THE POWER-SAVING MANIN-PEYRE CONJECTURE FOR A SENARY CUBIC

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Abstract. Using recent work of the first author [3], we prove a strong version of the Manin-Peyre's conjectures with a full asymptotic and a power-saving error term for the two varieties respectively in $\mathbb{P}^2 \times \mathbb{P}^2$ with bihomogeneous coordinates $[x_1 : x_2 : x_3], [y_1 : y_2, y_3]$ and in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with multihomogeneous coordinates $[x_1 : y_1], [x_2 : y_2], [x_3 : y_3]$ defined by the same equation $x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0$. We thus improve on recent work of Blomer, Brüdern and Salberger [8] and provide a different proof based on a descent on the universal torsor of the conjectures in the case of a del Pezzo surface of degree 6 with singularity type $\mathbf{A}_1$ and three lines (the other existing proof relying on harmonic analysis [17]). Together with [7] or with recent work of the second author [21], this settles the study of the Manin-Peyre's conjectures for this equation.

Contents

1. Introduction

In the late 80s, Manin and his collaborators [22] proposed a precise conjecture predicting, for smooth Fano varieties, the behaviour of the number of rational points of bounded height (with respect to an anticanonical height function) in terms of geometric invariants of the variety. The conjecture was later generalised by Peyre [26] to "*almost Fano*" varieties in the sense of [26, Définition 3.1].

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Conjecture 1.1 (Manin, 1989). *Let V be an "almost Fano" variety in the sense of* [26, *Définition 3.1*] *with $V(\mathbb{Q}) \neq \varnothing$ and let H be an anticanonical height function on $V(\mathbb{Q})$. Then there exists a Zariski open subset U of V and a constant $c_{H,V}$ such that, for $B \geqslant 1$,*

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) \mid H(x) \leqslant B\} = c_{H,V} B \log(B)^{\rho-1}(1+o(1)),$$

where $\rho = \mathrm{rank}(\mathrm{Pic}(V))$.

Peyre [26], and then Batyrev and Tschinkel [2] and Salberger [28] in a more general setting, also proposed a conjectural expression for the constant $c_{H,V}$ in terms of geometric invariants of the variety. We do not record this conjecture here in any more details and refer the interested reader to [27] for example. There are a number of refinements of the Manin-Peyre conjectures and we will focus throughout this paper on the following one [15].

Conjecture 1.2 (Refinement of the Manin-Peyre's conjectures). *Let V be an "almost Fano" variety in the sense of* [26, *Définition 3.1*] *with $V(\mathbb{Q}) \neq \varnothing$ and let H be an anticanonical height function on $V(\mathbb{Q})$. Then there exists a Zariski open subset U of V, a polynomial $P_{U,H}$ of degree ρ and $\delta \in]0,1[$ such that, for $B \geqslant 1$*

$$N_{U,H}(B) = BP_{U,H}(\log B) + O\big(B^{1-\delta}\big),$$

where $\rho = \mathrm{rank}(\mathrm{Pic}(V))$ and the leading coefficient of $P_{U,H}$ agrees with Peyre's prediction.

There has been very little investigations on the lower order coefficients and this seems to be a difficult question but the examples we study in this paper might be an interesting testing ground.

These two conjectures have been the center of numerous investigations in the past few years using techniques from harmonic analysis in the case of equivariant compactifications of some algebraic groups (see for example [1, 30]) or from analytic number theory and more specifically the circle method in the case where the number of variables is large enough with respect to the degree (see for example [5, 16]). In the remaining cases, the only available method relies on a combination of analytic number theory or geometry of number and on a descent on some quasi-versal torsors in the sense of [18]. Most of these investigations (especially in cases relying on a descent) are concerned with surfaces (see for example works of Browning, La Bretèche, Derenthal and Peyre [11, 12, 14]), whereas very little is known in higher dimensions. In particular there are only very few examples of varieties in higher dimension for which Conjecture 1.2, or even Conjecture 1.1, is known to hold using such a descent argument (see [9, 29, 7, 21]). The goal of this paper is to give another such example.

In this paper we shall consider the solutions to the equation

$$x_1y_2y_3\cdots y_n + x_2y_1y_3\cdots y_n + \cdots + x_ny_1y_2\cdots y_{n-1} = 0. \tag{1.1}$$

Notice that, upon excluding the points for which $y_1\cdots y_n = 0$, one can also rewrite the above equation as a linear equation between fractions

$$\frac{x_1}{y_1} + \cdots + \frac{x_n}{y_n} = 0.$$

We shall focus on the case $n = 3$ in the present paper. The cases $n \geqslant 4$ will be the subject of future work.

One can view equation (1.1) in three natural ways. First, one can consider the singular projective hypersurface of $\mathbb{P}^{2n-1}$ with homogeneous coordinates $[x_1 : \cdots : x_n : y_1 : \cdots : y_n]$ defined by (1.1). This was done in 2014 by Blomer, Brüdern and Salberger who in [7] proved Conjecture 1.2 for $n = 3$ using a combination of lattice point counting and analytic counting by multiple Mellin integrals. This setting was also studied by the second author [21] who, by elementary counting methods, proved Conjecture 1.1 when $n \geq 2$ for the following anticanonical height function

$$H([x_1 : \cdots : x_n : y_1 : \cdots : y_n]) = \max_{1\leqslant i\leqslant n} \max\{|x_i|, |y_i|\}^n.$$

It is worth noticing that in this case, the varieties under consideration are equivariant compactifications of the algebraic groups $\mathbb{G}_a^{n-1} \times \mathbb{G}_m^{n-1}$. Harmonic analysis techniques might also be able to handle this case and to prove Conjecture 1.2 for every $n \geqslant 4$. To our knowledge this hasn't been done so far, but it would be interesting to compare this approach with a generalization of the methods of [7] or of the present paper.

One can also think of (1.1) as defining the singular biprojective variety $\widetilde{W}_n$ of $\left(\mathbb{P}^{n-1}\right)^2$ with bihomogeneous coordinates $[x_1 : \cdots : x_n], [y_1 : \cdots : y_n]$ defined by the equation (1.1). An anticanonical height function is then given by

$$\widetilde{H}([x_1 : \cdots : x_n], [y_1 : \cdots : y_n]) = \max_{1 \leqslant i \leqslant n} |x_i|^{n-1} \max_{1 \leqslant i \leqslant n} |y_i|.$$

In this case, the varieties under consideration are not equivariant compactifications, the rank of the Picard group of $\widetilde{W}_n$ is $2^n - n$ and the subset where $x_1 \cdots x_n y_1 \cdots y_n = 0$ is an accumulating subset. In recent work [8], Blomer, Brüdern and Salberger showed that Conjecture 1.1 holds for $\widetilde{W}_3$ using Fourier analysis. Using recent results of the first author [4], we are able to refine the aforementioned result [8] proving the stronger Conjecture 1.2 for $\widetilde{W}_3$.

Theorem 1. *Let $\widetilde{U}$ be the Zariski open subset of $\widetilde{W}_3$ given by the condition $x_1x_2x_3y_1y_2y_3 \neq 0$. There exist $\xi_1 > 0$ and a polynomial P_1 of degree 4 such that*

$$N_{\widetilde{W}_3, \widetilde{H}}(B) := \#\Big\{([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]) \in \widetilde{U}(\mathbb{Q}) \mid \widetilde{H}(\mathbf{x}, \mathbf{y}) \leqslant B\Big\} = BP_1(\log B) + O\big(B^{1-\xi_1}\big).$$

The leading coefficient of P_1 is equal to $\frac{\mathfrak{S}_1 \cdot \mathcal{I}}{144}$, where

$$\mathfrak{S}_1 := \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}\right),$$

$$\mathcal{I} := \iint_{[-1,1]^3 \times [0,1]^2} \chi_{[0,1/|z|]}\left(\frac{x_1}{y_1} + \frac{x_2}{y_2}\right) dx_1 dx_2 dz \cdot \frac{dy_1 dy_2}{y_1 y_2} = \pi^2 + 24 \log 2 - 3$$

and χ_X denotes the characteristic function of a set X.

The work of [8] shows that $\frac{\mathfrak{S}_1 \cdot \mathcal{I}}{144}$ coincides with Peyre's prediction for this variety and so Theorem 1 gives Conjecture 1.2 for $\widetilde{W}_3$.

Finally, a third interpretation of (1.1) is as the singular subvariety $\widehat{W}_n$ of $\left(\mathbb{P}^1\right)^n$ with multihomogeneous coordinates $[x_1 : y_1], \ldots, [x_n : y_n]$. The only record of study of an analogous equation is from La Bretèche [10] but with a non anticanonical height function. An anticanonical height is given in this setting by

$$\widehat{H}([x_1 : y_1], \ldots, [x_n : y_n]) = \prod_{i=1}^n \max\{|x_i|, |y_i|\}.$$

We prove the two following theorems which, combined, give that Conjecture 1.2 holds for $\widehat{W}_3$.

Theorem 2. *Let $\widehat{U}$ be the Zariski open of $\widehat{W}_3$ defined by the condition $y_1y_2y_3 \neq 0$. Then there exist $\xi_2 > 0$ and a polynomial P_2 of degree 3 such that*

$$N_{\widehat{W}_3, \widehat{H}}(B) := \#\Big\{([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]) \in \widehat{U}(\mathbb{Q}) \mid \widehat{H}(\mathbf{x}, \mathbf{y}) \leqslant B\Big\} = BP_2(\log B) + O\big(B^{1-\xi_2}\big).$$

The leading coefficient of P_2 is equal to $\frac{\mathfrak{S}_2 \cdot \mathcal{I}}{144}$, where $\mathcal{I}$ is as in Theorem 1 and

$$\mathfrak{S}_2 := \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right).$$

Theorem 3. *The variety $\widehat{W}_3$ is isomorphic to a del Pezzo surface of degree 6 with singularity type $\mathbf{A}_1$ and 3 lines over $\mathbb{Q}$ and the leading constant of the polynomial P_2 in Theorem 2 agrees with Peyre's prediction.*

We remark that by Theorem 3 and [25] one has that $\widehat{W}_3$ is an equivariant compactification of $\mathbb{G}_a^2$. In particular Theorem 2 follows from the more general work of Chambert-Loir and Tschinkel [17].

The purpose of giving a new independent proof of Theorem 2 is double. First, the method presented here uses a descent on the versal torsor and thus it is different from the method in [17] which relies on harmonic analysis techniques and the study of the height zeta function. To our knowledge this is the first time that a full asymptotic with a power-saving error term is obtained on this del Pezzo surface by means of a descent on the versal torsor. The best result using such a method can be found in [15, Chapter 5] where Browning obtains a statement somewhere in between Conjectures 1.1 and 1.2.

Secondly, following the same approach for proving Theorem 1 and 2 allows one to appreciate the difference in the structure of the main terms in these two cases, showing how the extraction of the main term in the first case becomes substantially harder as well as allowing the use the proof of Theorem 2 as a guide for that of Theorem 1.

Remark. *We prove Theorem 2 for any $\xi_2 < 0.00228169\ldots$ One can easily give an explicit power saving also in the case of Theorem 1 as well as improving the allowed range for ξ_2, but in order to simplify the presentation we choose not to do so, since in any case the values obtained could be greatly improved by tailoring the methods of* [4] *to these specific problems.*

The proofs of Theorem 1 and 2 roughly proceed as follows. We use the same unique factorization as in recent work of the second author [21] to parametrize the counting problem combined with recent work of the first author [4]. More precisely, by means of a descent on the versal torsor we can transform the problem of counting solutions to (1.1) to that of counting solutions to $a_1x_1z_1 + a_2x_2z_3 + a_3x_3z_3 = 0$ with some coprimality conditions, with certain restraints on the sizes of x_i, z_j (depending on the height we had originally chosen), and with a_1, a_2, a_3 that can be thought of being very small. By [4] (see also [3]) we have the meromorphic continuation for the "parabolic Eisenstein series"

$$\sum_{\substack{m_1,m_2,m_3\in\mathbb{Z}_{>0}\\ a_1m_1+a_2m_2+a_3m_3=0}} \frac{\tau_{\alpha_1,\beta_1}(m_1)\tau_{\alpha_2,\beta_2}(m_3)\tau_{\alpha_3,\beta_3}(m_3)}{(m_1m_2m_3)^s}, \quad \Re(s) > \tfrac{2}{3} - \min(\Re(\alpha_i), \Re(\beta_i)) \ \ \forall i = 1,2,3$$

where $\tau_{\alpha_i,\beta_i}(m) = \sum_{d_1d_2=m} d_1^{-\alpha_i} d_2^{-\beta_i}$ for $(\alpha_1,\alpha_2,\alpha_3), (\beta_1,\beta_2,\beta_3) \in \mathbb{C}^3$. Using this we obtain that the counting problem in both cases is given by a certain multiple complex integral of the products of Γ and ζ functions, up to a power saving error term. The main part of the paper is then devoted to the use of complex analytic methods to extract the main terms from such integrals. This process is reminiscent of the work [19] of La Bretèche, where he showed how to deduce asymptotic formulas for generic arithmetic averages from the analytic properties of their associated Dirichlet series. However, his work is not directly applicable to our case. Indeed, in his setting all variables are summed in boxes, whereas in our case the main action happens at complicated hyperbolic spikes. Of course, one could use La Bretèche's work in combination with some suitable version of the hyperbola method, but in fact this would not simplify substantially the problem and would still eventually require arithmetic and complex-analytic computations essentially equivalent to ours. For this reason, we prefered to approach the relevant sums in a more direct way.

The paper is structured as follows. First, in Section 2 we reparametrize the solutions to (1.1) using a descent on the versal torsor. In Section 3 we prove Theorem 3. In Section 4 we state the main Lemma on the parabolic Eisenstein series and a smoothing lemma useful to avoid problems of sharp cut-offs. Then, in Section 5, 6 and 7 we prove Theorems 2 and 1 in three steps of increasing difficulties: first Theorem 2 without the aforementioned coprimality conditions, then we include these conditions and finally we prove Theorem 1.

Notations

We use the vector notation $\boldsymbol{v} = (v_1, \ldots, v_k)$ where the dimension is clear from the context. Also, given a vector $\boldsymbol{v} \in \mathbb{C}^k$ and $c \in \mathbb{C}$ with $\boldsymbol{v} + c$ we mean $(v_1 + c, \ldots, v_k + c)$. With $\iint$ we indicate the integration with respect to several variables, whose number is clear from the context. For $c \in \mathbb{R}$, with

$\int_{(c)}$ we indicate that the integral is taken along the vertical line from $c-i\infty$ to $c+i\infty$. Also, we indicate with c_z the line of integration corresponding to the variable z. Given, $a_1,\dots,a_k \in \mathbb{Z}$ we indicate the GCD and the HCF of $a_1,\dots,a_k$ by $(a_1,\dots,a_k)$ and $[a_1,\dots,a_k]$ respectively.

We indicate the real and imaginary part of a complex number $s \in \mathbb{C}$ by σ and t respectively, so that $s = \sigma + it$. Also, ε will denote an arbitrary small and positive real number, which is assumed to be sufficiently small and upon which all bounds are allowed to depend. Finally, in Section 7, we denote by $C_1, C_2, C_3, \dots$ a sequence of fixed positive real numbers.

2. The descent on the versal torsor

For $n \geqslant 2$, we let $N = 2^n - 1$. For every $h \in \{1,\dots,N\}$, we denote its binary expansion by

$$h = \sum_{1\leqslant j\leqslant n} \epsilon_j(h)2^{j-1},$$

with $\epsilon_j(h) \in \{0,1\}$. We will let $s(h) = \sum_{j\geqslant 1}\epsilon_j(h)$ be the sum of the bits of h. We will say that a integer h is dominated by ℓ if for every $j \in \mathbb{N}$, we have $\epsilon_j(h) \leqslant \epsilon_j(\ell)$. We will use the notation $h \preceq \ell$ to indicate that h is dominated by ℓ. We will say that an $N-$tuple $(z_1,\dots,z_N)$ is reduced if $\gcd(z_h,z_\ell) = 1$ when $h \npreceq \ell$ and $\ell \npreceq h$.

We give the following lemma which gives a unique factorization for the variables y_i inspired by [24, 13] and [10] and which will be very useful to parametrize rational solutions of (1.1).

Lemma 1 ([10])**.** *There is a one-to-one correspondence between the $n-$tuples of non negative integers $(y_i)_{1\leqslant i\leqslant n}$ and the reduced $N-$tuples $(z_h)_{1\leqslant h\leqslant N}$ of non negative integers such that*

$$\forall j \in [\![1,n]\!], \quad y_j = \prod_{1\leqslant h\leqslant N} z_h^{\epsilon_j(h)} \quad \textit{and} \quad [y_1,\dots,y_n] = \prod_{1\leqslant h\leqslant N} z_h.$$

2.1. **The case of $\widehat{W}_n$.** Let $n \geqslant 2$. We want to estimate, for $B \geqslant 1$, the quantity

$$N_{\widehat{W}_n,\widehat{H}}(B) := \#\Big\{([x_1:y_1],\dots,[x_n:y_n]) \in \widehat{U}(\mathbb{Q}) \mid \widehat{H}(\mathbf{x},\mathbf{y}) \leqslant B\Big\}.$$

Clearly, we have

$$N_{\widehat{W}_n,\widehat{H}}(B) = \#\left\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^n \times \mathbb{Z}^n_{>0} \ : \ \begin{array}{c} \widehat{H}(\mathbf{x},\mathbf{y}) \leqslant B \\ (\mathbf{x},\mathbf{y}) \text{ satisfies } \ (1.1), \ \gcd(x_i,y_i) = 1 \end{array}\right\}.$$

Using Lemma 1 the equation (1.1) can be rewritten as

$$\sum_{j=1}^n d_jx_j = 0 \quad \text{with} \quad d_i = \prod_{1\leqslant h\leqslant N} z_h^{1-\epsilon_i(h)} \quad \forall i \in [\![1,n]\!]. \tag{2.1}$$

We then obtain the divisibility relation $z_{2^{j-1}} \mid x_j$ for every $j \in \{1,\dots,n\}$. Since we have the conditions $\gcd(x_j,y_j) = 1$ and $z_{2^{j-1}} \mid y_j$, we can deduce that for every $j \in \{1,\dots,n\}$, $z_{2^{j-1}} = 1$. Finally, we have

$$N_{\widehat{W}_n,\widehat{H}}(B) = \#\left\{(\mathbf{x},\mathbf{z}) \in \mathbb{Z}^n \times \mathbb{Z}^{N-n}_{>0} \ : \ \begin{array}{c} \displaystyle\prod_{i=1}^n \max\Big\{|x_i|, \prod_{1\leqslant h\leqslant N}|z_h|^{\epsilon_i(h)}\Big\} \leqslant B \\ (z_h)_{1\leqslant h\leqslant N} \text{ reduced}, \quad \displaystyle\sum_{i=1}^n x_id_i = 0 \end{array}\right\}.$$

In the case $n=3$, renaming for simplicity z_6 by z_1, z_5 by z_2 and z_7 by z_4, one gets the following expression for $N_{\widehat{W}_3,\widehat{H}}(B)$

(2.2)
$$N_{\widehat{W}_3,\widehat{H}}(B) = \#\left\{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^3\times\mathbb{Z}^4_{>0} : \begin{array}{l}\gcd(x_1,z_2z_3z_4)=\gcd(x_2,z_1z_3z_4)=\gcd(x_3,z_1z_2z_4)=1\\ \gcd(z_1,z_2)=\gcd(z_1,z_3)=\gcd(z_2,z_3)=1\\ \max\{|x_1|,z_2z_3z_4\}\times\max\{|x_2|,z_1z_3z_4\}\times\max\{|x_3|,z_1z_2z_4\}\leqslant B\\ x_1z_1+x_2z_2+x_3z_3=0\end{array}\right\}.$$

As explained in Section 3, the open subvariety of $\mathbb{A}^7$ given by the equation $x_1z_1+x_2z_2+x_3z_3=0$ along with the conditions
$$(x_1,z_2z_3z_4)\neq(0,0),\quad (x_2,z_1z_3z_4)\neq(0,0),\quad (x_3,z_1z_2z_4)\neq(0,0)$$
and
$$(z_1,z_2)\neq(0,0),\quad (z_1,z_3)\neq(0,0),\quad (z_2,z_3)\neq(0,0)$$
is the versal torsor of the minimal desingularisation of $\widehat{W}_3$ and hence, through this parametrization, we just performed a descent on the versal torsor of this minimal desingularisation of $\widehat{W}_3$.

