TORSORS ON THE COMPLEMENT OF A SMOOTH DIVISOR

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Abstract. We complete the proof of the Nisnevich conjecture in equal characteristic: for a smooth algebraic variety X over a field k, a k-smooth divisor $D \subset X$, and a reductive X-group G whose base change G_D is totally isotropic, we show that each generically trivial G-torsor on $X \backslash D$ trivializes Zariski semilocally on X. In mixed characteristic, we show the same when k is a replaced by a discrete valuation ring $\mathcal{O}$, the divisor D is the closed $\mathcal{O}$-fiber of X, and either G is quasi-split or G is only defined over $X \backslash D$ but descends to a quasi-split group over $\mathrm{Frac}(\mathcal{O})$ (a Kisin–Pappas type variant). Our arguments combine Gabber–Quillen style presentation lemmas with excision and reembedding dévissages to reduce to analyzing generically trivial torsors over a relative affine line. We base this analysis on the geometry of the affine Grassmannian, and we we give a new proof for the Bass–Quillen conjecture for reductive group torsors over $\mathbb{A}^d_R$ in equal characteristic. As for the affine Grassmannian itself, we show that for totally isotropic G it is the *presheaf* quotient LG/L^+G.

1. The corrected statement of the Nisnevich conjecture and our main results

In [Nis89, Conjecture 1.3], Nisnevich proposed a common generalization of the Quillen conjecture [Qui76, (2) on page 170] that had grown out of Serre's problem about vector bundles on affine spaces and of the Grothendieck–Serre conjecture [Ser58, page 31, Remarque], [Gro58, pages 26–27, Remarques 3] about Zariski local triviality of generically trivial torsors under reductive groups. In its geometric case, the Nisnevich conjecture predicts that, for a reductive group scheme G over a smooth variety X over a field k and a k-smooth divisor $D \subset X$, every generically trivial G-torsor on

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$X \setminus D$ trivializes Zariski locally on X. Recent counterexamples of Fedorov [Fed22b, Proposition 4.1] show that this fails for anisotropic G, so, to bypass them, one considers the following isotropicity condition whose relevance for problems about torsors has been observed already in [Rag89].

Definition 1.1 ([Česnavičius22a, Definition 8.1]). Let S be a scheme and let G be a reductive S-group scheme. We say that G is *totally isotropic* at a point $s \in S$ if each factor $\widetilde{G}_i$ in the canonical decomposition

$$G^{\mathrm{ad}}_{\mathscr{O}_{S,s}} \cong \prod_i \widetilde{G}_i \quad \text{with} \quad \widetilde{G}_i := \mathrm{Res}_{R_i/\mathscr{O}_{S,s}}(G_i) \tag{1.1.1}$$

of [SGA $3_{\mathrm{III\ new}}$, exposé XXIV, proposition 5.10 (i)] has a proper parabolic subgroup; here i is a type of connected Dynkin diagrams, R_i is a finite étale $\mathscr{O}_{S,s}$-algebra, and G_i is an adjoint semisimple R_i-group with simple geometric R_i-fibers of type i. If this holds for all s, then G is *totally isotropic*.

Intuitively, G is totally isotropic if its simple factors are isotropic. Recall from [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 6.12] that, since $\mathscr{O}_{S,s}$ and each R_i are semilocal, it is equivalent to require in Definition 1.1 that each $\widetilde{G}_i$ contains $\mathbb{G}_{m,\mathscr{O}_{S,s}}$ as a subgroup, equivalently, each G_i contains an R_i-fiberwise proper parabolic R_i-subgroup, equivalently, each G_i contains $\mathbb{G}_{m,R_i}$ as an R_i-subgroup. For instance, every quasi-split, so also every split, group is totally isotropic, as is any torus.

With the total isotropicity in place, the Nisnevich conjecture becomes the following statement.

Conjecture 1.2 (Nisnevich). *For a regular semilocal ring R, an $r \in R$ that is a regular parameter in the sense that $r \notin \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m} \subset R$, and a reductive R-group scheme G such that $G_{R/(r)}$ is totally isotropic, every generically trivial G-torsor over $R[\frac{1}{r}]$ is trivial, that is,*

$$\mathrm{Ker}(H^1(R[\tfrac{1}{r}], G) \to H^1(\mathrm{Frac}(R), G)) = \{*\}.$$

For instance, in the case when r is a unit, the total isotropicity condition holds for every reductive R-group G and we recover the Grothendieck–Serre conjecture. The condition also holds in the case when G is a torus, and this case follows from the known toral case of the Grothendieck–Serre conjecture, see [Čes22b, Section 3.4.2 (1)]. In [Fed22b], Fedorov settled the Nisnevich conjecture in the case when R contains an infinite field and G itself is totally isotropic. Other than this, some low dimensional cases are known, see [Čes22b, Section 3.4.2]—for instance, the case when R is local of dimension $\leqslant 3$ and G is either GL_n or PGL_n is a result of Gabber [Gab81, Chapter I, Theorem 1].

We settle the Nisnevich conjecture in equal characteristic and in some mixed characteristic cases.

Theorem 1.3. *Let R be a regular semilocal ring, let $r \in R$ be a regular parameter in the sense that $r \notin \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m} \subset R$, and let G be a reductive $R[\frac{1}{r}]$-group. In the following cases,*

$$\mathrm{Ker}(H^1(R[\tfrac{1}{r}], G) \to H^1(\mathrm{Frac}(R), G)) = \{*\},$$

in other words, in the following cases every generically trivial G-torsor over $R[\frac{1}{r}]$ is trivial:

(1) (§8.2) *if R contains a field and G extends to a reductive R-group $\mathcal{G}$ with $\mathcal{G}_{R/(r)}$ totally isotropic;*

(2) (§5.4) *if R is geometrically regular*[1] *over a Dedekind subring $\mathcal{O}$ containing r and G either lifts to a quasi-split reductive R-group or descends to a quasi-split reductive $\mathcal{O}[\frac{1}{r}]$-group.*

[1] For a ring A, recall that an A-algebra B is *geometrically regular* if it is flat and the base change of each of its A-fibers to any finite field extension of the corresponding residue field of A is regular, see [SP, Definition 0382]. For instance, R could be a semilocal ring of a smooth algebra over a discrete valuation ring $\mathcal{O}$ with r as a uniformizer.

The mixed characteristic case (2) is new already for vector bundles, that is, for $G = \mathrm{GL}_n$. In contrast, at least for local R, the vector bundle case of the equicharacteristic (1) is due to Bhatwadekar–Rao [BR83, Theorem 2.5], with exceptions when the ground field is finite that have since been removed. When $r \in R^\times$, Theorem 1.3 recovers the equal and mixed characteristic cases of the Grothendieck–Serre conjecture settled in [FP15], [Pan20], [Česnavičius22a], and we reprove these cases along the way.

The case of (2) in which G descends to an $\mathcal{O}[\frac{1}{r}]$-group but need not extend to a reductive R-group was inspired by Kisin–Pappas [KP18, Section 1.4, especially, Lemma 1.4.6], who used such a statement for some 2-dimensional R under further assumptions on G.

The geometric version of Theorem 1.3 (1) is the following statement announced in the abstract.

Theorem 1.4. *For a field k, a smooth k-scheme X, a k-smooth divisor $D \subset X$, and a reductive X-group scheme G such that G_D is totally isotropic, every generically trivial G-torsor E over $X \backslash D$ is trivial Zariski semilocally on X, that is, for every $x_1, \ldots, x_m \in X$ that lie in a single affine open, there is an affine open $U \subset X$ containing all the x_i such that $E|_{U \backslash D}$ is trivial.*

Theorem 1.4 follows by applying Theorem 1.3 (1) to the semilocal ring of X at $x_1, \ldots, x_m$ (built via prime avoidance, see [SP, Lemma 00DS]) and spreading out. Even when X is affine, the stronger statement that E extends to a G-torsor over X is false: for $G = \mathrm{GL}_n$, this had been a question of Quillen [Qui76, (3) on page 170] that was answered negatively by Swan in [Swa78, Section 2]. Even for GL_n, Theorem 1.4 typically fails if D is singular or if X is singular, see [Lam06, pages 34–35].

We use Theorem 1.3 to reprove the following equal characteristic case of the generalization of the Bass–Quillen conjecture to torsors under reductive group schemes [Čes22b, Conjecture 3.6.1].

Theorem 1.5 (§9.1)**.** *For a regular ring R containing a field and a totally isotropic reductive R-group scheme G, every generically trivial G-torsor over $\mathbb{A}^d_R$ descends to a G-torsor over R, equivalently,*

$$H^1_{\mathrm{Zar}}(R, G) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\mathbb{A}^d_R, G) \quad \textit{or, if one prefers,} \quad H^1_{\mathrm{Nis}}(R, G) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(\mathbb{A}^d_R, G).$$

The equivalence of the three formulations follows from the Grothendieck–Serre conjecture, more precisely, by Theorem 1.3, a G-torsor over $\mathbb{A}^d_R$ is generically trivial, if and only if it is Zariski locally trivial, if and only if it is Nisnevich locally trivial. The generic triviality assumption is needed because, for instance, for every separably closed field k that is not algebraically closed, there are nontrivial PGL_n-torsors over $\mathbb{A}^1_k$, see [CTS21, Theorem 5.6.1 (vi)]. The total isotropicity assumption is needed because of [BS17, Proposition 4.9], where Balwe and Sawant show that a Bass–Quillen statement cannot hold beyond totally isotropic G. For earlier counterexamples to generalizations of the Bass–Quillen conjecture beyond totally isotropic reductive groups, see [Par78] and [Fed16, Theorem 3 (ii) (whose assumptions can be met thanks to Remark 2.6 (i))].

Theorem 1.5 was established by Stavrova in [Sta22, Corollary 5.5] by a different method, and in the case when R contains an infinite field already in the earlier [Sta19, Theorem 4.4]. Prior to that, the case when R is smooth over a field k and G is defined and totally isotropic over k was settled by Asok–Hoyois–Wendt: they used methods of $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky to verify axioms of Colliot-Thélène–Ojanguren [CTO92] that were known to imply the statement, see [AHW18, Theorem 3.3.7] for infinite k and [AHW20, Theorem 2.4] for finite k. As was explained in [Li21], one could also check these axioms directly, without $\mathbb{A}^1$-homotopy theory. For regular R of mixed characteristic, Theorem 1.5 is only known in sporadic cases, for instance, when G is a torus, see [CTS87, Lemma 2.4], as well as [Čes22b, Section 3.6.4] for an overview.

We obtain Theorem 1.3 by refining the Grothendieck–Serre type strategies used in [Fed22b] and [Čes22a]. In fact, we use the geometry of the affine Grassmannian Gr_G through self-contained inputs from the survey [Čes22b] that mildly generalized corresponding results of Fedorov from [Fed22a] to establish the following new type of Grothendieck–Serre result that is valid over arbitrary base rings.

Theorem 1.6 (Theorem 4.5)**.** *For a reductive group G over a ring A, every G-torsor over a smooth affine A-curve C that is trivial away from some A-finite $Z \subset C$ trivializes Zariski semilocally on C.*

Theorem 1.6, more precisely, its finer version given in Theorem 4.5, is our ultimate source of triviality of torsors under reductive groups. Armed with it we quickly reprove the cases of the Grothendieck–Serre conjecture that have been settled in [FP15], [Pan20], [Čes22a]: more precisely, we use Popescu approximation and presentation lemmas in the style of Gabber–Quillen to reduce these cases to the relative curve setting of Theorem 1.6, and in this way we dissect the overall argument into a part that works over arbitrary rings and a part that is specific to regular rings.

As for the affine Grassmannian itself, Theorem 1.6 implies that Gr_G is simply the Zariski sheafification of the quotient of the loop functor LG by the positive loop subfunctor L^+G as follows, in other words, that the étale or even fpqc sheafifications usually used to define Gr_G are overkills.

Theorem 1.7 (Corollary 4.6)**.** *For a reductive group G over a ring A, the affine Grassmannian Gr_G agrees with the Zariski sheafification of the presheaf quotient LG/L^+G, more precisely, if A is semilocal, then no nontrivial G-torsor over A, equivalently, over $A[\![t]\!]$, trivializes over $A(\!(t)\!)$ and*

$$\mathrm{Gr}_G(A) \cong G(A(\!(t)\!))/G(A[\![t]\!]). \qquad (1.7.1)$$

To deduce the conclusions about the affine Grassmannian from the statement about torsors, see the discussion in [Čes22b, Section 5.3.1]. Compare also with [Bac19, Proposition 14] for an earlier variant of the Zariski sheafification claim that restricted to smooth A over a field and deduced it from the Grothendieck–Serre conjecture. We do not know whether the Zariski sheafification is needed for general G, in fact, for totally isotropic G we have the following finer result that improves [Fed22a, Theorem 5] and simultaneously resolves [Čes22b, Conjecture 3.5.1] proposed by Ning Guo.

Theorem 1.8 (Theorem 2.1)**.** *Let G be a totally isotropic reductive group over a ring A.*

(a) *No nontrivial G-torsor over $\mathbb{A}^1_A$ trivializes away from some A-finite closed subscheme $Z \subset \mathbb{A}^1_A$.*

(b) *No nontrivial G-torsor over A, equivalently, over $A[\![t]\!]$, trivializes over $A(\!(t)\!)$ and*

$$G(A(\!(t)\!)) = G(A[t^{\pm 1}])G(A[\![t]\!]). \qquad (1.8.1)$$

(c) *The affine Grassmannian Gr_G agrees with the presheaf quotient LG/L^+G and*

$$\mathrm{Gr}_G(A) \cong G(A(\!(t)\!))/G(A[\![t]\!]) \cong G(A[t^{\pm 1}])/G(A[t]).$$

To deduce Theorem 1.8 (c) from (b), one again uses [Čes22b, Section 5.3.1] and the equality

$$G(A[t]) = G(A[t^{\pm 1}]) \cap G(A[\![t]\!]) \quad \text{in} \quad G(A(\!(t)\!)).$$

Theorem 1.8 (a) answers [Fed16, Question 2] because in *op. cit.* Fedorov already found counterexamples for the corresponding statement beyond totally isotropic reductive G. More precisely, in [Fed16, Theorem 3 and what follows], he gave examples of regular local rings A, anisotropic reductive A-groups G, and nontrivial G-torsors over $\mathbb{A}^1_A$ that trivialize away from some A-(finite étale) closed $Z \subset \mathbb{A}^1_A$. Special cases of Theorem 1.8 (a) were settled in [PSV15, Theorem 1.3], [Fed21, Theorem 2], and [Čes22a, Proposition 8.4], see also [Čes22b, Section 3.5.2] for an overview.

