Intersection local times, loop soups and permanental Wick powers

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Abstract

Several stochastic processes related to transient Lévy processes with potential densities $u(x, y) = u(y - x)$, that need not be symmetric nor bounded on the diagonal, are defined and studied. They are real valued processes on a space of measures $\mathcal{V}$ endowed with a metric $d$. Sufficient conditions are obtained for the continuity of these processes on $(\mathcal{V}, d)$. The processes include $n$-fold self-intersection local times of transient Lévy processes and permanental chaoses, which are ‘loop soup $n$-fold self-intersection local times’ constructed from the loop soup of the Lévy process. Loop soups are also used to define permanental Wick powers, which generalizes standard Wick powers, a class of $n$-th order Gaussian chaoses. Dynkin type isomorphism theorems are obtained that relate the various processes.

Poisson chaos processes are defined and permanental Wick powers are shown to have a Poisson chaos decomposition. Additional properties of Poisson chaos processes are studied and a martingale extension is obtained for many of the processes described above.

1 Introduction

We define and study several stochastic processes related to transient Lévy processes with potential densities $u(x, y) = u(y - x)$ that need not be symmetric nor bounded on the diagonal. We are particularly interested in the case when

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$u(x, y)$ is not symmetric since some of the processes we consider have already been studied in the case when it is symmetric. Significantly the results we obtain give the known results when $u(x, y)$ is symmetric.

The processes we consider are real valued processes on a space of measures $\mathcal{V}$ endowed with a metric $d$. We obtain sufficient conditions for the continuity of these processes on $(\mathcal{V}, d)$. Specifically, we study $n$-fold self-intersection local times of transient Lévy processes and permanental chaoses, which are ‘loop soup $n$-fold self-intersection local times’ constructed from the loop soup of the Lévy process. We also use loop soups to define permanental Wick powers, which, are generalizations of standard Wick powers, a class of $n$-th order Gaussian chaoses. We develop the concept of Poisson chaos decompositions and describe the Poisson chaos decomposition of permanent Wick powers. This illuminates the relationship between permanent Wick powers and the permanental chaos processes constructed from self-intersection local times. Lastly we define and study the exponential Poisson chaos and show that the processes described above have a natural extension as martingales.

Let $Y = \{Y_t, t \in R^+\}$ be a Lévy process in $R^d$, $d = 1, 2$, with characteristic exponent $\bar{\kappa}$, i.e.,

$$E(e^{i\xi Y_t}) = e^{-t\bar{\kappa}(\xi)}. \quad (1.1)$$

We assume that for some $\gamma \geq 0$,

$$|\frac{1}{\gamma + \bar{\kappa}(\xi)}| \leq \frac{1}{\varrho(\xi)}, \quad (1.2)$$

for some function $\varrho(\xi)$ that is regularly varying at infinity with index

$$\left(1 - \frac{1}{2n}\right)d < \alpha \leq d, \quad (1.3)$$

for some $n \geq 2$. (The precise value of $n$ depends on the results we prove.)

We assume that

$$\int_K (\varrho(\xi))^{-1} d\xi < \infty \quad \text{for all compact sets } K \in R^d. \quad (1.4)$$

and that

$$\int_R (\varrho(\xi))^{-1} d\xi = \infty. \quad (1.5)$$

We also assume that $\bar{\kappa}$ satisfies the sectorial condition,

$$|\text{Im } \bar{\kappa}(\xi)| \leq C(\gamma + \text{Re } \bar{\kappa}(\xi)) \quad \forall \xi \in R^d, \quad (1.6)$$
for some $C < \infty$.

Let $X = \{X_t, t \in \mathbb{R}^+\}$ be the process obtained by killing $Y$ at an independent exponential time with mean $1/\gamma$, $\gamma > 0$. When $\gamma = 0$, we take $X = Y$. In the rest of this paper we simply refer to the transient Lévy process $X$ without specifying whether or not it is an exponentially killed Lévy process.