2.2. **The case of $\widetilde{W}_n$.** Let $n\geqslant 2$. We now want to estimate, for $B\geqslant 1$, the quantity
$$N_{\widetilde{W}_n,\widetilde{H}}(B)=\frac{1}{4}\#\left\{(\mathbf{x},\mathbf{y})\in\mathbb{Z}^n\times\mathbb{Z}^n_{\neq 0} : \begin{array}{l}\widetilde{H}(\mathbf{x},\mathbf{y})\leqslant B\\ (\mathbf{x},\mathbf{y})\text{ satisfy } (1.1)\\ \gcd(x_1,\dots,x_n)=\gcd(y_1,\dots,y_n)=1\end{array}\right\}.$$

Clearly, we have
$$N_{\widetilde{W}_n,\widetilde{H}}(B)=2^{n-2}\#\left\{(\mathbf{x},\mathbf{y})\in\mathbb{Z}^n\times\mathbb{Z}^n_{>0} : \begin{array}{l}\widetilde{H}(\mathbf{x},\mathbf{y})\leqslant B\\ (\mathbf{x},\mathbf{y})\text{ satisfy } (1.1)\\ \gcd(x_1,\dots,x_n)=\gcd(y_1,\dots,y_n)=1\end{array}\right\}.$$

We can still rewrite the equation (1.1) as (2.1) using Lemma 1 but we can no longer deduce that $z_{2^{j-1}}=1$. We only have the divisibility relation $z_{2^{j-1}}\mid x_j$. However, we have $z_N=1$.

Finally, one gets
$$N_{\widetilde{W}_3,\widetilde{H}}(B)=2\,\#\left\{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^n\times\mathbb{Z}^{N-1}_{>0} : \begin{array}{l}\max_{1\leqslant i\leqslant n}|x_i|^{n-1}\max_{1\leqslant i\leqslant n}\left|\prod_{1\leqslant h\leqslant N-1} z_h^{\epsilon_i(h)}\right|\leqslant B\\ \sum_{i=1}^n x_id_i=0\\ \gcd(x_1,\dots,x_n)=1,\ (z_h)_{1\leqslant h\leqslant N-1}\text{ reduced}\end{array}\right\}$$
and particularly, in the case $n=3$, we obtain
$$N_{\widetilde{W}_3,\widetilde{H}}(B)=2\#\left\{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^3\times\mathbb{Z}^6_{>0} : \begin{array}{l}\max_{1\leqslant i\leqslant 3}|x_i|^2\max\{z_1z_3z_5,z_2z_3z_6,z_4z_5z_6\}\leqslant B\\ x_1z_2z_4z_6+x_2z_1z_4z_5+x_3z_1z_2z_3=0\\ \gcd(x_1,x_2,x_3)=1,\ (z_h)_{1\leqslant h\leqslant 6}\text{ reduced}\end{array}\right\}$$
$$=2\#\left\{(\mathbf{x}',\mathbf{z})\in\mathbb{Z}^3\times\mathbb{Z}^6_{>0} : \begin{array}{l}\max\{z_1|x_1'|,z_2|x_2'|,z_4|x_3'|\}^2\max\{z_1z_3z_5,z_2z_3z_6,z_4z_5z_6\}\leqslant B\\ x_1'z_6+x_2'z_5+x_3'z_3=0\\ \gcd(z_1x_1',z_2x_2',z_4x_3')=1,\ (z_h)_{1\leqslant h\leqslant 6}\text{ reduced}\end{array}\right\}. \tag{2.3}$$

It is easily seen that the coprimality conditions given by $\gcd(z_1x_1',z_2x_2',z_4x_3')=1$, and $(z_h)_{1\leqslant h\leqslant 6}$ reduced are equivalent to
$$\gcd(x_1',x_2',x_3')=\gcd(x_1',x_2',z_3)=\gcd(x_1',z_5,x_3')=\gcd(z_6,x_2',x_3')=1$$

together with the fact that $(z_h)_{1\leqslant h\leqslant 6}$ is reduced. It then follows from [6] that the open subvariety of $\mathbb{A}^9$ given by the equation $x_1'z_6 + x_2'z_5 + x_3'z_3 = 0$ along with the conditions

$$(x_1', x_2', x_3') \neq (0,0,0), \quad (x_1', x_2', z_3) \neq (0,0,0, \quad (x_1', z_5, x_3') \neq (0,0,0), \quad (z_6, x_2', x_3') \neq (0,0,0)$$

and

$$(z_1,z_2) \neq (0,0), \quad (z_1,z_4) \neq (0,0), \quad (z_1,z_6) \neq (0,0), \quad (z_2,z_4) \neq (0,0), \quad (z_2,z_5) \neq (0,0),$$
$$(z_3,z_4) \neq (0,0), \quad (z_3,z_5) \neq (0,0), \quad (z_3,z_6) \neq (0,0), \quad (z_5,z_6) \neq (0,0)$$

is the versal torsor of the minimal desingularisation of $\widetilde{W}_3$ and hence, through this parametrization, we just performed a descent on the versal torsor of this minimal desingularisation of $\widetilde{W}_3$.

3. Geometry and the constant in the case $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We give in this section the proof of the Theorem 3. For example, [15] yields that the surface $S \subseteq \mathbb{P}^6$ cut out by the following 9 quadrics

$$\begin{aligned} X_1^2 - X_2X_4 &= X_1X_5 - X_3X_4 = X_1X_3 - X_2X_5 = X_1X_6 - X_3X_5 \\ &= X_2X_6 - X_3^2 = X_4X_6 - X_5^2 = X_1^2 - X_1X_4 + X_5X_7 \\ &= X_1^2 - X_1X_2 - X_3X_7 = X_1X_3 - X_1X_5 + X_6X_7 = 0 \end{aligned}$$

is a del Pezzo surface of degree 6 of singularity type $\mathbf{A}_1$ with three lines, the lines being given by

$$X_1 = X_2 = X_3 = X_5 = X_6 = 0, \quad X_1 = X_3 = X_4 = X_5 = X_6 = 0$$

and

$$X_3 = X_5 = X_6 = X_1 - X_4 = X_1 - X_2 = 0.$$

The maps $f : \widetilde{W}_3 \to S$ given by

$$\begin{cases} X_1 = -y_3x_1x_2 \\ X_2 = -x_1(x_2y_3 + x_3y_2) \\ X_3 = -y_2y_3x_1 \\ X_4 = -x_2(x_1y_3 + x_3y_1) \\ X_5 = y_1y_3x_2 \\ X_6 = y_1y_2y_3 \\ X_7 = x_1x_2x_3 \end{cases}$$

and $g : S \to \widetilde{W}_3$ given by

$$g([X_1 : \cdots : X_7]) = ([X_1 : -X_5], [X_5 : X_6], [X_7 : -X_1])$$

are well defined and inverse from each other. Thus $\widetilde{W}_3 \cong S$ and is therefore a del Pezzo surface of degree 6 of singularity type $\mathbf{A}_1$ with three lines, the lines being given by $y_i = y_j = 0$ for $1 \leqslant i \neq j \leqslant 3$. As mentioned in the introduction, it follows then from [25] and from this isomorphism that $\widehat{W}_3$ is an equivariant compactification of $\mathbb{G}_a^2$ and Theorem 2 can be derived from the more general work of Chambert-Loir and Tschinkel [17]. However, the method presented here using a descent on the versal torsor is different from the method in [17] and it is always interesting to unravel a different proof.

Let us denote by $\widetilde{W}_3^*$ the minimal desingularisation of $\widetilde{W}_3$. The fact that the open subvariety $O \subseteq \mathbb{A}^7$ given by

$$x_1z_1 + x_2z_2 + x_3z_3 = 0$$

with the conditions

$$(x_1, z_2z_3z_4) \neq 0, \quad (x_2, z_1z_3z_4) \neq 0, \quad (x_3, z_1z_2z_4) \neq 0$$

and

$$(z_1, z_2) \neq 0, \quad (z_1, z_3) \neq 0, \quad (z_2, z_3) \neq 0$$

is the versal torsor of $\widetilde{W}_3^*$ is a consequence of work of Derenthal [20].

To conclude, let us briefly justify why the leading constant of Theorem 2

$$\frac{1}{144}\left(\pi^2+24\log(2)-3\right)\prod_p\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)$$

agrees with Peyre's prediction.

First of all, the variety $\widetilde{W}_3$ being rational, we know that $\beta(\widetilde{W}_3^*)=1$ and work from Derenthal [20] immediately yields $\alpha(\widetilde{W}_3^*)=\frac{1}{144}$. We now have that

$$\omega_H\left(\widetilde{W}_3^*(\mathbb{A}_{\mathbb{Q}})\right)=\omega_\infty\prod_p\omega_p$$

with ω_p and ω_∞ being respectively the p-adic and archimedean densities. It is now easy to get that

$$\omega_p=\frac{\#O(\mathbb{F}_p)}{p^6}=\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)$$

either by direct computation or by calling upon a more general result of Loughran [25]. Turning to the archimedean density and reasoning like in [7] one gets that ω_∞ is given by the archimedean density on the open subset $y_1\neq 0$, $y_2\neq 0$ and $y_3\neq 0$ of $\widetilde{W}_3$. This is the affine variety given by the equation

$$u_1+u_2+u_3=0.$$

Using a Leray form to parametrize in u_3, one finally obtains

$$\omega_\infty=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u_1\mathrm{d}u_2}{\max(|u_1|,1)\max(|u_2|,1)\max(|u_1+u_2|,1)}.$$

An easy computation now yields

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u_1\mathrm{d}u_2}{\max(|u_1|,1)\max(|u_2|,1)\max(|u_1+u_2|,1)}=\pi^2+24\log(2)-3$$

which finally shows that the conjectural value of Peyre's constant is

$$\alpha(\widetilde{W}_3^*)\beta(\widetilde{W}_3^*)\omega_H\left(\widetilde{W}_3^*(\mathbb{A}_{\mathbb{Q}})\right)=\frac{1}{144}\left(\pi^2+24\log(2)-3\right)\prod_p\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)$$

and hence that the leading constant in Theorem 2 agrees with Peyre's prediction.

4. The parabolic Eisenstein series and smooth approximations

We quote the following lemma from [4, Lemma 4 and Remark 2].

Lemma 2. *Let*

$$\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta}):=\frac{1}{8}\sum_{\substack{n_1,n_2,n_3,m_1,m_2,m_3\in\mathbb{Z}_{\neq0},\\ a_1n_1m_1+a_2n_2m_2+a_3n_3m_3=0}}\frac{1}{|n_1|^{\alpha_1}|m_1|^{\beta_1}|n_2|^{\alpha_2}|m_2|^{\beta_2}|n_3|^{\alpha_3}|m_3|^{\beta_3}},\tag{4.1}$$

where $\boldsymbol{a}=(a_1,a_2,a_3)\in\mathbb{Z}^3_{\neq0}$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3),\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\in\mathbb{C}^3$ such that $\Re(\alpha_i),\Re(\beta_i)>1$ for all $i\in\{1,2,3\}$. Then $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ converges absolutely if $\Re(\alpha_i),\Re(\beta_i)>\frac{2}{3}$ for all $i\in\{1,2,3\}$. Moreover for $\frac{2}{3}+\varepsilon<\Re(\alpha_i),\Re(\beta_i)\leqslant\frac{11}{12}$ it satisfies $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})\ll 1$ and

$$\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})=\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})\tag{4.2}$$

where

$$\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})=\sum_{\substack{\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*\in\mathbb{C}^3\\ \{\alpha_i^*,\beta_i^*\}=\{\alpha_i,\beta_i\}\\ \forall i\in\{1,2,3\}}}\frac{2\sqrt{\pi}\,S_{\boldsymbol{a}}(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)}{\alpha_1^*+\alpha_2^*+\alpha_3^*-2}\left(\prod_{i=1}^3\frac{\zeta(1-\alpha_i^*+\beta_i^*)}{|a_i|^{-\alpha_i^*+\frac{1+\alpha_1^*+\alpha_2^*+\alpha_3^*}{3}}}\frac{\Gamma(-\frac{\alpha_i^*}{2}+\frac{1+\alpha_1^*+\alpha_2^*+\alpha_3^*}{6})}{\Gamma(\frac{1+\alpha_i^*}{2}-\frac{1+\alpha_1^*+\alpha_2^*+\alpha_3^*}{6})}\right),$$

with

$$S_{\boldsymbol{a}}(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*):=\sum_{\ell\geqslant 1}\frac{(a_1,\ell)^{1-\alpha_1^*+\beta_1^*}(a_2,\ell)^{1-\alpha_2^*+\beta_2^*}(a_3,\ell)^{1-\alpha_3^*+\beta_3^*}}{\ell^{3-\sum_{i=1}^3(\alpha_i^*-\beta_i^*)}}\varphi(\ell),$$

and where $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is an holomorphic function on

$$\Omega_\varepsilon:=\left\{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathbb{C}^6\mid \Re(\alpha_i),\Re(\beta_i)\in[\tfrac{5}{12}+\varepsilon,\tfrac{11}{12}-\varepsilon]\ \forall i\in\{1,2,3\},\ \eta<\tfrac{2}{9}-\varepsilon\right\}\tag{4.3}$$

for all $\varepsilon>0$ with $\eta:=\sum_{i=1}^3(|\Re(\alpha_i)-\frac{2}{3}|+|\Re(\beta_i)-\frac{2}{3}|)$. Moreover, for $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega_\varepsilon$ one has

$$\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})\ll\left(\big(\max_{1\leqslant i\leqslant 3}|a_i|\big)^{14}\Big(1+\max_{1\leqslant i\leqslant 3}(|\Im(\alpha_i)|+|\Im(\beta_i)|)\Big)^{21}\right)^{\frac{9\eta+18\varepsilon}{4-9\eta}}.\tag{4.4}$$

Remark. *Note that the sum over $\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*\in\mathbb{C}^3$ such that $\{\alpha_i^*,\beta_i^*\}=\{\alpha_i,\beta_i\}$ for all $i\in\{1,2,3\}$ appearing in the definition of the quantity $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ contains eight terms given by*

$$\begin{cases}(\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\alpha_1,\alpha_2,\alpha_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\beta_1,\beta_2,\beta_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\beta_1,\alpha_2,\alpha_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\alpha_1,\beta_2,\beta_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\alpha_1,\beta_2,\alpha_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\beta_1,\alpha_2,\beta_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\alpha_1,\alpha_2,\beta_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\beta_1,\beta_2,\alpha_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\beta_1,\beta_2,\alpha_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\alpha_1,\alpha_2,\beta_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\beta_1,\alpha_2,\beta_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\alpha_1,\beta_2,\alpha_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\alpha_1,\beta_2,\beta_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\beta_1,\alpha_2,\alpha_3)\\ (\alpha_1^*,\alpha_2^*,\alpha_3^*)=(\beta_1,\beta_2,\beta_3) & \textit{and} \quad (\beta_1^*,\beta_2^*,\beta_3^*)=(\alpha_1,\alpha_2,\alpha_3).\end{cases}$$

Proof. Let $\boldsymbol{a}=(a_1,a_2,a_3)\in\mathbb{Z}_{\neq 0}^3$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3),\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\in\mathbb{C}^3$ such that $\Re(\alpha_i),\Re(\beta_i)>1$ for all $i\in\{1,2,3\}$. It is easy to see that $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ converges absolutely if $\Re(\alpha_i),\Re(\beta_i)>\frac{2}{3}$ for all $i\in\{1,2,3\}$ by alluding to inequalities of the form $x+y\geqslant 2\sqrt{xy}$ for $x,y\geqslant 0$.

