s map to the zero section of $\mathbb{A}^1_R$, we make the pullback of $\mathscr{E}$ along the zero section be E. Finally, we apply Lemma 3.1 to $\mathbb{P}^1_R$ to enlarge $Z \subset \mathbb{A}^1_R$ to be a hypersurface in $\mathbb{P}^1_R$. This ensures that $\mathbb{A}^1_R\backslash Z$ is affine, so that, due to the filtration of $\mathscr{R}_u(B)$ by vector groups as in Example 7.3 and the vanishing of quasi-coherent cohomology of affine schemes, our $\mathscr{E}_{\mathbb{A}^1_R\backslash Z}$ is even trivial. $\qquad\square$

8. The analysis of torsors over $\mathbb{A}^1_R$

Our final task is to study generically trivial torsors over $\mathbb{A}^1_R$, which may be viewed as a problem of Bass–Quillen type beyond the case of vector bundles (that is, beyond GL_n-torsors). Although one may carry it out much more generally, this analysis is significantly simpler for semisimple, simply-connected, totally isotropic reductive groups G. Total isotropicity, defined in Definition 8.1, is not quite unexpected, for instance, it also seems necessary for generalizing the Bass–Quillen conjecture to G-torsors, see [AHW18, Theorem 3.3.7] and [AHW20, Theorem 2.4] for such results when G comes from a ground field.

Definition 8.1. A reductive group scheme G over a scheme S is *totally isotropic* if for every $s \in S$ and the canonical decomposition

$$G^{\mathrm{ad}}_{\mathcal{O}_{S,s}} \cong \prod_i G_i$$

from [SGA 3$_{\text{III new}}$, Exposé XXIV, Proposition 5.10 (i)] of $G^{\mathrm{ad}}_{\mathcal{O}_{S,s}}$ into the product of Weil restrictions G_i of adjoint groups with simple geometric fibers over connected finite étale covers of $\mathcal{O}_{S,s}$, each G_i is isotropic in the sense that it contains a nontrivial split torus $\mathbb{G}_{m,\mathcal{O}_{S,s}}$ (equivalently, each G_i has a proper parabolic $\mathcal{O}_{S,s}$-subgroup, see [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 6.12]).

Example 8.2. The parenthetical reformulation implies that every quasi-split G is totally isotropic.

By the following lemma, which generalizes the main result of [Tsy19], for analyzing a torsor over $\mathbb{A}^1_R$ the key is to extend it to a torsor over $\mathbb{P}^1_R$ in such a way that the latter be trivial on closed R-fibers.

Lemma 8.3. *For a semilocal ring R and a reductive R-group G such that $\mathrm{rad}(G)$ splits over a finite étale cover of $\mathrm{Spec}(R)$ (a vacuous condition if R is normal, or if G is split, or if $\mathrm{rad}(G)$ is of rank $\leqslant 1$, for instance, if G is semisimple), every $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ whose base change to $\mathbb{P}^1_{k_{\mathfrak{m}}}$ is trivial for every maximal ideal $\mathfrak{m} \subset R$ is the base change of a G-torsor.*

Proof. By [Gil21, Theorem 1.1 and Corollary 4.3], the assumption on $\mathrm{rad}(G)$ is equivalent to requiring that G may be embedded as a closed subgroup of some $\mathrm{GL}_{n,R}$ and it holds in the indicated parenthetical cases. For the rest, we first use a limit argument to reduce to Noetherian R and then pass to connected components to also assume that $\mathrm{Spec}(R)$ is connected. Moreover, we begin with the case $G = \mathrm{GL}_{n,R}$, in which we may regard $\mathscr{E}$ as a vector bundle of rank n.

In this vector bundle case,

$$\mathscr{V} := \mathscr{H}om_{\mathscr{O}_{\mathbb{P}^1_R}}(\mathscr{O}^{\oplus n}_{\mathbb{P}^1_R}, \mathscr{E}) \cong \mathscr{E}^{\oplus n}$$

is also a vector bundle on $\mathbb{P}^1_R$. By [EGA III$_1$, Théorème 3.2.1], the R-module $V := \Gamma(\mathbb{P}^1_R, \mathscr{V})$ is finite. By assumption, $\mathscr{E}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ is trivial for every maximal ideal $\mathfrak{m} \subset R$, so for such $\mathfrak{m}$ we choose an isomorphism

$$\mathscr{O}^{\oplus n}_{\mathbb{P}^1_{k_{\mathfrak{m}}}} \xrightarrow{\sim} \mathscr{E}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}, \quad \text{which corresponds to some} \quad v_{\mathfrak{m}} \in \Gamma(\mathbb{P}^1_{k_{\mathfrak{m}}}, \mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}).$$

Likewise, each $\mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ is trivial, so $H^1(\mathbb{P}^1_{k_{\mathfrak{m}}}, \mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}) \cong 0$. Thus, by cohomology and base change [EGA III$_1$, Proposition 4.6.1], there is a $\widetilde{v}_{\mathfrak{m}} \in V/\mathfrak{m}V$ that maps to $v_{\mathfrak{m}}$. Since R is semilocal and $\mathfrak{m}$ ranges over its maximal ideals, we may then find a $v \in V$ that maps to all the $\widetilde{v}_{\mathfrak{m}}$, so also to all the $v_{\mathfrak{m}}$. By construction and the Nakayama lemma [SP, Lemma 00DV], the $\mathscr{O}_{\mathbb{P}^1_R}$-module homomorphism

$$\mathscr{O}^{\oplus n}_{\mathbb{P}^1_R} \to \mathscr{E}$$

corresponding to v is surjective at every closed point, so it is surjective. Cayley–Hamilton [SP, Lemma 05G8] then ensures that this homomorphism is an isomorphism, so $\mathscr{E}$ is trivial, as desired.

