Université Paris-Sud

École doctorale de mathématiques Hadamard (ED 574)
Laboratoire de mathématique d'Orsay (UMR 8628 CNRS)

Mémoire présenté pour l'obtention du

Diplôme d'habilitation à diriger les recherches

Discipline : Mathématiques

par

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Invariants spectraux et morphisme de Seidel (avec applications)

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Date de soutenance : 11 décembre 2018

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List of personal publications

List of publications since Ph.D. thesis

Leclercq, Rémi. 2009. "The Seidel morphism of Cartesian products". *Algebr. Geom. Topol.* 9 (4): 1951–1969.

Hu, Shengda, François Lalonde, and Rémi Leclercq. 2011. "Homological Lagrangian monodromy". *Geom. Topol.* 15 (3): 1617–1650.

Buss, Guy, and Rémi Leclercq. 2012. "Pseudo-distances on symplectomorphism groups and applications to flux theory". *Math. Z.* 272 (3-4): 1001–1022.

Humilière, Vincent, Rémi Leclercq, and Sobhan Seyfaddini. 2015a. "Coisotropic rigidity and C^0-symplectic geometry". *Duke Math. J.* 164 (4): 767–799.

— . 2015b. "New energy-capacity-type inequalities and uniqueness of continuous Hamiltonians". *Comment. Math. Helv.* 90 (1): 1–21.

— . 2016. "Reduction of symplectic homeomorphisms". *Ann. Sci. Éc. Norm. Supér. (4)* 49 (3): 633–668.

Anjos, Sílvia, and Rémi Leclercq. 2017. "Noncontractible Hamiltonian loops in the kernel of Seidel's representation". *Pacific J. Math.* 290 (2): 257–272.

— . 2018. "Seidel's morphism of toric 4–manifolds". *J. Symplect. Geom.* 16:1–68.

Leclercq, Rémi, and Frol Zapolsky. 2018. "Spectral invariants for monotone Lagrangians". *J. Topology Anal.* No. 10: 627–700.

Publication from Ph.D. thesis

Leclercq, Rémi. 2008. "Spectral invariants in Lagrangian Floer theory". *J. Mod. Dyn.* 2 (2): 249–286.

Contents

List of personal publications 3

Remerciements 7

Introduction (légèrement détaillée) en français 9

1. Les outils principaux (des “techniques dures”) 11
2. Propriétés de rigidité de trucs symplectiques 13
3. Calculs d’invariants symplectiques 15

Introduction 17

1. The main tools (via “hard” techniques) 18
2. Rigidity properties of symplectic stuff 19
3. Computing symplectic invariants 21

I. Homology theories 25

1. The symplectic setting 26
2. The finite dimensional model: Morse homology 30
3. From Morse to quantum homology 32
4. From Morse to Floer homologies 40
5. Interactions between quantum and Floer homologies 46
6. Action(s) of the symplectomorphism group 49

II. Around spectral invariants 53

1. Spectral invariants: definition, main properties 55
2. Applications to (smooth) symplectic geometry 66
3. Continuous symplectic geometry 69
4. On-going work (1): From spectral invariants to barcodes 79
5. On-going work (2): From symplectic to contact geometry 86

III. Around the Seidel representation 91

1. Homological rigidity of Lagrangian monodromy 92
2. Symplectic invariants of toric 4-manifolds 96
3. Further developments and prospects 109

Bibliography 113

Remerciements

Je tiens à remercier chaleureusement les membres du jury d'avoir accepté d'en faire partie. J'ai également envie de remercier chacun d'eux pour le rôle qu'il a joué à un moment ou un autre de mon développement mathématique et personnel. Je veux par exemple remercier Jean-François Barraud pour ses multiples invitations à exposer mes résultats et à parler de mathématiques sous formes imagées. Merci à Frédéric Bourgeois et Felix Schlenk pour m'avoir proposé un post-doc à Bruxelles, que j'ai amèrement regretté d'avoir refusé pendant la première année de mon séjour à Leipzig. Merci à Alberto Abbondandolo pour avoir illuminé la seconde moitié de ce séjour, et pour son invitation à Pise l'année suivante autour d'une question liant dynamique et géométrie symplectique. Merci à Patrick Massot de toujours prendre le temps de répondre à mes questions diverses et variées, et pour ses conseils éclairés à plusieurs moments-clés (en particulier juste avant et après mon arrivée à Paris-Sud). Merci à Alexandru Oancea dont les attentions tant mathématiques que personnelles, discrètes et constantes, ont souvent été réconfortantes.

Je tiens également à remercier Dusa McDuff, Leonid Polterovich, et Claude Viterbo qui m'ont fait l'honneur de prendre le temps de lire ce mémoire et d'écrire un rapport. Cela m'a donné une nouvelle occasion d'admirer leur disponibilité malgré des emplois du temps que j'imagine extrêmement chargés. J'en profite pour avouer que Claude est la personne dont les travaux m'ont le plus influencé : évidemment via l'introduction des invariants spectraux, mais aussi pour avoir formulé une conjecture qui m'empêche de dormir plusieurs semaines par an, pour son article avec Alfonso Sorrentino résolvant la question qui motivait mon séjour à Pise deux semaines avant ledit séjour (et qui a montré que mon intuition était bonne !), pour son travail en géométrie symplectique C^0 qui a motivé ma première collaboration sur le sujet, etc. Bien évidemment, les travaux de Dusa, et ceux de Leonid ne viennent pas loin derrière en termes d'influence, ce que le présent mémoire devrait illustrer.

Il va sans dire que je dois beaucoup à Octav Cornea qui a non seulement été mon directeur de thèse pendant quatre ans à Montréal mais m'a aussi constamment encouragé et soutenu, notamment pendant les années qui ont séparé la fin de mon doctorat de mon poste à Paris-Sud.

Le *Laboratoire de Mathématique d'Orsay* de l'Université Paris-Sud a été pour moi, et ce dès mon arrivée en septembre 2011, un environnement de travail exceptionnel (et je pèse mes mots), tant sur le plan humain que mathématique.

Cet environnement s'est trouvé parfaitement complété par mes collaborateurs, Sílvia Anjos, Vincent Humilière, Sobhan Seyfaddini, et Frol Zapolsky avec qui j'ai pris énormément de plaisir à travailler. Il en va de même d'Alberto Abbondandolo (sus-remercié), de Georgios Dimitroglou-Rizell, et de Sheila Sandon, même si aucun article n'a (jusqu'ici !) sanctionné le plaisir que je prends à travailler avec eux.

Pour terminer sur une note très personnelle, voici une pensée pour mon père que je remercie d'être là comme il l'est et pour ma mère qui n'en finit pas de nous avoir quittés trop tôt. Merci aussi à Milena qui rend mon quotidien incomparablement facile, et à Thomas qui n'est pas là depuis longtemps mais qui a déjà su se rendre indispensable.

Introduction (légèrement détaillée) en français

Le contexte général de ce travail est la *géométrie symplectique* grossièrement décrite ci-dessous à travers quatre dichotomies.

Flexibilité vs rigidité

Une variété symplectique est une variété M de dimension paire, munie d'une 2-forme fermée non-dégénérée ω. Une telle structure est très *flexible*. Par exemple, le théorème de Darboux assure qu'elle n'admet aucun invariant géométrique local puisque *tout* point de *toute* variété symplectique admet un voisinage symplectiquement équivalent à une boule de l'espace euclidien de la même dimension, muni de sa forme symplectique standard.

Cependant, une structure symplectique présente également un caractère rigide inattendu. Par exemple, un automorphisme φ d'une variété symplectique (M,ω), appelé *symplectomorphisme*, est un difféomorphisme de M qui préserve ω, c'est-à-dire dont la différentielle satisfait $\varphi^*\omega=\omega$. Un théorème célèbre dû à Gromov et Eliashberg affirme que cette condition passe à la limite C^0 malgré son caractère C^1, au sens où un homéomorphisme obtenu comme limite C^0 d'une suite de symplectomorphismes est lui-même un symplectomorphisme dès qu'il est lisse.

Automorphismes vs sous-variétés

Les objets principalement étudiés par la géométrie symplectique classique sont les symplectomorphismes, certaines sous-variétés naturelles des variétés symplectiques, ainsi que les relations qui les unissent.

Par exemple, le *groupe des difféomorphismes hamiltoniens* est un sous-groupe du groupe des symplectomorphismes, dénoté $\mathrm{Ham}(M,\omega)$, qui est formé des symplectomorphismes engendrés par le flot de fonctions lisses définies sur M, possiblement dépendantes du temps. Une illustration frappante de l'importance de ce sous-groupe est donnée par un résultat de Banyaga qui montre que la structure algébrique du groupe des difféomorphismes hamiltoniens détermine la structure symplectique de la variété. Plus précisément, si ω et ω' sont deux variétés symplectiques dont les groupes de difféomorphismes hamiltoniens $\mathrm{Ham}(M,\omega)$ et $\mathrm{Ham}(M,\omega')$ sont isomorphes, alors (M,ω) est symplectomorphe à $(M,c\omega')$ pour une certaine constante c.

Il y a plusieurs types de sous-variétés naturelles d'une variété symplectique : les sous-variétés isotropes, coisotropes, lagrangiennes et symplectiques. Une *sous-variété symplectique* de (M,ω) est une sous-variété sur le tangent de laquelle la restriction de ω est non-dégénérée. “Au contraire”, une sous-variété est *lagrangienne* si elle est de dimension $\frac{\dim M}{2}$ et si la restriction de ω s'annule identiquement sur son fibré tangent. Les sous-variétés lagrangiennes sont surprenamment rigides pour des sous-variétés d'une si grande codimension.

Difféomorphismes hamiltoniens et sous-variétés lagrangiennes interagissent de nombreuses façons et ces interactions conduisent à certains phénomènes fascinants. Par exemple, une lagrangienne L d'une variété symplectique (M,ω) dont le second groupe d'homotopie relative $\pi_2(M,L)$ s'annule *ne peut être déplacée d'elle-même par un difféomorphisme hamiltonien* : L intersecte son image par tout tel difféomorphisme. Ci-dessous, l'ensemble des lagrangiennes qui sont obtenues comme image de L par un difféomorphisme hamiltonien est dénoté par $\mathcal{L}_{\mathrm{Ham}}(L)$.

Géométrie vs topologie

De nombreux outils importants ont été mis au point pour étudier la géométrie et la topologie des groupes de difféomorphismes hamiltoniens ainsi que des ensembles de lagrangiennes précités. Pour justifier (de manière subtile !) le choix du titre de ce mémoire, mentionnons (par exemple !) les *invariants spectraux*, qui conduisent en particulier à une (pseudo-)distance sur ces ensembles, et le *morphisme de Seidel* qui a permis d'obtenir des informations sur leurs groupes fondamentaux.

Mou vs dur

Cette dernière dichotomie concerne les outils utilisés en vue d'obtenir les résultats désirés. L'adjectif "dur" se réfère aux outils basés sur l'utilisation de techniques de courbes pseudo-holomorphes, introduites en géométrie symplectique par Gromov en 1985. Par exemple, l'homologie quantique et l'homologie de Floer sont des parangons de techniques "dures", à partir desquelles les invariants spectraux et le morphisme de Seidel sont construits. Par opposition une technique est qualifiée de "molle" si elle n'est pas dure.

Au vu de ces dichotomies, il s'avère que j'ai étudié avec persévérance des

Propriétés de rigidité de trucs symplectiques grâce à des techniques dures.

Malheureusement, ceci semblait inapproprié comme titre pour ce mémoire.

Une autre conséquence malheureuse d'avoir étudié à la fois les propriétés géométriques et topologiques des objets divers regroupés sous l'appellation "trucs symplectiques" est que cela complique l'organisation du mémoire. En particulier, si *le reste de cette introduction* est organisée pour illustrer l'affirmation encadrée ci-dessus, le *mémoire* lui-même est organisé différemment, en fonction des outils utilisés.

- Le chapitre I présente les homologies quantique et de Floer, nécessaires aux constructions des invariants spectraux et du morphisme de Seidel.
- Le chapitre II est centré sur les invariants spectraux. Leurs définition et propriétés sont expliquées en section 1. Les sections 2 et 3 présentent certaines de leurs conséquences en termes de rigidité en géométrie symplectique "classique" (ou "lisse") et en géométrie symplectique "continue" respectivement. Finalement, les sections 4 et 5 abordent plusieurs travaux en cours.
- Le chapitre III est centré sur le morphisme de Seidel. En section 1, ce morphisme est utilisé pour établir une certaine rigidité homologique de lagrangiennes particulières. La section 2 montre comment il peut être explicitement calculé dans certaines situations[1]. La section 3 contient des pistes pour des travaux ultérieurs dans la continuité des sections 1 et 2.

Terminons l'introduction de cette introduction par deux avertissements.

Avertissement 1. Bien que l'article (Leclercq 2008) faisant suite à ma thèse de doctorat ne soit pas discuté dans ce mémoire, il apparaît à différents endroits puisqu'il a bien évidemment inspiré pour partie les travaux présentés dans ces pages.

L'article (Leclercq 2009) n'est mentionné que deux fois dans ce mémoire. Ceci reflète le fait qu'il est très technique et prouve, sous des hypothèses raisonnables, une propriété naturelle et prévisible du morphisme de Seidel des produits de variétés symplectiques. Il peut être utilisé par exemple, comme c'est le cas plus bas, pour obtenir plus d'exemples en prenant des produits.

Finalement, cette phrase contient la seule référence à l'article (Buss et Leclercq 2012) de tout le mémoire. Ceci reflète le fait que cet article est déconnecté du reste des travaux présentés ici.

Avertissement 2. Dans le but de garder cette introduction courte et lisible, le nombre de références à d'autres travaux a été réduit au minimum. Le lecteur est prié de consulter les sections appropriées du mémoire pour plus de détails.

1. C'est la seule partie du mémoire qui ne conduit pas évidemment à des propriétés de rigidité...

1. Les outils principaux (des "techniques dures")

Les résultats présentés dans ce mémoire concernent et/ou ont été obtenus grâce aux *invariants spectraux* et au *morphisme de Seidel*. Nous introduisons ces deux outils rapidement ici et spécifions nos contributions à leur développement.

Ils reposent tous les deux sur les théories d'*homologies quantique et de Floer*. Ces homologies dénotées respectivement HQ et HF ci-dessous, peuvent être associées à une variété symplectique ou à une sous-variété lagrangienne d'une variété symplectique. Elles sont toutes deux un mélange de théorie de Morse et de techniques de courbes pseudo-holomorphes (mais selon des recettes très différentes). La définition de ces homologies et des structures additionnelles dont elles jouissent, ainsi que les relations qui les lient sont décrites longuement au chapitre I.

Invariants spectraux (Leclercq et Zapolsky 2018)

Ils ont été introduits par Viterbo (Viterbo 1992) pour les lagrangiennes des fibrés cotangents via la théorie des fonctions génératrices. La construction a été ensuite adaptée à l'homologie de Floer dans le même contexte par Oh (Oh 1997). Puis, Schwarz (Schwarz 2000) et Oh (Oh 2005) ont adapté la construction aux variétés symplectiques compactes (sans mention de lagrangienne). Le cas des lagrangiennes des variétés symplectiques compactes a été initié dans l'article (Leclercq 2008) sous une hypothèse assez restrictive dite d'*asphéricité symplectique* et pour des théories d'homologie à coefficients dans $\mathbb{Z}/2\mathbb{Z}$.

Dans un travail commun avec Zapolsky (Leclercq et Zapolsky 2018), nous avons étendu cette dernière construction à des lagrangiennes satisfaisant l'hypothèse beaucoup plus faible de *monotonie* et pour des théories d'homologie à coefficients dans des anneaux beaucoup plus généraux. Étant donnée une lagrangienne monotone[2] L d'une variété symplectique (M, ω), les invariants spectraux sont donnés sous la forme d'une fonction

$$\ell\colon\ \mathrm{HQ}_*(L) \times C^0\big(M \times [0,1]\big) \longrightarrow \mathbb{R} \cup \{-\infty\}$$

qui associe un nombre réel à toute classe d'homologie (quantique) non nulle de L et à toute fonction continue dépendant du temps définie sur M. L'ingrédient principal de la construction est une filtration naturelle de l'homologie de Floer. La version Morse de la fonction ℓ, qui lui sert de modèle, est par ailleurs assez facile à décrire. Étant donnée une fonction de Morse f sur M et une classe d'homologie (de Morse) α non nulle, la filtration est donnée par les valeurs de f et $\ell(\alpha, f)$ est définie de sorte que $\{x \in M \mid f(x) \leq \ell(\alpha, f)\}$ soit le plus petit sous-niveau de f qui contienne un représentant de α (voir aussi la figure II.1 pour une illustration).

La définition de la fonction ℓ est détaillée en section 1.1 du chapitre II. Dans les sections 1.1 et 1.2 sont établies ses propriétés, dont les principales sont rassemblées dans le théorème suivant.

Théorème 1. *Soient α et β des classes d'homologie quantique de L non nulles, et H et K des fonctions continues sur $M \times [0,1]$.*

[Finitude] *$\ell(\alpha\,; H)$ est fini.*

[Spectralité] *Si H est lisse, $\ell(\alpha\,; H)$ appartient aux valeurs critiques d'une fonctionnelle $\mathcal{A}_H$.*

[Continuité] *$\ell(\alpha\,;\cdot)$ est continue par rapport à la norme $L^{1,\infty}$, plus précisément*

$$\int_0^1 \min_M (K_t - H_t)\, dt \leq \ell(\alpha; K) - \ell(\alpha; H) \leq \int_0^1 \max_M (K_t - H_t)\, dt\,.$$

[Monotonie] *Si $H \leq K$, alors $\ell(\alpha\,; H) \leq \ell(\alpha\,; K)$.*

[Inégalité triangulaire] *Soit $*$ le produit (d'intersection) de $\mathrm{HQ}_*(L)$ et $\sharp$ la concaténation,*

$$\ell(\alpha * \beta; H\sharp K) \leq \ell(\alpha; H) + \ell(\beta; K)\,.$$

2. La monotonie d'une lagrangienne donne un contrôle sur l'aire symplectique des disques dans M, à bord dans L. Voir la section 1.2.1 pour une définition précise.

[Décalage temporel] *Si* c *est une fonction du temps,* $\ell(\alpha\,;H+c)=\ell(\alpha\,;H)+\int_0^1 c(t)\,dt$*.*

[Valuation quantique] $\ell(\alpha\,;0)$ *est une valuation sur* $\mathrm{HQ}_*(L)$ *et* $\ell([L]\,;0)=0$*.*

[Contrôle lagrangien] $\ell(\cdot\,;H)$ *est contrôlée par la restriction de* H *à* L *:*

$$\int_0^1 \min_L H_t\,dt \le \ell(\alpha;H)-\ell(\alpha;0)\le \int_0^1 \max_L H_t\,dt\,.$$

[Action de Novikov] *Soit* $A\in\pi_2(M\,,L)$*, on a* $\ell(A\cdot\alpha\,;H)=\ell(\alpha\,;H)-\omega(A)$*.*

[Positivité] $\ell([L]\,;H)+\ell([L]\,;\overline{H})\ge 0$ *où* $\overline{H}$ *est définie par* $\overline{H}_t(x)=-H_{1-t}(x)$*.*

[Maximum] $\ell(\alpha\,;H)\le\ell([L]\,;H)+\ell(\alpha\,,0)$*.*

Ces propriétés rendent la fonction ℓ très pertinente, par exemple en vue de l'étude de la géométrie de $\mathrm{Ham}(M\,,\omega)$ et $\mathcal{L}_{\mathrm{Ham}}(L)$, ainsi que de leurs revêtements universels, munis de distances naturelles introduites par Hofer (Hofer 1990). En effet, les propriétés de *spectralité* et de *continuité* ci-dessus montrent que la fonction ℓ induit une fonction

$$\ell\colon\ \mathrm{HQ}_*(L)\times\widetilde{\mathrm{Ham}}(M,\omega)\longrightarrow\mathbb{R}\cup\{-\infty\}\,.$$

À ce niveau, l'expression $\ell([L]\,;H)+\ell([L]\,;\overline{H})$ se lit $\ell([L]\,;\widetilde{\phi}_H)+\ell([L]\,;\widetilde{\phi}_H^{-1})$ où $\widetilde{\phi}_H$ est la classe d'équivalence de l'isotopie hamiltonienne engendrée par H. Les propriétés de *positivité* et d'*inégalité triangulaire* montrent alors que $\|\widetilde{\phi}_H\|=\ell([L]\,;\widetilde{\phi}_H)+\ell([L]\,;\widetilde{\phi}_H^{-1})$ définit une pseudo-norme sur $\widetilde{\mathrm{Ham}}(M\,,\omega)$. Finalement, la propriété de *continuité* montre que la pseudo-distance obtenue est bornée par la distance de Hofer. Utiliser la propriété de *contrôle lagrangien* montre que ceci s'adapte naturellement au cas des lagrangiennes, définissant une pseudo-distance sur $\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$ également majorée par la distance de Hofer appropriée.

Ceci conduira à un résultat (Theorem II.12) dont la conséquence principale peut être exprimée dans le cadre de cette introduction de la façon suivante.

Théorème 2. *Soit* Θ *dénotant* $\widetilde{\mathrm{Ham}}(M\,,\omega)$ *ou* $\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$*. Pour toute classe non nulle* $\alpha\in\mathrm{HQ}_*(L)$*, la fonction spectrale* ℓ *induit des fonctions* $\ell^\alpha\colon\ \Theta\times\widetilde{\mathrm{Ham}}(M\,,\omega)\to\mathbb{R}$*, qui sont lipschitziennes par rapport à la distance de Hofer naturelle sur* $\Theta\times\widetilde{\mathrm{Ham}}(M\,,\omega)$*.*

Le morphisme / la représentation de Seidel (Hu, Lalonde et Leclercq 2011)

L'autre outil important étayant les résultats présentés dans ces pages est dû à Seidel (Seidel 1997). Il a deux descriptions équivalentes : l'une *géométrique* et l'autre *algébrique*, respectivement comme morphismes

$$\pi_1\big(\mathrm{Ham}(M,\omega)\big)\longrightarrow\mathrm{HQ}_*(M)^\times\quad\text{et}\quad\pi_1\big(\mathrm{Ham}(M,\omega)\big)\longrightarrow\mathrm{Aut}\big(\mathrm{HF}_*(M)\big),$$

où $\mathrm{HQ}_*(M)^\times$ dénote le groupe des inversibles de $\mathrm{HQ}_*(M)$.

Shengda Hu et François Lalonde (Hu et Lalonde 2010) ont adapté la construction algébrique à l'homologie de Floer d'une lagrangienne monotone L. Le morphisme obtenu est défini sur le groupe fondamental de $\mathrm{Ham}(M\,,\omega)$ *relativement* à $\mathrm{Ham}(M\,,\omega\,;L)$, son sous-groupe formé des difféomorphismes hamiltoniens qui préservent globalement L :

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big)\longrightarrow\mathrm{Aut}\big(\mathrm{HF}_*(L)\big).$$

Dans un travail en collaboration avec eux (Hu, Lalonde et Leclercq 2011), nous avons adapté la construction géométrique, définissant ainsi un morphisme

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big)\longrightarrow\mathrm{HQ}_*(L)^\times$$

dont nous avons prouvé l'équivalence avec sa contrepartie algébrique.

Au vu des applications discutées dans ce mémoire, la version algébrique du morphisme de Seidel d'une lagrangienne est présentée en section 1.1 du chapitre III, tandis que la version géométrique du morphisme de Seidel d'une variété symplectique (sans mention d'une lagrangienne) est présentée en section 2.1 (*ibid.*).

2. Propriétés de rigidité de trucs symplectiques

Voici à présent trois types de rigidité très différents discutés dans ce mémoire.

Lagrangiennes super-lourdes (Leclercq et Zapolsky 2018)

Les invariants spectraux lagrangiens peuvent être appliqués à la théorie des ensembles *lourds* et *super-lourds* d'Entov et Polterovich (Entov et Polterovich 2009). Ce sont des sous-ensembles de M qui sont *très* rigides en termes d'intersection. En particulier, ils ne peuvent être disjoints d'eux-mêmes par un symplectomorphisme (non nécessairement hamiltonien). Ce phénomène de rigidité est bien plus fort que celui mentionné dans l'introduction. Considérons par exemple le tore $\mathbb{T}^2 = S^1 \times S^1$ muni de la forme volume produit. C'est une variété symplectique et tout lacet fermé plongé en est une sous-variété lagrangienne compacte. Soit L un méridien de $\mathbb{T}^2$. Comme $\pi_2(\mathbb{T}^2, L) = 0$, aucun difféomorphisme hamiltonien ne peut disjoindre L de lui-même. Au contraire, toute rotation (non triviale !) de $\mathbb{T}^2$ le long d'une longitude est un symplectomorphisme (la forme volume étant préservée) qui y parvient. L'exemple le plus simple de lagrangienne ne pouvant être disjointe d'elle-même par un symplectomorphisme est l'équateur de S^2 (munie de la forme volume standard) puisqu'il divise la sphère en deux partie d'aires égales.

En section 2.2 du chapitre II, nous prouvons que le tore de Chekanov de $\mathbb{CP}^2$, ainsi que le tore exotique de $S^2 \times S^2$ sont super-lourds. Ces exemples ne procurent pas de nouvelles lagrangiennes super-lourdes, la super-lourdeur du tore de Chekanov ayant été prouvée par Wu (Wu 2012), et celle du tore exotique par Eliashberg et Polterovich (Eliashberg et Polterovich 2010). Cependant, ils illustrent à quel point il est facile de prouver de telles propriétés une fois que les invariants spectraux *lagrangiens* ont été définis et leurs propriétés établies. En effet, la preuve se réduit à établir une inégalité qui découle directement des propriétés de *valuation quantique* et de *contrôle lagrangien* du théorème 1, ainsi que d'une propriété additionnelle (la propriété de *structure de module* de la section 1.4.2 du chapitre II) qui exprime le fait que les invariants spectraux se comportent bien avec un structure algébrique additionnelle de l'homologie quantique.

Trivialité de la monodromie lagrangienne (Hu, Lalonde et Leclercq 2011)

Ce second résultat de rigidité est basé sur le morphisme de Seidel. Supposons que ϕ soit un difféomorphisme hamiltonien d'une variété symplectique, *qui préserve une lagrangienne* L. La restriction de ϕ à L induit donc un isomorphisme de l'homologie de L et se pose la question naturelle de savoir quels isomorphismes peuvent être obtenus de la sorte. Par exemple, Yau (Yau 2009) a prouvé que les générateurs du premier groupe d'homologie du tore de Chekanov de $\mathbb{CP}^2$ peuvent être inter-changés via un difféomorphisme hamiltonien de $\mathbb{CP}^2$. Il s'avère que la *monotonie* du tore de Chekanov est cruciale ici. En effet, nous avons montré que lorsque la lagrangienne est *symplectiquement asphérique*, c'est-à-dire quand l'aire symplectique des disques de la variété à bord dans L est identiquement nulle, la situation est autrement plus rigide...

Théorème 3 (Theorem III.2). *Soit L une lagrangienne symplectiquement asphérique de (M, ω), seule l'identité de $\mathrm{H}_*(L)$ peut être induite par un difféomorphisme hamiltonien de (M, ω) préservant L.*

De manière équivalente, ceci assure que si un difféomorphisme de L n'induit pas l'identité en homologie, il ne peut être étendu en un difféomorphisme hamiltonien de la variété ambiante. Bien sûr, si l'homologie de L s'injecte dans celle de l'espace total, un difféomorphisme hamiltonien (qui est isotope à l'identité) ne peut produire un automorphisme non trivial de $\mathrm{H}_*(L)$. Par exemple, l'homologie du produit de deux méridiens dans $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ est fixée par tout difféomorphisme hamiltonien de $\mathbb{T}^4$. Par contre, un lacet L plongé dans la surface Σ_2 orientée fermée de genre 2, dont la classe d'homotopie est non nulle mais dont la classe d'homologie est nulle, est une lagrangienne (symplectiquement) asphérique (puisque $\pi_2(M, L) = 0$), dont l'homologie disparaît dans celle de Σ_2. Aucune raison *a priori* n'empêche donc un difféomorphisme hamiltonien de $\Sigma_2 \times \Sigma_2$ d'inter-changer les générateurs de $\mathrm{H}_1(L \times L)$; ce que le théorème 3 interdit pourtant.

Rigidité C^0 des sous-variétés coisotropes (Humilière, Leclercq et Seyfaddini 2015b, 2015a, 2016)

Le résultat inattendu de rigidité dû à Gromov et Eliashberg mentionné dans la partie liminaire de cette introduction a marqué la naissance de la *géométrie symplectique* C^0, "première merveille" de la géométrie symplectique d'après (Polterovich et Rosen 2014). La section 3 du chapitre II rassemble plusieurs résultats que j'ai obtenus dans ce domaine, en collaboration avec Vincent Humilière et Sobhan Seyfaddini.

Le résultat central est dans l'esprit du théorème de Gromov–Eliashberg, mais plutôt que de considérer une suite de symplectomorphismes $(\psi_k)_{k\in\mathbb{N}}$ qui converge C^0 vers un difféomorphisme ψ, nous fixons une sous-variété coisotrope C et nous nous concentrons sur la suite $(\psi_k(C))_{k\in\mathbb{N}}$. Rappelons qu'une sous-variété *coisotrope* C d'une variété symplectique est une variété telle qu'en chacun de ses points, son espace tangent contient son orthogonal symplectique. Les plus grandes (au sens de la dimension) sous-variétés coisotropes propres sont les hypersurfaces, les plus petites sont les lagrangiennes. Symétriquement, il existe des sous-variétés *isotropes* dont l'orthogonal symplectique du tangent en tout point contient le tangent. Les variétés coisotropes admettent un feuilletage naturel, dit *feuilletage caractéristique*, dont les feuilles sont isotropes.

Théorème 4 (Theorem II.39). *Soit C une sous-variété coisotrope de (M,ω) et $(\psi_k)_{k\in\mathbb{N}}$ une suite de difféomorphismes symplectiques qui converge C^0 vers un* homéomorphisme *ψ. Si l'image $\psi(C)$ est lisse, elle est coisotrope et ψ envoie le feuilletage caractéristique de C sur celui de $\psi(C)$.*

Ce résultat est remarquable sous différents points de vue. Tout d'abord, il montre que des résultats qui n'étaient *a priori* pas reliés, de Laudenbach et Sikorav (Laudenbach et Sikorav 1994) pour les lagrangiennes et de Opshtein (Opshtein 2009) pour les hypersurfaces, sont en fait les cas *extrêmes* (au sens de la dimension) d'un même phénomène de rigidité. De plus, le théorème 4 est local au sens où C n'est pas nécessairement fermée ou, de manière équivalente, $(\psi_k)_{k\in\mathbb{N}}$ n'est pas requise d'être définie globalement sur M. Finalement, il a aussi comme conséquence surprenante le fait qu'il suffit que la variété $\psi(C)$ soit lisse, pour qu'il en soit de même de son feuilletage caractéristique.

Le fait que ce dernier soit préservé sous limite C^0 pose une question très naturelle. Quotienter une sous-variété coisotrope C par les feuilles de son feuilletage caractéristique définit (au moins localement) une nouvelle variété symplectique. Dans la situation du théorème 4, la restriction à C de ψ descend en un homéomorphisme entre les variétés symplectiques quotients et l'on peut alors se demander "à quel point" cet homéomorphisme est symplectique.

Si la réponse générale semble hors de portée, nous sommes parvenus à répondre partiellement, dans un cas particulier. En effet, pour un difféomorphisme la propriété d'être symplectique (ou anti-symplectique) est équivalente à la préservation de quantités appelées *capacités symplectiques*. En considérant la capacité symplectique qui peut être naturellement définie via les invariants spectraux, dite *capacité spectrale*, dans le cas des tores symplectiques, nous avons établi le résultat suivant.

Théorème 5 (Theorem II.44). *Si un homéomorphisme symplectique du tore symplectique standard $\mathbb{T}^{2(k_1+k_2)}$ préserve un sous-tore coisotrope standard : $\mathbb{T}^{2k_1+k_2}\times\{0\}\subset\mathbb{T}^{2(k_1+k_2)}$, alors l'homéomorphisme induit sur la réduction préserve la capacité spectrale.*

La preuve du théorème 4 est aussi basée sur l'utilisation des invariants spectraux et de certaines capacités. En particulier, nous établissons des inégalités de type énergie-capacité pour difféomorphismes hamiltoniens (Theorem II.30) et pour lagrangiennes (Theorem II.27), qui s'inspirent d'inégalités classiques en géométrie symplectique. Elles montrent que les capacités peuvent être utilisées pour borner inférieurement des distances naturelles (dont la distance de Hofer) entre difféomorphismes hamiltoniens et entre certaines lagrangiennes.

Ces inégalités ont des conséquences en termes de *dynamique hamiltonienne* C^0. Dans l'esprit du théorème de Gromov–Eliashberg mentionné plus haut, Müller et Oh (Oh et Müller 2007) ont défini une notion d'*homéomorphisme hamiltonien*. Un tel homéomorphisme est engendré (en un sens approprié) de manière unique par certaines fonctions continues appelées *hamiltoniens continus*. En utilisant nos inégalités de type énergie-capacité, nous avons montré entre autres le résultat suivant, étape essentielle de la preuve du théorème 4.

Théorème 6 (Theorem II.37 (1)). *Un hamiltonien continu $H\colon M\times[0,1]\to\mathbb{R}$ engendre un flot qui préserve une sous-variété coisotrope C et se propage le long des feuilles de son feuilletage caractéristique si et seulement si H est une fonction du temps (indépendante de la composante M).*

3. Calculs d'invariants symplectiques

La dernière section du mémoire est de nature différente du reste (et en particulier ne conduit a priori à aucun phénomène de rigidité). Il s'agit de *calculer* le morphisme de Seidel de certaines variétés et d'en déduire leur anneau d'homologie quantique. Plus précisément, McDuff et Tolman (McDuff et Tolman 2006) ont calculé le terme de plus haut degré et spécifié la structure des autres termes des images, par le morphisme de Seidel, de certains lacets de difféomorphismes hamiltoniens de variétés presque complexes (M, J) n'admettant aucune sphère pseudo-holomorphe de premier nombre de Chern[3] strictement négatif, dites *NEF*. Lorsque toutes les sphères pseudo-holomorphes ont un nombre de Chern strictement positif, tous les termes d'ordre inférieur disparaissent ; au contraire, l'existence d'une sphère à nombre de Chern nul peut provoquer la présence d'une infinité de termes.

Dans un travail en collaboration avec Sílvia Anjos (Anjos et Leclercq 2018), nous avons exprimé ces termes additionnels sous la forme de formules fermées, lorsque la variété est de dimension 4 et *torique*. De telles variétés peuvent être représentées par un polytope convexe de $\mathbb{R}^2$, et nous montrons également comment lire ces formules combinatoirement sur le polytope. La formule précise et la façon de la lire sur le polytope est le contenu de la section 2.2 du dernier chapitre.

Le groupe fondamental du groupe des difféomorphismes hamiltoniens de certaines variétés symplectiques toriques de dimension 4 étant connu, ceci nous a permis de déterminer *totalement* le morphisme de Seidel dans ces cas particuliers (même pour certaines variétés "même pas" NEF). Grâce à ces calculs, nous sommes parvenus dans (Anjos et Leclercq 2017) à détecter des éléments non-triviaux dans le noyau du morphisme de Seidel de certains éclatements de $\mathbb{CP}^1\times\mathbb{CP}^1$.

Théorème 7 (Theorem III.18). *Soit $\mathbb{X}^c$ la variété symplectique obtenue de $\mathbb{CP}^1\times\mathbb{CP}^1$, munie de sa forme symplectique produit pour laquelle le volume de chaque facteur vaut 1, après deux éclatements de capacité c. Le morphisme de Seidel de $\mathbb{X}^c$ n'est pas injectif.*

De plus, nous décrivons explicitement un élément du noyau en termes des actions en cercle dont est munie $\mathbb{X}^c$. Nous avons aussi déterminé qu'une telle situation ne pouvait se présenter sur $\mathbb{CP}^1\times\mathbb{CP}^1$ quelle que soit la forme symplectique dont elle est munie.

Théorème 8 (Theorem III.19). *Le morphisme de Seidel est injectif sur toutes les surfaces d'Hirzebruch.*

Nos calculs de l'image du morphisme de Seidel nous ont également permis d'exprimer l'homologie quantique de certaines variétés symplectiques toriques NEF de dimension 4, toujours en suivant des idées de McDuff et Tolman (*ibid.*). Ceci nous a ensuite permis d'établir l'expression du super-potentiel de Landau–Ginzburg de ces variétés qui s'avère lisible combinatoirement sur le polytope. Ces applications sont développées

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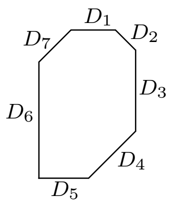
3. Il s'agit en fait de la valeur du premier nombre de Chern de (TM, J) appliqué à la classe d'homologie de la dite sphère.

en section 2.3 et illustrées sur des exemples naturels en section 2.4. Plutôt que d'énoncer le résultat final (Theorem III.15), terminons cette introduction par l'esquisse d'un des exemples de la section 2.4, illustrant comment extraire un invariant *a priori* très compliqué directement du polytope d'une telle variété (et donc – espérons-le – justifier la pertinence de cette dernière partie).

Exemple 9. Le but est de calculer l'anneau d'homologie quantique de la variété $(M\,,\omega)$ de dimension 4 obtenue de $(S^2 \times S^2\,,\omega_\mu)$ après trois éclatements de capacités respectives c_1, c_2 et c_3 (ω_μ est la forme symplectique produit telle que le premier facteur ait aire $\mu \geq 1$ et le second aire 1). La théorie générale des variétés symplectiques toriques assure que $(M\,,\omega)$ correspond de manière unique (à une relation d'équivalence raisonnable près) au polytope $P \subset \mathbb{R}^2$ suivant :



Normals	Constants
$\eta_1 = (0,1)$	$\kappa_1 = \mu$
$\eta_2 = (1,1)$	$\kappa_2 = \mu - c_3$
$\eta_3 = (1,0)$	$\kappa_3 = 0$
$\eta_4 = (1,-1)$	$\kappa_4 = -c_1$
$\eta_5 = (0,-1)$	$\kappa_5 = 0$
$\eta_6 = (-1,0)$	$\kappa_6 = 1$
$\eta_7 = (-1,1)$	$\kappa_7 = \mu + 1 - c_2$

qui est donné comme l'ensemble des points $x \in \mathbb{R}^2$ intérieurs (largement) à P, *i.e.* les points satisfaisant pour tout i, $\langle x\,,\eta_i\rangle \leq \kappa_i$ où η_i est la normale entière et primitive, extérieure à l'arête D_i. Chaque arête D_i du polytope correspond à une sphère de M.

La variété de dimension 4 $(M\,,\omega)$ est NEF et donc son anneau d'homologie quantique est engendré par ses éléments de degré 4, et $\mathrm{HQ}_4(M\,,\omega)$ est isomorphe à un quotient d'un anneau de polynômes par un idéal. L'anneau de polynômes est l'ensemble des polynômes de Laurent en deux variables z_1, z_2, à coefficients dans un anneau de séries de Laurent généralisées en une variable t. L'idéal est engendré par toutes les dérivées partielles de l'un de ces polynômes W, le *super-potentiel de Landau–Ginzburg*. Pour obtenir une description explicite de l'anneau d'homologie quantique, il "suffit" donc de calculer W. Dans le cas présent, il est donné à partir de P sous la forme

$$W = \sum_{i=1}^{7} z^{\eta_i} t^{\kappa_i} + \text{corrections quantiques}$$

où les termes de corrections quantiques proviennent des arêtes de P qui correspondent à des sphères de M dont le premier nombre de Chern est nul.

Orientons les arêtes dans le sens anti-trigonométrique et associons à D_i un poids w_i défini comme le vecteur de $\mathbb{R}^2$ entier et primitif porté par D_i. L'arête D_i correspond à une sphère de M de premier nombre de Chern nul si et seulement si son poids associé est la moyenne des poids de ses voisines, *i.e.* si $2w_i = w_{i-1} + w_{i+1}$. En parcourant le polytope, on conclut aisément que seules les arêtes D_1 et D_3 correspondent à de telles sphères. Notre résultat (Theorem III.15) s'applique dans ce cas et assure qu'il n'y a que deux termes additionnels de corrections quantiques, $q(D_i) = z^{\eta_i} t^{\kappa_{i+1}+\kappa_{i-1}-\kappa_i}$ avec $i = 1$ et 3, et donc finalement

$$W = \left(z_2 t^{\mu} + z_1 z_2 t^{\mu - c_3} + z_1 + z_1 z_2^{-1} t^{-c_1} + z_2^{-1} + z_1^{-1} t + z_1^{-1} z_2 t^{\mu+1-c_2}\right) + z_2 t^{\mu+1-c_2-c_3} + z_1 t^{\mu-c_1-c_3}\,.$$

Introduction

The general context of the work presented here is that of *symplectic geometry*, whose description I will loosely divide into four dichotomies.

Flexibility vs rigidity

A symplectic manifold is an even-dimensional manifold M, endowed with a nondegenerate closed 2–form ω. Such a structure is quite *flexible*. For example, it is a classical result by Darboux that there are no local geometric invariants, as *any* point of *any* symplectic manifold admits a neighborhood symplectically equivalent to a small ball in the Euclidean space of the same dimension, endowed with its standard symplectic form.

On the other hand, a symplectic structure also presents unexpectedly strong rigidity properties. For example, an automorphism φ of a symplectic manifold (M,ω), also known as symplectomorphism, is a diffeomorphism of M which preserves ω, *i.e.* whose differential satisfies $\varphi^*\omega = \omega$. A famous theorem due to Gromov and Eliashberg asserts that this condition goes through C^0–limits despite its C^1 nature, in the sense that a *smooth* C^0–limit of symplectomorphisms is a symplectomorphism.

Automorphisms vs submanifolds

The main objects of study are natural subgroups of the automorphism group and natural submanifolds of symplectic manifolds, and the relationships between those.

For example, there is a subgroup of the group of symplectomorphisms, called the *Hamiltonian diffeomorphism group* and denoted $\mathrm{Ham}(M,\omega)$, consisting of symplectomorphisms generated by the flow of time-dependent functions on M. A striking illustration of the importance of this subgroup is given by a result of Banyaga, which shows that the algebraic structure of the Hamiltonian diffeomorphism group determines the symplectic structure of the manifold. Namely, if ω and ω' are two symplectic forms on a manifold M so that the Hamiltonian diffeomorphism groups $\mathrm{Ham}(M,\omega)$ and $\mathrm{Ham}(M,\omega')$ are isomorphic, then (M,ω) is symplectomorphic to $(M,c\omega')$ for some constant c.

There are several natural types of submanifolds of a given symplectic manifold : isotropic, coisotropic, Lagrangian, and symplectic submanifolds. At one end of the spectrum, a submanifold of (M,ω) is symplectic if the restriction of ω to its tangent bundle is nondegenerate. At the other end, a submanifold is Lagrangian if the restriction of ω to its tangent bundle vanishes and if it is of maximal dimension among submanifolds with this property. What is quite surprising about Lagrangians is that they are *small* as their dimension is half that of M, but they are very *rigid*.

Hamiltonian diffeomorphisms and Lagrangians interact in many ways and these interactions led to fascinating phenomena. For example, a Lagrangian L of a symplectic manifold (M,ω) such that $\pi_2(M,L) = 0$ *cannot be displaced from itself by Hamiltonian diffeomorphisms*, that is L intersects its image by any Hamiltonian diffeomorphism. We denote the set of all those Lagrangians which are obtained from L by a Hamiltonian diffeomorphism by $\mathcal{L}_{\mathrm{Ham}}(L)$.

Geometry vs topology

Several beautiful tools were built in order to study the geometry and the topology of Hamiltonian diffeomorphism groups as well as the aforementioned sets of Lagrangians. In order to justify in a subtle way the choice of title for these memoirs, let me mention (for example !) spectral invariants, which yield a (pseudo-)distance on these groups, and the Seidel morphism which gives information on their fundamental group.

Soft vs hard

Finally, this last dichotomy concerns the tools used to get to the desired results. The adjective “hard” refers to tools relying on pseudo-holomorphic curves techniques, introduced in symplectic geometry by Gromov. For example, quantum and Floer homology are parangons of “hard” techniques, on top of which the Seidel morphism and the version of spectral invariants used here were built. The adjective “soft” refers to techniques which are not hard.

In view of these dichotomies, it turns out that I consistently studied

Rigidity properties of symplectic stuff via hard techniques.

Unfortunately, this seemed inappropriate as a HDR title.

Another unfortunate consequence of studying both geometrical and topological properties of the various objects included in “symplectic stuff” is that it made the present memoirs harder to organize. Hence, even though the *remaining of this introduction* is organized as to reflect the boxed sentence above, *the memoir* is not. It is organized around (yet) another dichotomy, based on the main tools I used.

- Chapter I presents necessary background on quantum and Floer homologies.
- Chapter II is centered on spectral invariants. Their definition and properties are explained in Section 1. Sections 2 and 3 address some of their consequences to “*smooth*” and *continuous* symplectic geometry respectively. Finally, Sections 4 and 5 sketch works in progress.
- Chapter III is centered on the Seidel morphism. In Section 1, this morphism is used to establish some homological rigidity of certain Lagrangians. Section 2 shows how it can be explicitly computed in particular situations[4]. Section 3 contains ideas for future works.

Let me close the introduction of the introduction with two disclaimers.

Disclaimer 1. While the article (Leclercq 2008), subsequent to my Ph.D. thesis, is not discussed in here, it will appear at several places as, unsurprisingly, it inspired some of the work presented here.

The article (Leclercq 2009) is only cited twice in this memoirs. This reflects the fact that it is quite technical and proves, under a reasonable assumption, a natural property of the Seidel morphism of product manifolds. It can for example be used, as it is the case here, to obtain more examples by taking products.

Finally, this sentence contains the only reference to (Buss and Leclercq 2012) in the memoirs. This reflects the fact that it is quite disconnected from the rest.

Disclaimer 2. In order to keep the introduction short and readable, I kept the number of references to other works at a minimum level (and a *very* small one at that). Please, refer to the appropriate sections for more details.

1. The main tools (via “hard” techniques)

The results presented in this HDR memoirs concern or have been obtained thanks to *spectral invariants* and *the Seidel morphism*. Let me quickly introduce them here and explain my contribution to their respective development.

They both rely on *quantum and Floer homology*. These homologies, respectively denoted by HQ and HF below, can be associated with a symplectic manifold as well as with a Lagrangian submanifold. They are both a mix between Morse homology and pseudo-holomorphic curves techniques (though, in very different fashions). The definition of these homologies and of additional algebraic structures they enjoy, as well as the relationships they share, are described at length in Chapter I.

4. This is somehow the only part of the present memoirs which does not obviously lead to rigidity properties...

Spectral invariants (Leclercq and Zapolsky 2018)

They were introduced by Viterbo (Viterbo 1992) for Lagrangians of cotangent bundles via the theory of generating functions. The construction was later adapted to Floer homology, in the same setting by Oh (Oh 1997). Then, Schwarz (Schwarz 2000) and Oh (Oh 2005) adapted the construction to compact symplectic manifolds, without mention of a Lagrangian. The case of Lagrangians of compact symplectic manifolds was first dealt with in (Leclercq 2008) under a fairly restrictive technical assumption of *asphericity* and with respect to homology theories with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

In a joint work with Frol Zapolsky (Leclercq and Zapolsky 2018), we extended the latter construction to Lagrangians satisfying the much weaker assumption of *monotonicity* and for homology theories with a much wider range of coefficients. Given such a Lagrangian L of a symplectic manifold $(M\,,\omega)$, the resulting object is a function

$$\ell\colon\ \mathrm{HQ}_*(L)\times C^0\big(M\times[0,1]\big)\longrightarrow\mathbb{R}\cup\{-\infty\}$$

which associates a real number with any non-zero (quantum) homology class of L and any continuous time-dependent function on M. The main ingredient in the construction is a natural filtration of Floer homology. The Morse-theoretic version of ℓ, which serves as toy-model, is rather easy to describe. For a (Morse) homology class α and a Morse function f on M, the filtration is given by the values of f and $\ell(\alpha\,,f)$ is defined so that $\{x\in M\,|\,f(x)\leq\ell(\alpha\,,f)\}$ is the smallest sublevel set of f which contains a representative of α (see also Figure II.1).

The definition of the function ℓ as well as its properties are presented in Section 1 of Chapter II. Then, in the following section, we explain how the aforementioned properties make ℓ well-suited to study the geometry of $\mathrm{Ham}(M\,,\omega)$ and $\mathcal{L}_{\mathrm{Ham}}(L)$, with respect to quite natural distances on these sets, introduced by Hofer (Hofer 1990). This is the content of Theorem II.12.

The Seidel morphism / representation (Hu, Lalonde, and Leclercq 2011)

The other important tool on which the present work relies is due to Seidel (Seidel 1997). It has two equivalent descriptions : a *geometric* and an *algebraic* description, respectively as morphisms

$$\pi_1\big(\mathrm{Ham}(M,\omega)\big)\longrightarrow\mathrm{HQ}_*(M)^\times\quad\text{and}\quad\pi_1\big(\mathrm{Ham}(M,\omega)\big)\longrightarrow\mathrm{Aut}\big(\mathrm{HF}_*(M)\big),$$

where $\mathrm{HQ}_*(M)^\times$ denotes the multiplicative group of the invertible elements of $\mathrm{HQ}_*(M)$.

Shengda Hu and François Lalonde (Hu and Lalonde 2010) adapted the algebraic construction to the Floer homology of a monotone Lagrangian L of the symplectic manifold $(M\,,\omega)$. The resulting morphism is defined on the fundamental group of $\mathrm{Ham}(M\,,\omega)$ *relative* to $\mathrm{Ham}(M\,,\omega\,;L)$, its subgroup consisting of those Hamiltonian diffeomorphisms which preserve L globally :

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big)\longrightarrow\mathrm{Aut}\big(\mathrm{HF}_*(L)\big).$$

In a joint work with them (Hu, Lalonde, and Leclercq 2011), we then adapted the geometric construction by defining a morphism

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big)\longrightarrow\mathrm{HQ}_*(L)^\times$$

which we proved to be equivalent to its algebraic counterpart.

In view of the applications discussed in this memoir, the algebraic version of the Seidel morphism of a Lagrangian is presented in Section 1.1 of Chapter III and the geometric version of the Seidel morphism of the ambient manifold in Section 2.1 (*ibid.*).

2. Rigidity properties of symplectic stuff

Here are three quite different types of rigidity results which are discussed in this HDR memoirs.

Super-heaviness of Lagrangians (Leclercq and Zapolsky 2018)

Lagrangian spectral invariants can be applied to the theory of heavy and super-heavy sets of Entov and Polterovich (Entov and Polterovich 2009). These are subsets of M which are *very* rigid in terms of intersections. In particular, they cannot be displaced by any symplectomorphism (non necessarily Hamiltonian). This is a much stronger rigidity phenomenon than the one mentioned in the introduction. For example, consider the 2–torus $\mathbb{T}^2 = S^1 \times S^1$ together with the product volume form. This is a symplectic manifold and any closed embedded loop is a Lagrangian submanifold. Let L be a meridian of $\mathbb{T}^2$. Since $\pi_2(\mathbb{T}^2, L) = 0$, no Hamiltonian diffeomorphisms can displace it from itself. However, "rotating $\mathbb{T}^2$ along a longitude" is a symplectomorphism (the volume form is preserved!) which displaces L from itself. The toy-example of a Lagrangian which is not displaceable by any symplectomorphism is the equator of S^2 (endowed with the round volume form). Since it divides the sphere into two parts of equal areas, no symplectomorphisms can displace it from itself.

In Section 2.2 of Chapter II, we prove that the Chekanov torus of $\mathbb{CP}^2$, as well as some exotic Lagrangian torus of $S^2 \times S^2$ are super-heavy. These examples do not provide new super-heavy Lagrangians, the Chekanov torus was proved to be super-heavy by Wu (Wu 2012), and the exotic torus by Eliashberg and Polterovich (Eliashberg and Polterovich 2010). However, they illustrate how easy it is to prove (super-) heaviness once *Lagrangian* spectral invariants have been defined and their properties established. Indeed, the proof boils down to an estimate which directly follows from the facts that these invariants

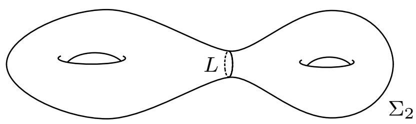
(i) are "controlled" by the values of *the restriction to the Lagrangian L* of the continuous function which enters the spectral invariant function ℓ as second variable, and
(ii) behave well with respect to their non-Lagrangian counterparts, through an additional algebraic structure of quantum homology

(see the proof of Proposition II.16).

Vanishing of Lagrangian monodromy (Hu, Lalonde, and Leclercq 2011)

This second rigidity result is based on the Lagrangian Seidel morphism. Assume that ϕ is a Hamiltonian diffeomorphism of a symplectic manifold *which preserves a Lagrangian L*. Then the restriction of ϕ to L induces an isomorphism of the homology of L and a natural question is to determine which isomorphisms can be obtained this way. For example, it was shown by Yau (Yau 2009) that the generators of the first homology group of the Chekanov Lagrangian torus of $\mathbb{CP}^2$ can be interchanged by a Hamiltonian diffeomorphism of $\mathbb{CP}^2$. It turns out that the *monotonicity* of the Chekanov torus is crucial here. Indeed, Theorem III.2 shows that, when the Lagrangian is *aspherical*, the situation exhibits surprising rigidity as *only the identity* of $\mathrm{H}_*(L)$ can be obtained like this. Equivalently, this ensures that if a diffeomorphism of L is not homologically trivial, it *cannot* be extended to M as a Hamiltonian diffeomorphism.

Of course, if the homology of the Lagrangian injects in that of the total space, a Hamiltonian diffeomorphism (which is isotopic to the identity) cannot induce any non-trivial automorphism of $\mathrm{H}_*(L)$. For example, the homology of the product of meridians in $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ is fixed by any Hamiltonian diffeomorphism of the ambient manifold. On the other hand, consider the following Lagrangian loop of the closed oriented surface Σ_2 of genus 2 :



There is no *a priori* reason why a Hamiltonian diffeomorphism of $\Sigma_2 \times \Sigma_2$ could not switch the generators of $\mathrm{H}_1(L \times L)$. And yet, it cannot, by Theorem III.2.

C^0 *rigidity of coisotropic submanifolds* (Humilière, Leclercq, and Seyfaddini 2015b, 2015a, 2016)

The surprising rigidity result of Gromov and Eliashberg mentioned above constituted the birth of C^0–symplectic geometry, the "first wonder" of symplectic geometry according to (Polterovich and Rosen 2014). Section 3 of Chapter II gathers several results obtained in this area, in joint works with Vincent Humilière and Sobhan Seyfaddini.

The central result, Theorem II.39 below, is in the spirit of the Gromov–Eliashberg theorem, but rather than looking at a sequence of symplectomorphisms $(\psi_k)_{k\in\mathbb{N}}$ which C^0 converges to a diffeomorphism ψ, we fix a coisotropic submanifold C and we restrict our attention to the sequence $(\psi_k(C))_{k\in\mathbb{N}}$. Our result is that, if $\psi(C)$ is smooth (ψ however can be "only" continuous), then it is coisotropic. Moreover, a coisotropic submanifold admits a foliation called *characteristic foliation* and we showed that the characteristic foliation of the limit is the C^0–limit of the characteristic foliations.

This result is remarkable for several reasons. First, it connects earlier *a priori* unrelated results, for Lagrangians by Laudenbach and Sikorav (Laudenbach and Sikorav 1994) and for hypersurfaces by Opshtein (Opshtein 2009), as "extreme" cases of the same rigidity phenomenon[5]. Second, it is local in the sense that C is not required to be closed, or equivalently the sequence $(\psi_k)_{k\in\mathbb{N}}$ is not required to be globally defined. Third, it shows that the smoothness of the image of C automatically yields the smoothness of the limiting foliation.

The preservation under C^0–limits of the characteristic foliation also raises another interesting question. Quotienting a coisotropic C by the leaves of its characteristic foliation yields (at least locally) another symplectic manifold. In the situation of Theorem II.39, the restriction to C of ψ descends to a homeomorphism of the resulting symplectic manifold and one might wonder "how symplectic" it is. Theorem II.44 provides a partial answer to this question as it shows, in the particular case of tori, that the resulting homeomorphism "preserves a capacity" built from spectral invariants.

The proof of the central theorem also relies on spectral invariants and capacities. In particular, we establish new energy-capacity inequalities for Hamiltonian diffeomorphisms Theorem II.27 as well as for Lagrangians Theorem II.30, which are inspired by the classical ones. They show that capacities provide lower bounds on certain natural distances on Hamiltonian diffeomorphism groups and sets of Lagrangians (Hamiltonian-isotopic to a fixed Lagrangian).

These inequalities have several consequences on C^0 *Hamiltonian dynamics*. Roughly speaking, as a homeomorphism which is a C^0–limit of symplectomorphisms can be thought of as a *symplectic homeomorphism* in view of the Gromov–Eliashberg Theorem, Müller and Oh (Oh and Müller 2007) defined a notion of *Hamiltonian homeomorphism*. They are uniquely generated (in some appropriate sense) by certain continuous functions called *continuous Hamiltonians*. These consequences are presented in Sections 3.2 and 3.3 of Chapter II. For example, Theorem II.37 shows that a continuous Hamiltonian H generates a flow which preserves a coisotropic C and flows along the leaves of its characteristic foliation if and only if H is a function of time (that is, H is a time-dependent constant function on M !). This result is a main step in the proof of Theorem II.39.

3. Computing symplectic invariants

The last part of this HDR memoirs is of different nature as it consists of *computations* of all *Seidel elements* (*i.e.* the invertible quantum classes in the image of the geometric description of Seidel's morphism) of certain symplectic manifolds and, in turn, of their quantum homology ring.

5. The adjective "extreme" refers to dimensions : Lagrangians are the smallest coisotropic submanifolds, while hypersurfaces are the greatest (proper) ones.

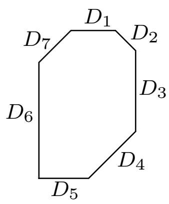
More precisely, McDuff and Tolman (McDuff and Tolman 2006) computed the highest order term and specified the structure of the other terms of the Seidel elements associated with certain loops of Hamiltonian diffeomorphisms for manifolds endowed with an almost complex structure *which does not admit pseudo-holomorphic spheres with negative first Chern number* (such manifolds are called *NEF*). When all pseudo-holomorphic spheres have positive first Chern number, all lower order terms vanish. In contrast, the existence of even one sphere with vanishing first Chern number might yield infinitely many terms.

In joint work with Sílvia Anjos (Anjos and Leclercq 2018), we express these additional terms by closed formulae, when the manifold is 4–dimensional and *toric*. Toric manifolds can be represented by convex polytopes of $\mathbb{R}^2$, and we also explain how one can read these formulae from the polytope. The specific formulae and how they can be extracted from the polytope is the content of Section 2.2 in the last chapter.

The fundamental group of the group of Hamiltonian diffeomorphisms of some of these 4–dimensional, toric symplectic manifolds are known, and we were able to compute *all* their Seidel elements (even for some non-NEF manifolds). As applications of these computations in the subsequent (Anjos and Leclercq 2017), we first observe that Seidel's morphism is injective on all Hirzebruch surfaces, see Theorem III.19. Then, turning to certain 3–point blow-ups of $\mathbb{CP}^2$, we are able to determine *explicitly* an element in the kernel of Seidel's morphism, see Theorem III.18. As far as we know, this is the only such example.