We also clearly have

$$\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})=\mathcal{A}_{(-\varepsilon_1a_1,\varepsilon_2a_2,\varepsilon_3a_3)}(\boldsymbol{\alpha},\boldsymbol{\beta})=\mathcal{A}^*_{\mathbf{a},\boldsymbol{\alpha}-\frac{2}{3},\boldsymbol{\beta}-\frac{2}{3}}\left(\frac{2}{3}\right).$$

with the notations of [4, Section 2, (2.7)]. Therefore, [4, Lemma 4] with $k=3$ and 3ε instead of ε implies the first part of the lemma, namely that for $\frac{2}{3}+\varepsilon<\Re(\alpha_i),\Re(\beta_i)\leqslant\frac{11}{12}$ we have $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})\ll 1$ and

$$\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})=\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})+\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})\tag{4.5}$$

with $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ being given by the quantity $\mathcal{M}^*_{\mathbf{a},\boldsymbol{\alpha}-\frac{2}{3},\boldsymbol{\beta}-\frac{2}{3}}\left(\frac{2}{3}\right)$ in [4, Remark 2] with $k=3$ and $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ being defined by $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})-\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Now, the expression at then end of [4, Remark 2] in which one has only one summand corresponding to $\mathcal{I}=\{1,2,3\}$ and $\mathcal{J}=\emptyset$ in the case $k=3$ immediately yields the expression of $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ given in the statement of the lemma.

The final part of the lemma follows from [4, Theorem 3] for $k=3$, with 3ε instead of ε and after noticing that, contrary to [4], we don't have the fraction $\frac{3}{2}$ in front of the sum in the definition of η. Hence, in particular, η is given by $\frac{3}{2}\eta_{\boldsymbol{\alpha}-\frac{2}{3},\boldsymbol{\beta}-\frac{2}{3}}$ with the notations of [4, Theorem 3]. □

Since Lemma 2 constitutes the main tool for our proof of Theorems 1 and 2, we say a few words about its proof. First, one divides the variables $r_i=n_im_i$ in various ranges and eliminates the largest one (say r_1) using the linear relation among them. In order to do this one has to write $\sum_{n_1m_1=r_1}|n_1|^{-\alpha_1}|m_1|^{-\beta_1}$ in an efficient way in terms of the remaining variables. This is done by using (a shifted version of) the identity of Ramanujan for the divisor function τ in terms of Ramanujan sums, in combination with a careful use of Mellin transforms to separate variables in expressions such as $(r_2\pm r_3)^s$. After the variables are completely separated, one applies Voronoi's summation formula to the sums over r_2 and r_3. The main terms will then give the polar structure, whereas the error term will produce functions which are holomorphic on the stated range.

The following lemma allows us to replace the characteristic function of the interval $[0,1]$ by a smooth approximation at a cost of a controlled error.

Lemma 3. *Let* $f(x)=e^{-1/(x-x^2)}$ *for* $0<x<1$ *and* $f(x)=0$ *otherwise. Let* $C:=\int_0^1 f(y)\mathrm{d}y$ *and for* $0<\delta<1/2$*, let*

$$F_\delta^{\pm}(x):=\frac{1}{\delta C}\int_x^{+\infty} f\Big(\frac{y-1+(1\mp 1)\delta/2}{\delta}\Big)\,\mathrm{d}y,\qquad x\in\mathbb{R}^+.$$

Then $F_\delta^{\pm}\in\mathcal{C}^\infty(\mathbb{R}^+)$, $F_\delta^{\pm}(x)=1$ *for* $x\leqslant 1-\delta$, $F_\delta^{\pm}(x)=0$ *for* $x\geqslant 1+\delta$ *and, for* $x\geqslant 0$,

$$0\leqslant F_\delta^{-}(x)\leqslant \chi_{[0,1]}(x)\leqslant F_\delta^{+}(x),$$

where $\chi_{[0,1]}$ *is the indicator function of the interval* $[0,1]$. *Moreover, the Mellin transform* $\tilde{F}_\delta^{\pm}(s)$ *of* $F_\delta^{\pm}(x)$ *is holomorphic in* $\mathbb{C}\setminus\{0\}$ *with a simple pole of residue one at* $s=0$ *and for all* $n\geqslant 0$ *it satisfies for all* $s\in\mathbb{C}\setminus\{0\}$

$$\tilde{F}_\delta^{\pm}(s)\ll_n \frac{1}{\delta^n(1+|s|)^{n+1}},\qquad \tilde{F}_\delta^{\pm}(s)-\frac{1}{s}\ll \min(\delta,(1+|s|)^{-1}). \tag{4.6}$$

Proof. The statements on $F_\delta^{\pm}$ are immediate from the definitions. Moreover, assuming $\Re(s)>0$ and integrating by parts we have

$$\tilde{F}_\delta^{\pm}(s)=\frac{1}{\delta C s}\int_0^{+\infty} x^s f\Big(\frac{x-1+(1\mp 1)\delta/2}{\delta}\Big)\,\mathrm{d}x=\frac{1}{sC}\int_0^1 (1+\delta x-(1\mp 1)\delta/2)^s f(x)\,\mathrm{d}x,$$

the last inequality resulting from a change of variable. This already shows that $\tilde{F}_\delta^{\pm}$ is holomorphic in $\mathbb{C}\setminus\{0\}$ with a simple pole of residue one at $s=0$. Integrating by parts n times then yields $\tilde{F}_\delta^{\pm}(s)\ll_n \delta^{-n}(1+|s|)^{-n-1}$, which also implies the second bound in (4.6) if $\delta\geqslant 1/|s|$. Finally, if $\delta<1/|s|$ we have

$$\tilde{F}_\delta^{\pm}(s)-\frac{1}{s}=\frac{1}{sC}\int_0^1 \big((1+\delta x-(1\mp 1)\delta/2)^s-1\big)f(x)\,\mathrm{d}x\ll\delta$$

since $(1+x)^s=1+O(|sx|)$ for $|sx|<1$, $|x|\leqslant\frac{1}{2}$. □

5. Proof of Theorem 2 neglecting the coprimality conditions

By Section 2.1, we need to count the integer solutions to

$$x_1z_1+x_2z_3+x_3z_3=0 \tag{5.1}$$

satisfying the inequality $\max\{|x_1|,z_2z_3z_4\}\times\max\{|x_2|,z_1z_3z_4\}\times\max\{|x_3|,z_1z_2z_4\}\leqslant B$ and the coprimality conditions

$$\gcd(z_1,z_2)=\gcd(z_1,z_3)=\gcd(z_2,z_3)=1,\quad \gcd(x_1,z_2z_3z_4)=\gcd(x_2,z_1z_3z_4)=\gcd(x_3,z_2z_3z_4)=1 \tag{5.2}$$

with $z_1,z_2,z_3,z_4>0$. The case where $x_1x_2x_3=0$ can be dealt with easily and we postpone its treatment to section 6, so we focus on the case where $x_1x_2x_3\neq 0$. We start with the following proposition which gives an asymptotic formula for the number of solutions to the more general equation $a_1x_1z_1+a_2x_2z_3+a_3x_3z_3=0$ without imposing any coprimality condition. These conditions do not factor out immediately at the beginning of the argument, so one cannot deduce Theorem 2 directly from the Proposition 1, however it is instructive to prove this result first, as all the analytic difficulties are exactly the same but the notations and the arithmetic are simplified. In Section 6 we shall give the proof of Theorem 2 by performing the required arithmetic computations and indicating the minor differences in the analytic argument.

Proposition 1. *Let $B \geqslant 1$ and $\varepsilon > 0$. Let $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3_{\neq 0}$ and*

(5.3)
$$K_{\boldsymbol{a}}(B) := \#\left\{ (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{Z}^3_{\neq 0} \times \mathbb{Z}^4_{>0} \left| \begin{array}{l} a_1x_1z_1 + a_2x_2z_2 + a_3x_3z_3 = 0 \\ \max\{|x_1|, z_2z_3z_4\} \times \max\{|x_2|, z_1z_3z_4\} \times \max\{|x_3|, z_1z_2z_4\} \leqslant B \end{array} \right. \right\}.$$

Then there exists a polynomial $P_{\mathbf{a}}$ of degree 3 such that

$$K_{\boldsymbol{a}}(B) := BP_{\mathbf{a}}(\log B) + O_\varepsilon\Big(B^{\frac{296}{297}+\varepsilon} \max_{1 \leqslant i \leqslant 3} |a_i|^{14}\Big), \tag{5.4}$$

where the implied constant only depends on ε. The coefficients of the polynomial $P_{\mathbf{a}}$ are $O(\max_{1\leqslant i\leqslant 3} |a_i|^5)$ and the leading coefficient is $\frac{1}{144}\mathcal{I}_{\boldsymbol{a}}\mathfrak{S}'_{\boldsymbol{a}}$, where

$$\mathcal{I}_{\boldsymbol{a}} := \iint_{[-1,1]^3\times[0,1]^2} \chi_{[0,|a_3/z|]}\Big(a_1\frac{x_1}{y_1} + a_2\frac{x_2}{y_2}\Big)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z\frac{\mathrm{d}y_1\mathrm{d}y_2}{|a_3|y_1y_2} \tag{5.5}$$

and

$$\mathfrak{S}'_{\boldsymbol{a}} := \sum_{\ell=1}^{\infty} \frac{(a_1,\ell)(a_2,\ell)(a_3,\ell)\varphi(\ell)}{\ell^3}.$$

Proof. In the set defining $K_{\boldsymbol{a}}(B)$ we have 8 inequalities coming from all the possible values taken by the maxima. In other words, given each subset $I \subseteq S_3 := \{1, 2, 3\}$ we have the condition

$$\frac{(z_1z_2z_3z_4)^{|J|}}{B} \prod_{i\in I} |x_i| \prod_{j\in J} z_j^{-1} \leqslant 1$$

where $J := S_3 \setminus I$. Now, let $0 < \delta < \frac{1}{2}$ and $F^{\pm}_\delta$ be as in Lemma 3. Then we have $K^-_{\boldsymbol{a}}(B) \leqslant K_{\boldsymbol{a}}(B) \leqslant K^+_{\boldsymbol{a}}(B)$, where

$$K^{\pm}_{\boldsymbol{a}}(B) := \sum_{\substack{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0} \\ a_1x_1z_1+a_2x_2z_2+a_3x_3z_3=0}} \prod_{I\subseteq S_3} F^{\pm}_\delta\left(\frac{(z_1z_2z_3z_4)^{|J|}}{B}\prod_{i\in I}|x_i|\prod_{j\in J} z_j^{-1}\right).$$

Clearly it is sufficient to show that (5.4) holds for both $K^-_{\boldsymbol{a}}(B)$ and $K^+_{\boldsymbol{a}}(B)$ with the same polynomial P. We now write each $F^{\pm}_\delta$ in terms of its Mellin transform using the variable s_I for the cut-off function corresponding to the set I. For brevity we shall often indicate for example with s_{123} the variable $s_{\{1,2,3\}}$ and with $c_{\{1,2,3\}}$ or c_{123} the corresponding line of integration, and similarly for the other variables. In particular, we will denote by s the variable $s_\emptyset$. As lines of integration, we take $c_I = \frac{|I|}{12} + \varepsilon$ for all I for a fixed $\varepsilon > 0$ small enough. Doing so we obtain

$$K^{\pm}_{\boldsymbol{a}}(B) = \sum_{\substack{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0} \\ a_1x_1z_1+a_2x_2z_2+a_3x_3z_3=0}} \frac{1}{(2\pi i)^8}\iint_{(c_I)} \frac{B^{\sum_I s_I}}{z_4^{\sum_I s_I(3-|I|)}} \prod_{i=1}^{3} |x_i|^{-\sum_{I,i\in I} s_I} z_i^{-\sum_I s_I(2+\delta_{i\in I}-|I|)} \prod_I \tilde{F}^{\pm}_\delta(s_I)\mathrm{d}s_I$$

where $\delta_{i\in I} = 1$ if $i \in I$ and $\delta_{i\in I} = 0$ otherwise and where the sums inside the integrals are over $I \subseteq S_3$. Notice that with this choice we have

$$\sum_{I\subseteq S_3} c_I = 1 + 8\varepsilon, \quad \sum_{\substack{I\subseteq S_3 \\ i\in I}} c_I = \frac{2}{3} + 4\varepsilon, \quad \sum_{I\subseteq S_3} c_I(2 + \delta_{i\in I} - |I|) = \frac{2}{3} + 8\varepsilon, \quad \sum_{I\subseteq S_3} c_I(3 - |I|) = 1 + 12\varepsilon.$$

In particular the above series are absolutely convergent by Lemma 2. Now, write

$$
\begin{aligned}
&\xi := \tfrac{1}{2}(2s+s_1+s_2+s_3-s_{123})\\
&\alpha_1 := \tfrac{1}{2}(2s+3s_1+s_2+s_3+2s_{12}+2s_{13}+s_{123}) = \sum_{1\in I} s_I + \xi = \sum_I s_I(2+\delta_{1\in I}-|I|)-\xi\\
&\alpha_2 := \tfrac{1}{2}(2s+s_1+3s_2+s_3+2s_{12}+2s_{23}+s_{123}) = \sum_{2\in I} s_I + \xi = \sum_I s_I(2+\delta_{2\in I}-|I|)-\xi\\
&\alpha_3 := \tfrac{1}{2}(2s+s_1+s_2+3s_3+2s_{13}+2s_{23}+s_{123}) = \sum_{3\in I} s_I + \xi = \sum_I s_I(2+\delta_{3\in I}-|I|)-\xi
\end{aligned} \tag{5.6}
$$

where we are neglecting here the dependencies on the variables s_I in the notations in order to simplify the exposition. Notice then that

$$
\sum_{I\subseteq S_3} s_I = \tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3-\xi), \qquad \sum_{I\subseteq S_3} s_I(3-|I|) = \tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi).
$$

Thus, summing the Dirichlet series we have

$$
K_{\boldsymbol{a}}^{\pm}(B) = \frac{1}{(2\pi i)^8}\iint_{(c_I)} B^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-\xi)}\zeta(\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi))\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)\prod_I \tilde{F}_\delta^{\pm}(s_I)\mathrm{d}s_I
$$

with the notation of Lemma 2. By Lemma 2 and using the notations (4.5), we can split $\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)$ into

$$
\mathcal{A}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi) = \mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)+\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)
$$

thus obtaining the corresponding decomposition $K_{\boldsymbol{a}}^{\pm}(B) = M_{\boldsymbol{a}}^{\pm}(B)+E_{\boldsymbol{a}}^{\pm}(B)$, with

$$
M_{\boldsymbol{a}}^{\pm}(B) := \frac{1}{(2\pi i)^8}\iint_{(c_I)} B^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-\xi)}\zeta(\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi))\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)\prod_I \tilde{F}_\delta^{\pm}(s_I)\mathrm{d}s_I \tag{5.7}
$$

and

$$
E_{\boldsymbol{a}}^{\pm}(B) := \frac{1}{(2\pi i)^8}\iint_{(c_I)} B^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-\xi)}\zeta(\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi))\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)\prod_I \tilde{F}_\delta^{\pm}(s_I)\mathrm{d}s_I.
$$

In the latter integral we move the line of integration c_{S_3} to $c_{S_3} = \frac{1}{4}-\frac{2}{27}+6\varepsilon = \frac{19}{108}+6\varepsilon$. Notice that doing so, in the new lines of integration, we have $\Re(\alpha_i+\xi) = \frac{2}{3}+8\varepsilon$ and $\Re(\alpha_i-\xi) = \frac{2}{3}-\frac{2}{27}+9\varepsilon = \frac{16}{27}+9\varepsilon$ for all $i\in\{1,2,3\}$. In particular we have $\eta = \sum_{i=1}^3(|\Re(\alpha_i-\xi)-\frac{2}{3}|+|\Re(\alpha_i+\xi)-\frac{2}{3}|) = \frac{2}{9}-3\varepsilon$ on the new lines of integration. Moreover, since

$$
\frac{\alpha_1+\alpha_2+\alpha_3+3\xi}{2} = 3s+2(s_1+s_2+s_3)+s_{12}+s_{13}+s_{23} \tag{5.8}
$$

is independent of s_{123}, we have $\Re(\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi)) = 1+12\varepsilon$ on the range $\frac{19}{108}+6\varepsilon \leqslant c_{S_3} \leqslant \frac{1}{4}+\varepsilon$ and hence we stay on the right of the pole of the ζ-function. By (4.4) we then have for ε small enough,

$$
E_{\boldsymbol{a}}^{\pm}(B) \ll A^{14}B^{\frac{25}{27}+13\varepsilon}\iint_{(c_I)}\big(1+\max_I|s_I|\big)^{21}\prod_I\big|\tilde{F}_\delta^{\pm}(s_I)\big|\mathrm{d}s_I
$$

where $A := \max_{1\leqslant i\leqslant 3}|a_i|$. Now, we have

$$
\begin{aligned}
&\int_1^{+\infty}\min\Big(\frac{1}{x},\frac{1}{\delta x^2}\Big)\mathrm{d}x \ll |\log\delta| \ll_\varepsilon \delta^{-\varepsilon},\\
&\int_1^{+\infty}x^{21}\min\Big(\frac{1}{x},\frac{1}{\delta^{22}x^{23}}\Big)\mathrm{d}x \leqslant \int_1^{1/\delta}x^{20}\,\mathrm{d}x+\delta^{-22}\int_{1/\delta}^{+\infty}x^{-2}\,\mathrm{d}x \ll \delta^{-21}
\end{aligned} \tag{5.9}
$$

and so, using Lemma 3 we find

$$
E_{\boldsymbol{a}}^{\pm}(B) \ll A^{14}B^{\frac{25}{27}+13\varepsilon}\delta^{-21-\varepsilon}. \tag{5.10}
$$

Now, we consider the main term $M^{\pm}_{\boldsymbol{a}}(B)$ defined in (5.7). We can write $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)$ as

$$\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi) = \sum_{k=0}^{3} \frac{\zeta(1+2\xi)^k\zeta(1-2\xi)^{3-k}}{\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi-2}\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi)$$

with

$$\begin{aligned}\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi) := &\sum_{\substack{\boldsymbol{\epsilon}\in\{\pm1\}^3\\ \#\{i|\epsilon_i=1\}=k}} \sum_{\ell=1}^{+\infty} \frac{(a_1,\ell)^{1+2\epsilon_1\xi}(a_2,\ell)^{1+2\epsilon_2\xi}(a_3,\ell)^{1+2\epsilon_3\xi}}{\ell^{3+2(2k-3)\delta}}\varphi(\ell)\\ &\times 2\pi^{\frac12}\prod_{i=1}^{3}\frac{\Gamma(\frac{-\alpha_i+\epsilon_i\xi}{2}+\frac{1+\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi}{6})}{|a_i|^{-\alpha_i+\epsilon_i\xi+\frac{1+\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi}{3}}\Gamma(\frac{1+\alpha_i-\epsilon_i\xi}{2}-\frac{1+\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi}{6})}\end{aligned} \tag{5.11}$$

and where the sum is over $\boldsymbol{\epsilon}=(\epsilon_1,\epsilon_2,\epsilon_3)\in\{\pm1\}^3$. Notice that in the region

$$0 \leqslant \Re\left(-\alpha_i+\epsilon_i\xi+\frac{1+\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi}{3}\right)\leqslant\frac12 \quad \forall i\in\{1,2,3\}, \qquad |\Re(\xi)|\leqslant\tfrac16-\varepsilon \tag{5.12}$$

we have that $\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi)$ is holomorphic and satisfies

$$\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi)\ll A^{3+6\Re(\xi)} \tag{5.13}$$

uniformly in $\boldsymbol{a}$, by the bound $|(a,\ell)^s/\ell^s|\leqslant(|a|/\ell)^{\Re(s)}$ for $\Re(s)\geq 0$, and since

$$\Gamma(\tfrac{s}{2})/\Gamma(\tfrac{1-s}{2})\ll_\sigma(1+|t|)^{\sigma-\frac12} \tag{5.14}$$

by Stirling's formula [23, (8.328.1)].