Coming back to the Nisnevich conjecture itself, a key novelty of our approach is the following extension result for G-torsors over smooth relative curves.

Theorem 1.9 (Proposition 7.3 and Theorem 6.1)**.** *Let R be a regular semilocal ring containing a field and let G be a reductive R-group. For a smooth affine R-scheme C of pure relative dimension* 1 *and an R-(finite étale) closed $Y \subset C$ such that G_Y is totally isotropic, every G-torsor E over $C\backslash Y$ that is trivial away from some R-finite closed $Z \subset C$ extends to a G-torsor over C.*

Roughly, extending a G-torsor to all of C in Theorem 1.9 corresponds to extending a G-torsor in Theorem 1.3 (1) to all of R, in effect, to reducing the Nisnevich conjecture to the Grothendieck–Serre conjecture—this is why Theorem 1.9 is crucial for us. Conversely, to reduce Theorem 1.3 (1) to Theorem 1.9 we use a presentation lemma that extends its variants due to Quillen and Gabber: we first use Popescu theorem to pass to the geometric setting of Theorem 1.4 and then show in Lemma 8.1 that, up to replacing X by an affine open neighborhood of $x_1, \ldots, x_m$, we can express X as a smooth relative curve over some affine open of $\mathbb{A}^{d-1}_k$ in such a way that D is relatively finite étale and our generically trivial G-torsor over X is trivial away from a relatively finite closed subscheme.

As for Theorem 1.9, in §7 we present a series of excision and patching dévissages to reduce to when $C = \mathbb{A}^1_R$ and $C\backslash Y$ descends to a smooth curve defined over a subfield $k \subset R$. In this "constant" case, we show that our G-torsor over $C\backslash Y$ is even trivial by the "relative Grothendieck–Serre" theorem of Fedorov from [Fed22a] (with an earlier version due to Panin–Stavrova–Vavilov [PSV15]) that we reprove in Theorem 6.1: for every k-algebra W, no nontrivial G-torsor over $R \otimes_k W$ trivializes over $\mathrm{Frac}(R) \otimes_k W$; the total isotropicity assumption is crucial for this beyond the "classical" case $W = \mathrm{Spec}(k)$. As for the excision and patching techniques, we overcome known finite field difficulties with novel versions of the Lindel style embedding in Proposition 3.4 and of Panin's "finite field tricks" in Lemma 3.6. The wide scope of these techniques makes our overall approach to Theorem 1.3 quite axiomatic, and although we do not pursue this here, it would be interesting to have similar results for other functors, for instance, for the unstable K_1-functor studied by Stavrova and her coauthors, compare, for instance, with [Sta22], [Sta19] and earlier articles cited there.

1.10. Notation and conventions. All the rings we consider are commutative and unital. For a point s of a scheme (resp., for a prime ideal $\mathfrak{p}$ of a ring), we let k_s (resp., $k_\mathfrak{p}$) denote its residue field. For a global section s of a scheme S, we write $S[\frac{1}{s}] \subset S$ for the open locus where s does not vanish. For a semilocal regular ring R, we say that an $r \in R$ is a *regular parameter* if $r \notin \mathfrak{m}^2$ for every maximal ideal $\mathfrak{m} \subset R$. For a ring A, we let $\mathrm{Frac}(A)$ denote its total ring of fractions. For a parabolic subgroup P of a reductive group scheme G, we let $\mathscr{R}_u(P)$ denote its unipotent radical constructed in [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, proposition 1.6 (i)]. We say that a torus T over a scheme S is *isotrivial* if it splits over some finite étale cover over S, and that this condition always holds if S is locally Noetherian and geometrically unibranch (in the sense that the map from the normalization of S_{red} to S is a universal homeomorphism), see [SGA 3_{II}, exposé X, théorème 5.16], or if T is of rank $\leqslant 1$.

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2. Torsors over $\mathbb{A}^1_A$ via the affine Grassmannian

Our eventual source of triviality of torsors is the following general result whose part (a) was conjectured in [Česnavičius22b, Conjecture 3.5.1] and generalizes [Fed22a, Theorem 5] along with several earlier results in the literature, and whose part (b) was suggested by results from [BČ22]. We ultimately deduce it from geometric properties of the affine Grassmannian Gr_G and from the unramified nature of the Whitehead group, both of which come through references to the survey [Čes22b].

Theorem 2.1. *Let A be a ring, let G be a totally isotropic reductive A-group scheme, and let $d > 0$.*

(a) *Every G-torsor over $\mathbb{A}^d_A$ that is trivial away from some A-finite closed subscheme is trivial, more generally, every G-torsor over $\mathbb{A}^d_A$ that reduces to an $\mathscr{R}_u(P)$-torsor away from some A-finite $Z \subset \mathbb{A}^d_A$ for some parabolic $(\mathbb{A}^d_A \backslash Z)$-subgroup $P \subset G_{\mathbb{A}^d_A \backslash Z}$ is trivial.*

(b) *No nontrivial G-torsor over A, equivalently, over $A[\![t]\!]$, trivializes over $A(\!(t)\!)$.*

(c) *Letting $A\{t\}$ denote the t-Henselization of $A[t]$, we have*

$$G(A(\!(t)\!)) = G(A[t^{\pm 1}])G(A[\![t]\!]), \qquad G(A\{t\}[\tfrac{1}{t}]) = G(A[t^{\pm 1}])G(A\{t\}),$$
$$G((A[t]_{1+tA[t]})[\tfrac{1}{t}]) = G(A[t^{\pm 1}])G(A[t]_{1+tA[t]}).$$

Remarks.

2.2. We state Theorem 2.1 for any $d > 0$, but one should keep focused on its main case of interest $d = 1$. Indeed, for $d > 1$, the first part of Theorem 2.1 (a) is an immediate consequence of [EGA IV_4, Proposition 19.9.8] and holds for any affine A-group G. Similar remarks apply to other results below: the key case to focus on is always that of a relative curve when $d = 1$.

2.3. As we already mentioned in the introduction, Theorem 2.1 (a) is sharp in that it fails if the reductive A-group G is no longer totally isotropic, see [Fed16, Theorem 3 and what follows].

We recall from [Čes22b, Section 3.5.2] that Theorem 2.1 (a) was known when G is either semisimple simply connected, or split, or a torus, and from [BČ22, Theorems 2.1.24 and 3.1.7] that Theorem 2.1 (b) was known when G is either a pure inner form of GL_n or a torus. For a general totally isotropic G, we first show in Lemma 2.5 (b) that a torsor over $\mathbb{A}^1_A$ that is trivial away from some A-finite $Z \subset \mathbb{A}^1_A$ trivializes after pulling back along any map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \to t^m$ for a sufficiently divisible m. For this, we follow Fedorov's strategy from [Fed22a] that is based on the geometry of the affine Grassmannian. The latter enters through (self-contained) citations to the survey [Čes22b] that mildly generalized Fedorov's steps. We will also use the following general form of Quillen patching.

Lemma 2.4 (Gabber, see [Čes22b, Corollary 5.1.5 (b)])**.** *For a ring A and a locally finitely presented A-group algebraic space G, a G-torsor (for the fppf topology) over $\mathbb{A}^d_A$ descends to a G-torsor over A if and only if it does so Zariski locally on* $\mathrm{Spec}(A)$. ☐

Lemma 2.5. *Let A be a ring, let G be a totally isotropic reductive A-group scheme, and let E be a G-torsor over $\mathbb{A}^1_A$ that is trivial away from some A-finite closed subscheme $Z \subset \mathbb{A}^1_A$.*

(a) *If, for some extension of E to a G-torsor $\widetilde{E}$ over $\mathbb{P}^1_A$ obtained by glueing E with the trivial torsor over $\mathbb{P}^1_A \backslash Z$ and for every prime ideal $\mathfrak{p} \subset A$, the G^{ad}-torsor over $\mathbb{P}^1_{k_\mathfrak{p}}$ induced by $\widetilde{E}$ lifts to a generically trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{k_\mathfrak{p}}$, then E is trivial.*

(b) *For any $m > 0$ divisible by the A-fibral degrees of the isogeny $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$, the pullback of E along any finite flat map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ of degree m that extends to a map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ is trivial.*

Proof. In (b), we extend E to a G-torsor $\widetilde{E}$ over $\mathbb{P}^1_A$ as in (a) and consider the pullback of this extension under our map $\mathbb{P}^1_A \to \mathbb{P}^1_A$. By [Čes22b, Lemma 5.3.5] (or [Fed22a, Proposition 2.3]), the choice of m ensures that the fibral condition of (a) holds for this pullback, so (b) follows from (a).

In (a), it suffices to show that both E and the restriction of $\widetilde{E}$ to the complementary affine line $\mathbb{P}^1_A \backslash \{t = 0\}$ descend to G-torsors over A: both of these descents will agree with the restriction of $\widetilde{E}$ to $t = 1$, which will agree with the restriction of $\widetilde{E}$ to $t = \infty$ and hence be trivial, and then E will also be trivial. By Quillen patching of Lemma 2.4, for the descent claim we may replace A by its localization at a maximal ideal to reduce to the case of a local A.

Once A is local, we will directly show that both E and the restriction of $\widetilde{E}$ to $\mathbb{P}^1_A \backslash \{t = 0\}$ are trivial. For this, we first show that we may modify Z so that it does not meet $t = 0$. Namely, if the residue field k of A is infinite, then there is some $s \in (\mathbb{A}^1_A \backslash (Z \cup \{t = 0\}))(A)$ and, by [Česnavičius22b, Proposition 5.3.6] (which uses the total isotropicity assumption, the fibral assumption on $\widetilde{E}$, and is based on geometric input about the affine Grassmannian in the style of [Fed22a, Theorem 6]), the restriction of $\widetilde{E}$ to $\mathbb{P}^1_A \backslash s$ is a trivial G-torsor, so that we may replace Z by s to arrange the desired $Z \cap \{t = 0\} = \varnothing$. In contrast, if the residue field k of A is finite, then there is some large n such that $\mathbb{A}^1_k \backslash (Z_k \cup \{t = 0\})$ contains a finite étale subscheme y that is the union of a point valued in the field extension of k of degree n and a point valued in the field extension of k of degree $n + 1$. Both of these components of y are cut out by separable monic polynomials with coefficients in k, so y lifts to an A-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_A \backslash (Z \cup \{t = 0\})$ that is a disjoint union of an A-(finite étale) closed subscheme of degree n and an A-(finite étale) closed subscheme of degree $n + 1$. In particular, both $\mathscr{O}(n)$ and $\mathscr{O}(n + 1)$ restrict to trivial line bundles on $\mathbb{P}^1_A \backslash Y$, and hence so does $\mathscr{O}(1)$. Thus, by [Čes22b, Proposition 5.3.6] once more, $\widetilde{E}$ is trivial on $\mathbb{P}^1_A \backslash Y$, to the effect that in the case when k is finite we may replace Z by Y to again arrange that $Z \cap \{t = 0\} = \varnothing$.

Once our $Z \subset \mathbb{A}^1_A$ does not meet $\{t = 0\}$, it suffices to apply [Čes22b, Proposition 5.3.6] twice to conclude that $\widetilde{E}$ restricts to the trivial torsor both on $\mathbb{P}^1_A \backslash \{t = \infty\}$ and on $\mathbb{P}^1_A \backslash \{t = 0\}$, as desired. ☐

2.6. Proof of Theorem 2.1. We have a ring A and a totally isotropic reductive A-group G. In (a), we have a G-torsor E over $\mathbb{A}^d_A$ that reduces to an $\mathscr{R}_u(P)$-torsor away from an A-finite closed subscheme $Z \subset \mathbb{A}^d_A$ for some parabolic $(\mathbb{A}^d_A \backslash Z)$-subgroup $P \subset G_{\mathbb{A}^d_A \backslash Z}$, and we need to show that E is trivial. For this it suffices to show that its pullback under any section $s \in \mathbb{A}^d_A(A)$ is trivial: indeed, as Gabber pointed out, by applying this after base change to the coordinate ring $A[t]$ of $\mathbb{A}^1_A$ and to the "diagonal" section of $\mathbb{A}^1_{A[t]} \to \operatorname{Spec}(A[t])$, we would get that E itself is trivial. Any A-point s of $\mathbb{A}^d_A$ factors through some $\mathbb{A}^{d-1}_A$-point, so we may replace A by $A[t_1, \ldots, t_{d-1}]$ to reduce to $d = 1$. In the case $d = 1$, since the coordinate ring of Z is a finite A-module, some monic polynomial in $A[t]$ vanishes on Z, so we may replace Z by this vanishing locus to arrange that $\mathbb{A}^1_A \backslash Z$ be affine. The advantage of this is that then [SGA $3_{\text{III new}}$, exposé XXVI, corollaire 2.2] ensures that E is even trivial over $\mathbb{A}^1_A \backslash Z$, in other words, we have reduced to the case when $d = 1$ and the parabolic P is G itself. We then change variables on $\mathbb{A}^1_A$ to transform s into the origin $t = 0$. This makes s lift to an A-point along every map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \mapsto t^d$. In particular, we may pull back along such a map for a sufficiently divisible d and apply Lemma 2.5 (b) to conclude that $s^*(E)$ is trivial.

For (b), we first recall from [BČ22, Theorem 2.1.6] that pullback along $t \mapsto 0$ is a bijection between the sets of isomorphism classes of G-torsors over $A[[t]]$ and over A. Thus, we need to show that every G-torsor over $A[[t]]$ that trivializes over $A((t))$ is trivial. However, patching of [BČ22, Lemma 2.2.11 (b)] (already of Lemma 3.8 below if $A[[t]]$ is $A[t]$-flat) ensures that we may glue such a G-torsor over $A[[t]]$ with the trivial G-torsor over $A[t, t^{-1}]$ to descend it to a G-torsor over $\mathbb{A}^1_A$ that trivializes over $\mathbb{G}_{m,A}$. It then suffices to apply part (a), according to which this descended G-torsor is trivial.

