RATIONAL POINTS AND PRIME VALUES OF POLYNOMIALS IN MODERATELY MANY VARIABLES

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Abstract. We derive the Hasse principle and weak approximation for fibrations of certain varieties in the spirit of work by Colliot-Thélène–Sansuc and Harpaz–Skorobogatov–Wittenberg. Our varieties are defined through polynomials in many variables and part of our work is devoted to establishing Schinzel's hypothesis for polynomials of this kind. This last part is achieved by using arguments behind Birch's well-known result regarding the Hasse principle for complete intersections with the notable difference that we prove our result in 50% fewer variables than in the classical Birch setting. We also study the problem of square-free values of an integer polynomial with 66.6% fewer variables than in the Birch setting.

Contents

1. Introduction

1.1. **Prime values of polynomials and rational points.** Let $n \geqslant 1$ be an integer and assume that $f \in \mathbb{Q}[t_1, \dots, t_n]$. Let $K_1, \dots, K_r$ be cyclic extensions of $\mathbb{Q}$, denote the degree $[K_i : \mathbb{Q}]$ by d_i and fix a basis $\{\omega_{1,i}, \dots, \omega_{d_i,i}\}$ for K_i as a vector space over $\mathbb{Q}$. We will denote

$$\mathbf{N}_{K_i/\mathbb{Q}}(\mathbf{x}_i) = N_{K_i/\mathbb{Q}}(x_{1,i}\omega_{1,i} + \cdots + x_{d_i,i}\omega_{d_i,i}), \quad (1 \leqslant i \leqslant r)$$

where $N_{K_i/\mathbb{Q}}$ denotes the field norm. Let now the quasi-affine variety $X \subset \mathbb{A}^n \times \mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_r}$ be defined via

$$X : \big(0 \neq f(t_1, \dots, t_n) = \mathbf{N}_{K_1/\mathbb{Q}}(\mathbf{x}_1) = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\mathbf{x}_r)\big) \tag{1.1}$$

and let V be a smooth proper model of the affine subvariety of $\mathbb{A}^n \times \mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_r}$ given by

$$f(t_1, \dots, t_n) = \mathbf{N}_{K_1/\mathbb{Q}}(\mathbf{x}_1) = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\mathbf{x}_r). \tag{1.2}$$

The Hasse principle and weak approximation for varieties of this kind have been the object of intensive study. There are cases where the Hasse principle and weak approximation hold

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and there are examples for which they fail, [9]. However, it has been conjectured by Colliot-Thélène [8] that all such failures are accounted for by the Brauer–Manin obstruction.

The main objective of this paper is to study the Hasse principle and weak approximation for the class of varieties defined by (1.1) and (1.2) under the restriction that the polynomial f is an irreducible form and has many variables compared to its degree, but only moderately so as it will appear in due course. To this end, information on prime values assumed by integer polynomials can be exploited. The prototypical example is due to Hasse [21], whose proof of the Hasse principle for smooth quadratic forms in four variables relies on Dirichlet's theorem on primes in arithmetic progressions combined with the global reciprocity law and the Hasse principle for non-singular quadratic forms in three variables. This fibration argument was later generalized in an important work by Colliot-Thélène and Sansuc [9] to establish that, conditionally under Schinzel's hypothesis, various pencils of varieties over $\mathbb{Q}$ satisfy the Hasse principle and weak approximation. Their result was then extended by many authors, see the introduction of [20] for a list of relevant references. Theorem 1.3 below will allow us to replace Schinzel's hypothesis in order to prove unconditionally the Hasse principle and weak approximation for the varieties defined by (1.1) and (1.2) in the case of an irreducible form f with moderately many variables compared to its degree. Let us conclude by mentioning that unconditional proofs in this subject exist in cases where the underlying polynomials have small degree (see for instance [10, Th. 9.3]) or special factorisation over $\mathbb{Q}$. For example, polynomials that are completely split over $\mathbb{Q}$ are treated in [5]. Our main result provides an example where the polynomial needs to have moderately many variables compared to its degree but has no restriction on its shape.

For a homogeneous polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ that is irreducible in $\mathbb{Q}[x_1, \dots, x_n]$ we let σ_f be the dimension of the singular locus of $f = 0$, namely the dimension of the affine variety cut out by the system of equations $\nabla f(x_1, \dots, x_n) = 0$ (see [2, pg. 250]). Observe that $0 \leqslant \sigma_f \leqslant n-1$ and that $\sigma_f = 0$ if and only if the projective variety defined by f is non-singular.

Theorem 1.1. *Let f, K_i and X be as in (1.1) with f an irreducible form and assume that*

$$n - \sigma_f \geqslant \max\{4, 1 + 2^{\deg(f)-1}(\deg(f) - 1)\}.$$

Then X satisfies the Hasse principle and weak approximation. In particular, $X(\mathbb{Q})$ is Zariski dense as soon as it is non-empty.

Note that, thanks to the fact that our Theorem 1.3 below (which is the main ingredient of the proof of Theorem 1.1) holds in half as many variables as in the work of Birch, a direct application of [2, §7,Th. 1] would not prove Theorem 1.1. Our strategy will be to establish an analogue of [20, Prop. 1.2] and then to adapt the argument in the proof of [20, Th. 1.3].

Like in the proof of [10, Th. 9.3], one can deduce weak approximation for the variety V defined by (1.2) from Theorem 1.1, since weak approximation is a birational invariant of smooth varieties and hence it is enough to establish the result for the smooth model of V provided by X. We then conclude the proof of Corollary 1.2 by alluding to the fact that the Hasse principle and Zariski density by the existence of a rational point are consequences of weak approximation.

Corollary 1.2. *Keep the assumptions of Theorem 1.1 and let V be as in (1.2). Then V satisfies the Hasse principle and weak approximation. In particular, $V(\mathbb{Q})$ is Zariski dense as soon as it is non-empty.*

1.2. **Primes represented by polynomials.** As mentioned above, a key tool in our proof of Theorem 1.1 is a generalization of Schinzel's hypothesis for polynomials in moderately many variables compared to its degree. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a, not necessarily homogeneous, polynomial that is irreducible in $\mathbb{Q}[x_1, \ldots, x_n]$ and denote by f_0 the top degree part of f. We define $\sigma_f := \sigma_{f_0}$. For a non-empty compact box $\mathscr{B} \subset \mathbb{R}^n$ with the property that $f_0(\mathscr{B}) \subset (1, \infty)$, we define

$$\pi_f(\mathscr{B}) := \#\{\mathbf{x} \in \mathbb{Z}^n \cap \mathscr{B} : f(\mathbf{x}) \text{ is a positive prime}\} \quad \text{and} \quad \mathrm{Li}_f(\mathscr{B}) := \int_{\mathscr{B}} \frac{\mathrm{d}\mathbf{x}}{\log f_0(\mathbf{x})}. \tag{1.3}$$

Our main result in this section is the following theorem.

Theorem 1.3. *Assume that $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is any integer polynomial which is irreducible in $\mathbb{Q}[x_1, \ldots, x_n]$ and let $\mathscr{B} \subset \mathbb{R}^n$ be any non-empty compact box with $f_0(\mathscr{B}) \subset (0, \infty)$. If*

$$n - \sigma_f \geqslant \max\{4, (\deg(f) - 1)2^{\deg(f)-1} + 1\}, \tag{1.4}$$

then for every fixed $A > 0$ the following holds for all sufficiently large P,

$$\pi_f(P\mathscr{B}) = \left(\prod_{p \text{ prime}} \frac{(1 - p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f(\mathbf{x}) = 0\})}{(1 - 1/p)} \right) \mathrm{Li}_f(P\mathscr{B}) + O_{A,\mathscr{B},f}\left(\frac{P^n}{(\log P)^A} \right)$$

where the implied constant depends at most on $A, \mathscr{B}$ and f.

Note that the assumption $f_0(\mathscr{B}) \subset (0, \infty)$ shows that for all sufficiently large P every $\mathbf{x} \in P\mathscr{B}$ satisfies $f_0(\mathbf{x}) \gg P^{\deg(f)} > 1$, thus $\mathrm{Li}_f(P\mathscr{B})$ is well-defined. We shall see in Lemma 3.22 that $\mathrm{Li}_f(P\mathscr{B}) - \mathrm{vol}(\mathscr{B})P^n/\log(P^{\deg(f)}) \ll_{f,\mathscr{B}} P^n/(\log P)^2$, hence

$$\pi_f(P\mathscr{B}) = \frac{\mathrm{vol}(\mathscr{B})}{\deg(f)} \left(\prod_{p \text{ prime}} \frac{(1 - p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f(\mathbf{x}) = 0\})}{(1 - 1/p)} \right) \frac{P^n}{\log P} + O_{f,\mathscr{B}}\left(\frac{P^n}{(\log P)^2} \right). \tag{1.5}$$

Bateman and Horn [1] provided heuristics that led to a conjecture regarding the prime values of an integer polynomial in a single variable. One can modify their heuristics in the case that the polynomial has arbitrarily many variables, thus resulting in an analogous conjecture regarding the prime values of an integer polynomial in many variables. We refer the reader to Appendix A for a quick overview of Schinzel's hypothesis, the Bateman-Horn conjecture and their generalisations. Theorem 1.3 then establishes the analogous conjecture provided that the polynomial has sufficiently many variables compared to its degree.

There are currently no available techniques capable of settling any case of the Bateman–Horn conjeture in one variable apart from the case of one linear polynomial, which is Dirichlet's theorem for primes in arithmetic progressions. Efforts have therefore focused on settling such problems for polynomials in more variables. Notable examples in cases with $n = 2$ are Iwaniec's work [25] for quadratic polynomials, Fouvry–Iwaniec's work [15] for $x_1^2 + x_2^2$ with x_2 prime, Friedlander–Iwaniec's work [16] for $x_1^2 + x_2^4$, Heath-Brown's work [22] for $x_1^3 + 2x_2^3$, Heath-Brown–Moroz's work [24] for binary cubic forms and the recent work of Heath-Brown–Li [23] on $x_1^2 + x_2^4$ with x_2 prime. The special shape of these polynomials plays a central rôle in the proofs of these results; they are all related to norms of a number field. In cases with $n > 2$ it should be noted that Green–Tao–Ziegler [19] studied simultaneous prime values of certain linear polynomials by a variety of methods, Friedlander and Iwaniec [17] studied the prime values of $x_1^2 + x_2^2 + x_3^2$ via the class number formula of Gauss, while Maynard's work [28] employs geometry of numbers to cover the case of incomplete norm forms.

It is therefore a natural question whether the problem of representing primes by polynomials can be studied for polynomials with no special shape. Let us recall here that one of the important theorems in the frontiers between analytic number theory and Diophantine geometry concerns the Hasse principle for systems of polynomials in many variables and with no special shape by Birch [2]. To prove Theorem 1.3 we shall employ the Hardy–Littlewood circle method in the form used by Birch and use several of his estimates.

While Birch's work applies to every non-singular homogeneous polynomial f having at least $n \geqslant (\deg(f)-1)2^{\deg(f)}+1$ variables (which was recently improved by Browning and Prendiville [6] to $n \geqslant (\deg(f) - \sqrt{\deg(f)}/2)2^{\deg(f)}$), the assumption (1.4) of our Theorem 1.3 is less restrictive, as it allows for half as many variables. The improved range is due to the use of L_2-norm inequalities in the minor arcs, as well as bounds for exponential sums due to Browning–Heath-Brown [4] and Deligne [13] to show that the singular series in Birch's work converges absolutely in the range (1.4).

Let us finally give a direct consequence of Theorem 1.3.

Corollary 1.4. *Let $f \in \mathbb{Z}[x_1,\dots,x_n]$ be an integer homogeneous polynomial which is irreducible in $\mathbb{Q}[x_1,\dots,x_n]$ and assume that $n-\sigma_f \geqslant \max\{4,(\deg(f)-1)2^{\deg(f)-1}+1\}$. Then $f(\mathbf{x})$ takes infinitely many distinct positive prime values as $\mathbf{x}$ ranges over $\mathbb{Z}^n$ if and only if $f(\mathbb{R}^n)$ is not included in $(-\infty,0]$ and for every prime p the set $f(\mathbb{Z}^n)$ is not included in $p\mathbb{Z}$.*

Proof. We clearly need to focus only on the sufficiency. If $f(\mathbb{R}^n) \not\subset (-\infty,0]$ holds, then we can obviously find a non-empty box $\mathscr{B} \subset \mathbb{R}^n$ with $f(\mathscr{B}) \subset (0,+\infty)$, so that $\mathrm{vol}(\mathscr{B}) \neq 0$. If $f(\mathbb{Z}^n) \not\subset p\mathbb{Z}$ holds, then the p-adic factor in (1.5) is strictly positive and we shall see in Lemma 3.17 that the product over p is absolutely convergent. Hence by (1.5) we deduce that $\pi_f(P) \asymp_{f,\mathscr{B}} P^n/\log P$. If $f(\mathbf{x})=q$ was soluble only for finitely many primes q, say $q_1,\dots,q_r$, then the standard estimate $\#\{\mathbf{x} \in \mathbb{Z}^n \cap P\mathscr{B} : f(\mathbf{x})=q\} \ll_{f,q,\mathscr{B}} P^{n-1}$ would lead to

$$\frac{P^n}{\log P} \asymp_{f,\mathscr{B}} \pi_f(B) = \sum_{i=1}^{r} \#\{\mathbf{x} \in \mathbb{Z}^n \cap P\mathscr{B} : f(\mathbf{x})=q\} \ll_{f,q_1,\dots,q_r,\mathscr{B}} P^{n-1},$$

which is a contradiction. □

1.3. **Square-free integers represented by polynomials.** An integer m is called square-free if for every prime p we have $p^2 \nmid m$. In particular, 0 is not square-free and m is square-free if and only if $-m$ is.