We first consider the $n$-fold self-intersections of $X$ in $\mathbb{R}^d$, $d = 1, 2$. This entails studying functionals of the form

$$
\alpha_{n,\epsilon}(\nu, t) \overset{\text{def}}{=} \int_0^t \prod_{j=1}^n f_\epsilon(X(t_j) - y) \, dt_1 \cdots dt_n \, d\nu(y),
$$

where $f_\epsilon$ is an approximate $\delta$-function at zero and $\nu$ is a finite measure on $\mathbb{R}^d$. Ideally we would like to take the limit of $\alpha_{n,\epsilon}(\nu, t)$ as $\epsilon \to 0$, but if the potential density of $X$ is unbounded at the origin, which is always the case in dimension $d \geq 2$, the limit is infinite for all $n \geq 2$. To deal with this we use a technique called renormalization, which consists of forming a linear combination of the $\{\alpha_{k,\epsilon}(\nu, t)\}_{k=1}^n$ which has a finite limit, $L_n(\nu, t)$, as $\epsilon \to 0$.

We study the behavior of $L_n(\nu) := L_n(\nu, \infty)$.

We set $L_{1,\epsilon}(\nu) = \alpha_{1,\epsilon}(\nu, \infty)$ and define recursively

$$
L_{n,\epsilon}(\nu) = \alpha_{n,\epsilon}(\nu, \infty) - \sum_{j=1}^{n-1} c_{n,j,\epsilon} L_{j,\epsilon}(\nu),
$$

where the $c_{n,j,\epsilon}$ are constants which diverge as $\epsilon \to 0$; see (2.17) and (2.21).

We introduce a $\sigma$-finite measure $\mu$ on the path space of $X$, called the loop measure, and show that for a certain class of positive measures $\nu$

$$
L_n(\nu) := \lim_{\epsilon \to 0} L_{n,\epsilon}(\nu) \quad \text{exists in } L^p(\mu), \text{ for all } p \geq 2. \quad (1.9)
$$

We refer to $L_n(\nu)$ as the $n$-fold renormalized self-intersection local time of $X$, with respect to $\nu$.

We show in the beginning of Section 2 that $X$ has potential densities $u(x, y) = u(y - x)$. For $\gamma \geq 0$ we define

$$
\hat{u}(\xi) = \frac{1}{\gamma + \hat{\kappa}(\xi)},
$$

and

$$
|\hat{u}(\xi)| \leq \frac{1}{\varrho_\alpha(|\xi|)}. \quad (1.11)
$$
Let 
\[ \tau_n(\xi) := \underbrace{\hat{u} * \hat{u} * \cdots * \hat{u}}_{n\text{-times}}(\xi) \] 
(1.12)
denote the \( n \)-fold convolutions of \( \hat{u} \). We define
\[ \| \nu \|_{2,\tau_n^2} := \left( \int \tau_{2n}(|\lambda|)^2 d\lambda \right)^{1/2} < \infty. \] 
(1.13)

Let \( f \) and \( g \) be functions on \( \mathbb{R}^d_+ \). By the rotation invariance of Lebesgue measure on \( \mathbb{R}^d \), \( \int_{\mathbb{R}^d} f(|\eta - \xi|)g(|\eta|) d\eta \) depends only on \( |\xi| \). We let \( f *_d g \) denote the function on \( \mathbb{R}^d_+ \) which satisfies
\[ f *_d g(|\xi|) = \int_{\mathbb{R}^d} f(|\eta - \xi|)g(|\eta|) d\eta. \] 
(1.14)

We refer to \( f *_d g \) as the \( d \)-convolution of \( f \) and \( g \). (The letter \( d \) refers to the dimension of the space we are integrating on.)