For (c), it suffices to note that if one of the displayed equalities was only a proper inclusion, then we could use the same patching as in the proof of (b) to produce a nontrivial G-torsor over $\mathbb{A}^1_A$ that trivializes over $\mathbb{G}_{m,A}$. This would contradict the already settled part (a). $\square$

3. Reembeddings into $\mathbb{A}^1_A$

To progress from torsors over the affine line $\mathbb{A}^1_A$ treated in §2 to those over an arbitrary smooth relative A-curve C in §4, we aim to build an étale map $C \to \mathbb{A}^1_A$ that would excisively embed a given A-finite closed subscheme $Z \subset C$ into $\mathbb{A}^1_A$. We build such excisive reembeddings in this section, and to accommodate for their construction we begin by first reembedding Z itself.

Proposition 3.1. *For a semilocal ring A, a finite A-scheme Z, a closed subscheme $Y \subset Z$, and compatible closed immersions $\iota_Y \colon Y \hookrightarrow \mathbb{A}^d_A$ and $\iota_{\mathfrak{m}} \colon Z_{k_{\mathfrak{m}}} \hookrightarrow \mathbb{A}^d_{k_{\mathfrak{m}}}$ for every maximal ideal $\mathfrak{m} \subset A$ and $d > 0$, there is a closed immersion $\iota \colon Z \hookrightarrow \mathbb{A}^d_A$ over A that extends the fixed ι_Y and $\iota_{\mathfrak{m}}$.*

Proof. Let $\widetilde{A}$ be the coordinate ring of Z. Since the closed immersions ι_Y and the $\iota_{\mathfrak{m}}$ are compatible, there are $a_1, \dots, a_d \in \widetilde{A}$ such that a_i on each $Z_{k_{\mathfrak{m}}}$ (resp., on Y) is the $\iota_{\mathfrak{m}}$-pullback (resp., ι_Y-pullback) of the i-th standard coordinate of $\mathbb{A}^d_{k_{\mathfrak{m}}}$ (resp., of $\mathbb{A}^d_A$). By sending the i-th standard coordinate of $\mathbb{A}^d_A$ to a_i we obtain a map $\iota \colon Z \to \mathbb{A}^d_A$. To check that this ι is our desired closed immersion it suffices to apply the Nakayama lemma [SP, Lemma 00DV] and to note that each $\iota_{\mathfrak{m}}$ is a closed immersion. $\square$

To apply Proposition 3.1 effectively, we need a practical criterion for the existence of the closed immersions $\iota_{\mathfrak{m}}$. For this, we first have to wrestle with the following finite field obstruction.

Definition 3.2. For a ring A, a quasi-finite A-scheme Z, and an A-scheme X, there is no *finite field obstruction* to embedding Z into X if for each maximal ideal $\mathfrak{m} \subset A$ with $k_{\mathfrak{m}}$ finite,

$$\#\{z \in Z_{k_{\mathfrak{m}}} \mid [k_z : k_{\mathfrak{m}}] = m\} \leqslant \#\{z \in X_{k_{\mathfrak{m}}} \mid [k_z : k_{\mathfrak{m}}] = m\} \quad \text{for every} \quad m \geqslant 1. \qquad (\dagger)$$

In practice, Z occurs as a closed subscheme of a smooth affine A-scheme, so the following lemma gives an applicable criterion for the existence of the closed immersions $\iota_{\mathfrak{m}} \colon Z_{k_{\mathfrak{m}}} \hookrightarrow \mathbb{A}^d_{k_{\mathfrak{m}}}$ in Proposition 3.1.

Lemma 3.3. *For a finite scheme Z over a field k and a nonempty open $U \subset \mathbb{A}^d_k$ with $d > 0$, there is a closed immersion $\iota \colon Z \hookrightarrow U$ if and only if Z is a closed subscheme of* some *smooth k-scheme C of pure dimension d and there is no finite field obstruction to embedding Z into U, in which case we may choose ι to extend any fixed embedding $\iota_Y \colon Y \hookrightarrow U$ of any closed subscheme $Y \subset Z$.*

Proof. The 'only if' is clear, so we fix closed immersions $Z \subset C$ and ι_Y as in the statement and assume that there is no finite field obstruction. We may build ι one connected component of Z at a time and shrink U at each step, so we may assume that Z is connected with unique closed point z.

In the case when the extension k_z/k is separable, [EGA IV_4, proposition 17.5.3] and the invariance of the étale site under nilpotents ensure that the n-th infinitesimal neighborhood of z in C is isomorphic to $\mathrm{Spec}(k_z[x_1, \dots, x_d]/(x_1^{n+1}, \dots, x_d^{n+1}))$ over k. In particular, this neighborhood does not depend on C and we may extend any fixed embedding $z \hookrightarrow U$, which exists by the assumption on the finite field obstruction, to a similar embedding of the n-th infinitesimal neighborhood of Z in C compatibly with ι_Y. This suffices because Z lies in this n-th infinitesimal neighborhood for every large enough n.

In the case when k_z is infinite and $Y = \varnothing$, we use [Čes22b, Proposition 4.1.4] (whose proof uses a presentation theorem similar to Lemma 8.1). By *loc. cit.*, there is an étale map $f \colon C \to \mathbb{A}^d_k$ such that $k_{f(z)} \xrightarrow{\sim} k_z$. In particular, since étale sites are insensitive to nilpotents, f embeds Z as a closed subscheme of $\mathbb{A}^d_k$. To force Z land in U, we note that since k_z is infinite, so is k, and for infinitely many changes of coordinates $t_1 \mapsto t_1 + \alpha_1, \ldots, t_d \mapsto t_d + \alpha_d$ with $\alpha_i \in k$ the image of Z will lie in U.

In the remaining case when k_z (equivalently, k) is infinite and $Y \neq \varnothing$, it suffices to show that the given closed immersion $\iota_Y : Y \hookrightarrow U$ extends to a closed immersion of the square-zero infinitesimal neighborhood ε_Y of Y in C: by iterating this with Y replaced by ε_Y and eventually restricting to Z, we will obtain the desired ι. By deformation theory, more precisely, by [Ill05, Theorem 8.5.9 (a)], the k-morphisms $\varepsilon_Y \to U$ that restrict to ι_Y are parametrized by some affine space $\mathbb{A}^N_k$. Since ε_Y is k-finite, the Nakayama lemma [SP, Lemma 00DV] ensures that the locus parametrizing those $\varepsilon_Y \to U$ that are closed immersions is an open $\mathscr{V} \subset \mathbb{A}^N_k$. Moreover, $\mathscr{V} \neq \varnothing$: indeed, we may check this after base change to any field extension of k, and a suitable such base change reduces us to the already settled case when k_z/k is separable. Since k is infinite and $\mathscr{V}$ is nonempty, $\mathscr{V}(k) \neq \varnothing$. Any k-point of $\mathscr{V}$ corresponds to a sought closed immersion $\varepsilon_Y \hookrightarrow U$ that restricts to ι_Y. □

To transform Proposition 3.1 into a statement that we will use for patching, we now extend [Čes22a, Lemma 6.3] (so also earlier versions due to Panin and Fedorov, see *loc. cit.*) to arrange that the closed immersion $\iota\colon Z \hookrightarrow \mathbb{A}^d_A$ built there be excisive as follows.

Proposition 3.4. *Let A be a semilocal ring, let $U \subset \mathbb{A}^d_A$ with $d > 0$ be an A-fiberwise nonempty open, and let Z be a finite A-scheme.*

(a) *There is a closed immersion $\iota\colon Z \hookrightarrow U$ iff there is no finite field obstruction to embedding Z into U and Z is a closed subscheme of* some *A-smooth affine scheme C of relative dimension d.*

(b) *If the conditions of (a) hold, then ι may be chosen to be excisive: then there are an affine open $D \subset C$ containing Z and an étale A-morphism $f\colon D \to U$ that fits into a Cartesian square*

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ \sim\downarrow & & \downarrow f \\ Z' & \hookrightarrow & U, \end{array} \tag{3.4.1}$$

in particular, such that f embeds Z as a closed subscheme $Z' \subset U$; for every A-finite closed subscheme $Y \subset Z$ and an embedding $\iota_Y\colon Y \hookrightarrow U$, there are D and f as above with $f|_Y = \iota_Y$.

Proof. For the 'only if' in (a), it suffices to note that if there is a closed immersion $\iota\colon Z \hookrightarrow U$, then, by Proposition 3.1, there is also a closed immersion $Z \hookrightarrow \mathbb{A}^d_A$. Thus, we focus on the 'if' in its stronger form (b). In particular, we fix an embedding $Z \subset C$ as in (a) and we let $\varepsilon_Z \subset C$ be the first infinitesimal neighborhood of Z in C, so that ε_Z is also finite over A.

By Lemma 3.3 and Proposition 3.1, there is a closed immersion $\widetilde{\iota}\colon \varepsilon_Z \hookrightarrow U$ that extends the fixed ι_Y. By lifting the $\widetilde{\iota}$-pullbacks of the standard coordinates of $\mathbb{A}^d_A$, we may extend $\widetilde{\iota}$ to an A-morphism $\widetilde{f}\colon C \to \mathbb{A}^d_A$. By construction, the *a priori* open locus of C where $\widetilde{f}$ is quasi-finite (see [SP, Lemma 01TI]) contains the points of Z. Thus, since Z has finitely many closed points, we may use prime avoidance [SP, Lemma 00DS] to shrink C around Z to arrange that $\widetilde{f}$ is quasi-finite. The flatness criteria [EGA IV_2, Proposition 6.1.5] and [EGA IV_3, Corollaire 11.3.11] then ensure that $\widetilde{f}$ is flat at the points of Z, so, by construction, $\widetilde{f}$ is even étale at the points of Z. Consequently, we may shrink C further around Z to arrange that $\widetilde{f}$ is étale and factors through U. A section of a separated étale morphism, such as $\widetilde{f}^{-1}(\widetilde{f}(Z)) \to \widetilde{f}(Z)$, is an inclusion of a clopen subset, so, by shrinking C around Z once more, we arrange that $Z = \widetilde{f}^{-1}(\widetilde{f}(Z))$. This equality means that the square (3.4.1) is Cartesian, so, granted all the shrinking above, it remains to set $D := C$ and $\widetilde{f} := f$. □

The following corollary is useful for embedding a *finite étale* Z into U without an ambient scheme C.

Corollary 3.5. *For a semilocal ring A and an A-fiberwise nonempty open $U \subset \mathbb{A}^d_A$ with $d > 0$, an A-(finite étale) scheme Z embeds into U if and only if there is no finite field obstruction to it.*

Proof. The 'only if' is clear. For the 'if,' by Proposition 3.4 (a), it is enough to embed Z into $\mathbb{A}^d_A$, so we may assume that $U = \mathbb{A}^d_A$. It then suffices to show that $Z \cong \operatorname{Spec}(A')$ with an A' that may be generated by d elements as an A-algebra. Thus, since A' is a finite A-module and A is semilocal, the Nakayama lemma [SP, Lemma 00DV] allows us to replace A by the product of the residue fields of its maximal ideals, so we may assume that A is a field k. In this case, Z is a disjoint union of spectra of finite separable field extensions k and, since there is no finite field obstruction to embedding Z into $\mathbb{A}^d_k$, such an embedding exists by the primitive element theorem. □

To bypass the finite field obstruction in practice, we will modify Z via the following lemma. It extends [Čes22a, Lemma 6.1] (so also earlier versions by Panin) and is built on Panin's "finite field tricks."

Lemma 3.6. *Let A be a semilocal ring, let Z be a quasi-finite, separated A-scheme, let $Y \subset Z$ be an A-finite closed subscheme, and let X be an A-scheme such that for every maximal ideal $\mathfrak{m} \subset A$ some closed subscheme of $X_{k_\mathfrak{m}}$ is of finite type over $k_\mathfrak{m}$, positive dimensional, and geometrically irreducible.*

(a) *There is a finite étale surjection $\widetilde{Z} \twoheadrightarrow Z$ such that there is no finite field obstruction to embedding $\widetilde{Z}$ into X, moreover, for every large $N > 0$ we may find such a $\widetilde{Z}$ of the form $\widetilde{Z}_0 \sqcup \widetilde{Z}_1$ with $\widetilde{Z}_i \cong \underline{\operatorname{Spec}}(\mathscr{O}_Z[t]/(f_i(t))) \twoheadrightarrow Z$ surjective and f_i monic of constant degree $N + i$.*

(b) *Fix any sufficiently divisible $n \geqslant 0$ and suppose that $Y = Y_0 \sqcup Y_1$ such that there is no finite field obstruction to embedding Y_0 into X. Then (a) holds with the requirement that*

$$\widetilde{Y} := Y \times_Z \widetilde{Z} \quad \text{is a disjoint union} \quad \widetilde{Y} = \widetilde{Y}_0 \sqcup \widetilde{Y}_1$$

such that $\widetilde{Y}_0 \xrightarrow{\sim} Y_0$ and each connected component of $\widetilde{Y}_1$ is a scheme over a finite $\mathbb{Z}$-algebra B each of whose residue fields k of characteristic $p \mid n$ satisfies

$$\#k > n \cdot \deg(\widetilde{Z}/Z).$$

Part (b) is a critical statement that we will use in §7 to bypass finite field difficulties of [Fed22b], and a typical case is when $Y = Y_0 = \operatorname{Spec}(A)$ is an A-rational point of Z. To be clear, in (b) the $\mathbb{Z}$-algebra B depends on the connected component of $\widetilde{Y}_1$ in question.