Another consequence of our computations is that we can compute the quantum homology of certain 4–dimensional NEF toric symplectic manifolds, still following ideas of McDuff and Tolman (*ibid.*). In turn, we establish the expression of the Landau–Ginzburg superpotential of such manifolds, which in this case happens to be readable from the associated polytope, see Section 2.3. We finish this introduction with an example which illustrates how easy it is to read such an *a priori* complicated information directly from the polytope (and thus – hopefully – illustrates the relevance of this work). This is part of several examples presented in Section 2.4.

Example 3. We want to compute the quantum homology ring of the 4–dimensional toric manifold (M, ω), obtained from $(S^2 \times S^2, \omega_\mu)$ by performing three blow-ups of respective capacities c_1, c_2, and c_3 (ω_μ is the split symplectic form such that the first factor has area $\mu \geq 1$ while the second has area 1). General theory of toric symplectic geometry asserts that (M, ω) corresponds uniquely (up to reasonable equivalence relations) to the following polytope $P \subset \mathbb{R}^2$:



Normals	Constants
$\eta_1 = (0, 1)$	$\kappa_1 = \mu$
$\eta_2 = (1, 1)$	$\kappa_2 = \mu - c_3$
$\eta_3 = (1, 0)$	$\kappa_3 = 0$
$\eta_4 = (1, -1)$	$\kappa_4 = -c_1$
$\eta_5 = (0, -1)$	$\kappa_5 = 0$
$\eta_6 = (-1, 0)$	$\kappa_6 = 1$
$\eta_7 = (-1, 1)$	$\kappa_7 = \mu + 1 - c_2$

which consists of the points $x \in \mathbb{R}^2$ such that for all i, $\langle x, \eta_i \rangle \leq \kappa_i$. Each edge D_i corresponds to a sphere in M.

The 4–dimensional manifold (M, ω) is NEF so its quantum homology ring is generated by its elements of degree 4, and $\mathrm{HQ}_4(M, \omega)$ is isomorphic to a quotient of a polynomial ring by an ideal. The polynomial ring consists of Laurent polynomials in two variables, z_1, z_2, over a ring of generalized Laurent series in a variable t. The ideal is generated by all the partial derivatives of one of these polynomials, W, called the *Landau–Ginzburg superpotential*. To get an explicit description of the quantum homology ring of (M, ω), we "only" need to compute W. In our case, it is given from P by

$$W = \sum_{i=1}^{7} z^{\eta_i} t^{\kappa_i} + \text{quantum correction terms}$$

where the quantum correction terms arise from edges which correspond to spheres in the manifold with vanishing first Chern number.

We orient all edges in the clockwise direction and we associate with D_i a weight w_i defined as its integral primitive direction vector. The sphere in M corresponding to the edge D_i has vanishing first Chern number if and only if its weight is the mean of the weights of its neighbors, *i.e.* if $2w_i = w_{i-1} + w_{i+1}$. Going around the polytope, we can easily check that only D_1 and D_3 correspond to spheres with vanishing first Chern number. Then Theorem III.15 asserts that there are only two additional quantum correction terms, $q(D_i) = z^{\eta_i} t^{\kappa_{i+1}+\kappa_{i-1}-\kappa_i}$ for $i = 1$ and 3, so that finally

$$W = \left(z_2 t^{\mu} + z_1 z_2 t^{\mu - c_3} + z_1 + z_1 z_2^{-1} t^{-c_1} + z_2^{-1} + z_1^{-1} t + z_1^{-1} z_2 t^{\mu+1-c_2}\right) + z_2 t^{\mu+1-c_2-c_3} + z_1 t^{\mu-c_1-c_3}\,.$$

CHAPTER I

Homology theories

Once upon a time, Gromov... (1)

"The paper under review opens a new effective approach to fundamental problems of symplectic topology." It is by this sweet euphemism that Yakov Eliashberg starts the MathSciNet® review of Gromov's foundational article (Gromov 1985). The use of pseudo-holomorphic curves did have and still has striking consequences in symplectic geometry, two of which are described below.

In the end of the 1980's, Floer (Floer 1989b, 1988a, 1988b, 1989a) constructed a homology theory, subsequently called *Floer homology*, in order to tackle the Arnol'd conjecture. This theory *is* built as "an infinite dimensional analogue of Morse homology" where the main piece of the construction (the analogue of the gradient flow lines) are 2-dimensional objects satisfying the pseudo-holomorphic equation, perturbed by a Hamiltonian function. The fact that the moduli spaces of such objects can be efficiently used in the construction is due to pseudo-holomorphic curves techniques, developed by Gromov (*ibid.*). Since Floer's work, this theory has been developed in a great variety of contexts, see the introduction of Section 4 for details and references.

Another (not unrelated!) striking consequence is the theory of *Gromov–Witten invariants*, relying also on Kontsevich's work on stable maps (Kontsevich 1995). They in particular yielded the definition of *quantum homology*, another homology theory for symplectic (sub-) manifolds. It was introduced in the context of topological quantum field theory by Vafa (Vafa 1991) and Witten (Witten 1991), then by Ruan and Tian (Ruan and Tian 1995) in symplectic geometry, together with Kontsevich and Manin (Kontsevich and Manin 1994) in algebraic geometry. This theory is also a mixed construction involving (the usual, finite-dimensional) Morse theory and pseudo-holomorphic 2-dimensional objects.

The description of these homology theories for symplectic manifolds and their Lagrangian submanifolds, as well as the relationships they share constitute the content of this chapter.

Organization of Chapter I

In Section 1, we first introduce the main objects and notions which will be studied in this HDR memoirs, we set up notation, and present important technical assumptions mentioned above and under which we will perform the different constructions.

In Section 2, we briefly describe the construction of Morse homology since it serves as an example for the two types of homology successively introduced afterward :

- the quantum homology ring in Section 3, which *is* Morse homology whose product (and even its differential in the Lagrangian setting) is twisted by the use of pseudo-holomorphic objects,
- the Floer homology ring in Section 4, where the ideas of the Morse construction are adapted to the infinite-dimensional setting.

Both Sections 3 and 4 are divided into 3 subsections since, after the construction of *Lagrangian* homology, we quickly describe that of the homology of the ambient space, and then show how to view the former as a module over the latter.

In Section 5, we describe interactions between quantum and Floer homologies, while Section 6 explains how to use specific classes of diffeomorphisms to act on these homologies.

Our contribution to this chapter is not much greater than the value of the $\overline{\partial}$ operator on a Floer trajectory. Namely, the isomorphism of Section 5.3, between the Floer (respectively quantum) homology of a symplectic manifold and the Floer (respectively quantum) homology of the Lagrangian diagonal in the product, was proved under asphericity by Biran, Polterovich, and Salamon (Biran, Polterovich, and Salamon 2003) and extended to the monotone setting in a joint work with Zapolsky (Leclercq and Zapolsky 2018).

A slightly more original result, Theorem I.22 below, which shows in particular that exact Lagrangian isotopies act on Lagrangian quantum homology (and that the resulting isomorphisms only depend on the homotopy class of the isotopy) also appeared in the aforementioned paper.

Finally, one might also hold the author responsible for the viewpoints of Sections 5.1 and 6 and, of course, for all the (hopefully rare and small) remaining mistakes...

1. The symplectic setting

1.1. Main objects of study

Let (M, ω) be a symplectic manifold. We will be interested in understanding certain properties of the group of its Hamiltonian diffeomorphisms as well as specific sets of its Lagrangian submanifolds. This section is dedicated to presenting in more details the objects, setting the notation used in the rest of the memoirs, as well as explaining which properties will be of interest.

1.1.1. *The objects*

A *symplectic manifold* is a $2n$-dimensional (smooth) manifold M endowed with a nondegenerate closed 2-form, ω. The automorphisms of (M, ω), *i.e.* diffeomorphisms ψ of M such that $\psi^*\omega = \omega$, are called *symplectomorphisms* and form a group, the *symplectomorphism group*, which is usually denoted by $\mathrm{Symp}(M, \omega)$.

A *Hamiltonian function* on M is a smooth, time-dependent function $H\colon M \times [0,1] \to \mathbb{R}$. For a fixed $t \in [0,1]$, we will often denote the function $H(t, \cdot)\colon M \to \mathbb{R}$ by H_t. The symplectic form being nondegenerate, it establishes a diffeomorphism between tangent and cotangent bundles of M, so that the differential of H_t corresponds to a unique vector field X_H^t satisfying

$$\text{for all } p \in M \text{ and all } \xi \in T_pM, \quad \omega(X_H^t(p), \xi) = -d_pH_t(\xi)$$

and called *Hamiltonian vector field* generated by H. We will denote by $\phi_H = \{\phi_H^t\}_t$ the *Hamiltonian isotopy* generated by H, defined by $\phi_H^0 = \mathrm{Id}$ and for all t in $[0,1]$, $\partial_t\phi_H^t = X_H^t(\phi_H^t)$. For all t, ϕ_H^t is a symplectomorphism.

The set of time-1 diffeomorphisms of such isotopies is called the *Hamiltonian group* and is denoted by $\mathrm{Ham}(M, \omega) = \{\psi \in \mathrm{Symp}(M, \omega) \mid \exists H \text{ so that } \psi = \phi_H^1\}$. It is easy to see that it is indeed a subgroup of the symplectomorphism group. For given Hamiltonians H and K, the Hamiltonian isotopy $\phi_H^t \circ \phi_K^t$ is generated by the Hamiltonian $H \circ K$ defined by $(H \circ K)_t(p) = H_t(p) + K_t((\phi_H^t)^{-1}(p))$. This immediately shows that $\widehat{H}$ defined by $\widehat{H}_t(p) = H_t((\phi_H^t)^{-1}(p))$ generates the isotopy $(\phi_H^t)^{-1}$.

Since adding a function of time to a Hamiltonian function does not alter the Hamiltonian vector field (and thus its flow), we need a normalization of Hamiltonian functions so that each of them uniquely corresponds to a Hamiltonian isotopy. The standard normalization condition depends on the compactness of the ambient manifold.

Definition I.1. A Hamiltonian is said to be *normalized* if for all t, $\int_M H_t\,\omega^n = 0$ in case M is compact or if it has compact support otherwise.

There is an equivalence relation on the set of normalized Hamiltonians which naturally arises when one is interested in studying Hamiltonian diffeomorphism groups.

Definition I.2. We say that normalized Hamiltonians H^0 and H^1 generating the same Hamiltonian diffeomorphism ϕ are *equivalent* if they are the extremities of a homotopy H^s, $s \in [0\,,1]$, such that for all s, H^s is normalized and $\phi^1_{H^s} = \phi$.

It was proved by Banyaga (Banyaga 1978) that a smooth path of Hamiltonian diffeomorphisms starting at identity, $\{\phi_t\}_{t\in[0,1]}$, is a Hamiltonian isotopy, *i.e.* there exists a Hamiltonian $H\colon\ M \times [0\,,1] \to \mathbb{R}$ such that for all t, $\phi_t = \phi^t_H$. Moreover, notice that two normalized Hamiltonians generate isotopies which are homotopic relative to endpoints if and only if they are equivalent. It follows that the set of equivalence classes of normalized Hamiltonians coincides with the universal cover of the Hamiltonian group, $\widetilde{\mathrm{Ham}}(M\,,\omega)$, whose elements will be denoted by $\widetilde{\phi}$.

Example I.3. Consider the concatenation of two Hamiltonians H and K, usually denoted $H\sharp K$, and defined by $(H\sharp K)_t(x) = 2K_{2t}(x)$ for $t \in [0\,,\frac{1}{2}]$ and by $(H\sharp K)_t(x) = 2H_{2t-1}(x)$ for $t \in [\frac{1}{2}\,,1]$. It generates a flow whose time-1 diffeomorphism is $\phi^1_H \circ \phi^1_K$. If H and K are normalized, then $H\sharp K$ and $H \circ K$ are normalized and equivalent to one another. Similarly, $\overline{H}$, defined by $\overline{H}_t(x) = -H_{1-t}(x)$, generates an isotopy whose time-1 diffeomorphism is $(\phi^1_H)^{-1}$, and is equivalent to $\widehat{H}$. In short, $\widetilde{\phi}_{H\circ K} = \widetilde{\phi}_{H\sharp K}$ and $\widetilde{\phi}_{\widehat{H}} = \widetilde{\phi}_{\overline{H}}$ in $\widetilde{\mathrm{Ham}}(M\,,\omega)$.

We will also be interested in certain sets of Lagrangians. Recall that an n-dimensional submanifold L_0 of M is *Lagrangian* if $\omega|_{TL_0} = 0$. We will denote by $\mathcal{L}_{\mathrm{Ham}}(L_0)$ the set of all Lagrangians isotopic to L_0 via a Hamiltonian isotopy. We will denote its universal cover by $\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L_0)$. Notice that by construction of the universal cover, the endpoint of one of its elements, $\widetilde{L}$, is a well-defined Lagrangian which we will denote by $\widetilde{L}(1)$.

We will also need to "reverse the perspective", so that we denote by $\mathrm{Ham}(M\,,\omega\,;L_0)$, the subgroup of $\mathrm{Ham}(M\,,\omega)$ which preserves L_0 globally. We denote by $\mathcal{P}_{L_0}\mathrm{Ham}(M\,,\omega)$ the set of smooth paths of Hamiltonian diffeomorphisms starting at identity and ending in $\mathrm{Ham}(M\,,\omega\,;L_0)$. The set of homotopy classes relative to endpoints of such paths is the pullback of the universal cover of the Hamiltonian diffeomorphism group by the inclusion :

$$\begin{array}{ccc} \pi_0\big(\mathcal{P}_{L_0}\mathrm{Ham}(M,\omega)\big) & \longrightarrow & \widetilde{\mathrm{Ham}}(M,\omega) \\ \big\downarrow & & \big\downarrow \mathrm{ev}_1 \\ \mathrm{Ham}(M,\omega;L_0) & \overset{i}{\hookrightarrow} & \mathrm{Ham}(M,\omega) \end{array}$$

which is why this set will appear later on.

1.1.2. *Hofer's distances*

There is a very natural quantity which can be associated with a Hamiltonian diffeomorphism ϕ, expressing the minimum amount of energy required to generate it. Namely, define the *energy* of a compactly supported Hamiltonian function H as its $L^{(1,\infty)}$-norm

$$\|H\| = \int_0^1 \operatorname*{osc}_M H_t\,dt = \int_0^1 \big(\max_M H_t - \min_M H_t\big)\,dt\,. \tag{I.1}$$

(This quantity can also be interpreted as the *length* of the induced Hamiltonian isotopy ϕ_H.) It induces a *norm* on the group of Hamiltonian diffeomorphisms

$$\|\phi\| = \inf\{\|H\| \mid \phi^1_H = \phi\}$$

which, in turn, induces a distance on $\mathrm{Ham}(M\,,\omega)$ by setting $d(\phi\,,\psi) = \|\phi\psi^{-1}\|$; the resulting distance is bi-invariant with respect to the action of $\mathrm{Ham}(M\,,\omega)$.

The fact that this defines a *genuine* distance, *i.e.* that $d(\phi\,,\psi) > 0$ as soon as ϕ differs from ψ, is highly non-trivial. It was first proved by Hofer for $\mathrm{Ham}(\mathbb{R}^{2n}\,,\omega_0)$ in (Hofer 1990) and was then extended to more general manifolds by Polterovich (Polterovich 1993) and to the general case by Lalonde and McDuff (Lalonde and McDuff 1995).

This distance, now called *Hofer's distance*, is nothing short of fascinating. It gave rise to (arguably countably) many interesting questions, deep results, and powerful tools. Let us give two examples here and encourage the interested reader to start additional readings with (Polterovich 2001).

First, Hofer's geometry is essentially unique. By this we mean that constructing a bi-invariant pseudo-distance on the Hamiltonian diffeomorphism group *the Finsler way*[1] produces either one of two things : either 0 or a distance equivalent to Hofer's distance. This deep result was proved by Buhovsky and Ostrover (Buhovsky and Ostrover 2011) building on an earlier result by Ostrover and Wagner (Ostrover and Wagner 2005). It generalized a (much) earlier result by Eliashberg and Polterovich (Eliashberg and Polterovich 1993) who proved that replacing the L^∞-norm by the L^p-norm for any p in Equation (I.1) produces 0, the trivial pseudo-distance.

Second, the development of the theory of spectral invariants, which is the focus of Chapter II, has been greatly motivated by the will to understand Hofer's geometry. For example, Viterbo introduced spectral invariants via the theory of generating functions in (Viterbo 1992) and used them to give a different proof of the nondegeneracy of Hofer's distance for $\mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$. The idea is that spectral invariants also yield a bi-invariant pseudo-distance which is smaller than Hofer's. Thus the nondegeneracy of the former immediately ensures the nondegeneracy of the latter. We will illustrate this fact in Section 2.1 of Chapter II.

A version of Hofer's distance for Lagrangians was defined and studied by Chekanov (Chekanov 2000). Define for $\widetilde{L} \in \widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$,

$$\|\widetilde{L}\| = \inf\{\|H\| \mid \{\phi_H^t(L)\}_{t\in[0,1]} \in \widetilde{L}\}$$

and for Lagrangian submanifolds themselves, set

$$\delta(L, L') = \inf\{\|H\| \mid \phi_H^1(L) = L'\}$$

for any L and $L' \in \mathcal{L}_{\mathrm{Ham}}(L_0)$. (When $L = L_0$, $\delta(L_0, L') = \inf\{\|\widetilde{L}\| \mid \widetilde{L}(1) = L'\}$.) Chekanov proved the nondegeneracy of δ on any symplectic manifold (M, ω) admitting an almost complex structure J such that $\omega(\cdot, J\cdot)$ is a Riemannian metric with bounded sectional curvature and with injectivity radius bounded away from 0.[2]

1.1.3. *Symplectic and pseudo-complex toolbox*

As explained in the introduction, most of the constructions which are presented in this first chapter are based on pseudo-holomorphic curves techniques. This necessitates some basic tools and notions which we gather here ; a good reference for this is (McDuff and Salamon 1998).

First, we will need *almost complex structures*, that is endomorphisms of TM which square to $-\mathrm{Id}$. They will be chosen to be *compatible* with ω, in the sense that

a) $\omega(J\cdot, J\cdot) = \omega$, b) for all $v \neq 0$, $\omega(v, Jv) > 0$.

An important feature of the set of almost complex structures, compatible with a given symplectic form, is that it is contractible (and non-empty !).

The compatibility condition ensures that the bilinear form $g = \omega(\cdot, J\cdot)$ is a Riemannian metric. It is used to define the *energy* of a 2-dimensional surface (immersed) in M. For example, the energy of a (half-) cylinder in M, $u\colon \mathbb{R}\times[0,1] \to M$, is defined as

$$E(u) = \int_{\mathbb{R}\times[0,1]} \|\partial_s u\|_g^2 \, dsdt = \int_{\mathbb{R}\times[0,1]} \omega(\partial_s u, J_u \partial_s u) \, dsdt \,. \tag{I.2}$$

---

1. The set of normalized autonomous Hamiltonians can be identified with the Lie algebra of the infinite dimensional Lie group $\mathrm{Ham}(M, \omega)$ endowed with the C^∞ topology. Thus by picking a norm on this set (*e.g.* the oscillation norm), we can "as usual" define the length of isotopies starting at identity by (I.1), and then infer a distance on $\mathrm{Ham}(M, \omega)$ by taking the infimum over the set of appropriate paths.

2. Such a manifold is called "tame" in (Chekanov 2000) while "tame" usually refers only to the fact that $\omega(\cdot, J\cdot)$ is positive definite, see *e.g.* (McDuff and Salamon 1998) and also below for quite related notions.

We will also often refer to the *first Chern class* of (M, ω) which is short for "the first Chern class of the tangent bundle of M with respect to (any) almost complex structure compatible with ω". It will be denoted by c_1 when there will be no ambiguity.

We will also need assumptions on the *minimal Chern number* of M, which will be denoted by C_M. It is defined as the positive generator of $c_1(\pi_2(M)) = C_M\mathbb{Z}$ if the latter is not trivial and it is set to $C_M = \infty$ otherwise.

On the Lagrangian side, additionally to spheres in M there will also be discs in M with boundary on a Lagrangian L, and the *Maslov index* (Viterbo 1987), $\mu : \pi_2(M, L) \to \mathbb{Z}$, plays the role of the first Chern class. Roughly writing, a disc in M with boundary in L, u, gives a trivialization of u^*TM thanks to which we get a loop of Lagrangians in $\mathbb{R}^{2n}$ along ∂u, the boundary of u. The Maslov index quantifies the non-triviality of this loop. This leads to the notion of *minimal Maslov number*, N_L, defined as the positive generator of $\mu(\pi_2(M, L)) = N_L\mathbb{Z}$ if it exists, or set to $N_L = \infty$ otherwise.

Finally, and as is already partially clear from the last paragraph, we will need discs (and half-discs) in M. The discs will be maps from D, the standard unit disc in $\mathbb{C}$, to M. They will be equipped with the usual complex structure coming from $\mathbb{C}$, also known as i, whenever we will need them to be pseudo-holomorphic mappings to (M, J), for a given almost complex structure on TM. They will usually have boundary either in L or along a Hamiltonian orbit for which they will serve as *cappings*.

The half-discs will be cappings of Hamiltonian chords from L to itself. They will be maps from $D_- = \{z \in D \,|\, \mathrm{im}(z) \leq 0\}$ to M, the "round" part of their boundary will be mapped to L, and the "straight" part to a chord.

1.2. Symplectic (and almost complex) assumptions

The different constructions we present below can be performed under different technical assumptions. We now quickly review the most common ones and explain the choices made in the rest of the present memoirs.

1.2.1. *Common technical assumptions*

A symplectic manifold (M, ω) is

(1) *exact* if the symplectic form is exact, i.e there exists a 1-form λ such that $\omega = d\lambda$,
(2) *(symplectically) aspherical* if the symplectic form, ω, and the first Chern class, c_1, vanish on $\pi_2(M)$,
(3) *monotone* if there exists a constant $\nu \geq 0$ such that for all $A \in \pi_2(M)$, $\omega(A) = \nu c_1(A)$; ν is called the *monotonicity constant*,
(4) *rational* if the group of periods $\omega(\pi_2(M))$ is discrete.

There are many other assumptions – with quite subtle variations of adjectives – which appeared in the literature during the development of Floer homology, for example

(5) *strong semi-positivity* : for any $A \in \pi_2(M)$, $2 - n \leq c_1(A) < 0 \Rightarrow \omega(A) \leq 0$,
(6) *weak monotonicity* : for any $A \in \pi_2(M)$, $3 - n \leq c_1(A) < 0 \Rightarrow \omega(A) \leq 0$.

Strong semi-positivity is equivalent to : either monotonicity, or the vanishing of c_1 on $\pi_2(M)$, or the minimal Chern number being greater than or equal to $n - 1$. The same holds for weak monotonicity, with $n - 1$ replaced by $n - 2$. They are quite natural (if not from a geometric viewpoint) in terms of the difficulties encountered to define Floer homology since a lower bound on the first Chern number of pseudo-holomorphic spheres is quite helpful to ensure the compactness of moduli spaces which is itself necessary to the construction.

The existence of pseudo-holomorphic spheres with vanishing first Chern number will also be in the center of Section 2 of Chapter III, so that we introduce here two additional definitions. We do emphasize that they concern almost complex manifolds rather than symplectic manifolds. An almost complex manifold (M, J) is *Fano* if all pseudo-holomorphic spheres have positive first Chern number, while it is *NEF* (for Numerically EFfective) if such spheres have non-negative first Chern number.

Let L be a Lagrangian submanifold of a symplectic manifold (M,ω). Similarly to the previous definitions, L is

(1) *exact* if (M,ω) is exact and the restriction to L of the primitive of ω is itself exact, *i.e.* $\lambda|_L = df$ for some function $f\colon L \to \mathbb{R}$,
(2) *(relatively symplectically) aspherical* or *weakly exact* if the symplectic form, ω, and the Maslov number, μ, vanish on $\pi_2(M,L)$,
(3) *monotone* if there exists a constant $\tau \geq 0$ such that for all $A \in \pi_2(M,L)$, $\omega(A) = \tau\mu(A)$; τ is called the *monotonicity constant*.

It is easy to see that the monotonicity of L implies that the restrictions to $\pi_2(M)$ of ω and c_1 either both vanish, that is (M,ω) is symplectically aspherical (this is for example the case when $\pi_2(M) = 0$), or satisfy $\omega|_{\pi_2(M)} = 2\tau c_1|_{\pi_2(M)}$, in which case (M,ω) is monotone with monotonicity constant $\nu = 2\tau$.

Definition I.4. We will often work under the assumptions that L is monotone with constant $\tau > 0$ and that its minimal Maslov number N_L satisfies $N_L \geq 2$. In order to ease the reading we will call such a Lagrangian *monotone*$^+$.

Disclaimer I.5. When not specified, all the considered symplectic manifolds and Lagrangian submanifolds are connected, and closed (compact, without boundary). ▮

1.2.2. *In this memoirs…*

Unlike in a talk where it is common (and much easier) to present a general theory under the most restrictive assumptions and then shamelessly appeal to the imagination of the audience, opportunity is given here to expose many constructions in a fairly general situation.

This being written, there are choices to be made so that the present memoirs can still be readable (I gave up all hope of an enjoyable reading after writing down the titles of all subsections of Chapter I...). This is mainly due to the fact that there are many, subtly different, versions of the main constructions which will be needed in the following chapters : pretty much all possible combinations of [“absolute” or Lagrangian], [quantum or Floer] homology under the [asphericity or monotonicity or NEF] assumption will appear later[3]. Moreover, the coefficients used for these homology theories is also a sensitive matter and many different cases will be evoked, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Q}$, Novikov rings of all sorts...

So in the reminder of this chapter, choice was made to present the main constructions with a focus on

the Lagrangian setting, under the monotonicity condition, and for $\mathbb{Z}/2\mathbb{Z}$ coefficients.

This being written, let us immediately add that we will also briefly describe the “absolute” versions of the constructions, and explain how to pass from monotonicity to asphericity. Moreover, we will also briefly describe the “absolute” quantum homology ring under the NEF assumption, as this will be needed to understand the second half of Chapter III.

2. The finite dimensional model : Morse homology

Let us start with the mother of all homology theories presented below, also known as *Morse homology*. (This is a very quick overview, for more details on the topic, *in the perspective of shifting to Floer theory*, we recommend (Schwarz 1993) together with (Audin and Damian 2014), or the quite enjoyable recent survey (Abbondandolo and Schlenk 2017).)

Let W be a smooth compact manifold and $f\colon W \to \mathbb{R}$ a Morse function. Let Crit(f) denote the set of its critical points. As f is Morse and W compact, Crit(f) is finite. Let

3. This is even without mentioning Floer homology for intersections of *pairs* of Lagrangians...

ρ be a Riemannian metric on W. It allows us to define, for any $p \in \mathrm{Crit}(f)$ the stable and unstable manifolds of p, that is, respectively

$$\mathcal{W}^s(p; f, \rho) = \{x \in W \mid \lim_{t \to \infty} \gamma_x(t) = p\} \quad \text{and} \quad \mathcal{W}^u(p; f, \rho) = \{x \in W \mid \lim_{t \to -\infty} \gamma_x(t) = p\}$$

where $t \mapsto \gamma_x(t)$ is the flow line of the *negative* gradient flow of f with respect to ρ, going through x at time $t = 0$. This in turn yields the following two definitions. A pair (f, ρ) is *Morse–Smale* if for any pair of critical points p and q of f, $\mathcal{W}^s(p; f, \rho)$ and $\mathcal{W}^u(q; f, \rho)$ intersect transversely. The *(Morse) index* of a critical point p of f, denoted $|p|_f$, is defined as the dimension of $\mathcal{W}^u(p; f, \rho)$. We now fix such a Morse–Smale pair (f, ρ) and since there is no ambiguity, we will denote respectively $\mathcal{W}^s(p; f, \rho)$ by $\mathcal{S}(p)$ and $\mathcal{W}^u(p; f, \rho)$ by $\mathcal{U}(p)$.

Example I.6. Consider the sphere S^2 endowed with the metric ρ induced by its embedding into $\mathbb{R}^3$ illustrated on the left-most part of Figure I.1. Let f be the height function. It has 4-critical points p_0, p_1, p_2^1, and p_2^2.

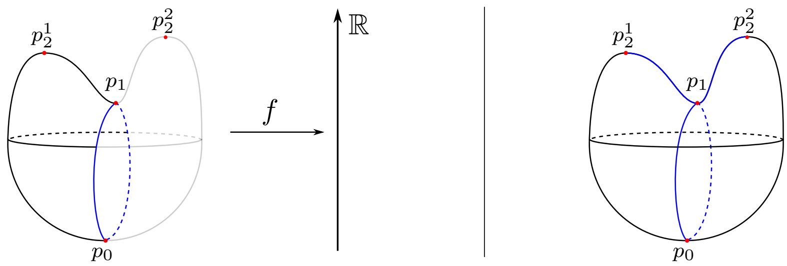


FIGURE I.1. Illustration of the construction of Morse homology

The black 2-dimensional open disc on the left is the unstable manifold of p_2^1, while the unstable manifold of p_1 is represented in blue. Of course, the unstable manifold of p_2^2 is the gray open 2-disc on the right, and p_0 has a 0-dimensional unstable manifold. (Notice that the subscript of a critical point is the dimension of its unstable manifold, *i.e.* its index.)

Now, we consider the $\mathbb{Z}/2\mathbb{Z}$-vector space generated by the critical points of f, $\mathrm{CM}_*(W; f) = \mathbb{Z}/2\mathbb{Z}\langle \mathrm{Crit}(f) \rangle$, which is graded by the Morse index. Let us define a map of degree -1 on $\mathrm{CM}_*(W; f)$ by setting

$$\partial_{(f,\rho)}\, p = \sum_{q \text{ s.t. } |q|_f = |p|_f - 1} \#_2 \mathcal{M}(p, q; f, \rho) \cdot q \quad \text{with } \mathcal{M}(p, q; f, \rho) = \big(\mathcal{U}(p) \cap \mathcal{S}(q)\big)/\,\mathbb{R}$$

for all critical points p of f and extending it by linearity to the whole space. The symbol $\#_2$ denotes the cardinality mod 2, and $\mathbb{R}$ acts on the *connecting manifold* of p and q, $\mathcal{U}(p) \cap \mathcal{S}(q)$, by reparameterization of flow lines. Hence, $\#_2\mathcal{M}(p, q; f, \rho)$ actually counts (mod 2) the number of *geometric* flow lines going from p to q.

To ease the reading, we remove all references to the fixed f and ρ from the notation. The essential fact which makes the whole construction work is that, for general critical points p and q, $\mathcal{M}(p, q)$ is a manifold of dimension

$$\dim(\mathcal{M}(p, q)) = \dim(\mathcal{U}(q) \cap \mathcal{S}(q)) - 1 = |p| - |q| - 1$$

which can be compactified in a very specific way. In particular, when $|q| = |p| - 1$, $\mathcal{M}(p, q)$ is a compact 0-dimensional manifold so that the map ∂ is well-defined.

Moreover, when $|r| = |p| - 2$, $\mathcal{M}(p, r)$ can be compactified in such a way that the boundary of its compactification splits as the union of *broken flow lines*

$$\partial\overline{\mathcal{M}}(p, r) = \bigcup_{q \text{ s.t. } |q| = |p| - 1} \mathcal{M}(p, q) \times \mathcal{M}(q, r)\,.$$

Since the number of boundary components of a compact 1-dimensional manifold vanishes mod 2, we deduce that

$$\partial^2_{(f,\rho)}\, p = \sum_{q,r} \#_2\mathcal{M}(p,q)\cdot \#_2\mathcal{M}(q,r) = \sum_r \#_2\, \partial\overline{\mathcal{M}}(p,r) = 0\,.$$

This ensures that the map $\partial_{(f,\rho)}$ squares to 0. It is a differential on $\mathrm{CM}_*(W;f)$ and the pair $(\mathrm{CM}_*(W;f)\,,\partial_{(f,\rho)})$ is the *Morse complex* of M associated with the Morse–Smale pair $(f\,,\rho)$. By taking the homology of this complex, we get the *Morse homology* of W, relatively to the Morse–Smale pair $(f\,,\rho)$: $\mathrm{HM}_*(W;f\,,\rho) = \mathrm{H}_*(\mathrm{CM}(W;f)\,,\partial_{(f,\rho)})$.

Example I.7. Back to Example I.6 above, we immediately see that

$$\mathrm{CM}_0(S^2;f) = \mathbb{Z}/2\mathbb{Z}\langle p_0\rangle\,,\quad \mathrm{CM}_1(S^2;f) = \mathbb{Z}/2\mathbb{Z}\langle p_1\rangle\,,\quad \text{and}\ \mathrm{CM}_2(S^2;f) = \mathbb{Z}/2\mathbb{Z}\langle p_2^1, p_2^2\rangle\,.$$

We represented on the right-most part of Figure I.1 the elements of the moduli space of connecting flow lines which belong to its 0-dimensional component. Notice that $\mathcal{M}(p_2^1\,,p_1)$ and $\mathcal{M}(p_2^2\,,p_1)$ consist of a single element (each), while $\mathcal{M}(p_1\,,p_0)$ consists of two elements. (In particular, $\mathcal{M}(p_2^1\,,p_0)$ is of dimension 1 and can indeed be compactified so that the boundary of its compactification corresponds to the two trajectories "broken at p_1".)

Thus, we get that $\partial p_2^2 = \partial p_2^1 = p_1$, while $\partial p_1 = 2p_0 = 0 = \partial p_0$. This shows that there are three cycles $p_2^1 + p_2^2$, p_1, and p_0, one of which is a boundary. In short,

$$\mathrm{HM}_0(S^2;f,\rho) = \mathbb{Z}/2\mathbb{Z}\langle p_0\rangle \ \text{and}\ \mathrm{HM}_2(S^2;f,\rho) = \mathbb{Z}/2\mathbb{Z}\langle p_2^1 + p_2^2\rangle$$

while $\mathrm{HM}_*(S^2\,;f\,,\rho) = 0$ in any other degree $* \neq 0$ and 2, which turns out to be isomorphic to the (whichever natural) homology of S^2 with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

The last observation is no coincidence, it is well-known that the homology of the Morse complex is independent of the Morse–Smale pair and coincides with the (cellular) homology of W. Thus we will usually denote it simply by $\mathrm{HM}_*(W)$. Moreover, it admits a product which is defined by counting a slightly more complicated type of connecting manifolds involving flow lines of several Morse–Smale pairs. This product corresponds to (*i.e.* is the Morse-theoretic version of) the *intersection product*, which turns the homology of W into a ring with unit. The unit is denoted $[W]$, by analogy with the fundamental class of W. (This will be explained in some details below, in the exposition of the quantum adaptation of the theory, see respectively Sections 3.1.2 and 3.1.3 below.)

Remark I.8 (Action of the diffeomorphism group of W). The diffeomorphism group of W, $\mathrm{Diff}(W)$, acts on Morse homology as follows. Let $(f\,,\rho)$ be a Morse–Smale pair for W and, for any diffeomorphism h of W, define the pair $(f^h\,,\rho^h)$ by $f^h = f\circ h^{-1}$ and $\rho^h = (h^{-1})^*\rho$. It is also Morse–Smale and mapping $p\in \mathrm{Crit}(f)$ to $p^h = h(p) \in \mathrm{Crit}(f^h)$ induces a bijection

$$h_*\colon\ \mathrm{CM}_*(W;f,\rho) \longrightarrow \mathrm{CM}_*(W;f^h,\rho^h)\,,\quad \sum_i p_i \longmapsto \sum_i p_i^h\,.$$

Given any two critical points of f, p and q, a flow line of f with respect to ρ and connecting p to q is mapped to a flow line of f^h with respect to ρ^h, connecting p^h to q^h, by $\gamma \longmapsto \gamma^h = h\circ\gamma$. This shows that h_* is a degree-0 chain morphism which then descends to an isomorphism $h_*\colon\ \mathrm{HM}_*(W;f\,,\rho) \to \mathrm{HM}_*(W;f^h\,,\rho^h)$.

This is the Morse theoretic version of the usual action of $\mathrm{Diff}(W)$ on $\mathrm{H}_*(W)$ by automorphisms, $\mathrm{Diff}(W) \to \mathrm{Aut}(\mathrm{H}_*(W))$. ▮

3. From Morse to quantum homology

Beyond presenting the construction of quantum homology (and many of the additional structures it enjoys), this section will hopefully show that quantum homology is somehow the most efficient way to get a Morse-type homology which also encodes the symplectic / almost complex data by taking into account pseudo-holomorphic objects (discs in M with boundary in L for the Lagrangian case, spheres in M in the absolute case).

In Section 4, we will present another way to do it, called Floer homology, which relies also on the additional data of a Hamiltonian. Then we will explain the relationships between the two theories and in particular, in Section 5.1, we will explain how one can see quantum homology as a Morse–Bott version of Floer homology, when the Hamiltonian function vanishes.

3.1. Lagrangian quantum homology

Lagrangian quantum homology was defined by Biran and Cornea (Biran and Cornea 2009, 2012)[4]. It was reformulated by Zapolsky (Zapolsky 2015) in a way allowing orientations of the various moduli spaces to be handled canonically, which can be understood as "allowing to define these homology theories with respect to a very large choice of coefficients". While the article (Leclercq and Zapolsky 2018), whose results are exposed in Sections 1 and 2 of the next chapter, follows the convention of the latter, we base the exposition of the homology theories on the former, more standard, since choice was made not to give any account on orientations by restricting the exposition to $\mathbb{Z}/2\mathbb{Z}$ coefficients.

To end this introduction, let us make the pretty whimsical remark that this section can also be used to get more details on some aspects of the construction and properties of Morse homology, simply by throwing away all pseudo-holomorphic discs... In particular, all quantum constructions below reduce to their Morse counterparts under the assumption of symplectic asphericity.

3.1.1. *The complex*

Let L denote a Lagrangian of a symplectic manifold (M, ω). Recall that we work here under the monotone$^+$ assumption (see Definition I.4). We fix a *quantum datum* for L, that is a triple $\mathcal{D} = (f, \rho, J)$ where (f, ρ) is a Morse–Smale pair for L and J is an almost complex structure on TM which is compatible with ω.

Introduce a variable t whose degree is set to $-N_L$ and denote by $\Lambda = \mathbb{Z}/2\mathbb{Z}[t^{-1}, t]$, the ring of Laurent polynomials in t. Now define the *Lagrangian quantum complex* as the $\mathbb{Z}/2\mathbb{Z}$-vector space $\mathrm{CQ}_*(L;\mathcal{D}) = \mathbb{Z}/2\mathbb{Z}\langle \mathrm{Crit}(f)\rangle \otimes \Lambda$. The differential is defined by counting *strings of pearls*, that is combinations of Morse flow lines of f with respect to ρ, and (i, J)-pseudo-holomorphic discs. Namely, for $A \in \pi_2(M, L)$, let $\widehat{\mathcal{M}_Q}(p, q; \mathcal{D}, A)$ consists of r-tuples of non-constant J-pseudo-holomorphic discs with boundary in L, $\underline{u} = (u_1, \dots u_r)$ with $r \geq 1$, such that

(1) $u_1(-1) \in \mathcal{U}(p)$ (in particular, $u_1(-1) = p$ is allowed),
(2) for all $1 \leq i \leq r-1$, $u_{i+1}(-1) = \gamma_{u_i(1)}(t_i)$ for some $t_i > 0$,
(3) $u_r(1) \in \mathcal{S}(q)$ (in particular, one might have $q = u_r(1)$),
(4) the total class $[\underline{u}] = [u_1] + \dots + [u_r] = A \in \pi_2(M, L)$.

Recall that for a point x in L which is not a critical point of f, γ_x denotes the flow line of f with respect to ρ which passes through x at $t = 0$, and recall that for a critical point x, $\mathcal{S}(x)$ and $\mathcal{U}(x)$ respectively denote the stable and unstable manifolds of x. A typical element is depicted in Figure I.2.

Remark I.9. It is important in the above definition to allow p and q to lie respectively on the first and last pseudo-holomorphic discs, but to require the flow line between any two successive discs to have positive length. This will be used to compensate for the phenomenon of bubbling off of certain pseudo-holomorphic discs at the boundary of the moduli space. ∎

In the Morse case, in order to count geometric objects we had to take the quotient of the connecting manifolds $\mathcal{U}(p) \cap \mathcal{S}(q)$ by the action of $\mathbb{R}$ corresponding to time shift. In the quantum case, we also need to do that for the possible non-trivial initial and final

4. The proofs of the main statements of (Biran and Cornea 2009) are carried out with (a dreadful but necessary amount of) details in (Biran and Cornea 2007).

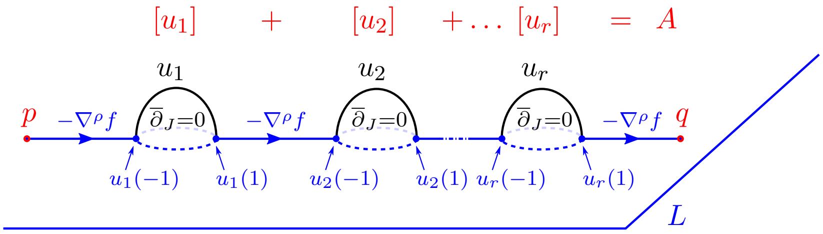


FIGURE I.2. A typical element $\underline{u} \in \widehat{\mathcal{M}_Q}(p\,,q\,;\mathcal{D}\,,A)$

flow lines, but we also need to mod out by the automorphism group of the set of pearls $\underline{u}$. Two such sets, $\underline{u}^1$ and $\underline{u}^2$, are equivalent if they have same length $r_1 = r_2 = r$ and if, for each pearl, there exists an automorphism σ_i of the domain, fixing ± 1 and such that $u_i^2 = u_i^1 \circ \sigma_i$ $(1 \leq i \leq r)$. We denote the quotient by $\mathcal{M}_Q(p\,,q\,;\mathcal{D}\,,A)$ and define the differential by

$$\partial_{\mathcal{D}}\, p = \sum_{q,A} \#_2 \mathcal{M}_Q(p, q; \mathcal{D}, A)\, q \otimes t^{\frac{\mu(A)}{N_L}} \tag{I.3}$$

on (Morse) generators and extend it by linearity and tensor product. In this definition, $\#_2$ denotes the cardinality mod 2 of $\mathcal{M}_Q(p\,,q\,;\mathcal{D}\,,A)$ and the sum runs over the set of $q \in \mathrm{Crit}(f)$ and $A \in \pi_2(M\,,L)$ such that $\dim \mathcal{M}_Q(p\,,q\,;\mathcal{D}\,,A) = |p|_f - |q|_f + \mu(A) - 1 = 0$.

Disclaimer I.10 (Transversality). There is a fairly big approximation implied by the notation $\dim \mathcal{M}_Q$ here as the moduli spaces are not (even only generically) smooth manifolds. The usual reason for this is the fact that some curves appearing in the moduli spaces are not simple. For spheres, for example, such a curve decomposes as a simple curve composed with a branched covering of the domain which allows one to deal with this issue by replacing the initial curve by the simple one. However, this decomposition does not hold in the case of discs with boundary on a Lagrangian and one has to be more subtle. The idea, which has been independently developed by Lazzarini (Lazzarini 2000, xxxx) and Kwon and Oh (Kwon and Oh 2000), is to start by a division of the domain into subdomains on which the restriction of the initial map decomposes into simple map and branched covering. The division preserves the total class $[u]$ and one can conclude as above. For generic choices, the moduli spaces of dimension 0 and 1 can thus be assumed to be smooth manifolds. ∎

As the 0-dimensional component of $\mathcal{M}_Q$ is compact, (I.3) defines a morphism of the graded vector space $\mathrm{CQ}_*(L\,;\mathcal{D})$ whose square can be shown to vanish. In order to do that, we proceed as in the Morse case : we need to understand the boundary of the compactification of the 1-dimensional component of the moduli space $\mathcal{M}_Q$. Gromov's compactness theorem (Gromov 1985) states that *up to possible bubbling off phenomena*, $\mathcal{M}_Q$ can be compactified by adding "broken" strings of pearls. The bubbling off phenomena are then taken care of by index considerations and the observation made in Remark I.9.

More precisely, the boundary of the compactification of $\mathcal{M}_Q(p\,,r\,;\mathcal{D}\,,A)$ might *a priori* consist of elements of the following various types (and combinations of such) :

(1) a Morse breaking, *i.e.* the products $\cup_{q,B,C} \mathcal{M}_Q(p\,,q\,;\mathcal{D}\,,B) \times \mathcal{M}_Q(q\,,r\,;\mathcal{D}\,,C)$ (with $B + C = A$) appear in $\partial\overline{\mathcal{M}_Q}(p\,,r\,;\mathcal{D}\,,A)$,
(2) the shrinking-to-nothing of a Morse flow line,
(3) the bubbling off of a J-pseudo-holomorphic disc (on ± 1 or elsewhere),
(4) the bubbling off of a J-pseudo-holomorphic sphere.

It is easy to see that, generically, the latter case does not happen. Indeed, the bubbling off of a sphere v requires the existence of an element in $\mathcal{M}_Q(p\,,r\,;\mathcal{D}\,,A - [v])$. However by the formula giving the (virtual) dimension of the moduli space, this specific component has dimension $\dim \mathcal{M}_Q(p\,,r\,;\mathcal{D}\,,A) - c_1([v]) = 1 - c_1([v])$ which is negative since $c_1([v]) \geq C_M$ which is greater than or equal to 2 (since C_M is infinite or $N_L | C_M$).

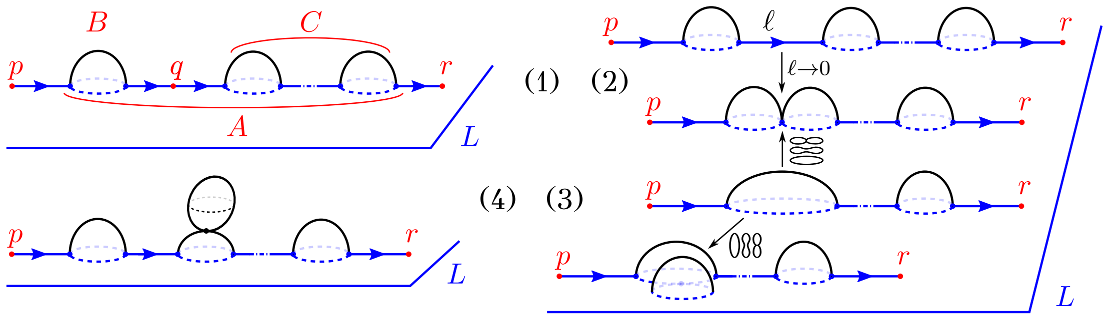


FIGURE I.3. Compactifying the 1-dimensional component of $\widehat{\mathcal{M}}_Q(p,q;\mathcal{D},A)$

The same observation applies to the bubbling off of a *side* disc, that is a disc which bubbles off of a sequence of pseudo-holomorphic discs at a boundary point different from ± 1. Thus, such bubbling off (illustrated by the lower right side of Figure I.3) generically does not happen.

What *does* happen is the bubbling off of discs *at* ± 1. However, this is compensated by the fact that such an element actually appears *twice* in the compactification, since it is also the limit of a sequence of elements for which one of the Morse flow lines entirely collapses (as illustrated on the upper right side of Figure I.3).

Proving that not only these are the only possibilities but that they actually appear results from standard gluing arguments which will not be exposed here (and not only because the proof is too large to fit in the margin).

This procedure shows that *mod* 2 (or with suitable orientations but Hush!) the Morse breakings sum to 0 which proves, as in the Morse case, that the differential squares to 0. We can thus define the *Lagrangian quantum homology* of L as the homology of this complex :

$$\mathrm{HQ}_*(L) = \mathrm{H}_*(\mathrm{CQ}(L;\mathcal{D}), \partial_{\mathcal{D}}) .$$

3.1.2. *Independence on the data*

As suggested by the notation of the homology, even though the complex itself heavily depends on the auxiliary choice of quantum datum $\mathcal{D}$, its homology does not. A standard proof of this fact is inspired by cobordism techniques from Morse homology. Pick two admissible quantum data $\mathcal{D}_i = (f_i, \rho_i, J_i)$ for $i = 0$ and 1 and a smooth 1-parameter family of admissible interpolating triples $t \longmapsto \mathcal{D}_t = (f_t, \rho_t, J_t)$ for $t \in [0,1]$ such that $\mathcal{D}_t$ indeed coincides with $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively for $t = 0$ and 1.

Considering $f\colon [0,1] \times L \to \mathbb{R}$, defined by $f(t,\cdot) = f_t$ for all t, one additionally requires that $\frac{\partial f}{\partial t} < 0$ (which might impose to first globally shift f_1 by a big enough constant) so that $\mathrm{Crit}(f) = \big(\{0\} \times \mathrm{Crit}(f_0)\big) \cup \big(\{1\} \times \mathrm{Crit}(f_1)\big)$. This allows us to identify $(i,p) \in \mathrm{Crit}(f)$ with $p \in \mathrm{Crit}(f_i)$ for $i = 0$ and 1. Under this identification, the indices satisfy $|p|_f = |p|_{f_0} + 1$ if $p \in \mathrm{Crit}(f_0)$ and $|p|_f = |p|_{f_1}$ if $p \in \mathrm{Crit}(f_1)$.

Now, define a morphism by formula (I.3) where the moduli space $\mathcal{M}_Q$ is slightly altered :

(1) $p \in \mathrm{Crit}(f_1)$ while the sum runs on $q \in \mathrm{Crit}(f_2)$,
(2) the Morse–Smale pair is replaced by (f, ρ), and non-trivial flow lines hit discs inside $(0,1) \times L$,
(3) each disc u_i is J_{τ_i}-pseudo-holomorphic for some $\tau_i \in [0,1]$.

In view of the indices, counting mod 2 the 0-dimensional component of these new moduli spaces yields a degree-0 morphism $\Phi_{\mathcal{D}}\colon \mathrm{CQ}_*(L;\mathcal{D}_0) \to \mathrm{CQ}_*(L;\mathcal{D}_1)$. Thanks to the same argument as above (however slightly harder as for example gluing arguments need to take into account that the almost complex structure is not constant anymore), analyzing the boundary of the compactification of the 1-dimensional component of the moduli spaces ensures that this chain morphism induces a morphism in homology.

Finally, one can show the following :

(1) $\Phi_{\mathcal{D}}$ does not depend on the choice of admissible cobordism triple $\mathcal{D}$ from $\mathcal{D}_0$ to $\mathcal{D}_1$, and we will thus denote it $\Phi_{\mathcal{D}_1 \leftarrow \mathcal{D}_0}$,

(2) if $\mathcal{D}$ and $\mathcal{D}'$ are admissible cobordism triples respectively from $\mathcal{D}_0$ to $\mathcal{D}_1$ and from $\mathcal{D}_1$ to $\mathcal{D}_2$, then $\Phi_{\mathcal{D}_2 \leftarrow \mathcal{D}_1} \circ \Phi_{\mathcal{D}_1 \leftarrow \mathcal{D}_0} = \Phi_{\mathcal{D}_2 \leftarrow \mathcal{D}_0}$,

(3) if $\mathcal{D}$ is the constant cobordism from $\mathcal{D}_0$ to itself, then $\Phi_{\mathcal{D}}$ is the identity.

The first point is proved along the exact same lines as the construction of $\Phi_{\mathcal{D}}$ except that one needs one more parameter in order to consider a cobordism triple between two given cobordism triples $\mathcal{D}$ and $\mathcal{D}'$ with identical extremities $\mathcal{D}_0$ and $\mathcal{D}_1$. Considering the indices one now constructs by counting the 0-dimensional component of these "new" moduli spaces a degree-1 map, $\xi\colon \ \mathrm{CQ}_*(L\,;\mathcal{D}_0) \to \mathrm{CQ}_{*+1}(L\,;\mathcal{D}_1)$, which can then be seen to be a chain homotopy, $\Phi_{\mathcal{D}} - \Phi_{\mathcal{D}'} = \xi\,\partial_{\mathcal{D}_1} + \partial_{\mathcal{D}_0}\,\xi$, by analyzing the boundary of the compactification of the 1-dimensional component.

The last two points are straightforward and obviously show that the canonical morphism constructed that way is actually an isomorphism. It is usually called the *continuation morphism* of quantum homology.

3.1.3. *Quantum product*

There exists a Lagrangian quantum version of the intersection product which turns it into an algebra over Λ. This enhancement, inspired by its Morse-theoretic counterpart, is realized at the chain level by counting the 0-dimensional component of suitable moduli spaces and is shown to induce a product in homology by considering the boundary of the compactification of the 1-dimensional component of the same moduli spaces. Hence, I will only describe the moduli spaces (as the deep arguments which make the whole machinery work – by ensuring transversality and compactness of the moduli spaces – are the exact same ones as the ones I did not describe earlier).

Consider three admissible quantum data $\mathcal{D}_i$ for $i = 0$, 1, and 2, consisting of three Morse functions f_i, a common metric ρ such that the three pairs $(f_i\,,\rho)$ are Morse–Smale, and a common ω-compatible almost complex structure J. We will reserve the subscript i for critical points of f_i and, to avoid notational redundancy, we will denote by $|\cdot|$ their respective indices (*i.e.* $p_i \in \mathrm{Crit}(f_i)$ and $|p_i|$ will stand for $|p_i|_{f_i}$).

The moduli space used to define the product consists of "Y" configurations for which each branch is a string of pearls with respect to a quantum datum $\mathcal{D}_i$, all three of them meeting at a (possibly trivial) J-pseudo-holomorphic disc. More precisely, for three critical points p_0, p_1, q_2, and a class $A \in \pi_2(M\,,L)$, define $\mathcal{M}_{*_Q}(p_0\,,p_1\,,q_2\,;\mathcal{D}_0\,,\mathcal{D}_1\,,\mathcal{D}_2\,,A)$ as the set of quadruples $\{\underline{u}^0\,,\underline{u}^1\,,\underline{u}^2\,,v\}$ such that

(1) v is a possibly constant J-pseudo-holomorphic disc, on which three points are marked : $p_0' = v(e^{\frac{2\pi}{3}})$, $p_1' = v(e^{\frac{-2\pi}{3}})$, and $q_2' = v(1)$,

(2) for $i = 0$ and 1, $\underline{u}^i \in \mathcal{M}_Q(p_i\,,p_i'\,;\mathcal{D}_i\,,A_i)$, while $\underline{u}^2 \in \mathcal{M}_Q(q_2'\,,q_2\,;\mathcal{D}_2\,,A_2)$, with A_0, A_1, and $A_2 \in \pi_2(M\,,L)$, such that $A = A_0 + A_1 + A_2 + [v]$.

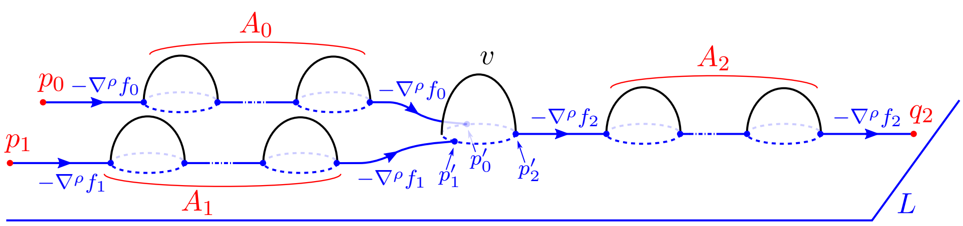


FIGURE I.4. A typical element $\underline{u} \in \mathcal{M}_{*_Q}(p_0\,,p_1\,,q_2\,;\mathcal{D}_0\,,\mathcal{D}_1\,,\mathcal{D}_2\,,A)$

The virtual dimension of $\mathcal{M}_{*_Q}(p_0\,,p_1\,,q_2\,;\mathcal{D}_0\,,\mathcal{D}_1\,,\mathcal{D}_2\,,A)$ is $|p_0|+|p_1|-|q_2|+\mu(a)-n$ (with n being the dimension of L) and its 0- and 1-dimensional components are smooth manifolds for generic choices of the data $\mathcal{D}_i$. The 0-dimensional component is compact, thus the formula

$$p_0 * p_1 = \sum_{q_2, A} \#_2 \mathcal{M}_{*_Q}(p_0, p_1, q_2; \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, A)\, q_2 \otimes t^{\frac{\mu(A)}{N_L}}$$

where the sum runs over all the critical points q_2 of f_2 and all classes $A \in \pi_2(M, L)$ such that $|q_2| - \mu(a) = |p_0| + |p_1| - n$, extended by bi-linearity and tensor product, defines a chain map $\mathrm{CQ}_*(L, \mathcal{D}_0) \times \mathrm{CQ}_{*'}(L, \mathcal{D}_1) \to \mathrm{CQ}_{*+*'-n}(L, \mathcal{D}_2)$. The latter is shown to induce a product in homology,

$$* : \ \mathrm{HQ}_*(L) \times \mathrm{HQ}_{*'}(L) \longrightarrow \mathrm{HQ}_{*+*'-n}(L)\,,$$

by considering the boundary of the compactification of the 1-dimensional component of $\mathcal{M}_{*_Q}$.

Remark I.11. It is possible and sometimes very useful to "only" consider two admissible quantum data by choosing $f_1 = f_2$ in the construction we just described. (This slightly simplifies some arguments which have not been made explicit in this memoirs, but somehow breaks the symmetry.) ∎

Let us finish this section with a useful exercise.

Exercise I.12. Show that the algebra $(\mathrm{HQ}_*(L), *)$ admits a unit which is of degree n. (*Hint.* This is one of the occurrences where choosing $f_1 = f_2$ might help. Choose also f_0 with a single maximum, m_0. Then show that for all $p_1 \in \mathrm{Crit}(f_1)$, $m_0 * p_1 = p_1$, thanks to clever considerations on the respective dimensions of the moduli spaces $\mathcal{M}_{*_Q}$ and $\mathcal{M}_Q$.)

This element is denoted $[L]$ and called *(quantum) fundamental class* of L by analogy.

3.2. Quantum homology

There is also a quantum homology theory of the ambient manifold without mention of an auxiliary Lagrangian. It was defined earlier than the Lagrangian version described above, and served as model. The main difference between both versions is that in this "absolute" setting, pseudo-holomorphic objects only alter non-trivially the intersection product (and not the differential of the complex itself, compare with the definition of $\mathcal{M}_Q$ of Section 3.1.1).

Disclaimer I.13. Exceptionally, we describe this construction in the more general setting of *strongly semi-positive manifolds* and with coefficients in the field $\mathbb{Q}$ rather than $\mathbb{Z}/2\mathbb{Z}$. This is due to the fact that we will have to work under this assumption and with these coefficients in Section 2 of Chapter III. This is also why we follow the presentation of McDuff and Tolman (McDuff and Tolman 2006).

In Remark I.14 below, we briefly explain how the construction gets simplified under the more restrictive assumption of monotonicity and with $\mathbb{Z}/2\mathbb{Z}$ coefficients since this case will also be of interest (as for example in Section 3.3). We also harmonize the presentation with that of the Lagrangian quantum homology above. ∎

The *quantum homology* of (M, ω) is defined as the tensor product $\mathrm{HQ}_*(M;\Gamma) = \mathrm{H}_*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma$ where Γ is the ring $\Gamma^{\mathrm{univ}}[q, q^{-1}]$: the variable q is of degree 2 and Γ^{univ} is a ring of generalized Laurent series in a variable of degree 0,

$$\Gamma^{\mathrm{univ}} = \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \,\middle|\, r_\kappa \in \mathbb{Q},\ \text{and for all } c \in \mathbb{R},\ \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty \right\}. \tag{I.4}$$

The quantum homology $\mathrm{HQ}_*(M;\Gamma)$ is $\mathbb{Z}$-graded in such a way that for all $a \in \mathrm{H}_k(M;\mathbb{Z})$, all $d \in \mathbb{Z}$ and all $\kappa \in \mathbb{R}$, $\deg(a \otimes q^d t^\kappa) = k + 2d$.