Let \( \tilde{\tau}_n \) denote the \( n \)-fold \( d \)-convolution of \( (\varrho_\alpha)^{-1} \). (We use the notation \( (\varrho_\alpha)^{-1} \) for \( 1/\varrho_\alpha \)). That is,
\[ \tilde{\tau}_n(|\xi|) := \underbrace{(\varrho_\alpha)^{-1} *_d (\varrho_\alpha)^{-1} *_d \cdots *_d (\varrho_\alpha)^{-1}}_{n\text{-times}}(|\xi|). \] 
(1.15)

It follows from (1.3) and Lemma 10.2 that \( \tilde{\tau}_{2n}(|\xi|) < \infty \) and \( \lim_{|\xi| \to \infty} \tilde{\tau}_{2n}(|\xi|) = 0. \)

Let \( \mathcal{B}_{2n}(\mathbb{R}^d) \) denote the set of finite signed measures \( \nu \) on \( \mathbb{R}^d \) such that
\[ \| \nu \|_{2,\tilde{\tau}_{2n}} := \left( \int \tilde{\tau}_{2n}(|\lambda|)^2 d\lambda \right)^{1/2} < \infty. \] 
(1.16)

**Theorem 1.1** Let \( X = \{ X(t), t \in \mathbb{R}^+ \} \) be a Lévy process in \( \mathbb{R}^d, d = 1, 2 \), as described above and let \( \nu \in \mathcal{B}_{2n}(\mathbb{R}^d), n \geq 1 \). Then (1.9) holds.

We are also concerned with the continuity of \( \{ L_n(\nu), \nu \in \mathcal{V} \} \), where \( \mathcal{V} \) is some metric space. Here is a particularly straightforward example of our results.

For any finite positive measure \( \nu \) on \( \mathbb{R}^d \), let \( \nu_x(A) = \nu(A - x) \).
Theorem 1.2 Under the hypotheses of Theorem 1.1
\[
\int_1^\infty \left( \int_{|\xi|\geq x} \tau_{2n}(\xi)|\hat{\nu}(\xi)|^2 \, d\xi \right)^{1/2} \frac{(\log x)^{n-1}}{x} \, dx < \infty, \tag{1.17}
\]
is a sufficient condition for \( \{L_n(\nu_x), x \in \mathbb{R}^d\} \) to be continuous \( P^y \) almost surely, for all \( y \in \mathbb{R}^d \).

(As usual, \( P^y \) denotes the probability of the Lévy process \( X \) starting at \( y \in \mathbb{R}^d \).

Several concrete examples are given at the end of Section 3. As a sample, we note that when (1.11) holds for \( 1/\varrho \alpha(\xi) = O(\xi^{-d}) \) as \( |\xi| \to \infty \), \( \{L_n(\nu_x), x \in \mathbb{R}^d\} \) exists for all \( n \geq 2 \), and is continuous almost surely when
\[
|\hat{\nu}(\xi)|^2 = O \left( \frac{1}{(\log |\xi|)^{4n+\delta}} \right) \quad \text{as} \quad |\xi| \to \infty, \tag{1.18}
\]
for any \( \delta > 0 \).

When \( \varrho(\xi) = |\xi|^\alpha \), for \( d(1 - \frac{1}{2n}) < \alpha < d \), \( \{L_n(\nu_x), x \in \mathbb{R}^d\} \) exists and is continuous almost surely when
\[
|\hat{\nu}(\xi)|^2 = O \left( \frac{1}{|\xi|^{2n(d-\alpha)}(\log |\xi|)^{2n+1+\delta}} \right) \quad \text{as} \quad |\xi| \to \infty, \tag{1.19}
\]
for any \( \delta > 0 \).

Theorems 1.1 and 1.2 follow easily from the proof of the next theorem.