Proof. We may replace Z by any A-finite scheme containing Z as an open, so we use the Zariski Main Theorem [EGA IV$_4$, Corollaire 18.12.13] to assume that $Z = \operatorname{Spec}(A')$ for an A-finite A'. To explain the role of the assumption on X, recall that by the Weil conjectures [Poo17, Theorem 7.7.1 (ii)], it implies that for every $m > 0$, every maximal ideal $\mathfrak{m} \subset A$ with $k_\mathfrak{m}$ finite, and every large $d > 0$,

$$\{z \in X_{k_\mathfrak{m}} \mid [k_z : k_\mathfrak{m}] = d\} \geqslant m. \tag{3.6.1}$$

In (a), we let $N > 2$ be sufficiently large and choose the following monic polynomials: for each closed point $z \in Z$ with k_z finite (resp., infinite), a monic $f_z(t) \in k_z[t]$ that is irreducible of degree N (resp., that is the product of N distinct monic linear polynomials). We let $f_0(t) \in A'[t]$ be a monic polynomial that simultaneously lifts all the $f_z(t)$, and we define a monic $f_1(t) \in A'[t]$ analogously with N replaced by $N + 1$. Granted that N is large enough, by (3.6.1), the resulting $\widetilde{Z}_i$ settle (a).

In (b), to satisfy the "sufficiently divisible" requirement it suffices to make sure that n is divisible by all the positive residue characteristic of A. Granted this, for each $N > 2$ we choose

- an $f_{Y_0}(t) \in \mathbb{Z}[t]$ that is the product of t and a monic polynomial of degree $N-1$ whose reduction modulo every prime $p \mid n$ is irreducible;
- a monic $f_{Y_1}(t) \in \mathbb{Z}[t]$ of degree N whose reduction modulo every prime $p \mid n$ is irreducible;
- for each closed point $z \in Z$ not in Y with k_z finite (resp., infinite), an $f_z(t) \in k_z[t]$ that is irreducible of degree N (resp., that is the product of N distinct monic linear polynomials).

We write $Y_i = \mathrm{Spec}(A'_i)$ and view $f_{Y_i}(t)$ as an element of $A'_i[t]$. Since Y and the closed points of Z not in Y form a closed subscheme of Z, there is a monic polynomial $f_0(t) \in A'[t]$ whose image in $A'_i[t]$ (resp., in $k_z[t]$ for each closed point $z \in Z$ not in Y) is $f_{Y_i}(t)$ (resp., $f_z(t)$). With the resulting $\widetilde{Z}_0$ defined by this $f_0(t)$ as in (a), let $\widetilde{Y}_0$ be component of $Y_0 \times_Z \widetilde{Z}_0$ cut out by the factor t of $f_{Y_0}(t)$ to arrange that $\widetilde{Y}_0 \xrightarrow{\sim} Y_0$. By the choice of the $f_{Y_i}(t)$, each connected component of the complement of $\widetilde{Y}_0$ in $Y \times_Z \widetilde{Z}_0$ is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k of characteristic $p > 0$ with $p \mid n$ has degree either $N-1$ or N over $\mathbb{F}_p$.

We repeat the construction with N replaced by $N+1$, except that now we let $f_{Y_0}(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $N+1$ whose reduction modulo every prime $p \mid n$ is irreducible, to build a monic $f_1(t) \in A'[t]$ of degree $N+1$. For the resulting $\widetilde{Z}_1$, by construction, each connected component of $Y \times_Z \widetilde{Z}_1$ is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k of characteristic $p > 0$ with $p \mid n$ has degree $N+1$ over $\mathbb{F}_p$. Overall, with the resulting $\widetilde{Y}_1$ complementary to $\widetilde{Y}_0$, every connected component of $\widetilde{Y}_1$ is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k of characteristic $p \mid n$ has degree $N-1$, N, or $N+1$ over $\mathbb{F}_p$. For large N, such a k satisfies

$$\#k > nN(N+1) = n \cdot \deg(\widetilde{Z}/Z).$$

By construction, the number of closed points of $\widetilde{Z}$ not in $\widetilde{Y}_0$ with a finite residue field is bounded as N grows and the degree of the residue field of every such closed point over the corresponding $\mathbb{F}_p$ is $\geqslant \varepsilon N$ for some $\varepsilon > 0$ (that depends on the degrees of k_z over the $\mathbb{F}_p$, but not on N). In particular, for large N, by (3.6.1), there is no finite field obstruction to embedding $\widetilde{Z}$ into X. □

Remark 3.7. The A-finite Z that is to be modified as in Lemma 3.6 to avoid the finite field obstruction to embedding it into X often occurs as a closed subscheme of a smooth affine A-scheme C, and it is useful to lift the resulting $\widetilde{Z} \twoheadrightarrow Z$ to a finite étale cover $\widetilde{C} \twoheadrightarrow D$ of an affine open neighborhood $D \subset C$ of Z. Since the $\widetilde{Z}_i$ are explicit, this is possible to arrange: it suffices to lift each $f_i(T)$ to a monic polynomial with coefficients in the coordinate ring of the semilocalization of C at the closed points of Z (built via prime avoidance [SP, Lemma 00DS]) and to spread out.

Throughout this article, we will analyze torsors that are trivial away from a closed subscheme Z. For this, the following basic glueing technique of Moret-Bailly [MB96] (with a more restrictive version implicit already in [FR70, Proposition 4.2]) will let us take advantage of excisive squares like (3.4.1).

Lemma 3.8 ([Čes22b, Proposition 4.2.1])**.** *For a scheme S, a closed subscheme $Z \subset S$ that is locally cut out by a finitely generated ideal, an affine, flat map f that fits into a Cartesian square*

$$\begin{array}{ccc} Z & \hookrightarrow & S \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \hookrightarrow & S' \end{array}$$

and embeds Z as a closed subscheme $Z' \subset S'$ (so $Z \cong Z' \times_{S'} S$ by the Cartesianness requirement), and a quasi-affine, flat, finitely presented S'-group G, base change induces an equivalence of categories

$$\{G\text{-torsors over } S'\} \xrightarrow{\sim} \{G\text{-torsors over } S\} \times_{\{G\text{-torsors over } S\setminus Z\}} \{G\text{-torsors over } S'\setminus Z'\},$$

in particular, a G-torsor over S descends to S' if and only if it does so away from Z'. □

Remark 3.9. If the flat map f is locally of finite presentation, then the excisive condition on Z and Z' implies that f is étale at the points of Z. This means that it then induces an isomorphism between the formal completion of S along Z and that of S' along Z'.

4. Grothendieck–Serre for smooth relative curves over arbitrary rings

We use the reembedding techniques discussed above to present a Grothendieck–Serre phenomenon over arbitrary base rings: in Theorem 4.5 we show that torsors under reductive groups over smooth relative curves are Zariski semilocally trivial as soon as they are trivial away from some relatively finite closed subscheme. To approach this beyond constant G, we first establish Lemma 4.3 about equating reductive groups, which is a variant of [PSV15, Theorem 3.6] of Panin–Stavrova–Vavilov and combines ideas from [Čes22a, Lemma 5.1] with those from the survey [Čes22b, Chapter 6].

Definition 4.1 ([Čes22b, ($\star$) in the beginning of Section 6.2])**.** For a ring A and an ideal $I \subset A$, we consider the following property of a set-valued functor $\mathscr{F}$ defined on the category of A-algebras:

$$\begin{aligned}&\text{for every } x \in \mathscr{F}(A/I)\text{, there are a faithfully flat, finite, étale } A\text{-algebra } \widetilde{A},\\ &\text{an } A/I\text{-point } a\colon \widetilde{A} \twoheadrightarrow A/I\text{, and an } \widetilde{x} \in \mathscr{F}(\widetilde{A}) \text{ whose } a\text{-pullback is } x.\end{aligned} \qquad (\bigstar)$$

Remark 4.2. Let $f\colon \mathscr{F} \to \mathscr{F}'$ be a map of functors on the category of A-algebras and, for a $y \in \mathscr{F}'(A)$, let $\mathscr{F}_y \subset \mathscr{F}$ denote the f-fiber of y. If $\mathscr{F}'$ has property ($\bigstar$) with respect to $I \subset A$ and, for every faithfully flat, finite, étale A-algebra $\widetilde{A}$ and every $y \in \mathscr{F}'(\widetilde{A})$, the fiber $(\mathscr{F}|_{\widetilde{A}})_y$ has property ($\bigstar$) with respect to any ideal $\widetilde{I} \subset \widetilde{A}$ with $\widetilde{A}/\widetilde{I} \cong A/I$, then $\mathscr{F}$ itself has property ($\bigstar$) with respect to $I \subset A$. This straight-forward dévissage is useful in practice for dealing with short exact sequences.

Lemma 4.3. *For a semilocal ring A, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibers have the same type and whose maximal central tori $\mathrm{rad}(G)$ and $\mathrm{rad}(G')$ are isotrivial, maximal A-tori $T \subset G$ and $T' \subset G'$, and an A/I-group isomorphism*

$$\iota\colon G_{A/I} \xrightarrow{\sim} G'_{A/I} \quad \textit{such that} \quad \iota(T_{A/I}) = T'_{A/I},$$

there are a faithfully flat, finite, étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a\colon \widetilde{A} \twoheadrightarrow A/I$ and an $\widetilde{A}$-group isomorphism $\widetilde{\iota}\colon G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ whose a-pullback is ι and such that $\widetilde{\iota}(T_{\widetilde{A}}) = T'_{\widetilde{A}}$.

Proof. By passing to connected components, we may assume that $\mathrm{Spec}(A)$ is connected, so that the types of the geometric fibers of G and G' are constant. The claim is that the functor

$$X := \underline{\mathrm{Isom}}_{\mathrm{gp}}((G,T),(G',T'))$$

that parametrizes those group scheme isomorphisms between base changes of G and G' that bring T to T' has property ($\bigstar$) with respect to $I \subset A$. By [SGA $3_{\mathrm{III\ new}}$, exposé XXIV, corollaires 1.10 et 2.2 (i)], the normalizer $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ of the A-torus $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ induced by T acts freely on X and, thanks to the assumption about the geometric fibers of G and G', the quotient

$$\overline{X} := X/N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$$

is a faithfully flat A-scheme that becomes constant étale locally on A. We claim that $\overline{X}$ has property ($\star$) with respect to $I \subset A$, more generally, that each quasi-compact subset of $\overline{X}$ is contained is some A-(finite étale) clopen subscheme of $\overline{X}$. The advantage of this last claim is that it suffices to argue it after base change along any finite étale cover of A. Thus, we may combine our assumption on $\mathrm{rad}(G)$ and $\mathrm{rad}(G')$ with [SGA $3_{\mathrm{III\ new}}$, exposé XXIV, théorème 4.1.5] to assume that both G and G' are split. In this case, however, [SGA $3_{\mathrm{III\ new}}$, exposé XXIV, théorème 1.3 (iii) et corollaire 2.2 (i)] ensure that $\overline{X}$ is a constant A-scheme, so the claim is clear.

With the property ($\star$) of $\overline{X}$ in hand, by Remark 4.2, we may replace A by a finite étale cover to reduce to showing that every $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$-torsor has property ($\star$). However, $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ is an extension of a finite étale A-group scheme by T^{ad} (see, for instance, [Čes22b, Section 1.3.2]), so we may repeat the same reduction based on Remark 4.2 and be left with showing that every T^{ad}-torsor has property ($\star$) with respect to $I \subset A$. This, however, is a special case of [Čes22b, Corollary 6.3.2] (based on building an equivariant projective compactification of the A-torus T^{ad} using toric geometry). ☐

Remark 4.4. Lemma 4.3 continues to hold if instead of the maximal A-tori $T \subset G$ and $T' \subset G'$, the groups G and G' come equipped with fixed quasi-pinnings extending Borel A-subgroups $B \subset G$ and $B' \subset G'$, and if ι and $\tilde{\iota}$ are required to respect these quasi-pinnings, see [Čes22a, Lemma 5.1].

We are ready for the following promised Grothendieck–Serre type result over arbitrary base rings.

Theorem 4.5. *Let A be a ring, let B be an A-algebra, let C be a smooth affine A-scheme of pure relative dimension $d > 0$, let $\mathscr{G}$ be a reductive $(C \otimes_A B)$-group scheme that lifts to a reductive C-group $\widetilde{\mathscr{G}}$ whose maximal central torus $\mathrm{rad}(\widetilde{\mathscr{G}})$ is isotrivial Zariski semilocally on C (resp., that descends to a reductive B-group G), and let $\mathscr{P} \subset \mathscr{G}$ be a parabolic $(C \otimes_A B)$-subgroup that lifts to a parabolic C-subgroup $\widetilde{\mathscr{P}} \subset \widetilde{\mathscr{G}}$ (resp., that descends to a parabolic B-subgroup $P \subset G$). Suppose either that*

(i) $B = A$; *or that*

(ii) $\mathscr{G}$ *is totally isotropic.*

Then every $\mathscr{G}$-torsor $\mathscr{E}$ over $C \otimes_A B$ whose restriction to $(C \backslash Z) \otimes_A B$ for some A-finite $Z \subset C$ reduces to a $\mathscr{R}_u(\mathscr{P})$-torsor trivializes Zariski semilocally on C, that is, for every $c_1, \ldots, c_n \in C$, there is an affine open $C' \subset C$ containing all the c_i such that $\mathscr{E}$ trivializes over $C' \otimes_A B$.

The core case to keep in mind is when $\mathscr{P} = \mathscr{G}$ and $B = A$: then the claim is that every $\mathscr{G}$-torsor $\mathscr{E}$ on C that is trivial away from some A-finite closed subscheme $Z \subset C$ is Zariski semilocally trivial. This case generalizes [Fed22a, Theorem 4], as well as several earlier results in the literature.

Proof. Let A' be the semilocal ring of C at $c_1, \ldots, c_n$, so that, by a limit argument, it suffices to show that $\mathscr{E}$ trivializes over $A' \otimes_A B$. After base change to A' the map $\mathrm{Spec}(A') \to C$ induces a "diagonal" section of C, so, by performing such a base change and replacing B by $B \otimes_A A'$, we reduce to showing that, when A semilocal, the pullback of $\mathscr{E}$ under $s \otimes_A B$ for any $s \in C(A)$ is trivial. In addition, we enlarge Z if necessary to ensure that $s \in Z(A)$.