To deduce the general case, we use our closed embedding $G \hookrightarrow \mathrm{GL}_{n,R}$. Namely, the settled case of $\mathrm{GL}_{n,R}$ and the nonabelian cohomology sequence [Gir71, Chapitre III, Proposition 3.2.2] show that our $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ comes from a some $\mathbb{P}^1_R$-point of $\mathrm{GL}_{n,R}/G$. However, G is reductive, so, by [Alp14, Theorem 9.4.1 and Theorem 9.7.5], this quotient is affine. By [MFK94, Proposition 6.1] (to reduce to an R-fiber), this means that the only R-morphisms from $\mathbb{P}^1_R$ to $\mathrm{GL}_{n,R}/G$ are constant, in particular, that our $\mathbb{P}^1_R$-point comes from an R-point. This then implies that our $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ is the base change of a G-torsor, as desired. $\square$

The preceding lemma leads to the triviality of generically trivial reductive group torsors over $\mathbb{A}^1_R$ as follows. Examples from [Fed16] show that without some isotropicity condition on G such triviality does not hold.

Proposition 8.4. *For a semilocal ring R and a totally isotropic, semisimple, simply connected reductive R-group G, every $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ is trivial.*

Proof. Due to the canonical decomposition [SGA 3$_{\text{III new}}$, Exposé XXIV, Sections 5.2–5.3, Proposition 5.10 (i)], we may assume that

$$G \cong \text{Res}_{R'/R} G'$$

for a finite étale R-algebra R' and a semisimple, simply-connected R'-group G' whose geometric R'-fibers are simple (in the sense that the Dynkin diagrams of these geometric fibers are connected). By [SGA 3$_{\text{III new}}$, Exposé XXIV, Proposition 8.4], we may then replace R and G by R' and G', respectively, to assume that the geometric R-fibers of G are simple.

We let t be the inverse of the coordinate on $\mathbb{A}^1_R$ and consider $R[\![t]\!]$ as the completion of $\mathbb{P}^1_R$ along infinity. Due to its R-finiteness, Z is closed in $\mathbb{P}^1_R$, so its pullback to $\text{Spec}(R[\![t]\!])$ is also closed and hence is even empty because it does not meet the locus $\{t = 0\}$. Thus, we may use formal glueing supplied by, for instance, [BČ21, Lemma 2.2.11 (b)] (or Lemma 7.1 when R is Noetherian) to extend $\mathscr{E}$ to a $G_{\mathbb{P}^1_R}$-torsor $\overline{\mathscr{E}}$ by glueing $\mathscr{E}$ with the trivial $G_{R[\![t]\!]}$-torsor. It suffices to argue that we can glue like this so that $\overline{\mathscr{E}}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ be trivial for every maximal ideal $\mathfrak{m} \subset R$: Lemma 8.3 will then imply that $\overline{\mathscr{E}}$ is the base change of its pullback by the section at infinity, and hence that $\overline{\mathscr{E}}$ and $\mathscr{E}$ are trivial.

Explicitly, the elements of $G(R(\!(t)\!))/G(R[\![t]\!])$ give rise to all the possible glueings of $\mathscr{E}$ and the trivial $G_{R[\![t]\!]}$-torsor to a $G_{\mathbb{P}^1_R}$-torsor, and likewise over the residue fields $k_{\mathfrak{m}}$ of R. We will first build a trivial $G_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$-bundle $\overline{\mathscr{E}}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ from $\mathscr{E}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ by such a glueing for every maximal ideal $\mathfrak{m} \subset R$ and then argue that these glueings come from a single glueing over R. These steps reduce, respectively, to the following:

(1) for every maximal ideal $\mathfrak{m} \subset R$, the $G_{\mathbb{A}^1_{k_{\mathfrak{m}}}}$-torsor $\mathscr{E}_{\mathbb{A}^1_{k_{\mathfrak{m}}}}$ is trivial;

(2) letting $\mathfrak{m} \subset R$ range over all the maximal ideals, the following map is surjective:

$$G(R(\!(t)\!))/G(R[\![t]\!]) \twoheadrightarrow \prod_{\mathfrak{m}} G(k_{\mathfrak{m}}(\!(t)\!))/G(k_{\mathfrak{m}}[\![t]\!]).$$

For (1), since $\mathscr{E}_{\mathbb{A}^1_{k_{\mathfrak{m}}}}$ is trivial away from $Z_{k_{\mathfrak{m}}}$, we may glue it arbitrarily with the trivial $G_{k_{\mathfrak{m}}[\![t]\!]}$-torsor to obtain a $G_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$-torsor whose pullback along the infinity section is trivial. By [Gil02, Lemme 3.12] (see also [Gil05]), such torsors are trivial over $\mathbb{A}^1_{k_{\mathfrak{m}}}$, so $\mathscr{E}_{\mathbb{A}^1_{k_{\mathfrak{m}}}}$ is trivial, that is, (1) holds.