Theorem 1.3 Let \( X \) be as in Theorem 1.1 and let \( n = n_1 + \cdots + n_k, k \geq 2, \) and \( \nu_i \in B_{2n_i}(\mathbb{R}^d) \). Then
\[
\mu \left( \prod_{i=1}^k L_{n_i}(\nu_i) \right) = \frac{\prod_{i=1}^k (n_i)!}{n} \sum_{\pi \in \mathcal{M}_a} \int \prod_{j=1}^n u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{i=1}^k d\nu_i(x_i)
\leq \frac{\left| \mathcal{M}_a \right|}{n} \prod_{i=1}^k n_i! \|\nu_i\|_{2, \tau_{2n_i}}, \tag{1.20}
\]
where \( \pi(n+1) = \pi(1) \) and \( \mathcal{M}_a \) is the set of maps \( \pi : [1, n] \mapsto [1, k] \) with \( |\pi^{-1}(i)| = n_i \) for each \( i \) and such that, if \( \pi(j) = i \) then \( \pi(j+1) \neq i \). (The subscript ‘a’ in \( \mathcal{M}_a \) stands for alternating).
(Note that although the hypothesis of Theorem 1.3 requires that $\|\nu_i\|_{2,\tau_{n_i}} < \infty$, the bound in (1.20) is in terms of the possibly smaller norms $\|\nu_i\|_{2,\tau_{2n_i}}$.)

In [13] we study self-intersection local times of Lévy processes with symmetric potential densities. We obtain sufficient conditions for continuity, such as Theorem 1.2 above, by associating the self-intersection local time $L_n(\nu)$ with a $2n$-th order Gaussian chaos $G^{2n}_\nu$, called a $2n$-th Wick power, that is constructed from the Gaussian field with covariance

$$E(\langle G^{2n}\nu : G^{2n}_\mu \rangle) = (2n)! \int \int (u(x,y))^{2n} \, d\nu(x) \, d\mu(y). \tag{1.21}$$

(In (1.21) $u$ must be symmetric.) The association is by a Dynkin type isomorphism theorem that allows us to infer results about $\{L_n(\nu), \nu \in \mathcal{V}\}$ from results about $\{G^{2n}\nu : \nu \in \mathcal{V}\}$. The advantage here is that the continuity results we consider are known for the Gaussian chaoses, so once we have the isomorphism theorem it is easy to extend them to the associated intersection local times.

In this paper $u$ need not be symmetric. We define and obtain continuity results about self-intersection local times directly, without relating them to any other stochastic process. Nevertheless, the question remains, is there a Dynkin type isomorphism theorem that relates them to another process and more specifically to what other process. We give two answers to this question. In the first we relate the intersection local times to the ‘loop soup $n$-fold self-intersection local time’ $\psi_n(\nu)$, which we also call an $n$-th order permanental chaos. To construct $\psi_n(\nu)$ we take a Poisson process $L_\alpha$ with intensity measure $\alpha \mu$, $\alpha > 0$, on $\Omega_\Delta$, the space of the paths of $X$. This process is called a loop soup. The loop soup self-intersection local time $\psi_n(\nu)$ is the renormalized sum of the $n$-fold self-intersection local times, i.e., the $L_n(\nu)$, of the paths in $L_\alpha$. In Theorem 1.5 we give a Dynkin type isomorphism theorem that relates $\{L_n(\nu), \nu \in \mathcal{V}\}$ and $\{\psi_n(\nu), \nu \in \mathcal{V}\}$.

Analogous to (1.20) we have the following joint moment formula for $\psi_{n_1}(\nu), \ldots, \psi_{n_k}(\nu)$, which is proved in Section 4.

**Theorem 1.4** Let $X$ be as in Theorem 1.1 and let $n = n_1 + \cdots + n_k$ and $c(\pi)$ equal to the number of cycles in the permutation $\pi$. Then

$$E_{L_\alpha} \left( \prod_{i=1}^k \psi_{n_i}(\nu_i) \right) = \sum_{\pi \in \mathcal{P}_0} \alpha^{c(\pi)} \int \prod_{j=1}^n \int u(z_j, z_{\pi(j)}) \prod_{i=1}^k d\nu_i(x_i), \tag{1.22}$$

where $z_1, \ldots, z_{n_1}$ are all equal to $x_1$, the next $n_2$ of the $\{z_j\}$ are all equal to $x_2$, and so on, so that the last $n_k$ of the $\{z_j\}$ are all equal to $x_k$ and $\mathcal{P}_0$ is the set
of permutations π of \([1, n]\) with cycles that alternate the variables \(\{x_i\}\); (i.e.,
for all \(j\), if \(z_j = x_i\) then \(z_{\pi(j)} \neq x_i\), and in addition, for each \(i = 1, \ldots, k\), all
the \(\{z_j\}\) that are equal to \(x_i\) appear in the same cycle.