For two classes $a \in \mathrm{H}_i(M;\mathbb{Z})$ and $b \in \mathrm{H}_j(M;\mathbb{Z})$, their *quantum intersection product*, $a * b \in \mathrm{HQ}_{i+j-\dim M}(M;\Gamma)$, has the form

$$a * b = \sum_{B \in \mathrm{H}_2^S(M;\mathbb{Z})} (a * b)_B \otimes q^{-c_1(B)} t^{-\omega(B)}\,, \tag{I.5}$$

where $\mathrm{H}_2^S(M;\mathbb{Z})$ is the image of $\pi_2(M)$ under the Hurewicz map. The homology class $(a*b)_B \in \mathrm{H}_{i+j-\dim M+2c_1(B)}(M;\mathbb{Z})$ is defined by the requirement that

$$(a*b)_B *_M c = \mathrm{GW}^M_{B,3}(a,b,c) \quad \text{for all } c \in \mathrm{H}_*(M)\,.$$

In this formula, $*_M$ is the intersection product in M, and $\mathrm{GW}^M_{B,3}(a\,,b\,,c) \in \mathbb{Q}$ denotes the Gromov–Witten invariant which, *roughly*, counts the number of J-pseudo-holomorphic spheres in M in class B that meet cycles representing the classes a, b, and $c \in \mathrm{H}_*(M;\mathbb{Z})$. The product $*$ is extended to $\mathrm{HQ}_*(M;\Gamma)$ by tensor product with Γ. It is associative. It also respects the grading and gives $\mathrm{HQ}_*(M;\Gamma)$ the structure of a graded commutative algebra over Γ, with unit $[M] \in \mathrm{H}_{2n}(M;\mathbb{Z})$, seen as $[M]\otimes 1 \in \mathrm{HQ}_{2n}(M;\Gamma)$.

Remark I.14. First, note that we will use this theory in Chapter III for blow-ups of $\mathbb{CP}^2$ which are simply connected so that their second homotopy and homology groups are isomorphic and thus $\mathrm{H}_2^S \simeq \pi_2$.

Second, if we replace $\mathrm{H}_*(M;\mathbb{Z})$ by its Morse theoretic version, we see that we need a Morse–Smale pair for M, $(f\,,\rho)$, together with an almost complex structure J which is ω-compatible. The triple $\mathcal{D} = (f\,,\rho\,,J)$ constitutes the equivalent "absolute" quantum datum. Moreover, we see that in this case, the quantum intersection product corresponds in spirit to its Lagrangian counterpart (again, except for the fact that pseudo-holomorphic objects *do not* alter the differential of the Morse complex), roughly counting the 0-dimensional component of moduli spaces of pseudo-holomorphic spheres meeting the unstable manifolds of critical points representing a and b, and the stable manifold of a critical point representing c (compare, with Figure I.5).

Finally, we see that when the manifold is not monotone, we need two variables, t and q, which respectively keep track of the symplectic area and the first Chern number of the involved pseudo-holomorphic spheres. When $(M\,,\omega)$ is monotone and the coefficient field is chosen to be $\mathbb{Z}/2\mathbb{Z}$ rather than $\mathbb{Q}$, the quantum homology of M can be taken with coefficients in $\Gamma = \mathbb{Z}/2\mathbb{Z}[q^{-1}\,,q]$. To harmonize the presentation with that of the Lagrangian case above, change the variable q for the variable $s = q^{-C_M}$ whose degree is thus $-2C_M$ and define the quantum homology of M as $\mathrm{HQ}_*(M;\Gamma) = \mathrm{H}_*(M;\mathbb{Z}/2\mathbb{Z})\otimes_{\mathbb{Z}/2\mathbb{Z}} \Gamma$, with $\Gamma = \mathbb{Z}/2\mathbb{Z}[s^{-1}\,,s]$. (There are of course a few additional adaptations to be made, for example $q^{-c_1(B)}$ should be replaced by $s^{\frac{c_1(B)}{C_M}}$ in the formula defining the product of classes a and b, (I.5) above.) ∎

3.3. A quantum module structure

There is one more algebraic structure which will be useful in Section 1 of the next chapter : the Lagrangian quantum homology of L is a module over the quantum homology ring of M.[5] This additional structure is defined via the procedure described above already several times...

Namely, suitable moduli spaces consisting of geometric objects relating generators of the different complexes will be defined thanks to auxiliary data. For generic choices, the low dimensional components of these moduli spaces are smooth manifolds which dimension is expressed in terms of the respective indices of the generators. The 0-dimensional component of the moduli spaces is compact and allows to define a chain map via a mod 2 count. The 1-dimensional component of the moduli space can be compactified, and analyzing the boundary of its compactification will show that this chain map induces a structure on the homologies. Finally, via cobordism arguments, one can show that it does not depend on the auxiliary data chosen for its construction and commutes with the adequate continuation morphisms.

As this is standard procedure by now (in this memoirs also), I will only here describe the relevant moduli spaces.

5. Actually, both rings being algebras over their respective Novikov rings, this yields a structure of super-algebra of $\mathrm{HQ}_*(L)$ over $\mathrm{HQ}_*(M)$.

Remark I.15. As we need both quantum homology theories to interact, let us recall from Section 1.2 that since L satisfies the monotone$^+$ assumption, either $\omega|_{\pi_2(M)} = 0$ or (M, ω) is monotone of monotonicity constant twice that of L. Below, *we work in the latter case, when* (M, ω) *is monotone*. The construction in the aspherical case, when ω vanishes on $\pi_2(M)$, can easily be deduced by simply removing all pseudo-holomorphic spheres from our description (since these spheres do not exist / have to be trivial). In particular, in the absence of pseudo-holomorphic spheres in M, the minimal Chern number is set to be ∞, the Novikov ring is nothing but the field $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and the quantum homology of M then reduces to its Morse homology. ∎

Since (M, ω) is assumed to be monotone, we consider the construction of its quantum homology as described in Remark I.14 above. In particular, $\Gamma = \mathbb{Z}/2\mathbb{Z}[s^{-1}, s]$ with s of degree $-2C_M$. Now, since C_M is a multiple of N_L, we see that there is a natural inclusion of Γ into $\Lambda = \mathbb{Z}/2\mathbb{Z}[t^{-1}, t]$ given by $s \mapsto t^{\frac{2c_M}{N_L}}$. In turn, this allows us to consider Λ as a Γ-module and the quantum homology of M, with coefficients in Λ by setting

$$\mathrm{HQ}_*(M;\Lambda) = \mathrm{H}_*(M;\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \Lambda = \mathrm{HQ}_*(M;\Gamma) \otimes_\Gamma \Lambda\,.$$

Let (f_M, ρ_M) be a Morse–Smale pair for M and denote the Morse complex with Λ coefficients by $\mathrm{CM}_*(M; f_M, \rho_M; \Lambda) = \mathbb{Z}/2\mathbb{Z}\langle \mathrm{Crit}(f_M)\rangle \otimes \Lambda$. There is an obvious Λ-module isomorphism between the homology of this complex and $\mathrm{HQ}_*(M;\Lambda)$.

We pick a Morse–Smale pair (f_L, ρ_L) for L and an ω-compatible almost complex structure J. We denote by $\mathcal{D}_M = (f_M, \rho_M, J)$ and $\mathcal{D}_L = (f_L, \rho_L, J)$ the respective quantum data. Let p and $q \in \mathrm{Crit}(f_L)$, $a \in \mathrm{Crit}(f_M)$, and $A \in \pi_2(M, L)$. First, consider pairs $(\underline{u}, k)$ where $\underline{u} = (u_1, \dots u_r)$, for some integer r, is a *generalized* string of pearls from p to q, *i.e.* an element of $\mathcal{M}_Q(p, q; \mathcal{D}_L, A)$ which satisfies Items 1 to 4 above Remark I.9 *except for the following facts* :

(1) the J-pseudo-holomorphic disc u_k, with $1 \le k \le r$, is allowed to be constant,
(2) $u_k(0)$ lies in $\mathcal{U}_{(f_M,\rho_M)}(a)$, the unstable manifold of a for f_M with respect to ρ_M.

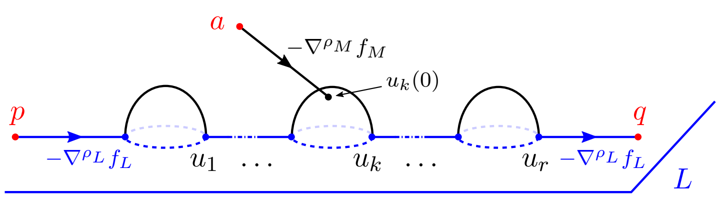


FIGURE I.5. A typical element $(\underline{u}, k) \in \mathcal{M}_{\odot_Q}(a, p, q; \mathcal{D}_M, \mathcal{D}_L; A)$

Now, define $\mathcal{M}_{\odot_Q}(a, p, q; \mathcal{D}_M, \mathcal{D}_L; A)$ as the set of equivalence classes of such pairs $(\underline{u}, k)$, where $(\underline{u}^1, k^1)$ and $(\underline{u}^2, k^2)$ are equivalent if $k^1 = k^2$ and $\underline{u}^1$ is equivalent to $\underline{u}^2$ as generalized strings of pearls. The slight difference with Section 3.1.1 is that this reduces here to the existence of $r-1$ automorphisms σ_i of D (r denotes $r^1 = r^2$) preserving ± 1, such that $u_i^2 = u_i^1 \circ \sigma_i$ for all $i \ne k$, as u_k^j has already three marked points.

As mentioned in introduction of the section, the standard procedure shows that

$$a \odot p = \sum_{y,A} \#_2 \mathcal{M}_{\odot_Q}(a, p, q; \mathcal{D}_M, \mathcal{D}_L; A)\, q \otimes t^{\frac{\mu(A)}{N_L}}$$

where the sum runs over all $y \in \mathrm{Crit}(f_L)$ and $A \in \pi_2(M, L)$ such that $|y|_{f_L} - \mu(A) = |x|_{f_L} + |a|_{f_M} - 2n$, extended by linearity and tensor product, defines a chain map. The latter yields the external product

$$\odot : \ \mathrm{HQ}_*(M) \otimes \mathrm{HQ}_{*'}(L) \longrightarrow \mathrm{HQ}_{*+*'-2n}(L)$$

which is independent of all auxiliary data and turns $\mathrm{HQ}_*(L)$ into a $\mathrm{HQ}_*(M)$-module.

4. From Morse to Floer homologies

We now present the other striking consequence of Gromov's work on pseudo-holomorphic curves (Gromov 1985) mentioned in the introduction of the present chapter, that is *Floer homology*. This homology theory was constructed by Floer (Floer 1989b, 1988a, 1988b, 1989a) in his successful quest of a proof to the Arnol'd conjecture. Since this seminal work, Floer homology has been extended to more general situations by Hofer and Salamon (Hofer and Salamon 1995) for example for weakly monotone manifolds and by Oh (Oh 1993, 1995), Biran and Cornea (Biran and Cornea 2009, 2012), Seidel (Seidel 2008), Fukaya, Oh, Ohta, and Ono (Fukaya et al. 2009a, 2009b) for the Lagrangian counterpart. Recently, Zapolsky wrote an extensive account of the theory (Zapolsky 2015) incorporating canonical orientations. As for the quantum case, we will not describe the theory beyond $\mathbb{Z}/2\mathbb{Z}$ coefficients and the exposition will be based on Oh (op.cit.), with notation adapted from (Leclercq and Zapolsky 2018).

4.1. Lagrangian Floer homology

We now describe Floer homology. It is "an infinite dimensional analogue of Morse homology" which makes extensive use of the pseudo-holomorphic tools and techniques exposed in the previous sections on quantum homology.

4.1.1. *The complex*

As above, L denotes a compact monotone$^+$ Lagrangian of a compact symplectic manifold (M,ω). Let Ω_L be the set of homotopically trivial chords from L to itself :

$$\Omega_L = \{\gamma\colon [0,1] \to M \mid \gamma(0), \gamma(1) \in L, [\gamma] = 0 \in \pi_1(M,L)\}\,.$$

A *capping* of a path $\gamma \in \Omega_L$ is a smooth half-disc $\widehat{\gamma}\colon D_- \to M$ such that the image of the "straight" part of its boundary, $\{ti \mid t \in [-1,1]\}$, is mapped to γ while the "round" part of its boundary, $D_- \cap \partial D$, lies in L. Given $\gamma \in \Omega_L$, such a capping exists as γ is required to be homotopically trivial with respect to L. Two pairs $(\gamma,\widehat{\gamma})$ and $(\gamma',\widehat{\gamma}')$ are equivalent if $\gamma = \gamma'$ and $\omega(\widehat{\gamma}) = \omega(\widehat{\gamma}')$, that is if the piecewise smooth disc in M with boundary in L obtained as the concatenation $u = \widehat{\gamma}\sharp(-\widehat{\gamma}')$ has zero symplectic area (here, $-\widehat{\gamma}'$ denotes the disc γ' with reversed orientation). Notice that this is equivalent, under the monotone$^+$ assumption, to $\mu(u) = 0$. We will denote $\widetilde{\gamma} = [\gamma,\widehat{\gamma}]$ the equivalence class of the pair $(\gamma,\widehat{\gamma})$ under this relation, and by $\widetilde{\Omega}_L$ the set of these equivalence classes.

Let $H\colon [0,1] \times M \to \mathbb{R}$ be a time-dependent Hamiltonian on M. Floer's *action functional* is defined by

$$\mathcal{A}_{H:L}\colon \widetilde{\Omega}_L \longrightarrow \mathbb{R}\,, \quad \mathcal{A}_{H:L}(\widetilde{\gamma}) = \int_0^1 H_t\big(\gamma(t)\big)\,dt - \int_{D_-} \widehat{\gamma}^*\omega\,.$$

Its critical points are classes $[\gamma,\widehat{\gamma}]$ for which γ is a *Hamiltonian chord* of H, *i.e.* satisfying the condition $\partial_t\gamma(t) = X_H^t(\gamma(t))$, which can also be expressed as $\gamma(t) = \phi_H^t(\gamma(0))$.

The Hamiltonian H is said to be *nondegenerate* if for every critical point $\widetilde{\gamma} \in \mathrm{Crit}(\mathcal{A}_{H:L})$, the linearized map $(\phi_H^1)_*\colon T_{\gamma(0)}M \to T_{\gamma(1)}M$ maps $T_{\gamma(0)}L$ transversely to $T_{\gamma(1)}L$.

It is well-known that for such a Hamiltonian, there exists a well-defined function

$$m_{H:L}\colon \mathrm{Crit}(\mathcal{A}_{H:L}) \longrightarrow \mathbb{Z}$$

called the *Conley–Zehnder index* which we normalize so that when H is a lift to a Weinstein neighborhood of L of a C^2-small Morse function f on L, and $\widehat{q}$ is the trivial capping of some $q \in \mathrm{Crit}(f)$, $m_{H:L}\big([q,\widehat{q}]\big) = |q|_f$ (and in general $m_{H:L}\big([\gamma, A\sharp\widehat{\gamma}]\big) = m_{H:L}\big([\gamma,\widehat{\gamma}]\big) - \mu(A)$ for $A \in \pi_2(M,L)$). The critical points of $\mathcal{A}_{H:L}$ (which play the role of the critical points of f) generate the *Lagrangian Floer complex*, $\mathrm{CF}_*(L;H) = \mathbb{Z}/2\mathbb{Z}\langle\mathrm{Crit}(\mathcal{A}_{H:L})\rangle$, whose graduation is given by the Conley–Zehnder index (which plays the role of the Morse index).

Now we proceed with defining the differential. Fix a 1-parameter family of almost complex structures compatible with ω on TM, J, and define *Floer half-tubes* between generators $\widetilde{x}_\pm \in \mathrm{Crit}(\mathcal{A}_{H:L})$ as maps $u\colon \mathbb{R} \times [0,1] \to M$ such that

(1) u converges uniformly in t : $u(\pm\infty\,,\cdot\,) = x_\pm$,
(2) it has boundary in L : $u(\mathbb{R}\times\{0\,,1\}) \subset L$,
(3) it satisfies the equation $\overline{\partial}_{J,H}u := \partial_s u + J(u)(\partial_t u - X_H(u)) = 0$,
(4) $\widetilde{x}_+ = [x_+\,,\widehat{x}_-\sharp u]$, or equivalently $\omega\big(\widehat{x}_-\sharp u\sharp(-\widehat{x}_+)\big) = 0$.

(Figure I.6 illustrates a typical Floer half-tube.)

Remark I.16. In comparison with Morse theory, let us mention that such a Floer tube can be considered as a negative gradient flow line of $\mathcal{A}_{H:L}$ going from $\widetilde{x}_-$ to $\widetilde{x}_+$, with respect to the scalar product on

$$T_{\widetilde{\gamma}}\widetilde{\Omega}_L = C^\infty\big([0,1],\{0,1\};\gamma^*TM,(\gamma|_{\{0,1\}})^*TL\big)$$

defined by $\langle\xi\,,\eta\rangle = \int_0^1 \omega\big(\xi(t)\,,J_t\eta(t)\big)\,dt$. The gradient of $\mathcal{A}_{H:L}$ at a point $\widetilde{\gamma}$ then reads

(I.6) $$\nabla\mathcal{A}_{H:L}(\widetilde{\gamma}) = J_\gamma(\partial_t\gamma - X_H(\gamma))\,.$$

Thus, viewing a Floer half-tube as a map $u\colon \mathbb{R}\to\Omega_L$, we see that $\partial_s u = -\nabla\mathcal{A}_{H:L}(u)$ is equivalent to the Floer equation $\overline{\partial}_{J,H}u = 0$. ▮

Again, $\mathbb{R}$ acts on such a tube by shift of the variable s and the moduli spaces $\mathcal{M}_F(\widetilde{x}_-\,,\widetilde{x}_+\,;L\,;H\,,J)$ is defined as the set of equivalence classes of Floer half-tubes.

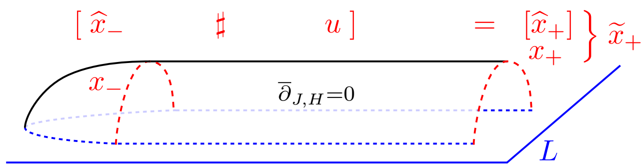


FIGURE I.6. A typical element $u\in\widehat{\mathcal{M}_F}(\widetilde{x}_-\,,\widetilde{x}_+\,;L\,;H\,,J)$

The differential of $\mathrm{CF}_*(L\,;H)$ is defined, formally as in the Morse case, by setting

(I.7) $$\partial_{J,H}\,\widetilde{x}_- = \sum_{\widetilde{x}_+\,:\,|x_+|=|x_-|-1} \#_2\mathcal{M}_F(\widetilde{x}_-,\widetilde{x}_+;L;H,J)\,\widetilde{x}_+$$

for generators and extending it by linearity. Of course, there is a Morse–Smale type regularity condition required in order to ensure that the component of dimension 0 (respectively 1) of $\mathcal{M}_F$ is a compact (respectively compactifiable) smooth manifold.[6] Given a nondegenerate Hamiltonian H, there exists a residual subset of ω-compatible almost complex structures J such that the pair $(H\,,J)$ satisfies this condition. Such a pair $(H\,,J)$ is said to be *regular*.

So, picking such a regular pair ensures that (I.7) defines a degree -1 chain morphism. Showing that it squares to 0 requires to play the same game than in the Morse case as well as handling possible bubbling off as in the Lagrangian quantum case. Indeed, analyzing the boundary of the compactification of the 1-dimensional component of $\mathcal{M}_F(\widetilde{x}_-\,,\widetilde{x}_+\,;L\,;H\,,J)$, *i.e.* when $|\widetilde{x}_+| = |\widetilde{x}_-| - 2$, leads to

(1) Floer breakings of the type $\mathcal{M}_F(\widetilde{x}_-\,,\widetilde{y}\,;L\,;H\,,J)\times\mathcal{M}_F(\widetilde{y}\,,\widetilde{x}_+\,;L\,;H\,,J)$, where $\widetilde{y}\in\mathrm{Crit}(\mathcal{A}_{H:L})$ has index $|\widetilde{y}| = |\widetilde{x}_-| - 1 = |\widetilde{x}_+| + 1$,
(2) possible bubbling off of J-pseudo-holomorphic spheres,
(3) possible bubbling off of J-pseudo-holomorphic discs with boundary on L, attached to a half-tube along one of its sides.

However, under the monotone$^+$ assumption, the two possible bubbling off phenomena can not happen in the boundary of the compactification of the 1-dimensional component of the moduli spaces for obvious dimension considerations as in Section 3.1.1.

This allows to conclude directly that the morphism defined by (I.7) squares to 0. Thus we can define the *Lagrangian Floer homology* of L by taking the homology of the Floer complex : $\mathrm{HF}_*(L) = \mathrm{H}_*(\mathrm{CF}(L\,;H)\,,\partial_{H,J})$.

6. This condition is expressed by requiring that the linearization of the operator $\overline{\partial}_{J,H}$ is surjective for all u in $\mathcal{M}_F(\widetilde{x}_-\,,\widetilde{x}_+\,;L\,;H\,,J)$ (for any two elements $\widetilde{x}_\pm\in\mathrm{Crit}(\mathcal{A}_{H:L})$).

Remark I.17. There is an equivalent description of the complex $(\mathrm{CF}_*(L\,;H)\,,\partial_{H,J})$ which requires more arbitrary choices but makes the construction looks more similar to Lagrangian quantum homology. Indeed, let arbitrarily assign a capping u_x to any Hamiltonian chord x *i.e.* we arbitrarily select a unique $\widetilde{x} = [x\,, u_x] \in \widetilde{\Omega}_L$ lifting x, for each Hamiltonian chord in Ω_L. Observe that any capping of a Hamiltonian chord x can be obtained from u_x by concatenation with a disc in M with boundary in L.

Let $\pi_L : \widetilde{\Omega}_L \to \Omega_L$ denote the projection $[\gamma\,,\widehat{\gamma}] \longmapsto \gamma$. We can now re-define the complex as generated by Hamiltonian chords only and incorporate the cappings in the Novikov ring Λ (the ring of Laurent polynomials in t from Section 3.1.1) by setting $\mathrm{CF}'_*(L\,;H) = \mathbb{Z}/2\mathbb{Z}\big\langle \pi_L\big(\mathrm{Crit}(\mathcal{A}_{H:L})\big)\big\rangle \otimes \Lambda$. The observation above shows that $\mathrm{CF}_*(L\,;H)$ and $\mathrm{CF}'_*(L\,;H)$ are isomorphic. In this slightly different viewpoint, the differential takes the form

$$\partial'_{J,H}\, x_- = \sum_{x_+,A} \#_2\mathcal{M}_F(x_-,x_+;L;H,J,A)\, x_+ \otimes t^{\frac{\mu(A)}{N_L}}$$

where the sum runs over all Hamiltonian chords x_+ and all classes $A \in \pi_2(M\,,L)$ such that $|x_+| - \mu(A) = |x_-| - 1$. ▮

4.1.2. *Independence on the data*

Again, even though the complex heavily relies on the regular Floer datum $(H\,,J)$, its homology does not. There are *Floer continuation morphisms* defined via cobordism techniques as described for Lagrangian quantum homology in Section 3.1.2. Here, two regular Floer pairs are fixed $(H_i\,,J_i)$ for $i=0$ and 1, and a pair of interpolating homotopies $t \longmapsto (H_t\,,J_t)$ is chosen. Consider slightly amended moduli spaces $\mathcal{M}_F$ consisting of negative gradient flow lines (Floer half-tubes) *for the* 1*-parameter family of Floer data*, from $\widetilde{x}_0 \in \mathrm{Crit}(\mathcal{A}_{H_0:L})$ to $\widetilde{x}_1 \in \mathrm{Crit}(\mathcal{A}_{H_1:L})$, *i.e.* satisfying

$$\overline{\partial}_{J_s,H_s} u = \partial_s u + J_s(u)(\partial_t u - X_{H_s}(u)) = 0$$

(*i.e.* the only difference with Item 3 in the definition of the Floer tubes composing $\mathcal{M}_F$, is the fact that J and H now depend on the parameter s).

For adequate choices of the cobordism pair, the low dimensional components of these moduli spaces are manifolds of dimension $m_{H_1:L}(\widetilde{x}_1) - m_{H_0:L}(\widetilde{x}_0)$. The component of dimension 0 is compact and the continuation morphism, $\mathrm{CF}_*(L\,;H_0) \to \mathrm{CF}_*(L\,;H_1)$, is defined by counting its cardinality as in Equation (I.7). A careful analysis of the boundary of the compactification of the 1-dimensional component shows that this morphism induces a morphism in homology, $\Phi_{H,J}\colon\ \mathrm{HF}_*(L) \to \mathrm{HF}_*(L)$.

By pulling over (and over) the same techniques, one can show that the continuation morphism does not depend on the choice of the homotopy used to define it and enjoys the same properties as its quantum counterpart. Thus it is a canonical isomorphism of Lagrangian Floer homology which we will denote by $\Phi_{(H_1,J_1)\leftarrow(H_0,J_0)}$.

4.1.3. *The product*

Similarly to the Morse and quantum theories, there exists a Floer-theoretic version of the intersection product, called the *half-pair of pants product*. It is defined in the same fashion, by considering appropriate moduli spaces which we now describe.

Remark I.18. The central object below, called *strip with a slit*, was introduced by Abbondandolo and Schwarz (Abbondandolo and Schwarz 2010). The product itself pre-existed and was defined via somewhat "rougher" pants. The action estimates which naturally follows from the construction were not sharp then and the introduction of these fancier pants drastically simplified the proof of the triangle inequality property of spectral invariants, see Section 1.2. ▮

Let H^1, H^2, and H^3 be time-dependent nondegenerate Hamiltonian functions on M and choose almost complex structures J^1, J^2, and J^3 so that $(H^i\,,J^i)$, $i=1$, 2, and

3 are regular Floer data for L. By a simple time reparameterization, we can harmlessly assume that H_t^i vanish for t close to 0 and 1. Let us denote the strip with a slit by

$$\Sigma_* = (\mathbb{R} \times [-1,0] \sqcup \mathbb{R} \times [0,1])/\sim\,,$$

where the equivalence relation is $(s,0^-) \sim (s,0^+)$ for all $s \geq 0$. The interior of this Riemann surface is naturally identified with $\mathbb{R} \times (-1,1) \setminus (-\infty,0] \times \{0\}$. Its boundary consists of three components : $\mathbb{R} \times \{-1\}$, $\mathbb{R} \times \{1\}$, and $(-\infty,0] \times \{0^-,0^+\}$. Except at $(0,0)$, the inclusion of Σ_* into $\mathbb{C}$ induces the standard complex structure $(s,t) \longmapsto s+it$. At $(0,0)$, the complex structure is given by the map $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 0\} \to \Sigma_*$, defined by $z \longmapsto z^2$. We also define a pair (K,I) where K is the family of Hamiltonian functions

$$K(s,t,x) = \begin{cases} H^1(t+1,x) & \text{if } s \leq -1,\ t \in [-1,0], \\ H^2(t,x) & \text{if } s \leq -1,\ t \in [0,1], \\ \frac{1}{2}H^3(\frac{t+1}{2},x) & \text{if } s \geq 1,\ t \in [-1,1] \end{cases}$$

and $I_{(s,t)}$ a family of almost complex structures on M for $s \in \mathbb{R}$ and $t \in [-1,1]$ such that (K,I) is regular and

$$I(s,t,x) = \begin{cases} J^1(t+1,x) & \text{if } s \leq -1,\ t \in [-1,0], \\ J^2(t,x) & \text{if } s \leq -1,\ t \in [0,1], \\ J^3(\frac{t+1}{2},x) & \text{if } s \geq 1,\ t \in [-1,1]. \end{cases}$$

Now, for $\widetilde{x}^i \in \mathrm{Crit}(\mathcal{A}_{H^i:L})$, we define the moduli spaces $\mathcal{M}_{*_F}(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3;K,I)$ as the set of maps $u\colon \Sigma_* \to M$ such that

(1) for all $t \in [-1,0]$, $u(-\infty,t) = x^1(t+1)$, for all $t \in [0,1]$, $u(-\infty,t) = x^2(t)$,
(2) for all $t \in [-1,1]$, $u(+\infty,t) = x^3(\frac{t+1}{2})$,
(3) $u(\partial\Sigma_*) = u(\mathbb{R} \times \{-1,1\} \cup (-\infty,0] \times \{0^-,0^+\}) \subset L$,
(4) $\widetilde{x}^3 = [x^3, \widetilde{x}^1 \sharp u \sharp \widetilde{x}^2]$ *i.e.* the disc obtained by concatenating u and the cappings $\widetilde{x}^i$ for $i = 1$, 2 and 3 has zero symplectic area (and Maslov).

Figure I.7 illustrates three variations of the strip with a slit, Σ_*, Σ_*', and $\Sigma_\odot$ respectively used to construct the product on Lagrangian Floer homology (the present section), the product on "absolute" Floer homology (see Section 4.2), and the Floer module structure (see Section 4.3). Let us emphasize the fact that the blue segments are mapped to the Lagrangian L, while the bent double arrows identify pairs of segments in M. Moreover, *on the figure* we denote lifts of Hamiltonian chords by $\widetilde{x}$ while $\widetilde{\gamma}$ denotes lifts of periodic orbits.

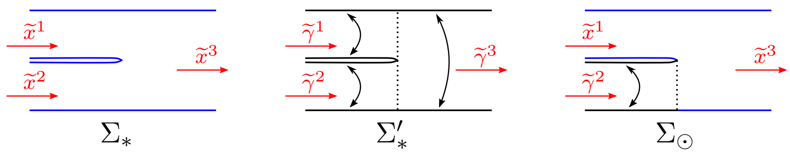


FIGURE I.7. Strips with a slit for Floer products and Floer module structure

The low dimensional component of $\mathcal{M}_{*_F}(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3;K,I)$ are smooth manifolds of dimension $m_{H^1:L}(\widetilde{x}^1) + m_{H^2:L}(\widetilde{x}^2) - m_{H^3:L}(\widetilde{x}^3) - n$. The 0-dimensional component is compact so that one can define a bi-linear map by setting

$$\widetilde{x}^1 * \widetilde{x}^2 = \sum_{\widetilde{x}^3} \#_2\mathcal{M}_{*_F}(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3;K,I)\,\widetilde{x}^3$$

where the sum runs over all $\widetilde{x}^3 \in \mathrm{Crit}(\mathcal{A}_{H^3:L})$ such that $m_{H^3:L}(\widetilde{x}^3) = m_{H^1:L}(\widetilde{x}^1) + m_{H^2:L}(\widetilde{x}^2) - n$. The 1-dimensional component can be compactified and analyzing the boundary of its compactification shows that this map induces a well-defined product on homology

$$*:\ \mathrm{HF}_*(L) \times \mathrm{HF}_{*'}(L) \longrightarrow \mathrm{HF}_{*+*'-n}(L)\,.$$

Finally, it can be shown via cobordism arguments that this product commutes with the relevant continuation morphisms, and does not depend on the choice of regular pair

$(K\,,I)$ used for its construction. It can also be shown that it admits a unit of degree n, denoted $[L]$ and called *(Floer) fundamental class* of L by analogy.

4.1.4. *Duality*

There is a Floer-theoretic version of Poincaré duality whose construction we sketch here. (There is also Morse and quantum versions, based on the same observation which we leave to the reader.) The idea is that, given a regular Floer datum $(H\,,J)$, one can turn the resulting complex upside down to get $(\mathrm{CF}^*(L;H),\partial_{H,J})$, and interpret its homology as the Lagrangian Floer cohomology of L, $\mathrm{HF}^*(L) = \mathrm{H}_*\big((\mathrm{CF}(L;H),\partial_{H,J}\big)$.

More precisely, we define another Floer datum, $(\overline{H}\,,\overline{J})$ as $\overline{H}_t(x) = -H_{1-t}(x)$ and $\overline{J}_t = J_{1-t}$, which is also regular. There is an identification between $\mathrm{Crit}(\mathcal{A}_{H:L})$ and $\mathrm{Crit}(\mathcal{A}_{\overline{H}:L})$ given by $\widetilde{x} = [x\,,\widehat{x}] \mapsto \overline{\widetilde{x}} = [\overline{x}\,,\overline{\widehat{x}}]$ where $\overline{x}$, $\overline{\widehat{x}}$ satisfy $\overline{x}(t) = x(1-t)$ and $\overline{\widehat{x}} = \widehat{x}\circ\tau$ with $\tau\colon\ D_- \to D_-$ the symmetry with respect to $\mathbb{R}$. We have that $\mathcal{A}_{H:L}(\widetilde{x}) = -\mathcal{A}_{\overline{H}:L}(\overline{\widetilde{x}})$ and $m_{H:L}(\widetilde{x}) = n - m_{\overline{H}:L}(\overline{\widetilde{x}})$.

Proceeding in the same fashion, we can identify further the moduli spaces involved in the definition of the Floer differential : $\mathcal{M}_F(\widetilde{x}\,,\widetilde{y}\,;L\,;H\,,J) \simeq \mathcal{M}_F(\overline{\widetilde{x}}\,,\overline{\widetilde{y}}\,;L\,;\overline{H}\,,\overline{J})$ by mapping u to $\overline{u}$ defined by $\overline{u}(s\,,t) = u(-s\,,1-t)$. Hence there is an identification of complexes, $\mathrm{CF}_*(L\,;\overline{H}\,,\overline{J}) = \mathrm{CF}^{n-*}(L\,;H\,,J)$ which leads to a canonical isomorphism in homology, $\mathrm{PD}\colon\ \mathrm{HF}_*(L) \to \mathrm{HF}^{n-*}(L)$.

It is easy to see that the identification above extends to all possible types of moduli spaces which we encountered thus far on the Floer side of the theory. This yields the compatibility of all morphisms and structures with this duality. For example, it is easy to show that the diagram of complexes

$$
\begin{array}{ccc}
\mathrm{CF}_*(L;H^0,J^0) & \xrightarrow{\Phi_{H,J}} & \mathrm{CF}_*(L;H^1,J^1) \\
\big\downarrow{\scriptstyle \mathrm{PD}} & & \big\downarrow{\scriptstyle \mathrm{PD}} \\
\mathrm{CF}^{n-*}(L;H^0,J^0) & \xrightarrow{\Phi_{K,I}} & \mathrm{CF}^{n-*}(L;H^1,J^1)
\end{array}
\tag{I.8}
$$

commutes as soon as the homotopy used to define the continuation morphism of the line below,

$$\Phi_{K,I}\colon\ \mathrm{CF}_*(L;\overline{H^0},\overline{J^0}) \longrightarrow \mathrm{CF}_*(L;\overline{H^1},\overline{J^1})\,,$$

is obtained from that of the upper line, $\Phi_{H,J}\colon\ \mathrm{CF}_*(L\,;H^0\,,J^0) \to \mathrm{CF}_*(L\,;H^1\,,J^1)$, by "duality", *i.e.* in such a way that $K = \overline{H}$ and $I = \overline{J}$ as above. This shows that Floer continuation morphisms are compatible with duality.

4.2. (Ambient) Floer homology

As for quantum homology, there is a version of Floer homology concerned with the ambient symplectic manifold with no reference to a Lagrangian submanifold. This Floer homology is often called "Hamiltonian", "absolute" (in that case the Lagrangian theory is called "relative" Floer homology), or "periodic orbit" Floer homology (non exhaustive list, in increasing level of accuracy). Its construction is similar to that of Lagrangian Floer homology except that all half-objects with boundary on L (Hamiltonian chords, half-disc cappings, Floer half-tubes, half-pairs of pants) are replaced by full objects (respectively : periodic orbits, disc cappings, tubes, pairs of pants).[7]

Namely, define Ω_M as the set of free contractible loops (parameterized by $[0\,,1]$) in M. A capping of a loop $\gamma\in\Omega$ is a smooth disc $\widehat{\gamma}\colon\ D \to M$ which maps ∂D to γ. Two capped loops $(\gamma,\widehat{\gamma})$ and $(\gamma'\,,\widehat{\gamma}')$ are equivalent if $\gamma=\gamma'$ and $\omega(\widehat{\gamma}) = \omega(\widehat{\gamma}')$, that is if the piecewise smooth sphere $u = \widehat{\gamma}\sharp(-\widehat{\gamma}')$ lies in the kernel of ω (and c_1 by monotonicity). Again, $\widetilde{\gamma} = [\gamma,\widehat{\gamma}]$ denotes the equivalence class of the pair $(\gamma,\widehat{\gamma})$ and $\widetilde{\Omega}_M$ denotes the set of these equivalence classes.

---

7. This suggests yet another pair of names for these theories as *Floer homology* and *half-Floer homology*.

Given a 1-periodic Hamiltonian $H\colon M\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}$, Floer's *action functional* is defined by

$$\mathcal{A}_H\colon \widetilde{\Omega}_M\to\mathbb{R}\,,\quad \mathcal{A}_H(\widetilde{\gamma})=\int_0^1 H_t\big(\gamma(t)\big)\,dt-\int_D\widehat{\gamma}^*\omega$$

so that its critical points are the classes $[x\,,\widehat{x}]$ where x is a periodic orbit of the Hamiltonian isotopy ϕ_H. In this setting, H is *nondegenerate* if for every critical point $\widetilde{x}\in\mathrm{Crit}(\mathcal{A}_{H:L})$, the linearized map $(\phi_H^1)_*\colon T_{x(0)}M\to T_{x(0)}M$ does not admit 1 as eigenvalue. In this case, we have a well-defined Conley–Zehnder index

$$m_H\colon \mathrm{Crit}(\mathcal{A}_H)\to\mathbb{Z}$$

(which we similarly normalize so that it coincides, when H is a C^2-small Morse function, with the Morse index on critical points endowed with their trivial capping and so that $m_H\big([x\,,A\sharp\widehat{x}]\big)=m_H\big([x\,,\widehat{x}]\big)-2c_1(A)$ for $A\in\pi_2(M)$). This yields the graded *Floer complex*, defined as $\mathrm{CF}_*(M;H)=\mathbb{Z}/2\mathbb{Z}\langle\mathrm{Crit}(\mathcal{A}_H)\rangle$.

The differential is defined as above. We fix a 1-parameter family of ω-compatible almost complex structures on TM, and we consider *Floer tubes*, that is maps $u\colon S^1\times[0\,,1]\to M$ such that

(1) u converges uniformly in t towards periodic orbits : $u(\pm\infty\,,\cdot)=x_\pm$,
(2) it satisfies the equation $\overline{\partial}_{J,H}u=\partial_s u+J(u)(\partial_t u-X_H(u))=0$,
(3) $\widetilde{x}_+=[x_+\,,\widehat{x}_-\sharp u]$.

Moduli spaces $\mathcal{M}_F(\widetilde{x}_-\,,\widetilde{x}_+\,;H\,,J)$ are then defined as the set of equivalence classes of Floer tubes and the differential is defined by the same formula (I.7) as its Lagrangian counterpart. As above, if the pair $(H\,,J)$ is regular (which has formally the exact same definition as in the Lagrangian case), this defines a morphism on $\mathrm{CF}_*(M;H)$ which squares to 0; the *Floer homology* of M is the homology of this complex $\mathrm{HF}_*(M)=\mathrm{H}_*(\mathrm{CF}(M;H)\,,\partial_{(H,J)})$.

As in the Lagrangian case, there exist continuation maps which provide canonical isomorphisms of Floer homologies built from different pairs of Floer data. Moreover, there is also a product on the resulting homology, constructed as its Lagrangian counterpart, starting from a (full) pair of pants

$$\Sigma'_*=(\mathbb{R}\times[-1,0]\sqcup\mathbb{R}\times[0,1])/\sim\,,$$

where the equivalence relation is, in this case, $(s\,,0^-)\sim(s\,,0^+)$ and $(s\,,-1)\sim(s\,,1)$ for all $s\geq 0$, while for all $s\leq 0$, $(s\,,\pm1)\sim(s\,,0^\pm)$ (see Figure I.7, middle situation). The rest of the construction is formally similar and yields a product on this version of Floer homology as well :

$$*\colon\ \mathrm{HF}_*(M)\times\mathrm{HF}_{*'}(M)\longrightarrow\mathrm{HF}_{*+*'-2n}(M)\,,$$

turning it into an algebra with unit, denoted $[M]\in\mathrm{HF}_{2n}(M)$.

4.3. Module structure on Floer homology

There is a Floer counterpart of the quantum module structure of Section 3.3. It is based on moduli spaces which are very similar to those used to define the product in Lagrangian Floer homology, described in Section 4.1.3. Below, we only outline the differences.

The first difference is a small alteration of the Riemann surface as it has to "connect" two Hamiltonian chords and a periodic orbit. So we denote

$$\Sigma_\odot=(\mathbb{R}\times[-1,0]\sqcup\mathbb{R}\times[0,1])/\sim\,,$$

where the equivalence relation is given by $(s\,,0^-)\sim(s\,,0^+)$ for all $s\geq 0$, and $(s\,,-1)\sim(s\,,0^-)$ for $s\leq 0$ (see Figure I.7, right-most situation). The second difference is that H^1

is a *periodic* Hamiltonian function and that $\widetilde{x}^1 \in \mathrm{Crit}(\mathcal{A}_{H^1})$. The rest of the construction is formally identical and leads to :

$$\odot : \ \mathrm{HF}_*(M) \otimes \mathrm{HF}_{*'}(L) \longrightarrow \mathrm{HF}_{*+*'-2n}(L)\,,$$

turning $\mathrm{HF}_*(L)$ into a $\mathrm{HF}_*(M)$-module.

5. Interactions between quantum and Floer homologies

The fact that the algebraic structures described above naturally appear and are constructed in a similar way in both the quantum and Floer settings is not an accident as these theories are isomorphic. Specific isomorphisms were first defined in (Piunikhin, Salamon, and Schwarz 1996) and (Schwarz 1998), and then extended to various settings (Katić and Milinković 2005; Albers 2008; Biran and Cornea 2009; Zapolsky 2015). They are commonly referred to as "PSS" morphisms. As for all the maps whose construction we have described so far, these isomorphisms are induced in homology by chain maps built thanks to suitable moduli spaces. We briefly describe these spaces in Section 5.2.

In Section 5.3, we explain the following deep relation between absolute and relative theories. Given a symplectic manifold (M,ω), the diagonal $\Delta \subset M\times M$ is a Lagrangian submanifold of $\big(M\times M, \omega\oplus(-\omega)\big)$. The quantum and Floer homologies of M *as ambient manifold* are respectively isomorphic to the quantum and Floer homologies of $M \simeq \Delta$ *as a Lagrangian*. Moreover, these isomorphisms agree with the main features of the involved homology theories.

However, we begin this section with the observation that Lagrangian quantum homology can be interpreted as a Morse–Bott Lagrangian Floer theory for (the action functional associated with) the zero Hamiltonian.

5.1. Quantum theory as a Morse–Bott Floer theory

We first briefly recall the idea behind Morse–Bott theory. In comparison with a Morse function which has nondegenerate isolated critical points, the critical set of a Morse–Bott function φ decomposes as a disjoint union of connected submanifolds. These critical submanifolds are *nondegenerate* in the sense that the kernel of the Hessian of φ at a critical point is the tangent space to the critical component at this point. The flow lines of φ thus "connect" one critical component to another and we need a way to flow *on* the critical components to connect the critical points themselves. In order to do this, we pick additional Morse functions, one for each critical component.

The objects playing the role of Morse flow lines in the definition of the differential are *cascades*. A cascade is a finite succession of flow lines of φ going from one critical submanifold to another alternating with pieces of flow lines of the Morse perturbations on the critical manifolds. Moduli spaces of cascades were introduced in the symplectic setting by Frauenfelder (Frauenfelder 2004), to which we refer for more details on the construction.

Disclaimer I.19. As an additional piece of notation to that of Sections 3.1.1 and 4.1.1, we denote by $\widetilde{\pi}_2(M,L)$ the quotient of the second homotopy group of M relative to L by the kernel of the symplectic form seen as a morphism $\omega\colon \pi_2(M,L)\to\mathbb{R}$. Since L is monotone, with monotonicity constant $\tau>0$, $\ker\omega = \ker\mu$. ▮

Let us slightly shift the viewpoint on quantum homology along the lines of Remark I.17 about Floer homology (but with reversed orientation !). Namely, we can see the quantum complex as generated by pairs (p,A), with $p\in\mathrm{Crit}(f)$ and $A\in\widetilde{\pi}_2(M,L)$. The equivalence between the two viewpoints is given by replacing elementary tensors $p\otimes t^n$ by pairs $(p,\widetilde{A})$ for $\widetilde{A}\in\widetilde{\pi}_2(M,L)$ defined by $\mu(\widetilde{A}) = nN_L$. (As for Remark I.17, the two descriptions only differ by *arbitrary choices of a disc* u_p *for each* p *in* $\mathrm{Crit}(f)$. However, in the quantum setting, there is a canonical choice consisting of picking u_p to be the constant disc at p.)

We now come back to Floer's theory for which we set $H = 0$ and consider the associated Floer action functional, $\varphi = \mathcal{A}_{0:L}\colon \widetilde{\Omega}_L \to \mathbb{R}$ defined by $\mathcal{A}_{0:L}(\widetilde{\gamma}) = -\int_{D_-} \widehat{\gamma}^* \omega$. Its critical points are the equivalence classes of pairs (p, A) where p is any point in M and A any disc in M with boundary in L. Two pairs (p, A) and (p', A') are equivalent if $p = p'$ and $\omega(A) = \omega(A')$. Thus, we can express these equivalence classes as pairs $(p, \widetilde{A})$ with $\widetilde{A} \in \widetilde{\pi}_2(M, L)$.

The critical point set of $\mathcal{A}_{0:L}$ decomposes as $\mathrm{Crit}(\mathcal{A}_{0:L}) = \coprod_{\widetilde{A}} L_{\widetilde{A}}$, the disjoint union indexed by $\widetilde{\pi}_2(M, L)$ of countably many smooth manifolds $L_{\widetilde{A}} = L \times \{\widetilde{A}\}$, each of them diffeomorphic to L. The Morse function f chosen to define quantum homology can then be seen as inducing the Morse perturbations on all critical submanifolds required by the Morse–Bott construction, by $f_{\widetilde{A}}\colon L_{\widetilde{A}} \to \mathbb{R}$ defined by $f_{\widetilde{A}}(p, \widetilde{A}) = f(p)$.

Moreover, from the expression (I.6) of the gradient of the action functional, we see that when $H = 0$ the equation defining the gradient flow of $\mathcal{A}_{0:L}$ reduces to the usual J-pseudo-holomorphic equation.

This shows that a typical element of $\mathcal{M}_Q(p, q; \mathcal{D}; A)$, appearing in the definition of the differential of the quantum complex as depicted in Figure I.2, can be seen as a Morse–Bott cascade : the negative gradient flow lines of f are the pieces of the gradient flow of the Morse perturbations $f_{\widetilde{A}}$, while the J-pseudo-holomorphic discs u_i, for $1 \leq i \leq r$, play the role of the gradient flow lines of $\mathcal{A}_{0:L}$. More precisely, starting from a critical point $(p, \widetilde{A}_0)$ of $\mathcal{A}_{0:L}$, we first flow to $(u_1(-1), \widetilde{A}_0)$ along the flow of $f_{\widetilde{A}_0}$, inside the critical component $L_{\widetilde{A}_0}$. Then, u_1 is a gradient flow line of $\mathcal{A}_{0:L}$ from $(u_1(-1), \widetilde{A}_0)$ to $(u_1(1), \widetilde{A}_1)$ where $\widetilde{A}_1 = \widetilde{A}_0 + \widetilde{U}_1$ ($\widetilde{U}_1$ denotes the class of $u_1 \in \widetilde{\pi}_2(M, L)$). Then, we flow via $f_{\widetilde{A}_1}$ to $(u_2(-1), \widetilde{A}_1)$ and, from there, via $\mathcal{A}_{0:L}$ to $(u_2(1), \widetilde{A}_2)$ where $\widetilde{A}_2 = \widetilde{A}_1 + \widetilde{U}_2$... In the end, an element of $\mathcal{M}_Q(p, q; \mathcal{D}; A)$ can be interpreted as a Morse–Bott cascade from $(p, \widetilde{A}_0)$ to $(q, \widetilde{A}_0 + \widetilde{A})$ (with $\widetilde{A}$ the class of A in $\widetilde{\pi}_2(M, L)$).

5.2. PSS morphisms

Let $\mathcal{D} = (f, \rho, J)$ be a quantum datum for L and H a nondegenerate Hamiltonian function such that both $\mathcal{D}$ and (H, J) are regular. With p in $\mathrm{Crit}(f)$ and $\widetilde{x} \in \mathrm{Crit}(\mathcal{A}_{H:L})$ are associated moduli spaces $\mathcal{M}_{\mathrm{PSS}}(p, \widetilde{x}; \mathcal{D}, H)$ composed of equivalence classes (up to automorphisms) of the following geometric objects. Consider a triple $(\underline{u}, m, v)$ where m is a point in L, $\underline{u}$ is a string of pearls of $\mathcal{M}_Q(p, m; \mathcal{D}; B)$ for some $B \in \pi_2(M, L)$, and v is a half-disc which satisfies an equation which interpolates between the J-pseudo-holomorphic and the Floer equations. Namely, v is a map $v\colon \mathbb{R} \times [0, 1] \to M$, with boundary $v(\mathbb{R} \times \{0, 1\})$ in L, asymptotics $v(-\infty, t) = m$ and $v(+\infty, t) = x(t)$, which satisfies

$$\text{for all } (s,t), \qquad \partial_s v(s,t) + J_{v(s,t)}\big(\partial_t v(s,t) - \beta(s) X_H^t(v(s,t))\big) = 0$$

where $\beta : \mathbb{R} \to [0, 1]$ is a smooth increasing cut-off function vanishing for $s \leq \frac{1}{2}$ and with value 1 for all $s \geq 1$. As usual, one also has to specify a relative homotopy condition : $[\underline{u}] = B$, $[v]$, and $\widehat{x}$ are related by $[\underline{u}] + [v \sharp (-\widehat{x})] = 0$.

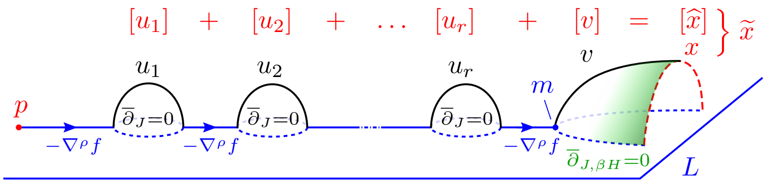


FIGURE I.8. A typical element $(\underline{u}, m, v) \in \mathcal{M}_{\mathrm{PSS}}(p, \widetilde{x}; \mathcal{D}, H)$

This allows to define a chain map by the formula

$$\mathrm{PSS}_{\mathcal{D},H}(p) = \sum_{\widetilde{x} \,|\, m_{H:L}(\widetilde{x}) = |p|_f} \#_2 \mathcal{M}_{\mathrm{PSS}}(p, \widetilde{x}; \mathcal{D}, H)\, \widetilde{x}$$

on generators and linear extension to $\mathrm{CQ}_*(L\,;\mathcal{D})$. Analyzing the boundary of the 1-dimensional component of these moduli spaces shows that it induces a morphism in homology.

Showing that this map is independent of the data and commutes with the continuation morphisms can be done by standard cobordism arguments. This leads to a canonical morphism

$$\mathrm{PSS}\colon\ \mathrm{HQ}_*(L) \longrightarrow \mathrm{HF}_*(L)\,.$$

To show that it is an isomorphism, one first define a chain map $\theta : \mathrm{CF}_*(L\,;H\,,J) \to \mathrm{CQ}_*(L\,;\mathcal{D})$ by counting mod 2 the cardinal of moduli spaces consisting of equivalence classes of objects mirror to the triples $(\underline{u}\,,m\,,v)$ above. This map induces (for the same reasons) a canonical morphism at the homology level. It remains to show that PSS and θ are inverse from each other which can easily be done by ... considering moduli spaces of similar types. This is also the standard way to show that the resulting PSS isomorphisms intertwine the product structures on the quantum and Floer homologies, *i.e.* PSS is a ring isomorphism between $(\mathrm{HQ}_*(L)\,,*)$ and $(\mathrm{HF}_*(L)\,,*)$. We leave the details to the very interested reader.

Finally, let us point out that there exists an absolute version of these morphisms, generating canonical ring isomorphisms

$$\mathrm{PSS}\colon\ (\mathrm{HQ}_*(M),*) \longrightarrow (\mathrm{HF}_*(M),*)\,.$$

5.3. The Lagrangian diagonal in the product

Recall that the diagonal $\Delta \subset M \times M$ is a Lagrangian submanifold of $\big(M \times M\,,\omega \oplus (-\,\omega)\big)$. It is monotone if and only if $(M\,,\omega)$ is a monotone symplectic manifold and, in this case, the minimal Maslov number of Δ coincides with twice the minimal Chern number of M. In this section we explain the construction of an isomorphism between the Floer homology of $(M\,,\omega)$ and the Lagrangian Floer homology of $\Delta \simeq M$. This is based on the following observation from (Biran, Polterovich, and Salamon 2003) in which the isomorphism was established under the asphericity assumption. A geometric object in $M \times M$ with boundary in Δ consists of two objects in M (one for each copy of M in the product) which can be "glued" together since their respective boundaries coincide. For example, a disc $u = (u_1\,,u_2)$ in $M \times M$ with boundary in Δ produces a sphere $v = u_1\sharp(-u_2)$ in M. Reciprocally, one can produce a relative object from an absolute one by "splitting" it in the middle and reversing its second half.

More precisely, let $(H\,,J)$ be a time-periodic regular Floer datum for M. We define a Floer datum $(\widehat{H}\,,\widehat{J})$ for $\Delta \subset M \times M$ as follows. First define

$$H^1_t = H_t\,,\quad H^2_t = H_{1-t}\,,\quad J^1_t = J_t\,,\quad J^2_t = -J_{1-t}\quad \text{for } t \in [0,\tfrac{1}{2}]\,,$$

and put

$$\widehat{H}_t(x,y) = H^1_t(x) + H^2_t(y)\,,\quad \widehat{J}_t(x,y) = J^1_t(x) \oplus J^2_t(y)\,.$$

Then $(\widehat{H}\,,\widehat{J})$ is a regular Floer datum for the diagonal Δ which can be used to construct $\mathrm{HF}_*(\Delta)$, up to very minor adjustments due to the fact that this datum is defined for $t \in [0\,,\frac{1}{2}]$.

Now we can build a canonical isomorphism of chain complexes

$$\big(\mathrm{CF}_*(H), \partial_{H,J}\big) = \big(\mathrm{CF}_*(\widehat{H} : \Delta), \partial_{\widehat{H},\widehat{J}}\big)\,, \tag{I.9}$$

preserving grading and action by "splitting" absolute objects into relative ones. We start with Hamiltonian orbits. A periodic orbit x of H gives rise to a Hamiltonian chord $X\colon\ [0\,,\frac{1}{2}] \to M \times M$ of $\widehat{H}$ with endpoints on Δ :

$$X(t) = \big(x_1(t), x_2(t)\big) \quad \text{where} \quad x_1(t) = x(t)\,,\quad x_2(t) = x(1-t)\,.$$

This yields a bijection between the periodic orbits of H and the Hamiltonian chords of $\widehat{H}$ with endpoints on Δ.

Let $\widehat{x}\colon (D, \partial D) \to (M, \mathrm{im}(x))$ be a capping of x. Up to reparameterization, we can assume that $\widehat{x}$ maps $-i$ to $x(0)$ and i to $x(\frac{1}{2})$. Then, we can define a capping $\widehat{X}\colon D_- \to M \times M$ of X by setting $\widehat{X} = [\widehat{X}_1, \widehat{X}_2]$ where $\widehat{X}_1 = \widehat{x}|_{D_+} \circ \sigma$ with $\sigma\colon D_+ \to D_-$ the symmetry with respect to $i\mathbb{R}$, and $\widehat{X}_2 = \widehat{x}|_{D_-}$.

We end up with a bijection $\mathrm{Crit}(\mathcal{A}_H) \to \mathrm{Crit}(\mathcal{A}_{\widehat{H}:\Delta})$, defined by $\widetilde{x} = [x, \widehat{x}] \mapsto \widetilde{X} = [X, \widehat{X}]$ preserving action values and grading.

This idea can be fairly straightforwardly adapted to the various geometric objects defining the moduli spaces entering the definition of all morphisms and structures described so far. Any such object in the Floer (respectively quantum) theory of M can be identified with an object in the Lagrangian Floer (respectively Lagrangian quantum) theory of Δ. This yields an identification of the Floer (respectively quantum) homology of M and the Lagrangian Floer (respectively Lagrangian quantum) homology of Δ which agrees with all their extra structures.

In particular, we get the following commutative diagram :

$$\begin{array}{ccc} \mathrm{HQ}_*(M) & = & \mathrm{HQ}_*(\Delta) \\ \downarrow{\scriptstyle \mathrm{PSS}} & & \downarrow{\scriptstyle \mathrm{PSS}} \\ \mathrm{HF}_*(M; H, J) & = & \mathrm{HF}_*(\Delta; \widehat{H}, \widehat{J}) \end{array} \tag{I.10}$$

which will be useful in our study of spectral invariants.

6. Action(s) of the symplectomorphism group

Like the diffeomorphism group of a manifold acts on its Morse homology, as briefly described in Remark I.8, the symplectomorphism group of a symplectic manifold acts on its Floer and quantum homologies. In this section, we explain in which ways.

6.1. Action of a symplectomorphism

Compared to the Morse case, where we only need a diffeomorphism of L in order to act on its homology, in the quantum case for example, we also need to address the issue of the pseudo-holomorphic discs in the ambient manifold with boundary in L. One obvious way to do that is to use diffeomorphisms of the ambient manifold which preserve L (globally). Then a J-pseudo-holomorphic disc with boundary in L is canonically mapped to a J'-pseudo-holomorphic disc for some adapted almost complex structure J'. In that case, and for obvious reasons of preservation of the symplectic area of discs for example, it is easy to see that we have to restrict to *symplectomorphisms* of the ambient manifold. As a consequence, we do not need to require L to be preserved : we can as well consider the quantum homology of L and that of its image.

Proposition I.20. *Any symplectomorphism ψ induces degree preserving isomorphisms*

$$\psi_*^Q\colon \mathrm{HQ}_*(L) \longrightarrow \mathrm{HQ}_*(\psi(L)), \quad \text{and} \quad \psi_*^F\colon \mathrm{HF}_*(L) \longrightarrow \mathrm{HF}_*(\psi(L))$$

which coincide via the suitable PSS morphisms, i.e. the diagram

$$\begin{array}{ccc} \mathrm{HQ}_*(L) & \xrightarrow{\psi_*^Q} & \mathrm{HQ}_*(\psi(L)) \\ \downarrow{\scriptstyle \mathrm{PSS}} & & \downarrow{\scriptstyle \mathrm{PSS}} \\ \mathrm{HF}_*(L) & \xrightarrow{\psi_*^F} & \mathrm{HF}_*(\psi(L)) \end{array} \tag{I.11}$$

is commutative. (In that respect, both isomorphisms will be denoted ψ_*.)

Let us start with the quantum side. Choose a regular quantum datum, $\mathcal{D} = (f, \rho, J)$, for L and $\psi \in \mathrm{Symp}(M, \omega)$. Since ψ maps a disc in M with boundary in L to a disc in M with boundary in $\psi(L)$ of identical symplectic area and Maslov index, $\psi(L)$ is a monotone$^+$ Lagrangian whose monotonicity constant is the same as that of L. We define $\mathcal{D}^\psi = (f^\psi, \rho^\psi, J^\psi)$ where (f^ψ, ρ^ψ) is the Morse–Smale pair for the

Lagrangian $\psi(L)$ defined by $f^\psi = f \circ \psi^{-1}$ and $\rho^\psi = (\psi^{-1})^*\rho$, and J^ψ the almost complex structure $\psi_* J$. The datum $\mathcal{D}^\psi$ is regular for $\psi(L)$. Mapping $p \in \mathrm{Crit}(f)$ to $p^\psi = \psi(p) \in \mathrm{Crit}(f^\psi)$ (extended to elementary tensors in the obvious way) induces a bijection $\psi_* \colon \mathrm{CQ}_*(L\,;\mathcal{D}) \to \mathrm{CQ}_*(\psi(L)\,;\mathcal{D}^\psi)$.