For the claim (2), we first note that the isotropicity assumption implies that G has a proper parabolic subgroup $P \subset G$ (see Definition 8.1). By [Gil09, Fait 4.3, Lemme 4.5] (this is where we use the assumptions on G), the Whitehead group of the base changes of G is of unramified nature, that is,

$$G(k_{\mathfrak{m}}(\!(t)\!)) = G(k_{\mathfrak{m}}(\!(t)\!))^+ G(k_{\mathfrak{m}}[\![t]\!]) \quad \text{for every maximal ideal} \quad \mathfrak{m} \subset R,$$

where $G(k_{\mathfrak{m}}(\!(t)\!))^+ \subset G(k_{\mathfrak{m}}(\!(t)\!))$ is the subgroup generated by $(\mathscr{R}_u(P))(k_{\mathfrak{m}}(\!(t)\!))$ and $(\mathscr{R}_u(P^-))(k_{\mathfrak{m}}(\!(t)\!))$ with $P^- \subset G$ being a parabolic opposite to P in the sense of [SGA 3$_{\text{III new}}$, Exposé XXVI, Définition 4.3.3, Corollaire 4.3.5 (i)]. To conclude (2), it suffices to show that the following pullback maps are surjective:

$$(\mathscr{R}_u(P))(R(\!(t)\!)) \twoheadrightarrow \prod_{\mathfrak{m}}(\mathscr{R}_u(P))(k_{\mathfrak{m}}(\!(t)\!)) \quad \text{and} \quad (\mathscr{R}_u(P^-))(R(\!(t)\!)) \twoheadrightarrow \prod_{\mathfrak{m}}(\mathscr{R}_u(P^-))(k_{\mathfrak{m}}(\!(t)\!)).$$

For this, we combine the surjectivity of the map $R(\!(t)\!) \twoheadrightarrow \prod_{\mathfrak{m}} k(\!(t)\!)$ with [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 2.5], according to which both $\mathscr{R}_u(P)$ and $\mathscr{R}_u(P^-)$ are isomorphic to affine spaces $\mathbb{A}^d_R$. $\square$

Remark 8.5. One difference between Proposition 8.4 and some of its versions in the literature is that we work directly with the $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ instead of first glueing it arbitrarily to a $G_{\mathbb{P}^1_R}$-torsor and then modifying this extension. Ultimately, this is an expository point, but it highlights that in (2) there is no need to pursue the analogous surjectivity before taking the quotients.

9. The quasi-split unramified case of the Grothendieck–Serre conjecture

We turn to the quasi-split case of the Grothendieck–Serre conjecture 1.1: we are ready to settle its semilocal version over unramified regular local rings. By choosing $\mathcal{O}$ to be either $\mathbb{Z}$, or $\mathbb{Q}$, or $\mathbb{F}_p$ for some prime p and R to be local, this recovers the first assertion in Theorem 1.2, see Example 9.2.

Theorem 9.1. *For a Dedekind ring $\mathcal{O}$, a semilocal, regular, flat $\mathcal{O}$-algebra R whose $\mathcal{O}$-fibers are geometrically regular,[6] and a quasi-split reductive R-group G, no nontrivial G-torsor trivializes over the total ring of fractions $K := \mathrm{Frac}(R)$ of R, that is,*

$$\mathrm{Ker}(H^1(R, G) \to H^1(K, G)) = \{*\}.$$

Proof. We pass to connected components to assume that $\mathrm{Spec}(R)$ is connected, so that R is a domain and, in particular, $R \neq 0$. We use Proposition 2.2 to replace G by $(G^{\mathrm{der}})^{\mathrm{sc}}$, and hence reduce to the case when our quasi-split G is also semisimple and simply connected.

Let E be a G-torsor that trivializes over K, so also over $R[\frac{1}{r}]$ for some $r \in R\backslash\{0\}$. By Popescu theorem [SP, Theorem 07GC], the ring R is a filtered direct limit of smooth $\mathcal{O}$-algebras. Thus, a limit argument allows us to assume that R is the semilocalization of a smooth $\mathcal{O}$-algebra at finitely many primes. In this case, Proposition 7.4 gives a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ whose pullback along the zero section is E such that $\mathscr{E}$ is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$. By Proposition 8.4 (with Example 8.2), this $\mathscr{E}$ is trivial, so E is also trivial, as desired. $\square$

Example 9.2. In the case when $\mathcal{O}$ is a perfect field, such as $\mathbb{Q}$ or $\mathbb{F}_p$, any regular $\mathcal{O}$-algebra is geometrically regular, so, for quasi-split G, Theorem 9.1 simultaneously reproves the equicharacteristic case of the Grothendieck–Serre conjecture settled in [FP15] and [Pan20a]. Similarly, in the case when $\mathcal{O} = \mathbb{Z}$, the $\mathbb{Z}$-fibers of a $\mathbb{Z}$-flat R are geometrically regular if and only if for every prime p and every maximal ideal $\mathfrak{m} \subset R$ of residue characteristic p, we have $p \notin \mathfrak{m}^2$, equivalently, p is a regular parameter for the regular local ring $R_{\mathfrak{m}}$. In particular, Theorem 9.1 recovers Theorem 1.2.

With our main result in hand, we are ready to settle the second assertion of Theorem 1.2.