Using standard results about Poisson processes, in Section 5, we obtain an
isomorphism theorem that relates the self-intersection local times \(L_n(\nu)\) of \(X\)
and the loop soup self-intersection local times \(\psi_n(\nu)\) of \(X\).

**Theorem 1.5 (Isomorphism Theorem I)** For any positive measures \(\rho, \phi \in \mathcal{B}_2(R^d)\) there exists a random variable \(\theta_{\rho,\phi}\) such that for any finite measures \(\nu_j \in \mathcal{B}_2(R^d)\), \(j = 1, 2, \ldots\), and bounded measurable functions \(F\) on \(R^\infty\),
\[
E_{\mathcal{L}_\alpha} \int Q^{x,x} (L_1(\phi) F(\psi_{n_j}(\nu_j) + L_{n_j}(\nu_j))) \, d\rho(x) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha} \left( \theta^\rho,\phi F(\psi_{n_j}(\nu_j)) \right).
\]
(1.23)
(Here we use the notation \(F(f(x_i)) := F(f(x_1), f(x_2), \ldots)\).)

It is interesting to compare (1.23) with [11, Theorem 1.3] in which all \(n_j = 1\), and with [3, Theorem 3.2] which is for local times.

The measure \(Q^{x,y}\), which is used to define the loop measure \(\mu\), is defined in (5.1). The term \(\theta_{\rho,\phi}\) is a positive random variable with \(E_{\mathcal{L}_\alpha} \left( (\theta_{\rho,\phi})^k \right) < \infty\)
for all integers \(k \geq 1\). In particular
\[
E_{\mathcal{L}_\alpha} \left( \theta^\rho,\phi \right) = \alpha \int Q^{x,x} (L_1(\phi)) \, d\rho(x).
\]
(1.24)

It is actually quite simple to obtain Theorems 1.4 and 1.5. However
they are not really generalizations of the results in [13]. The loop soup self-intersection local time \(\tilde{\psi}_n(\nu)\) is not a \(2n\)-th Wick power when the potential density of \(X\) is symmetric.

Our second answer to the questions raised in the paragraphs preceding Theorem 1.4 relates the self-intersection local times \(L_n(\nu)\) of \(X\) with a process \(\tilde{\psi}_n(\nu)\) which we call a permanental Wick power. The rationale for this name is that this process is a \(2n\)-th Wick power when \(u\) is symmetric and \(\alpha = 1/2\). It seems significant to have a generalization of Wick powers that does not require that the kernel that defines them is symmetric. This is done in Section 6 in which we give analogues of Theorems 1.4 and 1.5 for \(\tilde{\psi}\). (The analogue of Theorem 1.4 for \(\tilde{\psi}\) is exactly the same as Theorem 1.4 except that the final phrase “and in addition, for each \(i = 1, \ldots, k\), all the \(\{z_j\}\) that are equal to \(x_i\) appear in the same cycle.” is omitted.)

In Section 7 we develop the concept of Poisson chaos decompositions. In Section 8 we obtain a Poisson chaos decomposition of the permanental Wick
power $\tilde{\psi}_n(\nu)$ and relate it to the loop soup self-intersection local time $\psi_n(\nu)$. The process $\tilde{\psi}_n(\nu)$ incorporates the self-intersection local times $L_n(\nu)$ of each path in the loop soup. In addition to the self-intersection local times, the process $\tilde{\psi}_n(\nu)$ also incorporates the mutual intersection local times between different paths in the loop soup. In Theorem 8.3 we give an isomorphism theorem relating permanental Wick powers and self-intersection local times.

In Section 9 we define and study exponential Poisson chaoses and show that many of the processes we consider have a natural extension as martingales.