As in the Morse case, flow lines of f with respect to ρ are mapped to flow lines of f^ψ with respect to ρ^ψ by $\gamma \longmapsto \gamma^\psi = h \circ \gamma$. Moreover, a J-pseudo-holomorphic disc u with boundary in L is mapped to a J^ψ-holomorphic disc with boundary in $\psi(L)$ via $u \longmapsto u^\psi = \psi \circ u$. This shows that composing by ψ induces a bijection between the moduli spaces entering the definition of the differential of the quantum homology of L and of $\psi(L)$, respectively $\mathcal{M}_Q(p\,,q\,;\mathcal{D}\,,A)$ and $\mathcal{M}_Q(p^\psi\,,q^\psi\,;\mathcal{D}^\psi\,,\psi_* A)$. Since the symplectic area and Maslov index of u and u^ψ coincide, ψ provides *a complete identification of the complexes* which induces a degree preserving isomorphism in homology.

The Floer-theoretic version of this construction is as straightforward. A given Floer regular datum $(H\,,J)$ for L gives rise to another one, $(H^\psi\,,J^\psi)$ with J^ψ as above and $H^\psi = H \circ \psi^{-1}$. Mapping a capped orbit of H, $(x\,,\widehat{x})$ to its image by ψ, $(x^\psi = \psi(x)\,,\widehat{x}^\psi = \psi(\widehat{x}))$ induces a bijection between $\mathrm{Crit}(\mathcal{A}_{H:L})$ and $\mathrm{Crit}(\mathcal{A}_{H^\psi:\psi(L)})$ which preserves index and action. For $\widetilde{x}$ and $\widetilde{y}$ in $\mathrm{Crit}(\mathcal{A}_{H:L})$, the image of a Floer half-tube $u \in \mathcal{M}_F(\widetilde{x}\,,\widetilde{y}\,;L\,;H\,,J)$ by ψ is an element of $\mathcal{M}_F(\widetilde{x}^\psi\,,\widetilde{y}^\psi\,;\psi(L)\,;H^\psi\,,J^\psi)$ and this shows that both Floer complexes are, again, identified. The isomorphism induced by this identification in homology is the Floer version of ϕ_*.

The fact that they coincide via PSS morphisms is proved with the exact same method. With the choices above, the moduli spaces entering the definition of the respective PSS morphisms are pairwise identified so that, even *at the chain level*,

$$\begin{array}{ccc} \mathrm{CQ}_*(L;\mathcal{D}) & \xrightarrow{\psi_*^Q} & \mathrm{CQ}_*(\psi(L);\mathcal{D}^\psi) \\ \downarrow{\scriptstyle \mathrm{PSS}} & & \downarrow{\scriptstyle \mathrm{PSS}} \\ \mathrm{CF}_*(L;H,J) & \xrightarrow{\psi_*^F} & \mathrm{CF}_*(\psi(L);H^\psi,J^\psi) \end{array}$$

commutes. This yields the commutativity of (I.11).

Remark I.21. With the exact same method, it is easy to show that ψ_* agrees with *all the morphisms and extra structures* presented in this chapter, as this will hold at the chain level for careful choices of all the auxiliary data. In particular, ψ_* commutes with the continuation morphisms. ▮

6.2. Action of a Hamiltonian diffeomorphism / an exact Lagrangian isotopy

In case the symplectomorphism is actually Hamiltonian, there is another way to deal with the issue of the pseudo-holomorphic discs of the ambient manifold which appear in the definition of quantum homology. The idea is that a Hamiltonian isotopy ϕ_H going from identity to the desired Hamiltonian diffeomorphism provides a canonical identification between the discs appearing in the relevant moduli spaces, by attaching a tube (given by the isotopy ϕ_H).

More precisely, let v be a J-pseudo-holomorphic disc in M with boundary in L. Let $\partial v \colon S^1 \to L$ be a parameterization of its boundary, and denote by v_H the cylinder $(s\,,t) \longmapsto \phi_H^t(\partial v(s))$. Now observe that the symplectic area of v_H is zero since ϕ_H is a Hamiltonian isotopy, so that $(\phi_H^1(\partial v)\,,v \sharp v_H)$ and $(\phi_H^1(\partial v)\,,\phi_H^1(v))$ are equivalent in $\widetilde{\pi}_2(M\,,L)$ (the quotient of $\pi_2(M\,,L)$ by the kernel of ω).

This indicates that in case we act by a Hamiltonian diffeomorphism ϕ_H^1, all the information is contained in the *exact isotopy of Lagrangians* $\{\phi_H^t(L)\}_t$. This is the content of the following theorem from (Leclercq and Zapolsky 2018), which also shows that the resulting action is an invariant of the *homotopy class with fixed endpoints* of such an isotopy.

Theorem I.22. *Let $L' \in \mathcal{L}_{\mathrm{Ham}}(L)$ and H a Hamiltonian such that $L' = \phi_H^1(L)$. Then the isomorphism, $(\phi_H^1)_*\colon \mathrm{HQ}_*(L) \to \mathrm{HQ}_*(L')$, only depends on the equivalence class of the exact Lagrangian isotopy $\{\phi_H^t(L)\}_t$ in $\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$.*

Equivalently, any $\widetilde{L} \in \widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$ induces an isomorphism $\widetilde{L}_*\colon \mathrm{HQ}_*(L) \to \mathrm{HQ}_*(\widetilde{L}(1))$, which coincides with $(\phi_H^1)_*$ for any Hamiltonian isotopy ϕ_H such that $\phi_H(L) \in \widetilde{L}$.

Sketch of the proof of the theorem. Let $\{L_{s,t}\}_{(s,t)\in[0,1]^2}$ be a two-parameter family of Lagrangians in the space $\mathcal{L}_{\mathrm{Ham}}(L)$, such that for all s we have $L_{s,0} = L$ and $L_{s,1} = L'$. We wish to show that if ϕ^t and $\psi^t \in \mathrm{Ham}(M,\omega)$ are such that $\phi^0 = \psi^0 = \mathrm{Id}_M$ and $\phi^t(L) = L_{0,t}$, $\psi^t(L) = L_{1,t}$, then $\phi_*^1 = \psi_*^1\colon QH_*(L) \to QH_*(L')$.

We first start by the following observation that

$$\Pi\colon \mathrm{Ham}(M,\omega) \longrightarrow \mathcal{L}_{\mathrm{Ham}}(L)\,, \quad \phi \longmapsto \phi(L)$$

is a fibration. (Here, both $\mathrm{Ham}(M,\omega)$ and $\mathcal{L}_{\mathrm{Ham}}(L)$ are endowed with the C^∞-topologies. It is not hard to see that Π is actually a fiber bundle with fiber $\Pi^{-1}(L')$, the set of Hamiltonian diffeomorphisms mapping L to L'. Since this should be proved locally, we can work in a C^1-neighborhood of the 0-section of the cotangent bundle of L.)

This yields a 2-parameter family $\phi^{s,t} \in \mathrm{Ham}(M,\omega)$ such that $\phi^{s,0} = \mathrm{Id}_M$, $\phi^{0,t} = \phi^t$, $\phi^{1,t} = \psi^t$, and finally $\phi^{s,t}(L) = L_{s,t}$. Define $\chi^s = (\phi^1)^{-1}\phi^{s,1}$. Then $\chi^s(L) = L$, $\chi^0 = \mathrm{Id}_M$, and $\chi^1 = (\phi^1)^{-1}\psi^1$. Observe that for any σ_1 and $\sigma_2 \in \mathrm{Symp}(M,\omega)$ we have

$$(\sigma_2 \circ \sigma_1)_* = (\sigma_2)_* \circ (\sigma_1)_*\colon QH_*(L) \to QH_*(\sigma_2(\sigma_1(L)))\,.$$

Therefore $\chi_*^1 = (\phi_*^1)^{-1} \circ \psi_*^1$.

Finally, we claim that since $\chi^0 = \mathrm{Id}_M$ and $\chi^s(L) = L$ for all s, then $\chi_*^1 = \mathrm{Id}$ on $QH_*(L)$, which in turn shows that $\phi_*^1 = \psi_*^1$.

Concerning this last claim, let us insist on the fact that this only holds because χ^s is assumed to preserve L *for all* s.

The idea behind the phenomenon is that when $\phi_H^t(L) = L$ for all t, the whole tube v_H (defined in the introduction of the present section) is included in L, so that not only $v\sharp v_H$ and $\phi_H^1(v)$ are equivalent discs in $\widetilde{\pi}_2(M,L)$ but they also define the same element in $\pi_2(M,L)$. Morally, we have acted on the homology of L with a transformation whose restriction to L is isotopic to identity.

The idea behind the proof is to show that, under this assumption, χ_*^1 coincides with the continuation morphism. Via the PSS morphism and the commutativity of (I.11), it is equivalent to show that this holds on the Floer side. There, this is achieved *at the chain level* by similar methods as in Section 6.1.

Indeed, let u be a Floer half-tube designed to define the differential of the complex $\mathrm{CF}_*(L;H)$, *i.e.* such that the unparameterized half-tube $\mathrm{im}(u) \in \mathcal{M}_F(\widetilde{x},\widetilde{y};L;H,J)$. By using *the whole isotopy* χ^s we can produce an element $v\colon (s,t) \mapsto \chi^s(u(s,t))$ belonging to the moduli space $\mathcal{M}_F(\widetilde{x},\widetilde{y}^{\chi^1};L;K,I)$, designed to define the continuation morphism for carefully chosen auxiliary data (K,I).

This leads to an identification of $\mathcal{M}_F(\widetilde{x},\widetilde{y};L;H,J)$, *before moding out by the* $\mathbb{R}$*-action*, and $\mathcal{M}_F(\widetilde{x},\widetilde{y}^{\chi^1};L;K,I)$. Since we consider the 0-dimensional component of the latter moduli space, it means that the initial Floer half-tube u cannot be reparameterized. Thus it is the constant one and $u(s,\cdot) = x$ for all s. In turn this gives that $\widetilde{y} = \widetilde{x}$. By definition of the continuation morphism, this leads to $\Phi(\widetilde{x}) = \widetilde{x}^{\chi^1}$ which concludes. □

6.3. Action of a smooth path of Hamiltonian diffeomorphisms

Let us consider Floer theory. As explained above, there is an action of $\phi \in \mathrm{Ham}(M,\omega)$ on Floer homology which is based on the idea that if x is a Hamiltonian chord of the Hamiltonian H, then $x^\phi\colon t \mapsto \phi(x(t))$ is a Hamiltonian chord of $H^\phi = H \circ \phi$.

The next natural idea (in particular after reading the very end of the proof above!) would be to use a path of Hamiltonian diffeomorphisms to act on the Floer complex of L rather than just its time-1 extremity. Indeed, pick a smooth 1-parameter family

$\{\phi_t\}_t \subset \mathrm{Ham}(M, \omega)$ such that ϕ_0 is the identity and ϕ_1 preserves L. Why not mapping a Hamiltonian chord x to the chord $t \mapsto \phi_t(x(t))$, which will be Hamiltonian for some other well-chosen Hamiltonian function ? This is a great idea. The resulting morphism is the algebraic description of the *Lagrangian Seidel morphism*. It is described in Section 1.1 of Chapter III.

CHAPTER II

Around spectral invariants

Once upon a time, Viterbo...

In the seminal (Viterbo 1992), Viterbo introduced spectral invariants for Lagrangian submanifolds of cotangent bundles via the theory of generating functions. Then they were adapted to Floer's construction, for Lagrangian submanifolds in cotangent bundles by Oh in (Oh 1997) and, in (Milinković and Oh 1998), the authors proved that both the generating function and the Floer theoretical approaches lead to the same invariants, thanks to some Floer–Morse theory defined in (Milinković 2000), see also (Monzner, Vichery, and Zapolsky 2012) for another proof. Spectral invariants were then developed in the setting of Hamiltonian Floer homology, fairly simultaneously, by Schwarz (Schwarz 2000) and Oh (Oh 2005). In (Leclercq 2008), the Floer-type construction for Lagrangians is extended from cotangent bundles to the aspherical case[1].

Spectral invariants also appeared in the realm of contact geometry, again via the theory of generating functions, thanks to work by Chaperon (Chaperon 1995), Bhupal (Bhupal 2001), Sandon (Sandon 2011), and Zapolsky (Zapolsky 2013) for both Legendrian submanifolds and contactomorphisms of contact manifolds. Finally, Albers and Frauenfelder (Albers and Frauenfelder 2010) developed them in the context of Rabinowitz–Floer homology. This was the starting point of an on-going Floer-type construction of contact spectral invariants by Albers, Shelukhin, and Zapolsky, while another such on-going construction was started by Leclercq and Sandon, relying on Sandon's Floer-type theory of translated points (Sandon 201x).

The main idea

Consider a Morse function on a compact manifold $f\colon M \to \mathbb{R}$. For small enough values of f, the sublevel is empty and its homology is 0, while for big enough values the sublevel is the whole manifold and its homology is that of M. Every homology class α has to appear when one considers increasing values of f and the value for which α appears is the spectral invariant associated with α and f.

More precisely, this idea works because the values of f decrease along its negative gradient flow lines, so that given a Morse–Smale pair (f, ρ), for all $t \in \mathbb{R}$, we have a vector subspace

$$\mathrm{CM}_*^t(M;f) = \mathbb{Z}/2\mathbb{Z}\langle p \in \mathrm{Crit}(f) \,|\, f(p) < t\rangle \xhookrightarrow{\;i^t\;} \mathrm{CM}_*(M;f)$$

to which $\partial_{(f,\rho)}$ restricts as a differential. Thus, one can filter Morse homology by defining $\mathrm{HM}_*^t(M;f) = \mathrm{H}_*(\mathrm{CM}^t(M;f), \partial_{(f,\rho)})$ and, for any non-zero class $\alpha \in \mathrm{HM}_*(M)$,

$$c(\alpha; f) = \inf\{t \in \mathbb{R} \,|\, \alpha \in \mathrm{im}(i_*^t)\}$$

where i_*^t is the map induced in homology by i^t.

In this toy model, one can alternatively think of $c(\alpha; f)$ as the smallest value for which the (singular homology) class $\alpha \in \mathrm{H}_*(M)$ appears in the homology of the sublevel set of f, *i.e.* the infimum of the set of real numbers t such that $\alpha \in \mathrm{im}(\mathrm{H}_*(\{f < t\}) \hookrightarrow \mathrm{H}_*(M))$. This leads to the equivalent definition

$$c(\alpha; f) = \inf\big\{ \max_{p \in \mathrm{im}(a)} f(p) \,|\, a \in \alpha \big\}$$

1. The extension was two-fold as it was also extended from Floer homology to a Floer-type Leray–Serre spectral sequence machinery of Barraud and Cornea (Barraud and Cornea 2007).

which explains why these invariants are sometimes called *min-max invariants*. From this fact and Figure II.1, it is then quite easy to get convinced that $c(\alpha\,; f)$ is a critical value of f, that it is "monotone in f" (*i.e.* that if $f \leq g$ then $c(\alpha\,; f) \leq c(\alpha\,; g)$), and that it is "continuous with respect to f" (at least before any bifurcation appears in the set of critical values).

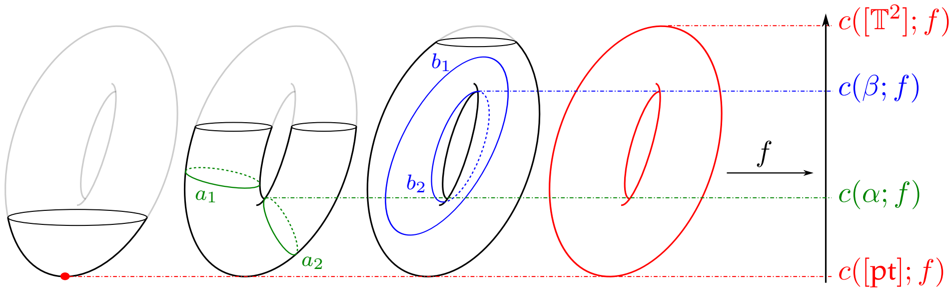


FIGURE II.1. Morse spectral invariants on $\mathbb{T}^2$

The goal of this chapter is to explain how this construction can be adapted to Floer theory, which properties the resulting invariants enjoy, and what they are good for.

Organization of and contribution to Chapter II

Section 1 consists in the exposition of joint work with Frol Zapolsky (Leclercq and Zapolsky 2018) in which we construct spectral invariants via Floer theory for *Lagrangian submanifolds*, under the assumption of *monotonicity* of the Lagrangians, and with respect to homology theories with coefficients in a very wide range of rings. (As mentioned above, this is the natural extension of the author's Ph.D. thesis and the subsequent article (Leclercq 2008), in which such invariants were defined under the quite restrictive assumption of *asphericity*, and with respect to homology theories with coefficients in the field $\mathbb{Z}/2\mathbb{Z}$.) Then we review all the properties satisfied by these invariants, which make them so useful in a great variety of situations. Finally, we briefly explain a variant of the standard construction of spectral invariants, namely spectral invariants of "conormal-type", which were constructed for closed aspherical Lagrangians in joint work with Vincent Humilière and Sobhan Seyfaddini (Humilière, Leclercq, and Seyfaddini 2016). They will also prove to have useful applications.

Section 2 illustrates the power of spectral invariants in two classical situations : to get information on the Hofer geometry of certain sets of Lagrangians, and to study rigidity phenomena of "smooth" symplectic geometry via quasimorphism techniques from Entov and Polterovich (Entov and Polterovich 2009). This is also part of the aforementioned work with Frol Zapolsky.

In Section 3, we explain original applications of the theory of spectral invariants to *continuous* symplectic geometry, obtained in collaboration with Vincent Humilière and Sobhan Seyfaddini (Humilière, Leclercq, and Seyfaddini 2015b, 2015a, 2016). This part is more involved in the sense that these were not *expected* applications of spectral invariants and that, additionally to standard techniques (like classical energy-capacity inequalities), we also had to use – and thus establish – neat intermediate results (on continuous Hamiltonian dynamics, and ... non-classical energy-capacity inequalities).

In Section 4, we sketch two ideas which we want to explore and which naturally extend what we have been doing with spectral invariants so far. Indeed, a particularly promising extension of the theory results from the introduction of topological data analysis techniques in symplectic geometry. The first idea is a very simple observation coming from the standard "cone construction" from algebraic topology which, however, might prove to be useful in this new language. The second is based on the Leray–Serre spectral sequence machinery introduced by Barraud and Cornea (Barraud and Cornea 2007) which we already used in conjunction with spectral invariants in (Leclercq 2008).

Finally, in Section 5 we review work in progress with Sheila Sandon in which we define spectral invariants *in contact geometry*, via a Floer-type homology due to Sandon.

The first step of our work consists in extending Sandon's homology to a less restrictive setting (this corresponds, in symplectic geometry, to going from the aspherical case to the monotone one). This already gives us a way to approach the contact Arnol'd conjecture. Then, we define contact spectral invariants and explore their applications to contact geometry and in particular to contact non-squeezing phenomena and orderability of contactomorphism groups.

1. Spectral invariants : definition, main properties

This section follows (Leclercq and Zapolsky 2018) and its content relies heavily on the homology theories exposed in Chapter I. It is divided as follows. In Section 1.1, we construct the spectral invariant function ℓ on $C^0\big(M\times[0\,,1]\big)$, whose main properties we review in Section 1.2. Then, in Section 1.3, we prove further invariance of ℓ, leading to additional properties and the definition of spectral invariants on the universal cover of $\mathrm{Ham}(M\,,\omega)$. Finally, we compare Lagrangian spectral invariants with their "absolute" counterparts in Section 1.4.

In addition to Disclaimer I.5 on compactness, let us emphasize that :

Disclaimer II.1. All the results stated here are stated for compact Lagrangians of compact symplectic manifolds. However, they can be straightforwardly extended to manifolds convex at infinity (Eliashberg and Gromov 1991), via techniques developed by Frauenfelder and Schlenk (Frauenfelder and Schlenk 2007) and Lanzat (Lanzat 2013).

Moreover, most of the time the finiteness of the minimal Maslov number, $N_L<\infty$, will be implied. The case $N_L=\infty$ is obtained by straightforward simplifications. ∎

1.1. Definition of spectral invariants

In what follows, $(M\,,\omega)$ is a compact $2n$-dimensional symplectic monotone manifold and L is a compact monotone$^+$ Lagrangian submanifold (of dimension n).

The goal is to define spectral invariants for *continuous time-dependent functions* on M. This is done in two steps. First, with any regular Floer datum $(H\,,J)$ for L we associate a function ℓ from the Lagrangian quantum homology of L to $\mathbb{R}\cup\{-\infty\}$, along the ideas sketched in the introduction of the present chapter for Morse functions. Then we show that this function satisfies some continuity property (for a given non-zero class in $\mathrm{HQ}_*(L)$) with respect to H. This will indicate that $\ell(\alpha\,;H\,,J)$ only depends on H and that ℓ can be extended to a function

$$\ell\colon\ \mathrm{HQ}_*(L)\times C^0\big(M\times[0,1]\big)\longrightarrow\mathbb{R}\cup\{-\infty\}\,.$$

1.1.1. *Nondegenerate Hamiltonians*

Pick a *regular* Floer datum $(H\,,J)$ for L. The associated Floer complex can be naturally filtered by the action, as for $a\in\mathbb{R}$ the following $\mathbb{Z}/2\mathbb{Z}$-vector subspaces

$$\mathrm{CF}^a_*(L;H)=\mathbb{Z}/2\mathbb{Z}\langle\widetilde{x}\in\mathsf{Crit}(\mathcal{A}_{H:L})\,|\,\mathcal{A}_{H:L}(\widetilde{x})<a\rangle$$

are preserved by the differential. Essentially, this comes from the fact that the boundary operator of the Floer complex is defined by considering Floer half-tubes which are negative gradient flow lines of $\mathcal{A}_{H:L}$. It can be proved thanks to the following elementary argument which will come up regularly in the proofs of the main properties of spectral invariants.

The argument is the following. Let $\widetilde{x}$ and $\widetilde{y}$ be generators so that $\widetilde{y}$ appears (non-trivially) in the image of $\widetilde{x}$ by $\partial_{(H,J)}$. This ensures that the moduli space used to define $\partial_{(H,J)}$, $\mathcal{M}_F(\widetilde{x}\,,\widetilde{y}\,;L\,;H\,,J)$, is not empty. Pick an element u and compute its *energy* (see Equation (I.2)) :

$$\begin{aligned}E(u)&=\int_{\mathbb{R}\times[0,1]}\|\partial_s u\|^2\,dsdt=\int_{\mathbb{R}\times[0,1]}\omega(\partial_s u,\partial_t u-X_H(u))\,dsdt\\&=\omega([u])-\int_{\mathbb{R}\times[0,1]}d_uH(\partial_s u)\,dsdt=\omega([u])-\int_{[0,1]}\big(H(y(t))-H(x(t))\big)dt\end{aligned}\tag{II.1}$$

where the first line reflects the (perturbed) pseudo-holomorphic nature of u and the second line the convergence condition required on $u \in \mathcal{M}_F(\widetilde{x}, \widetilde{y}; L; H, J)$. Now, the fact that u also has to agree with the cappings, *i.e.* the equivalence between $\widehat{y}$ and $\widehat{x}\sharp u$, shows that $E(u) = \mathcal{A}_{H:L}(\widetilde{x}) - \mathcal{A}_{H:L}(\widetilde{y})$ and we conclude that $\mathcal{A}_{H:L}(\widetilde{x}) \geq \mathcal{A}_{H:L}(\widetilde{y})$ since $E(u) \geq 0$.

This implies that $(\mathrm{CF}^a_*(L; H), \partial_{(H,J)})$ is a subcomplex of the Floer complex; we denote the inclusion by $i^a : \mathrm{CF}^a_*(L; H) \hookrightarrow \mathrm{CF}_*(L; H)$, and by i^a_* the induced morphism in homology. Now, using the canonical PSS isomorphism described in Section 5.2 of Chapter I,

$$\mathrm{PSS}_{H,J} : \mathrm{HQ}_*(L) \longrightarrow \mathrm{HF}_*(L; H, J),$$

we define the Lagrangian spectral invariant associated with a non-zero class $\alpha \in \mathrm{HQ}_*(L)$ to be

$$\ell(\alpha; H, J) = \inf\left\{a \in \mathbb{R} \,|\, \mathrm{PSS}_{H,J}(\alpha) \in \mathrm{im}(i^a_*)\right\}. \tag{II.2}$$

(Hopefully, this definition explains why spectral invariants are often called *action selectors*.)

Recall from Section 3.1 that there exists a canonical class $[L]$ in $\mathrm{HQ}_n(L)$. When $[L] \neq 0$, the spectral invariant associated with this specific class is of particular interest. It will be denoted $\ell_+(H, J) = \ell([L]; H, J)$.

1.1.2. *Main property of ℓ, its continuity*

The main property of ℓ is given by the following estimates.

Lemma II.2. *Let (H^i, J^i), $i = 0, 1$, be regular Floer data and $\alpha \in \mathrm{HQ}_*(L)$, non-zero. We have*

$$\int_0^1 \min_M (H^1_t - H^0_t)\, dt \;\leq\; \ell(\alpha; H^1, J^1) - \ell(\alpha; H^0, J^0) \;\leq\; \int_0^1 \max_M (H^1_t - H^0_t)\, dt. \tag{II.3}$$

The proof boils down to the same argument as above : *compute the energy of an element of some moduli space adapted to the situation.* As the goal is to compare constructions made with different Floer data, it seems natural to turn to the moduli spaces entering the definition of the continuation morphisms of Lagrangian Floer homology from Section 4.1.2.

Proof. Let consider a homotopy from H^0 to H^1 of the form $K^s_t(x) = H^0_t(x) + \beta(s)\big(H^1_t(x) - H^0_t(x)\big)$ where $\beta\colon \mathbb{R} \to [0, 1]$ is a smooth nondecreasing cut-off function which satisfies $\beta(s) = 0$ for $s \leq 0$ and $\beta(s) = 1$ for $s \geq 1$. There is a regular homotopy of Floer data $(H^s, J^s)_s$, stationary for $s \notin (0, 1)$, and with H^s ε-close to K^s.

Let $\widetilde{x}_i \in \mathrm{Crit}(\mathcal{A}_{H^i:L})$, $i = 0$ and 1, with common index and suppose that $\widetilde{x}_2$ appears in the image of $\widetilde{x}_1$ via $\Phi_{(H^s, J^s)}$. Pick $v \in \mathcal{M}_F(\widetilde{x}_0, \widetilde{x}_1; L; H, J)$ and compute its energy. Since v satisfies the pseudo-holomorphic equation perturbed by the vector field induced by the homotopy H, the computation is similar to (II.1), except that the dependence of H on s yields an additional term :

$$E(u) = \mathcal{A}_{H:L}(\widetilde{x}_0) - \mathcal{A}_{H:L}(\widetilde{x}_1) + \int_{\mathbb{R}\times[0,1]} \partial_s H^s_t\big(v(s,t)\big)\, ds dt.$$

One can thus straightforwardly conclude that

$$\mathcal{A}_{H:L}(\widetilde{x}_1) - \mathcal{A}_{H:L}(\widetilde{x}_0) \leq \int_0^1 \max_M (H^1_t - H^2_t)\, dt$$

since $E(u)$ is non-negative, K^s was chosen ε-close to H^s, and ε was arbitrarily small.

This computation proves the commutativity of

(II.4)
$$\begin{array}{ccccc}
\mathrm{HF}^a_*(H^0, J^0 : L) & \xrightarrow{\ i^a_*\ } & \mathrm{HF}_*(H^0, J^0 : L) & \xleftarrow{\ \mathrm{PSS}_{H^0,J^0}\ } & \mathrm{HQ}_*(L) \\
\downarrow \Phi & & \downarrow \Phi & \swarrow_{\mathrm{PSS}_{H^1,J^1}} & \\
\mathrm{HF}^{a+b}_*(H^1, J^1 : L) & \xrightarrow{\ i^{a+b}_*\ } & \mathrm{HF}_*(H^1, J^1 : L) & &
\end{array}$$

with $b = \int_0^1 \max_M (H^1_t - H^2_t)\, dt$, which yields the right-most inequality of Equation (II.3). The other inequality is obtained by exchanging the roles of H^0 and H^1. □

1.1.3. *Invariance of ℓ*

The first obvious consequence of Lemma II.2 is that the spectral invariants defined by (II.2) above are independent of the specific almost complex structure chosen for their construction ; hence, we remove it from the notation.

The second obvious consequence is that their definition can easily be extended to arbitrary continuous Hamiltonians $H : M \times [0\,,1] \to \mathbb{R}$. Indeed, pick a sequence of smooth nondegenerate Hamiltonians $(H_n)_{n\in\mathbb{N}}$ uniformly converging to a continuous H on $M \times [0\,,1]$ and define $\ell(\alpha\,; H)$ as the limit $\lim_{n\to\infty} \ell(\alpha\,; H_n)$. Equation (II.3) ensures that this limit exists and does not depend on the choice of the sequence $\{H_n\}_n$. Hence, we obtain a well-defined function ℓ on $\mathrm{HQ}_*(L) \times C^0\big(M \times [0\,,1]\big)$.

1.2. Main properties

Let L be a closed monotone$^+$ Lagrangian of $(M\,,\omega)$. We now present the main properties satisfied by the function ℓ defined on $\mathrm{HQ}_*(L) \times C^0\big(M \times [0\,,1]\big)$ in Section 1.1.

Finiteness For all $H \in C^0\big(M \times [0\,,1]\big)$, $\ell(\alpha\,; H) = -\infty$ if and only if $\alpha = 0$.

First, note that $\ell(0\,; H) = -\infty$ for all H *by convention*. Now, the fact that $\ell(\alpha\,; H) \in \mathbb{R}$ for any $\alpha \neq 0$ and any H will easily come from Quantum valuation and Continuity below.

Spectrality For $H \in C^\infty\big(M \times [0\,,1]\big)$ and $\alpha \neq 0$, $\ell(\alpha\,; H) \in \mathrm{Spec}(H : L)$.

The *action spectrum* of H, $\mathrm{Spec}(H : L)$, is defined as the set of critical values of the action functional associated with H, $\mathcal{A}_{H:L}$. The Spectrality property shades some light as to why ℓ is called *spectral* invariant. The proof of the statement is standard and goes back to (Oh 2005) in the setting of spectral invariants defined for monotone symplectic manifolds (without reference to a Lagrangian). The main argument is that $\omega(\pi_2(M\,,L)) = \tau N_L \mathbb{Z}$ is discrete. Together with the fact that the image of i^a_* does not change as long as a stays in the complement of $\mathrm{Spec}(H : L)$ which is open and dense when H is nondegenerate, this is already enough to handle the case of nondegenerate Hamiltonians.

For degenerate Hamiltonians, one has to be a bit more careful. We first fix a sequence of smooth nondegenerate Hamiltonians $(H_n)_{n\in\mathbb{N}}$ which converges to H. By definition of $\ell(\alpha\,; H)$ and the nondegeneracy case above, there exists a sequence of equivalence classes of capped orbits $\widetilde{x}_n = [x_n\,, \widehat{x}_n] \in \mathrm{Crit}(\mathcal{A}_{H_n:L})$ such that $\mathcal{A}_{H_n:L}(\widetilde{x}_n)$ converges to $\ell(\alpha\,, H)$. By Arzela–Ascoli we deduce the existence of a limiting Hamiltonian chord of H. Then, for n_0 big enough, it is easy to define an explicit capping for x, $\widehat{x}$, made of the concatenation of a well-chosen cylinder between x and x_{n_0} and $\widehat{x}_{n_0}$. Finally, by using again the discreteness of $\omega(\pi_2(M\,,L))$, we see that $\mathcal{A}_{H_n:L}(\widetilde{x}_n)$ converges to $\mathcal{A}_{H:L}([x\,,\widehat{x}])$ which concludes.

Continuity For any continuous functions H and K, and $\alpha \neq 0$

$$\int_0^1 \min_M (K_t - H_t)\, dt \leq \ell(\alpha; K) - \ell(\alpha; H) \leq \int_0^1 \max_M (K_t - H_t)\, dt\,.$$

This is an immediate consequence of Lemma II.2, since (II.3) obviously extends to continuous functions.

MONOTONICITY If $H \leq K$, then $\ell(\alpha\,; H) \leq \ell(\alpha\,; K)$.

This follows directly from CONTINUITY.

TRIANGLE INEQUALITY For all α and β, $\ell(\alpha * \beta\,; H\sharp K) \leq \ell(\alpha\,; H) + \ell(\beta\,; K)$.

First note that by CONTINUITY, it is enough to prove this for nondegenerate smooth Hamiltonians. Now, for such Hamiltonians the proof is similar to the proof of Lemma II.2, boiling down to *computing the energy of an element of some moduli space adapted to the situation*. It is natural to consider here the moduli space entering the definition of the Lagrangian Floer product, as described in Section 4.1.3 of Chapter I. Thus, we choose H^3 to be a nondegenerate Hamiltonian ε-close to the concatenation $H^1\sharp H^2$ and J^3 accordingly. Computing the energy of an element $u \in \mathcal{M}_{*_F}(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\,; K\,, I)$ leads to

$$\mathcal{A}_{H^1:L}(\widetilde{\gamma}^1) + \mathcal{A}_{H^2:L}(\widetilde{\gamma}^2) - \mathcal{A}_{H^3:L}(\widetilde{\gamma}^3) \geq -2\varepsilon\,.$$

From this, it follows that the restriction of $*$ to the filtered complexes satisfies :

$$* : \mathrm{CF}_*^{a_1}(H^1; L) \otimes \mathrm{CF}_*^{a_2}(H^2; L) \to \mathrm{CF}_*^{a_1+a_2+2\varepsilon}(H^3; L)$$

for any a_1 and $a_2 \in \mathbb{R}$. Since ε can be chosen arbitrarily small, we get the expected result

$$\ell(\alpha * \beta; H^1\sharp H^2) \leq \ell(\alpha; H^1) + \ell(\beta; H^2)$$

thanks to the commutativity of a diagram similar to (II.4).

TIME SHIFT If c is a function of time then

$$\ell(\alpha; H + c) = \ell(\alpha; H) + \int_0^1 c(t)\, dt\,.$$

This follows from the fact that, for smooth Hamiltonians, the spectrum of $H^s = H + sc$ for $s \in [0\,,1]$ coincides with the spectrum of H, shifted by $s \cdot \int_0^1 c(t)\, dt$. Thus the statement holds by CONTINUITY and SPECTRALITY properties and obviously extends to continuous functions.

QUANTUM VALUATION We have $\ell(\alpha\,; 0) = \nu(\alpha) w_L$ where ν is the *quantum valuation* and $\ell_+(0) = 0$.

In the above formula, w_L denotes the generator of the group of periods $\omega(\pi_2(M\,, L)) \subset \mathbb{R}$. Since L is monotone with monotonicity constant τ, we have $w_L = \tau N_L$.

The QUANTUM VALUATION property reflects the fact that *spectral invariants agree with the interpretation of the quantum homology of L as the Lagrangian "Morse–Bott Floer homology" of the zero Hamiltonian*, as described in Section 5.1 of Chapter I.

Let us first explain how to see that. Recall from (Leclercq and Zapolsky 2018) the definition of the Lagrangian quantum valuation, $\nu\colon\ \mathrm{HQ}_*(L) \to \mathbb{Z} \cup \{-\infty\}$, inspired by and analogous to its absolute counterpart defined by Entov and Polterovich (Entov and Polterovich 2003). First fix a regular quantum datum $\mathcal{D}$ for L. Now, for any finite sum $\sum_k p_k \otimes t^{n_k}$ defining a chain $C \in \mathrm{CQ}_*(L\,; \mathcal{D})$,[2] define its valuation as

$$\nu_{\mathcal{D}}(C) = \max\big\{ -n_k \,|\, C = \sum p_k \otimes t^{n_k} \big\}$$

and, if $\alpha \in \mathrm{HQ}_*(L\,; \mathcal{D})$ is non-zero, define its valuation as

$$\nu_{\mathcal{D}}(\alpha) = \inf\{\nu(C) \,|\, [C] = \alpha\} = \inf_{C \in \alpha} \Big\{ \max\big\{ -n_k \,|\, C = \sum p_k \otimes t^{n_k} \big\} \Big\}$$

2. Recall that we restricted ourselves to homology theories with coefficients in $\mathbb{Z}/2\mathbb{Z}$ so that every elementary tensor $p \otimes t^n$ appears with coefficient 1 (or does not appear).

while $\nu(0)$ is set to $-\infty$.

Now, let us consider this quantity, under the change of viewpoint from Section 5.1, where elementary tensors $p\otimes t^n$ are replaced by pairs $(p\,,A)$ with $A\in\widetilde{\pi}_2(M\,,L)$. From this viewpoint, the quantum valuation of C, formal sum of pairs $(p_k\,,A_k)$, is $\nu_{\mathcal{D}}(C)=\frac{1}{w_L}\max_k\big\{-\omega(A_k)\big\}$ and therefore, if $\alpha\in\mathrm{HQ}_*(L\,;\mathcal{D})$ is non-zero,

$$\nu_{\mathcal{D}}(\alpha)\,w_L=\inf_{C\in\alpha}\Big\{\max\big\{\mathcal{A}_{0:L}(A_k)\,|\,C=\sum(p_k,A_k)\big\}\Big\}. \tag{II.5}$$

The right-hand side term would be the spectral invariant associated with α and the zero Hamiltonian, $\ell(\alpha\,;0)$, if the latter was nondegenerate. However, $H=0$ is obviously degenerate and thus the definition of $\ell(\alpha\,;0)$ requires picking a sequence of smooth nondegenerate Hamiltonians converging to 0, and considering the limit of the sequence of associated spectral invariants.

Remark II.3. Alternatively, we can also define a quantum subcomplex $\mathrm{CQ}^a_*(L\,;\mathcal{D})$ generated by elements $(p\,,A)$ with $\nu_{\mathcal{D}}(A)<a$. We denote by i^a_Q the map induced in homology by the inclusion of this subcomplex in the whole complex. Then (II.5) can be reformulated as :

$$\nu_{\mathcal{D}}(\alpha)\,w_L=\inf\big\{a\in\mathbb{R}\,|\,\alpha\in\mathrm{im}(i^a_Q)\big\},$$

which shows even more similarities with (II.2). ▮

Let us now prove the Quantum valuation property. It is an obvious corollary of the following statement : for all continuous time-dependent functions H,

$$\int_0^1\min_M H_t\,dt\le\ell(\alpha;H)-\nu(\alpha)w_L\le\int_0^1\max_M H_t\,dt\,. \tag{II.6}$$

Proof. The proof is yet another occurrence of a (by now standard !) method : *compute the energy of an element of a well-chosen moduli space*. The idea here being to link the quantum and Floer situations, we consider an element entering the definition of the PSS morphism, as depicted in Figure I.8, $(\underline{u}\,,m\,,v)\in\mathcal{M}_{\mathrm{PSS}}(p\,,\widetilde{x}\,;\mathcal{D}\,,H)$. Recall that for p in $\mathrm{Crit}(f)$ and $\widetilde{x}\in\mathrm{Crit}(\mathcal{A}_{H:L})$, $(\underline{u}\,,m\,,v)$ consists of a point m in L, a string of pearls $\underline{u}$ in $\mathcal{M}_Q(p\,,m\,;\mathcal{D}\,;B)$ for some $B\in\pi_2(M\,,L)$, and a half-disc v satisfying an equation which interpolates between the J-pseudo-holomorphic and the Floer equations. Computing the energy of v gives in this case

$$\int_0^1 H_t\big(x(t)\big)\,dt-\int v^*\omega\le\int_0^1\max_M(H_t-0)\,.$$

(Alternatively, v can be considered as a Floer half-tube like those entering the definition of the continuation morphism, interpolating between H and 0. Thus, considering m as a Hamiltonian chord of the 0 Hamiltonian which we cap with the constant disc at m, we can rewrite

$$\int_0^1 H_t\big(x(t)\big)\,dt-\int v^*\omega=\mathcal{A}_{H:L}\big([x,v]\big)-\mathcal{A}_{0:L}\big([m,m]\big)$$

and Continuity immediately gives the result.)

Since the action of $\widetilde{x}$ is bounded from above by $\int_0^1 H_t\big(x(t)\big)\,dt-\int v^*\omega$ (recall that $\underline{u}$ consists of J-pseudo-holomorphic discs which thus have positive symplectic area), we end the proof by exploiting the commutativity of

$$\begin{array}{ccc}\mathrm{HQ}^a_*(L;\mathcal{D}) & \xrightarrow{\mathrm{PSS}} & \mathrm{HF}^{a+b}_*(L;H,J)\\ \big\downarrow{\scriptstyle i^a_Q} & & \big\downarrow{\scriptstyle i^{a+b}_*}\\ \mathrm{HQ}_*(L;\mathcal{D}) & \xrightarrow{\mathrm{PSS}} & \mathrm{HF}_*(L;H,J)\end{array}$$

with $b=\int_0^1\max_M H_t\,dt$ (see also Remark II.3). □

Now, Equation (II.6) obviously implies the first equality, $\ell(\alpha\,;0) = \nu_{\mathcal{D}}(\alpha)\,w_L$, which in turn implies that $\nu = \nu_{\mathcal{D}}$ does not depend on the regular quantum data $\mathcal{D}$. Thus, we get that $\ell_+(0) = \ell([L]\,;0) = \nu(0)\,w_L$ which is easily seen to be non-positive : consider a Morse function with a unique maximum, this maximum is a cycle representing $[L]$ with valuation 0. The other inequality can be obtained by performing the following meaningless trick[3], involving TRIANGLE INEQUALITY : $\ell([L]\,;0) \leq \ell([L]\,;0) + \ell([L]\,;0)$. In the end, we get that $\ell([L]\,;0) = 0$.

LAGRANGIAN CONTROL If for all t, $H_t|_L = c(t) \in \mathbb{R}$ (respectively $\leq$, $\geq$), then

$$\ell(\alpha; H) = \int_0^1 c(t)\,dt + \nu(\alpha) w_L \quad (\text{respectively } \leq, \geq)\,.$$

So that, for all H :

$$\int_0^1 \min_L H_t\,dt \leq \ell(\alpha; H) - \nu(\alpha) w_L \leq \int_0^1 \max_L H_t\,dt\,.$$

This property shows that the behavior of the Hamiltonian *on the Lagrangian* produces bounds on its spectral invariants. These bounds generalize Equation (II.6) and thus show that QUANTUM VALUATION is satisfied as soon as H vanishes on the Lagrangian. Its proof is based on another idea which is quite frequently used in the context of spectral invariants. The idea is to follow the evolution of the spectrum of the action functional associated with a 1-parameter family of Hamiltonians, and then to conclude with SPECTRALITY and CONTINUITY.

PROOF. Let H be a smooth Hamiltonian which restricts to L as a function of time, $c\colon [0\,,1] \to \mathbb{R}$ and define $H^s = sH$ for $s \in [0\,,1]$. Since L is Lagrangian, for any fixed s, the chords of H^s are contained in L. Each of these comes with a natural capping (itself), contained in L so that it has area 0. This shows that for all $s \in \mathbb{R}$, $\mathrm{Spec}(H^s) = \left\{ s \cdot \int_0^1 c(t)\,dt + k w_L \,\middle|\, k \in \mathbb{Z} \right\}$.

By SPECTRALITY and CONTINUITY, $\ell(\alpha\,; H^s) = s\int_0^1 c(t)\,dt + k_0 w_L$ for some $k_0 \in \mathbb{Z}$, independent of s. Since $H^0 = 0$, $k_0 = \nu(\alpha)$ by QUANTUM VALUATION, we get the desired result for $s = 1$.

If H satisfies $H_t|_L \leq c(t)$ (respectively $H_t|_L \geq c(t)$) for all t, pick any function K such that $K \geq H$ (respectively $K \leq H$) and $K_t|_L = c(t)$ and conclude by MONOTONICITY. This handles the case of smooth Hamiltonians and we conclude by CONTINUITY. ☐

NOVIKOV ACTION For $A \in \pi_2(M\,, L)$, we have $\ell(A \cdot \alpha\,; H) = \ell(\alpha\,; H) - \omega(A)$.

The Novikov action of $\pi_2(M\,,L)$ (or rather $\widetilde{\pi}_2(M\,,L)$) is induced by the attachment of a disc, of relative homotopy class A, to a chain $C \in \mathrm{CQ}_*(L\,;\mathcal{D})$:

$$A \cdot C = A \cdot \Big(\sum_k p_k \otimes t^{n_k}\Big) = \sum_k p_k \otimes t^{n_k - \frac{\mu(A)}{N_L}}\,.$$

Via the PSS morphism, this leads on the Floer side to a shift of the action by $-\omega(A)$ which yields the result.

SYMPLECTIC INVARIANCE Let $\psi \in \mathrm{Symp}(M\,,\omega)$ and $L' = \psi(L)$. Let

$$\ell'\colon\ \mathrm{HQ}_*(L') \times C^0\big(M \times [0,1]\big) \to \mathbb{R} \cup \{-\infty\}$$

be the corresponding spectral invariant. Then $\ell(\alpha\,; H) = \ell'(\psi_*(\alpha)\,; H \circ \psi^{-1})$.

The isomorphism $\psi_*\colon\ \mathrm{HQ}_*(L) \to \mathrm{HQ}_*(L')$ is induced from the symplectomorphism ψ as described in Section 6.1 of Chapter I.

3. Meaningless but which we did not manage to avoid nevertheless...

The proof of this property is straightforward. Indeed, the commutative diagram (I.11), which shows that the PSS morphism intertwines the isomorphisms ψ_* induced by ψ on both Floer and quantum homologies, is easily shown to agree with the filtration of Floer homology.

DUALITY Let $\alpha \in \mathrm{HQ}_*(L)$ and $\alpha^\vee \in \mathrm{HQ}^{n-*}(L)$ be dual classes, $-\ell(\alpha\,;\overline{H}) = \ell(\alpha^\vee\,;H)$.

This is a simplified version of the DUALITY property which can be found in (Leclercq and Zapolsky 2018). However, this is the part which will be used in the applications below.

Recall that the Floer version of Poincaré duality is presented in Section 4.1.4 of Chapter I. There also exists a quantum version of this duality, which is constructed thanks to the exact same idea, that is to consider the same geometric objects, with reversed orientation. Given a regular quantum datum $\mathcal{D}$ for L, we easily build another regular quantum datum $\overline{\mathcal{D}}$, such that $\mathrm{CF}_*(L\,;\overline{\mathcal{D}})$ is canonically identified with $\mathrm{CF}^{n-*}(L\,;\mathcal{D})$, yielding an isomorphism $\mathrm{PD}\colon\ \mathrm{HQ}_*(L) \to \mathrm{HQ}^{n-*}(L)$. It is also easy (see for example the proof of the commutativity of (I.8)) to show that the PSS morphism induces also an isomorphism between the quantum and Floer cohomologies of L.

The only observation to make in order to prove the DUALITY property was made in Section 4.1.4 : "dual" generators with respect to dual Floer data have opposite action, namely $\mathcal{A}_{H:L}(\widetilde{x}) = -\mathcal{A}_{\overline{H}:L}(\overline{\widetilde{x}})$.

1.3. Further invariance and consequences

We now show that, when computed for *normalized* Hamiltonians, the spectral invariants defined by Equation (II.2) above only depend on the equivalence class of the Hamiltonian defined in Definition I.2.

Proposition II.4. *Let H^0 and H^1 be normalized, equivalent Hamiltonians. Then, for all $\alpha \in \mathrm{HQ}_*(L)$, $\ell(\alpha\,;H^0) = \ell(\alpha\,;H^1)$.*

We will use this further invariance to add two important properties, NON-NEGATIVITY and MAXIMUM, to the long list of the previous section. Finally, we will derive from ℓ a function defined on the universal cover of the Hamiltonian group.

1.3.1. *Further invariance*

We want to prove that if H^0 and H^1 are equivalent normalized Hamiltonians in the sense of Definition I.2, their respective associated spectral invariant functions $\ell(\cdot\,;H^i)\colon\ \mathrm{HQ}_*(L) \to \mathbb{R}\cup\{-\infty\}$ coincide.

First, let us show that the desired invariance property will be a consequence of the following result on the spectrum of the action functional.

THEOREM II.5. *Let H be normalized, $\mathrm{Spec}(H:L)$ only depends on the equivalence class of H.*

Indeed, this lemma shows that for a normalized homotopy $(H^s)_s$, such that for all $s \in [0\,,1]$, $\phi^1_{H^s} = \varphi$, the spectrum of $\mathcal{A}_{H^s:L}$ does not depend on s. This, combined with the fact that *the spectrum is nowhere dense*[4] and with the SPECTRALITY and CONTINUITY properties of spectral invariants, shows that for every non-zero $\alpha \in \mathrm{HQ}_*(L)$, $\ell(\alpha\,;H^0) = \ell(\alpha\,;H^1)$.

Now, the proof of Theorem II.5 easily follows from the fact that, given a normalized homotopy $H = (H^s)_{s\in[0,1]}$ so that $\phi^1_{H^s} = \varphi$ for all s, there exists a map $\widetilde{\phi}_H\colon\ \mathrm{Crit}(\mathcal{A}_{H^0}) \to \mathrm{Crit}(\mathcal{A}_{H^1})$ such that $\mathcal{A}_{H^0:L}(\widetilde{x}_0) = \mathcal{A}_{H^1:L}(\widetilde{\phi}_H(\widetilde{x}_0))$.

The map itself is very easy to construct. First, given H^0 and H^1 generating the same Hamiltonian diffeomorphism φ, there is a canonical identification between their

4. This regularity property of Floer's action functional in the Lagrangian case can be proved similarly to the Hamiltonian case, see (Oh 2005).

respective Hamiltonian chords, via $x^0 \longmapsto x^1 = \phi^t_{H^1}(x^0(0))$. Next, as explained at the beginning of Section 6.2 of Chapter I, the homotopy H provides a lift of this map to the critical sets of the respective action functionals, by setting $\widehat{x}^0 \longmapsto \widehat{x}^1 = \widehat{x}^0 \sharp u_H$ where $u_H \colon [0\,,1] \times [0\,,1] \to M$, defined by $u_H(s\,,t) = \phi^t_{H^s}(x^0(0))$, connects x^0 to x^1.

The proof that this map preserves the action is standard and can be summarized as follows. Denote by I the interval $[0\,,1]$. Similarly to the proof of the inequalities (II.3),

$$\mathcal{A}_{H^1:L}(\widetilde{x}^1) - \mathcal{A}_{H^0:L}(\widetilde{x}^0) = \int_I \partial_s \mathcal{A}_{H^s:L}(\widetilde{x}^s)\, ds = \int_I d_{\widetilde{x}^s}\mathcal{A}_{H^s:L}(\partial_s \widetilde{x}^s)\, ds + \int_{I^2} \partial_s H^s_t \circ u_H \, dsdt$$

where $x^s = u_H(s\,,\cdot)$, $\widehat{x}^s = \widehat{x}^0 \sharp u_H|_{[0,s]\times[0,1]}$, and $\widetilde{x}^s = [x^s\,, \widehat{x}^s]$. Since $\widetilde{x}^s \in \mathrm{Crit}(\mathcal{A}_{H^s:L})$, the first summand in the last expression vanishes and we end up with

$$\mathcal{A}_{H^1:L}(\widetilde{x}^1) - \mathcal{A}_{H^0:L}(\widetilde{x}^0) = \int_{I^2} (\partial_s H^s_t) \circ u_H \, dsdt\,. \tag{II.7}$$

Making the same computation with H replaced by $K^s = \overline{H}^0 \sharp H^s$, and u_H replaced by the map u_p defined by $(s\,,t) \longmapsto \phi^t_{H^s}(p)$, yields

$$\mathcal{A}_{K^1}(\widetilde{\gamma}^1_p) = \int_0^1 d_{\widetilde{\gamma}^s_p}\mathcal{A}_{K^s}(\partial_s \widetilde{\gamma}^s_p)\, ds + \int_{I^2} (\partial_s H^s_t) \circ u_p \, dsdt = \int_{I^2} (\partial_s H^s_t) \circ u_p \, dsdt \tag{II.8}$$

where $\widetilde{\gamma}^s_p$ is the equivalence class of $\gamma^s_p = \overline{u_p(0,\cdot)} \sharp u_p(s\,,\cdot)$, capped by $\widehat{\gamma}^s_p = u_p|_{[0,s]\times[0,1]}$. (As such, it is a critical point of $\mathcal{A}_{K^s}$.) As $p \longmapsto \widetilde{\gamma}^1_p$ is a smooth embedding of M (which is connected) into $\mathrm{Crit}(\mathcal{A}_{K^1})$, we deduce that the quantity (II.8) *is independent of* p.

Starting from the normalization of H^0 and H^1,

$$0 = \int_0^1 dt \int_M \left(H^1_t(p) - H^0_t(p)\right) \omega^n_p = \int_{I^2} \left(\int_M \partial_s H^s_t(p)\, \omega^n_p \right) ds\, dt$$

and using the facts that $\phi^t_{H^s}$ is a symplectomorphism for all s and t, and the independence of (II.8) on p, we deduce that

$$0 = \int_{I^2} \left(\int_M \partial_s H^s_t\left(\phi^t_{H^s}(p)\right) \omega^n_p \right) ds\, dt = \int_{I^2} \partial_s H^s_t\left(\phi^t_{H^s}(p)\right) ds\, dt \cdot \int_M \omega^n$$

which, together with (II.7), allows to conclude that $\mathcal{A}_{H^1:L}(\widetilde{x}^1) - \mathcal{A}_{H^0:L}(\widetilde{x}^0) = 0$.

Remark II.6. After these computations it is not hard to see that *when* L *is aspherical*, the spectrum of $\mathcal{A}_{H:L}$ (for a normalized Hamiltonian H) is even more invariant as it only depends on $\varphi = \phi^1_H$. ∎

1.3.2. *Additional properties*

The following properties of the function $\ell_+ = \ell([L]\,;\cdot)$ are consequences of the invariance proved in the above section, and of the properties established in Section 1.2.

| Non-negativity | For all $H \in C^0\big(M \times [0\,,1]\big)$, $\ell_+(H) + \ell_+(\overline{H}) \geq 0$.

By Triangle inequality with $\alpha = \beta = [L]$, we have $\ell_+(H) + \ell_+(\overline{H}) \geq \ell_+(H \sharp \overline{H})$. By Time shift and the definition of $\overline{H}$ and $\sharp$, it suffices to prove the result for H normalized. In that case, $H \sharp \overline{H}$ is also normalized and is equivalent to 0, in the sense of Definition I.2. By invariance of spectral invariants, $\ell_+(H \sharp \overline{H}) = \ell_+(0)$ which vanishes by Quantum valuation.

| Maximum | For all $H \in C^0\big(M \times [0\,,1]\big)$, $\ell(\alpha\,;H) \leq \ell_+(H) + \nu(\alpha) w_L$.

If H is normalized, since $\alpha = [L] \star \alpha$ and $H \sharp 0$ is equivalent to H in the sense of Definition I.2, by invariance and Triangle inequality we obtain $\ell(\alpha\,;H) \leq \ell_+(H) + \ell(\alpha\,;0)$. Again, when H is not normalized, the result follows from the first case and Time shift.

1.3.3. *Invariants of the universal cover of the Hamiltonian group*

Since the set of equivalence classes of normalized smooth Hamiltonians can be identified with $\widetilde{\mathrm{Ham}}(M,\omega)$, the function ℓ defined by (II.2) induces a function

$$\ell\colon\ \mathrm{HQ}_*(L)\times\widetilde{\mathrm{Ham}}(M,\omega)\longrightarrow\mathbb{R}\cup\{-\infty\}$$

which enjoys properties similar (and induced by) the properties listed in Sections 1.2 and 1.3.2. We list them below to indicate the very few delicate points in the adaptation as well as for further use.

FINITENESS $\ell(\alpha\,;\widetilde{\phi})=-\infty$ if and only if $\alpha=0$.

SPECTRALITY For all $\alpha\neq 0$, $\ell(\alpha\,;\widetilde{\phi})\in\mathrm{Spec}(\widetilde{\phi}:L)$.

Recall from Section 1.3.1, that the spectrum of the action itself only depends on the equivalence class of normalized Hamiltonians and thus is naturally associated with elements of $\widetilde{\mathrm{Ham}}(M,\omega)$.

CONTINUITY Assume H and K are normalized, then

$$\int_0^1\min_M(H_t-K_t)dt\leq\ell(\alpha;\widetilde{\phi}_H)-\ell(\alpha;\widetilde{\phi}_K)\leq\int_0^1\max_M(H_t-K_t)dt\,.$$

MONOTONICITY If M is noncompact, and $H\leq K$ have compact support, $\ell(\alpha\,;\widetilde{\phi}_H)\leq\ell(\alpha\,;\widetilde{\phi}_K)$.

Here, M is assumed to be noncompact since isotopies naturally correspond to *normalized* Hamiltonians, and that there are no normalized Hamiltonians H and K satisfying $H\leq K$ on a compact manifold except if they coincide everywhere. On noncompact manifolds, isotopies also correspond to normalized Hamiltonians and the natural normalization condition is requiring the Hamiltonian to have compact support.

TRIANGLE INEQUALITY For all α and β, $\ell(\alpha*\beta\,;\widetilde{\phi}\widetilde{\psi})\leq\ell(\alpha\,;\widetilde{\phi})+\ell(\beta\,;\widetilde{\psi})$.

QUANTUM VALUATION We have $\ell(\alpha\,;\mathrm{Id})=\nu(\alpha)w_L$ and $\ell_+(\mathrm{Id})=0$.

LAGRANGIAN CONTROL If for all t, $H_t|_L=c(t)\in\mathbb{R}$ (respectively $\leq$, $\geq$), then

$$\ell(\alpha;\widetilde{\phi}_H)=\nu(\alpha)w_L+\int_0^1\left(c(t)-\int_M H_t\,\omega^n\right)dt\quad(\text{respectively}\ \ \leq,\geq\,)\,.$$

Thus, for all H :

$$\int_0^1\min_L H_t\,dt\leq\ell(\alpha;\widetilde{\phi}_H)-\nu(\alpha)w_L+\int_0^1\int_M H_t\,\omega^n\,dt\leq\int_0^1\max_L H_t\,dt\,.$$

Note that the Hamiltonian $H_t-\int_M H_t\,\omega^n$ is normalized and generates the same Hamiltonian flow as H_t, therefore

$$\ell(\alpha;\widetilde{\phi}_H)=\ell\big(\alpha;H_t-\textstyle\int_M H_t\,\omega^n\big)=\ell(\alpha;H)-\int_0^1\int_M H_t\,\omega^n\,dt\,,$$

and the property now follows from its counterpart on Hamiltonian functions.

NOVIKOV ACTION For $A\in\pi_2(M,L)$, $\ell(A\cdot\alpha\,;\widetilde{\phi})=\ell(\alpha\,;\widetilde{\phi})-\omega(A)$.

SYMPLECTIC INVARIANCE Let $\psi\in\mathrm{Symp}(M,\omega)$, $L'=\psi(L)$ and

$$\ell'\colon\ \mathrm{HQ}_*(L')\times\widetilde{\mathrm{Ham}}(M,\omega)\longrightarrow\mathbb{R}\cup\{-\infty\}$$

the associated spectral invariant function. Then $\ell(\alpha\,;\widetilde{\phi})=\ell'(\psi_*(\alpha)\,;\psi\widetilde{\phi}\psi^{-1})$.