Then, we write $M^{\pm}_{\boldsymbol{a}}(B)=\sum_{k=0}^{3}M^{\pm}_{\boldsymbol{a},k}(B)$ where

$$\begin{aligned}M^{\pm}_{\boldsymbol{a},k}(B)=\frac{1}{(2\pi i)^8}\iint_{(c_I)}&B^{\frac12(\alpha_1+\alpha_2+\alpha_3-\xi)}\zeta(\tfrac12(\alpha_1+\alpha_2+\alpha_3+3\xi))\frac{\zeta(1+2\xi)^k\zeta(1-2\xi)^{3-k}}{\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi-2}\\&\times\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi)\prod_I\tilde{F}^{\pm}_\delta(s_I)\mathrm{d}s_I\end{aligned}$$

and with all the lines of integration still at $c_I=\frac{|I|}{12}+\varepsilon$ for all I. If $k\in\{0,1\}$ we move the line of integration c_{S_3} and $c_\emptyset$ to $c_{S_3}=\frac14-\frac{8}{81}+\varepsilon$ and $c=\frac{2}{81}+\varepsilon$ without passing through any pole. Indeed, by (5.8), we have

$$1+12\varepsilon\leqslant\Re(\tfrac12(\alpha_1+\alpha_2+\alpha_3+3\xi))\leqslant\frac{29}{27}+12\varepsilon$$

and $\Re(\xi)=\Re(s)-\frac{\Re(s_{123})}{2}+\frac18+\frac32\varepsilon$ satisfies $\Re(\xi)\geqslant2\varepsilon$ in the range $\frac14+\varepsilon\leqslant c_{S_3}\leqslant\frac{49}{324}+\varepsilon$ and $\varepsilon\leqslant c_\emptyset\leqslant\frac{2}{81}+\varepsilon$. Thus, for $k\in\{0,1\}$ we have

$$M^{\pm}_{\boldsymbol{a},k}(B)\ll A^{\frac{31}{9}+12\varepsilon}B^{\frac{25}{27}+8\varepsilon}\iint_{(c_I)}(1+\max_I|s_I|)\prod_I\left|\tilde{F}^{\pm}_\delta(s_I)\right|\mathrm{d}s_I\ll A^{\frac{31}{9}+12\varepsilon}B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon} \tag{5.15}$$

since we are inside the region (5.12) and since $|\zeta(1-2\xi)|^3\ll1+|\xi|$ for $\Re(\xi)=\frac{2}{27}+\varepsilon$ by the convexity bound [32, (5.1.4)].

Now, consider the case $k\in\{2,3\}$. In those cases and for ε small enough we move the line of integration c_{S_3} to $c_{S_3}=\frac{19}{108}+\varepsilon$ passing through the pole at $\alpha_1+\alpha_2+\alpha_3+(3-2k)\xi-2=0$, namely $s_{123}=\frac{2}{k}-\frac1k\sum_{I\neq S_3}((2-|I|)(3-k)+|I|)s_I$ with respect to s_{123}, but without crossing the poles of the ζ functions by (5.8) and since we increased $\Re(\xi)$ from 2ε to $\frac{1}{27}+2\varepsilon$. The contribution of the integral on the new lines of integration is easily seen to be $O\Big(A^{\frac{29}{9}+12\varepsilon}B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon}\Big)$ and so we are left with examining the contribution of the residue.

First we consider the case $k=2$. As mentioned above, with respect to s_{123} the pole is located at $s_{123}=1-\sum_{I\neq S_3} s_I$. Also, we can replace each $\tilde{F}^{\pm}_{\delta}(s_I)$ with $\frac{1}{s_I}$ at a cost of committing an error which, by (4.6) and (5.13) since (5.12) is satisfied, is bounded by

$$\ll B \max_{I'\subseteq S_3} \iint_{(c_I)_{I\neq S_3}} A^{3+6\Re(\xi)} |\zeta(1-2\xi)| \min\Big(\delta, \frac{1}{|s_{I'}|}\Big) \prod_{I\neq I'} \frac{1}{|s_I|}\cdot \prod_{I\neq S_3} \mathrm{d}s_I$$

where $s_{123}:=1-\sum_{I\neq S_3} s_I$. In particular $\Re(s_{123})=\frac14-7\varepsilon$. Also, $\Re(\xi)=6\varepsilon$ and so $|\zeta(1-2\xi)|\ll|\xi|^{7\varepsilon}$ by the convexity bound [32, (5.1.4)]. Thus, the above is, for $I'\neq S_3$,

(5.16)
$$\ll \iint_{\substack{(c_I)_{I\neq S_3}\\ |s_{I'}|\leqslant \delta^{-1}}} \frac{\delta A^{3+36\varepsilon} B \max_I |s_I|^{7\varepsilon}}{|1-\sum_{I\neq S_3} s_I|} \prod_{I\neq I', S_3} \frac{1}{|s_I|}\cdot \prod_{I\neq S_3} \mathrm{d}s_I + \iint_{\substack{(c_I)_{I\neq S_3}\\ |s_{I'}|\geqslant \delta^{-1}}} \frac{A^{3+36\varepsilon} B \max_I |s_I|^{7\varepsilon}}{|1-\sum_{I\neq S_3} s_I|} \prod_{I\neq S_3} \frac{\mathrm{d}s_I}{|s_I|}$$
$$\ll A^{3+36\varepsilon} B \delta^{1-7\varepsilon}$$

and a similar argument gives the same bound also for $I=S_3$. It follows that

$$M^{\pm}_{\boldsymbol{a},2}(B) = W_{\boldsymbol{a}} B + O\Big(A^{\frac{31}{9}+12\varepsilon} B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon} + A^{3+36\varepsilon} B \delta^{1-8\varepsilon}\Big) \tag{5.17}$$

where

$$W_{\boldsymbol{a}} := \frac{1}{(2\pi i)^7} \iint_{(c_I)_{I\neq S_3}} \frac{\zeta(1+2\xi)^3 \zeta(1-2\xi)}{2\big(1-\sum_{I\neq S_3} s_I\big)} \mathcal{Q}_{\boldsymbol{a},2}(\boldsymbol{\alpha},\xi) \prod_{I\neq S_3} \frac{\mathrm{d}s_I}{s_I}$$

and where $\boldsymbol{\alpha}$ and ξ are given by (5.6) with s_{123} replaced by $1-\sum_{I\neq S_3} s_I$. Notice that by (5.13), we have $W_{\boldsymbol{a}}\ll A^5$.

Now, let us consider the case $k=3$. We proceed as above replacing $\tilde{F}^{\pm}_{\delta}(s_I)$ by s_I^{-1} for all $I\neq\emptyset$. We can't do the same for $I=\emptyset$ yet because the pole giving the residue is, for ε small enough, at $s_{123}=\frac23-\frac13\sum_{I\neq S_3}|I|s_I$ with respect to s_{123} which does not depend on s and thus the integral with $\tilde{F}^{\pm}_{\delta}(s)$ replaced by $1/s$ is not absolutely convergent. We arrive to

$$M^{\pm}_{\boldsymbol{a},3}(B) = \frac{1}{(2\pi i)^7} \iint_{(c_I)_{I\neq S_3}} B^{1+\xi} \frac{\zeta(1+3\xi)\zeta(1+2\xi)^3}{2-\sum_{I\neq S_3}|I|s_I} \mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},\xi)\, s\,\tilde{F}^{\pm}_{\delta}(s) \prod_{I\neq S_3} \frac{\mathrm{d}s_I}{s_I}$$
$$+ O\Big(A^{\frac{31}{9}+12\varepsilon} B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon} + A^{3+36\varepsilon} B^{1+4\varepsilon} \delta^{1-8\varepsilon}\Big),$$

where the lines of integration are still at $c_I=\frac{|I|}{12}+\varepsilon$ for all $I\neq S_3$. Next, we move the lines of integration c_I to $c_I=\varepsilon$ for all I satisfying $|I|=1$. This has the effect of moving $\Re(\xi)$ from 4ε to $-\frac16+4\varepsilon$ and $\Re(\frac23-\frac13\sum_{I\neq S_3}|I|s_I)$ from $\frac14-3\varepsilon$ to $\frac13-3\varepsilon$. In particular we stay on the right of the poles at $s_I=0$ for all I and we encounter a quadruple pole at $s:=\frac13-\frac13\sum_{I\neq\emptyset,S_3}(3-|I|)s_I$ with respect to s. Note that we have $\xi=0$ at the quadruple pole. The contribution of the integrals on the new lines of integration is, as in (5.9)

$$\ll A^{2+24\varepsilon} B^{\frac56+4\varepsilon} \iint_{(c_I)_{I\neq\emptyset,S_3}} \frac{\max_I |s_I|^{\frac34}}{|2-\sum_{I\neq S_3}|I|s_I|} |s\,\tilde{F}^{\pm}_{\delta}(s)| \prod_{I\neq S_3} \frac{\mathrm{d}s_I}{|s_I|} \ll A^{2+24\varepsilon} B^{\frac56+4\varepsilon}\delta^{-\frac34-\varepsilon},$$

since we are on the region (5.12) and since, by the convexity bound [32, (5.1.4)], $|\zeta(1+3\xi)\zeta(1+2\xi)|^3\ll |\xi|^{\frac34}$ for $\Re(\xi)=-\frac16+4\varepsilon$. As for the residue, we notice that we can replace $s\tilde{F}^{\pm}_{\delta}(s)$ by 1 at a cost of an error which is $O\big(\delta A^{3+6\varepsilon}B^{1+5\varepsilon}\big)$. Indeed, we can write the residue as an integral in s along a circle of radius ε around $\frac13-\frac13\sum_{I\neq\emptyset,S_3}(3-|I|)s_I=O(\varepsilon)$. We then use $s\tilde{F}^{\pm}_{\delta}(s)-1=O(|s|\delta)$ coming from

(4.6) and bound trivially the integrals. Thus, we have

$$M^{\pm}_{\boldsymbol{a},3}(B) = \frac{1}{(2\pi i)^6}\iint_{(c_I)_{I\neq S_3}} \operatorname*{Res}_{\xi=0}\Big(B^{1+\xi}\frac{\zeta(1+3\xi)\zeta(1+2\xi)^3}{3s_{123}\,s}\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},\xi)\Big)\prod_{I\neq\emptyset,S_3}\frac{\mathrm{d}s_I}{s_I} + O\Big(B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon}+B^{1+5\varepsilon}\delta\Big)$$

with

$$s := \frac{1}{3}-\frac{1}{3}\sum_{I\neq\emptyset,S_3}(3-|I|)s_I, \quad s_{123} := \frac{2}{3}-\frac{1}{3}\sum_{I\neq S_3}|I|s_I, \tag{5.18}$$

$\boldsymbol{\alpha}$ given by (5.6) with s_{123} and s replaced by (5.18) and lines of integration which we can take to be $c_I = \frac{1}{12}$ for all $I \neq S_3, \emptyset$. Note that computing the residue in ξ rather than in s doesn't change the result. Computing the residue then gives

$$M^{\pm}_{\boldsymbol{a},3}(B) = BP_{\boldsymbol{a}}(\log B) + O\Big(A^{\frac{31}{9}+12\varepsilon}B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon}+A^{3+36\varepsilon}B^{1+5\varepsilon}\delta^{1-7\varepsilon}\Big), \tag{5.19}$$

where $P_{\boldsymbol{a}}$ is a degree 3 polynomial with leading coefficient

$$P_{\boldsymbol{a},3} := \frac{1}{(2\pi i)^6}\iint_{(\frac{1}{12})}\frac{\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},0)}{432\,s_{123}\,s}\prod_{I\neq\emptyset,S_3}\frac{\mathrm{d}s_I}{s_I} \tag{5.20}$$

with s and s_{123} given by (5.18),

(5.21)

$$\alpha_2 = \tfrac{1}{3}(2+2s_2-s_1-s_3+s_{12}+s_{23}-2s_{13}), \qquad \alpha_3 = \tfrac{1}{3}(2+2s_3-s_1-s_2+s_{13}+s_{23}-2s_{12}),$$

and $\alpha_1 = 2-\alpha_2-\alpha_3$, where again we neglect the dependencies on s_I for $I \neq \emptyset, S_3$ in the notations. Note that we will establish later that $P_{\boldsymbol{a},3} = \frac{1}{144}\mathcal{I}_{\boldsymbol{a}}\mathfrak{S}'_{\boldsymbol{a}}$ with the notations of Proposition 1 and hence $P_{\boldsymbol{a},3} \neq 0$. Also, by (5.13) and the corresponding bound for the derivatives of $\mathcal{Q}_{\boldsymbol{a},k}(\boldsymbol{\alpha},\xi)$ with respect to ξ, we have that the coefficients of $P_{\boldsymbol{a},3}$ are $O(A^5)$.

Collecting the estimates (5.10), (5.15), (5.17) and (5.19) we have

$$\begin{aligned} K^{\pm}_{\boldsymbol{a}}(B) &= BP_{\boldsymbol{a}}(\log B) + W_{\boldsymbol{a}}B + O\Big(A^{3+36\varepsilon}B^{1+5\varepsilon}\delta^{1-7\varepsilon}+A^{14}B^{\frac{25}{27}+13\varepsilon}\delta^{-21-\varepsilon}\Big)\\ &= BP_{\boldsymbol{a}}(\log B) + W_{\boldsymbol{a}}B + O\Big(A^{14}B^{\frac{296}{297}+14\varepsilon}\Big), \end{aligned}$$

upon choosing $\delta = B^{-\frac{1}{297}}$. Thus, it remains to show that $P_{\boldsymbol{a},3} = \frac{1}{144}\mathcal{I}_{\boldsymbol{a}}\mathfrak{S}'_{\boldsymbol{a}}$ with the notations of Proposition 1.

First, we notice that, for $\alpha_1+\alpha_2+\alpha_3 = 2$ and $\xi = 0$, $\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},\xi)$ simplifies to

$$\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},0) = 2\pi^{\frac{1}{2}}\mathfrak{S}'_{\boldsymbol{a}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{|a_i|^{1-\alpha_i}\Gamma(\frac{\alpha_i}{2})}.$$

Next, we use α_2, α_3 as new variables, writing s_2 and s_3 as

$$s_2 = -2+2\alpha_2+\alpha_3+s_1+s_{13}-s_{23}, \qquad s_3 = -2+2\alpha_3+\alpha_2+s_1+s_{12}-s_{23}. \tag{5.22}$$

Note that with this change of variables, we also have

$$s_{123} = 2-\alpha_2-\alpha_3-s_1-s_{12}-s_{13}, \qquad s = 3-2\alpha_2-2\alpha_3-2s_1-s_{12}-s_{13}+s_{23} \tag{5.23}$$

and remind that $\alpha_1 = 2-\alpha_2-\alpha_3$. The lines of integration for α_2, α_3 are at real part equal to $\frac{2}{3}$. Since the Jacobian of the above change of variables is equal to 3 we find, with (5.22) and (5.23)

$$P_{\boldsymbol{a},3} = \frac{1}{(2\pi i)^2}\iint_{(\frac{2}{3})}\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},0)\frac{1}{(2\pi i)^4}\iint_{(\frac{1}{12})}\frac{\mathrm{d}s_1\mathrm{d}s_{12}\mathrm{d}s_{13}\mathrm{d}s_{23}}{144\,s_{123}\,ss_2s_3s_1s_{12}s_{13}s_{23}}\,\mathrm{d}\alpha_2\mathrm{d}\alpha_3.$$

The inner integrals can be evaluated by moving each integral to $-\infty$ (or, equivalently, to $+\infty$), repeatedly applying the residue theorem. For example, one can start by moving c_{s_1} to $-\infty$ encountering poles at $s_1 = 0$, $s_1 = 2 - 2\alpha_2 - \alpha_3 - s_2 - s_{13} + s_{23}$ and $s_1 = 2 - 2\alpha_3 - \alpha_2 - s_3 - s_{12} + s_{23}$. Inserting the contribution of the residues and moving the remaining integrals in the same way one finds, after a simple but tedious calculation which can be readily checked using Mathematica, that

$$\begin{aligned} P_{\boldsymbol{a},3} &= \frac{1}{(2\pi i)^2}\iint_{(\frac{2}{3})} \frac{-\mathcal{Q}_{\boldsymbol{a},3}(\boldsymbol{\alpha},0)}{144\alpha_1\alpha_2\alpha_3(\alpha_1-1)(\alpha_2-1)(\alpha_3-1)}\,\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \\ &= \frac{\mathfrak{S}'_{\boldsymbol{a}}}{(2\pi i)^2}\iint_{(\frac{2}{3})} \frac{2\pi^{\frac{1}{2}}}{144}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\alpha_i(1-\alpha_i)|a_i|^{1-\alpha_i}\Gamma(\frac{\alpha_i}{2})}\,\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \end{aligned}$$

with $\alpha_1 = 2 - \alpha_2 - \alpha_3$ and the result follows by the following lemma. □

Lemma 4. *For $\boldsymbol{a} \in \mathbb{Z}^3_{\neq 0}$ we have*

$$\mathcal{I}_{\boldsymbol{a}} = \frac{1}{(2\pi i)^2}\iint_{(\frac{2}{3})} 2\pi^{\frac{1}{2}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\alpha_i(1-\alpha_i)|a_i|^{1-\alpha_i}\Gamma(\frac{\alpha_i}{2})}\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \tag{5.24}$$

where $\alpha_1 := 2 - \alpha_2 - \alpha_3$. Moreover, $\mathcal{I}_{\mathbf{1}} = \pi^2 + 24\log 2 - 3$.