Theorem 9.3. *For a Dedekind ring $\mathcal{O}$ and a semilocal, regular, flat $\mathcal{O}$-algebra R whose $\mathcal{O}$-fibers are geometrically regular, a reductive R-group G is split if and only if so its generic fiber $G_{\mathrm{Frac}(R)}$.*

Proof. We pass to connected components to assume that $\mathrm{Spec}(R)$ is connected, so that R is a domain, and we set $K := \mathrm{Frac}(R)$. Only the 'if' part requires an argument, so we assume that G_K is split. The geometric fibers of G have a constant type (see [SGA 3$_{\mathrm{III\ new}}$, Exposé XXII, Définition 1.13]), and we let $\mathbf{G}$ be a split reductive R-group of this type, so that G is a form of $\mathbf{G}$ that corresponds to some $x \in H^1(R, \underline{\mathrm{Aut}}(\mathbf{G}))$ whose pullback to $H^1(K, \underline{\mathrm{Aut}}(\mathbf{G}))$ is trivial. We wish to show that x is trivial.

By [SGA 3$_{\mathrm{III\ new}}$, Exposé XXIV, Théorème 1.3], we have a short exact sequence of group schemes

$$1 \to \mathbf{G}^{\mathrm{ad}} \to \underline{\mathrm{Aut}}(\mathbf{G}) \to \underline{\mathrm{Out}}(\mathbf{G}) \to 1$$

that, via a fixed pinning of $\mathbf{G}$, is split by a homomorphism

$$\underline{\mathrm{Out}}(\mathbf{G}) \hookrightarrow \underline{\mathrm{Aut}}(\mathbf{G}),$$

whose source is a constant R-group. Any $\underline{\mathrm{Out}}(\mathbf{G})$-torsor E is constant étale locally on R, so, by [SGA 3$_{\mathrm{II}}$, Exposé X, Corollaire 5.14], its connected components are finite étale over R. Thus, by, for

---

[6] We recall from [SP, Definition 0382] that a Noetherian algebra over a field k is *geometrically regular* if its base change to every finite purely inseparable (equivalently, to every finitely generated) field extension of k is regular.

instance, [Čes17, Lemma 3.1.9], every K-point of E extends to an R-point, to the effect that no nontrivial $\underline{\text{Out}}(\mathbf{G})$-torsor trivializes over K.

The nonabelian cohomology exact sequence now lifts x to an $\widetilde{x} \in H^1(R, \mathbf{G}^{\text{ad}})$ and, since the map

$$\underline{\text{Aut}}(\mathbf{G})(K) \twoheadrightarrow \underline{\text{Out}}(\mathbf{G})(K)$$

is surjective due to the splitting, it also shows that the pullback of $\widetilde{x}$ to $H^1(K, \mathbf{G}^{\text{ad}})$ is trivial. Theorem 9.1 then implies that $\widetilde{x}$ itself is trivial, and then so is x. $\qquad\square$

The ideas of the preceding proof also give a version for quasi-split groups in Theorem 9.5. To put it into context, we recall the following conjecture, which may be traced to results of [CT79] or [Pan09]. Even though not formulated there explicitly, it is sometimes attributed to Colliot-Thélène or Panin.

Conjecture 9.4. *For a regular local ring R, if the generic fiber of a reductive R-group scheme G has a parabolic subgroup, then G itself has a parabolic subgroup of the same type.*

This conjecture "of Grothendieck–Serre type" seems to lie deeper than the Grothendieck–Serre conjecture: even in equicharacteristic, it is only known in few cases, see [CT79], [Pan09], [PP10], [PP15], [Scu18] for precise results. We use the ideas of this article to settle its equicharacteristic case for minimal parabolics, that is, for Borel subgroups, as follows.

Theorem 9.5. *Let R a regular semilocal ring, let $K := \text{Frac}(R)$ be its total ring of fractions, and let G be a reductive R-group such that every form $\mathcal{G}$ of G^{ad} satisfies*

$$H^1(R, \mathcal{G}) \hookrightarrow H^1(K, \mathcal{G})$$

(this holds for every $\mathcal{G}$ if R contains a field). Then G is quasi-split if and only if G_K is quasi-split.

Proof. The injectivity assumption is a special case of the Grothendieck–Serre conjecture and of the "change of origin" twisting bijections in nonabelian cohomology [Gir71, Chapitre III, Proposition 2.6.1 (i)], so the parenthetical assertion follows from the known equicharacteristic case of the Grothendieck–Serre conjecture, see §1.4. By [Guo20, Proposition 14] (whose proof is similar to that of Theorem 9.3 above), this assumption implies that G is the unique reductive model of its generic fiber, so all we need to do is to assume that G_K is quasi-split and to produce a quasi-split reductive R-model of G_K. By the properness of the scheme of Borel subgroups, there is an open subscheme $U \subset \text{Spec}(R)$ whose complement is of codimension $\geqslant 2$ such that even G_U has a Borel subgroup.

Analogously to the proof of Theorem 9.3, we reduce to the setting when $\text{Spec}(R)$ is connected, we have a split reductive R-group $\mathbf{G}$, and G corresponds to an element $x \in H^1(R, \underline{\text{Aut}}(\mathbf{G}))$. We fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ that arises from a pinning of $\mathbf{G}$, and we consider the subfunctor

$$\underline{\text{Aut}}(\mathbf{G}, \mathbf{B}) \subset \underline{\text{Aut}}(\mathbf{G})$$

that parametrizes those automorphisms that preserve $\mathbf{B}$. In a reductive group, any two Borels are Zariski locally conjugate, so we are reduced to showing that for our $x \in H^1(R, \underline{\text{Aut}}(\mathbf{G}))$ such that $x|_U$ lifts to $H^1(U, \underline{\text{Aut}}(\mathbf{G}, \mathbf{B}))$, the restriction $x|_K \in H^1(K, \underline{\text{Aut}}(\mathbf{G}))$ lifts to $H^1(R, \underline{\text{Aut}}(\mathbf{G}, \mathbf{B}))$.