We use the natural action of the group $\mathrm{Symp}(M, \omega)$ on $\widetilde{\mathrm{Ham}}(M, \omega)$ by conjugation. Recall that if $\widetilde{\phi} = \widetilde{\phi}_H$ then $\psi \widetilde{\phi}_H \psi^{-1} = \widetilde{\phi}_{H \circ \psi^{-1}}$. The result now follows from the previous case.

DUALITY For dual elements $\alpha \in \mathrm{HQ}_*(L)$ and $\alpha^\vee \in \mathrm{HQ}^{n-*}(L)$, $-\ell(\alpha\,; \widetilde{\phi}^{-1}) = \ell(\alpha^\vee\,; \widetilde{\phi})$.

NON-NEGATIVITY $\ell_+(\widetilde{\phi}) + \ell_+(\widetilde{\phi}^{-1}) \geq 0$.

MAXIMUM $\ell(\alpha\,; \widetilde{\phi}) \leq \ell_+(\widetilde{\phi}) + \ell(\alpha\,; \mathrm{Id}) = \ell_+(\widetilde{\phi}) + \nu(\alpha) w_L$.

Remark II.7. As we will see in Definition II.10, we can infer from ℓ a bi-invariant pseudo-distance on $\widetilde{\mathrm{Ham}}(M, \omega)$, thanks to its NON-NEGATIVITY, TRIANGLE INEQUALITY, and SYMPLECTIC INVARIANCE properties. It is called *spectral distance* or *Viterbo distance*. The CONTINUITY property immediately yields that it is smaller than Hofer's distance. ∎

1.4. Relationship with "absolute" spectral invariants

As mentioned in the introduction of this chapter, Schwarz (Schwarz 2000) and Oh (Oh 2005) adapted Viterbo's construction to get spectral invariants of symplectic manifolds via Floer theory under asphericity condition for the former and monotonicity for the latter. The definition is formally the same as that of their Lagrangian counterpart. Indeed, one can, similarly to what was done here, filter the Floer complex of (M, ω) with respect to the values of the action. Spectral invariants are then defined by Equation (II.2), with Lagrangian quantum and Floer homologies (and PSS morphism) replaced by their respective "absolute" versions. We will denote them by[5]

$$c\colon\ \mathrm{HQ}_*(M) \times C^0\big(M \times \mathbb{R}/\mathbb{Z}\big) \longrightarrow \mathbb{R} \cup \{-\infty\}\,.$$

1.4.1. *Spectral invariants and the Lagrangian diagonal*

An *a priori* different way to define such "absolute" invariants would be via the *diagonal construction* from Section 5.3 of Chapter I. Indeed, one could associate with a Hamiltonian function H of (M, ω), the *Lagrangian* spectral invariant relative to the Lagrangian diagonal $\Delta \subset (M \times M, \omega \oplus (-\omega))$ associated with the Hamiltonian $H \oplus 0$.

It turns out that both constructions coincide. Indeed, the canonical ring isomorphism between the Floer (respectively quantum) theory of (M, ω) and the Lagrangian Floer (respectively quantum) theory of Δ preserves both degree and action. This ensures that Diagram (I.10) can be filtered to get

$$\begin{array}{ccccc}
\mathrm{HF}^a_*(M; H, J) & \xrightarrow{\ i^a_*\ } & \mathrm{HF}_*(M; H, J) & \xleftarrow{\ \mathrm{PSS}_{H,J}\ } & \mathrm{HQ}_*(M) \\
\| & & \| & & \| \\
\mathrm{HF}^a_*(\Delta; \widehat{H}, \widehat{J}) & \xrightarrow{\ i^a_*\ } & \mathrm{HF}_*(\Delta; \widehat{H}, \widehat{J}) & \xleftarrow{\ \mathrm{PSS}_{\widehat{H},\widehat{J}}\ } & \mathrm{HQ}_*(\Delta)
\end{array}$$

for any periodic nondegenerate smooth Hamiltonian H on M. Recall from Section 5.3 that $\widehat{H}$ is defined from H as $H^1 \oplus H^2$ with H^1 and H^2 roughly corresponding to H, respectively on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. However, up to a time reparameterization of H, which is harmless to spectral invariants, one can assume that H vanishes on the second interval.

This is enough to prove that

THEOREM II.8. *Let α denote classes in $\mathrm{HQ}_*(M)$ and $\mathrm{HQ}_*(\Delta)$, corresponding to each other via the canonical isomorphism. For all $H \in C^0\big(M \times \mathbb{R}/\mathbb{Z}\big)$, $c(\alpha\,; H) = \ell(\alpha\,; H \oplus 0)$.*

This theorem was first proved under asphericity condition in (Leclercq 2008) and then under monotonicity in (Leclercq and Zapolsky 2018). It shows in particular that the "absolute" spectral invariants can be computed from their Lagrangian counterparts.

5. Recall that for periodic orbit Floer theory, the Hamiltonian is assumed to be periodic.

1.4.2. *Spectral invariants and the module structure on quantum homology*

There is another interaction between the "absolute" and Lagrangian versions of quantum and Floer homologies, as the absolute theory yields a ring over which the Lagrangian homology is a module (see Sections 3.1.3 and 4.3 of Chapter I). In view of the definition of the module structures and their compatibility with the PSS morphism, it is not surprising that "absolute" and Lagrangian spectral invariants can be compared tanks to this additional algebraic structure. Namely, they satisfy the following property :

MODULE STRUCTURE For all $a \in \mathrm{HQ}_*(M)$ and $\alpha \in \mathrm{HQ}_*(L)$,

$$\ell(a \odot \alpha; H^1 \sharp H^2) \leq c(a; H^1) + \ell(\alpha; H^2)$$

(with H^1 assumed to be 1-periodic).

The proof is formally similar to the proof of the TRIANGLE INEQUALITY except that the "typical element of a suitable moduli space" whose energy is computed is an element of the moduli space entering the definition of the module structure, *i.e.* a map from the Riemann surface $\Sigma_\odot$ (instead of Σ_*).

This also translates to invariants of isotopies as follows :

MODULE STRUCTURE For all $a \in \mathrm{HQ}_*(M)$ and $\alpha \in \mathrm{HQ}_*(L)$, $\ell(a \odot \alpha\,; \widetilde{\phi}\widetilde{\psi}) \leq c(a\,; \widetilde{\phi}) + \ell(\alpha\,; \widetilde{\psi})$.

1.5. Conormal-type spectral invariants

To finish the exposition of Lagrangian spectral invariants, we now present *conormal-type spectral invariants* which we introduced in (Humilière, Leclercq, and Seyfaddini 2016) in order to study the behaviour of spectral invariants under symplectic reduction. They are inspired by the *conormal spectral invariants* defined in a cotangent bundle T^*N via consideration of the Lagrangian Floer homology of the zero section 0_N and the conormal ν^*V of a submanifold $V \subset N$, see *e.g.* (Oh 1997). The idea that these invariants are well-suited to study symplectic reduction goes back to Viterbo (Viterbo 1992) in the context of generating functions. In Section 3.5, however, we will need to work with closed submanifolds which is why we implemented them in Floer theory.

Thanks to an adequate change of Floer datum, the Floer homology of a Lagrangian L can be viewed as generated by intersection points between L and $\phi_H^1(L)$, rather than Hamiltonian chords of ϕ_H from L to itself. (This is very much related to the construction of the Seidel morphism of Section 1.1 in Chapter III.) Lagrangian spectral invariants behave well under this shift of perspective, at least when L is aspherical. This was used in (Leclercq 2008) to define spectral invariants associated with any pair of Hamiltonian isotopic Lagrangians L and L', by $\ell(\alpha\,; L\,, L') = \ell(\alpha\,; H)$ for any H such that $\phi_H^1(L) = L'$. (The fact that the right-hand side only depends on $\phi_H^1(L)$ follows from the invariance of the action spectrum under asphericity, see Remark II.6.)

On the contrary, *conormal-type spectral invariants* are defined with respect to Lagrangians which are *not* Hamiltonian isotopic. Indeed, the case we have in mind for applications is the case of two aspherical Lagrangians, L_0 and L_1, which intersect transversely in a single point p. (This prevents L_1 to be Hamiltonian isotopic to L_0 because Arnol'd said so.)

Example II.9. Let M be the torus $\mathbb{T}^{2k_1} \times \mathbb{T}^{2k_1} \times \mathbb{T}^{2k_2} \times \mathbb{T}^{2k_2}$. Our Lagrangians are

$$L_0 = \mathbb{T}^{k_1} \times \{0\} \times \mathbb{T}^{k_1} \times \{0\} \times \mathbb{T}^{k_2} \times \{0\} \times \mathbb{T}^{k_2} \times \{0\}\,,$$
$$L_1 = \mathbb{T}^{k_1} \times \{0\} \times \mathbb{T}^{k_1} \times \{0\} \times \mathbb{T}^{k_2} \times \{0\} \times \{0\} \times \mathbb{T}^{k_2}\,.$$

As mentioned above, if we heuristically think of the torus M as a compact version of the cotangent bundle to $\mathbb{T}^{k_1} \times \mathbb{T}^{k_1} \times \mathbb{T}^{k_2} \times \mathbb{T}^{k_2}$, then L_0 corresponds to the zero section and L_1 to the conormal bundle of the submanifold $V = \mathbb{T}^{k_1} \times \mathbb{T}^{k_1} \times \mathbb{T}^{k_2} \times \{0\}$.

Given two Lagrangians L_0 and L_1 which intersect transversely, one can construct a Floer homology as in Section 4 of Chapter I, except that the generators are now chords of some Hamiltonian H starting on L_0 and ending on L_1. A capping $\widehat{\gamma}$ of a chord γ is again a half-disc $u\colon D_- \to M$, such that the straight part of ∂D_- is mapped to γ, and $\partial D_- \cap \{e^{i\theta}\,|\,\theta \in [-\pi\,, -\frac{\pi}{2}]\}$ (respectively $\theta \in [\frac{\pi}{2}\,, \pi]$) is mapped to L_0 (respectively L_1). Necessarily, $u(-1) \in L_0 \cap L_1$ and actually, we require that $u(-1)$ is mapped to some fixed point p of the intersection.

In this case, the asphericity assumption is manufactured such that any two cappings of γ have the same symplectic area, so that the action functional $\mathcal{A}_{H:L_0,L_1}$ can be defined on the set of paths from L_0 to L_1 by the formula

$$\mathcal{A}_{H:L_0,L_1}(\gamma) = -\int_{D_-} \widehat{\gamma}^*\omega + \int_0^1 H_t(\gamma(t))\,dt$$

for any capping $\widehat{\gamma}$ of γ. The whole construction of Section 4.1.1 can be slightly adapted in the same fashion and we denote by $\mathrm{HF}_*(L_0\,, L_1\,; p)$ the resulting homology.

Next, one can associate spectral invariants with any class $\alpha \neq 0$ in $\mathrm{HF}_*(L_0\,, L_1\,; p)$ as above : as the infimum of action values such that α appears in the filtration. Namely, from any smooth nondegenerate Hamiltonian H, we get a real number $\ell_{L_0,L_1}(\alpha\,; H)$. (Note that when L_0 and L_1 intersect in a single point, there exists exactly one non-vanishing class α.)

These spectral invariants share many of the properties of their more classical counterparts and in particular the Spectrality and Continuity properties. This allows us to define them for any continuous time-dependent function H. They also satisfy a Triangle inequality, which allows us to compare them to the more classical Lagrangian spectral invariants. Indeed, we have that for any non-zero class α, $\ell_{L_0,L_1}(\alpha\,; H) \leq \ell_+(H)$.

2. Applications to (smooth) symplectic geometry

Spectral invariants have been successfully used to approach such a *great* variety of questions that it is now quite unreasonable to try to list them all. In the context of this HDR memoirs, let us only mention the following questions, to which we made a modest contribution, and which are presented in the next sections of this chapter. Section 2 is concerned with applications to the study of metrics on infinite-dimensional diffeomorphism groups, the existence of quasimorphisms, and subsequent rigidity of subsets of symplectic manifolds. In Section 3, we present applications to the C^0 rigidity of the Poisson bracket of functions, to C^0 Hamiltonian dynamics, and to C^0 rigidity of coisotropic submanifolds and their characteristic foliation. Section 5 sketch future applications to the contact non-squeezing phenomenon and orderability of contactomorphism groups.

2.1. Spectral distance and Hofer's geometry

Let us start with the Hofer geometry of Hamiltonian diffeomorphism groups and some sets of Lagrangians. As mentioned in Remark II.7, spectral invariants yield a natural pseudo-distance on the various spaces on which they are defined. This distance is bi-invariant with respect to the action of the Hamiltonian diffeomorphism group and is *smaller* that the appropriate Hofer distance. Let us for example define it in the setting of Section 1.3.3.

Definition II.10. Let L be a monotone$^+$ Lagrangian of $(M\,,\omega)$ whose quantum fundamental class $[L] \in \mathrm{HQ}_n(L)$ does not vanish. The *spectral (pseudo-) norm* on $\widetilde{\mathrm{Ham}}(M\,,\omega)$ is given by

$$\|\widetilde{\phi}\|_{\gamma_L} = \ell_+(\widetilde{\phi}) + \ell_+(\widetilde{\phi}^{-1})\,.$$

It induces a (pseudo-) distance on the group by setting $\gamma_L(\widetilde{\phi}\,, \widetilde{\psi}) = \|\widetilde{\phi}\widetilde{\psi}^{-1}\|_{\gamma_L}$.

Remark II.11. While verifying that this defines a pseudo-norm is straightforward in view of the properties of ℓ, its nondegeneracy is harder to prove. This is usually done by means of "energy-capacity inequalities". For example, Theorem II.23 below yields the nondegeneracy of the Lagrangian spectral distance induced by $\|\cdot\|_{\gamma_L}$ on certain subsets of aspherical Lagrangians, see Section 3.1.2 for more details.

The nondegeneracy of $\|\cdot\|_{\gamma_L}$ is not proved in (Leclercq and Zapolsky 2018). This is work in progress by Kawasaki (Kawasaki 20xx) who is following another approach, via Poisson bracket techniques as in (Polterovich and Rosen 2014, Proposition 4.6.2). ∎

Because of the Continuity property of ℓ, the spectral distance is smaller than Hofer's. We now illustrate this fact. As several Lagrangians appear in the statement, we add them as a subscript to the notation of the Lagrangian spectral invariant function to avoid ambiguity (*e.g.* $\ell_{L'}$ is the spectral invariant associated with the Lagrangian L').

Theorem II.12. *Let L be monotone$^+$ and let Θ denote either $\mathrm{Ham}(M,\omega)$ or $\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$. Let $\theta\in\Theta$ and denote by L_θ the Lagrangian obtained from L via θ, i.e. respectively $L_\theta=\theta(L)$ or $L_\theta=\theta(1)$. Via symplectic action, θ induces an isomorphism $\theta_*\colon\ \mathrm{HQ}_*(L)\to\mathrm{HQ}_*(L_\theta)$ such that for all non-zero $\alpha\in\mathrm{HQ}_*(L)$ and all $H\in C^0(M\times[0,1])$,*

$$|\ell_L(\alpha;H)-\ell_{L_\theta}(\theta_*(\alpha);H)|\leq\|\theta\|_{\mathrm{Hof}}\,. \tag{II.9}$$

Hence, each non-zero $\alpha\in\mathrm{HQ}_*(L)$ induces a function

$$\ell^\alpha\colon\ \Theta\times\widetilde{\mathrm{Ham}}(M,\omega)\longrightarrow\mathbb{R}\,,\qquad(\theta,\widetilde{\phi})\longmapsto\ell_{L_\theta}(\theta_*(\alpha);\widetilde{\phi})$$

which is Lipschitz with respect to the natural Hofer metric on $\Theta\times\widetilde{\mathrm{Ham}}(M,\omega)$.

The Lipschitz nature of ℓ^α is a straightforward consequence of the Continuity property of ℓ_L, together with (II.9). The latter is itself a consequence of Continuity, via the following estimate.

Lemma II.13. *Let $H\in C^0(M\times[0,1])$ and $\alpha\neq0$ in $\mathrm{HQ}_*(L)$. For any Hamiltonian diffeomorphism ϕ and any Hamiltonian K such that $\phi^1_K=\phi$,*

$$|\ell(\alpha;H)-\ell(\alpha;H\circ\phi)|\leq\int_0^1\operatorname*{osc}_M K_t\,dt\,. \tag{II.10}$$

Before proving the lemma, let us show how it yields the theorem. First, let us point out that the isomorphism θ_* comes from the action of the symplectomorphism group on quantum cohomology, as described in Section 6.1 of Chapter I : by Proposition I.20 when $\theta\in\mathrm{Ham}(M,\omega)$ and by Theorem I.22 for $\theta\in\widetilde{\mathcal{L}_{\mathrm{Ham}}}(L)$.

Now, by Symplectic invariance, we have that $\ell_{L_\theta}(\theta_*(\alpha);H)=\ell(\alpha;H\circ\phi)$ so that we get the desired result by taking the infimum of the right-hand side of (II.10) : respectively over the set of Hamiltonians K such that $\phi^1_K=\phi$ if $\theta\in\mathrm{Ham}(M,\omega)$, and over the set of Hamiltonians K such that $\phi_K(L)\in\widetilde{L}$ otherwise. The aforementioned results from Section 6.1 show that the left-hand side is independent of the choice of such Hamiltonians K and the result follows.

As a final remark, let us recall that, under the stronger assumption of asphericity, the maps ℓ^α of Theorem II.12 descend to $\mathcal{L}_{\mathrm{Ham}}(L)\times\mathrm{Ham}(M,\omega)$, see for example (Leclercq 2008) and (Monzner, Vichery, and Zapolsky 2012).

Proof of the main estimate, Lemma II.13. Since adding the same function of time to both H and $H\circ\phi$ does not affect their difference, by Time normalization and the invariance of ℓ proved in Section 1.3.1, we get

$$|\ell(\alpha;H)-\ell(\alpha;H\circ\varphi)|=|\ell(\alpha;\widetilde{\phi}_H)-\ell(\alpha;\varphi^{-1}\widetilde{\phi}_H\varphi)|$$

where $\widetilde{\phi}_H$ denotes the equivalence class of the isotopy ϕ_H in $\widetilde{\mathrm{Ham}}(M,\omega)$. Notice that the Hamiltonian isotopies $t\mapsto\phi^t_{H\circ\phi}=\phi^{-1}\phi^t_H\phi$ and $t\mapsto(\phi^t_K)^{-1}\phi^t_H\phi^t_K$ are homotopic relative to endpoints and thus define the same element $\varphi^{-1}\widetilde{\phi}_H\varphi=\widetilde{\phi}_K^{-1}\widetilde{\phi}_H\widetilde{\phi}_K$ in $\widetilde{\mathrm{Ham}}(M,\omega)$.

Using this and the fact that the fundamental class $[L]$ is the unit of the quantum homology ring, Triangle inequality leads to

$$\ell(\alpha;\varphi^{-1}\widetilde{\phi}_H\varphi) = \ell([L]*\alpha*[L];\widetilde{\phi}_K^{-1}\widetilde{\phi}_H\widetilde{\phi}_K) \leq \ell_+(\widetilde{\phi}_K^{-1}) + \ell(\alpha;\widetilde{\phi}_H) + \ell_+(\widetilde{\phi}_K)$$

from which we deduce that

$$\ell(\alpha;\widetilde{\phi}_H) - \ell(\alpha;\varphi^{-1}\widetilde{\phi}_H\varphi) \geq -\ell_+(\widetilde{\phi}_K^{-1}) - \ell_+(\widetilde{\phi}_K)\,.$$

Similarly, by writing $\ell(\alpha\,;\widetilde{\phi}_H) = \ell(\alpha\,;\widetilde{\phi}_K(\widetilde{\phi}_K^{-1}\widetilde{\phi}_H\widetilde{\phi}_K)\widetilde{\phi}_K^{-1})$, we get that

$$\ell(\alpha;\widetilde{\phi}_H) - \ell(\alpha;\varphi^{-1}\widetilde{\phi}_H\varphi) \leq \ell_+(\widetilde{\phi}_K^{-1}) + \ell_+(\widetilde{\phi}_K)$$

and thus conclude that

$$|\ell(\alpha;H) - \ell(\alpha;H\circ\varphi)| \leq \ell_+(\widetilde{\phi}_K^{-1}) + \ell_+(\widetilde{\phi}_K)\,.$$

By Continuity of spectral invariants, since $\ell_+(\widetilde{\phi}_K) \leq \int_0^1 \max_M K_t\,dt$ and $\ell_+(\widetilde{\phi}_K^{-1}) \leq \int_0^1 \max_M \overline{K}_t\,dt = -\int_0^1 \min_M K_t\,dt$, we deduce

$$|\ell(\alpha;H) - \ell(\alpha;H\circ\varphi)| \leq \int_0^1 \operatorname*{osc}_M K_t\,dt$$

which concludes. □

2.2. Symplectic rigidity

We now show that Lagrangian spectral invariants are also well-fitted to detect rigid subsets via techniques introduced by Entov and Polterovich.

If $e \in \mathrm{HQ}_*(M)$ is any nonzero idempotent, Entov and Polterovich (Entov and Polterovich 2009) defined two classes of rigid subsets of M, with respect to e.

Definition II.14. A closed subset $X \subset M$ is called *e-heavy* if, for any smooth function F on M,

$$\lim_{k\to\infty} \frac{c(e;kF)}{k} \geq \min_X F\,,$$

and it is called *e-superheavy* if for any F,

$$\lim_{k\to\infty} \frac{c(e;kF)}{k} \leq \max_X F\,.$$

Recall that c denotes the "absolute" spectral invariants of (Oh 2005), which can also be seen as Lagrangian spectral invariants via the Lagrangian diagonal construction of Chapter I, Section 5.3, as explained in Section 1.4.1.

In (Entov and Polterovich 2006), they also introduced the notion of *symplectic quasi-states*. In the following definition $\{\cdot\,,\cdot\}$ stands for the Poisson bracket.

Definition II.15. A *quasi-state* on M is a functional $\zeta\colon C^0(M)\to\mathbb{R}$ satisfying

Normalization. $\zeta(1) = 1$.

Quasi-linearity. For $F, G \in C^\infty(M)$ with $\{F,G\} = 0$, we have $\zeta(F+G) = \zeta(F) + \zeta(G)$.

Monotonicity. For $F, G \in C^0(M)$ with $F \leq G$, we have $\zeta(F) \leq \zeta(G)$.

They developed a construction of *(nonlinear) symplectic quasi-states* on M using idempotents in $\mathrm{HQ}_*(M)$. We refer the reader to (Entov and Polterovich 2008) for details. Briefly, if $\mathrm{HQ}_*(M) \simeq \mathcal{F}\oplus\mathcal{Q}$ as an algebra where $\mathcal{F}$ is a field, then the spectral invariant $c(e\,;\cdot)$, where $e\in\mathcal{F}$ is the unit, has the property that the functional

$$F \longmapsto \zeta_e(F) = \lim_{k\to\infty}\frac{c(e;kF)}{k}$$

is a symplectic quasi-state. They showed that, for such a class e, the notions of e-heaviness and e-superheaviness coincide (Entov and Polterovich 2009).

Here is how the Lagrangian spectral invariants of Section 1 can be used to prove the (super-) heaviness of certain Lagrangians.

Proposition II.16. *Let $e \in \mathrm{HQ}_*(M)$ be an idempotent. If $e \odot [L] \neq 0$, then L is e-heavy ; L will thus be e-superheavy if, furthermore, e gives rise to a non-linear symplectic quasi-state as above.*

This proposition is analogous to, and in certain cases follows from, Proposition 8.1 of (Entov and Polterovich 2009). Its (direct) proof elementarily relies on the properties of ℓ. Indeed, let $F \in C^\infty(M)$, by Module structure, Quantum valuation, and Lagrangian control we have for $k \in \mathbb{N}$:

$$c(e; kF) + \ell([L]; 0) \geq \ell(e \odot [L]; kF) \geq k \min_L F + \nu(e \odot [L])\, w_L\,,$$

which yields $\lim_{k\to\infty} c(e\,; kF)/k \geq \min_L F$, proving that L is e-heavy.

Disclaimer II.17. Thanks to this proposition, we were able to determine explicit examples of superheavy Lagrangians. However, this requires the use of more complicated coefficients than those presented so far. We need both *twisted coefficients* (instead of $\widetilde{\pi}_2(M\,, L)$, allowed by the more general construction) over a ground ring different from $\mathbb{Z}/2\mathbb{Z}$ (in particular for the second example).

Since this is the only part of Chapter II where more general coefficients were needed, it seemed reasonable to present the whole theory with coefficients in $\mathbb{Z}/2\mathbb{Z}$. However, while it permitted a less abstract presentation, this choice would make intermediate computations in the proof below impossible to follow.

Hence, we only present the geometric setting and sketch the proofs so that it will be clear how Proposition II.16 can be used. Brave readers are encouraged to check (Leclercq and Zapolsky 2018, Section 2.6) or (Zapolsky 2015, Section 8). ▮

The first example appeared in (Eliashberg and Polterovich 2010). Consider the monotone product $M = S^2 \times S^2$. View $S^2 \subset \mathbb{R}^3$ as the set of unit vectors. Then $L = \big\{(x\,, y) \in S^2 \times S^2 \,|\, x \cdot y = -\frac{1}{2}\,,\ x_3 + y_3 = 0\big\}$, where $x \cdot y$ is the Euclidean scalar product, is a monotone Lagrangian torus in M.

Over the ground ring $\mathbb{C}$, the quantum homology $\mathrm{HQ}_*(M)$ contains two idempotents $e_\pm$ so that $[M] = e_+ + e_-$. Moreover, there is a choice of coefficients for which $\mathrm{HQ}_*(L) \neq 0$ (and thus $[L] \neq 0$). As L is disjoint from the Lagrangian antidiagonal $\overline{\Delta} \subset M$ which is known to be e_--superheavy, it is not itself e_--superheavy, since superheavy sets must intersect (Entov and Polterovich 2009). Therefore, Proposition II.16 implies $e_- \odot [L] = 0$.

Gathering all these facts, we get that

$$e_+ \odot [L] = e_- \odot [L] + e_+ \odot [L] = (e_- + e_+) \odot [L] = [M] \odot [L] = [L] \neq 0\,.$$

A final use of Proposition II.16 shows that L is e_+-superheavy.

The second example is the Chekanov monotone torus $L \subset \mathbb{CP}^2$ (Chekanov and Schlenk 2010) which can be defined as follows. Consider the degree 2 polarization of $\mathbb{CP}^2$ by a conic. This conic is a complex (and thus symplectic) sphere, in which the equator is a monotone Lagrangian. The Lagrangian circle bundle construction (Biran and Cieliebak 2001 ; Biran 2006) in this situation yields L.

Again, one can show that there is a choice of coefficients (which requires the characteristic of the ground ring to be different from 2) so that $\mathrm{HQ}_*(L) \neq 0$. Since $[M] = [\mathbb{CP}^2]$ is the unit of $\mathrm{HQ}_*(\mathbb{CP}^2)$, it follows that $[M] \odot [L] = [L] \neq 0$ and we conclude with Proposition II.16 that the Chekanov torus L is superheavy with respect to the fundamental class $[\mathbb{CP}^2]$, taken over the ground ring $\mathbb{C}$.

3. Continuous symplectic geometry

Once upon a time, Gromov... (2)

The starting point of *continuous symplectic geometry* is the Gromov Alternative (Gromov 1986) which states that the group of symplectomorphisms of a symplectic manifold $(M\,, \omega)$ is either C^0-closed or C^0-dense (up to a subgroup of finite order) in the group of diffeomorphisms of M. The alternative itself is fascinating as, whichever option holds, it is a manifestation of extreme behavior : extreme rigidity in the first case, extreme flexibility in the second. Moreover, while rigidity seems too much to ask (preserving a symplectic form is a property of differential nature), flexibility, adding to the fact that symplectic manifolds have no local invariants, seems to doom symplectic geometry all together. Fortunately for symplectic geometry, Eliashberg (Eliashberg 1987) proved that rigidity holds, leading to the following celebrated result, known as *Gromov–Eliashberg Theorem*.

THEOREM II.18 (Gromov–Eliashberg). *If a sequence of symplectomorphisms of a symplectic manifold C^0-converges to a diffeomorphism ψ, then ψ is a symplectomorphism.*

This naturally suggests a definition of symplectic homeomorphisms.

Definition II.19. The C^0-limit of a sequence of symplectic diffeomorphisms of a symplectic manifold (M, ω) is called a *symplectic homeomorphism*.

And in turn, these symplectic homeomorphisms allow one to define many other C^0 objects as, for example, C^0 Lagrangian submanifolds (of smooth or C^0) symplectic manifolds and to study their rigidity properties.

In this section, are explained joint works with Vincent Humilière and Sobhan Seyfaddini (Humilière, Leclercq, and Seyfaddini 2015b, 2015a, 2016), in which we established such rigidity phenomena, the most striking one being that coisotropic submanifolds, together with their characteristic foliations, are C^0 rigid, see Section 3.4 below.

However, one should not believe that all smooth rigidity phenomena have continuous counterparts. Thanks to work by Buhovsky and Opshtein (Buhovsky and Opshtein 2016), we now know that there is also a good part of C^0 flexibility. For example, they showed that in dimension $2n \geq 6$, there exist symplectic homeomorphisms which map a symplectic disc of $\mathbb{C} \times \{0\}^{n-1}$ to a disc of different symplectic area! In order to prove such flexibility, they developed *quantitative h-principle techniques* which proved to be very useful.

In the author's opinion, the most striking C^0 flexibility phenomenon established so far is undoubtedly the counterexample to the C^0 version of the Arnol'd conjecture, due to Buhovsky, Humilière, and Seyfaddini (Buhovsky, Humilière, and Seyfaddini 2016). The construction of this counterexample is based on many ingenious ideas, including the aforementioned quantitative h-principle techniques.

Before going back to rigidity, let us refer the reader to (Humilière 2017) which contains in particular an excellent survey of C^0 symplectic geometry (in English, despite the title of the memoirs).

Organization of this section

In what follows, we present the main results of (Humilière, Leclercq, and Seyfaddini 2015b, 2015a, 2016). We start with presenting several symplectic capacities (one of which naturally coming from spectral invariants) and *energy-capacity inequalities* in the Lagrangian and Hamiltonian settings in Section 3.1. Then, we explain what the Hamiltonian energy-capacity inequalities can tell us in terms of continuous Hamiltonian isotopies, also known as *hameotopies*, whose definition we also explain in Section 3.2. We then turn to the Lagrangian situation and use the energy-capacity inequalities to establish dynamical properties of hameotopies in Section 3.3. In turn, these have consequences in terms of C^0 rigidity of (coisotropic) submanifolds which we describe in Section 3.4. Finally, the latter rigidity phenomenon raises a very natural question about symplectic homeomorphisms which we (very) partially answer in Section 3.5.

3.1. Energy-capacity inequalities

An *energy-capacity inequality* is an inequality between a symplectic capacity and a quantity defined via the energy of suitable Hamiltonians. For example, the "most classical" one compares the Hofer–Zehnder capacity of an open set U to the Hofer energy of a Hamiltonian diffeomorphism displacing U. This is Theorem II.26 below.

Such comparisons, between quantities defined via quite different methods, have already had many interesting consequences. We refer to (Hofer and Zehnder 1994) for a much more detailed account on the basics, and for example to (Frauenfelder, Ginzburg, and Schlenk 2005) (and Sections 3.2 to 3.4 below !) for some applications.

3.1.1. *Symplectic capacities*

A symplectic capacity is a function c, which associates with $2n$-dimensional symplectic manifolds (not necessarily closed) elements of $[0\,,\infty]$, and which is

- Monotone. We have $c(M_1\,,\omega_1) \leq c(M_2\,,\omega_2)$, as soon as $(M_1\,,\omega_1)$ can be symplectically embedded into $(M_2\,,\omega_2)$.
- Conformal. For any real number $\lambda \neq 0$, $c(M\,,\lambda\,\omega) = |\lambda|\, c(M\,,\omega)$.
- Nontrivial. The standard symplectic[6] ball and cylinder of radius 1 have identical capacity, $c(B^{2n}(1)\,,\omega_0) = \pi = c(C^{2n}(1)\,,\omega_0)$.

The notion of capacity arose thanks to *Gromov's non-squeezing Theorem* (Gromov 1985) which states that if $B^{2n}(r)$ can be symplectically embedded into $C^{2n}(R)$, then $r \leq R$ (see also Section 5.2.1 below where we discuss the analogous statement in contact geometry). This suggested the definition of what is now called the *Gromov width* of a symplectic manifold as

$$w_{\mathrm{G}}(M,\omega) = \sup\{\pi r^2 \mid \text{there exists a symplectic embedding } (B^{2n}(r),\omega_0) \hookrightarrow (M,\omega)\}\,.$$

Capacities are particularly well-suited to detect the symplectic nature of a diffeomorphism. Indeed, a corollary of a result of Ekeland and Hofer (Ekeland and Hofer 1989) shows that, given a capacity, a diffeomorphism is symplectic (or anti-symplectic) if and only if it preserves the capacity of small ellipsoids. (This is the central step of modern proofs of The Gromov–Eliashberg Theorem, Theorem II.18 above.)

Since the Gromov width, several other capacities have been constructed. In what follows, we will use three of them which we now introduce. The first one is very natural in the context of this chapter as it is defined via spectral invariants. Recall from Section 1.4 that we denoted by $c(\alpha\,;\cdot)\colon\; C^0(\mathbb{R}/\mathbb{Z}\times M)\to\mathbb{R}\cup\{+\infty\}$, the spectral invariant associated with $\alpha \in \mathrm{HQ}_*(M)$. Let us pick $\alpha = [M]$, the quantum fundamental class of M : $[M] \neq 0$ and we denote $c([M]\,;\cdot)$ by c_+. Now, we define the spectral capacity of an open subset of M following Viterbo (Viterbo 1992, Definition 4.11).

Definition II.20. Let U be an open set of M. Its *spectral capacity* is the quantity

$$c_{\mathrm{V}}(U) = \sup\{c_+(H) \mid H \in C^0(\mathbb{R}/\mathbb{Z}\times M) \text{ s.t. } \forall t\in[0,1], \operatorname{supp}(H_t)\subset U\}\,.$$

The other two capacities which will be useful here are (absolute and relative) variants of the same construction initially due to Hofer and Zehnder (Hofer and Zehnder 1990). We need one more piece of notation (each !), preliminary to their definition.

Definition II.21. A Hamiltonian H is *slow* if it is autonomous and if its flow $\{\phi_H^t\}$ has no non-trivial orbits of period at most 1. For an open subset $U \subset M$, we denote by $\mathcal{S}^+(U)$ the set of non-negative, slow Hamiltonians with compact support included in U.

Similarly, given a Lagrangian L, an autonomous Hamiltonian H is *L-slow* if its flow has no non-trivial chords from L to itself of length at most 1. For an open set $U \subset M$, we denote by $\mathcal{S}_L^+(U)$ the set of non-negative, slow Hamiltonians with compact support included in U, which reach their maximum at a point of L.

We can now define the Hofer–Zehnder capacity as well as its Lagrangian counterpart, introduced by Lisi and Rieser (Lisi and Rieser 2016).

Definition II.22. Let U be an open set of M, its *Hofer–Zehnder capacity* is the quantity

$$c_{\mathrm{HZ}}(U) = \sup\{\max_M H \mid H \in \mathcal{S}^+(U)\}\,.$$

Similarly, let L be a Lagrangian, the *Lisi–Rieser capacity* of U with respect to L is

$$c_{\mathrm{LR}}(U;L) = \sup\{\max_M H \mid H \in \mathcal{S}_L^+(U)\}\,.$$

6. The standard symplectic forms on $B^{2n}(r)$ and $C^{2n}(r) = B^{2n}(r)\times\mathbb{R}^{2n-2}$ are inherited from their embedding in $\mathbb{R}^{2n}$.

Note that if U does not intersect L, $c_{\mathrm{LR}}(U;L)=0$.

3.1.2. *The "classical" energy-capacity inequality*

The first energy-capacity inequality which is presented here was proved in (Humilière, Leclercq, and Seyfaddini 2015a). It is a relative version of the standard energy-capacity inequality, see Remark II.25 below for more details.

THEOREM II.23. *Let L be an aspherical Lagrangian of a symplectic manifold (M,ω) and U be an open subset of M. Assume that L' is a Lagrangian Hamiltonian-isotopic to L such that $L'\cap U=\emptyset$. Then $\gamma(L',L)\geq c_{\mathrm{LR}}(U;L)$.*

First, a word of explanation. We discussed how spectral invariants could be used to produce various norms and we defined $\|\cdot\|_{\gamma_L}$, the one obtained from ℓ on $\widetilde{\mathrm{Ham}}(M,\omega)$, see Definition II.10. Now, *when one considers aspherical Lagrangians*, the quantity $\|\widetilde{\phi}\|_{\gamma_L}$ only depends on the Lagrangian obtained from L at time 1, $L'=\widetilde{\phi}_1(L)$. This was proved in (Leclercq 2008) and shows that in the aspherical case, γ_L descends as a distance to the set of Lagrangians which are Hamiltonian-isotopic to L: for any two such Lagrangians L_0 and L_1,

$$\gamma_L(L_0,L_1)=\|\widetilde{\phi}\|_{\gamma_L} \qquad \text{for any } \widetilde{\phi} \text{ such that } \widetilde{\phi}_1(L_0)=L_1,$$

$$\text{i.e.}\quad \gamma_L(L_0,L_1)=\ell_+(H)+\ell_+(\overline{H}) \qquad \text{for any } H \text{ such that } L_1=\phi_H^1(L_0).$$

Hence, Theorem II.23 reads: *if a Hamiltonian H displaces an aspherical Lagrangian L from an open set U, its spectral energy has to be greater than or equal to the relative Hofer–Zehnder capacity of U.*

Second, the CONTINUITY property of the function ℓ implies that the *spectral* energy of H is smaller than its energy as defined by Equation (I.1). This has the following immediate consequence.

Corollary II.24. *Let L be aspherical and L' be Hamiltonian-isotopic to L. If $L'\cap U=\emptyset$ then $\delta(L,L')\geq c_{\mathrm{LR}}(U;L)$, where δ denotes the Lagrangian Hofer distance.*

It seems that we could obtain a sharper estimate by using LAGRANGIAN CONTROL instead of CONTINUITY, as the former gives $\int_0^1 \mathrm{osc}_L H_t\,dt\geq c_{\mathrm{LR}}(U;L)$ instead of the oscillations of H on the whole manifold M. This *a priori* intriguing fact was explored by Usher (Usher 2015) who showed that the extra sharpness gets lost in the infimum process, namely he proved that $\delta(L,L')=\inf\{\int_0^1 \mathrm{osc}_L H_t\,dt \mid \phi_H^1(L)=L'\}$.

Remark II.25. In the Hamiltonian setting, the counterpart of Theorem II.23 was proved by Viterbo (Viterbo 1992) in cotangent bundles and generalized by Usher (Usher 2010) to the general case. The classical energy-capacity inequality is the following counterpart of Corollary II.24.

THEOREM II.26. *If a Hamiltonian diffeomorphism ϕ displaces an open set U from itself, then $c_{\mathrm{HZ}}(U)\leq\|\phi\|$, where $\|\cdot\|$ denotes the Hofer norm.*

It was initially proved by Hofer (Hofer 1990) in $\mathbb{R}^{2n}$. ▌

Finally, before sketching the proof of Theorem II.23, let us mention that a special case of this inequality appeared in Barraud–Cornea (Barraud and Cornea 2006) and Charette (Charette 2012), while a similar inequality was worked out in Borman–McLean (Borman and McLean 2014).

SKETCH OF PROOF OF THEOREM II.23. Let H be a Hamiltonian such that $\phi_H^1(L)=L'$ and choose any function $f\in\mathcal{S}_L^+(U)$. We consider the family of Hamiltonian diffeomorphisms $\psi_s=\phi_{sf}^1\phi_H^1$. Recall from Remark II.6 that when L is aspherical, for any normalized Hamiltonian F, the spectrum of $\mathcal{A}_{F:L}$ only depends on ϕ_F^1. It is easy to see that, here, the spectrum of ψ_s does not change with s. By SPECTRALITY and CONTINUITY of spectral invariants, this shows that ℓ_+ also remains constant.

Then, by Triangle inequality and Duality, one gets that

$$\ell_+(\phi_f^1) \leq \ell_+(\phi_f^1\phi_H^1) + \ell_+((\phi_H^1)^{-1}) = \ell_+(\phi_H^1) + \ell_+((\phi_H^1)^{-1}) = \gamma(\phi_H^1)\,.$$

We then conclude with the fact that for $f \in \mathcal{S}_L^+(U)$, $\ell_+(\phi_f^1) = \max f|_L$. (Note that this fact, as natural as it seems, is not obvious. Our proof relies on the study of geodesics of Lagrangians in Hofer's geometry by Milinković (Milinković 2001).) □

3.1.3. *The "dual" energy-capacity inequality*

This second energy-capacity inequality is not classical and was also proved in (Humilière, Leclercq, and Seyfaddini 2015a). The name "dual" was suggested in (Humilière 2017) and is quite suited because this inequality and the previous one will be used to prove several pairs of dual statements, see for example Theorem II.37 below.

Theorem II.27. *Let L be an aspherical Lagrangian and $U_\pm$ be open subsets of M which both intersect L. If a Hamiltonian H is constant on each subset, $H|_{U_\pm} = \pm C_\pm$, with values $C_\pm > c_{\mathrm{LR}}(U_\pm\,; L)$, then $\gamma_L(\phi_H^1(L)\,, L) \geq \min\{c_{\mathrm{LR}}(U_-\,; L)\,, c_{\mathrm{LR}}(U_+\,; L)\}$.*

Let us consider an easy example. Let U be an open ball in M, centered at a point of L, and of radius r such that locally the situation is symplectomorphic to the standard open ball of radius r in $\mathbb{R}^{2n}$ while L coincides with $\mathbb{R}^n \times \{0\}$. Then, the Lisi–Rieser capacity of U is $c_{\mathrm{LR}}(U; L) = \frac{\pi r^2}{2}$, see (Lisi and Rieser 2016).

So the theorem states that if a Hamiltonian is constant, equal to (at least) $\frac{\pi r^2}{2}$ on such a ball, and constant equal to (at most) $-\frac{\pi r^2}{2}$ on another, then the spectral distance between L and its image $\phi_H^1(L)$ is at least $\frac{\pi r^2}{2}$.

Note that, because of Continuity, the same holds when the spectral distance is replaced by Hofer's distance, δ.

Remark II.28. We omit the proof of Theorem II.27 as it is quite similar to that of the classical energy-capacity inequality. First, in its philosophy, as one also studies how the action spectrum changes along a homotopy (here along $H_s = H - sf$ for $s \in [0\,,1]$, with the same notation as in the proof of Theorem II.23). Second, in the properties of the spectral invariants involved, as Triangle inequality and Duality, then Spectrality and Continuity are also used here, as well as the fact that $\ell_+(\phi_f^1) = \max f|_L$ when $f \in \mathcal{S}_L^+(U)$.

It is only slightly harder because of the fact that the whole spectrum is not constant in this case, however, one can show that $\ell_+((\phi_{H_s}^1)^{-1})$ is constant. ▮

Remark II.29. Theorem II.27 is the Lagrangian version of a result which we proved earlier (Humilière, Leclercq, and Seyfaddini 2015b), namely

Theorem II.30. *Let $(M\,,\omega)$ be a monotone symplectic manifold and let $U_\pm$ be non-empty open subsets of M. If a Hamiltonian H is constant on each subset, $H|_{U_\pm} = \pm C_\pm$ with values $C_\pm > c_{\mathrm{HZ}}(U_\pm\,; L)$, then $\|\widetilde{\phi}_H\|_\gamma \geq \min\{c_{\mathrm{HZ}}(U_-\,; L)\,, c_{\mathrm{HZ}}(U_+\,; L)\}$.*

Several remarks are in order. First, $\|\cdot\|_\gamma$ denotes here the *Hamiltonian* spectral norm on $\widetilde{\mathrm{Ham}}(M\,,\omega)$ which is constructed as its Lagrangian counterpart $\|\cdot\|_{\gamma_L}$, however starting with the *Hamiltonian* spectral invariants of Section 1.4.

Second, this version is stated for monotone symplectic manifolds and not only for aspherical ones. Actually, up to requiring the constant $C_\pm$ to be small enough (*i.e.* smaller than a fourth of the generator of the group of periods w_L), the same statement holds for *any rational symplectic manifold* for which the spectral distance can be defined.

The ideas behind the proof of this result are the same as those yielding its aspherical-Lagrangian counterpart. However, it requires to be slightly more subtle when dealing with the non-trivial Novikov ring. We also omit the proof and refer to (Humilière, Leclercq, and Seyfaddini 2015b) for details. ▮

3.2. Uniqueness of continuous generators of humiotopies

As mentioned in the introduction of this section, the Gromov–Eliashberg Theorem naturally led to a definition of symplectic homeomorphisms, Definition II.19. Obviously, one would like to have a working definition of Hamiltonian homeomorphisms and homeotopies which will be exceptionally called in this section *humiomorphisms* and *humiotopies* respectively[7].

The most straightforward idea, *i.e.* to define them as C^0-limits of Hamiltonian diffeomorphisms, has the flaw that it is not obvious (nor even known actually) whether these are "generated" in any relevant sense by functions. The following fix was suggested by Oh and Müller (Oh and Müller 2007). We choose a distance d induced on M by a Riemannian metric ; the C^0-distance between two homeomorphisms ϕ, ψ is defined by $d_{C^0}(\phi,\psi) = \max_x d(\phi(x),\psi(x))$. Similarly, paths of homeomorphisms ϕ^t and ψ^t ($t \in [0,1]$) are at distance $d_{C^0}(\phi^t,\psi^t) = \max_{t,x} d(\phi^t(x),\psi^t(x))$. (A very enjoyable property of this distance is that if a sequence of homeomorphisms converges to a *homeomorphism*, then the sequence of their inverses converges to the inverse of the limit.)

Definition II.31. A path of homeomorphisms h^t is a *humiotopy* if there exists a sequence of smooth Hamiltonian functions $H_k \colon \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ ($k \in \mathbb{N}$) such that

a) $d_{C^0}(\phi^t_{H_k}, h^t) \to 0$, b) $(H_k)_{k\in\mathbb{N}}$ converges uniformly to a continuous function H.

Analogously to the smooth case, the limit function H is said to "*generate*" the isotopy h^t. This is justified by the facts that : Müller and Oh showed that a continuous function H generates at most one humiotopy, while Viterbo (Viterbo 2006), and later Buhovsky and Seyfaddini (Buhovsky and Seyfaddini 2013), proved that given a humiotopy, the generating continuous Hamiltonian is unique. The central result of (Humilière, Leclercq, and Seyfaddini 2015b) is analogous to the latter, with the C^0-distance replaced by the spectral pseudo-distance γ.

Theorem II.32. *Let (M,ω) be a rational symplectic manifold, let U be a non-empty open subset of M, I be a non-empty open interval in $\mathbb{R}$ and $(H_k)_{k\in\mathbb{N}}$ be a sequence of smooth Hamiltonians so that*

(i) for any $t \in I$, $\|\widetilde{\phi}^t_{H_k}\|_\gamma$ converges to zero when k goes to infinity,
(ii) $(H_k)_{k\in\mathbb{N}}$ converges uniformly on $I \times U$ to a continuous function H.

Then, the restriction of H to $I \times U$ only depends on the time variable.

Here, given $t \in \mathbb{R}/\mathbb{Z}$, $\widetilde{\phi}^t_H$ denotes the homotopy class with fixed endpoints of the path defined on $[0,1]$ by $s \longmapsto \phi^{st}_H$. Hence, Condition (i) expresses that $\widetilde{\phi}_H$ converges to Id *uniformly in time*. The norm $\|\cdot\|_\gamma$ is the *Hamiltonian* spectral norm on $\widetilde{\mathrm{Ham}}(M,\omega)$. However, by Continuity property of spectral invariants, the conclusion also holds if $\|\cdot\|_\gamma$ is replaced by the Hofer norm. The proof of Theorem II.32 mostly relies on the dual Hamiltonian energy-capacity inequality Theorem II.30 (and very careful choices of ε's and δ's – and η's and even a σ...).

Remark II.33. Among other consequences, it is interesting to note that we can recover the (initial) uniqueness of generator of a humiotopy proved by Viterbo. Thus, on one hand the latter can be seen as a consequence of the dual energy-capacity inequality Theorem II.30. On the other hand, the "dual" statement, *i.e.* the uniqueness of a humiotopy (given a continuous function), is an immediate consequence of the classical energy-capacity inequality Theorem II.26. ∎

Finally, let us also mention that Theorem II.32 has consequences in terms of C^0-rigidity of the Poisson bracket. This phenomenon was first discovered by Cardin and Viterbo (Cardin and Viterbo 2008) who proved the following result.

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7. As a small tribute to Vincent H. for the very natural names of "hameomorphisms" and "hameotopies".

THEOREM II.34. *Let $(H_k)_{k\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ be sequences of smooth Hamiltonians converging uniformly to the $C^{1,1}$ Hamiltonians H and G respectively. If the sequence of Poisson brackets $\{H_k\,,G_k\}$ uniformly converges to 0 when k goes to ∞, then H is a first integral of G.*

Recall that H is a *first integral* of G if it is constant along the flow of G. This is equivalent, when H and G are smooth, to $\{H\,,G\}=0$. Theorem II.32 shows that the same holds for continuous generators of humiotopies (the latter then play the roles of the respective flows).

3.3. Continuous Hamiltonian dynamics

In this section, we explain that characteristic dynamical properties of smooth Hamiltonian isotopies also hold for their continuous analogues. These properties feature some "big" natural submanifolds which carry a foliation and whose definition follows.

Definition II.35. A submanifold C of a symplectic manifold $(M\,,\omega)$ is called *coisotropic* if for any of its point p, its tangent space contains its symplectic orthogonal : $T_pC\supset(T_pC)^\omega$.

Similarly, a submanifold I is *isotropic* if $T_pC\subset(T_pC)^\omega$. A coisotropic submanifold admits a foliation $\mathcal{F}$, called *characteristic foliation*, with isotropic leaves, $\mathcal{F}(p)$ for all $p\in C$.

Obvious examples of coisotropic submanifolds are given by : the ambient manifold M itself (the leaves of its characteristic foliation corresponding to its points), or any hypersurface. The smallest ones are Lagrangian submanifolds (which are also the biggest possible isotropic manifolds). In this case the foliation consists of a unique leaf, the Lagrangian itself.

As for Darboux neighborhoods, there is a universal local model for any coisotropic manifold, see *e.g.* (Libermann and Marle 1987, Proposition 13.7) and (Gotay 1982).

Example II.36 (Coisotropic chart). Let C be a coisotropic submanifold of codimension k in a $2n$-dimensional symplectic manifold. For every point $p\in C$, there exists a chart $(\theta\,,U)$, such that U is an open neighborhood of p and $\theta\colon\ U\to V\subset\mathbb{R}^{2n}$ is a symplectic diffeomorphism which maps p to 0 and C to the standard coisotropic linear subspace

$$\mathcal{C}_0=\{(x_1,\dots,x_n,y_1,\dots,y_n)\,|\,(y_{n-k+1},\dots,y_n)=(0,\dots,0)\}\,.$$

Moreover θ sends the characteristic foliation of C to that of $\mathcal{C}_0$, whose leaf through a point $q=(a_1\,,\dots,a_n\,,b_1\,,\dots,b_{n-k}\,,0\,,\dots,0)\in\mathcal{C}_0$ is the affine subspace

$$\mathcal{F}_0(q)=\{(a_1,\dots,a_{n-k},x_{n-k+1},\dots,x_n,b_1,\dots,b_{n-k},0,\dots,0)\,|\,(x_{n-k+1},\dots,x_n)\in\mathbb{R}^k\}\,.$$

Now, recall the following two dynamical properties of a coisotropic submanifold C :

1. $H|_C$ is a function of time if and only if ϕ_H preserves C and flows along the leaves of its characteristic foliation, by which we mean that for any $p\in C$ and any $t\geq0$, $\phi_H^t(p)\in\mathcal{F}(p)$.
2. For each $p\in C$, $H|_{\mathcal{F}(p)}$ is a function of time if and only if ϕ_H preserves C.

In (Humilière, Leclercq, and Seyfaddini 2015a), we proved that these properties hold for continuous Hamiltonians. Let us denote by C^0_{Ham} the set of time-dependent continuous functions on M which generate a hameotopy in the sense of Definition II.31.

THEOREM II.37. *Let C be a properly embedded connected coisotropic submanifold of a symplectic manifold $(M\,,\omega)$. Let $H\in C^0_{\mathrm{Ham}}$ with induced hameotopy ϕ_H.*

(1) The restriction of H to C is a function of time if and only if ϕ_H preserves C and flows along the leaves of its characteristic foliation.
(2) The restriction of H to each leaf of the characteristic foliation of C is a function of time if and only if the flow ϕ_H preserves C.

The first item drastically generalizes the aforementioned uniqueness of generators Theorem : Indeed, if C is taken to be M, leaves of the characteristic foliation coincide with points in M and the theorem follows immediately :

Corollary II.38. *$H \in C^0_{\mathrm{Ham}}$ is a function of time if and only if $\phi_H^t = \mathrm{Id}$.*

When C is a Lagrangian, both items of Theorem II.37 coincide and state that : *The restriction of H to a Lagrangian L is a function of time if and only if $\phi_H^t(L) = L$ for all t.* In an interesting manifestation of Weinstein's creed, "Everything is a Lagrangian submanifold !", the general case of Theorem II.37 is essentially deduced from the *a priori* particular case of Lagrangians.

Let us focus on the "only if" direction of the first statement, *i.e.* the fact that if $H|_C$ is a function of time, ϕ_H preserves C and flows along the leaves of its characteristic foliation.

Ideas of the proof. We start with a Lagrangian L. We want to prove that if $H|_L$ is a function of time, then ϕ_H preserves L. The first observation is that by working *locally*, one can reduce the situation to the case of (a neighborhood of) a point p of the 0-section in T^*L or simply assume that L is aspherical.

Now, if $\phi_H^t(p)$ does not belong to L for some t, it means that ϕ_H^t displaces some ball B_ε centered at p from L. By definition, H is the C^0-limit of a sequence of smooth Hamiltonians $(H_k)_{k\in\mathbb{N}}$, so that for all k big enough, $\phi_{H_k}^t$ displaces some (possibly slightly smaller) ball $B_{\varepsilon'}$ from L.

By the classical Lagrangian energy-capacity inequality Theorem II.23, we deduce that for such integers k, the oscillations of H_k on L are greater than $c_{\mathrm{LR}}(B_{\varepsilon'}\,; 0_L)$. The oscillations on L of H itself are thus also bounded from below by the same quantity. This obviously contradicts the fact that $H|_L$ is a function of time.

Now that the statement is established for Lagrangians, the case of a coisotropic submanifold C follows (still locally !) by viewing the k-dimensional leaf of C at $p = 0$,

$$\mathcal{F}_0(0) = \{(0, \dots, 0, x_{n-k+1}, \dots, x_n, 0, \dots, 0) \,|\, (x_{n-k+1}, \dots, x_n) \in \mathbb{R}^k\}$$

as the intersection of the following $n-k$ Lagrangians

$$\Lambda_i = \{(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, 0, \dots, 0, y_i, 0, \dots, 0)\} \subset \mathbb{R}^{2n}$$

for $1 \le i \le n-k$. This gives the desired result *locally*, *i.e.* for small times, from which it is not hard to deduce the general case. □

Let us conclude this section with the following facts.

(1) The statement whose proof is sketched above is one of the main ingredients of the proof of the C^0-rigidity of coisotropic submanifolds, Theorem II.39 below.
(2) The proof of the converse statement is similar in spirit, as the general case is also deduced from the Lagrangian one ; the proof of the Lagrangian case relies in particular on the *dual* Lagrangian energy-capacity inequality of Theorem II.27.
(3) The proof of item *(2)* of Theorem II.37 is based on the Lagrangian case proved above, together with the aforementioned Theorem II.39, and the fact that the characteristic foliation of a coisotropic submanifold of M can be viewed as a Lagrangian in $M \times M$. We omit it.

3.4. Coisotropic rigidity

Building on the dynamical properties of hameotopies and their generators, we also established in (Humilière, Leclercq, and Seyfaddini 2015a) a strong rigidity property of coisotropic submanifolds.

Theorem II.39. *Let C be a smooth coisotropic submanifold of a symplectic manifold (M, ω). Let U be an open subset of M and $\theta\colon U \to V$ be a symplectic homeomorphism. If $\theta(C \cap U)$ is smooth, then it is coisotropic. Furthermore, θ maps the characteristic foliation of $C \cap U$ to that of $\theta(C \cap U)$.*

An important feature of this result is its locality : C is not assumed to be necessarily closed and θ is not necessarily globally defined.

Moreover, it has an immediate, but quite surprising, consequence.

Corollary II.40. *If the image of a coisotropic submanifold via a symplectic homeomorphism is smooth, then so is the image of its characteristic foliation.*

Finally, Theorem II.39 uncovers a link between two earlier rigidity results and demonstrates that they are in fact extreme manifestations of a single rigidity phenomenon.

One extreme case, where C is a hypersurface, was established by Opshtein (Opshtein 2009). Clearly, in this case, the interesting part is the assertion on rigidity of characteristics, as the first assertion is trivially true.

Lagrangians constitute the other extreme case. When C is Lagrangian, its characteristic foliation consists of one leaf, C itself. In this case the theorem reads : *If θ is a symplectic homeomorphism and $\theta(C)$ is smooth, then $\theta(C)$ is Lagrangian.* Laudenbach and Sikorav (Laudenbach and Sikorav 1994) proved a similar result : *Let L be a closed manifold and $(\iota_k)_{k\in\mathbb{N}}$ denote a sequence of Lagrangian embeddings, $\iota_k \colon L \to (M,\omega)$, which C^0-converges to an embedding ι. If $\iota(L)$ is smooth, then (under some technical assumptions) $\iota(L)$ is Lagrangian.* On one hand, their result only requires convergence of embeddings while Theorem II.39 requires convergence of symplectomorphisms. On the other hand, Theorem II.39 is local : It does not require the Lagrangian nor the symplectic manifold to be closed.

Before sketching the proof of the theorem, let us suggest the following exercise.

Exercise II.41. Assume Theorem II.39 and give a one-line proof of the Gromov–Eliashberg Theorem (Theorem II.18 above).

Sketch of proof. First, we show that the smooth version of the first item of Theorem II.37 (more precisely, the direction whose proof is sketched above) characterizes coisotropic submanifolds.

Lemma II.42. *If the flow of any smooth autonomous Hamiltonian function H which vanishes on a submanifold C preserves it, i.e. if for any such function $H|_C = 0$ yields $\phi_H^t(C) = C$, then C is coisotropic. Moreover, the leaf of its characteristic foliation at any point p is locally the union of such flows :*

$$\mathcal{F}(p) \cap W = \{\phi_H^t(p) \,|\, t \geq 0,\ H \in C_c^\infty(W),\ H|_C = 0\} \cap W$$

for some small open set W.

This is not hard to prove. *Infinitesimally speaking*, it comes down to the fact that H is constant on C if and only if $T_pC \subset \ker(d_pH)$, *i.e.* $X_H(p) \in (T_pC)^\omega$, while $\phi_H^t(C) = C$ shows that $X_H(p) \in T_pC$.