Here is a summary of the processes we study and a reference to the first place they appear. They are all related to a Lévy process $X$ and are functions of the potential density of $X$. 
\( \alpha_{n, \epsilon}(\nu, t) \) approximate \( n \)-fold self-intersection local time of \( X \), (1.7).

\( L_n(\nu) \) \( n \)-fold renormalized self-intersection local time (of \( X \)), (2.21), (1.9).

\( \mu \) loop measure on the path space of \( X \), (2.11).

\( \mathcal{L}_\alpha \) Poisson point process on the path space of \( X \) with intensity measure \( \alpha \mu \) called the loop soup of \( X \), Section 4.

\( \psi_n(\nu) \) loop soup \( n \)-fold self-intersection local time (of \( X \)). Also called a permanental chaos. \( \psi_1(\nu) \) is also called a permanental field, (4.4), (4.5).

\( \tilde{\psi}_n(\nu) \) \( n \)-th order renormalized permanental field (of \( X \)). Also called a \( 2n \)-th permanental Wick power, (6.5), Theorems 6.1 and 8.2.

\( I_n(g_1, \ldots, g_n) \) Poisson Wick product, page 57.

\( \mathcal{I}_{l_1, \ldots, l_k}(\nu) \) \( l_1 + \cdots + l_k = n \), \( k \)-path, \( n \)-fold intersection local time (of \( X \)), (8.2).

\( \oplus_{n=0}^{\infty} H_n \) Poisson chaos decomposition of \( L^2(\mathcal{P}_{\mathcal{L}_\alpha}) \), (7.36).

\( \sum \mathcal{I}_{|D_1|, \ldots, |D_l|}(\nu) \) Poisson chaos decomposition of \( \tilde{\psi}_n(\nu) \), Theorem 8.2.

\( \mathcal{E}(g) \) exponential Poisson chaos (9.10).

\( I^{(\alpha)}_n, \tilde{\psi}^{(\alpha)}_n \) etc. \( (\mathcal{E}_\mathcal{L}, \mathcal{F}_\alpha) \) martingales, page 78

Several relationships between these processes are given. For example, it follows immediately from in Theorem 8.2 and Corollary 8.1 that

\[
\tilde{\psi}_n(\nu) = \psi_n(\nu) + \sum_{D_1 \cup \cdots \cup D_l = [1, n], l \neq 1} \mathcal{I}_{|D_1|, \ldots, |D_l|}(\nu). \tag{1.25}
\]

Critical estimates are used in all the proofs that require understanding
properties of convolutions of regularly varying functions. These are studied in Section 10.

2 Loop measures and renormalized intersection local times

Let \( X = \{ X_t, t \in \mathbb{R}^+ \} \) be a Lévy process in \( \mathbb{R}^d \) as described in the Introduction. It follows from (1.6) that

\[
\frac{1}{\gamma + \text{Re} \bar{\kappa}(\xi)} \leq \frac{C'}{|\gamma + \bar{\kappa}(\xi)|}, \quad \forall \xi \in \mathbb{R}^d,
\]

for some constant \( C' < \infty \). Together with (1.2) this shows that for each \( t > 0 \)

\[
e^{-t\bar{\kappa}(\xi)} \in \mathcal{L}^1(\mathbb{R}^d),
\]

We define the continuous function

\[
p_t(x) = \frac{1}{(2\pi)^d} \int e^{-ix\xi} e^{-t(\gamma + \bar{\kappa}(\xi))} \, d\xi.
\]

Note that for any \( f \in \mathcal{S}(\mathbb{R}^d) \), the space of rapidly decreasing \( C^\infty \) functions,

\[
\int p_t(x)f(x) \, dx = \int \hat{f}(\xi) e^{-t(\gamma + \bar{\kappa}(\xi))} \, d\xi \quad (2.4)
\]

\[
= \int \hat{f}(\xi) E \left( e^{i\xi X_t} \right) \, d\xi
\]

\[
= E \left( f(X_t) \right).
\]