Assume that we are given a polynomial $f \in \mathbb{Z}[x_1,\dots,x_n]$ that is separable as an element of $\mathbb{Q}[x_1,\dots,x_n]$ and let f_0 and σ_f be as in §1.2. A similar approach to the one for Theorem 1.3 allows us to study the set $\mathbf{S}_f := \{\mathbf{x} \in \mathbb{Z}^n : f(\mathbf{x}) \text{ is square-free}\}$.

Theorem 1.5. *Assume that $f \in \mathbb{Z}[x_1,\dots,x_n]$ is any integer polynomial which is separable as an element of $\mathbb{Q}[x_1,\dots,x_n]$ and let $\mathscr{B} \subset \mathbb{R}^n$ be any non-empty closed box. If*

$$n-\sigma_f > \max\Big\{1, \frac{1}{3}(\deg(f)-1)2^{\deg(f)}\Big\}, \tag{1.6}$$

then there exists $\beta=\beta(f)>0$ such that for all $P \geqslant 2$ the equality

$$\frac{\#\{\mathbf{S}_f \cap P\mathscr{B}\}}{\#\{\mathbb{Z}^n \cap P\mathscr{B}\}} = \prod_{p \ prime} \Big(1-p^{-2n}\#\big\{\mathbf{x} \in (\mathbb{Z}/p^2\mathbb{Z})^n : f(\mathbf{x})=0\big\}\Big) + O_{f,\mathscr{B}}(P^{-\beta})$$

holds with an implied constant that depends at most on f and $\mathscr{B}$.

The problem of square-free values of integer polynomials has a very long history, see [3] for a list of references. Many cases are still open, for example, there is no irreducible quartic integer polynomial in one variable for which we know that it takes infinitely many square-free values. One of the most general results, conditional on the *abc* conjecture, is due to Poonen [29] where arbitrary polynomials are treated. Our Theorem 1.5 covers unconditionally arbitrary polynomials of fixed degree and number of variables with the proviso that the number of variables is suitably large compared to the degree. Theorem 1.5 features a saving of two thirds of the variables compared to the Birch setting [2]. This saving comes from the fact that exponential sums whose terms are restricted to square-free integers can be bounded in a satisfactory manner, this was done in the work of Brüdern, Granville, Perelli, Vaughan and Wooley [7] and Keil [27].

Notation. We shall use the notation $\mathbf{x}$ to refer to n-tuples $\mathbf{x} = (x_1, ..., x_n)$. We will also make use of the classical von Mangoldt function denoted Λ and of the classical Möbius function denoted μ. The letter d will refer exclusively to the degree of the polynomial f in Theorem 1.3. Finally, throughout the paper, we shall make use of the notation

$$\mathrm{e}(z) := \exp(2\pi i z), z \in \mathbb{C}. \tag{1.7}$$

The polynomial f and the box $\mathscr{B}$ will be considered fixed throughout. This is taken to mean that, although each implied constant in the big O notation will depend on several quantities related to f, we shall avoid recording these dependencies. The list of the said quantities consists of

$$f_0, n, d, \sigma_f, \theta_0, \delta, \eta, \lambda_1, A, \lambda,$$

whose meaning will become evident in due course. The symbol ε will be used for a small positive parameter whose value may vary, allowing, for example, inequalities of the form $x^\varepsilon \ll x^{\varepsilon/4}$. Further dependency of the implied constants on other quantities will be recorded explicitly via an appropriate use of subscript.

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2. The proof of Theorem 1.1

Denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ with respect to the place v, let $|\cdot|_p$ be the p-adic norm defined by $|x|_p = p^{-\nu_p(x)}$ for $x \in \mathbb{Q}_p$ if $v = p$ is finite and define $|\cdot|_\infty$ as the classical absolute value for the real place. We will use the notation $\mathbb{Z}_S$ for the ring of S-integers for any finite set of finite places S and we will say that a prime p is a fixed prime divisor of a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ if, for all $(x_1, \dots, x_n) \in \mathbb{Z}^n$, we have $p \mid f(x_1, \dots, x_n)$.

2.1. **Preliminary lemmas.**

We begin by establishing the following analogue of [9, Lem. 2].

Lemma 2.1. *Let $f \in \mathbb{Z}[x_1, \dots, x_n]$ be a non-zero polynomial with content equal to* 1*. If p is such that $f(\mathbb{Z}^n) \subseteq p\mathbb{Z}$, then $p \leqslant \deg(f)$.*

Proof. Define $d := \deg(f)$ and let p be a prime such that $f(\mathbb{Z}^n) \subseteq p\mathbb{Z}$. On one hand, we have by assumption that

$$\#\{\mathbf{x} \in (\mathbb{Z}/p\mathbb{Z})^n \ : \ f(\mathbf{x}) = 0\} = p^n.$$

On the other hand, since f has content one, [33, Eq.(2.7)] implies that

$$\#\{\mathbf{x}\in(\mathbb{Z}/p\mathbb{Z})^n \ : \ f(\mathbf{x})=0\}\leqslant dp^{n-1}$$

and hence $p\leqslant d$, thus concluding the proof of the lemma. □

We now use Lemma 2.1 to verify the following analogue of [20, Prop. 1.2] and of the hypothesis (H_1) of [11] over $\mathbb{Q}$.

Proposition 2.2. *Let $f\in\mathbb{Q}[x_1,\dots,x_n]$ be an irreducible homogeneous polynomial satisfying the assumptions (1.4) and $f(1,0,\dots,0)>0$. Let C be a positive real constant and $\varepsilon>0$. Suppose we are given $(\lambda_{1,p},\dots,\lambda_{n,p})\in\mathbb{Q}_p^n$ for p in a finite set of finite places S containing all primes $p\leqslant\deg(f)$ and all primes p such that f does not have p-integral coefficients as well as all primes p such that $\nu_p(f(1,0,\dots,0))>0$. Then there exists infinitely many $(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}_S^n$ such that $\lambda_1>C\lambda_i>0$ for all $i\in\{2,\dots,n\}$, $|\lambda_i-\lambda_{i,p}|_p<\varepsilon$ for all $i\in\{1,\dots,n\}$ and $p\in S$ and $f(\lambda_1,\dots,\lambda_n)=\ell u$ for a prime $\ell\notin S$ and $u\in\mathbb{Z}_S^\times$, $u>0$.*

Proof. Up to multiplication of $(\lambda_1,\dots,\lambda_n)$ and $(\lambda_{1,p},\dots,\lambda_{n,p})$ by a product of powers of primes in S, we can assume without loss of generality that $(\lambda_{1,p},\dots,\lambda_{n,p})\in\mathbb{Z}_p^n$ for $p\in S$. The assumption that $f(1,0,\dots,0)>0$ provides with $a_{\mathbf{i}}\in\mathbb{Q}$ and $a>0$ such that

$$f(x_1,\dots,x_n)=ax_1^d+\sum_{\substack{\mathbf{i}=(i_1,\dots,i_n)\in\mathbb{N}^n\\ i_1+\cdots+i_n=d\\ 0\leqslant i_1,\dots,i_n\leqslant d\\ i_1\neq d}}a_{\mathbf{i}}x_1^{i_1}\cdots x_n^{i_n}. \tag{2.1}$$

Let $N_{<0}$ be the number of indices $\mathbf{i}$ with $a_{\mathbf{i}}<0$. We can assume that $C>1$ and that $C>\frac{N_{>0}|a_{\mathbf{i}}|}{a}$ whenever $a_{\mathbf{i}}<0$. As in the proof of [20, Prop. 1.2], we can now find $(\lambda_{0,1},\dots,\lambda_{0,n})\in\mathbb{Z}^n$ such that $|\lambda_{0,i}-\lambda_{i,p}|_p<\varepsilon/2$ for all $p\in S$. We can choose them such that $\lambda_{0,1}>C\lambda_{0,i}>0$ for all $i\in\{2,\dots,n\}$. We can now see that $f(\lambda_{0,1},\dots,\lambda_{0,n})>0$ by alluding to

$$f(\lambda_{0,1},\dots,\lambda_{0,n})\geqslant a\lambda_{0,1}^d-\sum_{a_{\mathbf{i}}<0}|a_{\mathbf{i}}|\lambda_{0,1}^{i_1}\cdots\lambda_{0,n}^{i_n},$$

and the inequalities

$$a\lambda_{0,1}^d=\frac{a}{N_{>0}}\sum_{a_{\mathbf{i}}<0}\lambda_{0,1}^d=\frac{a}{N_{>0}}\sum_{a_{\mathbf{i}}<0}\lambda_{0,1}^{i_1}\cdots\lambda_{0,1}^{i_n}>\sum_{a_{\mathbf{i}}<0}\frac{a}{N_{>0}}C^{d-i_1}\lambda_{0,1}^{i_1}\cdots\lambda_{0,n}^{i_n}>\sum_{a_{\mathbf{i}}<0}|a_{\mathbf{i}}|\lambda_{0,1}^{i_1}\cdots\lambda_{0,n}^{i_n}.$$

Let $A=\prod_{p\in S}p$ and a fixed integer N big enough so that $|A^N|_p<\varepsilon/2$ for all $p\in S$. Now consider the polynomial $g\in\mathbb{Q}[x_1,\dots,x_n]$ given by

$$g(x_1,\dots,x_n)=f(\lambda_{0,1}+x_1A^N,\dots,\lambda_{0,n}+x_nA^N).$$

The polynomial g can be expressed as $g=t\tilde{g}$ for $t\in\mathbb{Z}_S^\times$ and $\tilde{g}$ a polynomial with integer coefficients which is irreducible over $\mathbb{Q}$. Let us denote by c the product of all fixed prime factors of $\tilde{g}$. We will now establish that if p is a prime factor of c then $p\in S$. Let $p\mid c$. Either p divides the content of $\tilde{g}$ and in particular, with the notation (2.1), $\nu_p(aA)\neq0$ which immediately implies that $p\in S$ or, denoting by $\tilde{c}$ the content of $\tilde{g}$, p is a fixed prime factor of the polynomial $\tilde{g}/\tilde{c}$ which has integral coefficients and content equal to one. By Lemma 2.1 this implies that $p\leqslant\deg(f)$ and hence that $p\in S$. Moreover, with the notation of §1.2, $\tilde{g}_0=A^{dN}f$ and the conditions $x_1>Cx_i>0$ define an open cone $\mathscr{C}$ in $\mathbb{R}^n$. In addition, when f is evaluated at $(\lambda_{0,1},\dots,\lambda_{0,n})\in\mathscr{C}$ it produces a strictly positive value, therefore we can find a box $\mathscr{B}\subseteq\mathscr{C}$ such that $f(\mathscr{B})\subset(0,\infty)$. Since for all P we have $P\mathscr{B}\subset\mathscr{C}$, we obtain from Theorem 1.3 that there exist infinitely many $\mathbf{x} \in \mathbb{Z}^n \cap \mathscr{C}$ such that $\tilde{g}(\mathbf{x})/c$ is prime. Introducing $\lambda_i = \lambda_{0,i} + x_i A^N$ for any $i \in \{1, \dots, n\}$ and for any such $\mathbf{x} \in \mathbb{Z}^n \cap \mathscr{C}$, we get that

$$f(\lambda_1, \dots, \lambda_n)/(ct) = g(x_1, \dots, x_n)/(ct)$$

is prime. This yields the result because $\lambda_{0,1} > C\lambda_{0,i} > 0$ and $x_1 > Cx_i > 0$, which implies $\lambda_1 = (\lambda_{0,1} + x_1 A^N) > C\lambda_i = C(\lambda_{0,i} + x_i A^N)$. Moreover, $|\lambda_i - \lambda_{i,0}|_p \leqslant |A^N|_p < \varepsilon/2$ and hence $|\lambda_i - \lambda_{i,p}|_p < \varepsilon$ for all $p \in S$ and all $i \in \{1, \dots, n\}$. □