Next, assume as in the theorem that a symplectic homeomorphism θ maps (a piece of) a coisotropic C to a smooth submanifold C'. Let H be a smooth autonomous Hamiltonian H which vanishes on C'. Even though the function $K = H \circ \theta$, resulting from pulling back H to C via θ, might be non-smooth, it is a C^0-Hamiltonian which generates the hameotopy $\theta^{-1}\phi_H^t\theta$. Thus, by item *(1)* of Theorem II.37, it preserves C, which shows that ϕ_H preserves C'. Since this holds for any such H, we conclude with Lemma II.42. □

3.5. Reduction of symplectic homeomorphisms

We saw in Sections 3.1 to 3.4 that spectral invariants (via energy-capacity inequalities) allowed us to obtain information on continuous Hamiltonian dynamics, and then in turn on C^0 rigidity of coisotropics. In this section, we will see how they can also suggest the symplectic nature of homeomorphisms which (so far) cannot be shown to be symplectic in the sense of Definition II.19.

The specific question in which we are interested here was naturally raised by the C^0 rigidity of coisotropic submanifolds. Indeed, let C and C' be smooth coisotropic submanifolds of a symplectic manifold (M,ω), and denote by $\mathcal{F}$ and $\mathcal{F}'$ their respective characteristic foliations. Recall that one can define reduced spaces $\mathcal{R} = C/\mathcal{F}$ and

$\mathcal{R}' = C'/\mathcal{F}'$ as the quotients of the coisotropic submanifolds by their characteristic foliations. These spaces are, at least locally, smooth manifolds and they naturally come with symplectic structures induced by ω. Assume that a symplectic homeomorphism ϕ maps C to C', by Theorem II.37 it also maps $\mathcal{F}$ to $\mathcal{F}'$. Thus, it induces a homeomorphism $\phi_R : \mathcal{R} \to \mathcal{R}'$ of the reduced spaces. It is a classical fact that when ϕ is smooth, and hence symplectic, the reduced map ϕ_R is a symplectic diffeomorphism as well. It is therefore natural to ask whether the homeomorphism ϕ_R remains symplectic when ϕ is not assumed to be smooth.

A first remark is that even starting from a non-smooth symplectic homeomorphism ϕ, *if we assume the reduced homeomorphism to be smooth*, then it is symplectic. This is quite easily seen to be a consequence of item *(1)* of Theorem II.37 (again, of the specific part whose proof is sketched above), see (Humilière, Leclercq, and Seyfaddini 2016). A similar result, with a similar proof also appeared as (Buhovsky and Opshtein 2016, Proposition 6).

However, if the reduced homeomorphism is not assumed to be smooth, the question is still wide open (and possibly too hard for current techniques). Given the difficulty of this question, one could instead ask if there exist symplectic invariants which are preserved by the reduced homeomorphism. In this spirit, Opshtein formulated the following a priori easier problem :

Question II.43. Is the reduction ϕ_R of a symplectic homeomorphism ϕ preserving a coisotropic submanifold always a capacity preserving homeomorphism ?

This question is very meaningful, since a diffeomorphism preserving a capacity is symplectic (or anti-symplectic), as mentioned in Section 3.1.1. Partial positive results have been obtained by Buhovsky and Opshtein (Buhovsky and Opshtein 2016). They prove in particular that for a hypersurface C the map ϕ_R is "non-squeezing", in the sense that for every open set U containing a symplectic ball of radius r, the image $\phi_R(U)$ cannot be embedded in a symplectic cylinder over a 2-disk of radius $R < r$. This does not resolve Opshtein's question, but since capacity preserving maps are non-squeezing it does provide positive evidence for it. In the case of general coisotropic submanifolds, they conjecture that the same holds and indicate as to how one might approach this conjecture.

In (Humilière, Leclercq, and Seyfaddini 2016), we work in the specific setting where M is the torus $\mathbb{T}^{2(k_1+k_2)}$ equipped with its standard symplectic structure and C is the standard coisotropic subtorus $\mathbb{T}^{2k_1+k_2} \times \{0\}^{k_2}$. The reduction of C is $\mathbb{T}^{2k_1}$ with its usual symplectic structure. The theorem below shows that the reduced homeomorphism ϕ_R preserves appropriate spectral invariants, namely c_+, the "absolute" spectral invariant from Section 1.4 associated with the fundamental class of M.

Theorem II.44. *Let ϕ be a symplectic homeomorphism of $\mathbb{T}^{2(k_1+k_2)}$ equipped with its standard symplectic form. Assume that ϕ preserves the coisotropic submanifold $C = \mathbb{T}^{2k_1+k_2} \times \{0\}^{k_2}$. Denote by ϕ_R the induced homeomorphism on the reduced space $\mathcal{R} = \mathbb{T}^{2k_1}$.*

Then, for every time-dependent continuous function H on $\mathbb{R}/\mathbb{Z} \times \mathcal{R}$, we have $c_+(H \circ \phi_R) = c_+(H)$, with $H \circ \phi_R$ defined by $H \circ \phi_R(t, x) = H(t, \phi_R(x))$.

This answers Opshtein's question positively, as it follows immediately that the spectral capacity of Definition II.20 is preserved by ϕ_R.

Corollary II.45. *The map ϕ_R, from Theorem II.44, preserves the spectral capacity, i.e. $c(\phi_R(U)) = c(U)$ for any open set U.*

Ideas behind the proof of Theorem II.44. Let g_R be any continuous real-valued (autonomous for simplicity) function on $\mathcal{R}$ and denote by f_R the composition $f_R = g_R \circ \phi_R^{-1}$. We want to show that $c_+(g_R) = c_+(f_R)$. Let g and f be the respective lifts of g_R and f_R to $M = \mathbb{T}^{2k_1} \times \mathbb{T}^{2k_2}$ given by $g(z_1, z_2) = g_R(z_1)$ and $f(z_1, z_2) = f_R(z_1)$. By construction,

$g \circ \phi^{-1} = f$ on C. The situation can be summarized as follows

$$
\begin{array}{ccc}
\mathbb{T}^{2k_1+2k_2} & \xrightarrow{\phi} & \mathbb{T}^{2k_1+2k_2} \\
\uparrow & & \uparrow \\
C & \xrightarrow{\phi|_C} & C \\
\downarrow{\scriptstyle \text{red}} & & \downarrow{\scriptstyle \text{red}} \\
\mathcal{R} & \xrightarrow{\phi_R} & \mathcal{R}
\end{array}
\qquad
\begin{array}{ccccc}
g & \longrightarrow & g \circ \phi^{-1} & \neq & f \\
\downarrow & & \textit{restriction} & & \downarrow \\
g|_C & \longrightarrow & (g \circ \phi^{-1})|_C & = & f|_C \\
\downarrow & & \textit{reduction} & & \downarrow \\
g_R & \longrightarrow & g_R \circ \phi_R^{-1} & = & f_R
\end{array}
$$

To compare $c_+(g_R)$ to $c_+(f_R)$, we have to juggle "absolute" and Lagrangian spectral invariants, and "classical" and conormal-type Lagrangian spectral invariants. Somehow, we start at the lower right corner of the diagram above (f_R) and we compare spectral invariants, while going around the diagram in counterclockwise direction. Along the way, we also use most of their properties and standard arguments (like following the action spectrum along homotopies of Hamiltonians). Let us sketch *very* roughly the proof to illustrate this.

First, we replace $c_+(f_R)$ by its Lagrangian counterpart $\ell_+(f_R)$, via the diagonal construction of Section 1.4.1. Then, we compare $\ell_+(f_R)$ to the conormal-type spectral invariant $\ell_{L_0,L_1}(\alpha\,;f)$ (where L_0 and L_1 are the Lagrangians described in Example II.9). This is done via a *splitting formula* which ensures that both quantities coincide.

Remark II.46. This splitting formula was not introduced earlier as it is used only at this point and implies to first explain how to see the Künneth formula in Floer and quantum homology. Let us do this now in a very brief fashion.

Let L and L' be Lagrangians of respective symplectic manifolds $(M\,,\omega)$ and $(M'\,,\omega')$. Then $L \times L'$ is a Lagrangian of the product $(M \times M'\,, \omega \oplus \omega')$. Quantum and Floer homologies split, *i.e.* $H_*(L \times L') = H_*(L) \otimes H_*(L')$ with H_* being either HQ_* or HF_*. This splitting agrees with the involved PSS morphisms, and with the filtrations of the Floer complexes. Consequently it agrees with spectral invariants, in the sense that if $\alpha \otimes \alpha' \neq 0$ in $\mathrm{HQ}_*(L) \otimes \mathrm{HQ}_*(L')$ corresponds to β in $\mathrm{HQ}_*(L \times L')$:

$$\ell_{L \times L'}(\beta; H \sharp H') = \ell_L(\alpha; H) + \ell_{L'}(\alpha'; H')\,.$$

For classical spectral invariants, this holds even under the monotonicity assumption, see Theorem 40 in (Leclercq and Zapolsky 2018), with R being the field $\mathbb{Z}/2\mathbb{Z}$. For the conormal-type spectral invariants, the equivalent splitting formula was proved under asphericity assumption in (Humilière, Leclercq, and Seyfaddini 2016). ▮

The third step is to compare the conormal-type spectral invariant associated with f with that associated with $g \circ \phi^{-1}$. In order to do this, we consider the linear homotopy F between f and $g \circ \phi^{-1}$. Since these two functions coincide on C and are constant along the leaves of its characteristic foliation, the same holds for each F_s. This ensures that the action spectrum does not change along the homotopy F, so that, by Spectrality and Continuity, spectral invariants also remain unchanged.

Next, we compare the conormal-type spectral invariant $\ell_{L_0,L_1}(\alpha\,;g \circ \phi^{-1})$ to the classical $\ell_+(g \circ \phi^{-1})$. This time, we use the Triangle inequality for conormal-type invariants (see Section 1.5) which ensures that $\ell_{L_0,L_1}(\alpha\,;g \circ \phi^{-1}) \leq \ell_+(g \circ \phi^{-1})$.

Applying the diagonal construction again and using the fact that ϕ is symplectic, we get at this stage that $c_+(f_R) \leq c_+(g)$. Going "back" the first three steps, we conclude that $c_+(g) = c_+(g_R)$ which yields $c_+(f_R) \leq c_+(g_R)$.

By considering ϕ^{-1}, the same proof yields $c_+(f_R) \geq c_+(g_R)$ which concludes. □

4. On-going work and further perspective (1) : From spectral invariants to barcodes

Among natural extensions of spectral invariants, the most promising consists in persistence module and barcode techniques from topological data analysis. They were introduced in symplectic geometry by Polterovich and Shelukhin (Polterovich and Shelukhin 2016) who studied them further with Stojisavljević (Polterovich, Shelukhin, and Stojisavljević 2017), and also appeared in the framework of Morse–Novikov theory in work by Usher and Zhang (Usher and Zhang 2016), and in contact geometry in the work of Fraser (Fraser 2015).

The core idea is that the homology of a filtered chain complex is equivalent to a *persistence module* which can be coded in a *barcode*. In view of the construction of Morse, quantum, and Floer homology it is then natural to wonder which invariants can be read from such a representation ; it turns out that not only spectral invariants but also the boundary depth defined by Usher (Usher 2011b) naturally appear in the barcode.

This section discusses two directions which the author is exploring in this framework, both centered on understanding how barcodes get deformed to one another : via the various additional structures and morphisms presented in Chapter I for the first direction, and via a spectral sequence machinery introduced by Barraud and Cornea (Barraud and Cornea 2007) for the second.

4.1. Persistence modules and barcodes

In the context of this memoirs, we will consider *persistence modules* of $\mathbb{Z}/2\mathbb{Z}$-vector fields over $\mathbb{R}$ which can be defined as functors from $\mathbb{R}$ to the category $\mathrm{Vect}_{\mathbb{Z}/2\mathbb{Z}}$ of $\mathbb{Z}/2\mathbb{Z}$-vector spaces. Here, $\mathbb{R}$ is seen as a category whose objects are the real numbers and the morphism set between objects x and y is empty if $x > y$ and consists of a single arrow otherwise.

Equivalently, a persistence module (of $\mathbb{Z}/2\mathbb{Z}$-vector fields over $\mathbb{R}$) is a family $\mathbb{V} = (V^t)_{t\in\mathbb{R}}$ of vector spaces parameterized by $\mathbb{R}$ such that for all s and $t \in \mathbb{R}$ with $s \leq t$, there is a morphism $\iota_{t\leftarrow s}\colon V^s \to V^t$ such that

a) $\iota_{s\leftarrow s} = \mathrm{Id}$, b) for all r, s, and $t \in \mathbb{R}$, if $r \leq s \leq t$ then $\iota_{t\leftarrow r} = \iota_{t\leftarrow s} \circ \iota_{s\leftarrow r}$.

Example II.47. The main example in which we will be interested here is persistence modules which are obtained by taking the homology of $\mathbb{R}$-filtered chain complexes with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Indeed, let $(C\,,\partial)$ be such a chain complex, the filtration defines for all $t \in \mathbb{R}$ subvector spaces $C^t \hookrightarrow C$, with natural inclusions $C^s \hookrightarrow C^t$ for $s \leq t$, such that the restriction of ∂ to C^t defines a differential on C^t. The vector spaces $V^t = \mathrm{H}_*(C^t\,,\partial)$ together with the maps induced in homology by the natural inclusions above form a persistence module.

There is a classification theorem due to Zomorodian and Carlsson (Zomorodian and Carlsson 2005), and Crawley-Boevey (Crawley-Boevey 2015), which states that if the persistence module $\mathbb{V}$ is finite (in the sense that for all t in $\mathbb{R}$, the dimension of V^t is finite), then $\mathbb{V}$ is a direct sum $\mathbb{V} = \oplus_\alpha(\mathbb{Z}/2\mathbb{Z})_{I_\alpha}$ for some uniquely determined intervals I_α. (The notation $(\mathbb{Z}/2\mathbb{Z})_{I_\alpha}$ indicates that V^t admits a $\mathbb{Z}/2\mathbb{Z}$ factor if $t \in I_\alpha$.) The collection of these intervals form the *barcode*. Explicit examples of such barcodes are illustrated below, see Sections 4.2.2 and 4.3.1.

Both sets of persistence modules and barcodes can be endowed with a natural distance (respectively the interleaving and the bottleneck distance). Even though studying the properties of these sets with respect to the appropriate distance is one of the goal of what follows, the specific definition of the distances themselves does not appear explicitly below and we refer the interested reader to (Polterovich and Shelukhin 2016) or (Usher and Zhang 2016).

4.2. Morphisms and operations as barcodes

This is a project which the author is currently exploring with Alberto Abbondandolo. The idea behind it is quite simple. Most of the morphisms and additional structures built on Morse and Floer homology naturally lead to filtered chain complexes via the standard cone construction from algebraic topology. Thus, they also lead to persistence modules and barcodes. This gives a way to encode certain important morphisms between persistent modules into (persistence modules and thus) barcodes.

Indeed, let (A_*, ∂_*^A) and (B_*, ∂_*^B) be graded chain complexes and consider a chain map f of degree d between them, $f_*\colon A_* \to B_{*+d}$. The cone of f, denoted by $C(f)$, is the graded chain complex $(C_*(f), \partial_*^f)$ where

$$C_k(f) = A_k \oplus B_{k+d+1} \quad \text{and} \quad \partial_k^f = \begin{pmatrix} -\partial_k^A & 0 \\ f_k & \partial_{k+d+1}^B \end{pmatrix}.$$

It is designed so that its homology measures how far f is from inducing an isomorphism in homology. In particular the cone of f is acyclic (*i.e.* its homology vanishes), if and only if f is a quasi-isomorphism (*i.e.* induces an isomorphism in homology).

Now, if we assume that both A and B are filtered by $\mathbb{R}$, it is easy to see that the cone of a filtration-preserving chain morphism is naturally filtered.

Lemma II.48. *If the chain morphism $f_*\colon A_* \to B_{*+d}$ agrees with the respective filtrations, i.e. if there exists $\delta(f) \in \mathbb{R}$ such that $f(A^t) \subset B^{t+\delta(f)}$ for all $t \in \mathbb{R}$, then $C(f)$ admits a $\mathbb{R}$-filtration defined by $C_*^t(f) = A_*^{t-\delta(f)} \oplus B_{*+d-1}^t$ for all t.*

The filtered chain complex $C(f)$ yields a persistence module $\mathbb{V}(f)$ and a barcode $\mathcal{B}(f)$. Here is a tentative description (whose proof in progress we omit) of the associated barcode.

Proposition II.49 (Tentative statement...). *The barcode $\mathcal{B}(f)$ associated with a chain morphism $f\colon A \to B$ as above admits infinite bars, in one-to-one correspondence with the generators of the kernel and cokernel of the map induced by f in homology.*

Its finite bars are of three types :

(i) finite bars from $\mathcal{B}(A)$ shifted by $\delta(f)$, and from $\mathcal{B}(B)$,
(ii) bars corresponding to the identification of $[\alpha] \in \mathrm{H}_*(A)$ with $f_*[\alpha] \neq 0 \in \mathrm{H}_*(B)$,
(iii) bars announcing the birth of an element in $\ker f_*$ or $\operatorname{coker} f_*$.

Each bar (a, b) of the latter type is "paired" with an infinite bar which starts at b.

4.2.1. *The continuation morphism*

Let (f^1, ρ^1) and (f^2, ρ^2) be Morse–Smale pairs on M and pick a regular homotopy (f, ρ) between them. The continuation morphism induced by (f, ρ) is the degree 0 quasi-isomorphism defined for all critical points p_1 of f^1, by

$$\Phi_{f,\rho}\colon\ \mathrm{CM}_*(f^1, \rho^1) \longrightarrow \mathrm{CM}_*(f^2, \rho^2)\,, \quad \Phi_{f,\rho}(p_1) = \sum \#_2\mathcal{M}(p_1, p_2; f, \rho) \cdot p_2$$

where the sum runs over all critical points of f^2 of index $|p_2|_{f^2} = |p_1|_{f^1}$, and extended by linearity (see the description of the slightly more complicated quantum version in Section 3.1.2 of Chapter I).

Since the moduli space $\mathcal{M}(p_1, p_2; f, \rho)$ counts negative gradient flow lines of (f, ρ) connecting p_1 to p_2, it is easy to see that here Lemma II.48 immediately yields a filtration on the cone of the continuation morphism, where the shift $\delta(\Phi_{f,\rho})$ only depends on f^1 and f^2. The filtration is defined for all $t \in \mathbb{R}$ by

$$\begin{aligned} C_*^t(\Phi_{f,\rho}) &= \mathrm{CM}_*^{t-\delta(f^1,f^2)}(f^1) \oplus \mathrm{CM}_{*+1}^t(f^2) \quad \textit{i.e.} \\ C^t(\Phi_{f,\rho}) &= \mathbb{Z}/2\mathbb{Z}\big\langle p_1 \in \mathrm{Crit}(f^1) \,|\, f^1(p_1) < t + m(f^1, f^2)\big\rangle \\ &\qquad \oplus \mathbb{Z}/2\mathbb{Z}\big\langle p_2 \in \mathrm{Crit}(f^2) \,|\, f^2(p_2) < t\big\rangle \end{aligned}$$

with $\delta(f^1, f^2) = \max_M f^2 - \min_M f^1 + \varepsilon$.

4.2.2. *The intersection product*

Recall that the Morse-theoretical version of the intersection product is obtained by counting Y-configurations. More precisely let us briefly adapt Section 3.1.3 in Chapter I by forgetting all the almost complex data and by picking $f_2 = f_3$ (see Remark I.11). The intersection product

$$*_r\colon\ \big(\mathrm{CM}_*(f^1) \otimes \mathrm{CM}_*(f^2)\big)_r = \bigoplus_{k+l=r} \mathrm{CM}_k(f^1) \otimes \mathrm{CM}_l(f^2) \to \mathrm{CM}_{r-d}(f^2)$$

is defined on generators $p_1 \in \mathrm{Crit}(f^1)$ and $p_2 \in \mathrm{Crit}(f^2)$ (and then extended by bilinearity) by

$$p_1 * p_2 = \sum \#_2\mathcal{Y}(p_1, p_2, q_2; f^1, \rho^1, f^2; \rho^2) \cdot q_2$$

where the sum runs over all $q_2 \in \mathrm{Crit}(f^2)$ of index $|q_2|_{f^2} = |p_1 \otimes p_2| - d = |p_1|_{f^1} + |p_2|_{f^2} - d$.

A typical element of $\mathcal{Y}(p_1\,, p_2\,, q_2\,; f^1\,, \rho^1\,, f^2\,; \rho^2)$ is a pair of flow lines $(\gamma_1\,, \gamma_2)$ which meet at $t = 0$. Namely, γ_1 belongs to the unstable manifold of p_1 with respect to the Morse–Smale pair $(f^1\,, \rho^1)$, γ_2 to the connecting manifold from p_2 to q_2 with respect to $(f^2\,, \rho^2)$, and $\gamma_1(0) = \gamma_2(0)$.

The cone of $*$ is the chain complex

$$C_r(*) = \big(\mathrm{CM}(f^1) \otimes \mathrm{CM}(f^2)\big)_r \oplus \mathrm{CM}(f^2)_{r-d+1}\ , \quad \partial^* = \begin{pmatrix} -\partial^{\otimes}_r & 0 \\ *_r & \partial^{f_2}_{r-d+1} \end{pmatrix}$$

whose elements of degree r are of the form $(p_1 \otimes p_2) \oplus q_2$ with $|p_1|_{f^1} + |p_2|_{f^2} = r$ and $|q_2|_{f^2} = r - d + 1$. Recall that $\partial^{\otimes}$ is defined by the formula $\partial^{\otimes}(p_1 \otimes p_2) = \partial^{f^1} p_1 \otimes p_2 + (-1)^{|p_1|_{f^1}} p_1 \otimes \partial^{f^2} p_2$.

Filtering the tensor product by the maximum of the filtration on each factor, and applying Lemma II.48 to the situation, we get that the cone of $*$ admits a filtration defined for all $t \in \mathbb{R}$ by

$$C^t_*(*) = \big(\mathrm{CM}^t_*(f^1) \otimes \mathrm{CM}^t_*(f^2)\big) \oplus \mathrm{CM}^t_{*+1}(f^2)\,.$$

The fact that $\delta(*)$ can be chosen to be 0, comes from the observation that $p_1 * p_2$ is a linear combination of critical points of f^2 whose values by f^2 are strictly lower than $f^2(p_2)$.

4.2.3. *Perspectives and possible applications*

Encoding the intersection product into a barcode would for example allow us to detect different Morse functions which produce identical barcodes, as in (Polterovich, Shelukhin, and Stojisavljević 2017, Section 2.4) where this is done by considering the intersection product with a given homology class (which has to be cleverly chosen *a priori*).

It should also allow us to define *new* invariants of powers of Hamiltonian diffeomorphisms (by considering $f^1 = f^2 = F$).

Other structures can obviously be turned into barcodes via the same technique, as for example the various module structures which appear in Morse and Floer theory. This might be interesting if one manages to relate the properties of the resulting barcodes to the properties of the initial structure.

For example, and as for the continuity property of spectral invariants (II.3), one can show that the map which associates with a Morse function its barcode is continuous with respect to the C^∞ distance on Morse functions and the bottleneck distance on barcodes, by studying the continuation morphism.

It could be interesting to understand which part of this phenomenon can be read directly from the barcode associated with the continuation morphism itself, *e.g.* what is the relation between the bottleneck distance between two barcodes and the length of the finite bars of the barcode associated with the continuity morphism between the complexes defined for the respective Morse functions? (Note that this also concerns barcodes coming from module structures, as Shelukhin noticed that one can prove continuity with respect to the spectral and bottleneck distances by studying the latter rather than the continuation morphism.)

Another application would be to extract other well-known invariants from these barcodes. For example, it is not absurd to imagine that we might read (or bound) the cuplength of a manifold on the barcode of its intersection product.

However, the main reason why we would like to explore this direction is of a different nature. From the barcode perspective, a morphism of Morse homology deforms a barcode into another. Coding the deformation itself into a barcode makes it plausible to deduce the barcode obtained after deformation by studying the initial barcode and the barcode of the deformation. Somehow, this is also part of the motivation behind the idea developed in the next section.

4.3. Spectral sequences as degeneration of barcodes

As the description of the various decorations of the Morse, quantum, and Floer homologies from Chapter I hopefully made clear, most of them (starting with the differential of the respective complexes themselves) are defined at the chain level by counting the 0-dimensional component of suitable moduli spaces. Then, the study of the boundary of the compactification of the 1-dimensional component of the same moduli spaces shows that they do satisfy the main property which they are expected to satisfy (*e.g.* the differential squares to 0, the formula defining the continuation morphism is a chain morphism...).

Somehow, by using this process we do not take into account the information carried by the higher dimensional components of the moduli spaces. Barraud and Cornea (Barraud and Cornea 2007) found a way to encode the additional information by setting up Morse and Floer versions of the Leray–Serre spectral sequence of the path-loop fibration.

Let M be a smooth manifold, which we assume simply connected (to ease the construction and following statements), and from which we fix a point p. Its path-loop fibration is the fibration given by the evaluation at 1 of paths in M starting at p :

$$\Omega_p M \hookrightarrow \mathcal{P}_p M \xrightarrow{\mathrm{ev}_1} M$$

whose fiber $\Omega_p M$ is the set of loops in M based at p. Since p has been chosen once and for all, we forget it from the notation of the fiber ΩM.

As any fibration, the path-loop fibration of M induces a spectral sequence known as *the Leray–Serre spectral sequence* of M, which we will denote by $E(M)$. Let us describe this specific example of spectral sequences, by only pointing out some of its main properties. As for all homology theory in this memoirs, we work with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

(1) $E(M)$ consists of a finite numbers of *pages*, $E^r(M)$ with $2 \leq r \leq \dim M$.
(2) Each page is a bi-graded complex $(E^r_{p,q}(M), \partial^r)$ such that the $\mathbb{Z}/2\mathbb{Z}$-vector spaces $E^r_{p,q}(M) = 0$ unless $q \geq 0$ and $0 \leq p \leq \dim M$.
(3) The differential ∂^r as bi-degree $(-r, r-1)$.
(4) The vector spaces of the $(r+1)$-th page are the homology of those of page r, namely

$$E^{r+1}_{p,q}(M) = \ker\left(\partial^r\colon\ E^r_{p,q}(M) \to E^r_{p-r,q+r-1}(M)\right)/\mathrm{im}\left(\partial^r\colon\ E^r_{p+r,q-r+1}(M) \to E^r_{p,q}(M)\right).$$

(5) The vector spaces composing page 2 of the spectral sequence are

$$E^2_{p,q}(M) = \mathrm{H}_p(M) \otimes \mathrm{H}_q(\Omega M).$$

(6) The vector spaces of the last page vanish except for $E^{\dim M}_{0,0} = \mathbb{Z}/2\mathbb{Z}$.

Note that property (4) above is satisfied by any spectral sequence. It shows in particular that the vector spaces of the first page together with *all the differentials* determine the whole spectral sequence.

Let us now explain the Morse-theoretical version of this construction. Given a Morse–Smale pair (f, ρ) for M, Barraud and Cornea constructed a spectral sequence starting at page 1 which can be roughly described by the facts that

(7) its page 1 is given by $E^1_{p,q}(f) = \mathrm{CM}_p(M; f) \otimes \mathrm{H}_q(\Omega M)$,
(8) the differential at page 1 is defined as the differential of the Morse complex

$$\partial^1_{(f,\rho)}\, p = \sum_{q\,|\,|q|_f = |p|_f - 1} \#_2 \mathcal{M}(p,q; f,\rho)\, q \quad \text{with } \mathcal{M}(p,q;f,\rho) = \left(\mathcal{U}(p) \cap \mathcal{S}(q)\right)/_{\mathbb{R}},$$

(9) similarly, the differential at page r is defined by a suitable "count" of the elements of $\mathcal{M}(p, q; f, \rho)$ where $|q|_f = |p|_f - r$, *i.e.* the $(r-1)$-dimensional component of the moduli spaces.

Disclaimer II.50. The *roughness* of the previous description is two-fold. First, the generators of page r for $r \geq 2$ are not critical points of f *per se* but classes of linear combinations of such (even when we completely forget the generators of $\mathrm{H}_*(\Omega M)$).

More importantly, one should define "counting" the r-dimensional component of the moduli spaces. The idea behind it is to contract a path in M whose image contains all critical points. A flow line of f in M going from a critical point to another is then seen as a loop in the quotient. The r-dimensional components of $\mathcal{M}(p, q; f, \rho)$ can then be represented by elements in $\mathrm{H}_r(\Omega M)$. ▮

From (7) and (8) above it is clear that page 2 of this new spectral sequence is isomorphic (as bi-graded vector space) to page 2 of the Leray–Serre spectral sequence of M. Thus Barraud and Cornea's result, which states that both spectral sequences are isomorphic starting at page 2, shows that the additional information carried by the higher dimensional components of the connecting manifolds of f with respect to ρ are encoded in the differentials of the higher order pages of the Leray–Serre spectral sequence of M.

Now, the *rough* idea is that property (2) above, together with the fact that the Morse description comes naturally with a filtration, suggests that each page $E^r(f)$ *might* be encoded as a barcode $\mathcal{B}^r(f)$. In this perspective, property (5) then says that the barcode extracted from page 2 is mostly the barcode coming from the Morse complex, $\mathcal{B}^2(f) = \mathcal{B}(f)$. Property (6) indicates that the barcode corresponding to the last page is the trivial barcode consisting of only one infinite bar. In a sense, the succession of barcodes $\mathcal{B}^r(f)$ shows how the higher dimensional components of the connecting manifolds of f make $\mathcal{B}(f)$ degenerate to the trivial barcode.

Of course, this is again "rough" as, for example, one has to manage to "organize" the extra-data coming from the homology of the loop space of M. This seems doable, at least on easy examples as the following one.

4.3.1. *Example of $S^2 \times S^4$*

We build on computations of the Barraud–Cornea spectral sequence of $S^2 \times S^4$ which we made in (Leclercq 2008) to illustrate the ideas of the previous section.

The setting is as follows : we consider the sum of the height function of each of the factors. This function f has four critical points denoted p_i for $i = 0$, 2, 4, and 6 in such a way that the Morse index of p_i is i. The non-empty connecting manifolds are

$$\mathcal{M}_{p_6,p_4},\ \mathcal{M}_{p_6,p_2},\ \mathcal{M}_{p_6,p_0}, \mathcal{M}_{p_4,p_0},\ \text{and } \mathcal{M}_{p_2,p_0}.$$

Let α (respectively β) denote the generator of $\mathrm{H}_1(\Omega S^2)$ (respectively $\mathrm{H}_3(\Omega S^4)$) seen as homology classes of $\Omega S^2 \times \Omega S^4 = \Omega(S^2 \times S^4)$. It represents $\mathcal{M}_{p_6,p_4}$ and $\mathcal{M}_{p_2,p_0}$ (respectively $\mathcal{M}_{p_6,p_2}$ and $\mathcal{M}_{p_4,p_0}$).

The differential at page 2 vanishes except for $\partial^2 p_6 = p_4 \otimes \alpha$ and $\partial^2 p_2 = p_0 \otimes \alpha$. At page 3 the differential vanishes identically. At page 4, it vanishes except for $\partial^4 p_6 = p_2 \otimes \beta$ and $\partial p_4 = p_0 \otimes \beta$. The spectral sequence already collapses at page 5. After page 2, only the classes of $p_0 \otimes \beta^k$ and $p_4 \otimes \beta^k$ (with $k \geq 1$) survive. They die at page 4. Schematically, this is summarized by Figure II.2.

This yields the following series of barcodes. The first one corresponds to the usual barcode obtained from f : since the function is a perfect Morse function, we only get 4 infinite bars, each starting at a critical value. Then, we take into account $\partial^2_{(f,\rho)}$: $\mathcal{B}^2(f)$ has two finite bars, the left-most one corresponds to $p_0 \otimes \alpha^k$ for $k > 0$, while the right-most to $p_4 \otimes \alpha^k$ for $k > 0$. They reflect the fact that the 2-dimensional component of the connecting manifolds is responsible for the death of two classes. Two infinite bars consequently "disappear" in $\mathcal{B}^3(f)$, compared to $\mathcal{B}^1(f)$.

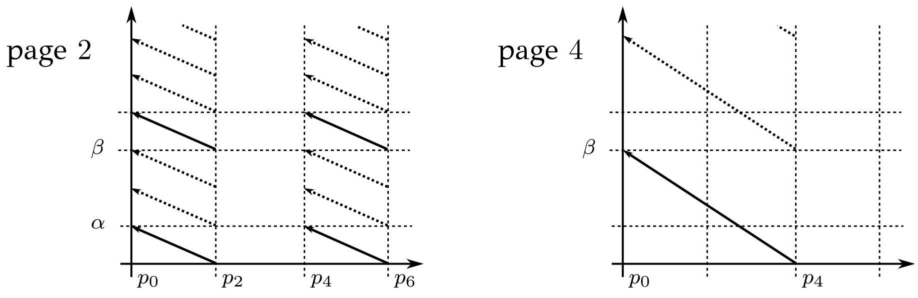
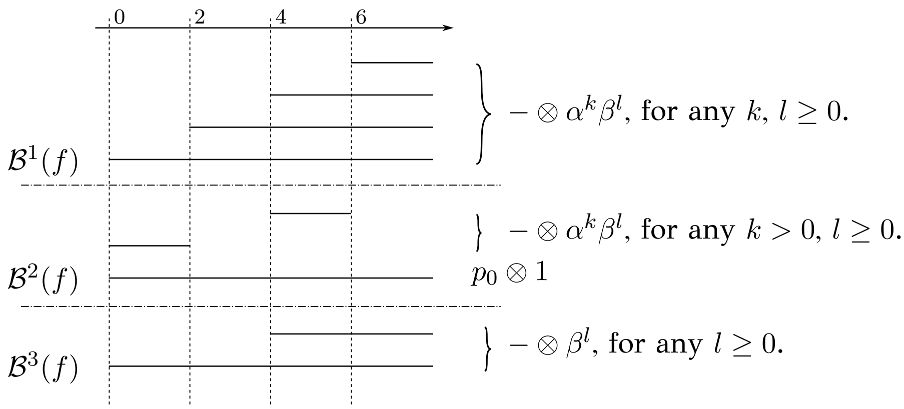
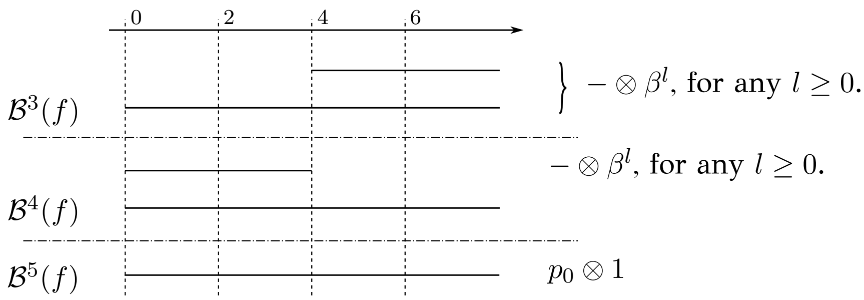


FIGURE II.2. The spectral sequence for $S^2 \times S^4$



The same phenomenon happens when we take into account $\partial^4_{(f,\rho)}$ and we end up with the trivial barcode.



4.3.2. *Interpretation and possible applications*

As in Section 4.2 above, this might be interesting as a way to encode the deformation of a given barcode (or the equivalent persistence module) to the trivial one. More precisely, as in the tentative statement of Proposition II.49, it happens that a finite bar $(a\,,b)$ of page r is *paired* with an infinite bar of page $r-1$ which starts at b. The barcode of page $r+1$ can then be deduced from the one at page $r-1$ by removing this infinite bar.

What makes these deformations relevant here is that they are given by (the higher dimensional components of) the connecting manifolds of f with respect to ρ. This could for example lead to distinguishing Morse functions which have the same barcode.

The example of $S^2 \times S^4$ illustrates another interesting phenomenon. Inspired by the *boundary depth*, the construction of Barraud and Cornea leads to the notion of r*-th boundary depth*, for $r \geq 1$. It can be defined as the length of the longest flow line making a non-trivial contribution to the differential of the r-th page of the Morse-theoretic Leray–Serre spectral sequence of f. As such, it can be read on the series of barcodes, as the length of the longest finite bar of $\mathcal{B}^r(f)$. For $r = 1$, it coincides with the usual boundary depth.

In the example above, while the boundary depth vanishes (since the Morse function is perfect!), the 2-nd and 4-th boundary depths are non zero.

5. On-going work and further perspective (2) : From symplectic to contact geometry

As mentioned in the introduction of the present chapter, spectral invariants were also introduced in the setting of contact geometry. However, they have been defined so far thanks to generating functions (as those initially defined by Viterbo in symplectic geometry). Together with Sheila Sandon, we are working on defining them via a Floer-type theory (in the process of having been) constructed by Sandon (Sandon 201x).

As for their symplectic counterparts, they are expected to be useful to study the geometry and topology of contactomorphism groups and certain sets of Legendrian submanifolds.

In this section, we present the main ideas of this project. In order to keep the presentation short, we will not define the objects but will rather compare them to their respective symplectic counterparts. For example, in the previous paragraph, a *contactomorphism* is the contact counterpart of a symplectomorphism (and they are all Hamiltonian), while a *Legendrian* is the counterpart of a Lagrangian.

5.1. Translated point homology and the contact Arnol'd conjecture

The first step of the program is to extend Sandon's construction from *hypertight* contact manifolds, which are the equivalent of aspherical symplectic manifolds, to a more general situation which we want equivalent to the monotone case.

5.1.1. *Going to the monotone case*

The main examples we have in mind are *prequantization bundles* which are principal S^1-bundles over a symplectic base (W, ω) obtained by a construction due to Boothby and Wang (Boothby and Wang 1958). In short, let (W, ω) be a symplectic manifold. Assume that ω is *integral*, *i.e.* it is the image of some integral cohomology class $e \in \mathrm{H}^2(W;\mathbb{Z})$. Consider the principal S^1-bundle

$$S^1 \hookrightarrow (M, \alpha) \xrightarrow{\pi} (W, \omega)$$

with Euler class e. Then there is a contact form α on M, whose Reeb flow generates the S^1-action, and which is a connection form with curvature form equal to ω (in particular, $d\alpha = \pi^*\omega$). The contact manifold $(M, \xi = \ker\alpha)$ is the *prequantization* of (W, ω).

Now, assume that a multiple γ of a fiber is contractible in the total space. Then there exists a capping A of γ in M. This disc descends as a sphere to W whose symplectic area is easily seen to coincide with the period of γ, $\omega(A) = \eta(\gamma)$. Since a contact form is *hypertight* if it admits no contractible Reeb orbits, it is not hard to get convinced that α is hypertight if and only if ω is symplectically aspherical.

Our goal is to be able to work with contact manifolds $(M, \xi = \ker\alpha)$ obtained as total spaces of such S^1-bundles over *monotone* manifolds (W, ω).

Adapting Sandon's construction should be rather straightforward, by mimicking the arguments used to construct Floer homology in monotone symplectic manifolds. This will allow us to define the *translated point homology* of M, $\mathrm{HT}_*(M)$, which results from a Floer-type construction. In particular, it comes from a chain complex which is naturally filtered by the values of an appropriate action functional.

Slightly more precisely, its closest symplectic counterpart is quantum homology *seen as a Morse–Bott Floer theory*, similarly to Section 5.1 of Chapter I. Recall that in the latter case (when we pick the zero Hamiltonian), any point of the manifold is an "orbit" of the Hamiltonian whose cappings are nothing but spheres. Moreover, the action of such a capped orbit reduces to the symplectic area of the sphere. Since we saw that the symplectic area of spheres in W corresponds to the period of capped Reeb orbits in M, one shall not be surprised that the generators of the translated point complex are Reeb orbits whose action is given by their periods.

5.1.2. *A Gysin sequence result*

In view of the construction above, it is quite natural to wonder what is the relation between the "symplectic" homology of (W,ω) and the "contact" homology of (M,ξ). It turns out that they are related in the most natural way.

Indeed, we are in the process of establishing the existence of a quantum-Floer Gysin long exact sequence, comparing the quantum homology of the base to the translated point homology of the total space. Namely, it takes the form of a long exact sequence

$$\cdots \longrightarrow \mathrm{HQ}^{k-1}(W) \xrightarrow{\cup e} \mathrm{HQ}^{k+1}(W) \xrightarrow{\pi^*} \mathrm{HT}^{k+1}(M) \xrightarrow{\pi_!} \mathrm{HQ}^{k}(W) \longrightarrow \cdots$$

where (as in the topological case) e denotes the Euler class of the bundle (which in our setting also corresponds to an integral lift of the symplectic form !).

Note that there were earlier adaptations to the symplectic and contact realms of the Gysin sequence, see for example (Biran and Khanevsky 2013) and (Bourgeois and Oancea 2013) which are the most relevant to our project.

5.1.3. *Application : the contact Arnol'd conjecture*

The main motivation for the development of Floer homology was to prove the Arnol'd conjecture (Arnol'd 1965) which predicts that Hamiltonian diffeomorphisms have *many* fixed points. There are several variants of the conjecture, here is one which is now proved thanks to Floer's seminal work (Floer 1989b) for monotone closed symplectic manifolds and generalizations to the general case by Hofer and Salamon (Hofer and Salamon 1995), Liu and Tian (Liu and Tian 1998), and Fukaya and Ono (Fukaya and Ono 1999).[8]

Conjecture II.51. *The number of fixed points of a nondegenerate Hamiltonian diffeomorphism of a closed symplectic manifold (M,ω) is at least the sum of the Betti numbers of M.*

This conjecture obviously follows from the facts that Floer homology is generated by orbits of a Hamiltonian isotopy and that it is isomorphic to the homology of M.

The naive contact analogue of this conjecture does not hold. For example, the Reeb vector field which is generated by the constant Hamiltonian 1 does not vanish, so for small times its flow does not admit any orbit. Morally speaking, being transported by the flow generated by a *constant* contact Hamiltonian should not be held against any point of a contact manifold. In this perspective, the relevant contact analogue to Hamiltonian periodic orbits, or equivalently to fixed points of Hamiltonian diffeomorphisms, was introduced by Sandon (Sandon 2011) :

Definition II.52. Let ϕ be a contactomorphism of a contact manifold (V,ξ). A point $p \in V$ is a *translated point* of ϕ if p and $\phi(p)$ belong to the same Reeb orbit, and if $d_p\phi$ preserves the contact form, $(\phi^*\alpha)_p = \alpha_p$ (concerning this condition, see Remark II.54 below).

Then Sandon (Sandon 2013) conjectured the contact analogue of the Arnol'd conjecture in terms of translated points and proved it in the case of $S^{2n-1}$and $\mathbb{RP}^{2n-1}$. Since then the conjecture has been proved to hold in several cases, mostly thanks to Floer–Rabinowitz techniques, see *e.g.* (Albers and Merry 2013) and (Meiwes and Naef 2015).

The power of Sandon's Floer-type homology in the perspective of proving this conjecture is that the generators of the complex are in one-to-one correspondence with translated points. Thus, as for the classical Arnol'd conjecture, it "only" remains to

8. The Arnol'd conjecture has been a great motivation not only for developing Floer's theory but also for writing *epic* MathSciNet® reviews, see *e.g.* those of (Liu and Tian 1998) by Jean-Claude Sikorav and of (Fukaya and Ono 1999) by David E. Hurtubise.

show that the resulting homology always has "many" generators. This is an important motivation for its generalization to the case of monotone prequantization bundles.

5.2. Contact spectral invariants and applications

Here is another important motivation. As mentioned above, the translated point homology comes from a complex which is filtered by the values of the action functional (which turns out in this case to be the period of the generating Reeb orbits). This allows us to define spectral invariants in a very similar fashion as the central construction of this chapter.

The first question (again, in view of the setting) will then be to compare them to the absolute spectral invariants of the base.

Concerning their properties, most of them (and in particular the most important one, their continuity) should easily go through. One major difference however concerns the triangle inequality. Indeed, thus far we have not been able to define a product structure on the translated point homology. This will be a major issue as triangle inequality is essential for some applications (as the definition of a spectral distance or spectral quasimorphisms).

However, there are already several applications of the theory which do not require spectral invariants to satisfy a triangle inequality. Below, I present two of these applications to questions which are currently extensively explored.

5.2.1. *Contact non-squeezing*

Recall from Section 3.1.1, that Gromov's non-squeezing Theorem states that if the standard $2n$-dimensional symplectic ball of radius r, $B^{2n}(r)$, can be symplectically embedded into the standard $2n$-dimensional symplectic cylinder $C^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$, then $r \leq R$.

In contact geometry, there is *a priori* no such rigidity. For example, in $\mathbb{R}^3$ endowed with its standard contact structure (which, by Darboux is the local model around any point of any contact manifold), the map $(x, y, z) \mapsto (cx, cy, c^2 z)$ is a contactomorphism which can squeeze any domain into an arbitrarily small one.

However, Eliashberg, Kim, and Polterovich (Eliashberg, Kim, and Polterovich 2006) showed that the contact (non-) squeezing problem was somehow more subtle than the previous example might have suggested.

Theorem II.53. *In $\mathbb{R}^{2n} \times S^1$ endowed with the contact form $dz - \sum_i (x_i dy_i - y_i dx_i)$,*

(1) if $R_2 \leq m \leq R_1$ for some integer m, no compactly supported contactomorphisms map the closure of $B^{2n}\left(\sqrt{\frac{R_1}{\pi}}\right) \times S^1$ into $C^{2n}\left(\sqrt{\frac{R_2}{\pi}}\right) \times S^1$,

(2) if $n \geq 2$, for any R_1 and R_2 in $(0, 1)$, the closure of $B^{2n}\left(\sqrt{\frac{R_1}{\pi}}\right) \times S^1$ can be mapped into $B^{2n}\left(\sqrt{\frac{R_2}{\pi}}\right) \times S^1$ by a compactly supported contactomorphism.

The first statement shows that Gromov's non-squeezing in $\mathbb{R}^{2n}$ extends to the prequantized objects if $\pi R^2 \leq m \leq \pi r^2$ for some integer m. The second statement shows that it does not extend if both πr^2 and $\pi R^2 < 1$. (More recently, Chiu (Chiu 2017) filled in the blanks, proving that one cannot squeeze the prequantization of $B(r)$ into that of $B(R)$ if $\pi r^2 > \pi R^2 \geq 1$, via completely different techniques coming from microlocal sheaf theory.)

In (Eliashberg, Kim, and Polterovich 2006), the non-squeezing result is obtained thanks to involved homology theories (cylindrical contact homology and symplectic homology). It was reproved, by Sandon (Sandon 2011) thanks to the theory of generating functions. More precisely, and following earlier work of Bhupal (Bhupal 2001), Sandon defines contact analogues of Viterbo's spectral invariants (in other words contact spectral invariants based on generating functions) associated with contactomorphisms, from which she infers a capacity for subsets (which is analogous to that of Definition II.20) which plays the main role in her proof of the non-squeezing result of Eliashberg, Kim, and Polterovich.

Remark II.54. Contactomorphisms were presented as contact counterparts of Hamiltonian diffeomorphisms. This is of course not accurate (Ô le bel euphémisme !) and here is a very big difference. Contrarily to symplectomorphisms which preserve the symplectic form, contactomorphisms preserve the contact *structure*. This means that they preserve the contact form *up to a conformal factor* : $\phi^*\alpha = e^f\alpha$ for some real-valued function f.

One consequence of this fact is that the contact version of Viterbo's spectral invariants are not conjugation invariant. However, Sandon proved that they satisfy $c(\psi\phi\psi^{-1}) = c(\phi)$ if (and only if) $c(\phi) \in \mathbb{N}$. Their integer part is thus conjugation invariant and this is why the condition that "$\pi R^2 \leq m \leq \pi r^2$ for some integer m" naturally appears in Sandon's proof. ∎

Since the definition of the spectral capacity does not require the spectral invariants to satisfy the triangle inequality, there should not be any major issues in giving (yet) an alternative proof of the non-squeezing result for $\mathbb{R}^{2n} \times S^1$.

5.2.2. *Orderability*

Another application which we have in mind concerns the orderability of contact manifolds. This notion was introduced by Eliashberg and Polterovich (Eliashberg and Polterovich 2000).

Remark II.55. Contactomorphisms were presented as contact counterparts of Hamiltonian diffeomorphisms, with the warning from Remark II.54 that they preserve the contact form *up ta a conformal factor*. This is still not accurate (ibid.) and here is another difference. Contrarily to a Hamiltonian isotopy whose generator is defined up to addition of a function of time (since the Hamiltonian vector field is defined as ω-dual to the differential of the Hamiltonian function), the generator of a contact isotopy is uniquely determined by $\alpha(X_H^t) = H_t$.

One consequence of this fact is that there is a notion of *positive* contact isotopy, which is a contact isotopy generated by a positive contact Hamiltonian function. ∎

This naturally led Eliashberg and Polterovich to the definition of a relation between contact isotopies which induced a relation on the universal cover of the identity component of the contactomorphism group of a contact manifold (V, ξ).

Definition II.56. Let $\widetilde{\phi}$ and $\widetilde{\psi} \in \widetilde{\mathrm{Cont}_0}(V, \xi)$. If $\widetilde{\phi} \cdot \widetilde{\psi}^{-1}$ can be generated by a non-negative contact Hamiltonian, then $\widetilde{\phi} \geq \widetilde{\psi}$.

This relation is easily seen to be reflexive and transitive ; *when* it is anti-symmetric (and thus defines a *partial order*), the contact manifold (V, ξ) is said to be *orderable*. As an example, let us mention (among the many results which are now known) that $\mathbb{RP}^{2n-1}$ is orderable, while S^{2n-1} is not. (Note that both manifolds can be obtained as prequantization bundle over $\mathbb{CP}^{n-1}$ endowed with the standard Fubini–Study symplectic form, multiplied by $\frac{1}{\pi}$ for the latter and by $\frac{2}{\pi}$ for the former.)

Both results were proved respectively in (Eliashberg and Polterovich 2000) and (Eliashberg, Kim, and Polterovich 2006) via a criterion established by Eliashberg and Polterovich which states that a manifold is orderable if and only if there are no positive contractible loops of contactomorphisms.

This is another occurrence where contact spectral invariants might prove to be useful. Indeed, it turns out that they detect the lack of such positive contractible loops, as we can show that, if the spectral invariant associated with any homology class is finite, then there are no positive contractible loops of contactomorphisms (so that the manifold is orderable).

Again, this fact does not rely on any type of triangle inequality but rather on the fact that we understand the behavior of these invariants along continuation morphisms.

Indeed, the proof consists in making sense of the naive idea that since the loop is *contractible* iterating it should not change the value of the spectral invariants, while since it is *positive* these values should strictly decrease (for convention reasons, the opposite of the spectral values strictly increase).

CHAPTER III

Around the Seidel representation

Once upon a time, Seidel...

The *Seidel representation* was introduced by Seidel (Seidel 1997) who described it in two quite different (but equivalent!) ways. On one side it can be described as a morphism

$$\mathcal{S}\colon\ \pi_1\big(\mathrm{Ham}(M,\omega)\big) \longrightarrow \mathrm{HQ}_*(M)^\times\,, \qquad \widetilde{\phi} \longmapsto \mathcal{S}(\widetilde{\phi}) \tag{III.1}$$

where $\mathrm{HQ}_*(M)^\times$ denotes the multiplicative group formed by the invertible elements of $\mathrm{HQ}_*(M)$. A quantum class in the image of $\mathcal{S}$ is called a *Seidel element*. This morphism is defined by counting pseudo-holomorphic sections of Hamiltonian fibrations over S^2 with fibre M. On the other side, it can be seen as a representation of the fundamental group of the Hamiltonian diffeomorphism group into the automorphisms of Floer homology,

$$\pi_1\big(\mathrm{Ham}(M,\omega)\big) \longrightarrow \mathrm{Aut}\big(\,\mathrm{HF}_*(M)\big)\,, \qquad \widetilde{\phi} \longmapsto \widetilde{\phi}_*\,.$$

The former interpretation will be called *geometric* and the latter *algebraic*.

Remark III.1. To be very specific, Seidel's representation is naturally defined on a covering of the fundamental group which will be explained below. Since (Lalonde, McDuff, and Polterovich 1999) and (McDuff 2000), one can work directly on $\pi_1(\mathrm{Ham}(M\,,\omega))$. This change of perspective corresponds to choosing a favourite lift of any element of the fundamental group; this is very similar and related to Remark I.17. It will be made explicit in the case of toric manifolds in Remark III.12. ∎

The idea behind the equivalence between the algebraic and geometric viewpoints is easy to describe. Let ϕ denote a loop of Hamiltonian diffeomorphisms based at identity. The automorphism of Floer homology $\widetilde{\phi}_*$, induced by the homotopy class of ϕ in the algebraic interpretation, is the pair-of-pants multiplication by an element of $\mathrm{HF}_*(M)$. This element is thus invertible; it corresponds, via the PSS morphism, to $\mathcal{S}(\widetilde{\phi})$, the invertible of $\mathrm{HQ}_*(M)$ associated with $\widetilde{\phi}$ by the geometric version. In short,

$$\widetilde{\phi}_*\colon\ \mathrm{HF}_*(M) \longrightarrow \mathrm{HF}_*(M)\,, \qquad \alpha \longmapsto \mathrm{PSS}\big(\mathcal{S}(\widetilde{\phi})\big) * \alpha\,.$$

Both descriptions have been adapted to the Lagrangian quantum and Floer homologies of a monotone$^+$ Lagrangian L. In this setting, the resulting morphisms are naturally defined on (a covering of) the fundamental group of $\mathrm{Ham}(M\,,\omega)$ *relative* to $\mathrm{Ham}(M\,,\omega\,;L)$, its subgroup consisting of those Hamiltonian diffeomorphisms which preserve L globally. The algebraic version

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big) \longrightarrow \mathrm{Aut}\big(\,\mathrm{HF}_*(L)\big) \tag{III.2}$$

appeared in (Hu and Lalonde 2010), and the geometric version

$$\pi_1\big(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)\big) \longrightarrow \mathrm{HQ}_*(L)^\times$$

in (Hu, Lalonde, and Leclercq 2011). The equivalence between the two Lagrangian viewpoints is also proved in the latter.

Because of the applications discussed in this HDR memoirs, we will present the Lagrangian algebraic description (III.2) under monotonicity assumption in Section 1.1 and the geometric description in the "absolute" case (III.1) under the more general NEF assumption in Section 2.2.

Organization of and contribution to Chapter III

This third chapter gathers results around the Seidel representation obtained in disjoint collaborations. The chapter is fairly straightforwardly divided into two parts. Since the topics of each of these two parts are quite different, they are only overviewed here and properly introduced at the beginning of each section.

Section 1 exposes a rigidity result obtained with Shengda Hu and François Lalonde (Hu, Lalonde, and Leclercq 2011). By definition, a Hamiltonian diffeomorphism is isotopic to identity and thus it acts trivially on the homology of the ambient manifold M. However, when it preserves a Lagrangian submanifold L whose homology does not inject into that of M, there is no reason why it should act trivially on the homology of L. As a matter of fact, there exist Hamiltonian diffeomorphisms which interchange the two generators of the degree 1 homology group of certain monotone Lagrangian tori in $\mathbb{CP}^2$. Our main result shows that this can not happen when L is assumed to be aspherical. Equivalently, it says that only those diffeomorphisms of L which induce the identity in homology can be extended to Hamiltonian diffeomorphism of the total space. The proof is based on the Lagrangian algebraic description (III.2) of the Seidel morphism.

Section 2 covers work in collaboration with Sílvia Anjos (Anjos and Leclercq 2017, 2018). In this work, we compute explicitly all the Seidel elements of certain toric symplectic 4-dimensional manifolds. Indeed, these manifolds can be represented by convex 2-dimensional polytopes. Edges of such a polytope correspond to Hamiltonian circle actions, which are generated by loops of Hamiltonian diffeomorphisms based at identity. Building on results of McDuff and Tolman (McDuff and Tolman 2006), we show that when the toric symplectic manifold admits a NEF almost complex structure, we can compute explicitly the Seidel elements associated with these Hamiltonian circle actions. This allows us for example to determine specific loops of Hamiltonian diffeomorphisms of certain blow-ups of $\mathbb{CP}^2$ whose homotopy class is in the kernel of the Seidel morphism. We also use our computations of the Seidel elements to get explicit descriptions of the quantum homology ring of NEF toric symplectic 4-dimensional manifolds. In particular, we can read the Landau–Ginzburg superpotential of certain of these manifolds directly from their associated polytopes.

1. Homological rigidity of Lagrangian monodromy

Let L be a Lagrangian submanifold of a symplectic manifold (M,ω). The natural question which motivated (Hu, Lalonde, and Leclercq 2011) is the following. Recall that we denote by $\mathrm{Ham}(M,\omega;L)$ the subgroup of the Hamiltonian diffeomorphism group consisting of diffeomorphisms which preserve L. Restriction to L yields the morphism

$$\mathrm{Ham}(M,\omega;L) \longrightarrow \mathrm{Diff}(L)\,, \quad \phi \longmapsto \phi|_L$$

whose image we denote by $\mathcal{G}$. It induces $\mathcal{G}_*$ a subgroup of $\mathrm{Aut}(\mathrm{H}_*(L))$ and the question is : How big is $\mathcal{G}_*$?

Theorem III.2. *If L is an aspherical Lagrangian in (M,ω), then $\mathcal{G}_*$ is trivial, i.e. if a Hamiltonian diffeomorphism preserves L, it acts trivially on its homology.*

Notice that the contrapositive version of the result is an interesting obstruction to extending a given diffeomorphism of a Lagrangian to a Hamiltonian diffeomorphism of the ambient space[1]. It was for example used by Varolgunes (Varolgunes 2016), in conjunction with Polterovich's Lagrangian surgery (Polterovich 1991), to prove that specific Dehn twists of certain immersed Lagrangian spheres do not extend to Hamiltonian diffeomorphisms.

Disclaimer III.3. Yet another disclaimer ... about coefficient. Theorem III.2 was proved for coefficient in $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{Q}$. As in the previous chapters, we focus here on $\mathbb{Z}/2\mathbb{Z}$

1. Of course, while "being interesting" is subjective, let us point out that it is objectively *not* interesting when the homology of L injects in the homology of M.

coefficient, *i.e.* $\mathcal{G}_*$ is here a subgroup of $\mathrm{Aut}(\mathrm{H}_*(L\,;\mathbb{Z}/2\mathbb{Z}))$. However, in order to discuss related results below, we denote by $\mathcal{G}_{*,R}$ the relevant subgroup of $\mathrm{Aut}(\mathrm{H}_*(L\,;R))$. ▮

Theorem III.2 is yet another manifestation of the fact that asphericity tends to rigidify drastically symplectic objects. Let us examine two different situations.

The most rigid case is that of *exact* Lagrangian submanifolds of cotangent bundles T^*W. Recall that the *nearby Lagrangian conjecture* states that any closed exact Lagrangian L in T^*W is Hamiltonian-isotopic to the 0-section. This conjecture is still wide open : it is known to hold only for S^2 by results of Hind (Hind 2004) and for $\mathbb{T}^2$ by results of Dimitroglou-Rizell, Goodman, and Ivrii (Dimitroglou Rizell, Goodman, and Ivrii 2016). However, its homological counterpart (with coefficients in $\mathbb{Q}$) was proved by Fukaya, Seidel, and Smith (Fukaya, Seidel, and Smith 2008) and by Nadler (Nadler 2009). More precisely, they showed that when W is simply connected and L has Maslov class zero, the canonical projection $\pi\colon T^*W \to W$ induces an isomorphism $(\pi|_L)_*\colon \mathrm{H}_*(L\,;\mathbb{Q}) \to \mathrm{H}_*(W;\mathbb{Q})$ and the triviality of $\mathcal{G}_{*,\mathbb{Q}}$ follows. The additional assumptions were later removed respectively by Abouzaid (Abouzaid 2012) and Kragh (Kragh 2013).

