Persistence modules and barcodes in symplectic geometry

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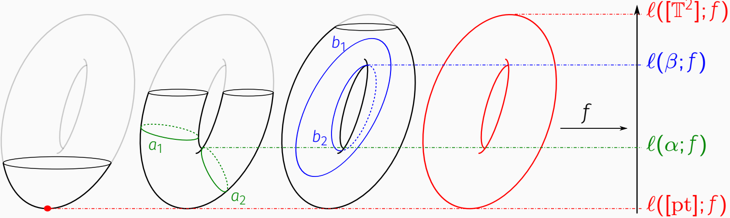
DataShape annual seminar, may 2023

Laboratoire de Mathématiques d’Orsay

Spectral invariants

Definition (Morse / finite dimension case)
Let $f\colon\ M \to \mathbb{R}$ be Morse and $\alpha \in \mathrm{H}_*(M)$, $\ell(\alpha, f)$ is defined by:
$\{x \in M \,|\, f(x) \leq \ell(\alpha, f)\}$ = smallest sublevel set containing a representative of α

Example



Remark

$\ell\colon\ \mathrm{H}_*(M) \times C^\infty_{\mathrm{Morse}}(M) \longrightarrow \mathbb{R} \cup \{-\infty\}$ + C^0 *continuity*

$$= \quad \ell\colon\ \mathrm{H}_*(M) \times C^0(M) \longrightarrow \mathbb{R} \cup \{-\infty\}$$

Persistence modules and barcodes

Persistence modules

Definition

$\mathbb{V} = (V^t)_{t\in\mathbb{R}} \subset \mathbb{Z}/2\mathbb{Z}$ is a persistence module if

- for all $s \leq t$ in $\mathbb{R}$, there exists a morphism $\iota_{t,s} : V^s \to V^t$,
- for all $r \leq s \leq t$ in $\mathbb{R}$, $\iota_{t,r} = \iota_{t,s} \circ \iota_{s,r}$,
- $\iota_{t,s}$ isomorphism up to finite spectrum $F \subset \mathbb{R}$.

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Example

A chain complex (C, ∂) is $\mathbb{R}$-filtered if

- for all t, C^t is a vector subspace of C s.t. $\partial(C^t) \subset C^t$,
- for all $s \leq t$, we have natural inclusions $\hat{\iota}_{t,s} : C^s \to C^t$.

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A filtered complex induces a persistence module $\mathbb{V}$ with $V^t = \mathrm{H}_*(C^t, \partial)$ and $\iota_{t,s} = \mathrm{H}_*(\hat{\iota}_{t,s})$.

$\rightsquigarrow$ $\mathrm{M}\mathbb{V}(f, g)$, $\mathrm{F}\mathbb{V}(H, J)$

Persistence modules

finite dim M	loop space $\mathcal{P}_0(M)$
$f \in C^0(M)$	$H \in C^0(\mathbb{S}^1 \times M)$
sublevel set M^t	
$M^t = f^{-1}(-\infty, t)$	??
$V^t = \mathrm{H}(M^t)$	

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$f \in C^0(M)$ sublevel set M^t $M^t = f^{-1}(-\infty, t)$ $V^t = \mathsf{H}(M^t)$	$H \in C^0(\mathbb{S}^1 \times M)$??
f **Morse** Morse subcomplex $\mathsf{MC}^t(f)$ $\mathsf{MC}^t(f) = \mathrm{Crit}(f) \cap f^{-1}(-\infty, t)$ $\partial(\mathsf{MC}^t(f)) \subset \mathsf{MC}^t(f)$ $\rightsquigarrow \mathrm{M}V^t = \mathsf{H}(\mathsf{MC}^t(f), \partial)$	

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f **Morse**	H **generic**
Morse subcomplex $\mathsf{MC}^t(f)$	Floer subcomplex $\mathsf{FC}^t(H)$
$\mathsf{MC}^t(f) = \mathrm{Crit}(f) \cap f^{-1}(-\infty, t)$	$\mathsf{FC}^t(H) = \mathrm{Crit}(\mathcal{A}_H) \cap \mathcal{A}_H^{-1}(-\infty, t)$
$\partial(\mathsf{MC}^t(f)) \subset \mathsf{MC}^t(f)$	$\partial(\mathsf{FC}^t(H)) \subset \mathsf{FC}^t(H)$
$\rightsquigarrow \mathrm{M}V^t = \mathsf{H}(\mathsf{MC}^t(f), \partial)$	$\rightsquigarrow \mathrm{F}V^t = \mathsf{H}(\mathsf{FC}^t(H), \partial)$