By [SGA 3$_{\text{III new}}$, Exposé XXIV, Théorème 1.3, Proposition 2.1], letting $\mathbf{B}^{\text{ad}} \subset \mathbf{G}^{\text{ad}}$ be the Borel subgroup of $\mathbf{G}^{\text{ad}}$ corresponding to $\mathbf{B}$, we have a morphism of short exact sequences of group schemes

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{B}^{\text{ad}} & \longrightarrow & \underline{\text{Aut}}(\mathbf{G}, \mathbf{B}) & \longrightarrow & \underline{\text{Out}}(\mathbf{G}) & \longrightarrow & 1 \\
 & & \cap\downarrow & & \cap\downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{G}^{\text{ad}} & \longrightarrow & \underline{\text{Aut}}(\mathbf{G}) & \longrightarrow & \underline{\text{Out}}(\mathbf{G}) & \longrightarrow & 1
\end{array}
$$

that, due to our fixed pinning, are compatibly split by some homomorphism

$$\underline{\mathrm{Out}}(\mathbf{G}) \hookrightarrow \underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B}).$$

We may first map x to an $\bar{x} \in H^1(R, \underline{\mathrm{Out}}(\mathbf{G}))$ and then map $\bar{x}$ via the splitting to obtain a

$$y \in H^1(R, \underline{\mathrm{Out}}(\mathbf{G}, \mathbf{B}))$$

whose image in $H^1(R, \underline{\mathrm{Out}}(\mathbf{G}))$ is also $\bar{x}$. Twisting by (the images of) y gives us the morphism of short exact sequences of R-groups of corresponding forms:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A}_0 & \longrightarrow & \mathcal{E} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \parallel & & \\
1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{E} & \longrightarrow & 1
\end{array}
$$

and, via the "change of origin" bijections [Gir71, Chapitre III, Proposition 2.6.1 (i)], we obtain an $x' \in H^1(R, \mathcal{A})$ such that $x'|_U$ lifts to $H^1(U, \mathcal{A}_0)$ for which we need to lift $x'|_K \in H^1(K, \mathcal{A})$ to $H^1(R, \mathcal{A}_0)$ or even to $H^1(R, \mathcal{B})$.

By the nonabelian cohomology sequence, $x'|_U$ even lifts to some $b \in H^1(U, \mathcal{B})$. By descent, $\mathcal{B} \subset \mathcal{G}$ is the inclusion of a Borel R-subgroup, and we let

$$\mathcal{T} := \mathcal{B}/\mathscr{R}_u(\mathcal{B})$$

be the indicated torus. The image of b is a $t \in H^1(U, \mathcal{T})$, which, by purity for torsors under tori [CTS79, Corollaire 6.9], extends uniquely to a

$$\widetilde{t} \in H^1(R, \mathcal{T}).$$

Any Levi R-subgroup of $\mathcal{B}$ splits the surjection $\mathcal{B} \twoheadrightarrow \mathcal{T}$, and then $\widetilde{t}$ gives a $\widetilde{b} \in H^1(R, \mathcal{B})$ whose image in $H^1(K, \mathcal{B})$, thanks to [SGA 3$_{\mathrm{III\ new}}$, Exposé XXVI, Corollaire 2.3], is nothing else but $b|_K$. In particular, the image of $\widetilde{b}$ in $H^1(K, \mathcal{A})$ is $x'|_K$, to the effect that $\widetilde{b}$ is the desired lift. $\qquad\square$

We thank Uriya First for pointing out the following further consequence about quadratic forms.

Corollary 9.6. *For a regular semilocal ring R as in Theorem* 9.1 *with $2 \in R^\times$, we have*

$$H^1(R, \mathrm{SO}_n) \hookrightarrow H^1(\mathrm{Frac}(R), \mathrm{SO}_n) \quad and \quad H^1(R, \mathrm{O}_n) \hookrightarrow H^1(\mathrm{Frac}(R), \mathrm{O}_n) \quad for\ all \quad n \geqslant 1;$$

moreover, no two nonisomorphic quadratic forms over R that are nondegenerate (in the sense that their associated symmetric bilinear forms are perfect) become isomorphic over $\mathrm{Frac}(R)$.

Proof. Every inner form of SO_n is an $\mathrm{SO}(E)$ for a nondegenerate quadratic space E over R of rank n. Thus, by twisting [Gir71, Chapitre III, Proposition 2.6.1 (i)], the injectivity assertion for SO_n reduces to showing that

$$\mathrm{Ker}(H^1(R, \mathrm{SO}(E)) \to H^1(\mathrm{Frac}(R), \mathrm{SO}(E))) = \{*\}.$$