On the other hand, monotonicity provides a much softer environment. Natural examples were constructed by Yau (Yau 2009) showing that it was possible to interchange the generators of the degree 1 homology group with integer coefficients, $\mathrm{H}_1(\mathbb{T}^2\,;\mathbb{Z})$, of certain monotone Lagrangian tori of $\mathbb{CP}^2$ (namely, the Chekanov torus and the standard monotone torus). Thus, in these cases $\mathcal{G}_{*,\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z}$.

1.1. The Seidel morphism (1 – algebraic description)

We now present the algebraic description of the Lagrangian Seidel representation.

Disclaimer III.4. While the proof of Theorem III.2 explained below is carried out under the assumption of asphericity, we present the construction for monotone Lagrangians. There are several reasons for this. Not only all required material was introduced in Chapter I, but also the *presentation* of the construction is not much more complicated than in the aspherical case. The main reason, however, is the following fact which is central in the proof of Theorem III.2 : when L is aspherical, the seidel morphism is trivial. Thus, it seemed rather pedantic to spend the rest of this section defining the constant map $\widetilde{\phi} \mapsto \mathrm{Id}$, in such a complicated way. ▮

Let L be a monotone$^+$ Lagrangian of $(M\,,\omega)$. Our goal is to define the Seidel representation relative to L which is a morphism

$$\widetilde{\pi}_1\big(\mathrm{Ham}(M,\omega), \mathrm{Ham}(M,\omega;L)\big) \longrightarrow \mathrm{Aut}\big(\mathrm{HF}_*(L)\big)\,, \tag{III.3}$$

where $\widetilde{\pi}_1$ is a covering of the fundamental group which we now explain.

The main idea behind the whole construction is very similar to what was explained about the symplectic action on quantum homology in Section 6.1 of Chapter I, the only difference being that we want to use Hamiltonian *isotopies* to act on Floer homology rather than “only” their time-1 map. Thus, while it is as easy to transform a Hamiltonian chord x^0 of a given Hamiltonian into a Hamiltonian chord x^1 of another, adapted, Hamiltonian, there is no canonical way to obtain a capping for x^1 from a capping of x^0. This is were the covering of the relative fundamental group above naturally appears : lifts of the Hamiltonian isotopy to the covering will correspond to (non-canonical) ways to do that.

Let us be more specific. Let $\mathcal{P}_L\mathrm{Ham}(M\,,\omega)$ be the set of Hamiltonian isotopies[2] starting at identity and ending in $\mathrm{Ham}(M\,,\omega\,;L)$, and pick $\phi \in \mathcal{P}_L\mathrm{Ham}(M\,,\omega)$. With a path γ in M with extremities in L, we associate the path γ^ϕ defined by $\gamma^\phi(t) = \phi_t(\gamma(t))$. This path starts at $\gamma(0) \in L$ and ends in L. Moreover, if $[\gamma] = 0$ in $\pi_1(M\,,L)$, then so does γ^ϕ ; this was initially proved in the aspherical case by Bialy and Polterovich (Bialy and Polterovich 1992). Hence, $\gamma \mapsto \gamma^\phi$ restricts to a map from Ω_L to itself.

2. We replace any given isotopy of Hamiltonian diffeomorphisms by a smooth isotopy (which is thus a Hamiltonian isotopy) in the same homotopy class relative to endpoints.

Moreover, the action of ϕ can be lifted to a homeomorphism of $\widetilde{\Omega}_L$ and we can define $\widetilde{\mathcal{P}}_L\mathrm{Ham}(M,\omega)$ as the set of pairs $\widetilde{\phi} = (\phi, \widehat{\phi})$ where $\widehat{\phi}$ lifts ϕ, *i.e.* for all $[\gamma, \widehat{\gamma}] \in \widetilde{\Omega}_L$, $\widehat{\phi}(\widehat{\gamma})$ is a capping of γ^{ϕ}. The covering $\widetilde{\pi}_1(\mathrm{Ham}(M,\omega), \mathrm{Ham}(M,\omega;L))$ is defined as $\pi_0(\widetilde{\mathcal{P}}_L\mathrm{Ham}(M,\omega))$.

Now, denote by $K\colon M \times [0,1] \to \mathbb{R}$ the generator of the isotopy $\phi = \phi_K$. Let (H,J) be a regular Floer datum for L and define the Lagrangian Floer datum (H^{ϕ}, J^{ϕ}) by setting $H^{\phi} = K + H \circ \phi_K^{-1}$ and $J^{\phi} = \phi_* J$. This new Floer datum is also regular for L. Moreover, recall from Section 1.1.1, that $H^{\phi} = K \circ H$ generates the Hamiltonian isotopy $\phi_K^t \circ \phi_H^t$. Thus, if x is a Hamiltonian chord of H, then x^{ϕ} is a Hamiltonian chord of H^{ϕ}. Hence, any lift $\widetilde{\phi}$ of ϕ induces a map

$$\widetilde{\phi}\colon \mathrm{Crit}(\mathcal{A}_{H:L}) \longrightarrow \mathrm{Crit}(\mathcal{A}_{H^{\phi}:L}), \quad [x, \widehat{x}] \longmapsto [x^{\phi}, \widehat{\phi}(\widehat{x})]$$

which yields an isomorphism of vector spaces $\widetilde{\phi}_*\colon \mathrm{CF}_*(L;H) \to \mathrm{CF}_{*'}(L;H^{\phi})$. It is not hard to see that starting from a Floer half-tube $u \in \mathcal{M}_F(\widetilde{x}, \widetilde{y}; L; H, J)$, the same procedure produces an element $u^{\phi}\colon (s,t) \mapsto \phi_t(u(s,t))$ in $\mathcal{M}_F(\widetilde{\phi}(\widetilde{x}), \widetilde{\phi}(\widetilde{y}); L; H^{\phi}, J^{\phi})$ so that $\widetilde{\phi}$ identifies the complexes $(\mathrm{CF}_*(L;H), \partial_{J,H})$ and $(\mathrm{CF}_*(L;H^{\phi}), \partial_{J^{\phi},H^{\phi}})$. This ensures that $\widetilde{\phi}_*$ descends to an isomorphism, also denoted $\widetilde{\phi}_*$, in homology. The Seidel morphism is then defined by $\widetilde{\phi} \mapsto \widetilde{\phi}_*$.

Remark III.5. The proof of the fact that it only depends on the homotopy class relative to endpoints of $\phi \in \mathcal{P}_L\mathrm{Ham}(M,\omega)$ is very similar to the last claim of the proof of Theorem I.22. We omit it completely. ∎

Remark III.6. We can identify further the various moduli spaces entering the definition of the morphisms and structures described in Chapter I for carefully chosen auxiliary data (Floer data, triple of data for the product, Floer and quantum data for the module structure and so on). This shows that the map $\widetilde{\phi}_*\colon \mathrm{HF}_*(L) \to \mathrm{HF}_{*'}(L)$ "agrees" with all these nice features.

Additionally, we can also prove with the same techniques that ϕ_* is a *module* morphism when $\mathrm{HF}_*(L)$ is seen as a module over $\mathrm{HQ}_*(L)$. This extra "mixed" module structure has a direct description by counting elements in suitable moduli spaces[3]. Moreover, it can also be seen as the composition of the PSS morphism together with the Floer product,

$$\begin{array}{ccc} \mathrm{HQ}_*(L) \otimes \mathrm{HF}_{*'}(L) & \xrightarrow{\quad\bullet\quad} & \mathrm{HF}_{*+*'-n}(L) \\ & {\scriptstyle \mathrm{PSS}\times\mathrm{Id}} \searrow \quad \nearrow {\scriptstyle *} & \\ & \mathrm{HF}_*(L) \otimes \mathrm{HF}_{*'}(L) & \end{array}$$

(but in this case the compatibility with ϕ_* is harder to show). ∎

There are obvious similarities between this construction and the symplectic action on Lagrangian quantum homology, presented in Section 6.1 of Chapter I. The interactions between the two are central in the proof of Theorem III.2 which occupies the next section.

1.2. Vanishing of the monodromy in the aspherical case

In this section, we consider a symplectic manifold (M,ω) and an aspherical Lagrangian submanifold L. Recall that under this assumption, there are no non-constant pseudo-holomorphic discs in M with boundary in L. In particular, the quantum complex is identified with the Morse complex. On the Floer side, the action functional is

3. Totally unsurprisingly, as we wish to relate generators of the quantum and Floer complexes to each other, the moduli spaces in question are formed of mixed objects, part string of pearls as in the definition of the quantum differential and part Floer half-tube as in the definition of the Floer differential.

well-defined on Ω_L since any two cappings of the same chord have same symplectic area, and the Floer complex is generated by the Hamiltonian chords of H themselves.

Let $\phi \in \mathcal{P}_L\mathrm{Ham}(M, \omega)$ be a Hamiltonian isotopy whose time-1 map is denoted ϕ^1. With L being aspherical and preserved by ϕ^1, the symplectic action of the latter (see Section 6.1 of Chapter I) coincides with the action of the diffeomorphism $\phi^1|_L$ on the Morse homology of L, *i.e.*, under asphericity, Diagram (I.11) reads

(III.4)
$$\begin{array}{ccc} \mathrm{HM}_*(L) & \xrightarrow{(\phi^1|_L)_*} & \mathrm{HM}_*(L) \\ \downarrow{\scriptstyle \mathrm{PSS}} & & \downarrow{\scriptstyle \mathrm{PSS}} \\ \mathrm{HF}_*(L) & \xrightarrow{\phi^1_*} & \mathrm{HF}_*(L)\,. \end{array}$$

Next, we link the symplectic action of ϕ^1 and the automorphism ϕ_* obtained by Seidel's construction. Recall from Section 4.1.4 of Chapter I that there is a Floer-theoretical version of Poincaré duality, which basically consists in considering the same geometric objects but with reversed orientation. The relation between ϕ_* and ϕ^1_* is illustrated on Figure III.1. The idea is that, in order to send a chord x to $\phi^1(x)$, one can first send it to x^ϕ via Seidel's construction. Then, up to reversing orientations (twice!) by duality, one gets $\phi^1(x)$ from the automorphism obtained as the Seidel image of $\overline{\phi}$, defined by $\overline{\phi}^t = \phi^1 \circ (\phi^{1-t})^{-1}$.

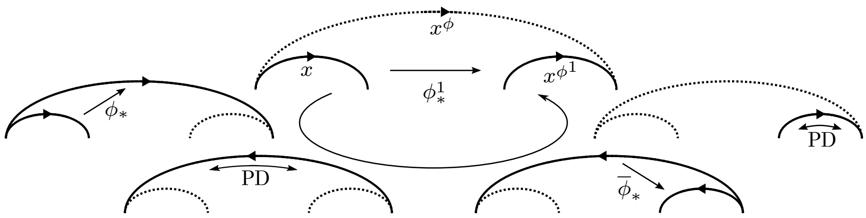


FIGURE III.1. Relation between Seidel's morphism and symplectic action.

It is straightforward to check that $(\overline{H^\phi})^{\overline{\phi}} = \overline{H^{\phi^1}} = \overline{H}^{\phi^1}$ and $(\overline{J^\phi})^{\overline{\phi}} = \overline{J^{\phi^1}} = \overline{J}^{\phi^1}$ (see notation from Section 4.1.4), and that all the involved pairs are regular as soon as (H, J) is. Applying the same ideas to the various required objects, one can show that moduli spaces are pairwise identified so that the following diagram of chain complexes commutes :

(III.5)
$$\begin{array}{ccccc} \mathrm{CF}_*(L; H, J) & \xrightarrow{\quad\phi^1_*\quad} & \mathrm{CF}_*(L; H^{\phi^1}, J^{\phi^1}) & \overset{\mathrm{PD}}{\to} & \mathrm{CF}^{n-*}(L; \overline{H}^{\phi^1}, \overline{J}^{\phi^1}) \\ & \searrow{\scriptstyle \phi_*} & \nearrow{\scriptstyle \cdots} & & \nearrow{\scriptstyle \overline{\phi}_*} \\ & & \mathrm{CF}_*(L; H^\phi, J^\phi) & \overset{\mathrm{PD}}{\to} & \mathrm{CF}^{n-*}(L; \overline{H}^\phi, \overline{J}^\phi) \end{array}$$

The dotted arrow, usually denoted $(\overline{\phi}_*)_!$, is defined by the commutativity of the right square.

Now comes the main point of the proof.

Lemma III.7. *When the Lagrangian L is aspherical, Seidel's morphism is trivial, i.e. for all $\phi \in \mathcal{P}_L\mathrm{Ham}(M, \omega)$, $\phi_* = \mathrm{Id}$.*

Note that this is equivalent to saying that ϕ_* acts as the usual Floer continuation morphism. Recall from Diagram (I.8) that continuation morphisms commute with duality. Thus, assuming Lemma III.7, we can replace ϕ_* and $\overline{\phi}_*$ by continuation morphisms in Diagram (III.5) which then shows that the symplectic action of ϕ^1 also coincides with continuation morphism on the complexes. Thus it induces the identity in homology and Theorem III.2 follows from Diagram (III.4).

It only remains to prove Lemma III.7. This can be done via a very simple algebraic trick. Recall from Remark III.6 that $\mathrm{HM}_*(L)$ is a ring over which $\mathrm{HF}_*(L)$ is a module and that ϕ_* is a module morphism. The PSS morphism is also a module morphism because, in view of the alternative description of the module structure given in Remark III.6, this is equivalent to the fact that it intertwines the quantum and Floer product structures. This yields the following diagram of $\mathrm{HM}_*(L)$-modules

$$\begin{array}{ccc} \mathrm{HM}_*(L) & \xrightarrow{\text{PSS}} & \mathrm{HF}_*(L) \\ {\scriptstyle \text{PSS}}\big\downarrow & \nearrow_{\phi_*} & \\ \mathrm{HF}_*(L) & & \end{array}$$

which we now show to commute. Let $\Phi = (\mathrm{PSS})^{-1} \circ \phi_* \circ \mathrm{PSS}$. Recall that $[L] \in \mathrm{HM}_n(L)$ is the unit of *the ring* $\mathrm{HM}_*(L)$. Since $[L]$ generates $\mathrm{HM}_n(L)$ (due to $\mathbb{Z}_2$ coefficients, signs are arbitrary), $\Phi([L]) = [L]$. On the other hand, since $[L]$ is also the unit, we have for any $a \in \mathrm{HM}_*(L)$:

$$\Phi(a) = \Phi(a \cdot [L]) = a \cdot \Phi([L]) = a \cdot [L] = a \,.$$

Thus Φ is the identity and the diagram above commutes. As the PSS morphism commutes with the usual comparison morphisms, the diagram shows that ϕ_* coincides with the continuation morphism on Floer homology. Lemma III.7 is proved.

2. Symplectic invariants of toric 4-manifolds

In this section, which covers works in collaboration with Sílvia Anjos (Anjos and Leclercq 2017, 2018), we explain how to effectively compute the Seidel morphism for certain toric manifolds.

A symplectic $(2n)$-dimensional manifold is said to be *toric* if it admits a Hamiltonian action of a n-dimensional torus. This action allows to represent the manifold by a particularly friendly convex polytope in $\mathbb{R}^n$, from which many invariants of the manifolds can be extracted. Since symplectic toric manifolds satisfying all types of symplectic and complex assumptions (monotone, Fano, NEF, ...) can easily be built, they form a family which is quite useful in order to perform explicit computations of abstract constructions.

Disclaimer III.8. In what follows, "the first Chern number of a sphere in M" loosely refers to $\langle c_1(TM\,,J)\,,[u(S^2)]\rangle$ where u is a map $u\colon\, S^2 \to M$ (aka "the sphere") and J is an almost complex structure on TM. If no such structure was specified, please pick any, compatible with ω. ∎

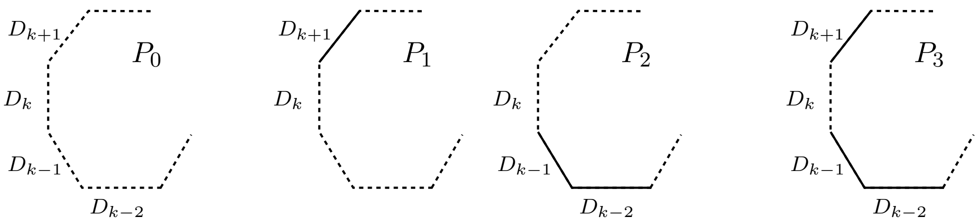
Our computations build on work of McDuff and Tolman (McDuff and Tolman 2006) who specified the structure of the Seidel element associated with a single Hamiltonian *circle* action whose certain fixed point component, denoted $F_{\max}$, is semifree[4]. When $F_{\max}$ has codimension 2, their result immediately ensures that if there exists an almost complex structure J on M so that $(M\,,J)$ is Fano, *i.e.* all J-pseudo-holomorphic spheres in M have positive first Chern number, the Seidel element consists of a single term involving the homology class of $F_{\max}$ (and all lower order terms vanish).

In the presence of J-pseudo-holomorphic spheres with vanishing first Chern number, there is a priori no reason why arbitrarily large multiple coverings of such objects should not contribute to the Seidel elements. As a matter of fact, McDuff and Tolman exhibited an example of such a phenomenon when M admits an almost complex structure so that $(M\,,J)$ is NEF, *i.e.* so that there are no J-pseudo-holomorphic spheres in M with negative first Chern number.

In Section 2.2 below, we explain that even though there *are* infinitely many contributions to the Seidel elements associated with the Hamiltonian circle actions of a NEF 4-dimensional toric manifold, not only these quantum classes can be expressed by explicit closed formulae, but also these formulae are readable from the polytope.

4. Recall that this condition means that the action is semifree on a neighborhood of $F_{\max}$, or equivalently here that the stabilizer of each point is trivial or the whole circle.

Let us for example consider four NEF polytopes which we schematically represent below. The actual polytopes are closed but we do not care about the edges which are not drawn. Among the drawn edges, only those represented with non-doted lines correspond to spheres in the manifold with vanishing first Chern number. The edge D_k is the edge corresponding to the Hamiltonian circle action Λ_k whose associated Seidel element we want to compute. We denote by A_i the homology class of the sphere in M corresponding to the edge D_i.



The left-most polytope, P_0, is Fano and McDuff and Tolman's result ensures that $\mathcal{S}_{P_0}(\Lambda_k) = A_k \otimes qt^{\Phi_{\max}} \in \mathrm{HQ}_4(M;\Gamma)^\times$ for some $\Phi_{\max} \in \mathbb{R}$.

The polytope P_1 is not Fano. Exactly one of the drawn edges corresponds to a sphere with vanishing first Chern number, D_{k+1}. Thus, not only A_k contributes to $\mathcal{S}_{P_1}(\Lambda_k)$ but also lA_{k+1}, for each integer $l > 0$. In (Anjos and Leclercq 2018), we showed that the contribution of lA_{k+1} is $-A_{k+1} \otimes qt^{\Phi_{\max}-\omega(lA_{k+1})} \in \mathrm{HQ}_4(M;\Gamma)$. Thus we obtain that the Seidel element is

$$\mathcal{S}_{P_1}(\Lambda_k) = A_k \otimes qt^{\Phi_{\max}} - A_{k+1} \otimes q\frac{t^{\Phi_{\max}-\omega(A_{k+1})}}{1-t^{-\omega(A_{k+1})}}\,.$$

In the polytope P_2, exactly two drawn edges correspond to spheres with vanishing first Chern number, D_{k-1} and D_{k-2}. Thus not only A_k and lA_{k-1} (for each $l > 0$) contribute to $\mathcal{S}_{P_2}(\Lambda_k)$ but also $lA_{k-1} + mA_{k-2}$, for each pair of positive integers $(l\,, m)$. Again, we computed each of these additional contributions : $lA_{k-1} + mA_{k-2}$ contributes by $B \otimes qt^{\Phi_{\max}-\omega(lA_{k-1}+mA_{k-2})}$, where $B = -A_{k-1}$ if $l \geq m$ and A_{k-2} otherwise. The Seidel element is thus given by

$$\begin{aligned}\mathcal{S}_{P_2}(\Lambda_k) &= A_k \otimes qt^{\Phi_{\max}} - A_{k-1} \otimes q\frac{t^{\Phi_{\max}-\omega(A_{k-1})}}{1-t^{-\omega(A_{k-1})}} \\ &\quad - \left(A_{k-1} \otimes q\frac{t^{\Phi_{\max}}}{1-t^{-\omega(A_{k-1})}} - A_{k-2} \otimes q\frac{t^{\Phi_{\max}-\omega(A_{k-2})}}{1-t^{-\omega(A_{k-2})}}\right)\frac{t^{-\omega(A_{k-1})-\omega(A_{k-2})}}{1-t^{-\omega(A_{k-1})-\omega(A_{k-2})}}\,.\end{aligned}$$

Notice that the first line coincides with the previous case while the terms of the second line are the additional contributions of the classes of the type $lA_{k-1} + mA_{k-2}$.

Exercise III.9. Based on the expressions of $\mathcal{S}_{P_1}(\Lambda_k)$ and $\mathcal{S}_{P_2}(\Lambda_k)$ above, guess the expression of $\mathcal{S}_{P_3}(\Lambda_k)$, the Seidel element associated with the circle action Λ_k in the manifold corresponding to the polytope P_3.

The next natural case to consider consists of polytopes for which the edge D_k itself corresponds to a sphere with vanishing first Chern number. To illustrate this situation, in Section 2.2, we explain how to get the formula for $\mathcal{S}(\Lambda_k)$ when D_k and D_{k+1} correspond to spheres with vanishing first Chern number (while D_{k-1} and D_{k+2} do not).

In Section 2.3, by building further on McDuff and Tolman's ideas, we use the computations of the Seidel elements to determine explicit presentations of the quantum cohomology ring[5] of NEF symplectic toric 4-dimensional manifolds. Indeed, they showed that there is a ring isomorphism

$$\mathrm{HQ}^*(M;\check{\Gamma}) \simeq \mathbb{Q}[Z_1,\ldots,Z_n] \otimes \check{\Gamma}/(\mathrm{Lin}(P) + \mathrm{SR}_Y(P))$$

5. One can safely think of it as dual to $\mathrm{HQ}_*(M;\Gamma)$ from Section 3.2 of Chapter I.

where n is the number of facets of the polytope, the ideal $\mathrm{Lin}(P)$ can easily be read on the polytope, while the expression of the ideal $\mathrm{SR}_Y(P)$ can be obtained thanks to the expressions of the Seidel elements associated with all circle actions.

By using this presentation, we can in turn compute the *Landau–Ginzburg superpotential* of certain NEF toric 4-dimensional manifolds. This superpotential, which in the 4-dimensional case is a Laurent polynomial in two variables z_1 and z_2, also yields a presentation of the quantum cohomology ring of the manifold : a result of Givental (Givental 1996) shows that there is an isomorphism

$$\mathrm{HQ}^*(M;\check{\Gamma}) \simeq \check{\Gamma}[z_1^{\pm}, z_2^{\pm}]/J_W$$

where J_W is the ideal generated by all partial derivatives of W. In the Fano case, the expression of W is known and can easily be extracted from the polytope. In the NEF case, there are again additional contributions coming from facets corresponding to spheres with vanishing first Chern number. These contributions are given by Theorem III.15, when such facets come in groups of at most two consecutive ones.

Remark III.10. This work is closely related to works by Fukaya, Oh, Ohta, and Ono (Fukaya et al. 2010), González and Iritani (González and Iritani 2012), and Chan, Lau, Leung, and Tseng (Chan et al. 2017). Roughly speaking, Fukaya et al. gave a presentation of the quantum homology of toric NEF symplectic manifolds using the Jacobian of some "open Gromov–Witten potential". Chan et al. proved that this potential actually coincides with the Hori–Vafa superpotential and that, in this presentation, the Seidel elements correspond to simple explicit monomials. This result which builds on a similar result by González and Iritani, via mirror symmetry, generalizes ours (this is not limited to dimension 4). However in dimension 4, our approach is more elementary and *stays on the symplectic side of the mirror*. This explains in particular why our results on the Landau–Ginzburg potential (luckily!) agree up to changes of variables with those obtained by Chan and Lau (Chan and Lau 2014). ▮

Finally, in Section 2.4, we describe several examples of toric manifolds for which we compute explicitly all Seidel elements, the Landau–Ginzburg superpotential, etc. This allows us to find examples of manifolds on which the Seidel morphism is injective, see Theorem III.19, and examples on which it is not, see Theorem III.18. In the latter case, we can also exhibit an explicit element of its kernel. As a final remark, let us emphasize that the examples of Section 2.4.3 for which we compute all Seidel elements are "not even" NEF.

But first, let us describe the geometric version of the Seidel morphism.

2.1. The Seidel morphism (2 – geometric description)

In this section, we explain the construction of the *geometric* version of Seidel's morphism, *i.e.* the morphism $\mathcal{S}\colon\ \widetilde{\pi}_1(\mathrm{Ham}(M,\omega)) \to \mathrm{HQ}_*(M,\omega)^{\times}$. It is defined by counting pseudo-holomorphic sections of a Hamiltonian fibre bundle over S^2 with fibre M.

The first step is the construction of the fibre bundle via the *clutching construction* : a fibration over S^2 consists of two trivial fibrations over both hemispheres which are "glued" together along the equator thanks to the choice of a loop of automorphisms of the fibre. Starting from a symplectic manifold (M,ω), and a loop of symplectomorphisms ϕ based at identity, this produces a locally trivial symplectic fibration

$$(M,\omega) \longleftrightarrow (M_\phi,\Omega_\phi) \xrightarrow{\ \pi\ } (S^2,\omega_0)\,,$$

hence a symplectic fibre bundle. Its isomorphism class only depends on the homotopy class of ϕ. The family Ω_ϕ of symplectic forms parameterized by S^2 is a symplectic form on the vertical sub-bundle $TM_\phi^{\mathrm{vert}} = \ker(d\pi)$ of the tangent bundle TM_ϕ. It can be extended to a closed 2-form $\widetilde{\Omega}_\phi$ on M_ϕ if and only if $\phi \subset \mathrm{Ham}(M,\omega)$. In such a case, (M_ϕ,Ω_ϕ) is said to be a *Hamiltonian fibre bundle* and we see that the isomorphism classes of such objects are identified with $\pi_1(\mathrm{Ham}(M,\omega))$.

Next, we fix almost complex structures j on S^2 and J on M_ϕ in such a way that the projection is pseudo-holomorphic, $d\pi \circ J = j \circ d\pi$, and for all $z \in S^2$, the restriction of J to the fiber $\pi^{-1}(z)$ is a $(\Omega_\phi)|_{\pi^{-1}(z)}$-compatible almost complex structure. We also require them to satisfy additional regularity conditions which we omit (even though these conditions make the following work).

For a section class S in $\pi_2(M_\phi)$, we denote the set of (j, J)-pseudo-holomorphic spheres in the class S by $\mathcal{S}(j, J, S)$. The evaluation map at $z_0 \in S^2$ allows to define a *pseudo-cycle* in M, which roughly means that $\mathrm{ev}_{z_0}(\mathcal{S}(j, J, S)) \subset M$ can be compactified by addition of finitely many submanifolds of codimension at least 2. This pseudo-cycle yields in turn an honest (Morse) homology class $[\mathrm{ev}_{z_0}(\mathcal{S}(j, J, S))]$ see *e.g.* (Schwarz 1998). This class only depends on the equivalence class of S with respect to the following relation : S and S' are equivalent if $\widetilde{\Omega}_\phi(S) = \widetilde{\Omega}_\phi(S')$ and $c_1(TM_\phi^{\mathrm{vert}}, \Omega_\phi)(S) = c_1(TM_\phi^{\mathrm{vert}}, \Omega_\phi)(S')$.

Via the long exact sequence given in homotopy by the fibration, it is easy to see that two section classes differ by an element of the second homotopy group of the fibre. Hence, given a fixed section class S_0, any other section class is given as $S_A = S_0 + A$, with $A \in \pi_2(M)$. Moreover, two section classes S_A and $S_{A'}$ are equivalent if and only if A and A' have same symplectic area and same first Chern number in M, *i.e.* if $[A] = [A'] \in \Gamma$. Notice that since (M, ω) might not be monotone, the kernels of ω and c_1 do not necessarily coincide; hence here Γ is defined as the quotient of $\pi_2(M)$ by $\ker\omega \cap \ker c_1$. This finally defines an element

$$Q(M_\phi, \omega_\phi, S_0) = \sum_{[A]\in\Gamma} [\mathrm{ev}_{z_0}(\mathcal{S}(j, J, S_0 + A))] \otimes q^{-c_1(A)} t^{-\omega(A)}$$

in $\mathrm{HQ}_d(M, \omega)$ with $d = 2n + 2c_1(TM_\phi^{\mathrm{vert}}, \Omega_\phi)(S_0)$.

The final step is to provide *a priori* a fixed section class S_0 as above. As for the algebraic description, this is where we introduce the covering $\widetilde{\pi}_1(\mathrm{Ham}(M, \omega))$. The idea is the same as in Section 1.1. The set $\widetilde{\Omega}_M$ is formed of equivalence classes of capped orbits $(\gamma, \widehat{\gamma})$ where $(\gamma, \widehat{\gamma})$ and $(\gamma', \widehat{\gamma}')$ are equivalent if $\gamma = \gamma'$ and $A = [\widehat{\gamma}\sharp(-\widehat{\gamma}')] = 0 \in \Gamma$.

Then define $\widetilde{\Omega}\mathrm{Ham}(M, \omega)$ as the set of pairs $\widetilde{\phi} = (\phi, \widehat{\phi})$ which lift ϕ to a homeomorphism of $\widetilde{\Omega}_M$. The covering $\widetilde{\pi}_1(\mathrm{Ham}(M, \omega))$ is defined as $\pi_0(\widetilde{\Omega}\mathrm{Ham}(M, \omega))$. Notice that given a lift $\widetilde{\phi} = (\phi, \widehat{\phi})$, we can easily get a fixed section class $S_{\widetilde{\phi}}$ by "clutching" a representative of *any* element $\widetilde{\gamma}$ in $\widetilde{\Omega}_M$ together with a representative of its image $\widetilde{\phi}(\widetilde{\gamma})$. The image of $\widetilde{\phi} \in \widetilde{\pi}_1(\mathrm{Ham}(M, \omega))$ by the Seidel morphism is defined as $\mathcal{S}(\widetilde{\phi}) = Q(M_\phi, \omega_\phi, S_{\widetilde{\phi}})$.

Remark III.11. As mentioned at the beginning of Section 1.1, the automorphism of Floer homology $\widetilde{\phi}_*$ provided by the Seidel representation is the pair-of-pants multiplication by the image of $Q(M_\phi, \omega_\phi, S_{\widetilde{\phi}})$ in $\mathrm{HF}_d(M, \omega)$ via the PSS morphism. This immediately shows that $Q(M_\phi, \omega_\phi, S_{\widetilde{\phi}})$ is invertible. ∎

2.2. The Seidel morphism of NEF symplectic toric manifolds

In this section, we explain how one can compute the image by the Seidel morphism of a circle action, coming from a Hamiltonian toric action, on a 4-dimensional NEF toric symplectic manifold, by staring (intensely) at its associated polytope. This is the content of Theorem 4.5 of (Anjos and Leclercq 2018). The complete proof is quite long and involved (and sprinkled with tedious computations) so we will only explain its main steps, in a particular case which already presents all the difficulties.

More precisely, we consider a 4-dimensional closed symplectic manifold (M, ω), endowed with a toric structure. We denote by $P \subset \mathbb{R}^2$ the corresponding Delzant polytope which is assumed to have at least 4 edges.

Let Λ be a Hamiltonian action generated by a circle subgroup of the acting torus, and denote its associated moment map by Φ_Λ. We assume additionally, that the fixed point component of Λ on which Φ_Λ is maximal is a 2-sphere, $F_{\max}$. We denote the homology class of this sphere by $A \in \mathrm{H}_2(M;\mathbb{Z})$, the corresponding edge of P by D, and by $\Phi_{\max} = \Phi_\Lambda(F_{\max})$.

We assume further that M admits a NEF almost complex structure so that D and a neighboring facet D' correspond to spheres in M with vanishing first Chern number, while the two other facets intersecting either D or D' correspond to spheres with positive first Chern number. We let $A' \in \mathrm{H}_2(M;\mathbb{Z})$ denote the homology class of the sphere in M corresponding to the edge D' of P.

Under these assumptions and with this notation, we will see that the only section classes which contribute to the Seidel element associated with Λ are $kA + lA'$ with $k \geq 0$ and $l > 0$. We will also explain how to prove that their respective contribution is given by $a_{kA+lA'} \otimes qt^{w(k,l)}$ with $a_{kA+lA'} = A$ if $k \geq l$ and $-A'$ otherwise (and $w(k,l) = \Phi_{\max} - k\,\omega(A) - l\,\omega(A')$). In turn, we get that the associated Seidel element is

$$S(\Lambda) = \left(A \otimes q\,\frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A)}} - A' \otimes q\,\frac{t^{\Phi_{\max} - \omega(A')}}{1 - t^{-\omega(A')}} \right) \frac{1}{1 - t^{-\omega(A) - \omega(A')}} \,.$$

2.2.1. *Toric symplectic manifolds*

A closed symplectic $(2m)$-dimensional manifold (M, ω) is *toric* if it is equipped with an effective Hamiltonian action of a m-dimensional torus T. By definition of a *Hamiltonian action*, this comes with a *moment map*, *i.e.* a map $\Phi\colon M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$.

There is a natural integral lattice $\mathfrak{t}_\mathbb{Z}$ in $\mathfrak{t}$ whose elements H exponentiate to circles Λ_H in T, and hence also a dual lattice $\mathfrak{t}^*_\mathbb{Z}$ in $\mathfrak{t}^*$. By a result of Atiyah (Atiyah 1982), and Guillemin and Sternberg (Guillemin and Sternberg 1982), the image $\Phi(M)$ is a convex polytope P in $\mathfrak{t}^*$. Moreover this polytope is *Delzant*, that is, it is

simple : at each vertex meet exactly m *facets*, that is, $(m-1)$-dimensional faces,
rational : each facet admits a normal vector which may be chosen integral, and
smooth : at each vertex v the outward normals to the facets meeting at v form a basis of $\mathfrak{t}_\mathbb{Z}$.

Such a polytope can be described as

$$P = P(\kappa) = \{x \in \mathfrak{t}^* | \langle \eta_i, x\rangle \leq \kappa_i,\ i = 1, \dots, n\}$$

where P has n facets $D_1, \dots, D_n$ with outward primitive integral normals $\eta_i \in \mathfrak{t}_\mathbb{Z}$ and *support constants* $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n$.

Delzant (Delzant 1988) proved that there is a one-to-one correspondence between toric manifolds and Delzant polytopes up to reasonable equivalence relations, *i.e.* equivariant symplectomorphism of (M, ω, T, Φ) and translation of the polytope $\Phi(M) \subset \mathfrak{t}^*$.

So here, we are given a 4-dimensional toric symplectic manifold (M, ω, T, Φ).

2.2.2. *Hamiltonian fibre bundle associated with a circle action* Λ

When we start with a toric symplectic $(2m)$-dimensional manifold (M, ω, T, Φ), it turns out that the total space of the Hamiltonian fibre bundle obtained via the clutching construction of Section 2.1 is a $(2m+2)$-dimensional toric manifold whose polytope can easily be obtained from $\Phi(M)$. This was for example already used in a related context by González and Iritani (González and Iritani 2012).

In our case, the loop of Hamiltonian diffeomorphisms based at identity we are interested in is the circle action Λ, generated by some element $b \in \mathfrak{t}_\mathbb{Z}$. Then $(M_\Lambda, \Omega_\Lambda)$ is a 6-dimensional toric manifold whose associated Delzant polytope and integral lattice are

$$P_\Lambda = \{(x, x_0) \in (\mathfrak{t} \times \mathbb{R})^* \mid x \in P,\ c + \langle x, b\rangle \leq x_0 \leq 0\}$$

and $\mathfrak{t}^\Lambda_\mathbb{Z} = \mathfrak{t}_\mathbb{Z} \times \mathbb{Z} \subset \mathfrak{t} \times \mathbb{R}$ where $c > \max\{\langle x, b\rangle, x \in P\}$ depends on the choice of closed 2-form $\widetilde{\Omega}_\Lambda$ extending Ω_Λ. Moreover, the outward normals η_Λ of P_Λ are given in terms of those of P, η, as follows : $\eta^\Lambda = \{\eta_i^\Lambda = (\eta_i, 0) \mid \eta_i \in \eta\} \cup \{\eta_t^\Lambda = (0, 1), \eta_b^\Lambda = (b, -1)\}$.

Roughly speaking, the polytope associated with M_Λ is obtained from that of M as follows : each facet (edge !) D_i of P yields a *vertical* facet (a 2-dimensional face) D_i^Λ of P_Λ. It only remains to "close" the 3-dimensional polytope P_Λ with an upper *horizontal* facet D_t^Λ and a lower facet D_b^Λ, which is normal to $(b\,,-1) \in \mathfrak{t}^* \times \mathbb{R}$.

2.2.3. *The Seidel element associated with a circle action* Λ

Recall that M admits an almost complex structure J so that $(M\,,J)$ is NEF, that the maximal fixed point component of the moment map Φ_Λ associated with Λ corresponds to a divisor, which we denoted by $F_{\max}$, and that $\Phi_{\max} = \Phi_\Lambda(F_{\max})$ denotes the maximal value of the moment map associated with Λ. Notice that $F_{\max}$ is semifree (and of dimension 2) so that we can use the expression of the Seidel morphism associated with Λ given in (McDuff and Tolman 2006).

Remark III.12. First, we need a bit of preparation. As mentioned in Remark III.1, there is a way to define the Seidel morphism directly on the fundamental group of the Hamiltonian diffeomorphism group (and not on a covering). In view of the description of the Seidel morphism from Section 2.1, it amounts to canonically associating (an equivalence class of) a section class S_0 with a loop of Hamiltonian diffeomorphisms based at identity.

When the loop in question is a circle action Λ, it is convenient to do it as follows. Pick a fixed point x lying in $F_{\max}$. The "constant" discs $\{x\} \times D_\pm$ ($D_\pm$ denote the hemispheres of S^2) are then naturally glued during the clutching construction, yielding a section $\sigma_{\max}$ of M_Λ. The Seidel element associated with Λ can then be computed with respect to the equivalence class of $\sigma_{\max}$: $\mathcal{S}(\Lambda) = Q(M_\Lambda\,,\omega_\Lambda\,,\widetilde{\sigma}_{\max})$. Below, we denote $A_{\max} = [\sigma_{\max}] \in \mathrm{H}_2(M;\mathbb{Z})$. Any other section class is given by $A_{\max} + B$ with $B \in \mathrm{H}_2(M;\mathbb{Z})$. ▮

Disclaimer III.13. In Seidel's construction, we are interested in (the equivalence class of) the *homotopy* class of sections. However, they contribute to Seidel's elements via their *homology* class. It is thus convenient to consider the subgroup of $\mathrm{H}_2(M)$ consisting of spherical classes, *i.e.* the image of $\pi_2(M)$ via the Hurewicz map. Since toric manifolds are simply connected, all these groups are isomorphic in the present situation and we can safely consider the section classes as elements of $\mathrm{H}_2(M)$. ▮

Next we state the result we need from (McDuff and Tolman 2006). It aggregates and adapts to the present case Theorem 1.10, Lemma 2.2, Definition 2.4, and Lemma 3.10 of the latter.

Theorem III.14. *Under our assumptions, the Seidel element associated with the circle action* Λ *is*

$$S(\Lambda) = [F_{\max}] \otimes q t^{\Phi_{\max}} + \sum_{B \in \mathrm{H}_2(M;\mathbb{Z})^{>0}} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max} - \omega(B)} \tag{III.6}$$

where $\mathrm{H}_2(M;\mathbb{Z})^{>0}$ *consists of all classes of positive symplectic area and the contribution* $a_B \in \mathrm{H}_*(M;\mathbb{Z})$ *of the section class* $A_{\max} + B$ *is defined by requiring that* $a_B *_M c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c)$ *for all* $c \in \mathrm{H}_*(M;\mathbb{Z})$. *Moreover*,

(1) *if* $a_B \neq 0$ *then* B *intersects* $F_{\max}$, $c_1(B) = 0$, *and* $a_B \in \mathrm{H}_2(M;\mathbb{Z})$,
(2) *if* $c_1(B') \geq 1$ *for all* J*-holomorphic spheres* B' *which intersect* $F_{\max}$, *then all the lower order terms vanish*,
(3) *If* $c_1(B') \geq 1$ *for all* J*-holomorphic spheres* B' *which intersect* $F_{\max}$ *but are not included in* $F_{\max}$, *then* $a_B \neq 0 \Rightarrow c_1(B) = 0$.

Recall that the polytope $P = \Phi(M)$ admits $n \geq 4$ facets, $D_1\,,\ldots,D_n$. These facets correspond to divisors whose homology classes we respectively denote by $A_1\,,\ldots,A_n$ (with the convention that $\omega(A_i) \geq 0$). We set $A_n = [F_{\max}]$: in other words the facets D and D' from the introduction of Section 2.2 are D_n and D_1. They correspond via the moment map to spheres in M with vanishing first Chern number, while the first Chern number of the spheres corresponding to D_{n-1} and D_2 are positive.

For any n-tuple $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, we denote by $A_{\bar{a}} = \sum_i a_i A_i$ the homology class of the union of (possibly multiply covered) spheres in M whose projection to P is given by $D_{\bar{a}} = \cup_i D_i$. With this notation, Theorem III.14 in particular yields that if $A_{\bar{a}}$ contributes to $\mathcal{S}(\Lambda)$, *i.e.* the coefficient a_B in (III.6) is non-zero for $B = A_{\bar{a}}$, then

(1) $D_{\bar{a}}$ is connected and intersects D_n,
(2) $c_1(A_{\bar{a}}) = 0$ *i.e.* for all i so that $a_i \neq 0$, $c_1(A_i) = 0$.

Thus, we need to compute the contribution to $\mathcal{S}(\Lambda)$ of the section class $A_{\max} + B_{k,l}$ with $B_{k,l} = kA_n + lA_1 \in \mathrm{H}_2(M;\mathbb{Z})$. *A priori*, this means to compute the 1-point Gromov–Witten invariants of M_Λ in these specific section classes for any possible constraint $c \in \mathrm{H}_*(M;\mathbb{Z})$. In the next section, we explain that we can reduce the problem to the computations of very few specific Gromov–Witten invariants.[6]

2.2.4. *Reducing the computation to that of few, simple Gromov–Witten invariants*

Since P is Delzant, we can chose D_n and D_1 to be orthogonal, and thus so are their respective outward normals η_n and η_1. Consecutive normals satisfy the following relation : $\eta_{i-1} + \eta_{i+1} = d_i \eta_i$ where $d_i = -D_i \cdot D_i$ (the indices are set "mod n"). From this, we deduce the expression of the normals η_i, and in turn η_i^Λ, for $i = n-1$, n, 1, and 2. We also know the expression of η_t^Λ and η_b^Λ, the "top" and "bottom" facets of P_Λ respectively.

Then, we turn to the cohomology of M_Λ. It can be extracted from P_Λ in the following way, see also *e.g.* Cox and Katz (Cox and Katz 1999), or Batyrev (Batyrev 1993). There is a ring isomorphism

$$\mathrm{H}^*(M_\Lambda;\mathbb{Q}) \simeq \mathbb{Q}[Z_1, \dots, Z_n, Z_t, Z_b]/(\mathrm{Lin}(P_\Lambda) + \mathrm{SR}(P_\Lambda)) \tag{III.7}$$

where $Z_i \in \mathrm{H}^2(M_\Lambda;\mathbb{Q})$ is Poincaré dual to A_i^Λ, the degree 4 integral homology class of M_Λ, induced by the pre-image of the facet D_i^Λ via the moment map Φ_Λ.

The ideal $\mathrm{Lin}(P_\Lambda)$ of *linear relations* is given by the columns of a matrix whose rows consist of the η_i^Λ, while the *Stanley–Reisner ideal* $\mathrm{SR}(P_\Lambda)$ cancels products of cohomology classes corresponding to facets which do not intersect, *e.g.* Z_1Z_3, Z_1Z_4, ... Z_2Z_4, ... Z_bZ_t. In our specific case, it is then easy to see that Z_{n-1}, Z_n, and Z_t are linear combinations of the other classes so that Z_1, ... Z_{n-2}, and Z_b generate $\mathrm{H}^2(M_\Lambda;\mathbb{Q})$.

Going back to homology, it is then possible to show that A_1, and A_4 to A_n generate $\mathrm{H}_2(M;\mathbb{Z})$. In particular, the contribution to $\mathcal{S}(\Lambda)$ of $B_{k,l} = kA_n + lA_1$ with $k > 0$ and $l \geq 0$ can be written as $a_{B_{k,l}} = \sum_{i=1}^n a_i(k,l) A_i$ where $a_2(k,l) = a_3(k,l) = 0$.

Next, remember that the contribution of A_B is defined by requiring that $a_B *_M c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c)$ for all $c \in \mathrm{H}_*(M;\mathbb{Z})$. In particular, for $B_{k,l}$ we get

$$a_4(k,l) = a_{B_{k,l}} *_M A_3 = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_3) = 0$$

since A_3 does not intersect $B_{k,l}$. Now since $a_4(k,l) = 0$, the coefficient $a_5(k,l) = a_{B_{k,l}} *_M A_4 = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_4)$ which vanishes for the same reason. Going around the polytope as above, we see that all coefficients $a_i(k,l)$ for $3 \leq i \leq n-2$ vanish, and we get an expression of $a_1(k,l)$ and $a_n(k,l)$ in terms of Gromov–Witten invariants :

a) $a_1(k,l) = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_2)$, b) $a_n(k,l) - 2a_1(k,l) = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_1)$.

We now only need to prove that the pair of coefficients $(a_n(k,l), a_1(k,l)) = (1,0)$ if $k \geq l$, and $(0,-1)$ otherwise to get the expected result. This is done by computing $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_1)$ and $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(A_2)$ by induction.

6. "very few" might seem optimistic as there will remain countably many Gromov–Witten invariants to compute... However, for a fixed pair (k,l) there will be only two.

2.2.5. *Computing Gromov–Witten invariants by induction : 1. Base case*

We computed all base cases for $k = 1$, and $l = 0$ or 1 thanks to Spielberg's formula (Spielberg 1999b, 1999a) which allows to count genus-0 Gromov–Witten invariants in toric manifolds, see also Liu (Liu 2013) for a more general result. Spielberg's formula was established on the algebraic side of toric actions, based on (cones and) fans of cones. In our particular case, where we "only" want to count 1-point Gromov–Witten invariants, the general formula can be simplified. Still... the involved computations remain rather tedious.

Basically, to compute the 1-point Gromov–Witten invariant in class A with constraint given by a class B, one needs to consider all possible graphs in the fan (whose vertices are given by cones of the fan) which represent the class A and contain one of the cones which are "dual" to B. The whole combinatorial structure of each of such graphs is encoded in a set of weights, and Spielberg's formula combines all the weights of all possible graphs.

For example, given the section class $A_{\max}+A_n+A_1$ (*i.e.* for $k = l = 1$) and constraint A_1, there are 5 such graphs, each of which made of 4 vertices and 3 edges. Moreover the number of graphs and the complexity of each of them seem to explode with the complexity of the section class. This explains why we do not add more detail on this construction here (and also why only the base cases were computed via this method !).

2.2.6. *Computing Gromov–Witten invariants by induction : 2. Inductive step*

This step also amounts to several pages of tedious computations of all sorts. Hence, we only explain the main steps.

First, we switch to cohomology so that we now want to compute the 1-point Gromov–Witten invariants $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B_{k,l},1}(Z)$ when Z is Poincaré dual to A_1 and A_2, *i.e.* respectively $Z = Z_1 Z_b$ and $Z_2 Z_b$.

The inductive step is based on the *splitting axiom* of Gromov–Witten invariants which, in our case, allows to "split" 4-point Gromov–Witten invariants as a sum of products of 3-point Gromov–Witten invariants. Namely, given cohomology classes $\alpha_i \in \mathrm{H}^*(M;\mathbb{Q})$ for $i \in \{1\,,2\,,3\,,4\}$,

$$\mathrm{GW}^{M_\Lambda}_{A,4}(\alpha_1,\ldots,\alpha_4;\mathrm{pt}) = \sum_{A=A_0+A_1}\sum_{\nu,\mu}\mathrm{GW}^{M_\Lambda}_{A_0,3}(\alpha_1,\alpha_2,e_\nu)\,g^{\nu\mu}\,\mathrm{GW}^{M_\Lambda}_{A_1,3}(e_\mu,\alpha_3,\alpha_4)$$

where $(e_\nu)_\nu$ is a basis of $\mathrm{H}^*(M;\mathbb{Q})$, $g_{\nu\mu}$ are the coefficients of the cup-product matrix : $g_{\nu\mu} = \int_M e_\nu \cup e_\mu$, and $g^{\nu\mu}$ the coefficients of its inverse. We also use repeatedly two other properties of the Gromov–Witten invariants, the *zero axiom* and the *divisor axiom*. However, these do not appear explicitly in this sketch of proof.

The first step is to compute $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1\,,Z_b\,,Z_b\,,Z_1\,;\mathrm{pt})$ via the splitting axiom by considering the partition $\{(Z_1\,,Z_b)\,,(Z_1\,,Z_b)\}$. An unexpected difficulty arises from the fact that we need to compute specific coefficients of the cup-product matrix and of its inverse, even though we *can not* know the whole matrix. Indeed, we only imposed conditions on the facets D_{n-1}, D_n, D_1, and D_2 so that we only know the local picture. This turns out to be possible, via easy linear algebra considerations. The end result is a first expression of $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1\,,Z_b\,,Z_b\,,Z_1\,;\mathrm{pt})$, in terms of $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_b)$.

We then follow the same strategy but considering the partition $\{(Z_1\,,Z_1)\,,(Z_b\,,Z_b)\}$. Here, another difficulty appears due to certain 0-point Gromov–Witten invariants which we have to compute independently. It also turns out to be possible, thanks to Liu's localization techniques. We get a second expression of $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1\,,Z_b\,,Z_b\,,Z_1\,;\mathrm{pt})$, now in terms of $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_2)$, which depends on whether $k \geq l$ or $k < l$.

Comparing in each case the expressions of $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1\,,Z_b\,,Z_b\,,Z_1\,;\mathrm{pt})$, we obtain a relation between $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_b)$ and $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_2)$ (again, depending on whether $k \geq l$ or $k < l$).

We proceed along the same lines : starting with

- $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_2\,,Z_b\,,Z_b\,,Z_2\,;\mathrm{pt})$, we get relations between the Gromov–Witten invariants $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_2Z_b)$ and $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1Z_2)$,
- $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1\,,Z_b\,,Z_b\,,Z_2\,;\mathrm{pt})$, we get relations between the invariants $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1Z_b)$, $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_2Z_b)$, and $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1Z_2)$.

Finally, we combine all these relations into simple systems of linear equations. Solving these gives us the desired Gromov–Witten invariants for all k and l, as well as those associated to the constraint Z_1Z_2 (whose base cases we compute via Spielberg's formula as above !).

2.3. Quantum cohomology and the Landau–Ginzburg superpotential

We now explain how the previous computations of the Seidel elements help determining the quantum homology ring, and in turn the Landau–Ginzburg superpotential. This also follows from (McDuff and Tolman 2006) together with work by Ostrover and Tyomkin (Ostrover and Tyomkin 2009), which were themselves developments of original ideas due to Batyrev (Batyrev 1993) and Givental (Givental 1996, 1998).

2.3.1. *Quantum cohomology of NEF toric manifolds*

Recall from (III.7) that the rational cohomology of a symplectic toric manifold M is isomorphic to the quotient of the polynomial ring $\mathbb{Q}[Z_1\,,\dots,Z_n]$ by the ideal of linear relations and the Stanley–Reisner ideal. The isomorphism is given by mapping Z_i to Z_i, *i.e.* the variable Z_i to the cohomology class of M Poincaré dual to the homology class A_i induced by the facet D_i.

McDuff and Tolman proved that this has the following quantum adaptation :

$$\mathrm{HQ}^*(M;\check\Gamma)\simeq\mathbb{Q}[Z_1,\dots,Z_n]\otimes\check\Gamma/(\mathrm{Lin}(P)+\mathrm{SR}_Y(P))\,.$$

The first difference is that we consider here the *quantum* cohomology of $(M\,,\omega)$. It is defined similarly to its homological counterpart (compare with (I.4)) by $\mathrm{HQ}^*(M;\check\Gamma)=\mathrm{H}^*(M;\mathbb{Q})\otimes\check\Gamma$ where $\check\Gamma$ is the ring $\check\Gamma=\check\Gamma^{\mathrm{univ}}[q\,,q^{-1}]$ with

$$\check\Gamma^{\mathrm{univ}}=\Big\{\sum_{\kappa\in\mathbb{R}}r_\kappa t^\kappa\,\big|\,r_\kappa\in\mathbb{Q},\ \text{and for all } c\in\mathbb{R},\ \#\{\kappa>c\mid r_\kappa\neq0\}<\infty\Big\}.$$

The second difference concerns the Stanley–Reisner ideal. The usual Stanley–Reisner ideal is generated by monomials $Z_I=Z_{i_1}\dots Z_{i_k}$ with $2\le k\le\frac12\dim M$, whose index sets $I=\{i_1\,,\dots,i_k\}$ are *primitive*, *i.e.* the facets $D_{i_1}\,,\dots,D_{i_k}$ do not intersect while any proper subset of $\{D_{i_1}\,,\dots,D_{i_k}\}$ does.

In the Fano case, the *quantum* Stanley–Reisner ideal $\mathrm{SR}_Y(P)$ is generated by differences of monomials $Z_I-Z_{I^*}^{c_{I^*}}\otimes q^{c_1(\beta_I)}t^{\omega(\beta_I)}$, for all primitive index sets I. Here, $I^*=\{j_1\,,\dots,j_l\}$ is a subset of $\{1\,,\dots,n\}$ disjoint from I, $c_{I^*}\in(Z_{>0})^l$, and $\beta_I\in\mathrm{H}_2(M;\mathbb{Z})$; they are all uniquely determined by I via the combinatorics of P.

In the NEF case, the expression of $\mathrm{SR}_Y(P)$ is the same except that the classes Z_i are replaced by classes Y_i determined by the Seidel elements associated with all circle actions. More precisely, denote by $\mathcal{S}^*(\Lambda_i)$ the cohomology class Poincaré dual to the Seidel element $\mathcal{S}(\Lambda_i)$. Define y_i as the unique element of $\mathrm{HQ}^*(M;\check\Gamma)$ such that $\mathcal{S}^*(\Lambda_i)=y_i\otimes q^{-1}t^{-\eta_i(D_i)}$. The element $Y_i\in\mathbb{Q}[Z_1\,,\dots,Z_n]\otimes\check\Gamma$ is a lift of y_i. It is unique up to "$\check\Gamma$-linear combinations" of the Z_i which lie in $\mathrm{Lin}(P)$.

The conclusion is that, in the Fano case (in which one can set $Y_i=Z_i$) and in the NEF case (in which Y_i is unique up to $\mathrm{Lin}(P)$, the quantum cohomology of $(M\,,\omega)$ is determined by the image $\mathcal{S}(\Lambda_i)$, of the circle sub-actions by the Seidel morphism.

2.3.2. *The Landau–Ginzburg superpotential of NEF toric manifolds*

Here is another presentation of the quantum cohomology ring, via the *Landau–Ginzburg superpotential*, obtained by Givental (Givental 1996). When (M, ω) is a NEF symplectic manifold, there are ring isomorphisms

$$\text{(III.8)} \qquad \mathrm{HQ}^*(M; \check{\Gamma}) \simeq \check{\Gamma}[z_1^\pm, \dots, z_m^\pm]/J_W \quad \text{and} \quad \mathrm{HQ}^0(M; \check{\Gamma}) \simeq \check{\Gamma}^{\mathrm{univ}}[z_1^\pm, \dots, z_m^\pm]/J_W$$

where J_W is the Jacobian of the Landau–Ginzburg superpotential W, *i.e.* W is a Laurent polynomial and J_W is the ideal generated by all its partial derivatives. A useful feature of W is that, when (M, ω) is toric, W can be extracted from the polytope. For example, when (M, ω) is not only NEF but *Fano*, $W = \sum_{i=1}^n z^{\eta_i} t^{\kappa_i}$, *i.e.*

(1) it is the sum of n monomials, one for each facet D_i of the polytope P,
(2) the factor z^{η_i} of each monomial is $z^{\eta_i} = z_1^{\eta_{i,1}} \dots z_m^{\eta_{i,m}}$ with $\eta_i = (\eta_{i,1}, \dots, \eta_{i,m}) \in \mathbb{Z}^m$ is the outward primitive vector normal to D_i,
(3) the superscript κ_i is the constant defining the half-space delimited by D_i in P (for all $x \in P$, $\langle \eta_i, x\rangle \leq \kappa_i$).

In the NEF case, additional terms appear from the existence of facets corresponding to spheres in M with first Chern number 0. When M is of dimension 4, we can give the form of these correction terms thanks to our computations of the Seidel elements.

THEOREM III.15. *With the notation above, the Landau–Ginzburg superpotential has the following expression.*

(1) If D_k is the only edge of P corresponding to a sphere with vanishing first Chern number, or equivalently if c_1 vanishes only on the class A_k, then

$$W = \sum_{j=1}^n z^{\eta_j} t^{\kappa_j} + z^{\eta_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k}.$$

(2) If c_1 vanishes only on the classes A_{k-1} and A_k then

$$W = \sum_{j=1}^n z^{\eta_j} t^{\kappa_j} + z^{\eta_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k} + z^{\eta_{k-1}} t^{\kappa_k + \kappa_{k-2} - \kappa_{k-1}} + z^{\eta_k} t^{\kappa_{k+1} + \kappa_{k-2} - \kappa_{k-1}} + z^{\eta_{k-1}} t^{\kappa_{k+1} + \kappa_{k-2} - \kappa_k}.$$

The proof of this result is adapted from the Fano case, as carried out in (Ostrover and Tyomkin 2009). The idea is to exhibit a surjective ring morphism

$$\Psi\colon\ \mathbb{Q}[Z_1, \dots, Z_n] \otimes \check{\Gamma} \to \check{\Gamma}[z_1^\pm, \dots, z_m^\pm]$$

such that $\mathrm{SR}_Y(P)$ is in the kernel of Ψ. Then the image of the additive relations yields the ideal J_W. It turns out that in the Fano case, defining Ψ by setting $\Psi(Z_i) = q z^{v_i} t^{\kappa_i}$ provides the desired morphism. This actually comes from the fact that the Stanley–Reisner ideal is generated by elements of the form $Z_I - Z_{I^*}^{c_{I^*}} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)}$. In view of the discussion concerning $\mathrm{SR}_Y(P)$ from Section 2.3.2, it is not surprising that defining Ψ by setting $\Psi(Y_i) = q z^{v_i} t^{\kappa_i}$, where Y_i lifts $\mathcal{S}^*(\Lambda_i) \otimes q t^{\eta_i(D_i)}$ to $Q[Z_1, \dots, Z_n] \otimes \check{\Gamma}$, provides the suitable morphism.

Remark III.16. Since our knowledge of W comes from its derivatives, we might forget possible "constant" terms (only containing the variable t). In the NEF case we know by (Chan et al. 2017) and (González and Iritani 2012) that such terms do not exist. In general, however, there might be infinitely many of these. ∎

2.4. Explicit computations and essential loops of Hamiltonian diffeomorphisms

We now present examples of manifolds for which we can *explicitly* compute the Seidel morphism and the quantum cohomology ring. We start with a NEF manifold (which is not Fano) to first illustrate the content of Sections 2.2 and 2.3. This is the case of the 4-point blow-ups of $\mathbb{CP}^2$ presented in Section 2.4.1. Then, in Section 2.4.2, we consider Fano manifolds, given as specific 3-point blow-ups of $\mathbb{CP}^2$, for which we can determine an explicit loop of Hamiltonian diffeomorphisms lying in the kernel of Seidel's morphism. Finally in Section 2.4.3, we consider the 1-point blow-ups of $\mathbb{CP}^2$, also known as Hirzebruch surfaces. Even though most of these manifolds are "not-even-NEF", we can compute all their Seidel elements. These computations yield the injectivity of the Seidel morphism of all Hirzebruch surfaces.