Persistence modules

Definition

A morphism of persistence modules $F : \mathbb{V} \to \mathbb{W}$ is a family of morphisms $F^t : V^t \to W^t$ s.t. for all $s \leq t$ in $\mathbb{R}$:

$$\begin{array}{ccc} V^s & \xrightarrow{\iota^{\mathbb{V}}_{t,s}} & V^t \\ \downarrow{\scriptstyle F^s} & & \downarrow{\scriptstyle F^t} \\ W^s & \xrightarrow{\iota^{\mathbb{W}}_{t,s}} & W^t \end{array}$$

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Definition

The interleaving distance between $\mathbb{V}$ and $\mathbb{W}$ is the infimum of ε s.t. there exist $\varphi : \mathbb{V} \to \mathbb{W}^{+\varepsilon}$ and $\psi : \mathbb{W} \to \mathbb{V}^{+\varepsilon}$ s.t.

$$\begin{array}{ccccccc} & & & \iota^{\mathbb{V}}_{s-\varepsilon,s+\varepsilon}: V^{s-\varepsilon} \to V^{s+\varepsilon} & & \iota^{\mathbb{W}}_{s,s+2\varepsilon}: W^{s} \to W^{s+2\varepsilon} & \\ V^{s-\varepsilon} & \xrightarrow{\varphi^{s-\varepsilon}} & W^s & \xrightarrow{\psi^s} & V^{s+\varepsilon} & \xrightarrow{\varphi^{s+\varepsilon}} & W^{s+2\varepsilon} \\ \downarrow \iota^{\mathbb{V}} & & \downarrow \iota^{\mathbb{W}} & & \downarrow \iota^{\mathbb{V}} & & \downarrow \iota^{\mathbb{W}} \\ V^{s-\varepsilon} & \xrightarrow{\varphi^{s-\varepsilon}} & W^s & \xrightarrow{\psi^s} & V^{s+\varepsilon} & \xrightarrow{\varphi^{s+\varepsilon}} & W^{s+2\varepsilon} \end{array}$$

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$$\begin{tikzcd} V^{s-\varepsilon} \arrow[r, "\varphi^{s-\varepsilon}"] \arrow[rr, bend left, "\iota^{\mathbb{V}}_{s-\varepsilon,s+\varepsilon}"] \arrow[d, "\iota^{\mathbb{V}}"] & W^s \arrow[r, "\psi^s"] \arrow[rr, bend left, "\iota^{\mathbb{W}}_{s,s+2\varepsilon}"] \arrow[d, "\iota^{\mathbb{W}}"] & V^{s+\varepsilon} \arrow[r, "\varphi^{s+\varepsilon}"] \arrow[d, "\iota^{\mathbb{V}}"] & W^{s+2\varepsilon} \arrow[d, "\iota^{\mathbb{W}}"] \\ V^{s-\varepsilon} \arrow[r, "\varphi^{s-\varepsilon}"] & W^s \arrow[r, "\psi^s"] & V^{s+\varepsilon} \arrow[r, "\varphi^{s+\varepsilon}"] & W^{s+2\varepsilon} \end{tikzcd}$$

Theorem (Chazal–Cohen-Steiner–Glisse–Guibas–Oudot, 09)
Pseudo-distance to $[0, +\infty]$ *s.t.* $d_{\mathrm{int}}(\mathbb{V}, \mathbb{W}) = 0 \Rightarrow \mathbb{V} \simeq \mathbb{W}$

Barcodes

Definition

A barcode $\mathbb{B}$ is an unordered, finite family of intervals of the type $(a, b]$ w/ $-\infty \leq a \leq b \leq +\infty$.

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- $(a_1, b_1]$ and $(a_2, b_2]$ are at distance $\max\{|a_1 - a_2|, |b_1 - b_2|\}$.
- B_1 and B_2 are ε-matched if there exists a bijection$^\varepsilon$ $\phi : \mathrm{B}_1 \to \mathrm{B}_2$, s.t. $d(I, \phi(I)) < \varepsilon$ for all I in B_1
  w/ bijection$^\varepsilon$ = bijection *up to intervals of length* $\leq 2\varepsilon$.
- The bottleneck distance between B_1 and B_2 is the infimum of such ε's.