By the analysis of the long exact cohomology sequence [CT79, page 17, proof of (D)$\Leftrightarrow$(E)], this triviality of the kernel is, in turn, equivalent to its analogue for $\mathrm{O}(E)$. Thus, by twisting again, we are reduced to the injectivity assertion for O_n, which itself, for varying n, is a reformulation of the assertion about quadratic forms. For the latter, however, due to the cancellation theorem for quadratic forms, specifically, due to [CT79, Proposition 1.2 (D)$\Leftrightarrow$(F)], we may assume that one of the forms is a sum of copies of the hyperbolic plane. In terms of O_n-torsors, this means that it suffices to show that

$$\mathrm{Ker}(H^1(R, \mathrm{O}_n) \to H^1(\mathrm{Frac}(R), \mathrm{O}_n)) = \{*\} \quad \text{for all even} \quad n \geqslant 1.$$

We then use [CT79, page 17, proof of (D)⇔(E)] again to replace O_n by SO_n in this display. With this replacement, however, the desired triviality of the kernel is a special case of Theorem 9.1. □

References

[AHW18] Aravind Asok, Marc Hoyois, and Matthias Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory, II: Principal bundles and homogeneous spaces*, Geom. Topol. **22** (2018), no. 2, 1181–1225, DOI 10.2140/gt.2018.22.1181. MR3748687

[AHW20] ———, *Affine representability results in $\mathbb{A}^1$-homotopy theory III: finite fields and complements*, Algebr. Geom. **7** (2020), no. 5, 634–644, DOI 10.14231/ag-2020-023. MR4156421

[Alp14] Jarod Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531, DOI 10.14231/AG-2014-022. See also https://arxiv.org/abs/1005.2398v2 for an updated post-publication version. MR3272912

[BB70] A. Białynicki-Birula, *Rationally trivial homogeneous principal fibrations of schemes*, Invent. Math. **11** (1970), 259–262, DOI 10.1007/BF01404652. MR276250

[BČ21] Alexis Bouthier and Kęstutis Česnavičius, *Torsors on loop groups and the Hitchin fibration*, Ann. Sci. École Norm. Sup., to appear (2021). Available at http://arxiv.org/abs/1908.07480v4.

[BFF17] Eva Bayer-Fluckiger and Uriya A. First, *Rationally isomorphic hermitian forms and torsors of some non-reductive groups*, Adv. Math. **312** (2017), 150–184, DOI 10.1016/j.aim.2017.03.012. MR3635808

[BFFH19] E. Bayer-Fluckiger, U. A. First, and M. Huruguen, *Orders that are étale-locally isomorphic*, Algebra i Analiz **31** (2019), no. 4, 1–15. MR3985253

[BFFP20] Eva Bayer-Fluckiger, Uriya A. First, and Raman Parimala, *On the Grothendieck–Serre conjecture for classical groups*, preprint (2020). Available at http://arxiv.org/abs/1911.07666v2.

[BouAC] Nicolas Bourbaki, *Éléments de mathématique. Algèbre commutative*, chap. I-VII, Hermann (1961, 1964, 1965); chap. VIII-X, Springer (2006, 2007) (French).

[BT87] F. Bruhat and J. Tits, *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 671–698 (French). MR927605

[BVG14] Sofie Beke and Jan Van Geel, *An isomorphism problem for Azumaya algebras with involution over semilocal Bézout domains*, Algebr. Represent. Theory **17** (2014), no. 6, 1635–1655, DOI 10.1007/s10468-013-9463-6. MR3284324

[Che10] V. Chernousov, *Variations on a theme of groups splitting by a quadratic extension and Grothendieck-Serre conjecture for group schemes F_4 with trivial g_3 invariant*, Doc. Math. **Extra vol.: Andrei A. Suslin sixtieth birthday** (2010), 147–169. MR2804252

[Con14] Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444 (English, with English and French summaries). MR3362641

[CT79] Jean-Louis Colliot-Thélène, *Formes quadratiques sur les anneaux semi-locaux réguliers*, Bull. Soc. Math. France Mém. **59** (1979), 13–31 (French). Colloque sur les Formes Quadratiques, 2 (Montpellier, 1977). MR532002

[CT95] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64. MR1327280

[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn, *The Bloch-Ogus-Gabber theorem*, Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 31–94. MR1466971

[CTO92] Jean-Louis Colliot-Thélène and Manuel Ojanguren, *Espaces principaux homogènes localement triviaux*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 97–122 (French). MR1179077

[CTS79] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134, DOI 10.1007/BF01420486 (French). MR550842

[CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), no. 1, 148–205, DOI 10.1016/0021-8693(87)90026-3. MR878473

[Čes17] Kęstutis Česnavičius, *A modular description of $\mathscr{X}_0(n)$*, Algebra Number Theory **11** (2017), no. 9, 2001–2089. MR3735461

[Čes21] Kęstutis Česnavičius, *Macaulayfication of Noetherian schemes*, Duke Math. J. **170** (2021), no. 7, 1419–1455, DOI 10.1215/00127094-2020-0063. MR4255056

[Čes22] Kęstutis Česnavičius, *Problems about torsors over regular rings (with an appendix by Yifei Zhao)*, Acta Math. Vietnam., to appear (2022). Available at https://arxiv.org/abs/2201.06424.

[ČS21] Kęstutis Česnavičius and Peter Scholze, *Purity for flat cohomology*, preprint (2021). Available at http://arxiv.org/abs/1912.10932v2.