2.2. **Conclusion of the proof of Theorem 1.1.**

Proof. We proceed by adapting the proof of [20, Th. 1.3]. We are given $1 > \varepsilon > 0$, a finite set of places S and a point $(\mathbf{t}_v, \mathbf{x}_{1,v}, \dots, \mathbf{x}_{r,v}) \in X(\mathbb{Q}_v)$ for every place v of $\mathbb{Q}$ and we want to find $(\mathbf{t}, \mathbf{x}_1, \dots, \mathbf{x}_r) \in X(\mathbb{Q})$ such that for all $v \in S$,

$$\begin{cases} |t_i - t_{i,p}|_v < \varepsilon & (i \in \{1, \dots, n\}), \\ |x_{j_i,i} - x_{j_i,i,v}|_v < \varepsilon & (i \in \{1, \dots, r\}, \quad j_i \in \{1, \dots, d_i\}). \end{cases}$$

2.2.1. *First step.* By density and continuity, we can assume that $\mathbf{t}_\infty \in \mathbb{Q}^n$ and by a linear change of variables, we can assume that $\mathbf{t}_\infty = (1, 0, \dots, 0)$. Note that the solubility over $\mathbb{R}$ implies that $f(1, 0, \dots, 0) > 0$ in the case where there is a totally imaginary K_i. In addition, it implies that $f(1, 0, \dots, 0)$ can be strictly positive or strictly negative when all K_i are totally real. We denote by $s \in \{-1, +1\}$ the sign of $f(1, 0, \dots, 0)$. We can enlarge S so that it contains the real place, the field K_i is unramified outside S for all $i \in \{1, \dots, r\}$, S contains all primes $p \leqslant \deg(f)$, all primes p such that f does not have p-integral coefficients as well as primes p such that $\nu_p(f(1, 0, \dots, 0)) > 0$.

2.2.2. *Second step.* Let $L = [d_1, \dots, d_r]$ denote the least common multiple of the degrees $d_1, \dots, d_r$ and $M = \max_{v \in S} \max_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j_i \leqslant d_i}} |x_{j_i,i,v}|_v$. By [14, Prop. 6.1], we know that the image of the map $N_{K_{i,p}/\mathbb{Q}_p} : K_{i,p}^\times \to \mathbb{Q}_p^\times$ is open and that a polynomial function is continuous for the p-adic topology. Hence there exists $\varepsilon' > 0$ such that for every $(\lambda_1, \dots, \lambda_n) \in \mathbb{Q}^n$ satisfying for every $i \in \{1, \dots, n\}$, the inequality $|\lambda_i - t_{i,p}|_p < \varepsilon'$, we have that $f(\lambda_1, \dots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for the place p and there exists $\varepsilon'' > 0$ such that for every $(\lambda_1, \dots, \lambda_n) \in \mathbb{Q}^n$ satisfying for every $i \in \{1, \dots, n\}$, the inequality $|\lambda_i - t_{i,p}|_p < \varepsilon''$, we have that

$$|f(\lambda_1, \dots, \lambda_n) - f(t_{1,p}, \dots, t_{n,p})|_p < \frac{\varepsilon}{2M} |f(t_{1,p}, \dots, t_{n,p})|_p |L|_p, \qquad (p \in S). \tag{2.2}$$

Then applying Proposition 2.2 yields $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_S^n$ such that $\lambda_1 > C\lambda_i > 0$ for all $i \in \{2, \dots, n\}$, $|\lambda_i - t_{i,p}|_p < \min\{\varepsilon/2, \varepsilon', \varepsilon''/2\}$ for all $i \in \{1, \dots, n\}$ and $p \in S$ and $f(\lambda_1, \dots, \lambda_n) = s\ell u$ for a prime $\ell \notin S$ and $u \in \mathbb{Z}_S^\times$ with $u > 0$. We thus obtain that $f(\lambda_1, \dots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for all places of S. This is also the case for the real place because $f(\lambda_1, \dots, \lambda_n) > 0$ in the case that there is a totally imaginary K_i.

2.2.3. *Third step.* Now, $f(\lambda_1, \dots, \lambda_n) = \ell s u$ is a unit in $\mathbb{Z}_p$ for every $\mathbb{Q}_p$ and $p \notin S \cup \{\ell\}$ and we know by [26, Prop. V.3.11] that this implies that $f(\lambda_1, \dots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for all $p \notin S \cup \{\ell\}$. By the global reciprocity law and the fact that $K_i/\mathbb{Q}$ is unramified outside S we see that $f(\lambda_1, \dots, \lambda_n)$ is also a local norm for $K_i/\mathbb{Q}$ at the place ℓ (see [26, Prop. V.12.9]). The conclusion is that $f(\lambda_1, \dots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ at every place of $\mathbb{Q}$ and then by the Hasse norm principle [26, Th. V.4.5], one gets that there exists $(\mathbf{x}_1, \dots, \mathbf{x}_r) \in \mathbb{Q}^{d_1} \times \cdots \times \mathbb{Q}^{d_r}$ such that $0 \neq f(\lambda_1, \dots, \lambda_n) = \mathbf{N}_{K_1/\mathbb{Q}}(\mathbf{x}_1) = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\mathbf{x}_r)$.

2.2.4. *Fourth step.* By continuity, there exists $\varepsilon_1 > 0$ such that for all $q_1 \in \mathbb{Q}^\times$ such that $|q_1 - \lambda_1|_\infty < \varepsilon_1$, then $\left|\frac{1}{q_1} - \frac{1}{\lambda_1}\right|_\infty < \frac{\varepsilon}{2\max_{1\leqslant i\leqslant n}|\lambda_i|_\infty}$. Writing $m = [L, \deg(f)]$, by weak approximation in $\mathbb{Q}$, since $\lambda_1 > 0$ one can find $\rho \in \mathbb{Q}$ such that $|\rho - 1|_p < \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon''}{2}\right\}\min\{1, |\lambda_i|_p\}$ for all $p \in S$ and $|\rho^{m/\deg(f)} - \lambda_1|_\infty < \min\left\{\frac{\varepsilon}{2}\min\left\{1, \lambda_1 - \frac{\varepsilon}{2}\right\}, \varepsilon_1\right\}$ where we can assume that $\varepsilon < 2\lambda_1$. In particular, this implies that $|\rho|_p = 1$ for all $p \in S$. We now make the following change of variables,

$$\lambda_i = \rho^{m/\deg(f)}\lambda_i', \quad i \in \{1, \dots, n\}, \quad \mathbf{x}_i = \rho^{m/d_i}\mathbf{x}_i', \quad i \in \{1, \dots, r\},$$

so that for all finite place $p \in S$ we have

$$|\lambda_i' - \lambda_i|_p = |\lambda_i'|_p|\rho^{m/\deg(f)} - 1|_p \leqslant |\lambda_i|_p|\rho - 1|_p < \frac{\varepsilon}{2}$$

and therefore $|\lambda_i' - t_{i,p}|_p < \varepsilon$ for all $i \in \{1, \dots, n\}$. Moreover, we have

$$0 \neq f(\lambda_1', \dots, \lambda_n') = \mathbf{N}_{K_1/\mathbb{Q}}(\mathbf{x}_1') = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\mathbf{x}_r'). \tag{2.3}$$

As for the real place, we have $|\lambda_1' - 1|_\infty < \varepsilon$ and $|\lambda_i' - \lambda_i/\lambda_1|_\infty < \varepsilon/2$. The treatment of the archimedian place is now concluded similarly as in [20], by alluding to $0 < \lambda_i/\lambda_1 < C^{-1}$ and by taking C big enough, namely $C > \frac{2}{\varepsilon}$.

2.2.5. *Fifth step.* To conclude the proof, it remains to find $(\mathbf{x}_1'', \dots, \mathbf{x}_r'') \in \mathbb{Q}^{d_1} \times \cdots \times \mathbb{Q}^{d_r}$ v-adically close to $(\mathbf{x}_{1,v}, \dots, \mathbf{x}_{r,v})$ for all $v \in S$ and such that $\mathbf{N}_{K_i/\mathbb{Q}}(\mathbf{x}_i'') = \mathbf{N}_{K_i/\mathbb{Q}}(\mathbf{x}_i')$ for all $i \in \{1, \dots, r\}$. By (2.2) and the choice of $\boldsymbol{\lambda}' = (\lambda_1', \dots, \lambda_n')$ above, one can write $f(\lambda_1', \dots, \lambda_n') = f(t_{1,p}, \dots, t_{n,p})\beta_p$ with $\beta_p \in \mathbb{Q}_p$ satisfying $|\beta_p - 1|_p < \frac{\varepsilon}{2M}|L|_p$ for all finite place $p \in S$. In particular, $|\beta_p|_p = 1$ and $\beta_p \in \mathbb{Z}_p^\times$ for all finite place $p \in S$. Now, Hensel's lemma implies that there exists $\alpha_p \in \mathbb{Z}_p$ such that $\beta_p = \alpha_p^L$ and

$$|\alpha_p - 1|_p = \left|\frac{\beta_p - 1}{L}\right|_p < \frac{\varepsilon}{2M}.$$

Of course, there exists α_∞ such that $\beta_\infty = \alpha_\infty^L$ and $|\alpha_\infty - 1|_\infty < \varepsilon/(2M)$ since one can always ensure that $f(1, 0, \dots, 0)$ and $f(\lambda_1', \dots, \lambda_n')$ have the same sign. Now alluding to the facts that $(\mathbf{t}_v, \mathbf{x}_{1,v}, \dots, \mathbf{x}_{r,v}) \in X(\mathbb{Q}_v)$ and to (2.3), we obtain that for all $v \in S$

$$0 \neq f(\lambda_1', \dots, \lambda_n') = \mathbf{N}_{K_1/\mathbb{Q}}(\alpha_v^{L/d_1}\mathbf{x}_{1,v}) = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\alpha_v^{L/d_r}\mathbf{x}_{r,v}).$$

In other words, for every $i \in \{1, \dots, r\}$, we have $\mathbf{N}_{K_i/\mathbb{Q}}\left(\alpha_v^{L/d_i}\mathbf{x}_{i,v}\right) = \mathbf{N}_{K_i/\mathbb{Q}}\left(\mathbf{x}_i'\right)$ for all $v \in S$. Thanks to the fact that weak approximation holds for the norm tori $N_{K_i/\mathbb{Q}}(\mathbf{z}) = 1$, one gets the existence of $\mathbf{x}_i'' \in \mathbb{Q}^{d_i}$ such that $|x_{j,i}'' - \alpha_v^{L/d_i}x_{j,i,v}|_v < \varepsilon/2$ for all $v \in S$ and $j \in \{1, \dots, d_i\}$ and $\mathbf{N}_{K_i/\mathbb{Q}}(\mathbf{x}_i'') = \mathbf{N}_{K_i/\mathbb{Q}}(\mathbf{x}_i')$. Therefore, we have

$$0 \neq f(\lambda_1', \dots, \lambda_n') = \mathbf{N}_{K_1/\mathbb{Q}}(\mathbf{x}_1'') = \cdots = \mathbf{N}_{K_r/\mathbb{Q}}(\mathbf{x}_r'')$$

along with

$$|x_{j,i}'' - x_{j,i,v}|_v \leqslant |x_{j,i}'' - \alpha_v^{L/d_i}x_{j,i,v}|_v + |x_{j,i,v}|_v|\alpha_v - 1|_v < \varepsilon,$$

thus concluding the proof of Theorem 1.1. ◻

3. The proof of Theorem 1.3

3.1. **First steps and auxiliary estimates.** The proof of Theorem 1.3 is initiated by using the following exponential sums for real α,

$$S(\alpha) := \sum_{\mathbf{x}\in\mathbb{Z}^n\cap P\mathscr{B}} \mathrm{e}\left(\alpha f(\mathbf{x})\right) \quad \text{and} \quad W(\alpha) := \sum_{\frac{1}{2}\min\{f_0(\mathscr{B})\}P^d\leqslant p\leqslant 2\max\{f_0(\mathscr{B})\}P^d} \mathrm{e}(\alpha p), \tag{3.1}$$

where we used that, in the setting of Theorem 1.3, the following quantities are positive

$$\min\{f_0(\mathscr{B})\} = \min\{f_0(\mathbf{x}) : \mathbf{x}\in\mathscr{B}\} \quad \text{and} \quad \max\{f_0(\mathscr{B})\} := \max\{f_0(\mathbf{x}) : \mathbf{x}\in\mathscr{B}\}.$$

The fact that $\int_0^1 \mathrm{e}(\alpha\{f(\mathbf{x})-p\})\mathrm{d}\alpha$ is 1 when $f(\mathbf{x})=p$ and is otherwise 0, shows that for all $P \gg_{f,\mathscr{B}} 1$, we have the equality

$$\pi_f(P\mathscr{B}) = \int_0^1 S(\alpha)\overline{W(\alpha)}\mathrm{d}\alpha. \tag{3.2}$$

This identity has the useful feature that it completely separates the problem of evaluating π_f into two problems, one regarding the evaluation of the sum S (that is only related to the values of the polynomial f) and one regarding the evaluation of the sum W (that is only related to the distribution of primes). Birch [2] has a similar identity, save for the factor $\overline{W(\alpha)}$. The main idea is that the presence of this extra factor can be turned to our advantage, as it attains small values for certain α for which $|S(\alpha)|$ is large. Let us comment that we could have defined W in an alternative way by replacing the range for the primes p by the condition $\min\{f_0(\mathscr{B})\}P^d \leqslant p \leqslant \max\{f_0(\mathscr{B})\}P^d$, however, our choice will make more transparent the proof of Lemma 3.19.