Proof. For $\alpha_1 + \alpha_2 + \alpha_3 = 2$, we have the Γ identity (see (2.8) in [4])

$$\pi^{\frac{1}{2}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\Gamma(\frac{\alpha_i}{2})} = \sum_{i=1}^{3}\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\alpha_i)\Gamma(\alpha_i)}$$

and, considering $\alpha_1 = 2 - \alpha_2 - \alpha_3$ as a function of α_2 with α_3 fixed, we have the Mellin transforms [31, (7.7.9) and (7.7.14-15)] for $x > 0$

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\alpha_i)\Gamma(\alpha_i)}x^{\alpha_2-1}\,\mathrm{d}\alpha_2 = \begin{cases}(1-x)^{-\alpha_3}\chi_{[0,1]}(x) & \text{if } i=1\\ (x-1)^{-\alpha_3}\chi_{[1,\infty)}(x) & \text{if } i=2\\ (1+x)^{-\alpha_3} & \text{if } i=3\end{cases} \tag{5.25}$$

for $c > 0$ and $\Re(\alpha_3) < 1$ if $i = 1$, $c > 0$, $\Re(\alpha_3) > 0$ if $i = 2$ and $0 < c < \Re(\alpha_3)$ if $i = 3$. Also, for $0 < \Re(\alpha_2), \Re(\alpha_3) < 1$, we have the identity

$$\int_{[0,1]^4}(x_1/y_1)^{1-\alpha_2-\alpha_3}(x_2/y_2)^{\alpha_2-1}(y_1y_2)^{-1}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2 = \frac{1}{\alpha_1(\alpha_1-1)\alpha_2(\alpha_2-1)}.$$

It follows that, in the case $i = 1$,

$$\begin{aligned} &\frac{1}{(2\pi i)^2}\iint_{(\frac{2}{3})}\frac{\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)|a_1|^{1-\alpha_2-\alpha_3}|a_2|^{\alpha_2-1}|a_3|^{\alpha_3-1}}{\alpha_1\alpha_2\alpha_3(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\Gamma(\alpha_1)}\,\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \\ &\quad= \int_{\substack{[0,1]^4\\ |a_1|x_1/y_1-|a_2|x_2/y_2\geqslant 0}}\frac{1}{2\pi i}\int_{(\frac{2}{3})}\frac{(|a_1|x_1/y_1-|a_2|x_2/y_2)^{-\alpha_3}|a_3|^{\alpha_3-1}}{\alpha_3(1-\alpha_3)}\,\mathrm{d}\alpha_3(y_1y_2)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad= \int_{\substack{[0,1]^5\\ 0\leqslant|a_1|x_1/y_1-|a_2|x_2/y_2\leqslant|a_3|/z}}(|a_3|y_1y_2)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z, \end{aligned}$$

since $\frac{1}{2\pi i}\int_{(\frac{2}{3})}x^{-s}\frac{\mathrm{d}s}{s(s-1)} = \int_0^1\chi_{[0,1/z)}(x)\mathrm{d}z$, for $x > 0$. One evaluates similarly the cases arising from $i \in \{2,3\}$ and (5.24) easily follows.

Finally, in the case where $\mathbf{a} = (1,1,1)$, we notice that after using the Gamma identity $\cos(\frac{\pi s}{2})\Gamma(s) =$

$\pi^{1/2}2^{s-1}\Gamma(\frac{s}{2})/\Gamma(\frac{1-s}{2})$, which follows from the reflection and duplication formulae for the Gamma function, the integral on the right hand side of (5.24) reduces to

$$\frac{8}{\pi}\times\frac{1}{(2i\pi)^2}\int\int_{(\frac{2}{3})}\prod_{i=1}^{3}\frac{\Gamma(1-\alpha_i)\cos\Big(\frac{\pi(1-\alpha_i)}{2}\Big)}{\alpha_i(1-\alpha_i)}\mathrm{d}\alpha_2\mathrm{d}\alpha_3$$
$$=\frac{8}{\pi}\times\frac{1}{(2i\pi)^2}\int\int_{(\frac{1}{3})}\frac{\Gamma(z_1)\Gamma(z_2)\Gamma(1-z_1-z_2)\cos\big(\frac{\pi z_1}{2}\big)\cos\big(\frac{\pi z_2}{2}\big)\cos\Big(\frac{\pi(1-z_1-z_2)}{2}\Big)}{z_1z_2(1-z_1-z_2)(1-z_1)(1-z_2)(z_1+z_2)}\mathrm{d}z_1\mathrm{d}z_2$$

after the change of variables $z_1=\alpha_2$ and $z_2=\alpha_3$ and remembering that $\alpha_1=2-\alpha_2-\alpha_3$. The last integral above is computed in Lemma 2.10 of [8], where it was shown to be equal to $\pi^2+24\log 2-3$ by means of a long calculation. One could also give a shorter proof of this identity (still requiring some computations) by writing $(\alpha_1(\alpha_1-1)\alpha_2(\alpha_2-1))^{-1}$ in terms of its (1-variable) Mellin transform, applying (5.25) and evaluating the resulting integrals. □

6. Proof of Theorem 2

We now move to the proof of Theorem 2, namely counting points satisfying (5.1) and the coprimality conditions (5.2). First, we give three lemmata which respectively remove the extra coprimality conditions by mean of Möbius inversion formula, show the convergence of the resulting sums, and compute the Euler product arising in the main term.

Lemma 5. *Let $f:\mathbb{R}^7\to\mathbb{C}$ a function of compact support. Then,*

$$\sideset{}{'}\sum_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0}} f(x_1,z_1,x_2,z_2,x_3,z_3,z_4)$$
$$=\sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3\\ \boldsymbol{d}\in\mathbb{N}^6}}\mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell})\sum_{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0}} f(b_1x_1,c_1z_1,b_2x_2,c_2z_2,b_3x_3,c_3z_3,c_4z_4),$$

where here and below $\sum'$ indicates that the sum is restricted to satisfy the coprimality conditions

$$(z_1,z_2)=(z_1,z_3)=(z_2,z_3)=1,\quad (x_1,z_2z_3z_4)=(x_2,z_1z_3z_4)=(x_3,z_1z_2z_4)=1$$

and where $\boldsymbol{e}:=(e_1,e_2,e_3)$, $\boldsymbol{d}:=(d_{12},d_{13},d_{21},d_{23},d_{31},d_{32})$, $\boldsymbol{\ell}:=(\ell_{12},\ell_{13},\ell_{23})$,

$$\begin{aligned}&b_1:=[e_1,d_{12},d_{13}],\ b_2:=[e_2,d_{21},d_{23}],\ b_3:=[e_3,d_{31},d_{32}]\\&c_1:=[d_{21},d_{31},\ell_{12},\ell_{13}],\ c_2:=[d_{12},d_{32},\ell_{12},\ell_{23}],\ c_3:=[d_{13},d_{23},\ell_{13},\ell_{23}],\ c_4:=[e_1,e_2,e_3]\end{aligned}\tag{6.1}$$

and, with a slight abuse of notation,

$$\mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell}):=\mu(\ell_{12})\mu(\ell_{13})\mu(\ell_{23})\prod_{1\leqslant i\leqslant 3}\mu(e_i)\prod_{\substack{1\leqslant i,j\leqslant 3\\ i\neq j}}\mu(d_{ij}).$$

Proof. This is just an immediate application of Möbius' inversion formula. □

Lemma 6. *With the notations of (6.1) and for real numbers $u_1,u_2,u_3,w_1,w_2,w_3,w_4$ satisfying*

$$\begin{aligned}&u_i,w_i,u_i+w_4-1-\varepsilon\geqslant\kappa,\ \forall i\in\{1,2,3\};\quad w_4\geqslant 0;\quad \kappa\geqslant 0;\\&u_i+w_j-\kappa>1+\varepsilon,\ \forall i,j\in\{1,2,3\},i\neq j;\quad w_i+w_j-\kappa>1+\varepsilon,\ \forall i,j\in\{1,2,3\},i<j;\end{aligned}\tag{6.2}$$

we have

$$\sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3\\ \boldsymbol{d}\in\mathbb{N}^6}}\frac{\max(b_1c_1,b_2c_2,b_3c_3)^\kappa}{b_1^{u_1}b_2^{u_2}b_3^{u_3}c_1^{w_1}c_2^{w_2}c_3^{w_3}c_4^{w_4}}\ll 1.\tag{6.3}$$

Proof. We have the formal Euler product formula

$$\sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3\\ \boldsymbol{d}\in\mathbb{N}^6}} \frac{1}{b_1^{u_1}b_2^{u_2}b_3^{u_3}c_1^{w_1}c_2^{w_2}c_3^{w_3}c_4^{w_4}} = \prod_p \Bigg(\sum_{\boldsymbol{e}',\boldsymbol{\ell}'\in\mathbb{Z}^3_{\geqslant 0},\boldsymbol{d}'\in\mathbb{Z}^6_{\geqslant 0}} p^{-(b_1'u_1+b_2'u_2+b_3'u_3+c_1'w_1+c_2'w_2+c_3'w_3+c_4'w_4)}\Bigg), \tag{6.4}$$

where

$$\begin{aligned} &b_1' = \max(e_1', d_{12}', d_{13}'),\ b_2' = \max(e_2', d_{21}', d_{23}'),\ b_3' = \max(e_3', d_{31}', d_{32}'),\ c_4' = \max(e_1', e_2', e_3')\\ &c_1' = \max(d_{21}', d_{31}', \ell_{12}', \ell_{13}'),\ c_2' = \max(d_{12}', d_{32}', \ell_{12}', \ell_{23}'),\ c_3' = \max(d_{13}', d_{23}', \ell_{13}', \ell_{23}'). \end{aligned} \tag{6.5}$$

We have that the p-factor of the Euler product is

$$1 + O\Big(\sum_{1\leqslant i\leqslant 3} p^{-(u_i+w_4)} + \sum_{\substack{1\leqslant i,j\leqslant 3,\\ i\neq j}} p^{-(u_i+w_j)} + \sum_{1\leqslant i<j\leqslant 3} p^{-(w_i+w_j)}\Big)$$

and thus both sides of (6.4) converge whenever each of the exponents above are smaller than -1.

As for (6.3), we notice that by symmetry it suffices to consider the contribution to the series coming from the terms with $b_1c_1 \geqslant b_2c_2, b_3c_3$. This is less or equal than the left hand side of (6.4) with (u_1, w_1) replaced by $(u_1-\kappa, w_1-\kappa)$ and the lemma follows. □

Lemma 7. *With the notations of (6.1) and for* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3_{\neq 0}$*, let*

$$\mathfrak{S}^*_{\boldsymbol{a}} := \sum_{q\geqslant 1} \frac{\varphi(q)}{q^3} \sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3\\ \boldsymbol{d}\in\mathbb{N}^6}} \mu(\mathbf{e},\mathbf{d},\ell)\frac{(a_1b_1c_1, q)(a_2b_2c_2, q)(a_3b_3c_3, q)}{b_1b_2b_3c_1c_2c_3c_4}. \tag{6.6}$$

Then $\mathfrak{S}^*_{(1,1,1)} = \mathfrak{S}_2$ *with* $\mathfrak{S}_2$ *as in Theorem 2.*

Proof. With the same notations as in (6.5), we have that for $\boldsymbol{a} = \mathbf{1}$, the right hand side of (6.6) is equal to

$$\prod_p \Bigg(\sum_{\substack{q'\in\mathbb{Z}_{\geqslant 0},\, \boldsymbol{d}'\in\{0,1\}^6\\ \boldsymbol{e}',\boldsymbol{\ell}'\in\{0,1\}^3}} \frac{(p-1)^{\rho_{q'}}(-1)^{e_1'+e_2'+e_3'+d_{12}'+d_{13}'+d_{21}'+d_{23}'+d_{31}'+d_{32}'+\ell_{12}'+\ell_{13}'+\ell_{23}'}}{p^{2q'+\rho_{q'}+b_1'+b_2'+b_3'+c_1'+c_2'+c_3'+c_4'-\min(q',b_1'+c_1')-\min(q',b_2'+c_2')-\min(q',b_3'+c_3')}}\Bigg)$$

where $\rho_0 = 0$ and $\rho_{q'} = 1$ if $q' \geqslant 1$. As in lemma 2.7 of [8] we observe that the terms with $q' \geqslant 2$ do not contribute. Indeed, if $q' \geqslant 2$, then $\min(q', b_i' + c_i') = b_i' + c_i'$ so that the exponent of p above is $2q' + c_4'$ and so it does not depend on d_{12}'. In particular the contributions of $d_{12}' = 0$ and $d_{12}' = 1$ cancel out. After restricting the sum over q' to $q' \in \{0, 1\}$, we are just left with performing a finite computation over the 2^{13} possible values of the variables. With the help of a mathematical software we then obtain the claimed Euler product formula for $\mathfrak{S}^*_{(1,1,1)}$. □

We are now ready to prove our Theorem 2.

Proof of Theorem 2. Let $K^*_{\boldsymbol{a}}(B)$ as in (5.3) but imposing also the coprimality conditions (5.2). In particular, by (2.2) we have

$$N_{\widehat{W}_3,\widehat{H}}(B) = K^*_{(1,1,1)}(B) + N'_{\widehat{W}_3,\widehat{H}}(B), \tag{6.7}$$

where $N'_{\widehat{W}_3,\widehat{H}}(B)$ counts the number of points in $\widehat{W_3}$ of height less than B which also satisfy $x_1x_2x_3 = 0$. Now, for $x_3 = 0$ (and thus $y_3 = 1$) then (1.1) reduces to $\frac{x_1}{y_1} + \frac{x_2}{y_2} = 0$. Since we have $(x_1, y_1) = (x_2, y_2) = 1$, then $x_1 = -x_2$, $y_1 = y_2$ and thus

$$N'_{\widehat{W}_3,\widehat{H}}(B) = 1 + 3\sum_{\substack{x\in\mathbb{Z}_{\neq 0},\, y\in\mathbb{N},\, (x,y)=1\\ |x|,\, y\leqslant B^{1/2}}} 1 = \frac{36}{\pi^2}B + O\Big(B^{\frac{1}{2}}\log B\Big). \tag{6.8}$$

In particular, it suffices to prove an asymptotic formula with power saving error term for $K^*_{(1,1,1)}(B)$. Since it doesn't introduce any difficulties, in the following we shall compute an asymptotic formula for $K^*_{\boldsymbol{a}}(B)$ for all $\boldsymbol{a} \in \mathbb{Z}^3_{\neq 0}$.

Let $0 < \delta < \frac{1}{2}$. Using the same approach used for Proposition 1 and with the notations of Lemma 5 and of the proof of Proposition 1, we need to compute an asymptotic formula for

$$K^{*\pm}_{\boldsymbol{a}}(B) = \sideset{}{'}\sum_{\substack{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0}\\ a_1x_1z_1+a_2x_2z_2+a_3x_3z_3=0}} \prod_{I\subseteq S_3} F^{\pm}_{\delta}\left(\frac{(z_1z_2z_3z_4)^{|J|}}{B}\prod_{i\in I}|x_i|\prod_{j\in J} z_j^{-1}\right).$$

With the notations of Lemma 5, we can rewrite this as

$$K^{*\pm}_{\boldsymbol{a}}(B) = \sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3,\\ \boldsymbol{d}\in\mathbb{N}^6}} \mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell}) \sum_{\substack{(\boldsymbol{x},\boldsymbol{z})\in\mathbb{Z}^3_{\neq 0}\times\mathbb{Z}^4_{>0}\\ a^*_1x_1z_1+a^*_2x_2z_2+a^*_3x_3z_3=0}} \prod_{I\subseteq S_3} F^{\pm}_{\delta}\left(\frac{(c_1z_1c_2z_2c_3z_3c_4z_4)^{|J|}}{B}\prod_{i\in I}|b_ix_i|\prod_{j\in J}(c_jz_j)^{-1}\right),$$

where $\boldsymbol{a}^* := (a^*_1, a^*_2, a^*_3) = (a_1b_1c_1, a_2b_2c_2, a_3b_3c_3)$ with b_i, c_i as in (6.1). Thus, proceeding as in Proposition 1 and using the same notations and lines of integrations we find the following expression for $K^{*\pm}_{\boldsymbol{a}}(B)$

$$\sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3,\\ \boldsymbol{d}\in\mathbb{N}^6}} \frac{\mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell})}{(2\pi i)^8}\iint_{(c_I)} \frac{B^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3-\xi)}\zeta(\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi))\mathcal{A}_{\boldsymbol{a}^*}(\boldsymbol{\alpha}-\xi,\boldsymbol{\alpha}+\xi)}{c_4^{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+3\xi)}b_1^{\alpha_1-\xi}b_2^{\alpha_2-\xi}b_3^{\alpha_3-\xi}c_1^{\alpha_1+\xi}c_2^{\alpha_2+\xi}c_3^{\alpha_3+\xi}}\prod_I \tilde{F}^{\pm}_{\delta}(s_I)\mathrm{d}s_I.$$