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Theorem (Structure Theorem, Zomorodian–Carlson, 05)
There is a bijection: Persistence modules $\longleftrightarrow$ *Barcodes.*

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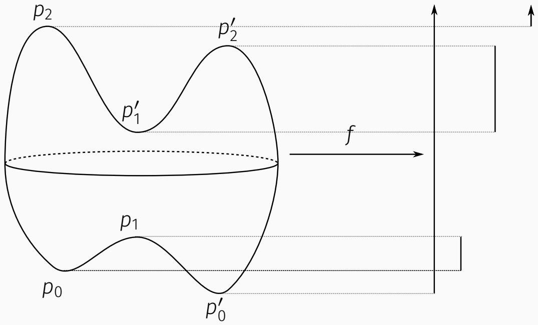
Theorem (Isometry Theorem, $*$)
There is an isometry: Persistence modules $\longleftrightarrow$ *Barcodes.*

$*$(Cohen-Steiner–Edelsbrunner–Harer 07, Chazal–daSilva–Glisse–Oudot, 16)

Example: the Morse case



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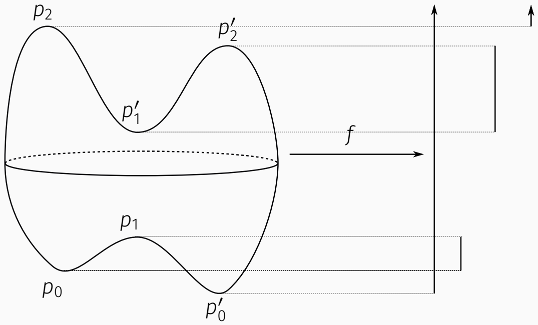

To symplectic geometry

- Polterovich–Shelukhin (Morse $\to$ aspherical) 2014
- Fraser (contact geometry) 2015

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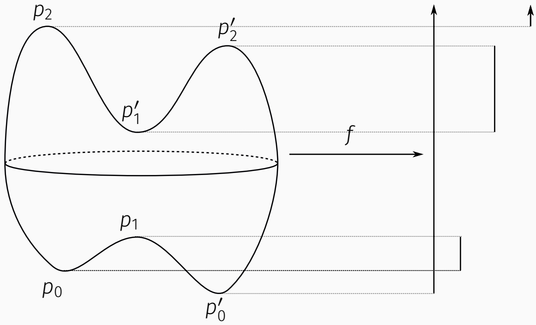

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- Polterovich–Shelukhin (Morse $\rightarrow$ aspherical) 2014
- Fraser (contact geometry) 2015
- Usher–Zhang (Morse $\rightarrow$ monotone) 2016
- Polterovich–Shelukhin–Stojisavljević 2017

The Barannikov complex

Definition (Simple Morse complex, Barannikov 1994)
A filtered complex (C, ∂) endowed with a basis $\mathcal{B}$ is simple if

- decomposition $C = C_- \oplus C_0 \oplus C_+$ which agrees with the filtration,
- partition $\mathcal{B} = \mathcal{B}_- \cup \mathcal{B}_0 \cup \mathcal{B}_+$ which agrees with decomposition and filtration,
- $\partial|_{\mathcal{B}_+} : \mathcal{B}_+ \to \mathcal{B}_-$ is bijection, and $\partial(\mathcal{B}_-) = \partial(\mathcal{B}_0) = 0$.

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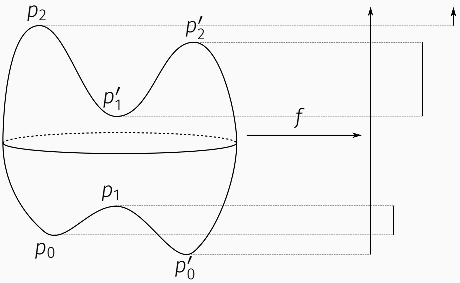
Popularized to the symplectic community (Chekanov–Pushkar, 2005), first used as such (Le Peutrec–Nier–Viterbo, 2013). Shown to be equivalent to persistence modules and barcodes (LeRoux–Seyfaddini–Viterbo, 2018).

C^0 symplectic geometry

Back to the Morse case



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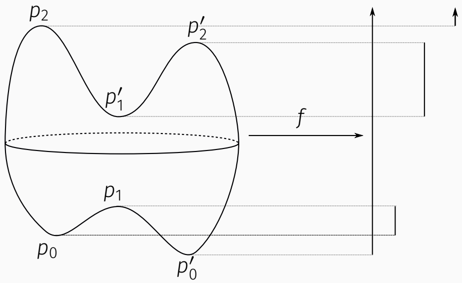