Before proceeding let us recall here the estimates from the work of Birch [2] that we shall need later. First, following [2, pg. 251,Eq.(5)], we let for $\theta\in(0,1]$ and $a\in\mathbb{Z}\cap[0,q)$ with $\gcd(a,q)=1$,

$$\mathscr{M}_{a,q}(\theta) := \left\{\alpha\in(0,1] : 2|q\alpha-a| \leqslant P^{-d+(d-1)\theta}\right\} \tag{3.3}$$

and

$$\mathscr{M}(\theta) = \bigcup_{1\leqslant q\leqslant P^{(d-1)\theta}} \bigcup_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} \mathscr{M}_{a,q}(\theta). \tag{3.4}$$

Birch then gives the following upper bound for the volume of $\mathscr{M}(\theta)$.

Lemma 3.1 (Birch [2, Lem. 4.2])**.** *$\mathscr{M}(\theta)$ has volume at most $P^{-d+2(d-1)\theta}$.*

Next, we choose any positive δ,θ_0 satisfying

$$1 > \delta + 6d\theta_0 \quad \text{and} \quad \frac{n-\sigma_f}{2^{d-1}} - (d-1) > \delta\theta_0^{-1}. \tag{3.5}$$

As in [2, pg.252,Eq.(13)-(14)] it is easy to see that there exists $T\in\mathbb{N}$ and positive real numbers $\theta_1,\ldots,\theta_T$ with the properties

$$\begin{cases} T \ll P^{\delta}, \\ \theta_T > \theta_{T-1} > \ldots > \theta_1 > \theta_0 > 0, \\ d = 2(d-1)\theta_T \\ \frac{1}{2}\delta > 2(d-1)(\theta_{t+1}-\theta_t) \text{ for } 0\leqslant t\leqslant T-1. \end{cases} \tag{3.6}$$

We next recall [2, Lem. 4.3]. Note that it was proved for homogeneous f, but, as noted by Schmidt [31, §9] a similar argument works for inhomogeneous f, because the Weyl differencing process is not affected by lower order terms.

Lemma 3.2 (Birch [2, Lem. 4.3])**.** *Let $0 < \theta \leqslant 1$ and $\varepsilon > 0$. Then if α is not in $\mathscr{M}(\theta)$ modulo 1,*

$$|S(\alpha)| \ll P^{n-\theta\left(\frac{n-\sigma_f}{2^{d-1}}\right)+\varepsilon}.$$

Following the notation in [2, pg. 253] and for θ, a, q as above we also let

$$\mathscr{M}'_{a,q}(\theta) := \left\{\alpha \in (0,1] : |q\alpha - a| \leqslant qP^{-d+(d-1)\theta}\right\} \tag{3.7}$$

and

$$\mathscr{M}'(\theta) = \bigcup_{1 \leqslant q \leqslant P^{(d-1)\theta}} \bigcup_{\substack{a \in \mathbb{Z} \cap [0,q) \\ \gcd(a,q)=1}} \mathscr{M}'_{a,q}(\theta). \tag{3.8}$$

With θ_0 as in (3.5), we let

$$\eta := (d-1)\theta_0 \tag{3.9}$$

and for $a \in \mathbb{Z}, q \in \mathbb{N}$ we define

$$S_{a,q} := \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} \mathrm{e}\left(\frac{af(\mathbf{x})}{q}\right). \tag{3.10}$$

Finally, for any $\gamma \in \mathbb{R}$ and any measurable $\mathscr{C} \subset [-1,1]^n$ we define

$$I(\mathscr{C};\gamma) := \int_{\mathbf{x} \in \mathscr{C}} \mathrm{e}\left(\gamma f_0(\mathbf{x})\right) \mathrm{d}\mathbf{x}. \tag{3.11}$$

The following result, due to Birch, gives an upper bound for the quantity $I(\mathscr{C};\gamma)$.

Lemma 3.3 (Birch [2, Lem. 5.2])**.** *Let $\mathscr{C}$ be a box contained in $[-1,1]^n$ with sidelength at most $\sigma < 1$. Then*

$$I(\mathscr{C},\gamma) \ll \sigma^n \min\left[1, (\sigma^d|\gamma|)^{-\frac{n-\sigma_f}{2^{d-1}(d-1)}+\varepsilon}\right].$$

The next result was proved by Birch with f instead of f_0 in the definition of $I(\mathscr{C};\gamma)$, however the following result holds in light of the remarks concerning $\mu(\infty,\mathscr{B})$ in Schmidt's work [31, §9].

Lemma 3.4 (Birch [2, Lem. 5.1])**.** *Assume that we are given coprime integers $q \in \mathbb{N}$ and $a \in \mathbb{Z} \cap [0,q)$ and let $\alpha \in \mathscr{M}'_{a,q}(\theta_0)$ with the notations (3.5) and (3.7). Denoting $\beta := \alpha - \frac{a}{q}$, we have*

$$S(\alpha) = q^{-n}P^n S_{a,q} I(\mathscr{B}; P^d\beta) + O(P^{n-1+2\eta}).$$

Let us now turn to the quantity $S_{a,q}$ defined in (3.10).

Lemma 3.5 (Birch [2, Lem. 5.4])**.** *For every $\varepsilon > 0$ and for $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $\gcd(a,q) = 1$ we have*

$$S_{a,q} \ll q^{n-\frac{n-\sigma_f}{2^{d-1}(d-1)}+\varepsilon}.$$

The next result is of key importance in the proof of Theorem 1.3. It is concerned with the convergence of the singular series and its proof relies on recent results on the estimation of exponential sums due to Browning–Heath-Brown [4] and Browning–Prendiville [6].

Lemma 3.6 (Browning–Heath-Brown [4, Lem. 25]). *We have*

$$S_{a,p^k} \ll_k p^{(k-1)n+\sigma_f}$$

for all $k \geqslant 2$.

Note that, as explained in [4, Eq. (6.1)], the quantity σ in [4, Lem. 25] coincides with $-1+\sigma_f$, with σ_f as in the present work.

Proposition 3.7. *Let* $f \in \mathbb{Z}[x_1, \ldots, x_n]$ *be an irreducible polynomial and define*

$$T_f(q) := q^{-n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |S_{a,q}|, \ \ q \in \mathbb{N}.$$

(1) *If* $n - \sigma_f \geqslant \max\{5, (\deg(f)-1)2^{\deg(f)-1}+2\}$ *then the abscissa of convergence of the Dirichlet series of* T_f *is strictly negative.*
(2) *If* $n - \sigma_f \geqslant \max\{4, (\deg(f)-1)2^{\deg(f)-1}+1\}$ *then there exists a constant* $C' = C'(f) > 0$ *such that* $\sum_{q \leqslant x} T_f(q) \ll (\log x)^{C'}$.

Proof. Part (1). It is sufficient to prove that there exists $\lambda_1 > 0$ such that $\sum_q q^{\lambda_1} T_f(q) < \infty$. By [2, §7], the function T_f is multiplicative, hence the series over q converges absolutely if the analogous Euler product converges absolutely, i.e.

$$\sum_{\substack{p \text{ prime} \\ k \in \mathbb{N}}} p^{k\lambda_1} T_f(p^k) = \sum_{\substack{p \text{ prime} \\ k \in \mathbb{N}}} p^{k(\lambda_1 - n)} \sum_{a \in (\mathbb{Z}/p^k\mathbb{Z})^*} |S_{a,p^k}| < \infty. \tag{3.12}$$

By Lemma 3.5 the terms with $k > 2^{d-1}(d-1)$ contribute

$$\ll \sum_p \sum_{k \geqslant 1+2^{d-1}(d-1)} p^{k\left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)}. \tag{3.13}$$

By the assumption $n - \sigma_f \geqslant (d-1)2^{d-1}+2$

$$\frac{n-\sigma_f}{(d-1)2^{d-1}} \geqslant 1 + \frac{2}{(d-1)2^{d-1}}$$

we have

$$p^{\left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)} \leqslant p^{\left(-\frac{2}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)},$$

thus taking ε, λ_1 sufficiently small we can ensure that this is at most $p^{-\frac{1}{2^{d-1}(d-1)}} \leqslant 2^{-\frac{1}{2^{d-1}(d-1)}}$, which is of the form $1-\delta$ for some $0 < \delta < 1$. Note that if $\delta \in (0,1)$ then for all $z \in \mathbb{R}$ with $0 \leqslant z \leqslant 1-\delta$ and all $k_0 \in \mathbb{N}$ we have

$$\sum_{k \geqslant k_0} z^k = \frac{z^{k_0}}{1-z} \leqslant \frac{z^{k_0}}{\delta}.$$

Therefore the sum in (3.13) is

$$\ll_d \sum_p p^{(1+2^{d-1}(d-1))\left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)} \ll \sum_p p^{-2+(\varepsilon+\lambda_1)(1+2^{d-1}(d-1))} \ll \sum_p p^{-3/2} < \infty,$$

where we have taken ε, λ_1 sufficiently small to ensure $(\varepsilon+\lambda_1)(1+2^{d-1}(d-1)) \leqslant 1/2$. Next, we study the contribution towards (3.12) of any $k \in [2, 2^{d-1}(d-1)]$. By [4, Lem. 25] we infer that the said contribution is

$$\ll \sum_p p^{k(\lambda_1-n)+k+(k-1)n+\sigma_f} = \sum_p p^{(1+\lambda_1)k-n+\sigma_f} \leqslant \sum_p p^{(1+\lambda_1)(2^{d-1}(d-1))-n+\sigma_f}.$$

The assumption $n-\sigma_f \geqslant (d-1)2^{d-1}+2$ shows that the exponent is $\leqslant \lambda_1(2^{d-1}(d-1))-2$ and for small λ_1 the sum converges. To conclude the proof of (3.12) it only remains to bound the contribution of terms with $k=1$. As noted in [6, §5], one can prove

$$T_f(p) = p^{-n} \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^*} |S_{a,p}| \ll p^{1-\frac{n-\sigma_f}{2}} \tag{3.14}$$

by Deligne's estimate and induction on σ_f. Taking small $\lambda_1 < 1/4$ and using the assumption $n-\sigma_f \geqslant 5$ shows that the terms with $k=1$ in (3.12) form a convergent series. This completes our proof.

Part (2). If $k \in [2, 2^{d-1}(d-1)]$ then Lemma 3.6 and $n-\sigma_f \geqslant 1+2^{d-1}(d-1)$ imply that $T_f(p^k) \ll p^{-1}$. Furthermore, using Lemma 3.5 and $n-\sigma_f \geqslant 1+2^{d-1}(d-1)$ we have that if $k \geqslant 1+2^{d-1}(d-1)$ then $T_f(p^k) \ll p^{-1-2^d(d-1)^{-1}}$. Finally, $n-\sigma_f \geqslant 4$, thus (3.14) ensures that $T_f(p) \ll p^{-1}$. Putting everything together yields $\sum_{k\geqslant 1} T_f(p^k) \leqslant C'p^{-1}$ for some $C' = C'(f) > 0$ and the proof is concluded by using $\sum_{q\leqslant x} T_f(q) \leqslant \prod_{p\leqslant x}(1+\sum_{k\geqslant 1} T_f(p^k))$. ☐

3.2. **The minor arcs.** For $\theta \in (0,1]$ and $a \in \mathbb{Z} \cap [0,q)$ with $\gcd(a,q)=1$ we use the sets $\mathscr{M}(\theta)$ and $\mathscr{M}_{a,q}(\theta)$ defined by (3.4) and (3.3). Next, we choose any positive δ, θ_0 satisfying (3.5).