Notice that by (6.3) with $\kappa = 0$ the outer series converges absolutely. We keep following the same approach as the proof of Proposition 1 splitting $\mathcal{A}_{\boldsymbol{a}}$ into $\mathcal{M}_{\boldsymbol{a}} + \mathcal{E}_{\boldsymbol{a}}$ and thus $K^{*\pm}_{\boldsymbol{a}}(B)$ into $M^{*\pm}_{\boldsymbol{a}}(B) + E^{*\pm}_{\boldsymbol{a}}(B)$. We can treat $E^{*\pm}_{\boldsymbol{a}}(B)$ as above with the only difference that in this case we move c_{S_3} to $\frac{1}{4} - 2\gamma + 6\varepsilon$, where $\gamma := \frac{391-\sqrt{152737}}{108}$ (this value is the smallest one can take under the condition that the inequalities (6.2) are satisfied). With this choice, (4.4) and (6.3) give the bound

$$\begin{aligned} E^{*\pm}_{\boldsymbol{a}}(B) &\ll \sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3,\\ \boldsymbol{d}\in\mathbb{N}^6}} \frac{\max(a_1b_1c_1,a_2b_2c_2,a_3b_3c_3)^{\frac{378\gamma}{2-27\gamma}}}{c_4^{1+12\varepsilon}(b_1b_2b_3)^{\frac{2}{3}-2\gamma+9\varepsilon}c_1^{\frac{2}{3}}c_2^{\frac{2}{3}}c_3^{\frac{2}{3}}}B^{1-2\gamma+13\varepsilon}\iint_{(c_I)}\left(1+\max_I|s_I|\right)^{\frac{567\gamma}{2-27\gamma}}\prod_I\left|\tilde{F}^{\pm}_{\delta}(s_I)\right|\mathrm{d}s_I \\ &\ll_A B^{1-2\gamma+13\varepsilon}\delta^{-\frac{567\gamma}{2-27\gamma}-\varepsilon}, \end{aligned}$$

where $A := \max_{1\leqslant i\leqslant 3}|a_i|$. As for $M^{*\pm}_{\boldsymbol{a}}(B)$, we treat it exactly as in Proposition 1, splitting it into $M^{*\pm}_{\boldsymbol{a}}(B) = \sum_{k=0}^3 E^{*\pm}_{\boldsymbol{a},k}(B)$. As above we have that $M^{*\pm}_{\boldsymbol{a},0}(B), M^{*\pm}_{\boldsymbol{a},1}(B) \ll_A B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon}$ where the sums in the error terms are immediately seen to be convergent by (5.13) and (6.3). For $M^{*\pm}_{\boldsymbol{a},3}(B)$ we find similarly as in the proof of Proposition 1

$$\begin{aligned} M^{*\pm}_{\boldsymbol{a},3}(B) = &\sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3,\\ \boldsymbol{d}\in\mathbb{N}^6}} \frac{\mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell})}{(2\pi i)^6}\iint_{(c_I)}\operatorname*{Res}_{\xi=0}\left(\frac{B^{1+\xi}\zeta(1+3\xi)\zeta(1+2\xi)^3\mathcal{Q}_{\boldsymbol{a}^*,3}(\boldsymbol{\alpha},\xi)}{3s_{123}\,s\,c_4^{1+3\xi}b_1^{\alpha_1-\xi}b_2^{\alpha_2-\xi}b_3^{\alpha_3-\xi}c_1^{\alpha_1+\xi}c_2^{\alpha_2+\xi}c_3^{\alpha_3+\xi}}\right)\prod_{I\neq\emptyset,S_3}\frac{\mathrm{d}s_I}{s_I} \\ &+ O_A\left(B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon} + B^{1+5\varepsilon}\delta^{1-7\varepsilon}\right) \end{aligned}$$

with $s := \frac{1}{3} - \frac{1}{3}\sum_{I\neq\emptyset,S_3}(3-|I|)s_I$, $s_{123} := \frac{2}{3} - \frac{1}{3}\sum_{I\neq S_3}|I|s_I$, lines of integration $c_I = \frac{1}{12}$ for all $I \neq S_3, \emptyset$, $\mathcal{Q}_{\boldsymbol{a}^*,3}$ as in (5.11) and $\boldsymbol{\alpha}$ and ξ given by (5.6) with s_{123} and s replaced by (5.18). Computing the residue then gives

$$M^{\pm}_{\boldsymbol{a},3}(B) = BP^*_{\boldsymbol{a}}(\log B) + O_A\left(B^{\frac{25}{27}+8\varepsilon}\delta^{-1-\varepsilon} + B^{1+5\varepsilon}\delta^{1-7\varepsilon}\right)$$

where $P^*_{\boldsymbol{a}}$ is a degree 3 polynomial with leading constant

$$P^*_{\boldsymbol{a},3} := \sum_{\substack{\boldsymbol{e},\boldsymbol{\ell}\in\mathbb{N}^3,\\ \boldsymbol{d}\in\mathbb{N}^6}} \frac{\mu(\boldsymbol{e},\boldsymbol{d},\boldsymbol{\ell})}{(2\pi i)^6} \iint_{(\frac{1}{12})} \frac{\mathcal{Q}_{\boldsymbol{a}^*,3}(\boldsymbol{\alpha},0)}{432\, s_{123}\, s\, c_4 b_1^{\alpha_1} b_2^{\alpha_2} b_3^{\alpha_3} c_1^{\alpha_1} c_2^{\alpha_2} c_3^{\alpha_3}} \prod_{I\neq\emptyset,S_3} \frac{\mathrm{d}s_I}{s_I}$$

and α_2,α_3 as in (5.21) and $\alpha_1 = 2-\alpha_2-\alpha_3$. Now,

$$\mathcal{Q}_{\boldsymbol{a}^*,3}(\boldsymbol{\alpha},0) = 2\pi^{\frac12} \sum_{\ell=1}^{\infty} \frac{(a_1b_1c_1,\ell)(a_2b_2c_2,\ell)(a_3b_3c_3,\ell)\varphi(\ell)}{\ell^3} \prod_{i=1}^{3} \frac{\Gamma(\frac{1-\alpha_i}{2})}{|a_i|^{1-\alpha_i}\Gamma(\frac{\alpha_i}{2})}.$$

and hence by the same computation as in Proposition 1 we find $P^*_{\boldsymbol{a},3} = \frac{1}{144}\mathfrak{S}^*_{\boldsymbol{a}}\mathcal{I}_{\boldsymbol{a}}$ with $\mathfrak{S}^*_{\boldsymbol{a}}$ as in Lemma 7.

Finally, we treat $M^{*\pm}_{\boldsymbol{a},2}(B)$ exactly as in Proposition 1 and so, collecting the various asymptotics and bounds, we arrive to

$$K^{*\pm}_{\boldsymbol{a}}(B) = BP^*_{\boldsymbol{a}}(\log B) + W^*_{\boldsymbol{a}}B + O_A\Big(B^{1+5\varepsilon}\xi^{1-8\varepsilon} + B^{1-2\gamma+13\varepsilon}\delta^{-\frac{567\gamma}{2-27\gamma}-7\varepsilon}\Big)$$

for a certain $W^*_{\boldsymbol{a}}\in\mathbb{R}$. Choosing $\delta = B^{-\frac{\gamma(2-27\gamma)}{1+270\gamma}}$ we obtain

$$K^*_{\boldsymbol{a}}(B) := BP^*_{\boldsymbol{a}}(\log B)^3 + W^*_{\boldsymbol{a}}B + O_A\Big(B^{1-\frac{\gamma(2-27\gamma)}{1+270\gamma}+13\varepsilon}\Big)$$

In particular, by (6.7), (6.8) and Lemma 7 and recalling that $\gamma = \frac{391-\sqrt{152737}}{108}$ we obtain Theorem 2 for all $\xi_2 \leqslant \frac{1165-3\sqrt{152737}}{3264} = 0.00228169\ldots$ □

7. Proof of Theorem 1

By (2.3) and renaming for simplicity $z_6 = d_1$, $z_5 = d_2$, $z_3 = d_3$ and z_4 by z_3 we have to count the solutions to

$$x_1d_1 + x_2d_2 + x_3d_3 = 0$$

where $\boldsymbol{x}\in\mathbb{Z}^3_{\neq0}, \boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3$ with $\boldsymbol{x} = (x_1,x_2,x_3), \boldsymbol{d} = (d_1,d_2,d_3), \boldsymbol{z} = (z_1,z_2,z_3)$ subject to the coprimality conditions

$$\begin{aligned}(d_i,d_j) = (z_i,z_j) = (d_k,z_k) = 1 \quad &\forall\ i,j,k\in\{1,2,3\},\ i\neq j,\\ (x_1,x_2,x_3) = (x_i,x_j,z_k) = 1 \quad &\forall i,j,k \text{ such that } \{i,j,k\} = \{1,2,3\},\end{aligned} \tag{7.1}$$

and

$$\max_{1\leqslant i,j\leqslant 3}\left\{|x_iz_i|^2 d_1d_2d_3\frac{z_j}{d_j}\right\} \leqslant B.$$

Let $0<\delta<\frac12$. This parametrization and (2.3) then imply that we just need to consider

$$N^{\pm}_{\delta}(B) := 2 \sideset{}{'}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^3_{\neq0},\boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3\\ x_1d_1+x_2d_2+x_3d_3=0}} \prod_{1\leqslant i,j\leqslant 3} F^{\pm}_{\delta}\Big((x_iz_i)^2\frac{d_1d_2d_3}{d_j}\frac{z_j}{B}\Big), \tag{7.2}$$

where $\sum'$ indicates the coprimality conditions (7.1), since for all $\delta>0$ we have

$$N^-_\delta(B) \leqslant N_{\widetilde{W}_3,\widetilde{H}}(B) \leqslant N^+_\delta(B).$$

We shall prove

$$N^{\pm}_{\delta}(B) = BP_1(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C\varepsilon} + B^{1-K}\delta^{-C}\big) \tag{7.3}$$

where P_1 is a polynomial of degree 4 with leading coefficient $\frac{\mathfrak{S}_1\cdot\mathcal{I}}{144}$ with the notations of Theorem 1 and for some $C,K>0$ and $\varepsilon>0$ small enough, so that choosing $\delta = B^{-K/(C+1)}$, we obtain Theorem 1.

7.1. **Initial manipulations.** We write the $F_\delta^\pm$ in term of its Mellin transform using the variable s_{ij} for the cut-off function corresponding to (i,j) and choosing

$$c_{s_{1j}} = \tfrac{1}{9} + \varepsilon, \quad c_{s_{2j}} = \tfrac{1}{9} + 4\varepsilon, \quad c_{s_{3j}} = \tfrac{1}{9} + 6\varepsilon$$

as lines of integration for all $j \in \{1,2,3\}$. We obtain

$$\begin{aligned} N_\delta^\pm(B) &= 2 \sideset{}{'}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^3_{\neq 0}, \boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3 \\ x_1d_1+x_2d_2+x_3d_3=0}} \frac{1}{(2\pi i)^9} \iint_{(c_{s_{ij}})} \frac{B^{\sum_{i,j} s_{ij}}}{\prod_k x_k^{2\sum_j s_{kj}} d_k^{\sum_{i,j\, j\neq k} s_{ij}} z_k^{2\sum_j s_{kj} + \sum_i s_{i,k}}} \prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \mathrm{d}s_{ij} \\ &= 2 \sideset{}{'}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^3_{\neq 0}, \boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3 \\ x_1d_1+x_2d_2+x_3d_3=0}} \frac{1}{(2\pi i)^9} \iint_{(c_{s_{ij}})} \frac{B^{s^*}}{\prod_k x_k^{\alpha_k-\xi_k} d_k^{\alpha_k+\xi_k} z_k^{s^*-2\xi_k}} \prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \mathrm{d}s_{ij} \end{aligned}$$

where

$$\alpha_k := \frac{1}{2} \sum_{1\leqslant i,j\leqslant 3} s_{ij} + \sum_{1\leqslant j\leqslant 3} s_{kj} - \frac{1}{2} \sum_{1\leqslant i\leqslant 3} s_{ik}, \qquad \xi_k := \frac{1}{2} \sum_{\substack{1\leqslant i,j\leqslant 3 \\ j\neq k}} s_{ij} - \sum_{1\leqslant j\leqslant 3} s_{kj}, \qquad \forall k \in \{1,2,3\}$$

$$s^* := \sum_{1\leqslant i,j\leqslant 3} s_{ij}$$

so that

$$\alpha_k - \xi_k = 2 \sum_{1\leqslant j\leqslant 3} s_{kj}, \qquad \alpha_k + \xi_k = \sum_{\substack{1\leqslant i,j\leqslant 3 \\ j\neq k}} s_{ij}.$$

Note that we have

$$\xi_1 + \xi_2 + \xi_3 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 2s^* \tag{7.4}$$

and that, like in the proof of Theorem 2, we are neglecting the dependencies of these notations on variables s_{ij} in order to simplify the exposition. Also, notice that with the above lines of integration we have

$$\begin{aligned} &\Re(\xi_1) = 8\varepsilon, \quad \Re(\xi_2) = -\varepsilon, \quad \Re(\xi_3) = -7\varepsilon, \qquad \Re(s^*) = 1 + 33\varepsilon \\ &\Re(\alpha_1) = \frac{2}{3} + 14\varepsilon, \quad \Re(\alpha_2) = \frac{2}{3} + 23\varepsilon, \quad \Re(\alpha_3) = \frac{2}{3} + 29\varepsilon \end{aligned} \tag{7.5}$$

so that in particular the above series are absolutely convergent by Lemma 2.

We make a change of variables, discarding the variables $s_{11}, s_{22}, s_{13}, s_{23}, s_{33}$, and introducing the variables $\alpha_1, \alpha_2, \xi_1, \xi_2$ and s^*. The inverse transformations are

$$\begin{aligned} &s_{11} = s^* - \alpha_1 - \xi_1 - s_{21} - s_{31}, && s_{22} = s^* - \alpha_2 - \xi_2 - s_{12} - s_{32}, \\ &s_{13} = -s^* + \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\xi_1 - s_{12} + s_{21} + s_{31}, && s_{23} = -s^* + \tfrac{3}{2}\alpha_2 + \tfrac{1}{2}\xi_2 + s_{12} - s_{21} + s_{32}, \\ &s_{33} = s^* - \tfrac{1}{2}(\alpha_1 + \alpha_2 - \xi_1 - \xi_2) - s_{31} - s_{32} \end{aligned} \tag{7.6}$$

and the Jacobian is equal to 1. In the following, to simplify the exposition, we shall keep using also the older variables (as well as ξ_3 and α_3 given by (7.4)), treating them as function of the new ones.

7.2. **Resolving the coprimality conditions.** For $\Re(\alpha_k \pm \xi_k) > \frac{2}{3}$ and $\Re(s^* - 2\xi_k) > 1$, using Möbius inversion formula to remove the coprimality conditions (7.1) (this is lemma 2.1 of [8]) we obtain

$$\begin{aligned}
&\sideset{}{'}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^3_{\neq 0},\boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3\\ x_1d_1+x_2d_2+x_3d_3=0}} \prod_{k=1}^{3} \frac{1}{x_k^{\alpha_k-\xi_k} d_k^{\alpha_k+\xi_k} z_k^{s^*-2\xi_k}}\\
&= \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^3_{\neq 0},\boldsymbol{d},\boldsymbol{z}\in\mathbb{N}^3\\ \sum_{k=1}^3 r_kx_kd_k=0}} \mu(h) \prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{(r_{1,k}x_k)^{\alpha_k-\xi_k}(r_{2,k}d_k)^{\alpha_k+\xi_k}(r_{3,k}z_k)^{s^*-2\xi_k}} \qquad (7.7)\\
&= \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \mu(h)\Bigg(\prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k-\xi_k} r_{2,k}^{\alpha_k+\xi_k} r_{3,k}^{s^*-2\xi_k}}\zeta(s^*-2\xi_k)\Bigg)\mathcal{A}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi}),
\end{aligned}$$

with the notation of Lemma 2 and where for $\{i,j,k\}=\{1,2,3\}$ we defined

$$r_{1,k} := [g_i,g_j,h], \qquad r_{2,k} := [b_i,b_j,f_k], \qquad r_{3,k} := [c_i,c_j,f_k,g_k], \qquad r_k := r_{1,k}r_{2,k}. \qquad (7.8)$$

For future use we also observe that for $\sigma \geqslant \frac{1}{2}+\varepsilon$, with $\varepsilon>0$, we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \prod_{k=1}^{3} \frac{1}{(r_{1,k}r_{2,k}r_{3,k})^{\sigma}} &= \prod_p \Bigg(\sum_{\substack{\boldsymbol{b}',\boldsymbol{c}',\boldsymbol{f}',\boldsymbol{g}'\in\mathbb{N}^3,\\ h'\in\mathbb{N}}} p^{-\sigma\sum_k\left(\max(g_i',g_j',h')+\max(b_i',b_j',f_k')+\max(c_i',c_j',f_k',g_k')\right)}\Bigg)\\
&= \prod_p (1+O(p^{-2\sigma})) \ll 1, \qquad (7.9)
\end{aligned}$$

where, in the sum over k in the first line, i,j are such that $\{i,j,k\}=\{1,2,3\}$.

Now, by (7.7), we have

$$\begin{aligned}
N_\delta^{\pm}(B) = 2\sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \frac{\mu(h)}{(2\pi i)^9} \iint_{(\cdots)} B^{s^*} \Bigg(\prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k-\xi_k} r_{2,k}^{\alpha_k+\xi_k} r_{3,k}^{s^*-2\xi_k}}\zeta(s^*-2\xi_k)\Bigg)&\\
\times \mathcal{A}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi})\Bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\Bigg)\mathrm{d}(\cdots),&
\end{aligned}$$

where, here and below, we indicate by $\int_{(\cdots)}\mathrm{d}(\cdots)$ an integral with respect to the variables s^*,α_1,α_2,δ_1,δ_2 and s_{12},s_{21},s_{31},s_{32}, along the lines of integration previously indicated, with the exclusion of the variables which have been eliminated by the computation of a residue.

7.3. **Applying Lemma 2.** We write $\mathcal{A}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi})$ as $\mathcal{M}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi})+\mathcal{E}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi})$ and we split accordingly $N_\delta^{\pm}(B)$ into

$$N_\delta^{\pm}(B) = M_\delta^{\pm}(B) + E_\delta^{\pm}(B). \qquad (7.10)$$

Differently from the case of Theorem 2, here $E_\delta^{\pm}(B)$ also contributes to a main term, of size B, which can be extracted as follow.

We move the lines of integration c_{ξ_1}, c_{ξ_2} and c_{s^*} in the integrals defining $E_\delta^{\pm}(B)$ to $c_{\xi_1}=2K$, $c_{\xi_2}=-K$ and $c_{s^*}=1-K$ for some fixed real number $K>0$ small enough, passing through the simple pole of the integrand at $s^*=1+2\xi_1$. If K is sufficiently small then we don't pass through any other pole and we stay inside the region (4.3) where $\mathcal{E}_{\boldsymbol{r}}$ is holomorphic and where the sums are absolutely convergent. For the integral on the new lines of integration we use (4.4) and a trivial bound for ζ and we obtain that, for K small enough, the integral is bounded by

$$\begin{aligned}&\ll B^{1-K} \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \iint_{(\cdots)} \left(\prod_{k=1}^{3} \frac{(r_k(|s^*|+|\xi_k|)(|\alpha_k|+|\xi_k|))^{C_1K}}{(r_{1,k}r_{2,k}r_{3,k})^{\frac{2}{3}-K}} \right) \Big| \prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \Big| \mathrm{d}(\cdots) \\ &\ll B^{1-K}\delta^{-C_2K},\end{aligned}$$

where the second line is obtained as for (5.10) using (4.6) and (7.9), after reintroducing the original variables s_{ij}. Also, we remind that, here and below, $C_1, C_2, C_3, \ldots$ will indicate fixed positive real numbers.