Granted this reformulation of the goal statement, we will reduce to the case when $\mathscr{G}$ descends to a reductive B-group G (which, being the pullback of $\mathscr{G}$ along $s \otimes_A B$, is totally isotropic in the case (ii)) and $\mathscr{P} \subset \mathscr{G}$ descends to a parabolic B-subgroup $P \subset G$. For this, it suffices to focus on the case when $\mathscr{G}$ lifts to a reductive C-group $\widetilde{\mathscr{G}}$ for which $\mathrm{rad}(\widetilde{\mathscr{G}})$ is isotrivial Zariski semilocally on C and $\mathscr{P} \subset \mathscr{G}$ lifts to a parabolic C-subgroup $\widetilde{\mathscr{P}} \subset \widetilde{\mathscr{G}}$, and to reduce this case to when $\widetilde{\mathscr{G}}$ descends to a reductive A-group $\widetilde{G}$ and $\widetilde{\mathscr{P}} \subset \widetilde{\mathscr{G}}$ descends to a parabolic A-subgroup $\widetilde{P} \subset \widetilde{G}$. We begin by defining the candidate $\widetilde{P} \subset \widetilde{G}$ simply as the s-pullback of $\widetilde{\mathscr{P}} \subset \widetilde{\mathscr{G}}$.

By shrinking C around the closed points of Z, we may assume that $\mathrm{rad}(\widetilde{\mathscr{G}})$ is isotrivial, that $\widetilde{\mathscr{G}}$ has a maximal torus $\widetilde{\mathscr{T}} \subset \widetilde{\mathscr{G}}$ defined over C (see [SGA 3_{II}, exposé XIV, corollaire 3.20]), and, by passing to clopens if needed, that the type of the geometric C-fibers of $\widetilde{\mathscr{G}}$ is constant. We let $\widetilde{T} \subset \widetilde{G}$ be the s-pullback of $\widetilde{\mathscr{T}}$. By Lemma 4.3 and spreading out, there are an affine open $D \subset C$ containing Z and a finite étale cover $\widetilde{C} \twoheadrightarrow D$ for which s lifts to some $\widetilde{s} \in \widetilde{C}(A)$ such that $\widetilde{\mathscr{G}}|_{\widetilde{C}} \simeq \widetilde{G}|_{\widetilde{C}}$ compatibly with the fixed identification of pullbacks along $\widetilde{s}$. Thus, we may replace C and s by $\widetilde{C}$ and $\widetilde{s}$, respectively, and reduce to the case when $\widetilde{\mathscr{G}}$ descends, that is, to when $\widetilde{\mathscr{G}} = \widetilde{G}_C$. To now likewise descend $\widetilde{\mathscr{P}}$, we first pass to clopens to assume that the type of $\widetilde{\mathscr{P}}$ as a parabolic subgroup of $\widetilde{G}_C$ is constant on C. Then $\widetilde{P}_C$ and $\widetilde{\mathscr{P}}$ are parabolic subgroups of $\widetilde{G}_C$ of the same type, so, by [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 5.5 (iv)] and a limit argument, they are conjugate over some affine open neighborhood of Z in C. Since parabolic subgroups are self-normalizing [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, proposition 1.2], the s-pullback of a conjugating section lies in $\widetilde{P}$, so we may adjust by this s-pullback to make the conjugating section pull back to the identity by s. Thus, by shrinking C and adjusting the identification between $\widetilde{\mathscr{G}}$ and $\widetilde{G}_C$ by an aforementioned conjugation, we achieve the promised reduction to the case when $\widetilde{\mathscr{P}} \subset \widetilde{\mathscr{G}}$ descends to $\widetilde{P} \subset \widetilde{G}$.

With $\mathscr{P} \subset \mathscr{G}$ now being the base change of $P \subset G$, our next goal is to reduce to the case when $C = \mathbb{A}^d_A$. For this, we begin with a closed immersion $Z \hookrightarrow C$ and combine Lemma 3.6 (b), Remark 3.7, and Proposition 3.4 to reduce to when there is an étale morphism $C \to \mathbb{A}^d_A$ and a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & C \\ \| & & \downarrow \\ Z & \hookrightarrow & \mathbb{A}^d_A. \end{array}$$

Since every A-point s of $\mathbb{A}^d_A$ factors through some $\mathbb{A}^{d-1}_A$-point, by restricting the square above to this $\mathbb{A}^{d-1}_A$-point of $\mathbb{A}^d_A$, we may decrease d to eventually reduce to $d = 1$. Once $d = 1$, however, the A-finite $Z \subset \mathbb{A}^1_A$ is monogenic in the sense that its coordinate ring is generated by a single element as an A-algebra. Since this coordinate ring is also a finite A-module, a lift of an A-algebra generator of Z to the coordinate ring of C satisfies a monic polynomial with coefficients in A when restricted to Z. Thus, by enlarging Z to be the vanishing locus of this monic polynomial, we reduce to the case when $C \backslash Z$ is affine at the cost of losing the Cartesian square above. Once $C \backslash Z$ is affine, [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 2.2] ensures that $\mathscr{E}$ restricts to the trivial G-torsor over $C \backslash Z$, in other words, we have reduced to the case when $P = G$.

Granted the reductions above, we now apply the same reembedding and patching technique based on Lemma 3.6 (b), Proposition 3.4, and Lemma 3.8 to $Z \sqcup \mathrm{Spec}(A) \hookrightarrow C \sqcup \mathbb{A}^1_A$ instead to reduce to the case when we still have the Cartesian square above with $d = 1$ such that, in addition, $(\mathbb{A}^1_A \backslash Z)(A) \neq \varnothing$. The square remains Cartesian after base change to B, so the patching Lemma 3.8 ensures that $\mathscr{E}$ descends to a G-torsor over $\mathbb{A}^1_A \otimes_A B$ that trivializes over $(\mathbb{A}^1_A \backslash Z) \otimes_A B$. In the totally isotropic case (ii), it then suffices to apply Theorem 2.1 (a) to conclude that $\mathscr{E}$ is trivial.

Since we are left with the case (i), we assume from now on that $B = A$. Moreover, we change coordinates to make s be the section $t = 0$, and we subsequently use the assumption $(\mathbb{A}^1_A \backslash Z)(A) \neq \varnothing$ to scale the standard coordinate of $\mathbb{A}^1_A$ to ensure that Z does not meet the section $t = 1$. Granted this, we glue $\mathscr{E}$ with the trivial torsor over $\mathbb{P}^1_A \backslash Z$ and hence extend it to a G-torsor $\widetilde{\mathscr{E}}$ over $\mathbb{P}^1_A$. We let m be the least common multiple of the A-fibral degrees of the isogeny $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$ and, as in the proof of Lemma 2.5 (b), replace $\widetilde{\mathscr{E}}$ by its pullback along the map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ given by $[x : y] \mapsto [x^m : y^m]$ to arrange that, for every maximal ideal $\mathfrak{m} \subset A$, the G^{ad}-torsor over $\mathbb{P}^1_{k_\mathfrak{m}}$ induced by $\widetilde{\mathscr{E}}$ lifts to a generically trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{k_{\mathfrak{m}}}$ (see [Čes22b, Lemma 5.3.5]). Both sections $t = 0$ and $t = 1$ of $\mathbb{A}^1_A$ lift along this map, so we retain other assumptions, in particular, we still have $Z \cap \{t = 1\} = \varnothing$. We will eventually obtain the conclusion from [Čes22b, Proposition 5.3.6], so, to prepare for applying it, we consider the canonical decomposition as in (1.1.1):

$$G^{\mathrm{ad}} \cong \textstyle\prod_i H_i \quad \text{with} \quad H_i \cong \mathrm{Res}_{A_i/A}(G_i),$$

where A_i is a finite étale A-algebra and G_i is an adjoint A_i-group scheme with simple geometric fibers. For each i, consider the projective, smooth A-scheme X_i parametrizing parabolic subgroups of H_i (see [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 3.5]). For each i, consider the closed subscheme $\mathrm{Spec}(A/I_i) \subset \mathrm{Spec}(A)$ that is the disjoint union of those maximal ideals $\mathfrak{m} \subset A$ such that $(H_i)_{k_{\mathfrak{m}}}$ is isotropic, in other words, such that $(H_i)_{k_{\mathfrak{m}}}$ has a proper parabolic subgroup (see [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 6.12]), and fix such parabolic subgroups to obtain an $x_i \in X_i(A/I_i)$. By [Čes22b, Lemma 6.2.2] (which is based on Bertini theorem), there are a faithfully flat, finite, étale A-scheme Y_i equipped with an A/I_i-point $y_i \in Y_i(A/I_i)$ and an A-morphism $Y_i \to X_i$ that maps y_i to x_i, so that, in particular, $(H_i)_{Y_i}$ is totally isotropic for every i.

By Lemma 3.6 (a), there is a finite étale cover $\pi\colon \widetilde{Y} \twoheadrightarrow \bigsqcup_i Y_i$ such that there is no finite field obstruction to embedding $\widetilde{Y}$ into $\mathbb{A}^1 \backslash (Z \cup \{t = 1\})$ and $\widetilde{Y} = \widetilde{Y}' \sqcup \widetilde{Y}''$ with $\widetilde{Y}'$ (resp., $\widetilde{Y}''$) surjective over $\bigsqcup_i Y_i$ of constant degree N (resp., $N + 1$) for some $N > 0$. By Corollary 3.5, we may therefore find an embedding $\widetilde{Y} \hookrightarrow \mathbb{A}^1_A$ whose image does not meet Z nor the section $t = 1$. By construction, for each i and each maximal ideal $\mathfrak{m} \subset A$ such that $(H_i)_{k_{\mathfrak{m}}}$ is isotropic, Y_i has a $k_{\mathfrak{m}}$-point, and so the $k_{\mathfrak{m}}$-fiber of the preimage $\widetilde{Y}_i := \pi^{-1}(Y_i)$ has two disjoint clopens that have degrees N and $N + 1$ over $k_{\mathfrak{m}}$. Consequently, for each such i and $\mathfrak{m}$, the line bundle $\mathscr{O}(1)$ is trivial over $(\mathbb{P}^1_A \backslash \widetilde{Y}_i)_{k_{\mathfrak{m}}}$. Thus, since $(\widetilde{Y} \sqcup \{t = 1\}) \cap Z = \varnothing$, we may apply [Čes22b, Proposition 5.3.6] to conclude that $\mathscr{E}$ is trivial over $\mathbb{A}^1_A \backslash (\widetilde{Y} \sqcup \{t = 1\})$. Since $\widetilde{Y}$ is disjoint from s, the pullback $s^*(\mathscr{E})$ is also trivial, and (i) follows. ☐

Corollary 4.6. *For a reductive group G over a semilocal ring A, no nontrivial G-torsor over A, equivalently, over $A[[t]]$, trivializes over $A((t))$.*

Proof. As in the proof of Theorem 2.1 (b) given in §2.6, every G-torsor $\mathscr{E}$ over $A[[t]]$ that trivializes over $A((t))$ descends to a G-torsor over $\mathbb{A}^1_A$ that trivializes over $\mathbb{G}_{m,A}$. Thus, Theorem 4.5 implies that $\mathscr{E}|_{\{t=0\}}$ is the trivial G-torsor over A. By [BČ22, Theorem 2.1.6], then $\mathscr{E}$ itself is trivial. ☐

5. The mixed characteristic cases of our main result on Nisnevich conjecture

We deduce the mixed characteristic cases of Theorem 1.3 from the Grothendieck–Serre phenomenon of Theorem 4.5. To arrive at its relative curve setting, we use the following presentation lemma.

Lemma 5.1 ([Čes22a, Proposition 4.1])**.** *For a smooth, affine scheme X of relative dimension $d > 0$ over a semilocal Dedekind ring $\mathcal{O}$, points $x_1, \ldots, x_m \in X$, and a closed subscheme $Z \subset X$ of codimension $\geqslant 2$, there are an affine open $X' \subset X$ containing $x_1, \ldots, x_m$, an affine open $S \subset \mathbb{A}^{d-1}_{\mathcal{O}}$, and a smooth morphism $f\colon X' \to S$ of relative dimension 1 such that $X' \cap Z$ is S-finite.* ☐

Remark 5.2. In the case when $\mathcal{O}$ is a field, the same statement holds under the weaker assumption that Z is merely of codimension $\geqslant 1$ in X, see [Čes22a, Remark 4.3] or Lemma 8.1 below (whose proof does not use any other results from the present article).

5.3. The abstract maximal torus. To every reductive group G over a scheme S one associates an S-torus T_G, the *abstract maximal torus* of G defined by étale descent on S as follows. Étale locally on S, the group G has a Borel $B \subset G$, and, letting $\mathscr{R}_u(B) \subset B$ denote the unipotent radical, one sets

$$T_G := B/\mathscr{R}_u(B).$$

Up to a canonical isomorphism, this T_G does not depend on the choice of B, and so it descends to the original S: indeed, any two Borels are Zariski locally conjugate and, up to multiplying by a section of B, the conjugating section is unique [SGA $3_{\text{III new}}$, exposé XXVI, proposition 1.2, corollaire 5.2], so it suffices to note the conjugation action of B on T_G is trivial because the latter is abelian.

5.4. Proof of Theorem 1.3 (2). We have a semilocal ring R that is flat and geometrically regular over a Dedekind subring $\mathcal{O}$, an $r \in \mathcal{O}$, a reductive $R[\frac{1}{r}]$-group G that either extends to a quasi-split reductive R-group or descends to a quasi-split reductive $\mathcal{O}[\frac{1}{r}]$-group, and a generically trivial G-torsor E over $R[\frac{1}{r}]$. We need to show that E is trivial, and we will do this by applying Theorem 4.5 (ii).

We use Popescu theorem [SP, Theorem 07GC] and a limit argument to reduce to the case when R is a semilocal ring of a smooth affine $\mathcal{O}$-scheme X. By passing to connected components if needed, we may assume that X is connected, of constant relative dimension d over $\mathcal{O}$. If $d = 0$, then R, and so also $R[\frac{1}{r}]$, is a semilocal Dedekind ring, and E is trivial by [Guo20, Theorem 1]; therefore, we lose no generality by assuming that $d > 0$. By shrinking X if needed, we may assume that G (resp., E) begins life over X (resp., over $X[\frac{1}{r}]$). In the case when our original G lifts to a quasi-split reductive R-group, we shrink X further to make G extend to a quasi-split reductive X-group $\widetilde{G}$ and we fix a Borel X-subgroup $B \subset \widetilde{G}$. In the case when our original G descends to a quasi-split $\mathcal{O}[\frac{1}{r}]$-group, we shrink X further to make sure that our new G over $X[\frac{1}{r}]$ still descends to a quasi-split reductive group over $\mathcal{O}[\frac{1}{r}]$, and we fix a Borel $\mathcal{O}[\frac{1}{r}]$-subgroup B of this descended group.