Lemma 3.8. *For any $0 < \theta \leqslant 1$ we have*

$$\left|\int_{\alpha\notin\mathscr{M}(\theta)} S(\alpha)\overline{W(\alpha)}\mathrm{d}\alpha\right| \ll \left(\int_{\alpha\notin\mathscr{M}(\theta)} |S(\alpha)|^2\mathrm{d}\alpha\right)^{1/2} P^{d/2}(\log P)^{-1/2}.$$

Proof. By Schwarz's inequality the integral on the left side is bounded by

$$\left(\int_{\alpha\notin\mathscr{M}(\theta)} |S(\alpha)|^2\mathrm{d}\alpha\right)^{1/2} \left(\int_0^1 |W(\alpha)|^2\mathrm{d}\alpha\right)^{1/2}.$$

The proof is concluded by noting that

$$\int_0^1 |W(\alpha)|^2\mathrm{d}\alpha = \sum_{\frac12 \min\{f_0(\mathscr{B})\}P^d \leqslant p \leqslant 2\max\{f_0(\mathscr{B})\}P^d} 1 \ll P^d/\log P.$$ ☐

Lemma 3.9. *Keep the assumptions of Theorem 1.3 and* (3.5). *Then we have,*

$$\left(\int_{\alpha\notin\mathscr{M}(\theta_0)} |S(\alpha)|^2\mathrm{d}\alpha\right)^{1/2} = O(P^{n-d/2-\delta/2}).$$

Proof. Using the entities $(\theta_i)_{i=0}^T$, given in (3.6), we have for sufficiently small $\varepsilon > 0$,

$$\int_{\alpha\notin\mathscr{M}(\theta_T)} |S(\alpha)|^2\mathrm{d}\alpha \ll P^{2\left(n-\left(\frac{n-\sigma_f}{2^{d-1}}\right)\theta_T\right)+\varepsilon} \leqslant P^{2n-d-\delta},$$

due to Lemma 3.2, the third equation of (3.6) and (1.4). For $t<T$ and $\varepsilon>0$ we get

$$\int_{\mathscr{M}(\theta_{t+1})\smallsetminus\mathscr{M}(\theta_t)} |S(\alpha)|^2\mathrm{d}\alpha \ll P^{-d+2(d-1)\theta_{t+1}+2\left(n-\left(\frac{n-\sigma_f}{2^{d-1}}\right)\theta_t\right)+\varepsilon}$$

by Lemmas 3.1 and 3.2. The proof can now be completed easily by using the last equation of (3.6), (3.5) and $T \ll P^{\delta}$, as in the last stage of the proof of [2, Lem. 4.4]. ☐

Lemma 3.10. *Keep the assumptions of Theorem 1.3 and* (3.5)*. Then we have,*

$$\left|\int_{\alpha\notin \mathscr{M}(\theta_0)} S(\alpha)\overline{W(\alpha)}\mathrm{d}\alpha\right| = O(P^{n-\delta/2}).$$

Proof. The proof follows immediately by tying together Lemmas 3.8 and 3.9. ☐

Recall the definition of $\mathscr{M}'(\theta_0)$ and $\mathscr{M}'_{a,q}(\theta_0)$ given in (3.8) and (3.7). The next lemma is analogous to [2, Lem. 4.5].

Lemma 3.11. *Keep the assumptions of Theorem 1.3 and* (3.5)*. Then we have*

$$\pi_f(P\mathscr{B}) = \sum_{q\leqslant P^{(d-1)\theta_0}} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} \int_{\mathscr{M}'_{a,q}(\theta_0)} S(\alpha)\overline{W(\alpha)}\mathrm{d}\alpha + O(P^{n-\delta/2}),$$

where C' is as in Proposition 3.7.

Before proceeding we note that one can take an arbitrarily small positive value for θ_0 in Lemma 3.11 because the system of inequalities (3.5) can be solved for any $\theta_0 > 0$ small enough. This will come at the cost of a worse error term in Lemma 3.11, however, it will still exhibit a power saving and it will thus be acceptable for the purpose of verifying Theorem 1.3.

3.3. **The intermediate range.** Under the assumptions of Theorem 1.3 and (3.5) we can use Lemmas 3.1, 3.4 and the trivial bound $W(\alpha) \ll P^d$ to evaluate the quantity $S(\alpha)$ in Lemma 3.11. This yields

$$\frac{\pi_f(P\mathscr{B})}{P^n} - \sum_{q\leqslant P^{(d-1)\theta_0}} q^{-n} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} S_{a,q}\int_{|\gamma|\leqslant P^{\eta}} I(\mathscr{B};\gamma)\frac{\overline{W(a/q+\gamma P^{-d})}}{P^d}\mathrm{d}\gamma \ll (\log P)^{-A}, \quad (3.15)$$

valid for all $A > 0$, where η, $S_{a,q}$ and $I(\mathscr{B};\gamma)$ are defined respectively in (3.9), (3.10) and (3.11).

For $A, q \in \mathbb{N}$ and $a \in \mathbb{Z} \cap [0, q)$ with $\gcd(a, q) = 1$ we let

$$\mathfrak{M}_{a,q}(A) := \{\alpha \in \mathbb{R} \ (\mathrm{mod}\ 1) : |\alpha - a/q| \leqslant P^{-d}(\log P)^A\}, \quad (3.16)$$

$$\mathfrak{M}(A) := \bigcup_{1\leqslant q\leqslant(\log P)^A} \bigcup_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} \mathfrak{M}_{a,q}(A) \quad (3.17)$$

and we observe that $\mathfrak{M}(A) \subset \mathscr{M}'(\theta_0)$ for all $P \gg 1$. We denote the difference by

$$\mathfrak{t}(A) := \mathscr{M}'(\theta_0)\backslash\mathfrak{M}(A). \quad (3.18)$$

The set $\mathfrak{t}(A)$ is therefore to be thought of as lying 'between' the major arcs $\mathscr{M}'(\theta_0)$ and the minor arcs $[0, 1)\backslash\mathscr{M}'(\theta_0)$. We shall see in §3.4 that $\mathfrak{M}(A)$ gives rise to the main term in Theorem 1.3.

Next, we observe that Lemma 3.3 and our assumption $n - \sigma_f \geqslant 1 + 2^{d-1}(d - 1)$ yield

$$\int_{|\gamma|\geqslant Q} |I(\mathscr{B};\gamma)|\mathrm{d}\gamma \ll Q^{-\frac{1}{2^d(d-1)}}, \quad (Q \geqslant 1), \quad (3.19)$$

in particular showing that $\int_{\mathbb{R}} |I(\mathscr{B};\gamma)|\mathrm{d}\gamma$ converges under assumption (1.4).

Lemma 3.12. *If* (1.4) *holds then*

$$\sum_{(\log P)^A<q\leqslant P^\eta} q^{-n} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} |S_{a,q}| \int_{|\gamma|\leqslant P^\eta} |I(\mathscr{B};\gamma)| \frac{|W(a/q+\gamma P^{-d})|}{P^d}\mathrm{d}\gamma \ll (\log P)^{-A/2+3+C'}, \tag{3.20}$$

where C' is as in Proposition 3.7.

Proof. If α is not in the union of the sets $\{\alpha \pmod 1) : |\alpha - a/q| \leqslant P^{-d+(d-1)\theta_0}\}$ taken over all $q \in \mathbb{N}\cap[1,(\log P)^A]$ and $a\in\mathbb{Z}\cap[0,q)$ with $\gcd(a,q)=1$, then by Dirichlet's approximation theorem there are coprime integers $1\leqslant a'\leqslant q'$ with $q'\leqslant P^{d-(d-1)\theta_0}$ and $|\alpha - a'/q'| \leqslant P^{-d+(d-1)\theta_0}/q'$. Thus we must have $q' > (\log P)^A$. Alluding to Vaughan's estimate [12, §25] and using partial summation we obtain

$$|W(\alpha)| \ll (P^d q'^{-1/2} + P^{4d/5} + (P^d q')^{1/2})(\log P)^3 \leqslant (P^d(\log P)^{-A/2} + P^{4d/5} + P^{d-\eta/2})(\log P)^3,$$

which is $\ll P^d(\log P)^{-A/2+3}$. For each a and q as in (3.20) we get by (3.19) that

$$\int_{|\gamma|\leqslant P^\eta} |I(\mathscr{B};\gamma)| \frac{|W(a/q+\gamma P^{-d})|}{P^d}\mathrm{d}\gamma \ll (\log P)^{-A/2+3},$$

hence by the second part of Proposition 3.7 we see that the sum over q in the lemma is

$$\ll \sum_{(\log P)^A<q\leqslant P^d} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} \frac{|S_{a,q}|}{q^n}(\log P)^{-A/2+3} \leqslant (\log P)^{-A/2+3+C'}. \qquad \square$$

Lemma 3.13. *Assume* (1.4)*. Then we have*

$$\sum_{q\leqslant(\log P)^A} q^{-n} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} |S_{a,q}| \int_{(\log P)^A<|\gamma|\leqslant P^\eta} |I(\mathscr{B};\gamma)| \frac{|W(a/q+\gamma P^{-d})|}{P^d}\mathrm{d}\gamma \ll \frac{(\log\log P)^{C'}}{(\log P)^{\frac{A}{2^d(d-1)}}}.$$

Proof. The proof follows immediately by combining the bound $W(\alpha)\ll P^d$, the inequality (3.19) for $Q=(\log P)^A$ and the second part of Proposition 3.7. $\square$

Tying Lemmas 3.12 and 3.13 proves the following lemma.

Lemma 3.14. *Keep the assumptions of Theorem 1.3. Then there exists a strictly positive constant $\lambda=\lambda(f)$ such that for every fixed sufficiently large $A>0$ we have*

$$\Big|\int_{\alpha\in\mathfrak{t}(A)} S(\alpha)\overline{W(\alpha)}\mathrm{d}\alpha\Big| \ll \frac{P^n}{(\log P)^{A\lambda}}.$$

3.4. **The major arcs.** Bringing together (3.15), (3.18), and Lemma 3.14 we see that under the assumptions of Theorem 1.3 there exists $\lambda>0$ such that for all large $A>0$ we have

$$\frac{\pi_f(P\mathscr{B})}{P^n} - \sum_{q\leqslant(\log P)^A} q^{-n} \sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} S_{a,q} \int_{|\gamma|\leqslant(\log P)^A} I(\mathscr{B};\gamma)\frac{\overline{W(a/q+\gamma P^{-d})}}{P^d}\mathrm{d}\gamma \ll (\log P)^{-A\lambda}. \tag{3.21}$$

Using the Siegel–Walfisz theorem as in [12, pg. 147] we can show that there exists $c=c(A)>0$ such that if $|\beta|\leqslant P^{-d}(\log P)^A$, $q\leqslant(\log P)^A$, a coprime to q and $x\in[P^{d/2},P^{2d}]$

then

$$\sum_{m\leqslant x}\Lambda(m)\mathrm{e}(m(a/q+\beta))=\frac{\mu(q)}{\varphi(q)}\left(\int_2^x\mathrm{e}(\beta t)\mathrm{d}t\right)+O\left((1+|\beta|x)x\exp\left(-c\sqrt{\log P}\right)\right),\quad(3.22)$$

where μ,φ and Λ denote the Möbius, Euler and von Mangoldt functions. We now see that

$$\sum_{p\leqslant x}(\log p)\mathrm{e}(p(a/q+\beta))=\frac{\mu(q)}{\varphi(q)}\left(\int_2^x\mathrm{e}(\beta t)\mathrm{d}t\right)+O\left((1+|\beta|x)x\exp\left(-c\sqrt{\log P}\right)\right)$$

due to the estimate $\sum_{\substack{m\leqslant x\\ m\neq p}}\Lambda(m)\ll x^{1/2}$. Partial summation shows that $W(a/q+\beta)$ equals

$$\frac{\mu(q)}{\varphi(q)}\left(\frac{\int_2^{2\max\{f_0(\mathscr{B})\}P^d}\mathrm{e}(\beta t)\mathrm{d}t}{\log(\frac12\max\{f_0(\mathscr{B})\}P^d)}-\frac{\int_2^{\frac12\min\{f_0(\mathscr{B})\}P^d}\mathrm{e}(\beta t)\mathrm{d}t}{\log(\frac12\min\{f_0(\mathscr{B})\}P^d)}-\int_{\frac12\min\{f_0(\mathscr{B})\}P^d}^{2\max\{f_0(\mathscr{B})\}P^d}\Big(\int_2^u\mathrm{e}(\beta t)\mathrm{d}t\Big)\Big(\frac{1}{\log u}\Big)'\mathrm{d}u\right)$$

up to an error of size $\ll(1+|\beta|P^d)P^d\exp\left(-c\sqrt{\log P}\right)$. Partial integration now yields

$$W(a/q+\gamma P^{-d})=\frac{\mu(q)}{\varphi(q)}\left(\int_{\frac12\min\{f_0(\mathscr{B})\}P^d}^{2\max\{f_0(\mathscr{B})\}P^d}\frac{\mathrm{e}(\gamma P^{-d}t)\mathrm{d}t}{\log t}\right)+O\left(\frac{(1+|\gamma|)P^d}{\exp\left(c\sqrt{\log P}\right)}\right).\quad(3.23)$$

The error term makes the following contribution towards (3.21),

$$\ll\exp\left(-c\sqrt{\log P}\right)\sum_{q\leqslant(\log P)^A}q^{-n}\sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}}|S_{a,q}|\int_{|\gamma|\leqslant(\log P)^A}|I(\mathscr{B};\gamma)|(1+(\log P)^A)\mathrm{d}\gamma$$

and, by the second part of Proposition 3.7 this is $\ll\exp\left(-c\sqrt{\log P}\right)(\log P)^{A+1}$, which is obviously $\ll\exp\left(-c/2\sqrt{\log P}\right)$. Hence, letting

$$\Xi_A(P):=\sum_{q\leqslant(\log P)^A}\frac{\mu(q)}{\varphi(q)q^n}\sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}}S_{a,q}$$

and

$$\Psi_A(P):=\int_{|\gamma|\leqslant(\log P)^A}I(\mathscr{B};\gamma)\left(\int_{\frac12\min\{f_0(\mathscr{B})\}P^d}^{2\max\{f_0(\mathscr{B})\}P^d}\frac{\mathrm{e}(-\gamma P^{-d}t)}{\log t}\mathrm{d}t\right)\mathrm{d}\gamma,$$

we obtain the following result via (3.21).