2.4.1. *The 4-point blow-ups of* $\mathbb{CP}^2$

The 4-point blow-ups of $\mathbb{CP}^2$ are also (symplectically) known as the 3-point blow-ups of $S^2 \times S^2$. Recall that in Example 3, we illustrate the "final" result of Section 2.3 by reading the expression of the Landau–Ginzburg superpotential of these manifolds from their polytopes. Below, we complete the example by making explicit the results of the intermediate steps appearing in Section 2.3. This example is a "strict" application of the results of Sections 2.2 and 2.3 as these manifolds admit a NEF almost complex structure, but no Fano ones.

Disclaimer III.17. This section contains explicit material which may not be suitable for all mathematicians. Reader discretion is advised. ▮

Let $\mathbb{X}_4$ be a 3-point blow-up of $S^2 \times S^2$ endowed with the split symplectic form ω_μ for which the symplectic area of the first factor is μ and the area of the second factor is 1. We denote the capacity of the blow-ups by c_i for $i = 1$ to 4. Let B and $F \in \mathrm{H}_2(\mathbb{X}_4\,;\mathbb{Z})$ denote the homology classes respectively defined by $B = [S^2 \times \{\mathrm{pt}\}]$ and $F = [\{\mathrm{pt}\} \times S^2]$. Let $E_i \in \mathrm{H}_2(\mathbb{X}_4\,;\mathbb{Z})$ be the exceptional class corresponding to the blow-up of capacity c_i. Consider $\mathbb{X}_4$ endowed with the standard action of the torus $T = S^1 \times S^1$ for which the moment polytope is given by

$$P = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \leq x_2 \leq \mu,\ x_2 + x_1 \leq \mu - c_3, \\ -1 \leq x_1 \leq 0,\ c_1 \leq x_2 - x_1 \leq \mu + 1 - c_2 \end{array} \right\}$$

so the primitive outward normals to the facets of P are as follows :

$$\eta_1 = (0,1),\ \eta_2 = (1,1),\ \eta_3 = (1,0),\ \eta_4 = (1,-1),\ \eta_5 = (0,-1),\ \eta_6 = (-1,0),\ \eta_7 = (-1,1)$$

(see also the figure in Example 3). Two constants appear in the expression of the normalized moment map of $\mathbb{X}_4$, namely

$$\epsilon_1 = \frac{c_1^3 + 3c_2^2 - c_2^3 + c_3^3 - 3\mu}{3(c_1^2 + c_2^2 + c_3^2 - 2\mu)} \quad \text{and} \quad \epsilon_2 = \frac{c_1^3 - c_2^3 - c_3^3 + 3c_2^2\mu + 3c_3^2\mu - 3\mu^2}{3(c_1^2 + c_2^2 + c_3^2 - 2\mu)}\,.$$

Finally, the homology classes $A_i = [\Phi^{-1}(D_i)]$ of the pre-images of the corresponding facets D_i are : $A_1 = F - E_2 - E_3$, $A_2 = E_3$, $A_3 = B - E_1 - E_3$, $A_4 = E_1$, $A_5 = F - E_1$, $A_6 = B - E_2$, and $A_7 = E_2$.

Let Λ_i be the circle action associated with η_i. Since the complex structure on $\mathbb{X}_4$ is NEF, we can use the techniques from Section 2.2 to compute the Seidel elements associated with the circle actions :

$$S(\Lambda_1) = (F - E_2 - E_3) \otimes q \frac{t^{\mu - \epsilon_2}}{1 - t^{c_2 + c_3 - 1}}\,,$$

$$S(\Lambda_2) = E_3 \otimes q t^{\mu - c_3 + \epsilon_1 - \epsilon_2} - (F - E_2 - E_3) \otimes q \frac{t^{\mu + c_2 - 1 + \epsilon_1 - \epsilon_2}}{1 - t^{c_2 + c_3 - 1}} - (B - E_1 - E_3) \otimes q \frac{t^{c_1 + \epsilon_1 - \epsilon_2}}{1 - t^{c_1 + c_3 - \mu}}\,,$$

$$S(\Lambda_3) = (B - E_1 - E_3) \otimes q \frac{t^{\epsilon_1}}{1 - t^{c_1 + c_3 - \mu}}\,,$$

$$S(\Lambda_4) = E_1 \otimes q t^{\epsilon_1 + \epsilon_2 - c_1} - (B - E_1 - E_3) \otimes q \frac{t^{\epsilon_1 + \epsilon_2 + c_3 - \mu}}{1 - t^{c_1 + c_3 - \mu}}\,,$$

$$S(\Lambda_5) = (F - E_1) \otimes q t^{\epsilon_2}\,,$$

$$S(\Lambda_6) = (B - E_2) \otimes q t^{1 - \epsilon_1}\,,$$

$$S(\Lambda_7) = E_2 \otimes q t^{\mu + 1 - c_2 - \epsilon_1 - \epsilon_2} - (F - E_2 - E_3) \otimes q \frac{t^{\mu + c_3 - \epsilon_1 - \epsilon_2}}{1 - t^{c_2 + c_3 - 1}}\,.$$

From these expressions we can deduce those of the y_i, coming from the cohomology class Poincaré dual to $S(\Lambda_i)$, thanks to which we can in turn then determine the lifts Y_i. We get

$$Y_1 = Z_1 \otimes \frac{1}{1-t^{1-c_2-c_3}}, \quad Y_2 = Z_2 - Z_1 \otimes \frac{t^{1-c_2-c_3}}{1-t^{1-c_2-c_3}} - Z_3 \otimes \frac{t^{\mu-c_1-c_3}}{1-t^{\mu-c_1-c_3}},$$
$$Y_3 = Z_3 \otimes \frac{1}{1-t^{\mu-c_1-c_3}}, \quad Y_4 = Z_4 - Z_3 \otimes \frac{t^{\mu-c_1-c_3}}{1-t^{\mu-c_1-c_3}}, \quad Y_5 = Z_5,$$
$$Y_6 = Z_6, \quad \text{and} \quad Y_7 = Z_7 - Z_1 \otimes \frac{t^{1-c_2-c_3}}{1-t^{1-c_2-c_3}}.$$

Then, we can write explicitly the relations generating the ideals $\mathrm{SR}_Y(P)$ and $\mathrm{Lin}(P)$. In order to be able to write them on a line (*not* to ease the reading though), let us inappropriately denote $t_{2,3} = 1 - t^{1-c_2-c_3}$ and $t_{1,3} = 1 - t^{\mu-c_1-c_3}$ (and decrease the font...). The quantum cohomology ring of the 4-point blow-ups of $\mathbb{CP}^2$ endowed with the symplectic form ω_μ can be described as

$$\boxed{\mathrm{HQ}^*(\mathbb{X}_4;\check{\Gamma}) \simeq \mathbb{Q}[Z_1,\dots,Z_7]\otimes\check{\Gamma}/(\mathrm{SR}_Y(P)+\mathrm{Lin}(P))}$$

where $SR_Y(P)$ is generated by

$$Z_1Z_3 = Z_2 \otimes qt^{c_3}t_{2,3}t_{1,3} - Z_1 \otimes qt^{1-c_2}t_{1,3} - Z_3 \otimes qt^{\mu-c_1}t_{2,3},$$
$$Z_1Z_4t_{1,3} = Z_1Z_3 \otimes t^{\mu-c_1-c_3} + Z_3 \otimes qt^{\mu-c_1}t_{2,3},$$
$$Z_1Z_5 = 1 \otimes q^2t^{\mu}t_{2,3},$$
$$Z_1Z_6 = Z_7 \otimes qt^{c_2}t_{2,3} - Z_1 \otimes qt^{1-c_3},$$
$$Z_2Z_4t_{2,3}t_{1,3} = Z_3(Z_2+Z_3+Z_4) \otimes t^{\mu-c_1-c_3}t_{2,3} + Z_1Z_4 \otimes t^{1-c_2-c_3}t_{1,3} - Z_1Z_3 \otimes t^{1+\mu-c_1-c_2-2c_3},$$
$$Z_2Z_5t_{2,3}t_{1,3} = Z_1Z_5 \otimes t^{1-c_2-c_3}t_{1,3} + Z_3Z_5 \otimes t^{\mu-c_1-c_3}t_{2,3} + Z_3 \otimes qt^{\mu-c_3}t_{2,3},$$
$$Z_2Z_6t_{2,3}t_{1,3} = Z_1Z_6 \otimes t^{1-c_2-c_3}t_{1,3} + Z_3Z_6 \otimes t^{\mu-c_1-c_3}t_{2,3} + Z_1 \otimes qt^{1-c_3}t_{1,3},$$
$$Z_2Z_7t_{2,3}t_{1,3} = Z_1(Z_1+Z_2+Z_7) \otimes t^{1-c_2-c_3}t_{1,3} + Z_3Z_7 \otimes t^{\mu-c_1-c_3}t_{2,3} - Z_1Z_3 \otimes t^{1+\mu-c_1-c_2-2c_3},$$
$$Z_3Z_5 = Z_4 \otimes qt^{c_1}t_{1,3} - Z_3 \otimes qt^{\mu-c_3},$$
$$Z_3Z_6 = 1 \otimes q^2tt_{1,3},$$
$$Z_3Z_7t_{2,3} = Z_1Z_3 \otimes t^{1-c_2-c_3} + Z_1 \otimes qt^{1-c_2}t_{1,3},$$
$$Z_4Z_6t_{1,3} = Z_5 \otimes qt^{1-c_1}t_{1,3} + Z_3Z_6 \otimes t^{\mu-c_1-c_3},$$
$$Z_4Z_7t_{2,3}t_{1,3} = Z_1Z_4 \otimes t^{1-c_2-c_3}t_{1,3} + Z_3Z_7 \otimes t^{\mu-c_1-c_3}t_{2,3} - Z_3Z_1 \otimes qt^{1+\mu-c_1-c_2-2c_3} + 1 \otimes q^2t^{\mu+1-c_1-c_2}t_{2,3}t_{1,3},$$
$$Z_5Z_7t_{2,3} = Z_1Z_5 \otimes t^{1-c_2-c_3} + Z_6 \otimes qt^{\mu-c_2}t_{2,3}$$

and $\mathrm{Lin}(P)$ is generated by $Z_6 = Z_1 + 2Z_2 + Z_3 - Z_5$ and $Z_7 = -Z_1 - Z_2 + Z_4 + Z_5$.

We now turn to the Landau–Ginzburg superpotential. We translate the situation to homology whose generators are more natural. Indeed, recall that the homology classes $A_i = [\Phi^{-1}(D_i)] \in H_2(\mathbb{X}_4;\mathbb{Z})$ are additive generators of $\mathrm{H}_2(\mathbb{X}_4;\mathbb{Z})$ and multiplicative generators of $\mathrm{HQ}_*(\mathbb{X}_4;\Gamma)$. Moreover the subring $\mathrm{HQ}_4(\mathbb{X}_4;\Gamma)$ is generated by the elements $A_i \otimes q$, *i.e.* $E_i \otimes q$, for $i = 1$ to 3, $(F - E_1) \otimes q$, $(B - E_2) \otimes q$, $(F - E_2 - E_3) \otimes q$, and $(B - E_1 - E_3) \otimes q$.

The presentation of the quantum cohomology above thus leads to the following presentation of $\mathrm{HQ}_4(\mathbb{X}_4;\Gamma)$:

$$\boxed{\mathrm{HQ}_4(\mathbb{X}_4;\Gamma) \simeq \Gamma^{\mathrm{univ}}[u,v]/J}$$

where

$$u = (F - E_2 - E_3) \otimes q\frac{1}{1-t^{c_2+c_3-1}}, \quad \text{and} \quad v = (B - E_1 - E_3) \otimes q\frac{1}{1-t^{c_1+c_3-\mu}}$$

and J is the ideal generated by the following two relations :

$$v(1+vt^{c_1}) = u^2t^{\mu}(v+t^{c_2-1})(1+vt^{c_3}), \quad \text{and} \quad u(1+ut^{c_2}) \ = v^2t(u+t^{c_1-\mu})(1+ut^{c_3}).$$

Now, the expression of the Landau–Ginzburg superpotential given in Example 3,

$$W = z_2t^{\mu} + z_1z_2t^{\mu-c_3} + z_1 + z_1z_2^{-1}t^{-c_1} + z_2^{-1} + z_1^{-1}t + z_1^{-1}z_2t^{\mu+1-c_2} + z_1t^{\mu-c_1-c_3} + z_2t^{\mu+1-c_2-c_3}$$

leads to the ideal J_W generated by

$$\partial_{z_1} W = z_2 t^{\mu-c_3} + 1 + z_2^{-1} t^{-c_1} - z_1^{-2} t - z_1^{-2} z_2 t^{\mu+1-c_2} + t^{\mu-c_1-c_3},$$
$$\partial_{z_2} W = t^{\mu} + z_1 t^{\mu-c_3} - z_1 z_2^{-2} t^{-c_1} - z_2^{-2} + z_1^{-1} t^{\mu+1-c_2} + t^{\mu+1-c_2-c_3}.$$

It is easy to see that J_W coincides with J by setting $u = z_2^{-1} t^{-\mu}$ and $v = z_1^{-1}$.

2.4.2. *The* 3*-point blow-ups of* $\mathbb{CP}^2$

Using the techniques presented above, we determined in (Anjos and Leclercq 2017) an explicit loop of Hamiltonian diffeomorphisms lying in the kernel of the Seidel morphism of certain 3-point blow-ups of $\mathbb{CP}^2$.

Namely, we consider the symplectic manifold $\mathbb{X}_3^{c_1,c_2}$ endowed with the symplectic form $\omega_1^{c_1,c_2}$, obtained from $(S^2 \times S^2, \omega_1)$ by performing two successive blow-ups of capacities c_1 and c_2 such that $0 < c_2 \leq c_1 < c_1 + c_2 \leq 1$. (The symplectic form ω_1 is the *monotone* split symplectic form for which the symplectic area of each factor is 1.) For the standard Hamiltonian torus action, the moment polytope is given by

$$P = \big\{(x_1,x_2) \in \mathbb{R}^2 \,|\, 0 \leq x_2 \leq \mu,\ -1 \leq x_1 \leq 0,\ c_1 \leq x_2 - x_1 \leq \mu + 1 - c_2\big\}$$

so the primitive outward normals to P are as follows :

$$\eta_1 = (0,1),\ \eta_2 = (1,0),\ \eta_3 = (1,-1),\ \eta_4 = (0,-1),\ \eta_5 = (-1,0),\ \text{and}\ \eta_6 = (-1,1)\,.$$

First, following the same steps as for $\mathbb{X}_4$ above, we can compute explicitly the Seidel elements associated with the circle actions Λ_i. From these expressions we can further compute the ideal $\mathrm{SR}_Y(P)$, and conclude that there is an isomorphism

$$\mathrm{HQ}_4(\mathbb{X}_3^{c_1,c_2};\Gamma) \simeq \Gamma^{\mathrm{univ}}[u,v]/I_{c_1,c_2}$$

where I_{c_1,c_2} is the ideal generated by

$$u^2v^2 + u^2 v t^{-c_2} = v t^{-1-c_2} + t^{-2+c_1-c_2} \quad \text{and} \quad u^2v^2 + u v^2 t^{-c_2} = u t^{-1-c_2} + t^{-2+c_1-c_2}\,.$$

Note that, alternatively, we can also read this result in (Entov and Polterovich 2008).

Second, recall from (Anjos and Pinsonnault 2013, Theorem 1.1) that if $c_2 < c_1$ then

$$\pi_1(\mathrm{Ham}(\mathbb{X}_3^{c_1,c_2}, \omega_1^{c_1,c_2})) \simeq \mathbb{Z}\langle x_0, x_1, y_0, y_1, z\rangle$$

where the generators x_0, x_1, y_0, y_1, and z correspond to circle actions contained in maximal tori of the Hamiltonian group. In particular, $x_0 = \Lambda_2$ and $y_0 = \Lambda_1$ are the circle actions associated with the primitive outward normals to the polytope P defined above, η_2 and η_1 respectively. Recall also that the case $c_1 = c_2$ is an interesting limit case in terms of the topology of the Hamiltonian group since y_1 disappears.

Using the explicit description of $\mathrm{HQ}_4(\mathbb{X}_3^{c_1,c_2};\Gamma)$ given above, and the explicit expressions of the Seidel elements $\mathcal{S}(x_0)$ and $\mathcal{S}(y_0)$ associated with the generators x_0 and y_0, it is then quite easy to prove that

Theorem III.18. *The class of* $2(x_0 + y_0)$ *belongs to* $\ker(\mathcal{S})$ *if and only if* $c_1 = c_2$.

Hence, we have determined an element of $\ker \mathcal{S}$ on $\mathbb{X}_3^{c,c}$ for all $c \in \mathbb{R}$. Note that for $c_0 = \frac{1}{2}$, $\mathbb{X}_3^{c_0,c_0}$ is monotone. We can build other examples from the latter, *e.g.* by products via (Leclercq 2009).

2.4.3. *The* 1*-point blow-ups of* $\mathbb{CP}^2$

Hirzebruch surfaces are particularly interesting examples. Recall that the toric "even" Hirzebruch surfaces $(\mathbb{F}_{2k}, \omega_\mu)$, $0 \leq k \leq \ell$ with $\ell \in \mathbb{N}$ and $\ell < \mu \leq \ell + 1$, can be identified with the symplectic manifolds $(S^2 \times S^2, \omega_\mu)$. The moment polytope of $\mathbb{F}_{2k}$ is

$$\big\{(x_1,x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1,\ x_2 + kx_1 \geq 0,\ x_2 - kx_1 \leq \mu - k\big\}$$

and its primitive outward normals are

$$\eta_1 = (1,0),\ \eta_2 = (-k,-1),\ \eta_3 = (-1,0),\ \text{and}\ \eta_4 = (-k,1).$$

Let Λ_1^{2k} and Λ_2^{2k} represent the circle actions whose moment maps are, respectively, the first and the second component of the moment map associated with the torus action T_{2k} acting on $\mathbb{F}_{2k}$. It follows from the classification of 4-dimensional Hamiltonian S^1-spaces established by Karshon (Karshon 1999) that Λ_1^{2k} and Λ_2^{2k} satisfy the relations $\Lambda_1^{2k} = k\Lambda_1^2 + (k-1)\Lambda_1^0$ and $\Lambda_2^{2k} = k\Lambda_1^0 + \Lambda_2^0$.

Since $\mathbb{F}_0$ is Fano and $\mathbb{F}_2$ is NEF, we can compute explicitly the Seidel elements associated with Λ_1^0, Λ_2^0, and Λ_1^2. Thus, we get those associated with the circle actions of $\mathbb{F}_{2k}$, *even though for all* $k \geq 2$, $\mathbb{F}_{2k}$ *is non-NEF*. In particular, we can give the explicit expressions of the Seidel elements of $\mathbb{F}_4$ which admits a pseudo-holomorphic sphere with negative first Chern number, representing the class $B - 2F$ where $B = [S^2 \times \{\mathrm{pt}\}]$, and $F = [\{\mathrm{pt}\} \times S^2]$.

We made these computations in (Anjos and Leclercq 2018) and used them to deduce the ideal $\mathrm{SR}_Y(P)$ and in turn the Landau–Ginzburg superpotential W up to "constant" terms (containing only powers of t), see Remark III.16. It is interesting to observe that even in this non-NEF example the number of quantum corrections in the Landau–Ginzburg superpotential is still finite. However, the number of constant terms is expected to be infinite.

Finally, let us recall from (Abreu and McDuff 2000) that the circle actions whose associated Seidel elements are computed above are known to generate the fundamental group of the Hamiltonian diffeomorphism group of the even Hirzebruch surfaces. Moreover, similar computations can be made for the "odd" Hirzebruch surfaces, so that we were also able to compute *all* their Seidel elements. This led us in (Anjos and Leclercq 2017) to prove the injectivity of the Seidel morphism.

THEOREM III.19. *On all Hirzebruch surfaces, the Seidel morphism is injective.*

3. Further developments and prospects

There are several extensions of the Seidel morphism for which there is hope to get explicit information in the setting of and with similar techniques as those presented in Section 2.

3.1. Homotopy of the Hamiltonian group in higher degrees

There exist invariants of the homotopy/homology groups of higher degree of Hamiltonian diffeomorphism groups generalizing Seidel's construction : the Floer-theoretic invariants for families defined by Hutchings in (Hutchings 2008) and the quantum characteristic classes introduced by Savelyev in (Savelyev 2008).

Briefly recall that the former are morphisms $\pi_*(\mathrm{Ham}(M,\omega)) \to \mathrm{End}_{*-1}(\mathrm{HQ}_*(M,\omega))$ obtained as higher continuation maps in Floer homology. The latter are defined via parametric Gromov–Witten invariants and lead to ring morphisms

$$\mathrm{H}_*(\Omega\mathrm{Ham}(M,\omega),\mathbb{Q}) \longrightarrow \mathrm{HQ}_{2n+*}(M,\omega)\,.$$

Both constructions reduce to the Seidel representation, respectively, in degree 1 and 0. It would be interesting to see if they can be computed, for example for some 2- and 3-point blow-ups of $\mathbb{CP}^2$. Indeed, the homotopy algebra of the Hamiltonian diffeomorphism groups of these manifolds are known, see (Pinsonnault 2008) and (Anjos and Pinsonnault 2013).

This extension is also related to an open question (raised by Alexandru Oancea) and concerning the Lagrangian Seidel morphism as described in Section 1.1. Indeed, in (Hu and Lalonde 2010), the authors showed that the Lagrangian and "absolute" versions of Seidel's morphism are involved in the following commutative diagram :

$$\begin{array}{ccccc} \widetilde{\pi}_1(\mathrm{Ham}(M,\omega)) & \longrightarrow & \widetilde{\pi}_1(\mathrm{Ham}(M,\omega),\mathrm{Ham}(M,\omega;L)) & \longrightarrow & \widetilde{\pi}_0(\mathrm{Ham}_L(M,\omega)) \\ \downarrow{\scriptstyle \mathcal{S}} & & \downarrow{\scriptstyle \mathcal{S}_L} & & \\ \mathrm{HQ}_*(M) & \xrightarrow{\quad\mathcal{A}\quad} & \mathrm{HQ}_*(L) & & \end{array} \tag{III.9}$$

The morphism $\mathcal{A}$ is the quantum counterpart of a morphism initially introduced by Albers (Albers 2008) in *Floer theory*, by counting chimneys. It can alternatively be described as the quantum product with the fundamental class of L, $\mathcal{A} = \cdot * [L]\colon \ \mathrm{HQ}_*(M) \to \mathrm{HQ}_*(L)$, as described in Section 3.3 of Chapter I.

The upper horizontal sequence (is exact and) comes from the usual long exact sequence of homotopy groups for pairs, so that one might wonder how to extend the diagram to the left. It seems quite reasonable to think that the answer to this question goes through a Lagrangian adaptation of Hutchings' or Savelyev's works.

3.2. Bulk extension

The version of *quantum homology* defined in Section 3.2 of Chapter I, and used in the present chapter, should be referred to as the *small* quantum homology ring. There is also a notion of *big* quantum homology ring, obtained by considering not only the usual quantum product but also a family of deformations via even-degree cohomology classes of M, see e.g Usher (Usher 2011a) and Fukaya, Oh, Ohta, and Ono (Fukaya et al. 2011) for a precise definition.

For $b \in H^{\mathrm{ev}}(M)$, one ends up with $\mathrm{HQ}^b_*(M, \omega)$ isomorphic to $\mathrm{HQ}_*(M, \omega)$ as a vector space but with a twisted product. Fukaya et al. extended Seidel's morphism to morphisms $\pi_1(\mathrm{Ham}(M, \omega)) \to \mathrm{HQ}^b_*(M, \omega)^\times$ and generalized in the toric case part of the results of McDuff and Tolman. It would be interesting to see which information on the big quantum homology can be extracted from the computations of Section 2.

3.3. Lagrangian setting

As we saw in Section 1 of the present chapter, the Seidel morphism has been extended to the Lagrangian setting in works by Hu and Lalonde (Hu and Lalonde 2010), and Hu, Lalonde, and Leclercq (Hu, Lalonde, and Leclercq 2011). Following McDuff and Tolman, Hyvrier (Hyvrier 2016) computed the leading term of the Lagrangian Seidel elements associated with circle actions preserving some given monotone Lagrangian. He showed that when the latter is the real Lagrangian of a Fano toric manifold, all lower order terms vanish.

It could be interesting to study the Lagrangian case in NEF toric manifolds. However the preliminary question of the structure of the lower order terms has to be tackled with different techniques than the ones used by Hyvrier since those require the use of almost complex structures which generically lacks regularity. Let us also mention that Hyvrier's work, as well as such an extension, provide examples where one can apprehend the categorical refinement of the Lagrangian Seidel morphism due to Charette and Cornea (Charette and Cornea 2016).

Finally, it would also be interesting to see if the examples (at least the *monotone* one when both blow-ups have capacity $\frac{1}{2}$) of essential loops of Hamiltonian diffeomorphisms undetected by Seidel's morphism which were constructed in (Anjos and Leclercq 2017) can be adapted to the Lagrangian setting, for example via the diagonal construction from Section 5.3 of Chapter I, or via the Albers comparison map (III.9) above.

3.4. Contact setting

There should be an extension of the algebraic description of Seidel's morphism to the contact setting, via Sandon's homology of translated points. Given a contact manifold (V, ξ), the resulting morphism should be naturally defined on the relative fundamental group $\pi_1(\mathrm{Cont}_0(V, \xi), \mathrm{Cont}^s_0(V, \alpha))$ up to necessary coverings. In the previous expression, $\mathrm{Cont}^s_0(V, \alpha)$ denotes the group of those contactomorphisms which preserve the contact form α and not only the contact structure $\xi = \ker \alpha$. (This restriction comes from the fact that *at* a translated point of a contactomorphism ϕ, ϕ is required to preserve the contact form, see Definition II.52.)

Because of our Lemma III.7 which asserts the triviality of the Lagrangian Seidel morphism under the asphericity condition, it is reasonable to expect that this contact counterpart of Seidel's morphism will turn out to be trivial on (for example) prequantization bundles over aspherical symplectic manifolds. However, there is no *a priori* reasons why it should be trivial when the base is monotone. This also motivates our work to define translated point homology in this case, see Section 5.1.1. Defining such a contact Seidel morphism on the total space of a prequantization bundle will immediately raise the question of its relationship with the (usual) Seidel morphism of the base.

Bibliography

Abbondandolo, Alberto, and Felix Schlenk. 2017. "Floer homologies, with applications". *ArXiv e-prints*. arXiv : 1709.00297. (Cit. on p. 30).

Abbondandolo, Alberto, and Matthias Schwarz. 2010. "Floer homology of cotangent bundles and the loop product". *Geom. Topol.* 14 (3) : 1569–1722. (Cit. on p. 42).

Abouzaid, Mohammed. 2012. "Nearby Lagrangians with vanishing Maslov class are homotopy equivalent". *Invent. Math.* 189 (2) : 251–313. (Cit. on p. 93).

Abreu, Miguel, and Dusa McDuff. 2000. "Topology of symplectomorphism groups of rational ruled surfaces". *J. Amer. Math. Soc.* 13 (4) : 971–1009. (Cit. on p. 109).

Albers, Peter. 2008. "A Lagrangian Piunikhin–Salamon–Schwarz morphism and two comparison homomorphisms in Floer homology". *Int. Math. Res. Not.* No. 4 : Art. ID rnm134, 56. (Cit. on pp. 46, 110).

Albers, Peter, and Urs Frauenfelder. 2010. "Spectral invariants in Rabinowitz–Floer homology and global Hamiltonian perturbations". *J. Mod. Dyn.* 4 (2) : 329–357. (Cit. on p. 53).

Albers, Peter, and Will J. Merry. 2013. "Translated points and Rabinowitz Floer homology". *J. Fixed Point Theory Appl.* 13 (1) : 201–214. (Cit. on p. 87).

Anjos, Sílvia, and Rémi Leclercq. 2017. "Noncontractible Hamiltonian loops in the kernel of Seidel's representation". *Pacific J. Math.* 290 (2) : 257–272. (Cit. on pp. 15, 22, 92, 96, 108–110).

— . 2018. "Seidel's morphism of toric 4–manifolds". *J. Symplect. Geom.* 16 :1–68. (Cit. on pp. 15, 22, 92, 96, 97, 99, 109).

Anjos, Sílvia, and Martin Pinsonnault. 2013. "The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of the projective plane". *Math. Z.* 275 (1-2) : 245–292. (Cit. on pp. 108, 109).

Arnol'd, Vladimir. 1965. "Sur une propriété topologique des applications globalement canoniques de la mécanique classique". *C. R. Acad. Sci. Paris* 261 :3719–3722. (Cit. on p. 87).

Atiyah, Michael. 1982. "Convexity and commuting Hamiltonians". *Bull. London Math. Soc.* 14 (1) : 1–15. (Cit. on p. 100).

Audin, Michèle, and Mihai Damian. 2014. *Morse theory and Floer homology*. xiv+596. Universitext. Translated from the 2010 French original by Reinie Erné. Springer, London ; EDP Sciences, Les Ulis. (Cit. on p. 30).

Banyaga, Augustin. 1978. "Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique". *Comm. Math. Helv.* 53 (2) : 174–227. (Cit. on p. 27).

Barraud, Jean-François, and Octav Cornea. 2006. "Homotopic dynamics in symplectic topology". In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, 217 :109–148. NATO Sci. Ser. II Math. Phys. Chem. Springer, Dordrecht. (Cit. on p. 72).

— . 2007. "Lagrangian intersections and the Serre spectral sequence". *Ann. of Math. (2)* 166 (3) : 657–722. (Cit. on pp. 53, 54, 80, 83).

Batyrev, Victor V. 1993. "Quantum cohomology rings of toric manifolds". Journées de Géométrie Algébrique d'Orsay (Orsay, 1992), *Astérisque*, no. 218 : 9–34. (Cit. on pp. 102, 104).

Bhupal, Mohan. 2001. "A partial order on the group of contactomorphisms of $\mathbb{R}^{2n+1}$ via generating functions". *Turkish J. Math.* 25 (1) : 125–135. (Cit. on pp. 53, 88).

Bialy, Misha, and Leonid Polterovich. 1992. "Hamiltonian diffeomorphisms and Lagrangian distributions". *Geom. Funct. Anal.* 2 (2) : 173–210. (Cit. on p. 93).

Biran, Paul. 2006. "Lagrangian non-intersections". *Geom. Funct. Anal.* 16 (2) : 279–326. (Cit. on p. 69).

Biran, Paul, and Kai Cieliebak. 2001. "Symplectic topology on subcritical manifolds". *Comment. Math. Helv.* 76 (4) : 712–753. (Cit. on p. 69).

Biran, Paul, and Octav Cornea. 2007. "Quantum structures for Lagrangian submanifolds". *ArXiv e-prints*. arXiv : 0708.4221. (Cit. on p. 33).

— . 2009. "A Lagrangian quantum homology". In *New perspectives and challenges in symplectic field theory*, 49 :1–44. CRM Proc. Lecture Notes. Amer. Math. Soc., Providence, RI. (Cit. on pp. 33, 40, 46).

— . 2012. "Lagrangian topology and enumerative geometry". *Geom. Topol.* 16 (2) : 963–1052. (Cit. on pp. 33, 40).

Biran, Paul, and Michael Khanevsky. 2013. "A Floer-Gysin exact sequence for Lagrangian submanifolds". *Comment. Math. Helv.* 88 (4) : 899–952. (Cit. on p. 87).

Biran, Paul, Leonid Polterovich, and Dietmar Salamon. 2003. "Propagation in Hamiltonian dynamics and relative symplectic homology". *Duke Math. J.* 119 (1) : 65–118. (Cit. on pp. 26, 48).

Boothby, William M., and Hsieu-chung Wang. 1958. "On contact manifolds". *Ann. of Math. (2)* 68 :721–734. (Cit. on p. 86).

Borman, Matthew Strom, and Mark McLean. 2014. "Bounding Lagrangian widths via geodesic paths". *Compos. Math.* 150 (12) : 2143–2183. (Cit. on p. 72).

Bourgeois, Frédéric, and Alexandru Oancea. 2013. "The Gysin exact sequence for S^1-equivariant symplectic homology". *J. Topol. Anal.* 5 (4) : 361–407. (Cit. on p. 87).

Buhovsky, Lev, Vincent Humilière, and Sobhan Seyfaddini. 2016. "A C^0 counterexample to the Arnold conjecture". *ArXiv e-prints*. arXiv : 1609.09192. (Cit. on p. 70).

Buhovsky, Lev, and Emmanuel Opshtein. 2016. "Some quantitative results in C^0 symplectic geometry". *Invent. Math.* 205 (1) : 1–56. (Cit. on pp. 70, 78).

Buhovsky, Lev, and Yaron Ostrover. 2011. "On the uniqueness of Hofer's geometry". *Geom. Funct. Anal.* 21 (6) : 1296–1330. (Cit. on p. 28).

Buhovsky, Lev, and Sobhan Seyfaddini. 2013. "Uniqueness of generating Hamiltonians for topological Hamiltonian flows". *J. Symplectic Geom.* 11 (1) : 37–52. (Cit. on p. 74).

Buss, Guy, and Rémi Leclercq. 2012. "Pseudo-distances on symplectomorphism groups and applications to flux theory". *Math. Z.* 272 (3-4) : 1001–1022. (Cit. on pp. 10, 18).

Cardin, Franco, and Claude Viterbo. 2008. "Commuting Hamiltonians and Hamilton–Jacobi multi-time equations". *Duke Math. J.* 144 (2) : 235–284. (Cit. on p. 74).

Chan, Kwokwai, and Siu-Cheong Lau. 2014. "Open Gromov-Witten invariants and superpotentials for semi-Fano toric surfaces". *Int. Math. Res. Not. IMRN*, no. 14 : 3759–3789. (Cit. on p. 98).

Chan, Kwokwai, et al. 2017. "Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds". *Duke Math. J.* 166 (8) : 1405–1462. (Cit. on pp. 98, 105).

Chaperon, Marc. 1995. "On generating families". In *The Floer memorial volume*, 133 :283–296. Progr. Math. Basel : Birkhäuser. (Cit. on p. 53).

Charette, François. 2012. "A geometric refinement of a theorem of Chekanov". *J. Symplectic Geom.* 10 (3) : 475–491. (Cit. on p. 72).

Charette, François, and Octav Cornea. 2016. "Categorification of Seidel's representation". *Israel J. Math.* 211 (1) : 67–104. (Cit. on p. 110).

Chekanov, Yuri. 2000. "Invariant Finsler metrics on the space of Lagrangian embeddings". *Math. Z.* 234 (3) : 605–619. (Cit. on p. 28).

Chekanov, Yuri, and Felix Schlenk. 2010. "Notes on monotone Lagrangian twist tori". *Electron. Res. Announc. Math. Sci.* 17 :104–121. (Cit. on p. 69).

Chiu, Sheng-Fu. 2017. "Nonsqueezing property of contact balls". *Duke Math. J.* 166 (4) : 605–655. (Cit. on p. 88).

Cox, David, and Sheldon Katz. 1999. *Mirror symmetry and algebraic geometry*. 68 :xxii+469. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI. (Cit. on p. 102).

Crawley-Boevey, William. 2015. "Decomposition of pointwise finite-dimensional persistence modules". *J. Algebra Appl.* 14 (5) : 1550066, 8. (Cit. on p. 80).

Delzant, Thomas. 1988. "Hamiltoniens périodiques et images convexes de l'application moment". *Bull. Soc. Math. France* 116 (3) : 315–339. (Cit. on p. 100).

Dimitroglou Rizell, Georgios, Elizabeth Goodman, and Alexander Ivrii. 2016. "Lagrangian isotopy of tori in $S^2 \times S^2$ and $\mathbb{C}P^2$". *Geom. Funct. Anal.* 26 (5) : 1297–1358. (Cit. on p. 93).

Ekeland, Ivar, and Helmut Hofer. 1989. "Symplectic topology and Hamiltonian dynamics". *Math. Z.* 200 (3) : 355–378. (Cit. on p. 71).

Eliashberg, Yakov. 1987. "A theorem on the structure of wave fronts and its application in symplectic topology". *Functional Anal. Appl.* 21 (3) : 227–232. (Cit. on p. 70).

Eliashberg, Yakov, and Mikhael Gromov. 1991. "Convex symplectic manifolds". In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, 52 :135–162. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI. (Cit. on p. 55).

Eliashberg, Yakov, Sang Seon Kim, and Leonid Polterovich. 2006. "Geometry of contact transformations and domains : orderability versus squeezing". *Geom. Topol.* 10 :1635–1747. (Cit. on pp. 88, 89).

Eliashberg, Yakov, and Leonid Polterovich. 1993. "Bi-invariant metrics on the group of Hamiltonian diffeomorphisms". *Internat. J. Math.* 4 (5) : 727–738. (Cit. on p. 28).

— . 2000. "Partially ordered groups and geometry of contact transformations". *Geom. Funct. Anal.* 10 (6) : 1448–1476. (Cit. on p. 89).

— . 2010. "Symplectic quasi-states on the quadric surface and Lagrangian submanifolds". *ArXiv e-prints*. arXiv : 1006.2501. (Cit. on pp. 13, 20, 69).

Entov, Michael, and Leonid Polterovich. 2003. "Calabi quasimorphism and quantum homology". *Int. Math. Res. Not.* No. 30 : 1635–1676. (Cit. on p. 58).

— . 2006. "Quasi-states and symplectic intersections". DOI : 10.4171/CMH/43, *Comm. Math. Helv.* 81 (1) : 75–99. (Cit. on p. 68).

— . 2008. "Symplectic quasi-states and semi-simplicity of quantum homology". In *Toric topology*, 460 :47–70. Contemp. Math. Amer. Math. Soc., Providence, RI. (Cit. on pp. 68, 108).

— . 2009. "Rigid subsets of symplectic manifolds". *Compos. Math.* 145 (3) : 773–826. (Cit. on pp. 13, 20, 54, 68, 69).

Floer, Andreas. 1988a. "Morse theory for Lagrangian intersections". *J. Differential Geom.* 28 (3) : 513–547. (Cit. on pp. 25, 40).

— . 1988b. "The unregularized gradient flow of the symplectic action". *Comm. Pure Appl. Math.* 41 (6) : 775–813. (Cit. on pp. 25, 40).

— . 1989a. "Witten's complex and infinite-dimensional Morse theory". *J. Differential Geom.* 30 (1) : 207–221. (Cit. on pp. 25, 40).

— . 1989b. "Symplectic fixed points and holomorphic spheres". *Comm. Math. Phys.* 120 (4) : 575–611. (Cit. on pp. 25, 40, 87).

Fraser, Maya. 2015. "Contact spectral invariants and persistence". *ArXiv e-prints*. arXiv : 1502.05979. (Cit. on p. 80).

Frauenfelder, Urs. 2004. "The Arnold-Givental conjecture and moment Floer homology". *Int. Math. Res. Not.* No. 42 : 2179–2269. (Cit. on p. 46).

Frauenfelder, Urs, Viktor L. Ginzburg, and Felix Schlenk. 2005. "Energy capacity inequalities via an action selector". In *Geometry, spectral theory, groups, and dynamics*, 387 :129–152. Contemp. Math. Providence, RI : Amer. Math. Soc. (Cit. on p. 71).

Frauenfelder, Urs, and Felix Schlenk. 2007. "Hamiltonian dynamics on convex symplectic manifolds". *Israel J. Math.* 159 :1–56. (Cit. on p. 55).

Fukaya, Kenji, and Kaoru Ono. 1999. "Arnold conjecture and Gromov-Witten invariant". *Topology* 38 (5) : 933–1048. (Cit. on p. 87).

Fukaya, Kenji, Paul Seidel, and Ivan Smith. 2008. "Exact Lagrangian submanifolds in simply-connected cotangent bundles". *Invent. Math.* 172 (1) : 1–27. (Cit. on p. 93).

Fukaya, Kenji, et al. 2009a. *Lagrangian intersection Floer theory : anomaly and obstruction. Part I.* 46 :xii+396. AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI ; International Press, Somerville, MA. (Cit. on p. 40).

— . 2009b. *Lagrangian intersection Floer theory : anomaly and obstruction. Part II.* Vol. 46, i–xii and 397–805. AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI ; International Press, Somerville, MA. (Cit. on p. 40).

— . 2010. "Lagrangian Floer theory on compact toric manifolds. I". *Duke Math. J.* 151 (1) : 23–174. (Cit. on p. 98).

— . 2011. "Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory". *ArXiv e-prints.* arXiv : 1105.5123. (Cit. on p. 110).

Givental, Alexander. 1996. "Equivariant Gromov-Witten invariants". *Internat. Math. Res. Notices*, no. 13 : 613–663. (Cit. on pp. 98, 104, 105).

— . 1998. "A mirror theorem for toric complete intersections". In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, 160 :141–175. Progr. Math. Birkhäuser Boston, Boston, MA. (Cit. on p. 104).

González, Eduardo, and Hiroshi Iritani. 2012. "Seidel elements and mirror transformations". *Selecta Math. (N.S.)* 18 (3) : 557–590. (Cit. on pp. 98, 100, 105).

Gotay, Mark J. 1982. "On coisotropic imbeddings of presymplectic manifolds". *Proc. Amer. Math. Soc.* 84 (1) : 111–114. (Cit. on p. 75).

Gromov, Mikhael. 1985. "Pseudo holomorphic curves in symplectic manifolds". *Invent. Math.* 82 (2) : 307–347. (Cit. on pp. 25, 34, 40, 71).

— . 1986. *Partial differential relations.* 9 :x+363. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Berlin : Springer-Verlag. (Cit. on p. 69).

Guillemin, Victor, and Shlomo Sternberg. 1982. "Convexity properties of the moment mapping". *Invent. Math.* 67 (3) : 491–513. (Cit. on p. 100).

Hind, Richard. 2004. "Lagrangian spheres in $S^2 \times S^2$". *Geom. Funct. Anal.* 14 (2) : 303–318. (Cit. on p. 93).

Hofer, Helmut. 1990. "On the topological properties of symplectic maps". *Proc. Roy. Soc. Edinburgh Sect. A* 115 (1-2) : 25–38. (Cit. on pp. 12, 19, 27, 72).

Hofer, Helmut, and Dietmar Salamon. 1995. "Floer homology and Novikov rings". In *The Floer memorial volume*, 133 :483–524. Progr. Math. Birkhäuser, Basel. (Cit. on pp. 40, 87).

Hofer, Helmut, and Eduard Zehnder. 1990. "A new capacity for symplectic manifolds". In *Analysis, et cetera*, 405–427. Academic Press, Boston, MA. (Cit. on p. 71).

— . 1994. *Symplectic invariants and Hamiltonian dynamics.* xiv+341. Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Basel : Birkhäuser Verlag. (Cit. on p. 70).

Hu, Shengda, and François Lalonde. 2010. "A relative Seidel morphism and the Albers map". *Trans. Amer. Math. Soc.* 362 (3) : 1135–1168. (Cit. on pp. 12, 19, 91, 109, 110).

Hu, Shengda, François Lalonde, and Rémi Leclercq. 2011. "Homological Lagrangian monodromy". *Geom. Topol.* 15 (3) : 1617–1650. (Cit. on pp. 12, 13, 19, 20, 91, 92, 110).

Humilière, Vincent. 2017. "Géométrie symplectique C^0 et sélecteur d'action". HDR memoirs, Université Pierre et Marie Curie. (Cit. on pp. 70, 73).

Humilière, Vincent, Rémi Leclercq, and Sobhan Seyfaddini. 2015a. "New energy-capacity-type inequalities and uniqueness of continuous Hamiltonians". *Comment. Math. Helv.* 90 (1) : 1–21. (Cit. on pp. 14, 21, 54, 70, 73, 74).

— . 2015b. "Coisotropic rigidity and C^0-symplectic geometry". *Duke Math. J.* 164 (4) : 767–799. (Cit. on pp. 14, 21, 54, 70, 72, 73, 75, 76).

— . 2016. "Reduction of symplectic homeomorphisms". *Ann. Sci. Éc. Norm. Supér. (4)* 49 (3) : 633–668. (Cit. on pp. 14, 21, 54, 65, 70, 78, 79).

Hutchings, Michael. 2008. “Floer homology of families. I”. *Algebr. Geom. Topol.* 8 (1) : 435–492. (Cit. on p. 109).

Hyvrier, Clément. 2016. “Lagrangian circle actions”. *Algebr. Geom. Topol.* 16 (3) : 1309–1342. (Cit. on p. 110).

Karshon, Yael. 1999. “Periodic Hamiltonian flows on four-dimensional manifolds”. *Mem. Amer. Math. Soc.* 141 (672) : viii+71. (Cit. on p. 109).

Katić, Jelena, and Darko Milinković. 2005. “Piunikhin–Salamon–Schwarz isomorphisms for Lagrangian intersections”. *Differential Geom. Appl.* 22 (2) : 215–227. (Cit. on p. 46).

Kawasaki, Morimichi. 20xx. “Lagrangian Heaviness, superheaviness, and their applications”. (in preparation). (Cit. on p. 67).

Kontsevich, Maxim. 1995. “Enumeration of rational curves via torus actions”. In *The moduli space of curves (Texel Island, 1994)*, 129 :335–368. Progr. Math. Birkhäuser Boston, Boston, MA. (Cit. on p. 25).

Kontsevich, Maxim, and Yuri Manin. 1994. “Gromov-Witten classes, quantum cohomology, and enumerative geometry”. *Comm. Math. Phys.* 164 (3) : 525–562. (Cit. on p. 25).

Kragh, Thomas. 2013. “Parametrized ring-spectra and the nearby Lagrangian conjecture”. With an appendix by Mohammed Abouzaid, *Geom. Topol.* 17 (2) : 639–731. (Cit. on p. 93).

Kwon, Daesung, and Yong-Geun Oh. 2000. “Structure of the image of (pseudo) holomorphic discs with totally real boundary condition”. Appendix 1 by Jean-Pierre Rosay, *Comm. Anal. Geom.* 8 (1) : 31–82. (Cit. on p. 34).

Lalonde, François, and Dusa McDuff. 1995. “The geometry of symplectic energy”. *Ann. of Math. (2)* 141 (2) : 349–371. (Cit. on p. 27).

Lalonde, François, Dusa McDuff, and Leonid Polterovich. 1999. “Topological rigidity of Hamiltonian loops and quantum homology”. *Invent. Math.* 135 (2) : 369–385. (Cit. on p. 91).

Lanzat, Sergei. 2013. “Quasi-morphisms and symplectic quasi-states for convex symplectic manifolds”. *Int. Math. Res. Not.* No. 23 : 5321–5365. (Cit. on p. 55).

Laudenbach, François, and Jean-Claude Sikorav. 1994. “Hamiltonian disjunction and limits of Lagrangian submanifolds”. *Internat. Math. Res. Notices*, no. 4 : 161 ff., approx. 8 pp. (electronic). (Cit. on pp. 14, 21, 77).

Lazzarini, Laurent. 2000. “Existence of a somewhere injective pseudo-holomorphic disc”. *Geom. Funct. Anal.* 10 (4) : 829–862. (Cit. on p. 34).

— . xxxx. “Decomposition of a J-holomorphic curve”. *Personal webpage*. Download : http://www.math.jussieu.fr/~lazzarin/Articles/net.ps. (Cit. on p. 34).

Leclercq, Rémi. 2008. “Spectral invariants in Lagrangian Floer theory”. *J. Mod. Dyn.* 2 (2) : 249–286. (Cit. on pp. 10, 11, 18, 19, 53, 54, 64, 65, 67, 72, 84).

— . 2009. “The Seidel morphism of Cartesian products”. *Algebr. Geom. Topol.* 9 (4) : 1951–1969. (Cit. on pp. 10, 18, 108).

Leclercq, Rémi, and Frol Zapolsky. 2018. “Spectral invariants for monotone Lagrangians”. *J. Topology Anal.* No. 10 : 627–700. (Cit. on pp. 11, 13, 19, 20, 26, 33, 40, 50, 54, 55, 58, 61, 64, 67, 69, 79).

Libermann, Paulette, and Charles-Michel Marle. 1987. *Symplectic geometry and analytical mechanics*. 35 :xvi+526. Mathematics and its Applications. Translated from the French by Bertram Eugene Schwarzbach. Dordrecht : D. Reidel Publishing Co. (Cit. on p. 75).

Lisi, Samuel, and Antonio Rieser. 2016. “Coisotropic Hofer-Zehnder capacities and non-squeezing for relative embeddings”. *ArXiv e-prints*. arXiv : 1312.7334. (Cit. on pp. 71, 73).

Liu, Chiu-Chu Melissa. 2013. “Localization in Gromov-Witten theory and orbifold Gromov-Witten theory”. In *Handbook of moduli. Vol. II*, 25 :353–425. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA. (Cit. on p. 103).

Liu, Gang, and Gang Tian. 1998. “Floer homology and Arnold conjecture”. *J. Differential Geom.* 49 (1) : 1–74. (Cit. on p. 87).

McDuff, Dusa. 2000. “Quantum homology of fibrations over S^2”. *Internat. J. Math.* 11 (5) : 665–721. (Cit. on p. 91).

McDuff, Dusa, and Dietmar Salamon. 1998. *Introduction to symplectic topology*. Oxford mathematical monographs. Oxford : Oxford University Press. (Cit. on p. 28).

McDuff, Dusa, and Susan Tolman. 2006. “Topological properties of Hamiltonian circle actions”. *IMRP Int. Math. Res. Pap.* 72826, 1–77. (Cit. on pp. 15, 22, 37, 92, 96, 101, 104).

Meiwes, Matthias, and Kathrin Naef. 2015. “Translated points on hypertight contact manifolds”. *ArXiv e-prints*. arXiv : 1510.00515. (Cit. on p. 87).

Milinković, Darko. 2000. “On equivalence of two constructions of invariants of Lagrangian submanifolds”. *Pacific J. of Math.* 195 :371–415. (Cit. on p. 53).

Milinković, Darko. 2001. “Geodesics on the space of Lagrangian submanifolds in cotangent bundles”. *Proc. Amer. Math. Soc.* 129 (6) : 1843–1851 (electronic). (Cit. on p. 73).

Milinković, Darko, and Yong-Geun Oh. 1998. “Generating functions versus action functional. Stable Morse theory versus Floer theory”. In *Geometry, topology, and dynamics (Montreal, PQ, 1995)*, 15 :107–125. CRM Proc. Lecture Notes. Amer. Math. Soc., Providence, RI. (Cit. on p. 53).

Monzner, Alexandra, Nicolas Vichery, and Frol Zapolsky. 2012. “Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization”. *J. Mod. Dyn.* 6 (2) : 205–249. (Cit. on pp. 53, 67).

Nadler, David. 2009. “Microlocal branes are constructible sheaves”. *Selecta Math. (N.S.)* 15 (4) : 563–619. (Cit. on p. 93).

Oh, Yong-Geun. 1993. “Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I”. *Comm. Pure Appl. Math.* 46 (7) : 949–993. (Cit. on p. 40).

— . 1995. “Addendum to : “Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I.” [Comm. Pure Appl. Math. **46** (1993), no. 7, 949–993]”. *Comm. Pure Appl. Math.* 48 (11) : 1299–1302. (Cit. on p. 40).

— . 1997. “Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle”. *J. Differential Geom.* 46 (3) : 499–577. (Cit. on pp. 11, 19, 53, 65).

— . 2005. “Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds”. In *The breadth of symplectic and Poisson geometry*, 232 :525–570. Progr. Math. Boston, MA : Birkhäuser Boston. (Cit. on pp. 11, 19, 53, 57, 61, 64, 68).

Oh, Yong-Geun, and Stefan Müller. 2007. “The group of Hamiltonian homeomorphisms and C^0-symplectic topology”. *J. Symplectic Geom.* 5 (2) : 167–219. (Cit. on pp. 15, 21, 74).

Opshtein, Emmanuel. 2009. “C^0-rigidity of characteristics in symplectic geometry”. *Ann. Sci. Éc. Norm. Supér. (4)* 42 (5) : 857–864. (Cit. on pp. 14, 21, 77).

Ostrover, Yaron, and Ilya Tyomkin. 2009. “On the quantum homology algebra of toric Fano manifolds”. *Selecta Math. (N.S.)* 15 (1) : 121–149. (Cit. on pp. 104, 105).

Ostrover, Yaron, and Roy Wagner. 2005. “On the extremality of Hofer’s metric on the group of Hamiltonian diffeomorphisms”. *Int. Math. Res. Not.* No. 35 : 2123–2141. (Cit. on p. 28).

Pinsonnault, Martin. 2008. “Symplectomorphism groups and embeddings of balls into rational ruled 4-manifolds”. *Compos. Math.* 144 (3) : 787–810. (Cit. on p. 109).

Piunikhin, Sergey, Dietmar Salamon, and Matthias Schwarz. 1996. “Symplectic Floer-Donaldson theory and quantum cohomology”. In *Contact and symplectic geometry (Cambridge, 1994)*, 8 :171–200. Publ. Newton Inst. Cambridge : Cambridge Univ. Press. (Cit. on p. 46).

Polterovich, Leonid. 1991. “The surgery of Lagrange submanifolds”. *Geom. Funct. Anal.* 1 (2) : 198–210. (Cit. on p. 92).

— . 1993. "Symplectic displacement energy for Lagrangian submanifolds". *Ergodic Theory Dynam. Systems* 13 (2) : 357–367. (Cit. on p. 27).

— . 2001. *The geometry of the group of symplectic diffeomorphisms*. xii+132. Lectures in Mathematics ETH Zürich. Basel : Birkhäuser Verlag. (Cit. on p. 28).

Polterovich, Leonid, and Daniel Rosen. 2014. *Function theory on symplectic manifolds*. 34 :xii+203. CRM Monograph Series. American Mathematical Society, Providence, RI. (Cit. on pp. 14, 21, 67).

Polterovich, Leonid, and Egor Shelukhin. 2016. "Autonomous Hamiltonian flows, Hofer's geometry and persistence modules". *Selecta Math. (N.S.)* 22 (1) : 227–296. (Cit. on p. 80).

Polterovich, Leonid, Egor Shelukhin, and Vukašin Stojisavljević. 2017. "Persistence modules with operators in Morse and Floer theory". *Mosc. Math. J.* 17 (4) : 757–786. (Cit. on pp. 80, 82).

Ruan, Yongbin, and Gang Tian. 1995. "A mathematical theory of quantum cohomology". *J. Differential Geom.* 42 (2) : 259–367. (Cit. on p. 25).

Sandon, Sheila. 2011. "Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions". *Ann. Inst. Fourier (Grenoble)* 61 (1) : 145–185. (Cit. on pp. 53, 87, 88).

— . 2013. "A Morse estimate for translated points of contactomorphisms of spheres and projective spaces". *Geom. Dedicata* 165 :95–110. (Cit. on p. 87).

— . 201x. "Floer homology for translated points". (Cit. on pp. 53, 86).

Savelyev, Yasha. 2008. "Quantum characteristic classes and the Hofer metric". *Geom. Topol.* 12 (4) : 2277–2326. (Cit. on p. 109).

Schwarz, Matthias. 1993. *Morse homology*. 111 :x+235. Progress in Mathematics. Birkhäuser Verlag, Basel. (Cit. on p. 30).

— . 1998. "A quantum cup-length estimate for symplectic fixed points". *Invent. Math.* 133 (2) : 353–397. (Cit. on pp. 46, 99).

— . 2000. "On the action spectrum for closed symplectically aspherical manifolds". *Pacific J. Math.* 193 (2) : 419–461. (Cit. on pp. 11, 19, 53, 64).

Seidel, Paul. 1997. "π_1 of symplectic automorphism groups and invertibles in quantum homology rings". *Geom. Funct. Anal.* 7 (6) : 1046–1095. (Cit. on pp. 12, 19, 91).

— . 2008. *Fukaya categories and Picard-Lefschetz theory*. viii+326. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich. (Cit. on p. 40).

Spielberg, Holger. 1999a. "The Gromov-Witten invariants of symplectic toric manifolds, and their quantum cohomology ring". *C. R. Acad. Sci. Paris Sér. I Math.* 329 (8) : 699–704. (Cit. on p. 103).

— . 1999b. *A formula for the Gromov-Witten invariants of toric varieties*. 1999/11 :103. Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research]. Thèse, Université Louis Pasteur (Strasbourg I), Strasbourg, 1999. Université Louis Pasteur, Département de Mathématique, Institut de Recherche Mathématique Avancée, Strasbourg. (Cit. on p. 103).

Usher, Michael. 2010. "The sharp energy-capacity inequality". *Commun. Contemp. Math.* 12 (3) : 457–473. (Cit. on p. 72).

— . 2011a. "Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasimorphisms". *Geom. Topol.* 15 (3) : 1313–1417. (Cit. on p. 110).

— . 2011b. "Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds". *Israel J. Math.* 184 :1–57. (Cit. on p. 80).

— . 2015. "Observations on the Hofer distance between closed subsets". *Math. Res. Lett.* 22 (6) : 1805–1820. (Cit. on p. 72).

Usher, Michael, and Jun Zhang. 2016. "Persistent homology and Floer-Novikov theory". *Geom. Topol.* 20 (6) : 3333–3430. (Cit. on p. 80).

Vafa, Cumrun. 1991. "Topological Landau-Ginzburg models". *Modern Phys. Lett. A* 6 (4) : 337–346. (Cit. on p. 25).

Varolgunes, Umut. 2016. “On the equatorial Dehn twist of a Lagrangian nodal sphere”. *ArXiv e-prints*. arXiv : 1612.00354. (Cit. on p. 92).

Viterbo, Claude. 1987. “Intersection de sous-variétés lagrangiennes, fonctionnelles d'action et indice des systèmes hamiltoniens”. *Bull. Soc. Math. France* 115 (3) : 361–390. (Cit. on p. 29).

— . 1992. “Symplectic topology as the geometry of generating functions”. *Math. Ann.* 292 (4) : 685–710. (Cit. on pp. 11, 19, 28, 53, 65, 71, 72).

— . 2006. “On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows”. *Int. Math. Res. Not.* Art. ID 34028, 9. (Cit. on p. 74).

Witten, Edward. 1991. “Two-dimensional gravity and intersection theory on moduli space”. In *Surveys in differential geometry (Cambridge, MA, 1990)*, 243–310. Lehigh Univ., Bethlehem, PA. (Cit. on p. 25).

Wu, Weiwei. 2012. “On an exotic Lagrangian torus in $\mathbb{C}P^2$”. *ArXiv e-prints*. arXiv : 1201.2446. (Cit. on pp. 13, 20).

Yau, Mei-Lin. 2009. “Monodromy and isotopy of monotone Lagrangian tori”. *Math. Res. Lett.* 16 (3) : 531–541. (Cit. on pp. 13, 20, 93).

Zapolsky, Frol. 2013. “Geometry of contactomorphism groups, contact rigidity, and contact dynamics in jet spaces”. *Int. Math. Res. Notices* 2013 (20) : 4687–4711. (Cit. on p. 53).

— . 2015. “The Lagrangian Floer-quantum-PSS package and canonical orientations in Floer theory”. *ArXiv e-prints*. arXiv : 1507.02253. (Cit. on pp. 33, 40, 46, 69).

Zomorodian, Afra, and Gunnar Carlsson. 2005. “Computing persistent homology”. *Discrete Comput. Geom.* 33 (2) : 249–274. (Cit. on p. 80).