By applying the valuative criterion of properness to $E/B_{X[\frac{1}{r}]}$, we may choose an open $U \subset X[\frac{1}{r}]$ with complement of codimension $\geqslant 2$ such that E_U to reduces to a generically trivial B-torsor $\mathcal{E}^B$ over U. By purity for torsors under tori [CTS79, corollaire 6.9], the T_G-torsor $\mathcal{E}^B/\mathscr{R}_u(B)$ over U extends to a generically trivial T_G-torsor over $X[\frac{1}{r}]$. To proceed, we use the following claim.

Claim 5.4.1. The abstract maximal torus of G has no nontrivial generically trivial torsors over $R[\frac{1}{r}]$:

$$H^1(R[\tfrac{1}{r}], T_G) \hookrightarrow H^1(\mathrm{Frac}(R[\tfrac{1}{r}]), T_G).$$

Proof. Thanks to our assumption on G, the torus $(T_G)_{R[\frac{1}{r}]}$ is the base change of a torus $\mathcal{T}$ defined over a ring A that is either R or $\mathcal{O}[\frac{1}{r}]$. By [CTS87, Proposition 1.3], this $\mathcal{T}$ has a flasque resolution

$$0 \to \mathcal{F} \to \mathrm{Res}_{A'/A}(\mathbb{G}_m) \to \mathcal{T} \to 0,$$

where A' is a finite étale A-algebra and $\mathcal{F}$ is a flasque A-torus. For now, all we need to know about flasque tori is that, by the regularity of $R[\frac{1}{r}]$ and [CTS87, Proposition 1.4, Theorem 2.2 (ii)],

$$H^2(R[\tfrac{1}{r}], \mathcal{F}) \hookrightarrow H^2(\mathrm{Frac}(R[\tfrac{1}{r}]), \mathcal{F}).$$

This reduces our desired claim to the vanishing $\mathrm{Pic}(R[\frac{1}{r}] \otimes_A A') \cong 0$, which we argue as follows. In the case $A = R$, the ring A' is again regular semilocal, so every line bundle on $A'[\frac{1}{r}]$ extends to a line bundle on A', and hence is trivial, to the effect that $\mathrm{Pic}(A'[\frac{1}{r}]) = 0$, as desired. In the case $A = \mathcal{O}[\frac{1}{r}]$, by [Ser79, Chapter I, Section 4, Proposition 8], the normalization of $\mathcal{O}$ in A' is a finite $\mathcal{O}$-algebra $\mathcal{O}'$, in particular, $\mathcal{O}'$ is again a Dedekind ring. Thus, $R \otimes_{\mathcal{O}} \mathcal{O}'$ is a finite R-algebra, and hence is semilocal, but is also flat and geometrically regular over $\mathcal{O}'$, so it is regular by [SP, Lemma 033A]. Since $R[\frac{1}{r}] \otimes_A A'$ is a localization of $R \otimes_{\mathcal{O}} \mathcal{O}'$, it again follows that $\mathrm{Pic}(R[\frac{1}{r}] \otimes_A A') \cong 0$, as desired. $\square$

Thanks to Claim 5.4.1, we may shrink X around $\mathrm{Spec}(R)$ to trivialize the T_G-torsor $\mathcal{E}^B/\mathscr{R}_u(B)$, in particular, to make E_U reduce to an $\mathscr{R}_u(B)$-torsor. Since the complement $X[\frac{1}{r}]\backslash U$ is of codimension $\geqslant 2$, its closure Z in X is also of codimension $\geqslant 2$. Thus, by Lemma 5.1, we may shrink X around $\mathrm{Spec}(R)$ to arrange that there exists an affine open $S \subset \mathbb{A}^{d-1}_{\mathcal{O}}$ and a smooth morphism $f\colon X \to S$ of relative dimension 1 such that Z is S-finite. We can now apply Theorem 4.5 (ii) with $A := \Gamma(S, \mathscr{O}_S)$ and $B := A[\frac{1}{r}]$ (and §1.10 for the isotriviality condition) to conclude that E is trivial over $R[\frac{1}{r}]$. □

6. The relative Grothendieck–Serre conjecture

In equal characteristic, the approach to Theorem 1.3 is based on the following relative version of the Grothendieck–Serre conjecture that is a mild improvement to [Fed22a, Theorem 1] (with an earlier more restrictive case due to Panin–Stavrova–Vavilov [PSV15, Theorem 1.1]). Its case (ii), included here for completeness, reproves the equal characteristic case of the Grothendieck–Serre conjecture.

Theorem 6.1. *For a regular semilocal ring R containing a field k, a reductive R-group G, and an affine k-scheme W, no nontrivial G-torsor over $W \otimes_k R$ trivializes over $W \otimes_k \mathrm{Frac}(R)$ if either*

(i) *G is totally isotropic*; *or*

(ii) *$W \otimes_k R$ is semilocal, for instance, if $W = \mathrm{Spec}(k)$.*

Proof. Let E be a G-torsor over $W \otimes_k R$ that trivializes over $W \otimes_k \mathrm{Frac}(R)$, let $\mathbb{F} \subset k$ be the prime subfield, and consider the k-algebra $k \otimes_{\mathbb{F}} R$. The composition $R \xrightarrow{a} k \otimes_{\mathbb{F}} R \xrightarrow{b} R$, in which the second map uses the k-algebra structure of R, is the identity. The base change of E along $\mathrm{id}_W \otimes_k a$ is a G-torsor over $W \otimes_{\mathbb{F}} R$ that trivializes over $W \otimes_{\mathbb{F}} \mathrm{Frac}(R)$. Thus, it suffices to settle the claim with $k = \mathbb{F}$ because, by then base changing further along $\mathrm{id}_W \otimes_k b$, we would get the desired triviality of E.

Since k is now perfect, Popescu theorem [SP, Theorem 07GC] expresses R as a filtered direct limit of smooth k-algebras. Thus, by passing to connected components of $\mathrm{Spec}(R)$ and doing a limit argument, we may assume that R is a semilocal ring of a smooth, affine, irreducible k-scheme X of dimension $d \geqslant 0$ and that G and E are defined over all of X. Since E trivializes over $W \otimes_k \mathrm{Frac}(X)$, is also trivializes over $W \times_k (X\backslash Z)$ for some closed $Z \subsetneq X$. If $d = 0$, then E is trivial, and if $d > 0$, then we may apply the presentation lemma of Remark 5.2 to shrink X around $\mathrm{Spec}(R)$ so that there exist an affine open $S \subset \mathbb{A}^{d-1}_k$ and a smooth morphism $X \to S$ of relative dimension 1 that makes Z finite over S. With such a fibration into curves in hand, however, the triviality of E over $W \otimes_k R$ is a special case of Theorem 4.5 (with §1.10 for the isotriviality condition) applied with $A = \Gamma(S, \mathscr{O}_S)$ and $B = \Gamma(W, \mathscr{O}_W)$ in case (i), and with $A = B = \Gamma(W \otimes_k R, \mathscr{O}_{W \otimes_k R})$ in case (ii). □

We will apply Theorem 6.1 with $W \subset \mathbb{A}^1_k$, in which case we may sharpen the assumptions as follows.

Lemma 6.2 ([Gil02, Corollaire 3.10])**.** *For a reductive group G over a field K and an open $U \subset \mathbb{P}^1_K$, each generically trivial G-torsor E over U reduces to a torsor under a maximal K-split subtorus of G; in particular, if $U \subset \mathbb{A}^1_K$, then, since U has no nontrivial line bundles, E is a trivial G-torsor.* □

Corollary 6.3. *For a regular semilocal ring R containing a field k, a totally isotropic reductive R-group, and a nonempty open $W \subset \mathbb{A}^1_k$, every generically trivial G-torsor on $W \otimes_k R$ is trivial.*

Proof. Thanks to Lemma 6.2, Theorem 6.1 (i) applies and gives the desired triviality. □

7. Extending G-torsors over a finite étale subscheme of a relative curve

A crucial preparation to the equicharacteristic case of the Nisnevich conjecture is a result about extending G-torsors over a finite étale closed subscheme of a smooth relative curve that we deduce in Proposition 7.3 from the reembedding techniques of §3. For wider applicability, we present this extension result axiomatically—it loosely amounts to a reduction of the Nisnevich conjecture to the Grothendieck–Serre conjecture. The equicharacteristic relative Grothendieck–Serre conjecture settled in Theorem 6.1 supplies the required axiomatic assumptions in our main case of interest.

Definition 7.1. For a ring A, a contravariant, set-valued functor F on the category of A-schemes of the form $S \backslash Z$ with S a smooth affine A-scheme of pure relative dimension d and $Z \subset S$ an A-quasi-finite closed subscheme, is *excisive* if for all $Z \subset S$ and $Z' \subset S'$ as above and all Cartesian squares

$$\begin{array}{ccc} Z & \hookrightarrow & S \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \hookrightarrow & S' \end{array}$$

with f étale that induces an indicated isomorphism $Z \xrightarrow{\sim} Z'$, we have

$$F(S') \twoheadrightarrow F(S) \times_{F(S \backslash Z)} F(S' \backslash Z').$$

For instance, by Lemma 3.8, for a quasi-affine, flat, finitely presented A-group G, the functor $H^1(-, G)$ is excisive. The following lemma is instrumental for the aforementioned 'excision tricks.'

Lemma 7.2. *Let A be a ring, let S be an A-scheme, let $Y \subset S$ be an A-(separated étale) closed subscheme that is locally cut out by a finitely generated ideal, and consider the decomposition*

$$Y \times_A Y = \Delta \sqcup Y'$$

in which $\Delta \subset Y \times_A Y$ is the diagonal copy of Y. The following square is Cartesian:

$$\begin{array}{ccc} \Delta & \hookrightarrow & S_Y \backslash Y' \\ {\scriptstyle\sim}\downarrow & & \downarrow \\ Y & \hookrightarrow & S, \end{array}$$

in particular, if S is as in Definition 7.1 and F is an excisive functor, then an element of $F(S \backslash Y)$ extends to $F(S)$ if and only if its pullback to $F((S \backslash Y)_Y)$ extends to $F(S_Y \backslash Y')$; for instance, for a quasi-affine, flat, finitely presented S-group scheme G, a G-torsor over $S \backslash Y$ extends to a G-torsor over S if and only if its base change to $(S \backslash Y)_Y$ extends to a G-torsor over $S_Y \backslash Y'$.

Proof. The claimed decomposition $Y \times_A Y = \Delta \sqcup Y'$ exists because any section of a separated étale morphism, such as the projection $Y \times_A Y \to Y$, is both a closed immersion and an open immersion. Thus, the square in question is Cartesian because the étale map $S_Y \backslash Y' \to S$ induces an isomorphism $\Delta \xrightarrow{\sim} Y$. The claim about F is then immediate from Definition 7.1. □

We are ready for our key axiomatic extension result, which extends Fedorov's [Fed22b, Proposition 2.6].

Proposition 7.3. *Let*

- *A be a semilocal ring,*
- *C be a smooth affine A-scheme of pure relative dimension $d > 0$,*

- *$Y \subset C$ be an A-(finite étale) closed subscheme, and*
- *F be an excisive, pointed set valued functor as in Definition 7.1.*

Suppose that for all finite étale Y-schemes $\mathscr{Y}$ and integers $m \leqslant \deg(Y/A)$ such that $\mathscr{Y}$ is a scheme over a finite $\mathbb{Z}$-algebra B for which $\mathbb{A}^d_B$ contains m disjoint copies of $\operatorname{Spec}(B)$*, some $\mathscr{Y}' \subset \mathbb{A}^d_{\mathscr{Y}}$ that is a union of m disjoint copies of $\mathscr{Y}$ and every $\mathscr{Y}$-finite closed subscheme $Z \subset \mathbb{A}^d_{\mathscr{Y}}$ containing $\mathscr{Y}'$,*

$$\operatorname{Ker}(F(\mathbb{A}^d_{\mathscr{Y}}) \to F(\mathbb{A}^d_{\mathscr{Y}}\backslash Z)) \twoheadrightarrow \operatorname{Ker}(F(\mathbb{A}^d_{\mathscr{Y}}\backslash \mathscr{Y}') \to F(\mathbb{A}^d_{\mathscr{Y}}\backslash Z)), \tag{7.3.1}$$

that is, every element of $F(\mathbb{A}^d_{\mathscr{Y}}\backslash \mathscr{Y}')$ that trivializes away from some $\mathscr{Y}$-finite $Z \subset \mathbb{A}^d_{\mathscr{Y}}$ extends to $F(\mathbb{A}^d_{\mathscr{Y}})$. Then, for every A-finite closed subscheme $Z \subset C$ containing Y,

$$\operatorname{Ker}(F(C) \to F(C\backslash Z)) \twoheadrightarrow \operatorname{Ker}(F(C\backslash Y) \to F(C\backslash Z)), \tag{7.3.2}$$

that is, every element of $F(C\backslash Y)$ that trivializes away from some A-finite $Z \subset C$ extends to $F(C)$.

Although a general $d > 0$ requires no extra work, the main case is $d = 1$. In this case, Corollary 6.3 supplies the assumption (7.3.1) when A is regular of equicharacteristic and $F(-) = H^1(-, G)$ for a reductive A-group G such that G_Y totally isotropic. Roughly, the point of Proposition 7.3 is to formally reduce the extendability property (7.3.2) to the case when $C = \mathbb{A}^d_A$ and Y is "constant."

Proof. For the proof, it is convenient to generalize our setup as follows. We assume that $C \subset C'$ is an open immersion of smooth affine A-schemes of pure relative dimension d such that $Y' := C'\backslash C$ is A-(finite étale), that our assumption (7.3.1) holds with $m \leqslant \deg((Y \cup Y')/A)$, and that we seek to show (7.3.2) for every A-finite $Z \subset C'$ containing Y and Y'. Of course, the case $C' = C$ recovers the original claim, and the formulation with an arbitrary C' is equivalent because after extending to an element of $F(C)$ we may extend further to an element of $F(C')$. For intermediate reductions, however, it is convenient to require that our A-finite Z lives in C' instead of the possibly smaller C.