- f Morse $\rightsquigarrow \mathsf{MC}^t(f) \rightsquigarrow \mathbb{MV}(f) \rightsquigarrow \mathsf{B}(f)$
- C^0 continuity: $\quad d_{\mathrm{bot}}(\mathsf{B}(f_1), \mathsf{B}(f_2)) \leq \|f_1 - f_2\|_\infty$ (CS-E-H07, C-dS-G-O16)

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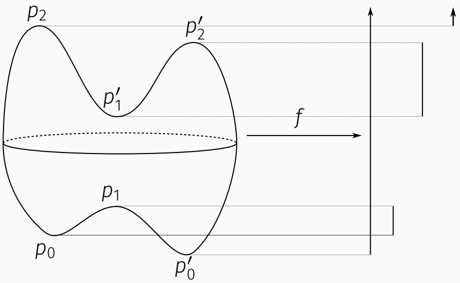

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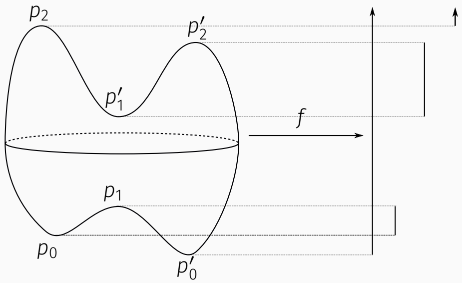

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- $f \in C^0(M)$ define $\mathsf{B}(f) = \lim \mathsf{B}(f_n)$ for Morse $f_n \to f$
- $\mathsf{B}(f) \in \overline{\mathcal{B}}$: bottleneck-completion of (finite) barcodes
  Finiteness condition: for all ε, $\#\{I \in \mathsf{B}(f) \mid \ell(I) > \varepsilon\} < \infty$
  (cf. also Chazal–Cohen-Steiner–Glisse–Guibas–Oudot, 09)

Toward C^0 Floer homology

Polterovich–Shelukhin 14:

$H \in C^\infty([0,1] \times M)$ generic $\rightsquigarrow \mathsf{FC}^t(H) \rightsquigarrow \mathrm{F}\mathbb{V}(H) \rightsquigarrow \mathsf{B}(H)$

Motivation (Arnold+):

$\#\mathrm{Fix}(\phi_H^1) \geq \#\mathrm{Endpts}(\mathsf{B}(H)) \geq \mathrm{rank}(\mathsf{H}_*(M))$

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Theorem

$\mathsf{B} : \mathrm{Ham}(M) \to \overline{\mathcal{B}}'$ *is continuous and extends to* $\overline{\mathrm{Ham}}(M)$.

(Le Roux–Seyfaddini–Viterbo for dim $= 2$;
Buhovsky–Seyfaddini–Humilière dim ≥ 4 aspherical ;
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$\rightsquigarrow$ C^0 Floer homology

<u>Question.</u> Why should we care?

C^0 symplectic geometry

Once upon a time ...

Theorem (Gromov's alternative)
$\mathrm{Ham}(M, \omega)$ *is either* C^0 *dense or* C^0 *closed in* $\mathrm{Diff}(M)$.

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Definition (Oh–Müller)
If $H_k \to H \in C^0$ and $\phi_{H_k} \to \phi$ (C^0, uniformly), then $\{\phi^t\}$ is an hameotopy generated by H.

Motivated by

- C^0 flux conjecture: is Ham C^0-closed in Symp?
- Fathi's question: is $\mathrm{Homeo}_c^{\omega}(\mathbb{D})$ simple?

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Study of flexi-rigidity prop of C^0 analogs of smooth objects

- Uniqueness of continuous generator (Viterbo 06, Buhovsky–Seyfaddini 12, Humilière–L.–Seyfaddini 14)

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Remark

Thus far: spectral invariants, γ distance, no persistence.

Theorem (Fathi 70's)
$\mathrm{Homeo}_c^{\mathrm{vol}}(\mathbb{D}^n)$ *is simple for* $n \geq 3$.

Question
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Theorem (Cristofaro-Gardiner–Humilière–Seyfaddini)
$\mathrm{Homeo}_c^{\mathrm{vol}}(\mathbb{D}^2)$ *is not simple.*

Proof.
Filtered *Periodic Floer homology* $\rightsquigarrow$ persistence modules and barcodes. Map $(\mathrm{Ham}_c(\mathbb{D}), d_{C^0}) \to (\overline{\mathcal{B}}, d_{\mathrm{bot}})$ is continuous and extends to $\overline{\mathrm{Ham}}(\mathbb{D}) = \mathrm{Homeo}_c^{\mathrm{vol}}(\mathbb{D}^2)$. □