[EGA I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960), 228. MR0217083 (36 #177a)

[EGA II] ———, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222 (French). MR163909 (29 #1208)

[EGA III$_1$] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. MR0217085 (36 #177c)

[EGA IV$_1$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. **20** (1964), 259 (French). MR0173675 (30 #3885)

[EGA IV$_2$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330)

[EGA IV$_3$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 255. MR0217086 (36 #178)

[EGA IV$_4$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361 (French). MR0238860 (39 #220)

[Fed16] Roman Fedorov, *Affine Grassmannians of group schemes and exotic principal bundles over $\mathbb{A}^1$*, Amer. J. Math. **138** (2016), no. 4, 879–906, DOI 10.1353/ajm.2016.0036. MR3538146

[Fed21a] ———, *On the Grothendieck–Serre Conjecture about principal bundles and its generalizations*, Algebra Number Theory, to appear (2021). Available at http://arxiv.org/abs/1810.11844v2.

[Fed21b] ———, *On the Grothendieck–Serre conjecture on principal bundles in mixed characteristic*, Trans. Amer. Math. Soc. **375** (2021), no. 01, 559–586, DOI 10.1090/tran/8490. MR4358676

[Fir22] Uriya A. First, *An 8-Periodic Exact Sequence of Witt Groups of Azumaya Algebras with Involution*, Manuscripta Math., to appear (2022). Available at http://arxiv.org/abs/1910.03232v2.

[FP15] Roman Fedorov and Ivan Panin, *A proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 169–193, DOI 10.1007/s10240-015-0075-z. MR3415067

[FR70] Daniel Ferrand and Michel Raynaud, *Fibres formelles d'un anneau local noethérien*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 295–311 (French). MR0272779

[Gab94] Ofer Gabber, *Gersten's conjecture for some complexes of vanishing cycles*, Manuscripta Math. **85** (1994), no. 3-4, 323–343, DOI 10.1007/BF02568202. MR1305746

[Gab01] O. Gabber, *On space filling curves and Albanese varieties*, Geom. Funct. Anal. **11** (2001), no. 6, 1192–1200, DOI 10.1007/s00039-001-8228-2. MR1878318

[Gil02] P. Gille, *Torseurs sur la droite affine*, Transform. Groups **7** (2002), no. 3, 231–245, DOI 10.1007/s00031-002-0012-3 (French, with English summary). MR1923972

[Gil05] ———, *Errata: "Torsors on the affine line" (French) [Transform. Groups 7 (2002), no. 3, 231–245; MR1923972]*, Transform. Groups **10** (2005), no. 2, 267–269, DOI 10.1007/s00031-005-1010-z (French, with English summary). MR2195603

[Gil09] Philippe Gille, *Le problème de Kneser-Tits*, Astérisque **326** (2009), Exp. No. 983, vii, 39–81 (2010) (French, with French summary). Séminaire Bourbaki. Vol. 2007/2008. MR2605318

[Gil21] ———, *When is a reductive group scheme linear?*, Michigan Math. J., to appear (2021). Available at https://arxiv.org/abs/2103.07305v3.

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971 (French). Die Grundlehren der mathematischen Wissenschaften, Band 179. MR0344253 (49 #8992)

[GLL15] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270, DOI 10.1215/00127094-2877293. MR3347315

[Gro58] Alexandre Grothendieck, *Torsion homologique et sections rationnelles*, Seminaire Claude Chevalley **3** (1958), no. 5, 1–29 (French).

[Gro68] Alexander Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 67–87 (French). MR0244270 (39 #5586b)

[Guo20] Ning Guo, *The Grothendieck–Serre conjecture over semilocal Dedekind rings*, Transform. Groups, to appear (2020). Available at https://arxiv.org/abs/1902.02315v3.

[Har67] Günter Harder, *Halbeinfache Gruppenschemata über Dedekindringen*, Invent. Math. **4** (1967), 165–191, DOI 10.1007/BF01425754 (German). MR225785

[Jou83] Jean-Pierre Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983 (French). MR725671

[Lin81] Hartmut Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981/82), no. 2, 319–323, DOI 10.1007/BF01389017. MR641133

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461 (90i:13001)

[MB96] Laurent Moret-Bailly, *Un problème de descente*, Bull. Soc. Math. France **124** (1996), no. 4, 559–585 (French, with English and French summaries). MR1432058

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR2242284

[Nis82] Yevsey A. Nisnevich, *Etale cohomology and arithmetic of semisimple groups*, ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–Harvard University. MR2632405

[Nis84] ———, *Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 1, 5–8 (French, with English summary). MR756297

[Nis89] Yevsey Nisnevich, *Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 10, 651–655 (English, with French summary). MR1054270

[Oja80] Manuel Ojanguren, *Quadratic forms over regular rings*, J. Indian Math. Soc. (N.S.) **44** (1980), no. 1-4, 109–116 (1982). MR752647

[Oja82] ———, *Unités représentées par des formes quadratiques ou par des normes réduites*, Algebraic K-theory, Part II (Oberwolfach, 1980), Lecture Notes in Math., vol. 967, Springer, Berlin-New York, 1982, pp. 291–299 (French). MR689397

[OP99] Manuel Ojanguren and Ivan Panin, *A purity theorem for the Witt group*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 1, 71–86, DOI 10.1016/S0012-9593(99)80009-3 (English, with English and French summaries). MR1670591

[OP01] ———, *Rationally trivial Hermitian spaces are locally trivial*, Math. Z. **237** (2001), no. 1, 181–198, DOI 10.1007/PL00004859. MR1836777

[OPZ04] M. Ojanguren, I. Panin, and K. Zainoulline, *On the norm principle for quadratic forms*, J. Ramanujan Math. Soc. **19** (2004), no. 4, 289–300. MR2125505

[Pan05] Ivan Panin, *Purity for multipliers*, Algebra and number theory, Hindustan Book Agency, Delhi, 2005, pp. 66–89. MR2193345

[Pan09] ———, *Rationally isotropic quadratic spaces are locally isotropic*, Invent. Math. **176** (2009), no. 2, 397–403, DOI 10.1007/s00222-008-0168-0. MR2495767

[Pan15] ———, *Proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing a finite field*, preprint (2015). Available at https://www.math.uni-bielefeld.de/lag/man/559.pdf.