In the setup with a C', we fix an F satisfying the assumptions and an $\alpha \in \operatorname{Ker}(F(C\backslash Y) \to F(C\backslash Z))$ that we wish to extend over Y. We then use Lemma 7.2 to base change along $Y \to \operatorname{Spec}(A)$ and shrink the base changed C by removing the off-diagonal part of $Y \times_A Y$ to reduce to the case when $Y \cong \operatorname{Spec}(A)$. Moreover, we decompose A to reduce to the case when $\operatorname{Spec}(A)$ is connected, so that $\deg(Y/A)$ is a well-defined integer. We then let n be the product of $\deg((Y \cup Y')/A)$ and all the primes p with either $p \leqslant \deg((Y \cup Y')/A)$ or $p \notin A^\times$. We combine Lemma 3.6 (b) with Remark 3.7 to find an affine open $D \subset C'$ containing Z as well as a finite étale cover $\widetilde{C}' \twoheadrightarrow D$ such that

$$\widetilde{Y} := Y \times_{C'} \widetilde{C}' \quad \text{decomposes as} \quad \widetilde{Y} = \widetilde{Y}_0 \sqcup \widetilde{Y}_1 \quad \text{where} \quad \widetilde{Y}_0 \xrightarrow{\sim} \operatorname{Spec}(A),$$

each component of $\widetilde{Y}_1$ or of $\widetilde{Y}' := \widetilde{C}'\backslash\widetilde{C}$ with $\widetilde{C} := (D\backslash Y') \times_{C'} \widetilde{C}'$ is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k' of characteristic $p \mid n$ satisfies

$$\#k' > \deg((\widetilde{Y} \cup \widetilde{Y}')/A),$$

and there is no finite field obstruction to embedding $\widetilde{Z} := Z \times_{C'} \widetilde{C}'$ into $\mathbb{A}^d_A$. By construction,

$$\begin{array}{ccc} \widetilde{Y}_0 & \hookrightarrow & \widetilde{C}\backslash\widetilde{Y}_1 \\ {\scriptstyle\sim}\downarrow & & \downarrow \\ Y & \hookrightarrow & C \cap D \end{array}$$

is a Cartesian square. Thus, since F is excisive, to extend α over Y we may first restrict to $C \cap D$ and then pass to $\widetilde{C}\backslash\widetilde{Y}_1$. That is, we may replace $Y \subset C \subset C'$ by $\widetilde{Y}_0 \subset \widetilde{C}\backslash\widetilde{Y}_1 \subset \widetilde{C}'$ and α by its pullback to $\widetilde{C}\backslash\widetilde{Y}$ to reduce to the case when each connected component of Y' is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k' of characteristic $p \mid n$ satisfies $\#k' > \deg((Y \cup Y')/A)$ and there is no finite field obstruction to embedding Z into $\mathbb{A}^d_A$. By Proposition 3.4, such an embedding exists, more precisely, there are an affine open $D \subset C'$ containing Z and a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ \sim\downarrow & & \downarrow f \\ Z' & \hookrightarrow & \mathbb{A}^d_A \end{array} \tag{7.3.3}$$

in which the map f is étale and embeds Z as a closed subscheme $Z' \subset \mathbb{A}^d_A$. The square remains Cartesian after passing to the complements of the A-(finite étale) $Y \cup Y'$ viewed inside Z (so also inside Z'). Thus, for the purpose of extending α over Y, we may use the excisive property of F to patch the restriction of $\alpha|_{D\backslash(Y\cup Y')}$ with the origin in $F(\mathbb{A}^d_A\backslash Z')$ to reduce to the case when $C' = \mathbb{A}^d_A$.

In conclusion, we reduced to the case when $C' = C = \mathbb{A}^d_A$ and $Y \cong \mathrm{Spec}(A) \sqcup y$ such that each connected component of y is an algebra over some finite $\mathbb{Z}$-algebra B each of whose residue fields k' of characteristic $p \mid n$ satisfies $\#k' > \deg(Y/A)$. Moreover, we may assume that A itself is an algebra over some such finite $\mathbb{Z}$-algebra B: indeed, once we settle this case, we may combine it with Lemma 7.2 to iteratively extend α over each connected component of y, and hence to reduce to the case when $y = \varnothing$, in which we may simply choose $B = \mathbb{Z}$.

Granted the reductions above, we now induct on the number of disjoint copies of $\mathrm{Spec}(A)$ contained in Y to reduce to when $Y \simeq \bigsqcup \mathrm{Spec}(A)$. Indeed, suppose that Y has a connected component W that does not map isomorphically to $\mathrm{Spec}(A)$, so that W is of degree $\geqslant 2$ over A. Since $W \times_A W$ contains the diagonal copy of W as a clopen (compare with Lemma 7.2), the W-(finite étale) closed subscheme $Y \times_A W \subset \mathbb{A}^d_W$ contains strictly more disjoint copies of W than Y contained disjoint copies of $\mathrm{Spec}(A)$. Thus, by the inductive hypothesis, the pullback of α to $\mathbb{A}^d_W\backslash(Y \times_A W)$ extends over $Y \times_A W$. By Lemma 7.2, this implies that α extends over W. By repeating this for each possible W, we effectively shrink Y until we reduce to the desired base case when $Y \simeq \bigsqcup \mathrm{Spec}(A)$.

To treat this last case, we set $m := \deg(Y/A)$ and note that, by Proposition 3.4, for any closed subscheme $\mathscr{Y}' \subset \mathbb{A}^d_A$ that is a disjoint union of m copies of $\mathrm{Spec}(A)$, there are an affine open $D \subset \mathbb{A}^d_A$ containing Z and a Cartesian square as in (7.3.3) such that f maps Y isomorphically onto $\mathscr{Y}'$. Since F is excisive, Lemma 3.8 then reduces us to the case when $Y = \mathscr{Y}'$ inside $\mathbb{A}^d_A$. At this point, we will finally use the assumption (7.3.1) on F. Namely, by the arranged condition on the residue fields of B, there is a B-(finite étale) closed subscheme of $\mathbb{A}^d_B$ that is a union of m disjoint copies of $\mathrm{Spec}(B)$, and its base change is then a closed subscheme of $\mathbb{A}^d_A$ that is a union of m disjoint copies of $\mathrm{Spec}(A)$. This means that (7.3.3) applies to *some* closed subscheme $\mathscr{Y}' \subset \mathbb{A}^d_A$ that is a disjoint union of m copies of $\mathrm{Spec}(A)$ and, as we have already argued, this implies our claim about extending over Y. ☐

Corollary 7.4. *For a regular semilocal ring R containing a field, a totally isotropic reductive R-group scheme G, and an R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$, no nontrivial G-torsor over $\mathbb{A}^1_R\backslash Y$ becomes trivial over $\mathbb{A}^1_R\backslash Z$ for some R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ containing Y, that is,*

$$\mathrm{Ker}(H^1(\mathbb{A}^1_R\backslash Y, G) \to H^1(\mathbb{A}^1_R\backslash Z, G)) = \{*\}.$$

Proof. By Proposition 7.3 (with Corollary 6.3), every G-torsor over $\mathbb{A}^1_R\backslash Y$ that is trivial over $\mathbb{A}^1_R\backslash Z$ extends to a G-torsor over $\mathbb{A}^1_R$. This $Y = \varnothing$ case, however, is covered by Theorem 2.1 (a). ☐

8. The Nisnevich conjecture over a field

The final preparation to the equicharacteristic case of the Nisnevich conjecture is the following geometric presentation lemma in the spirit of Gabber's refinement [Gab94, Lemma 3.1] of the

Quillen presentation lemma [Qui73, Section 7, Lemma 5.12], which itself is a variant of the Noether normalization theorem. For us, it is crucial to have its aspect about the smooth divisor D.

Lemma 8.1. *For a smooth, affine, irreducible scheme X of dimension $d > 0$ over a field k that is either finite or of characteristic 0,*[2] *points $x_1, \ldots, x_m \in X$, a proper closed subscheme $Z \subset X$, and a k-smooth divisor $D \subset X$, there are an affine open $X' \subset X$ containing $x_1, \ldots, x_m$, an affine open $S \subset \mathbb{A}_k^{d-1}$, and a smooth morphism*

$$f\colon X' \to S$$

of relative dimension 1 such that

$$X' \cap Z = f^{-1}(S) \cap Z \quad \text{is } S\text{-finite and} \quad X' \cap D = f^{-1}(S) \cap D \quad \text{is } S\text{-(finite étale)}.$$

Proof. In the case $d = 1$, we may choose $X' = X$ and $S = \mathrm{Spec}(k)$, so we assume that $d > 1$. We also replace each x_i by a specialization to reduce to x_i being a closed point (see [SP, Lemma 02J6]), and in this case we will force each $f(x_i)$ to be the origin of $\mathbb{A}_k^{d-1}$. We embed X into some projective space $\mathbb{P}_k^N$ and then form closures to arrange that X is an open of a projective $\overline{X} \subset \mathbb{P}_k^N$ of dimension d with $\overline{X} \backslash X$ of dimension $\leqslant d - 1$ and that there are

- a projective $\overline{D} \subset \overline{X}$ of dimension $d - 1$ with $\overline{D} \backslash D$ of dimension $\leqslant d - 2$, and
- a projective $\overline{Z} \subset \overline{X}$ of dimension $\leqslant d - 1$ with $\overline{Z} \backslash Z$ of dimension $\leqslant d - 2$.

We use the avoidance lemma [GLL15, Theorem 5.1] and postcompose with a Veronese embedding to build a hyperplane H_0 not containing any x_i such that $(\overline{X} \backslash X) \cap H_0$ is of dimension $\leqslant d - 2$ (to force the dimension drop, choose appropriate auxiliary closed points and require H_0 to not contain them). By the Bertini theorem [Poo04, Theorem 1.3] of Poonen if k is finite and by the Bertini theorem of [Česk22a, second paragraph of the proof of Lemma 3.2] applied both to X and to D in place of X if k is of characteristic 0, there is a hypersurface $H_1 \subset \mathbb{P}_k^N$ such that

- H_1 contains $x_1, \ldots, x_m$;
- $X \cap H_1$ (resp., $D \cap H_1$) is k-smooth of dimension $d - 1$ (resp., $d - 2$);
- $Z \cap H_1$ is (resp., $(\overline{D} \backslash D) \cap H_1$ and $(\overline{Z} \backslash Z) \cap H_1$ are) of dimension $\leqslant d - 2$ (resp., $\leqslant d - 3$);
- $(\overline{X} \backslash X) \cap H_0 \cap H_1$ is of dimension $\leqslant d - 2$.

In particular, by passing to intersections with H_1, we are left with an analogous situation with d replaced by $d - 1$. Therefore, by iteratively applying the Bertini theorem in this way, we build hypersurfaces $H_1, \ldots, H_{d-1}$ such that

(i) the $x_1, \ldots, x_m$ lie in $H_1 \cap \ldots \cap H_{d-1}$ but not in H_0;

(ii) $X \cap H_1 \cap \ldots \cap H_{d-1}$ (resp., $D \cap H_1 \cap \ldots \cap H_{d-1}$) is k-smooth of dimension 1 (resp., k-étale);

(iii) $(\overline{D} \backslash D) \cap H_1 \cap \ldots \cap H_{d-1} = (\overline{Z} \backslash Z) \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

(iv) $(\overline{X} \backslash X) \cap H_0 \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

By letting $1, w_1, \ldots, w_{d-1}$ be the degrees of the hypersurfaces $H_0, H_1, \ldots, H_{d-1}$ and choosing defining equations h_i of the H_i, we determine a projective morphism $\widetilde{f}\colon \widetilde{X} \to \mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ from the

[2]The assumption on k is likely not optimal but it will suffice and we do not wish to further complicate the proof.

weighted blowup $\widetilde{X} := \mathrm{Bl}(h_0, \ldots, h_{d-1})$ to the weighted projective space such that the diagram

$$\begin{array}{ccccc} \overline{X}\backslash H_0 & \hookrightarrow & \overline{X}\backslash(H_0 \cap \ldots \cap H_{d-1}) & \hookrightarrow & \widetilde{X} \\ \downarrow f & & \downarrow & & \downarrow \widetilde{f} \\ \mathbb{A}^{d-1}_k & \hookrightarrow & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) & = & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) \end{array}$$

commutes, where the bottom left arrow is the inclusion of the open locus where the first standard coordinate of $\mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ does not vanish, see [Čes22a, Sections 3.4 and 3.5]. By (i), each $f(x_i)$ is the origin of $\mathbb{A}^{d-1}_k$. By (ii) and the dimensional flatness criterion [EGA IV_2, Proposition 6.1.5], at every point of the fiber above the origin of $\mathbb{A}^{d-1}_k$, the map f is smooth of relative dimension 1 and its restriction to D is étale. Since $\widetilde{f}$ is projective, (iii)–(iv) and the openness of the quasi-finite locus [SP, Lemma 01TI] ensure that for some affine open neighborhood of the origin $S \subset \mathbb{A}^{d-1}_k$ both $f^{-1}(S) \cap Z$ and $f^{-1}(S) \cap D$ are S-finite (see also [SP, Lemma 02OG]). In conclusion, any affine open of $f^{-1}(S)$ that contains all the x_i and all the points of Z and D that lie above the origin of $\mathbb{A}^{d-1}_k$ becomes a sought X' after possibly shrinking S further. ☐

8.2. Proof of Theorem 1.3 (1). We have a regular semilocal ring R containing a field k, a regular parameter $r \in R$, a reductive R-group $\mathcal{G}$ with $\mathcal{G}_{R/(r)}$ totally isotropic, and a generically trivial $\mathcal{G}$-torsor E over $R[\frac{1}{r}]$. We need to show that E is trivial, equivalently, by the Grothendieck–Serre Theorem 6.1 (ii), we need to extend E to a $\mathcal{G}$-torsor $\mathcal{E}$ over R. For this, by patching supplied by Lemma 3.8 and a limit argument, we may semilocalize R along the union of those maximal ideals $\mathfrak{m} \subset R$ that contain r and reduce ourselves to the case when r lies in every maximal ideal $\mathfrak{m} \subset R$. Moreover, we may replace k by its prime subfield to assume that k is either $\mathbb{Q}$ or some $\mathbb{F}_p$.