Lemma 3.15. *Under the assumptions of Theorem 1.3 there exists $\lambda=\lambda(f)>0$ such that for every $A>0$ we have $\pi_f(P\mathscr{B})=\Xi_A(P)\Psi_A(P)P^{n-d}+O(P^n(\log P)^{-A\lambda})$ for all sufficiently large P.*

3.5. **The non-archimedean densities.** If $n-\sigma_f\geqslant3$ then (3.14) along with the multiplicativity of T_f [2, §7] gives

$$\sum_{q>x}\frac{|\mu(q)|}{\varphi(q)q^n}\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}|S_{a,q}|\leqslant\sum_{q>x}\frac{|\mu(q)|}{\varphi(q)}q^{1-\frac{(n-\sigma_f)}{2}+\varepsilon}.$$

Hence, for $q\in\mathbb{N}$, the estimate $q/\varphi(q)\ll\log\log(4q)$ that can be found for example in [36, Th. 5.6] implies

$$\sum_{q>x}\frac{|\mu(q)|}{\varphi(q)q^n}\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}|S_{a,q}|\ll x^{-1/2+\varepsilon}.$$

Therefore, we have

$$\Xi_A(P) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)q^n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} S_{a,q} + O((\log P)^{-A/4}).$$

The multiplicativity of the last sum over a shows that the above sum over q is $\prod_p \beta_p$, where

$$\beta_p := 1 - \frac{1}{(p-1)p^n} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^*} S_{a,p}.$$

Finally, the following lemma is obtained by observing that

$$\sum_{a \in (\mathbb{Z}/p\mathbb{Z})^*} S_{a,p} = \sum_{\mathbf{x} \in \mathbb{F}_p^n} \Big(-1 + \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \mathrm{e}(af(\mathbf{x})/p) \Big) = -p^n + p\#\{\mathbf{x} \in \mathbb{F}_p^n : f(\mathbf{x}) = 0\}. \tag{3.24}$$

Lemma 3.16. *If $n - \sigma_f \geqslant 3$ then*

$$\Xi_A(P) = \prod_p \left(\left(1 - \frac{\#\{\mathbf{x} \in \mathbb{F}_p^n : f(\mathbf{x}) = 0\}}{p^n}\right) \left(1 - \frac{1}{p}\right)^{-1} \right) + O((\log P)^{-A/4}).$$

Combining (3.14) and (3.24) yields $p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f(\mathbf{x}) = 0\} = 1/p + O(p^{-(n-\sigma_f)/2})$, thus verifying the following lemma.

Lemma 3.17. *If $n - \sigma_f \geqslant 3$ then the product in Theorem 1.3 converges absolutely.*

3.6. **The archimedean densities.** Letting for $P^d > \frac{1}{2}\min\{f_0(\mathscr{B})\}$

$$\Psi(P) := \int_{\gamma \in \mathbb{R}} I(\mathscr{B};\gamma) \left(\int_{\frac{1}{2}\min\{f_0(\mathscr{B})\}}^{2\max\{f_0(\mathscr{B})\}} \frac{\mathrm{e}(-\gamma\mu)}{\log(\mu P^d)} \mathrm{d}\mu \right) \mathrm{d}\gamma,$$

we see by (3.19) and our assumption (1.4) that there exists $\lambda_2 = \lambda_2(f) > 0$ such that

$$\Psi_A(P)P^{-d} = \Psi(P) + O_A((\log P)^{-\lambda_2 A}). \tag{3.25}$$

Now we observe that for all reals z, μ with $z > \mu > 0$ and $z \notin \{1/\mu, 1\}$ we have

$$\frac{1}{\log(\mu z)} = \frac{1}{\log z} \frac{1}{\left(1 + \frac{\log \mu}{\log z}\right)} = \frac{1}{\log z} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\log z)^k} (\log \mu)^k, \tag{3.26}$$

therefore, letting for $k \in \mathbb{Z}_{\geqslant 0}$,

$$J(k) := \int_{\gamma \in \mathbb{R}} I(\mathscr{B};\gamma) \left(\int_{\frac{1}{2}\min\{f_0(\mathscr{B})\}}^{2\max\{f_0(\mathscr{B})\}} \mathrm{e}(-\gamma\mu)(\log \mu)^k \mathrm{d}\mu \right) \mathrm{d}\gamma, \tag{3.27}$$

we infer that for all sufficiently large P we have

$$\Psi(P) = \frac{1}{\log(P^d)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\log(P^d))^k} J(k). \tag{3.28}$$

Let us furthermore introduce the following entity for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geqslant 0}$,

$$J_n(k) := \int_{\gamma \in \mathbb{R}} \exp\left(-\frac{\pi^2\gamma^2}{n^2} \right) I(\mathscr{B};\gamma) \left(\int_{\frac{1}{2}\min\{f_0(\mathscr{B})\}}^{2\max\{f_0(\mathscr{B})\}} \mathrm{e}(-\gamma\mu)(\log \mu)^k \mathrm{d}\mu \right) \mathrm{d}\gamma. \tag{3.29}$$

Lemma 3.18. *Under the assumption (1.4) we have $\lim_{n \to +\infty} J_n(k) = J(k)$ for every $k \in \mathbb{Z}_{\geqslant 0}$.*

Proof. We have $J(k) - J_n(k) \ll_k \int_{|\gamma|\leqslant \log n} \mathscr{H}_\dagger(\gamma)\mathrm{d}\gamma + \int_{|\gamma|>\log n} \mathscr{H}_\dagger(\gamma)\mathrm{d}\gamma$, where

$$\mathscr{H}_\dagger(\gamma) := \left(1 - \exp\left(-\frac{\pi^2\gamma^2}{n^2}\right)\right)|I(\mathscr{B};\gamma)|.$$

We have $I(\mathscr{B};\gamma) \ll 1$ due to Lemma 3.3, hence,

$$\int_{|\gamma|\leqslant \log n} \mathscr{H}_\dagger(\gamma)\mathrm{d}\gamma \ll (\log n)\left(1 - \exp\left(-\frac{\pi^2(\log n)^2}{n^2}\right)\right) = o(1).$$

By (3.19) we get $\int_{|\gamma|>\log n} \mathscr{H}_\dagger(\gamma)\mathrm{d}\gamma \ll (\log n)^{-\lambda_1} = o(1)$ for some positive $\lambda_1 = \lambda_1(f)$. ☐

Lemma 3.19. *Under the assumption* (1.4) *we have the following for every* $k \in \mathbb{Z}_{\geqslant 0}$,

$$\lim_{n\to+\infty} J_n(k) = \int_{\mathbf{t}\in\mathscr{B}} (\log f_0(\mathbf{t}))^k \mathrm{d}\mathbf{t}.$$

Proof. It is standard to see that the Fourier transform of the function $\varphi_n : \mathbb{R} \to \mathbb{R}$ defined through $\varphi_n(x) := \pi^{-1/2} n \exp(-n^2x^2)$ satisfies $\widehat{\varphi}_n(\gamma) = \exp(-\pi^2 n^{-2}\gamma^2)$. Therefore, the Fourier inverse formula yields $\varphi_n(x) = \int_{\mathbb{R}} \mathrm{e}(x\gamma)\widehat{\varphi}_n(\gamma)\mathrm{d}\gamma$. Using this for $x = f_0(\mathbf{t}) - y$ and rewriting (3.29) as

$$\int_{\mathbf{t}\in\mathscr{B}} \int_{\frac{1}{2}\min\{f_0(\mathscr{B})\}}^{2\max\{f_0(\mathscr{B})\}} (\log\mu)^k \left(\int_{\gamma\in\mathbb{R}} \exp\left(-\frac{\pi^2\gamma^2}{n^2}\right) \mathrm{e}((f(\mathbf{t})-\mu)\gamma)\mathrm{d}\gamma\right)\mathrm{d}\mu\mathrm{d}\mathbf{t},$$

we infer that $J_n(k) = \int_{\mathscr{B}} g_n(\mathbf{t})\mathrm{d}\mathbf{t}$, where

$$g_n(\mathbf{t}) := \int_{\frac{1}{2}\min\{f_0(\mathscr{B})\}}^{2\max\{f_0(\mathscr{B})\}} (\log\mu)^k \varphi_n(f(\mathbf{t}) - \mu)\mathrm{d}\mu.$$

It is obvious from [18, Ex. I.2] that for any reals $a < c < b$ and any continuous function $h : [a,b] \to \mathbb{R}$ one has

$$\lim_{n\to+\infty} \int_a^b h(\mu)\varphi_n(c-\mu)\mathrm{d}\mu = h(c).$$

Recalling that $f_0(\mathscr{B}) \subset (0,\infty)$ we infer that whenever $\mathbf{t} \in \mathscr{B}$ then the following inequality holds, $\frac{1}{2}\min\{f_0(\mathscr{B})\} < f_0(\mathbf{t}) < 2\max\{f(\mathscr{B})\}$. This gives $\lim_{n\to+\infty} g_n(\mathbf{t}) = (\log f_0(\mathbf{t}))^k$ and a use of the dominated convergence theorem concludes the proof of the lemma. ☐

Lemma 3.20. *Under the assumption* (1.4) *we have, for all sufficiently large* P, $\Psi(P) = P^{-n}\mathrm{Li}_f(P\mathscr{B})$.

Proof. Combining Lemmas 3.18 and 3.19 we get $J(k) = \int_{\mathscr{B}} (\log f_0(\mathbf{t}))^k \mathrm{d}\mathbf{t}$. Injecting this into (3.28) and interchanging the sum over k and the integral over $\mathbf{t}$ yields

$$\Psi(P) = \int_{\mathscr{B}} \left(\frac{1}{\log(P^d)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\log(P^d))^k} (\log f_0(\mathbf{t}))^k\right)\mathrm{d}\mathbf{t}. \tag{3.30}$$

The proof is concluded by alluding to (3.26) and making the change of variables $\mathbf{x} = P\mathbf{t}$. ☐

Combining Lemma 3.20 with (3.25) provides us with the following result.

Lemma 3.21. *Under the assumptions of Theorem 1.3 there exists* $\lambda_2 = \lambda_2(f) > 0$ *such that for every* $A > 0$ *and every sufficiently large* P *we have*

$$\Psi_A(P) = \mathrm{Li}_f(P\mathscr{B})P^{-(n-d)} + O_A(P^d(\log P)^{-A\lambda_2}).$$

Our final result offers an asymptotic expansion of $\mathrm{Li}_f(P\mathscr{B})$ in terms of $(\log P)^{-1}$.