[Pan18] ———, *On Grothendieck-Serre conjecture concerning principal bundles*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 201–221. MR3966763

[Pan19a] I. A. Panin, *Nice triples and moving lemmas for motivic spaces*, Izv. Math. **83** (2019), no. 4, 796–829, DOI 10.1070/IM8819. MR3985694

[Pan19b] I. Panin, *Notes on a Grothendieck-Serre conjecture in mixed characteristic case*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **484** (2019), no. Voprosy Teorii Predstavlenii Algebr i Grupp. 35, 138–148; English transl., J. Math. Sci. (N.Y.) **252** (2021), no. 6, 841–848. MR4053274

[Pan20a] I. A. Panin, *Proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing a field*, Izv. Ross. Akad. Nauk Ser. Mat. **84** (2020), no. 4, 169–186, DOI 10.4213/im8982 (Russian); English transl., Izv. Math. **84** (2020), no. 4, 780–795. MR4133391

[Pan20b] ———, *Two purity theorems and the Grothendieck–Serre's conjecture concerning principal* **G**-*bundles*, Mat. Sb. **211** (2020), no. 12, 123–142, DOI 10.4213/sm9393 (Russian). MR4181078

[Poo04] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127, DOI 10.4007/annals.2004.160.1099. MR2144974

[PP10] Ivan Panin and Konstantin Pimenov, *Rationally isotropic quadratic spaces are locally isotropic: II*, Doc. Math. **Extra vol.: Andrei A. Suslin sixtieth birthday** (2010), 515–523. MR2804263

[PP15] I. Panin and K. Pimenov, *Rationally isotropic quadratic spaces are locally isotropic. III*, Algebra i Analiz **27** (2015), no. 6, 234–241, DOI 10.1090/spmj/1433; English transl., St. Petersburg Math. J. **27** (2016), no. 6, 1029–1034. MR3589229

[PS97] I. A. Panin and A. A. Suslin, *On a conjecture of Grothendieck concerning Azumaya algebras*, Algebra i Analiz **9** (1997), no. 4, 215–223 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **9** (1998), no. 4, 851–858. MR1604322

[PS16] I. A. Panin and A. K. Stavrova, *On the Grothendieck-Serre conjecture concerning principal G-bundles over semi-local Dedekind domains*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **443** (2016), no. Voprosy Teorii Predstavlenii Algebr i Grupp. 29, 133–146, DOI 10.1007/s10958-017-3316-5; English transl., J. Math. Sci. (N.Y.) **222** (2017), no. 4, 453–462. MR3507770

[PSV15] I. Panin, A. Stavrova, and N. Vavilov, *On Grothendieck-Serre's conjecture concerning principal G-bundles over reductive group schemes: I*, Compos. Math. **151** (2015), no. 3, 535–567, DOI 10.1112/S0010437X14007635. MR3320571

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129

[Rag94] M. S. Raghunathan, *Principal bundles admitting a rational section*, Invent. Math. **116** (1994), no. 1-3, 409–423, DOI 10.1007/BF01231567. MR1253199

[RR84] M. S. Raghunathan and A. Ramanathan, *Principal bundles on the affine line*, Proc. Indian Acad. Sci. Math. Sci. **93** (1984), no. 2-3, 137–145, DOI 10.1007/BF02840656. MR813075

[Scu18] Stephen Scully, *The Artin-Springer theorem for quadratic forms over semi-local rings with finite residue fields*, Proc. Amer. Math. Soc. **146** (2018), no. 1, 1–13, DOI 10.1090/proc/13744. MR3723116

[Ser58] J.-P. Serre, *Espaces fibrés algébriques*, Seminaire Claude Chevalley **3** (1958), no. 1, 1–37 (French).

[SGA 3$_{\rm II}$] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970 (French). MR0274459 (43 #223b)

[SGA 3$_{\rm III\ new}$] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 8, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre; Revised and annotated edition of the 1970 French original. MR2867622

[SGA 4$_{\rm III}$] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4); Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. MR0354654 (50 #7132)

[SP] A. J. de Jong et al., *The Stacks Project*. Available at http://stacks.math.columbia.edu.

[Tsy19] A. E. Tsybyshev, *A step toward the non-equicharacteristic Grothendieck-Serre conjecture*, Algebra i Analiz **31** (2019), no. 1, 246–254, DOI 10.1090/spmj/1591 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **31** (2020), no. 1, 181–187. MR3932824

[Zai00] K. V. Zainullin, *On Grothendieck's conjecture on principal homogeneous spaces for some classical algebraic groups*, Algebra i Analiz **12** (2000), no. 1, 150–184 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **12** (2001), no. 1, 117–143. MR1758566

[Zai05] K. Zainoulline, *On Knebusch's norm principle for quadratic forms over semi-local rings*, Math. Z. **251** (2005), no. 2, 415–425, DOI 10.1007/s00209-005-0809-6. MR2191034