Popescu theorem [SP, Theorem 07GC] expresses R as a filtered direct limit of smooth k-algebras. Thus, by passing to connected components of $\mathrm{Spec}(R)$ and doing a limit argument, we may assume that R is a semilocal ring of a smooth, affine, irreducible k-scheme X of dimension $d \geqslant 0$, that r is a global section of X that cuts out a k-smooth divisor $D \subset X$ with complement $U := X\backslash D$, that $\mathcal{G}$ (resp., E) is defined over all of X (resp., U), and that $\mathcal{G}_D$ is totally isotropic. Since E is trivial over $\mathrm{Frac}(X)$, there is a closed $\mathscr{Z} \subsetneq X$ containing D such that E is trivial over $U\backslash\mathscr{Z}$. If $d = 0$, then E is trivial, so we assume that X is of dimension $d > 0$. Finally, we use [SGA 3_{II}, exposé XIV, corollaire 3.20] to shrink X further to make $\mathcal{G}$ have a maximal torus T defined over all of X.

With these preparations, Lemma 8.1 allows us to shrink X around $\mathrm{Spec}(R)$ to arrange that there exist an affine open $S \subset \mathbb{A}^{d-1}_k$ and a smooth morphism $f\colon X \to S$ of relative dimension 1 such that $\mathscr{Z}$ is S-finite and D is S-(finite étale). We base change f along the map $\mathrm{Spec}(R) \to S$ to obtain

- a smooth affine R-scheme C of pure relative dimension 1 (base change of X);
- an R-finite closed subscheme $Z \subset C$ (base change of $\mathscr{Z}$);
- an R-(finite étale) closed subscheme $Y \subset Z$ (base change of D);
- a section $s \in C(R)$ (induced by the “diagonal” section) such that $s|_{R[\frac{1}{r}]}$ factors through $C\backslash Y$;
- a reductive C-group $\mathscr{G}$ with $s^*(\mathscr{G}) \cong \mathcal{G}$ (base change of $\mathcal{G}$) such that $\mathscr{G}_Y$ is totally isotropic;
- a maximal C-torus $\mathscr{T} \subset \mathscr{G}$ (base change of T) with $s^*(\mathscr{T}) \cong T$; and
- a $\mathscr{G}$-torsor $\mathscr{E}$ over $C\backslash Y$ (base change of E) that is trivial over $C\backslash Z$ such that

$$(s|_{R[\frac{1}{r}]})^*(\mathscr{E}) \cong E \quad \text{as } \mathcal{G}\text{-torsors over } R[\tfrac{1}{r}].$$

We replace Z by $Z \cup s$ if needed to arrange that $s \in Z(R)$. By Lemma 4.3 (with §1.10 for the isotriviality aspect) and spreading out, there is a finite étale cover $\widetilde{C}$ of some affine open neighborhood of Z in C such that s lifts to some $\widetilde{s} \in \widetilde{C}(R)$ and $\mathscr{G}_{\widetilde{C}} \simeq \mathcal{G}_{\widetilde{C}}$, compatibly with an already fixed such isomorphism after pullback along $\widetilde{s}$. Thus, we may replace C and s by $\widetilde{C}$ and $\widetilde{s}$ and replace Z, Y, $\mathscr{G}$, $\mathscr{E}$ by their corresponding base changes to reduce to when $\mathscr{G}$ is $\mathcal{G}_C$. A similar reduction based on Lemma 3.6 (b) and Remark 3.7 instead allows us to assume that there is no finite field obstruction to embedding Z into $\mathbb{A}^1_R$. Thus, Proposition 3.4 gives an affine open $D \subset C$ containing Z and a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ \| & & \downarrow{\scriptstyle \pi} \\ Z & \hookrightarrow & \mathbb{A}^1_R \end{array}$$

in which the map π is étale and embeds Z as a closed subscheme $Z \subset \mathbb{A}^1_R$. These properties of the square persist after restricting to the open complements of the common closed subscheme Y of D and of $\mathbb{A}^1_R$. Thus, by Lemma 3.8, we may descend $\mathscr{E}|_{D\backslash Y}$ to a G-torsor $\mathscr{E}'$ over $\mathbb{A}^1_R\backslash Y$ that is trivial over $\mathbb{A}^1_R\backslash Z$ to reduce to the case when $C = \mathbb{A}^1_R$. In this case, however, by Proposition 7.3 (with Corollary 6.3 to check its main assumption), the $\mathcal{G}$-torsor $\mathscr{E}$ extends to a $\mathcal{G}$-torsor defined over all of $\mathbb{A}^1_R$. Thus, by pulling back along s, our $\mathcal{G}$-torsor E extends to a desired $\mathcal{G}$-torsor $\mathcal{E}$ over R. □

9. The generalized Bass–Quillen conjecture over a field

9.1. Proof of Theorem 1.5. We have a regular ring R containing a field, a totally isotropic reductive R-group G, and a generically trivial G-torsor E over $\mathbb{A}^d_R$. We need to show that E descends to a G-torsor over R. For this, by induction on d, we may assume that $d = 1$. By Quillen patching of Lemma 2.4, we may assume that R is local. In this key local case, we will show that E is trivial.

For this, by Theorem 2.1, it suffices to show that E is trivial on $\mathbb{A}^1_R\backslash Z$ for some R-finite closed subscheme $Z \subset \mathbb{A}^1_R$. By a limit argument, it therefore suffices to show that E becomes trivial over the localization of $R[t]$ obtained by inverting all the monic polynomials. By the change of variables $x := t^{-1}$, this localization is the localization of $\mathbb{P}^1_R$ along the section ∞, and hence is isomorphic to

$$(R[x]_{1+xR[x]})[\tfrac{1}{x}].$$

The ring $R' := R[x]_{1+xR[x]}$ is regular, local, and shares its fraction field with $\mathbb{A}^1_R$. In particular, the base change of E to R' is generically trivial. Thus, since x is a regular parameter of R', Theorem 1.3 (1) implies that this base change of E is trivial, as desired. □

References

[AHW18] Aravind Asok, Marc Hoyois, and Matthias Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory, II: Principal bundles and homogeneous spaces*, Geom. Topol. **22** (2018), no. 2, 1181–1225, DOI 10.2140/gt.2018.22.1181. MR3748687

[AHW20] ———, *Affine representability results in $\mathbb{A}^1$-homotopy theory III: finite fields and complements*, Algebr. Geom. **7** (2020), no. 5, 634–644, DOI 10.14231/ag-2020-023. MR4156421

[Bac19] Tom Bachmann, *Affine Grassmannians in $\mathbb{A}^1$-homotopy theory*, Selecta Math. (N.S.) **25** (2019), no. 2, Paper No. 25, 14, DOI 10.1007/s00029-019-0471-1. MR3925100

[BČ22] Alexis Bouthier and Kęstutis Česnavičius, *Torsors on loop groups and the Hitchin fibration*, Ann. Sci. École Norm. Sup., to appear (2022). Available at `http://arxiv.org/abs/1908.07480v4`.

[BR83] S. M. Bhatwadekar and R. A. Rao, *On a question of Quillen*, Trans. Amer. Math. Soc. **279** (1983), no. 2, 801–810. MR709584

[BS17] Chetan Balwe and Anand Sawant, *$\mathbb{A}^1$-connectedness in reductive algebraic groups*, Trans. Amer. Math. Soc. **369** (2017), no. 8, 5999–6015, DOI 10.1090/tran/7090. MR3646787

[CTO92] Jean-Louis Colliot-Thélène and Manuel Ojanguren, *Espaces principaux homogènes localement triviaux*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 97–122 (French). MR1179077

[CTS79] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134, DOI 10.1007/BF01420486 (French). MR550842

[CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), no. 1, 148–205, DOI 10.1016/0021-8693(87)90026-3. MR878473

[CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov, *The Brauer-Grothendieck group*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 71, Springer, Cham, 2021. MR4304038

[Čes22a] Kęstutis Česnavičius, *Grothendieck–Serre in the quasi-split unramified case*, Forum Math. Pi **10** (2022), Paper No. e9, 30, DOI 10.1017/fmp.2022.5. MR4403660

[Čes22b] Kęstutis Česnavičius, *Problems About Torsors over Regular Rings*, Acta Math. Vietnam. **47** (2022), no. 1, 39–107, DOI 10.1007/s40306-022-00477-y. MR4406561

[EGA IV_2] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330)

[EGA IV_3] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 255. MR0217086 (36 #178)

[EGA IV_4] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361 (French). MR0238860 (39 #220)

[Fed16] Roman Fedorov, *Affine Grassmannians of group schemes and exotic principal bundles over $\mathbb{A}^1$*, Amer. J. Math. **138** (2016), no. 4, 879–906, DOI 10.1353/ajm.2016.0036. MR3538146

[Fed21] ———, *On the Grothendieck–Serre conjecture on principal bundles in mixed characteristic*, Trans. Amer. Math. Soc. **375** (2021), no. 01, 559–586, DOI 10.1090/tran/8490. MR4358676

[Fed22a] ———, *On the Grothendieck-Serre conjecture about principal bundles and its generalizations*, Algebra Number Theory **16** (2022), no. 2, 447–465, DOI 10.2140/ant.2022.16.447. MR4412579

[Fed22b] ———, *On the purity conjecture of Nisnevich for torsors under reductive group schemes*, preprint (2022). Available at http://arxiv.org/abs/2109.10332v4.

[FP15] Roman Fedorov and Ivan Panin, *A proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 169–193, DOI 10.1007/s10240-015-0075-z. MR3415067

[FR70] Daniel Ferrand and Michel Raynaud, *Fibres formelles d'un anneau local noethérien*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 295–311 (French). MR0272779

[Gab81] Ofer Gabber, *Some theorems on Azumaya algebras*, The Brauer group (Sem., Les Plans-sur-Bex, 1980), Lecture Notes in Math., vol. 844, Springer, Berlin-New York, 1981, pp. 129–209. MR611868 (83d:13004)

[Gab94] ———, *Gersten's conjecture for some complexes of vanishing cycles*, Manuscripta Math. **85** (1994), no. 3-4, 323–343, DOI 10.1007/BF02568202. MR1305746

[Gil02] P. Gille, *Torseurs sur la droite affine*, Transform. Groups **7** (2002), no. 3, 231–245, DOI 10.1007/s00031-002-0012-3 (French, with English summary). MR1923972

[GLL15] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270, DOI 10.1215/00127094-2877293. MR3347315

[Gro58] Alexandre Grothendieck, *Torsion homologique et sections rationnelles*, Seminaire Claude Chevalley **3** (1958), no. 5, 1–29 (French).

[Guo20] Ning Guo, *The Grothendieck–Serre conjecture over semilocal Dedekind rings*, Transform. Groups, to appear (2020). Available at https://arxiv.org/abs/1902.02315v3.

[Ill05] Luc Illusie, *Grothendieck's existence theorem in formal geometry*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 179–233. With a letter (in French) of Jean-Pierre Serre. MR2223409

[KP18] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218, DOI 10.1007/s10240-018-0100-0. MR3905466

[Lam06] T. Y. Lam, *Serre's problem on projective modules*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006. MR2235330

[Li21] Shang Li, *The Bass–Quillen conjecture for general groups in equal characteristic*, Mémoire de Master 2, Sorbonne Université (2021).

[MB96] Laurent Moret-Bailly, *Un problème de descente*, Bull. Soc. Math. France **124** (1996), no. 4, 559–585 (French, with English and French summaries). MR1432058

[Nis89] Yevsey Nisnevich, *Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 10, 651–655 (English, with French summary). MR1054270

[Pan20] I. A. Panin, *Proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing a field*, Izv. Ross. Akad. Nauk Ser. Mat. **84** (2020), no. 4, 169–186, DOI 10.4213/im8982 (Russian); English transl., Izv. Math. **84** (2020), no. 4, 780–795. MR4133391

[Par78] S. Parimala, *Failure of a quadratic analogue of Serre's conjecture*, Amer. J. Math. **100** (1978), no. 5, 913–924, DOI 10.2307/2373953. MR517136

[Poo04] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127, DOI 10.4007/annals.2004.160.1099. MR2144974

[Poo17] ———, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254

[PSV15] I. Panin, A. Stavrova, and N. Vavilov, *On Grothendieck-Serre's conjecture concerning principal G-bundles over reductive group schemes: I*, Compos. Math. **151** (2015), no. 3, 535–567, DOI 10.1112/S0010437X14007635. MR3320571

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129

[Qui76] ———, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171, DOI 10.1007/BF01390008. MR0427303

[Rag89] M. S. Raghunathan, *Principal bundles on affine space and bundles on the projective line*, Math. Ann. **285** (1989), no. 2, 309–332, DOI 10.1007/BF01443521. MR1016097

[Ser58] J.-P. Serre, *Espaces fibrés algébriques*, Seminaire Claude Chevalley **3** (1958), no. 1, 1–37 (French).

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. MR554237 (82e:12016)

[SGA 3_{II}] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970 (French). MR0274459 (43 #223b)

[SGA $3_{\mathrm{III\ new}}$] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 8, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre; Revised and annotated edition of the 1970 French original. MR2867622

[SP] A. J. de Jong et al., *The Stacks Project.* Available at `http://stacks.math.columbia.edu`.

[Sta19] Anastasia Stavrova, *Isotropic reductive groups over discrete Hodge algebras*, J. Homotopy Relat. Struct. **14** (2019), no. 2, 509–524, DOI 10.1007/s40062-018-0221-7. MR3947969

[Sta22] ———, *$\mathbb{A}^1$-invariance of non-stable K_1-functors in the equicharacteristic case*, Indag. Math. (N.S.) **33** (2022), no. 2, 322–333, DOI 10.1016/j.indag.2021.08.002. MR4383113

[Swa78] Richard G. Swan, *Projective modules over Laurent polynomial rings*, Trans. Amer. Math. Soc. **237** (1978), 111–120, DOI 10.2307/1997613. MR0469906