Lemma 3.22. *For f and $\mathscr{B}$ as in Theorem 1.3 and P large enough we have*

$$\mathrm{Li}_f(P\mathscr{B}) = \frac{\mathrm{vol}(\mathscr{B})}{d}\frac{P^n}{\log P} + P^n \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{d^k}\left(\int_{\mathscr{B}} (\log f_0(\mathbf{t}))^{k-1}\mathrm{d}\mathbf{t}\right)\frac{1}{(\log P)^k}.$$

In particular, we have

$$\mathrm{Li}_f(P\mathscr{B}) = \frac{\mathrm{vol}(\mathscr{B})}{d}\frac{P^n}{\log P} + O_{f,\mathscr{B}}\left(\frac{P^n}{(\log P)^2}\right).$$

Proof. The first equality follows by combining Lemmas 3.18 and 3.19 with (3.28) and (3.30). To prove the second, note that if $\log P > 2$ then

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{d^k}\left(\int_{\mathscr{B}} (\log f_0(\mathbf{t}))^{k-1}\mathrm{d}\mathbf{t}\right)\frac{1}{(\log P)^k} \ll \sum_{k=2}^{\infty} \frac{1}{(\log P)^k} < \frac{1}{(\log P)^2}\sum_{k=2}^{\infty}\frac{1}{2^{k-2}},$$

thus concluding the proof. ∎

3.7. **The proof of Theorem 1.3.** It follows by merging Lemmas 3.15, 3.16 and 3.21. ∎

4. The proof of Theorem 1.5

4.1. **First steps and auxiliary estimates.** Similarly as in §3.1 we may write

$$\#\{\mathbf{S}_f \cap P\mathscr{B}\} = \int_0^1 S(\alpha)\overline{Q(\alpha)}\mathrm{d}\alpha,$$

where $S(\alpha)$ is defined in (3.1) and

$$Q(\alpha) := \sum_{\substack{m \text{ square-free}, m\neq 0 \\ \min\{f_0(\mathscr{B})\}-1 \leqslant mP^{-d} \leqslant \max\{f_0(\mathscr{B})\}+1}} \mathrm{e}(\alpha m).$$

We shall later need certain estimates concerning exponential sums taking values over square-free integers that we record here. For $\alpha \in \mathbb{R}$ and $N \in \mathbb{R}_{\geqslant 1}$ define

$$f_2(\alpha, N) := \sum_{1\leqslant n\leqslant N} \mu(n)^2 \mathrm{e}(\alpha n).$$

The following result is the very special case corresponding to the choices $k = 2$ and $p = 3/2$ in the work of Keil [27].

Lemma 4.1 (Keil [27, Th. 1.2])**.** *We have*

$$\int_0^1 |f_2(\alpha, N)|^{3/2}\mathrm{d}\alpha \ll N^{1/2}(\log N)^2.$$

For p prime and ℓ, m non-negative integers such that $m \leqslant \ell$, define $g(p^\ell, p^m)$ by

$$p^\ell(1-p^{-2})g(p^\ell, p^m) = \begin{cases} 0, & \text{if } \ell \geqslant m \geqslant 2, \\ 1, & \text{if } m < \min\{2, \ell\}, \\ 1 - p^{\ell-2}, & \text{if } \ell = m \leqslant 1. \end{cases}$$

We extend this definition by defining the following whenever $d, q \in \mathbb{N}$ are such that $d \mid q$,

$$g(q,d) := \prod_{p \mid q} g(p^{\nu_p(q)}, p^{\nu_p(d)}).$$

We can now introduce the following entity for $q \in \mathbb{N}$,

$$G(q) := \sum_{b=1}^{q} \mathrm{e}(b/q) g(q, \gcd(b,q)). \tag{4.1}$$

Brüdern, Granville, Perelli, Vaughan and Wooley studied $Q(\alpha)$ in [7].

Lemma 4.2. *There exist absolute positive constants δ_1, δ_2 such that for all $q \in \mathbb{N}$ with $q \leqslant P^{\delta_1}$, all $a \in \mathbb{Z} \cap [1,q)$, $d \in \mathbb{N}$, $\gamma \in \mathbb{R}$ with $|\gamma| \leqslant P^{\delta_1}$ and all $c_1 < c_2 \in \mathbb{R}$ we have*

$$\sum_{\substack{m \text{ square-free}, m \neq 0 \\ c_1 \leqslant m P^{-d} \leqslant c_2}} \mathrm{e}(m(a/q + \gamma P^{-d})) = \frac{G(q)}{\zeta(2)} \left(\int_{c_1 P^d}^{c_2 P^d} \mathrm{e}(\gamma P^{-d} t) \mathrm{d}t \right) + O_{c_1,c_2}\left((1+|\gamma|) P^{d-\delta_2} \right),$$

where ζ denotes the Riemann zeta function and the implied constant depends at most on c_1 and c_2.

Proof. We will show that there exists an absolute $\delta > 0$ such that if $|\beta| \leqslant P^{-d+\delta}$, $q \leqslant P^{\delta}$, a is coprime to q and $x \in [P^{d/2}, P^{2d}]$ then

$$\sum_{1 \leqslant m \leqslant x} \mu(m)^2 \mathrm{e}(m(a/q + \beta)) = \frac{G(q)}{\zeta(2)} \left(\int_2^x \mathrm{e}(\beta t) \mathrm{d}t \right) + O\left((1 + |\beta| x) x^{1-\delta} \right), \tag{4.2}$$

from which one can deduce the asymptotic stated in the lemma in the same way as we deduced (3.23) from (3.22). To prove (4.2) we first note that for all b and $q \in \mathbb{N}$ we have

$$\sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv b (\mathrm{mod}\ q) \\ m \text{ square-free}}} 1 = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv b (\mathrm{mod}\ q)}} \sum_{d^2 \mid m} \mu(d) = \sum_{\substack{1 \leqslant d \leqslant \sqrt{x} \\ \gcd(q,d^2) \mid b}} \mu(d) \left(\frac{x \gcd(q,d^2)}{q d^2} + O(1) \right) \tag{4.3}$$

and completing the sum over d gives $\frac{x}{\zeta(2)} g(q, \gcd(b,q)) + O(\sqrt{x})$. For $\gcd(a,q) = 1$ we let

$$Z(x;q,a) := \sum_{1 \leqslant m \leqslant x} \mu(m)^2 \mathrm{e}(ma/q) = \sum_{b=1}^{q} \mathrm{e}(mb/q) \sum_{\substack{1 \leqslant m \leqslant x \\ a \equiv b (\mathrm{mod}\ q)}} \mu(m)^2$$

and use (4.1) and (4.3) to get the following estimate with an absolute implied constant for $q \leqslant x$,

$$Z(x;q,a) - \frac{G(q) x}{\zeta(2)} \ll q \sqrt{x}. \tag{4.4}$$

We therefore obtain by partial summation that

$$\sum_{1 \leqslant m \leqslant x} \mu(m)^2 \mathrm{e}(m(a/q + \beta)) = \mathrm{e}(x\beta) Z(x;q,a) - 2\pi i \beta \int_1^x \mathrm{e}(u\beta) Z(u;q,a) \mathrm{d}u$$

and by (4.4) this becomes

$$\frac{G(q)}{\zeta(2)} \left(\int_1^x \mathrm{e}(\beta t) \mathrm{d}t \right) + O\left((1 + |\beta| x) q \sqrt{x} \right),$$

with an absolute implied constant. This proves (4.2) with $\delta = (2+d)^{-1}$. Indeed, if $q \leqslant P^\delta$ then the equality $P^\delta = P^{\frac{d}{2}(\frac{1}{2}-\delta)}$ and the bound $P^{\frac{d}{2}} \leqslant x$ yield $q \leqslant x^{\frac{1}{2}-\delta}$, i.e. $q\sqrt{x} \leqslant x^{1-\delta}$. □

Finally, the next result is shown in the proof of [7, Lem. 3.1].

Lemma 4.3 (Brüdern, Granville, Perelli, Vaughan and Wooley, [7, Lem. 3.1])**.** *The function G is multiplicative, supported in cube-free integers and satisfies for all prime p the identity*

$$G(p) = G(p^2) = -p^{-2}(1-p^{-2})^{-1}.$$

4.2. **Continuation of the proof.** Recalling the meaning of $\mathscr{M}(\theta)$ and $\mathscr{M}_{a,q}(\theta)$ in (3.4) and (3.3), we allude to Hölder's inequality and Lemma 4.1 to obtain

$$\begin{aligned}\left|\int_{\alpha\notin\mathscr{M}(\theta)} S(\alpha)\overline{Q(\alpha)}\mathrm{d}\alpha\right| &\leqslant \left(\int_{\alpha\notin\mathscr{M}(\theta)} |S(\alpha)|^3\mathrm{d}\alpha\right)^{1/3}\left(\int_0^1 |Q(\alpha)|^{3/2}\mathrm{d}\alpha\right)^{2/3} \\ &\leqslant \left(\int_{\alpha\notin\mathscr{M}(\theta)} |S(\alpha)|^3\mathrm{d}\alpha\right)^{1/3} P^{d/3}(\log P)^{4/3}.\end{aligned}$$

The proof of Lemma 3.9 can be adapted straightforwardly to show that if

$$1 > \delta + 6d\theta_0 \quad\text{and}\quad \frac{n-\sigma_f}{2^{d-1}} - \frac{2}{3}(d-1) > \delta\theta_0^{-1} \tag{4.5}$$

then

$$\left(\int_{\alpha\notin\mathscr{M}(\theta)} |S(\alpha)|^3\mathrm{d}\alpha\right)^{1/3} \ll P^{n-\frac{d}{3}-\frac{\delta}{9}}.$$

Let $\eta := (d-1)\theta_0$. Under the assumptions of Theorem 1.5 and for θ_0 as in (4.5), one obtains the following inequality that is in analogy with Lemma 3.11,

$$\#\{\mathbf{S}_f \cap P\mathscr{B}\} = \sum_{q\leqslant P^\eta}\sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}}\int_{\mathscr{M}'_{a,q}(\theta_0)} S(\alpha)\overline{Q(\alpha)}\mathrm{d}\alpha + O\left(P^{n-\frac{\delta}{10}}\right).$$

Similarly as in the proof of (3.15), one may now acquire some $\delta_1 = \delta_1(f) > 0$ such that

$$\frac{\#\{\mathbf{S}_f\cap P\mathscr{B}\}}{P^n} - \sum_{q\leqslant P^{\delta_1}} q^{-n}\sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} S_{a,q}\int_{|\gamma|\leqslant P^\eta} I(\mathscr{B};\gamma)\frac{\overline{Q(a/q+\gamma P^{-d})}}{P^d}\mathrm{d}\gamma \ll P^{-\delta_1}. \tag{4.6}$$

By Lemma 4.2 we see that for suitably small δ_1 and all a as in (4.6) and $q \leqslant P^{\delta_1}$ one has

$$Q(a/q+\gamma P^{-d}) = \frac{G(q)}{\zeta(2)}\left(\int_{(\min\{f_0(\mathscr{B})\}-1)P^d}^{(\max\{f_0(\mathscr{B})\}+1)P^d} \mathrm{e}(\gamma P^{-d}t)\mathrm{d}t\right) + O\left((1+|\gamma|)P^{d-\delta_2}\right).$$

Therefore, as in the proof of Lemma 3.15, we may infer that there exists a positive constant $\delta_3 = \delta_3(f)$ such that the quantity $\#\{\mathbf{S}_f \cap P\mathscr{B}\}$ equals

$$\frac{P^n}{\zeta(2)}\left(\sum_{q\leqslant P^{\delta_1}}\frac{G(q)}{q^n}\sum_{\substack{a\in\mathbb{Z}\cap[0,q)\\ \gcd(a,q)=1}} S_{a,q}\right)\left(\int_{|\gamma|\leqslant P^{\delta_1}}\frac{I(\mathscr{B};\gamma)}{P^d}\left(\int_{(\min\{f_0(\mathscr{B})\}-1)P^d}^{(\max\{f_0(\mathscr{B})\}+1)P^d}\mathrm{e}(-\gamma P^{-d}t)\mathrm{d}t\right)\mathrm{d}\gamma\right), \tag{4.7}$$

up to an error term which is $O(P^{n-\delta_3})$. We shall now use Lemma 4.3 to show that the sum over q forms an absolutely convergent series. Bringing into play (3.14) and [4, Lem. 25] we obtain the bounds

$$|G(p)T_f(p)| \ll p^{-1-(n-\sigma_f)/2} \quad \text{and} \quad |G(p^2)T_f(p^2)| \ll p^{-n+\sigma_f}.$$

Hence, assuming $n - \sigma_f \geqslant 2$, these two estimates allow to modify easily the proof of Proposition 3.7, thereby showing that the abscissa of convergence of the Dirichlet series of $|G(q)|T_f(q)$ is strictly negative. This provides $\delta_4 = \delta_4(f) > 0$ such that for all $x \geqslant 2$, one has $\sum_{q>x} |G(q)|T_f(q) \ll x^{-\delta_4}$, hence the sum over q in (4.7) is $\Pi' + O(P^{-\eta\delta_4})$, where Π' is

$$\sum_{q=1}^{\infty} \frac{G(q)}{q^n} \sum_{\substack{a \in \mathbb{Z} \cap [0,q) \\ \gcd(a,q)=1}} S_{a,q} = \prod_p \left(1 - p^{-2}(1-p^{-2})^{-1}\Big(\frac{1}{p^n} \sum_{a \in \mathbb{Z} \cap (0,p)} S_{a,p} + \frac{1}{p^{2n}} \sum_{\substack{a \in \mathbb{Z} \cap [0,p^2) \\ \gcd(a,p)=1}} S_{a,p^2}\Big)\right).$$

One can easily see, for example, by using orthogonality of characters of $\mathbb{Z}/p^2\mathbb{Z}$ to detect the condition $f(\mathbf{x}) = 0$, that

$$\#\big\{\mathbf{x} \in (\mathbb{Z}/p^2\mathbb{Z})^n : f(\mathbf{x}) = 0\big\} = p^{2(n-1)}\Big(1 + \frac{1}{p^n} \sum_{a \in \mathbb{Z} \cap (0,p)} S_{a,p} + \frac{1}{p^{2n}} \sum_{a \in \mathbb{Z} \cap [0,p^2), p \nmid a} S_{a,p^2}\Big),$$

from which we can show that $\Pi'/\zeta(2)$ is

$$\prod_p \left(1 - \frac{\#\big\{\mathbf{x} \in (\mathbb{Z}/p^2\mathbb{Z})^n : f(\mathbf{x}) = 0\big\}}{p^{2n}}\right).$$

This is in agreement with the infinite product in Theorem 1.5.