Collecting the contribution of the residue we obtain

$$\begin{aligned}E_\delta^\pm(B) = 2 \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \frac{\mu(h)}{(2\pi i)^8} \iint_{(\cdots)} B^{1+2\xi_1} \left(\prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k-\xi_k} r_{2,k}^{\alpha_k+\xi_k} r_{3,k}^{1+2\xi_1-2\xi_k}} \right) \zeta(1+2\xi_1-2\xi_2)\zeta(1+4\xi_1+2\xi_2) \\ \times \mathcal{E}_{\boldsymbol{r}}(\boldsymbol{\alpha}-\boldsymbol{\xi},\boldsymbol{\alpha}+\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \Big) \mathrm{d}(\cdots) + O\big(B^{1-K}\delta^{-C_2K}\big),\end{aligned}$$

since $\xi_3 = -\xi_1 - \xi_2$ and with the notations (7.6). We then move the line of integration c_{ξ_1} to $c_{\xi_1} = -K/2$ passing through the pole at $\xi_1 = -\xi_2/2$. The integral on the new lines of integration can be bounded as above, whereas in the integral coming from the contribution of the residue, we move c_{ξ_2} to $c_{\xi_2} = K$ passing through a simple pole at $\xi_2 = 0$ (in which case $\xi_3 = 0$). Bounding once again the contribution of the integral as above we arrive to

$$\begin{aligned}E_\delta^\pm(B) = \frac{B}{6} \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \frac{\mu(h)}{(2\pi i)^6} \iint_{(\cdots)} \left(\prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k} r_{2,k}^{\alpha_k} r_{3,k}} \right) \mathcal{E}_{\boldsymbol{r}}(\boldsymbol{\alpha},\boldsymbol{\alpha}) \Big(\prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \Big) \mathrm{d}(\cdots) \\ + O\big(B^{1-K}\delta^{-C_3K}\big).\end{aligned}$$

The product $\prod_{i,j} \tilde{F}_\delta^\pm(s_{ij})$ can now be replaced by $\prod_{i,j} \frac{1}{s_{ij}}$ at a cost of an error which is $O\big(B\delta^{1-C_5\varepsilon}\big)$. Indeed, by (4.4) and (7.9) we have

$$\begin{aligned}&\sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h\in\mathbb{N}}} \frac{\mu(h)}{(2\pi i)^6} \iint_{(\cdots)} \left(\prod_{k=1}^{3} \frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k} r_{2,k}^{\alpha_k} r_{3,k}} \right) \mathcal{E}_{\boldsymbol{r}}(\boldsymbol{\alpha},\boldsymbol{\alpha}) \Big(\prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) - \frac{1}{s_{ij}} \Big) \mathrm{d}(\cdots) \\ &\qquad \ll \iint_{(\cdots)} \Big(\prod_{i,j} \Big| \tilde{F}_\delta^\pm(s_{ij}) - \frac{1}{s_{ij}} \Big| |s_{ij}|^{C_4\varepsilon} \Big) \mathrm{d}(\cdots) \ll \delta^{1-C_5\varepsilon},\end{aligned}$$

by proceeding as in (5.16) after reintroducing six of the variables s_{ij} (with the remaining three variables kept as functions of those), since we now have the extra relation $\sum_{1\leqslant i,j\leqslant 3} s_{ij} = s^* = 1$. Collecting the above computations, we then get

$$E_\delta^\pm(B) = BP_0 + O\big(B\delta^{1-C_5\varepsilon} + B^{1-K}\delta^{-C_6K}\big) \tag{7.11}$$

for some $P_0 \in \mathbb{R}$.

We now move to the analysis of $M_\delta^\pm(B)$. Following the definition of $\mathcal{M}_{\boldsymbol{r}}$, we split $M_\delta^\pm(B)$ in the following way

$$M_\delta^\pm(B) = \sum_{\boldsymbol{\epsilon}\in\{\pm1\}^3} M_{\delta,\boldsymbol{\epsilon}}^\pm(B), \tag{7.12}$$

where the sum is over $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm1\}^3$ and

$$M_{\delta,\boldsymbol{\epsilon}}^\pm(B) := \frac{2}{(2\pi i)^9} \iint_{(\cdots)} B^{s^*} \frac{\prod_k \zeta(s^*-2\xi_k)\zeta(1+2\epsilon_k\xi_k)}{2s^* - \epsilon_1\xi_1 - \epsilon_2\xi_2 - \epsilon_3\xi_3 - 2} \mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^\pm(s_{ij}) \Big) \mathrm{d}(\cdots) \tag{7.13}$$

with lines of integration as given in Section 7.1 (in particular (7.5) is satisfied) and where

$$\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi}) := \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h,\ell\in\mathbb{N}}} \mu(h)\bigg(\prod_{k=1}^{3}\frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}^{\alpha_k-\xi_k} r_{2,k}^{\alpha_k+\xi_k} r_{3,k}^{s^*-2\xi_k}}\bigg)\frac{(r_1,\ell)^{1+2\epsilon_1\xi_1}(r_2,\ell)^{1+2\epsilon_2\xi_2}(r_3,\ell)^{1+2\epsilon_3\xi_3}}{\ell^{3+2\epsilon_1\xi_1+2\epsilon_2\xi_2+2\epsilon_3\xi_3}}\varphi(\ell)$$
$$\times 2\pi^{\frac12}\prod_{i=1}^{3}\frac{\Gamma(\frac{-\alpha_i+\epsilon_i\xi_i}{2}+\frac{1+2s^*-\epsilon_1\xi_1-\epsilon_2\xi_2-\epsilon_3\xi_3}{6})}{r_i^{-\alpha_i+\epsilon_i\xi_i+\frac{1+2s^*-\epsilon_1\xi_1-\epsilon_2\xi_2-\epsilon_3\xi_3}{3}}\Gamma(\frac{1+\alpha_i-\epsilon_i\xi_i}{2}-\frac{1+2s^*-\epsilon_1\xi_1-\epsilon_2\xi_2-\epsilon_3\xi_3}{6})}$$

where we recall that $2s^*=\alpha_1+\alpha_2+\alpha_3$ and the notations (7.6). Notice that by (7.9) and (5.14) we have that $\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})$ is holomorphic and bounded for

$$|\Re(s^*-1)|, |\Re(\xi_i)|, |\Re(\alpha_i-\tfrac23)| < 20K,$$

with K small enough.

Now, we move the lines of integration $c_{s^*}, c_{\xi_1}, c_{\xi_2}$ in (7.13) to $c_{s^*}=1-K$, $c_{\xi_1}=16K$, $c_{\xi_2}=-14K$ (so that on the new lines of integration $\Re(\xi_3)=-2K$ since $\xi_1+\xi_2+\xi_3=0$) passing through simple poles at $s^*=1+2\xi_1$ and, if $\boldsymbol{\epsilon}\neq(-1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, at $s^*=1+\frac12(\epsilon_1\xi_1+\epsilon_2\xi_2+\epsilon_3\xi_3)$. Indeed, the denominator has positive real part on the original lines of integrations whereas on the new lines of integrations it has real part equal to $2(-K-8\epsilon_1K+7\epsilon_2K+\epsilon_3K)$ which is negative if and only if $\epsilon_1=1$ or $\epsilon_1=\epsilon_2=\epsilon_3=-1$. Also note that we stay on the same side of the poles of the other ζ factors. Alluding to (4.6), a trivial bound for ζ and the fact that $\mathcal{Q}_{\boldsymbol{\epsilon}}$ is bounded, we can use the same argument used several times in Sections 5-6 in order to bound trivially the contribution of the integral on the new lines of integration obtaining that its contribution is $O\big(B^{1-K/2}\delta^{-C_7}\big)$.

It follows that

$$M^{\pm}_{\boldsymbol{\epsilon}}(B)=M^{\pm}_{\boldsymbol{\epsilon},1}(B)+M^{\pm}_{\boldsymbol{\epsilon},2}(B),$$

where $M^{\pm}_{\boldsymbol{\epsilon},1}$ denotes the contribution of the pole at $s^*=1+2\xi_1$ and $M^{\pm}_{\boldsymbol{\epsilon},2}(B)$ is the contribution of the pole at $s^*=1+\frac12(\epsilon_1\xi_1+\epsilon_2\xi_2+\epsilon_3\xi_3)$ if $\boldsymbol{\epsilon}\notin\{(-1,1,1),(-1,-1,1),(-1,1,-1)\}$ and $M^{\pm}_{\boldsymbol{\epsilon},2}(B):=0$ otherwise.

7.4. **The pole at $s^*=1+2\xi_1$ when $\boldsymbol{\epsilon}\neq(1,-1,1)$.** Using the fact that $\xi_3=-\xi_1-\xi_2$ we have

$$M^{\pm}_{\boldsymbol{\epsilon},1}(B)=\frac{2}{(2\pi i)^8}\iint_{(\cdots)}B^{1+2\xi_1}\frac{\prod_{k\neq1}\zeta(1+2\xi_1-2\xi_k)\prod_{k=1}^{3}\zeta(1+2\epsilon_k\xi_k)}{(4-\epsilon_1)\xi_1-\epsilon_2\xi_2-\epsilon_3\xi_3}$$
$$\times\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde F^{\pm}_{\delta}(s_{ij})\bigg)\,\mathrm{d}(\cdots)$$
$$=\frac{2}{(2\pi i)^8}\iint_{(\cdots)}B^{1+2\xi_1}\frac{\zeta(1+2\epsilon_1\xi_1)\zeta(1+2\xi_1-2\xi_2)\zeta(1+2\epsilon_2\xi_2)}{(4-\epsilon_1+\epsilon_3)\xi_1+(-\epsilon_2+\epsilon_3)\xi_2}\tag{7.14}$$
$$\times\zeta(1+4\xi_1+2\xi_2)\zeta(1-2\epsilon_3(\xi_1+\xi_2))\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde F^{\pm}_{\delta}(s_{ij})\bigg)\,\mathrm{d}(\cdots),$$

with $c_{\xi_1}=8\varepsilon$ and $c_{\xi_2}=-\varepsilon$. Notice that for $\boldsymbol{\epsilon}=(1,-1,1)$ one has a double pole when $4\xi_1+2\xi_2=0$ which causes some (mostly notational) issues when moving the lines of integration as we shall do throughout this section. For this reason, we prefer to defer to the next section the treatment of this term.

Next, we move the lines of integration c_{ξ_1} and c_{ξ_2} to $c_{\xi_1}=-K$ and $c_{\xi_2}=-4K$ passing through several poles. As before, the integral on the new lines of integration is $O\big(B^{1-K}\delta^{-C_8}\big)$. The poles we encounter are the following:

(a) a pole at $\xi_1=0$ which is simple if $\epsilon_2\neq\epsilon_3$ and is double if $\epsilon_2=\epsilon_3$;
(b) a simple pole at $\xi_1=-\frac12\xi_2$;
(c) a simple pole at $\xi_1=-\xi_2$;

(d) a simple pole at $\xi_1 = -\frac{1}{3}\xi_2$ if $\boldsymbol{\epsilon} = (-1,-1,1)$.

We now examine the contribution of the residue of each of these poles.

(a) We write the contribution of the residue at $\xi_1 = 0$ as a small circuit integral

$$\frac{2B}{(2\pi i)^7}\iint_{(\cdots)}\frac{1}{2\pi i}\oint_{|\xi_1|=\varepsilon/2}\Bigg((1+2\xi_1\log B)\frac{\zeta(1+2\epsilon_1\xi_1)\zeta(1+2\xi_1-2\xi_2)\zeta(1+2\epsilon_2\xi_2)}{(4-\epsilon_1+\epsilon_3)\xi_1+(-\epsilon_2+\epsilon_3)\xi_2} \times\zeta(1+4\xi_1+2\xi_2)\zeta(1\mp_3 2(\xi_1+\xi_2))\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\bigg)\Bigg)\mathrm{d}(\cdots)$$

where we can assume that the line of integration c_{ξ_2} is at $c_{\xi_2} = -\varepsilon$. The next step is to observe that we can replace $\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})$ by $\prod_{i,j}\frac{1}{s_{ij}}$ at a cost of an error which is

$$O\big(B^{1+\varepsilon}\delta^{1-C_9\varepsilon}\big). \tag{7.15}$$

To show this we first observe that, by the convexity bound [32, (5.1.4)], on the lines of integration the integrand is

$$\ll \log B\,(1+|\xi_1|+|\xi_2|)^{C_{10}\varepsilon}\prod_{i,j}|\tilde{F}_\delta^{\pm}(s_{ij})| \ll \log B\prod_{i,j}|\tilde{F}_\delta^{\pm}(s_{ij})|(1+|s_{ij}|)^{C_{11}\varepsilon}.$$

We go back to using the s_{ij} as variables (excluding for example the variable s_{11} because we have a variable less) and observe we have $s^* = \sum_{1\leqslant i,j\leqslant 3}s_{ij} = 1+2\xi_1 = 1+O(\varepsilon)$ and thus $s_{11} = 1-\sum_{(i,j)\neq(1,1)}s_{ij}+O(\varepsilon)$. Thus, proceeding as for (5.16) we can replace $\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})$ by $\prod_{i,j}\frac{1}{s_{ij}}$ at a cost of an error which is bounded by (7.15). In the end, we find that the contribution of the pole at $\xi_1 = 0$ is

$$BP_{\boldsymbol{\epsilon},1,1}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_9\varepsilon} + B^{1-K}\delta^{-C_8}\big)$$

where $P_{\boldsymbol{\epsilon},1,1}$ is a polynomial of degree 0 or 1 (not depending on the choice for δ and $\pm$ in $N_\delta^{\pm}(B)$) obtained by evaluating the above integral with the $\prod_{i,j}\frac{1}{s_{ij}}$ instead of $\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})$.

(b) The contribution of the pole at $\xi_1 = -\frac{1}{2}\xi_2$ is

$$\frac{-1}{(2\pi i)^7}\iint_{(\cdots)}B^{1-\xi_2}\frac{\zeta(1-\epsilon_1\xi_2)\zeta(1-3\xi_2)\zeta(1+2\epsilon_2\xi_2)\zeta(1-\epsilon_3\xi_2)}{(4-\epsilon_1+2\epsilon_2-\epsilon_3)\xi_2}\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\bigg)\mathrm{d}(\cdots).$$

We move the line of integration c_{ξ_2} to $c_{\xi_2} = K$ passing through a pole at $\delta_2 = 0$. The contribution of the integral on the new lines of integration is $O\big(B^{1-K}\delta^{-C_{12}}\big)$. For the contribution of the residue, we follow the same approach as above writing it as a small circle integral and observing that since again $s^* = \sum_{1\leqslant i,j\leqslant 3}s_{ij} = 1+2\xi_1 = 1-\xi_2 = 1+O(\varepsilon)$ we can replace $\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})$ by $\prod_{i,j}\frac{1}{s_{ij}}$ at the cost of an error which is $O\big(B^{1+\varepsilon}\delta^{1-C_{13}\varepsilon}\big)$. Then, computing the integral we find that the contribution to (7.14) from the pole at $\xi_1 = -\frac{1}{2}\xi_2$ is

$$BP_{\boldsymbol{\epsilon},1,2}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{13}\varepsilon} + B^{1-K}\delta^{-C_{12}}\big)$$

where $P_{\boldsymbol{\epsilon},1,2}$ is a polynomial of degree 4 of leading coefficient

$$-\frac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot 3\cdot(4-\epsilon_1+2\epsilon_2-\epsilon_3)}\mathcal{J},$$

where

$$\mathcal{J} := \frac{1}{4!}\frac{1}{(2\pi i)^6}\iint_{(\cdots)}\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\mathbf{0})\bigg(\prod_{i,j}\frac{1}{s_{ij}}\bigg)\mathrm{d}(\cdots). \tag{7.16}$$

It is noteworthy that, since $\boldsymbol{\xi} = \mathbf{0}$, $\mathcal{J}$ does not depend on $\boldsymbol{\epsilon}$. Moreover, we will see below with (7.17) that $\mathcal{J} \neq 0$.

(c) The contribution of the pole at $\xi_1=-\xi_2$ is

$$\frac{\epsilon_3}{(2\pi i)^7}\iint_{(\cdots)} B^{1-2\xi_2}\frac{\zeta(1-2\epsilon_1\xi_2)\zeta(1-4\xi_2)\zeta(1+2\epsilon_2\xi_2)\zeta(1-2\xi_2)}{(4-\epsilon_1+\epsilon_2)\xi_2}\mathcal{Q}_{\boldsymbol\epsilon}\big(\boldsymbol\alpha,(-\xi_2,\xi_2,0)\big)\bigg(\prod_{i,j}\tilde F_\delta^{\pm}(s_{ij})\bigg)\mathrm{d}(\cdots)$$

and proceeding as above we have that this is

$$BP_{\boldsymbol\epsilon,1,3}(\log B)+O\big(B^{1+\varepsilon}\delta^{1-C_{14}\varepsilon}+B^{1-K}\delta^{-C_{15}}\big)$$

where $P_{\boldsymbol\epsilon,1,3}$ is a polynomial of degree 4 of leading coefficient

$$\frac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot(4-\epsilon_1+\epsilon_2)}\mathcal{J},$$

with the notation (7.16).

(d) The contribution of the pole at $\xi_1=-\frac13\xi_2$ with $\boldsymbol\epsilon=(-1,-1,1)$ is

$$\frac13\frac{1}{(2\pi i)^7}\iint_{(\cdots)} B^{1-\frac23\xi_2}\zeta\big(1+\tfrac23\xi_2\big)\zeta\big(1-\tfrac83\xi_2\big)\zeta(1-2\xi_2)$$
$$\times\,\zeta\big(1+\tfrac23\xi_2\big)\zeta\big(1-\tfrac43\xi_2\big)\mathcal{Q}_{\boldsymbol\epsilon}(\boldsymbol\alpha,\boldsymbol\xi)\bigg(\prod_{i,j}\tilde F_\delta^{\pm}(s_{ij})\bigg)\mathrm{d}(\cdots)$$

and, proceeding as above, we have that this is

$$BP_{\boldsymbol\epsilon,1,4}(\log B)+O\big(B^{1+\varepsilon}\delta^{1-C_{16}\varepsilon}+B^{1-K}\delta^{-C_{17}}\big)$$

where $P_{\boldsymbol\epsilon,1,4}$ is a polynomial of degree 4 of leading coefficient

$$\frac{1}{2^4\cdot 3}\mathcal{J}.$$

Regrouping the above four contributions, we obtain that for $\boldsymbol\epsilon\neq(1,-1,1)$ we have

$$M^{\pm}_{\boldsymbol\alpha,\boldsymbol\epsilon,1}(B)=BP_{\boldsymbol\epsilon,1}(\log B)+O(B^{1+\varepsilon}\delta^{1-C_{18}\varepsilon}+B^{1-K}\delta^{-C_{19}})$$

where $P_{\boldsymbol\epsilon,1}$ is a degree 4 polynomial with leading coefficient

$$\mathcal{J}\times\begin{cases}-\dfrac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot3\cdot(4-\epsilon_1+2\epsilon_2-\epsilon_3)}+\dfrac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot(4-\epsilon_1+\epsilon_2)} & \text{if } \boldsymbol\epsilon\neq(-1,-1,1),(1,-1,1)\\[2mm] \dfrac{1}{2^4} & \text{if } \boldsymbol\epsilon=(-1,-1,1),\end{cases}$$

with the notation (7.16).

7.5. **The pole at $s^*=1+2\xi_1$ for $\boldsymbol\epsilon=(1,-1,1)$.** For brevity, in this section we write $\boldsymbol\epsilon^*:=(1,-1,1)$. We have

$$M^{\pm}_{\boldsymbol\epsilon^*,1}(B)=\frac{2}{(2\pi i)^8}\iint_{(\cdots)} B^{1+2\xi_1}\frac{\zeta(1+2\xi_1)\zeta(1+2\xi_1-2\xi_2)\zeta(1-2\xi_2)}{4\xi_1+2\xi_2}$$
$$\times\,\zeta(1+4\xi_1+2\xi_2)\zeta(1-2(\xi_1+\xi_2))\mathcal{Q}_{\boldsymbol\epsilon^*}(\boldsymbol\alpha,\boldsymbol\xi)\bigg(\prod_{i,j}\tilde F_\delta^{\pm}(s_{ij})\bigg)\mathrm{d}(\cdots).$$

Here, we move c_{ξ_2} to $c_{\xi_2}=2K$, passing through simple poles at $\xi_2=0$ and $\xi_2=\xi_1$. For the integral on the new lines of integration, we move c_{ξ_1} to $c_{\xi_1}=-K/2$ passing through a pole at $\xi_1=0$. If K is sufficiently small, the integral on these new lines of integration can then be estimated trivially by $O\big(B^{1-K}\delta^{-C_{20}}\big)$. Thus, overall we shall compute the following residues arising for the following poles

(a) a simple pole at $\xi_2=0$
(b) a simple pole $\xi_2=\xi_1$;
(c) a simple pole at $\delta_1=0$ (with $c_{\xi_2}>0$).

(a) The contribution of the residue at $\xi_2 = 0$ is

$$\frac{1}{(2\pi i)^7} \iint_{(\cdots)} B^{1+2\xi_1} \frac{\zeta(1+2\xi_1)^2\zeta(1+4\xi_1)\zeta(1-2\xi_1)}{4\xi_1} \mathcal{Q}_{\boldsymbol{\epsilon}^*}(\boldsymbol{\alpha},\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^{\pm}(s_{ij})\Big)\,\mathrm{d}(\cdots)$$

and, as in the previous section, one sees that this is

$$BP_{\boldsymbol{\epsilon}^*,1,4}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{21}\varepsilon} + B^{1-K}\delta^{-C_{22}}\big)$$

with $P_{\boldsymbol{\epsilon}^*,4,1}$ of degree 4 with leading coefficient

$$-\frac{1}{2^3}\mathcal{J},$$

with the notation (7.16).

(b) The contribution of the residue at $\xi_2 = \xi_1$ is

$$\frac{1}{(2\pi i)^7} \iint_{(\cdots)} B^{1+2\xi_1} \frac{\zeta(1+2\xi_1)\zeta(1-2\xi_1)\zeta(1+6\xi_1)\zeta(1-4\xi_1)}{6\xi_1} \mathcal{Q}_{\boldsymbol{\epsilon}^*}(\boldsymbol{\alpha},\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^{\pm}(s_{ij})\Big)\,\mathrm{d}(\cdots)$$
$$= BP_{\boldsymbol{\epsilon}^*,1,2}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{23}\varepsilon} + B^{1-K}\delta^{-C_{24}}\big)$$

with $P_{\epsilon^*,1,2}$ of degree 4 with leading coefficient

$$\frac{1}{2^2\cdot 3^2}\mathcal{J}.$$

(c) The contribution of the residue at $\xi_1 = 0$ is

$$\frac{1}{(2\pi i)^7} \iint_{(\cdots)} B \frac{\zeta(1-2\xi_2)^3\zeta(1+2\xi_2)}{2\xi_2} \mathcal{Q}_{\boldsymbol{\epsilon}^*}(\boldsymbol{\alpha},\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^{\pm}(s_{ij})\Big)\,\mathrm{d}(\cdots)$$
$$= P_{\boldsymbol{\epsilon}^*,1,3}B + O\big(B^{1+\varepsilon}\delta^{1-C_{25}\varepsilon} + B^{1-K}\delta^{-C_{26}}\big)$$

with $P_{\boldsymbol{\epsilon}^*,1,3} \in \mathbb{R}$.

Collecting the various terms, we then find that

$$M^{\pm}_{\boldsymbol{\epsilon}^*,1}(B) = BP_{\boldsymbol{\epsilon}^*,1}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{27}\varepsilon} + B^{1-K}\delta^{-C_{28}}\big)$$

where $P_{\boldsymbol{\epsilon}^*,1}$ is a polynomial of degree 4 with leading coefficient

$$-\frac{7}{2^3\cdot 3^2}\mathcal{J}.$$

7.6. **The pole at** $s^* = 1+\frac{1}{2}(\epsilon_1\xi_1+\epsilon_2\xi_2+\epsilon_3\xi_3)$**.** Recall that $M^{\pm}_{\boldsymbol{\epsilon},2}(B) := 0$ if $\boldsymbol{\epsilon} = (-1,1,1), (-1,-1,1), (-1,1,-1)$. In all other cases we have

$$M^{\pm}_{\boldsymbol{\epsilon},2}(B) = \frac{1}{(2\pi i)^8} \iint_{(\cdots)} B^{1+\frac{1}{2}(\epsilon_1\xi_1+\epsilon_2\xi_2+\epsilon_3\xi_3)} \prod_k \zeta\Big(1+\frac{1}{2}(\epsilon_1\xi_1+\epsilon_2\xi_2+\epsilon_3\xi_3) - 2\xi_k\Big)\zeta(1+2\epsilon_k\xi_k)$$
$$\times \mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi}) \Big(\prod_{i,j} \tilde{F}_\delta^{\pm}(s_{ij})\Big)\,\mathrm{d}(\cdots).$$

If $\epsilon_1 = \epsilon_2 = \epsilon_3$, then the exponent of B is 1. In particular, since in this case we have the relation $s^* = \sum_{i,j} s_{ij} = 1$ because $\xi_1+\xi_2+\xi_3 = 0$, we can replace $\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})$ by $\prod_{i,j} s_{ij}^{-1}$ at a cost of an error which is $O\big(B^{1+\varepsilon}\delta^{1-C_{25}\varepsilon}\big)$ like in Section 7.3. Thus, for $\boldsymbol{\epsilon} = (1,1,1), (-1,-1,-1)$ we have

$$M_{\boldsymbol{\epsilon},2^{\pm}(B)} = P_{\boldsymbol{\epsilon},2,1}B + O\big(B^{1+\varepsilon}\delta^{1-C_{29}\varepsilon}\big)$$

for some $P_{\boldsymbol{\epsilon},2,1} \in \mathbb{R}$.

Therefore, we are left with considering the cases where $\epsilon_r = -\epsilon_{k_1} = -\epsilon_{k_2}$ with $\{r, k_1, k_2\} = \{1, 2, 3\}$ and $k_1 < k_2$. Notice in particular that since $\xi_1 + \xi_2 + \xi_3 = 0$, then we have $\epsilon_1\xi_1 + \epsilon_2\xi_2 + \epsilon_3\xi_3 = 2\epsilon_r\xi_r$. Thus,

$$M_{\boldsymbol{\epsilon},2^{\pm}(B)} = \frac{1}{(2\pi i)^8}\iint_{(\cdots)} B^{1+\epsilon_r\xi_r}\zeta(1+(\epsilon_r-2)\xi_r)\zeta(1+2\epsilon_r\xi_r)\zeta(1+\epsilon_r\xi_r-2\xi_{k_1})\zeta(1+2\epsilon_{k_1}\xi_{k_1})$$
$$\times\,\zeta(1+(\epsilon_r+2)\xi_r+2\xi_{k_1})\zeta(1-2\epsilon_{k_2}(\xi_r+\xi_{k_1}))\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\bigg)\,\mathrm{d}(\cdots).$$

Notice that we made a change of variables (of jacobian ±1, since $\xi_1+\xi_2+\xi_3=0$) using ξ_r, ξ_{k_1} rather than ξ_1, ξ_2.

Next, we move the lines of integration c_{ξ_r}, $c_{\xi_{k_1}}$ to $c_{\xi_r} = c_{\xi_{k_1}} = -\epsilon_r K$. In doing so we pass through a double pole at $\xi_r = 0$ and, if $r = 1$, through the simple poles of the ζ factors on the second line at $\xi_1 = -\frac{2}{3}\xi_2$ and $\xi_1 = -\xi_2$. Thus, as in the previous sections, if $\boldsymbol{\epsilon} = (1,-1,1), (1,1,-1)$ we find

$$M_{\boldsymbol{\epsilon},2^{\pm}(B)} = BP_{\boldsymbol{\epsilon},2,2}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{30}\varepsilon} + B^{1-K}\delta^{-C_{31}}\big)$$

with $P_{\boldsymbol{\epsilon},2,2}(\log B)$ of degree at most 1 and the same holds for the contribution of the pole at $\xi_r = 0$ when $r = 1$. We are therefore left with computing the contributions of the two remaining poles when $\boldsymbol{\epsilon} = (1,-1,-1)$ and $(r, k_1, k_2) = (1, 2, 3)$. The contribution of the pole at $\xi_1 = -\frac{2}{3}\xi_2$ is

$$\frac{1}{3}\frac{1}{(2\pi i)^7}\iint_{(\cdots)} B^{1-\frac{2}{3}\xi_2}\zeta\big(1+\tfrac{2}{3}\xi_2\big)^2\zeta\big(1-\tfrac{4}{3}\xi_2\big)\zeta\big(1-\tfrac{8}{3}\xi_2\big)\zeta(1-2\xi_2)\mathcal{Q}_{(1,-1,-1)}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\bigg)\,\mathrm{d}(\cdots),$$

and this is $BP_{(1,-1,-1),2,3}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{31}\varepsilon} + B^{1-K}\delta^{-C_{32}}\big)$ with $P_{(1,-1,-1),2,3}$ a polynomial of degree 4 and leading coefficient

$$\frac{1}{2^4\cdot 3}\mathcal{J},$$

with the notation (7.16).

The contribution of the pole at $\xi_1 = -\xi_2$ is

$$\frac{1}{2}\frac{1}{(2\pi i)^7}\iint_{(\cdots)} B^{1-\xi_2}\zeta(1+\xi_2)\zeta(1-2\xi_2)^2\zeta(1-3\xi_2)\zeta(1-\xi_2)\mathcal{Q}_{(1,-1,-1)}(\boldsymbol{\alpha},\boldsymbol{\xi})\bigg(\prod_{i,j}\tilde{F}_\delta^{\pm}(s_{ij})\bigg)\,\mathrm{d}(\cdots).$$

and this is $BP_{(1,-1,-1),2,4}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{33}\varepsilon} + B^{1-K}\delta^{-C_{34}}\big)$ with $P_{(1,-1,-1),2,4}$ of degree 4 with leading coefficient

$$-\frac{1}{2^3\cdot 3}\mathcal{J}.$$

Thus, summarizing for all $\boldsymbol{\epsilon} \in \{1,-1\}^3$ we have

$$M^{\pm}_{\boldsymbol{\epsilon},2}(B) = BP_{\boldsymbol{\epsilon},2}(\log B) + O\big(B^{1+\varepsilon}\delta^{1-C_{35}\varepsilon} + B^{1-K}\delta^{-C_{36}}\big)$$

where $P_{\boldsymbol{\epsilon},2}$ is a polynomial of degree at most 1 unless $\boldsymbol{\epsilon} = (1,-1,-1)$ in which case $P_{\boldsymbol{\epsilon},2}$ is of degree 4 with leading coefficient

$$-\frac{1}{2^4\cdot 3}\mathcal{J}.$$

7.7. **Regrouping the various contributions.** By (7.10) and (7.11), (7.12) and regrouping the contributions from Sections 7.4, 7.5 and 7.6, we find

$$N^{\pm}_{\delta}(B) = B\sum_{\boldsymbol{\epsilon}\in\{1,-1\}^3}\big(P_{\boldsymbol{\epsilon},1}(\log B) + P_{\boldsymbol{\epsilon},2}(\log B)\big) + BP_0 + O\big(B^{1+\varepsilon}\delta^{1-C_{37}\varepsilon} + B^{1-K}\delta^{-C_{38}}\big)$$
$$= BP_1(\log B)) + O\big(B^{1+\varepsilon}\delta^{1-C_{37}\varepsilon} + B^{1-K}\delta^{-C_{38}}\big)$$

where P_1 is a polynomial of degree 4 with leading coefficient

$$\left(\frac{1}{16} - \frac{7}{72} - \frac{1}{48} + \sum_{\boldsymbol{\epsilon}\neq(-1,-1,1),(1,-1,1)} \left(\frac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot(4-\epsilon_1+\epsilon_2)} - \frac{\epsilon_1\epsilon_2\epsilon_3}{2\cdot 3\cdot(4-\epsilon_1+2\epsilon_2-\epsilon_3)}\right)\right)\mathcal{J} = \frac{1}{48}\mathcal{J}.$$

The estimate (7.3) and then the Theorem 1 then follows by the final next lemma.

Lemma 8. *With the notations of Theorem 1 and (7.16), we have* $\mathcal{J} = \frac{1}{3}\mathcal{I}\mathfrak{S}_1$.

Proof. First, we observe that for $s^* = 1$ we have

$$\begin{aligned}\mathcal{Q}_{\boldsymbol{\epsilon}}(\boldsymbol{\alpha}, \mathbf{0}) &= \sum_{\substack{\boldsymbol{b},\boldsymbol{c},\boldsymbol{f},\boldsymbol{g}\in\mathbb{N}^3,\\ h,\ell\in\mathbb{N}}} \mu(h)\left(\prod_{k=1}^{3}\frac{\mu(b_k)\mu(c_k)\mu(f_k)\mu(g_k)}{r_{1,k}\,r_{2,k}\,r_{3,k}}\right)\frac{(r_1,\ell)(r_2,\ell)(r_3,\ell)}{\ell^3}\varphi(\ell)\cdot 2\pi^{\frac{1}{2}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\Gamma(\frac{\alpha_i}{2})}\\ &= \mathfrak{S}_1\cdot 2\pi^{\frac{1}{2}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\Gamma(\frac{\alpha_i}{2})}\end{aligned}$$

where in the second row we computed the Euler product thanks to the lemma 2.7 of [8]. Therefore, we have

$$\mathcal{J} = \frac{\mathfrak{S}_1}{4!(2\pi i)^2}\iint_{(c_{\alpha_1},c_{\alpha_2})} 2\pi^{\frac{1}{2}}\left(\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\Gamma(\frac{\alpha_i}{2})}\right)\times\frac{1}{(2\pi i)^4}\int_{(c_{s_{12}},c_{s_{31}},c_{s_{32}},c_{s_{21}})}\frac{\mathrm{d}s_{12}\mathrm{d}s_{21}\mathrm{d}s_{31}\mathrm{d}s_{32}}{L(\boldsymbol{\alpha}, s_{12}, s_{21}, s_{31}, s_{32})}\mathrm{d}\alpha_1\mathrm{d}\alpha_2,$$

where the lines of integrations can be taken at $c_{s_{ij}} = \frac{1}{9}$, $c_{\alpha_1} = c_{\alpha_2} = \frac{2}{3}$ and, by (7.6),

$$\begin{aligned}L(\boldsymbol{\alpha}, s_{12}, s_{21}, s_{31}, s_{32}) &= (1-\alpha_1-s_{21}-s_{31})(1-\alpha_2-s_{12}-s_{32})(-1+\tfrac{3}{2}\alpha_1-s_{12}+s_{21}+s_{31})\\ &\quad\times(-1+\tfrac{3}{2}\alpha_2+s_{12}-s_{21}+s_{32})(1-\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2-s_{31}-s_{32})s_{12}s_{21}s_{31}s_{32}.\end{aligned}$$

In the same way as in the end of the proof of Proposition 1, one has that the inner integral over $s_{12}, s_{21}, s_{31}, s_{32}$ can be evaluated by moving each line of integration to $-\infty$ and collecting the residues of the poles encountered in the process. After this simple but a bit lengthy calculation, which can be easily performed with the help of Mathematica, one finds that the inner integral is equal to $8(\alpha_1\alpha_2\alpha_3(1-\alpha_1)(1-\alpha_2)(1-\alpha_3))^{-1}$, with $\alpha_3 = 2-\alpha_1-\alpha_2$. Thus, we finally get

$$\mathcal{J} = \frac{\mathfrak{S}_1}{3(2\pi i)^2}\iint_{(c_{\alpha_1},c_{\alpha_2})} 2\pi^{\frac{1}{2}}\prod_{i=1}^{3}\frac{\Gamma(\frac{1-\alpha_i}{2})}{\alpha_i(1-\alpha_i)\Gamma(\frac{\alpha_i}{2})}\,\mathrm{d}\alpha_2\mathrm{d}\alpha_3 = \frac{1}{3}\mathfrak{S}_1\cdot\mathcal{I} \tag{7.17}$$

by Lemma 4. □

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