To deal with the integral in (4.7) we observe that the transformation $t = P^d\mu$ gives

$$P^{-d} \int_{(\min\{f_0(\mathscr{B})\}-1)P^d}^{(\max\{f_0(\mathscr{B})\}+1)P^d} \mathrm{e}(-\gamma P^{-d}t)\mathrm{d}t = \int_{\min\{f_0(\mathscr{B})\}-1}^{\max\{f_0(\mathscr{B})\}+1} \mathrm{e}(-\gamma\mu)\mathrm{d}\mu \ll \min\{1, |\gamma|^{-1}\}, \tag{4.8}$$

hence Lemma 3.3 shows that the integral in (4.7) converges absolutely and equals

$$\int_{\gamma \in \mathbb{R}} I(\mathscr{B};\gamma)\left(\int_{\min\{f_0(\mathscr{B})\}-1}^{\max\{f_0(\mathscr{B})\}+1} \mathrm{e}(-\gamma\mu)\mathrm{d}\mu\right)\mathrm{d}\gamma + O(P^{-\delta_5}) \tag{4.9}$$

for some $\delta_5 = \delta_5(f) > 0$.

One can combine the bound (4.8) with Lemma 3.3 to show that the integral over γ in (4.9) equals $\mathrm{vol}(\mathscr{B})$ using arguments that are entirely analogous with the case $k = 0$ in Lemmas 3.18 and 3.19. Thereby alluding to the well-known estimate

$$\#\{\mathbb{Z}^n \cap P\mathscr{B}\} = \mathrm{vol}(\mathscr{B})P^n + O_{\mathscr{B}}(P^{n-1})$$

allows us to conclude the proof of Theorem 1.5.

Appendix A. The Bateman–Horn heuristics in many variables

In this section we extend the Bateman–Horn heuristics from the setting of univariate polynomials to that of polynomials with arbitrarily many variables; we do so because we were unable to find a reference for this extension in the literature.

In 1958, Schinzel [30] formulated the following conjecture concerning prime values of univariate polynomials.

Conjecture A.1 (**Schinzel's hypothesis H,** [30]). *Let $f_1, \dots, f_r \in \mathbb{Z}[x]$ be univariate irreducible polynomials with positive leading coefficient. If $\prod_{i=1}^r f_i$ has no repeated polynomial factors and, for every prime p, there exists $x_p \in \mathbb{Z}$ such that $p \nmid f_1(x_p) \cdots f_r(x_p)$, then there exist infinitely many integers m such that $f_1(m), \dots, f_r(m)$ are all primes.*

This conjecture was later refined by Bateman and Horn [1] who, based on the Cramér model and the heuristics behind the Hardy–Littlewood conjecture (see [34, pg. 6-8]), gave a quantitative version of Schinzel's conjecture.

Conjecture A.2 (**Bateman–Horn's conjecture,** [1]). *Keep the assumptions of Conjecture A.1. Then the number of integers $m \in [1, P]$ such that every $f_1(m), \dots, f_r(m)$ is prime is asymptotically equivalent to the following quantity as $P \to +\infty$,*

$$\left(\prod_{p \text{ prime}} \frac{(1 - p^{-1}\#\{x \in \mathbb{F}_p : f_1(x) \cdots f_r(x) = 0\})}{(1 - 1/p)^r} \right) \frac{1}{\deg(f_1) \cdots \deg(f_r)} \int_2^P \frac{\mathrm{d}x}{(\log x)^r}.$$

The convergence of the infinite product is established in [1] using the prime ideal theorem. These two conjectures lie very deep and imply a number of notoriously difficult conjectures as immediate corollaries (the twin primes conjecture among others; see [30] for a non exhaustive list of implications). There are applications to the arithmetic of algebraic varieties, see [9], [35] or [20], where Schinzel's hypothesis is assumed in order to prove that the Hasse principle and weak approximation holds.

Let us now record the multivariable version of the Bateman–Horn conjecture where we recall that f_0 is the top degree part of a polynomial f.

Conjecture A.3 (**Extension of the Bateman–Horn conjecture**). *Assume that we are given irreducible polynomials $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$ such that $\prod_{i=1}^r f_i$ has no repeated polynomial factors. Moreover, we assume that $\mathscr{B} \subset \mathbb{R}^n$ is a non-empty box such that $f_{i0}(\mathscr{B}) \subset (1, \infty)$ for all $i \in \{1, \dots, r\}$. Then the number $\pi_{f_1,\dots,f_r}(P\mathscr{B})$ of integer vectors $\mathbf{x} \in \mathbb{Z}^n \cap P\mathscr{B}$ for which every $f_1(\mathbf{x}), \dots, f_r(\mathbf{x})$ is a prime number is asymptotic to the following quantity as $P \to +\infty$,*

$$\left(\prod_{p \text{ prime}} \frac{(1 - p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\})}{(1 - 1/p)^r} \right) \int_{P\mathscr{B}} \frac{\mathrm{d}\mathbf{x}}{\prod_{i=1}^r \log f_{i0}(\mathbf{x})}.$$

Remark A.4. Before providing the heuristics behind Conjecture A.3 let us note that one can prove that the product over p converges. Indeed, a version of the prime number theorem for schemes over $\mathbb{Z}$ that can be found in the work of Serre [32, Cor. 7.13] for example ensures that

$$\sum_{p \leqslant x} \#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\} = r \left(\int_2^{x^n} \frac{\mathrm{d}t}{\log t} \right) + O\left(x^n \mathrm{e}^{-c\sqrt{\log x}} \right)$$

for some $c = c(f_1, \dots, f_r) > 0$. Now partial summation implies that

$$\sum_{p \leqslant x} p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\} = r \log\log x + C_1 + O\left(\frac{1}{\log x} \right)$$

for some constant C_1. Hence the series

$$\sum_p \left(p^{-n}\#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\} - \frac{r}{p} \right)$$

is convergent which in turn yields the convergence of the product over p appearing in the statement of the conjecture.

We end this section by adapting the heuristics behind Conjecture A.2 to the multivariate case. Recall that the Cramér model asserts that a random positive integer m of size X has probability $1/\log X$ of being a prime. An analogous statement can be made if the extra condition that m lies in a primitive arithmetic progression modulo q for some positive integer q is added, in this case the probability is $1/(\varphi(q)\log X)$ owing to Dirichlet's theorem on primes in arithmetic progressions. This implies that for coprime a, q, the conditional probability that a positive integer m of size X is prime provided that $m \equiv a \pmod{q}$ equals

$$\text{Prob}[m \sim X \text{ is a prime} \mid m \equiv a \pmod{q}] \approx \frac{1/(\varphi(q)\log X)}{1/q} = \frac{q}{\varphi(q)\log X}. \tag{A.1}$$

In the setting of Conjecture A.2 observe that for typical $\mathbf{x} \in \mathbb{Z}^n$ the integer $f_i(\mathbf{x})$ can be prime only if $f_i(\mathbf{x})$ is coprime to all small primes. Therefore, letting $z = z(P)$ be a function that slowly tends to infinity with P and letting $\mathscr{P} := \prod_{p\leqslant z} p$, we see that

$$\frac{\pi_{f_1,\dots,f_r}(P\mathscr{B})}{\#\{\mathbb{Z}^n \cap P\mathscr{B}\}} \approx \sum_{\substack{\mathbf{a}\in(\mathbb{Z}/\mathscr{P}\mathbb{Z})^n \\ \forall i\in\{1,\dots,r\},\ f_i(\mathbf{a})\in(\mathbb{Z}/\mathscr{P}\mathbb{Z})^\times}} \text{Prob}[x_i \equiv a_i \pmod{\mathscr{P}} \text{ for all } 1 \leqslant i \leqslant n] \cdot \mathbb{P}_{\mathbf{a},\mathscr{P}}, \tag{A.2}$$

where $\mathbb{P}_{\mathbf{a},\mathscr{P}}$ denotes the joint probability defined through

$$\mathbb{P}_{\mathbf{a},\mathscr{P}} := \text{Prob}[m_i \sim P^{\deg(f_i)} \text{ is a prime for all } 1 \leqslant i \leqslant r \mid m_i \equiv f_i(\mathbf{a}) \pmod{\mathscr{P}}].$$

This is because the integer $f_i(\mathbf{x})$ is typically of size $P^{\deg(f_i)}$ when $\mathbf{x} \in P\mathscr{B}$ and the values $f_i(\mathbf{x})$ are thought to behave like a random integer m_i lying in the arithmetic progression $f_i(\mathbf{a}) \pmod{\mathscr{P}}$, provided that $\mathbf{x} \equiv \mathbf{a} \pmod{\mathscr{P}}$. Note that for $i \neq j$ the polynomials f_i and f_j are coprime due to the assumption that $\prod_i f_i$ has no repeated factors, therefore it is reasonable to expect that for $i \neq j$ the integer values $f_i(\mathbf{x})$ and $f_j(\mathbf{x})$ behave independently. This suggests that

$$\mathbb{P}_{\mathbf{a},\mathscr{P}} = \prod_{i=1}^{r} \text{Prob}[m_i \sim P^{\deg(f_i)} \text{ is a prime} \mid m_i \equiv f_i(\mathbf{a}) \pmod{\mathscr{P}}]$$

and by (A.1) one now gets $\mathbb{P}_{\mathbf{a},\mathscr{P}} = \mathscr{P}^r \varphi(\mathscr{P})^{-r}(\log P)^{-r}\prod_{i=1}^r(\deg(f_i))^{-1}$. Substituting this into (A.2) and noting that $\text{Prob}[x_i \equiv a_i \pmod{\mathscr{P}}] = 1/\mathscr{P}$ yields

$$\frac{\pi_{f_1,\dots,f_r}(P\mathscr{B})}{\text{vol}(\mathscr{B})P^n} \approx \left(\frac{\mathscr{P}}{\varphi(\mathscr{P})\log P}\right)^r \frac{1}{\prod_{i=1}^r \deg(f_i)} \frac{1}{\mathscr{P}^n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}/\mathscr{P}\mathbb{Z})^n \\ \forall i\in\{1,\dots,r\},\ f_i(\mathbf{a})\in(\mathbb{Z}/\mathscr{P}\mathbb{Z})^\times}} 1.$$

The sum over $\mathbf{a}$ forms a multiplicative function of $\mathscr{P}$ that can be evaluated as

$$\prod_{p\leqslant z} \left(p^n - \#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\}\right).$$

Putting everything together shows that we expect $\pi_{f_1,\dots,f_r}(P\mathscr{B})$ to be approximated by

$$\frac{\text{vol}(\mathscr{B})P^n}{(\log P)^r \prod_{i=1}^r \deg(f_i)} \prod_{p\leqslant z} \left(\left(\frac{p}{p-1}\right)^r \left(\frac{p^n - \#\{\mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0\}}{p^n}\right)\right).$$

In view of Remark A.4 the product over $p \leqslant z(P)$ converges to the product in Conjecture A.3 as $P \to +\infty$. For $\mathbf{x} \in P\mathscr{B}$ we have $f_{i0}(\mathbf{x}) \asymp P^{\deg(f_{i0})}$ and using $\deg(f_i) = \deg(f_{i0})$ we get

$$\frac{\mathrm{vol}(\mathscr{B})P^n}{(\log P)^r \prod_{i=1}^r \deg(f_i)} = \frac{\int_{P\mathscr{B}} 1 \mathrm{d}\mathbf{x}}{\prod_{i=1}^r \log(P^{\deg(f_i)})} \asymp \int_{P\mathscr{B}} \frac{\mathrm{d}\mathbf{x}}{\prod_{i=1}^r \log f_{i0}(\mathbf{x})},$$

thereby concluding our explanation of the asymptotic in Conjecture A.3.

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