This shows that \( p_t(x) \) is a (sub)probability density function for \( X_t \). In particular it is integrable on \( \mathbb{R}^d \). Therefore, we can invert the transform in (2.3), and using the fact that \( \bar{\kappa}(\xi) \) is continuous, we obtain

\[
e^{-t(\gamma + \bar{\kappa}(\xi))} = \int e^{ix\xi} p_t(x) \, dx.
\]

We have

\[
\hat{p}_t * \hat{p}_s(\xi) = \hat{p}_t(\xi) \hat{p}_s(\xi) = e^{-(t+s)(\gamma + \bar{\kappa}(\xi))}.
\]

We define \( p_t(x, y) = p_t(y - x) \). It follows from (2.6) that \( \{ p_t(x, y), (x, y, t) \in \mathbb{R}^1_+ \times \mathbb{R}^{2d} \} \) is a jointly continuous semigroup of transition densities for \( X \). We define

\[
u(x, y) = \int_0^\infty p_t(x, y) \, dt.
\]

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Note that \( u(x, y) \) are potential densities for \( X \).

To justify the definition of \( \hat{u} \) in (1.10) note that for any \( f \in S(\mathbb{R}^d) \),

\[
\int p_t(x) f(x) \, dx = \int \hat{f}(\xi) e^{-t(\gamma + \bar{\kappa}(\xi))} \, d\xi, \tag{2.8}
\]

by (2.3). Consequently, by (2.7),

\[
\int u(x) f(x) \, dx = \int \frac{1}{\gamma + \bar{\kappa}(\xi)} \hat{f}(\xi) \, d\xi, \tag{2.9}
\]

in which the use of Fubini’s theorem is justified by (2.1) and (1.2). In this context, \((\gamma + \bar{\kappa}(\xi))^{-1}\) is the Fourier transform of \( u \), considered as a distribution in \( S' \).

Let \( X \) be a Lévy process in \( \mathbb{R}^d \) with transition densities \( p_t(x, y) = p_t(y - x) \) and potential densities \( u(x, y) = u(y - x) \) as described above. We assume that \( u(x) < \infty \) for \( x \neq 0 \), but since we are interested in Lévy processes that do not have local times we are primarily concerned with the case when \( u(0) = \infty \). We also assume that \( \int_0^\infty p_t(0) \, dt < \infty \) and that \( 0 < p_t(x) < \infty \). Most significantly, we do not require that \( p_t(x, y) \), and therefore that \( u(x, y) \), is symmetric.

Under these assumptions it follows, as in [4], that for all \( 0 < t < \infty \) and \( x, y \in \mathbb{R}^d \), there exists a finite measure \( Q^{x,y}_t \) on \( \mathcal{F}_t^- \), of total mass \( p_t(x, y) \), such that

\[
Q^{x,y}_t \left( 1_{\{\zeta > s\}} F_s \right) = P^x \left( F_s p_{t-s}(X_s, y) \right), \tag{2.10}
\]

for all \( F_s \in \mathcal{F}_s, s < t \).

We take \( \Omega \) to be the set of right continuous paths \( \omega \) in \( \mathbb{R}^d_{\Delta} = \mathbb{R}^d \cup \Delta \) where \( \Delta \notin \mathbb{R}^d \), and is such that \( \omega_t = \Delta \) for all \( t \geq \zeta = \inf\{ t > 0 | \omega_t = \Delta \} \). We set \( X_t(\omega) = \omega_t \) and define a \( \sigma \)-finite measure \( \mu \) on \((\Omega, \mathcal{F})\) by the following formula:

\[
\int F \, d\mu = \int_0^\infty \frac{1}{t} \int Q^{x,x}_t \left( F \circ k_t \right) \, dm(x) \, dt, \tag{2.11}
\]

for all \( \mathcal{F} \)-measurable functions \( F \) on \( \Omega \). Here \( k_t \) is the killing operator defined by \( k_t \omega(s) = \omega(s) \) if \( s < t \) and \( k_t \omega(s) = \Delta \) if \( s \geq t \), so that \( k_t^{-1} \mathcal{F} \subset \mathcal{F}_{t-} \). (We often write \( \mu(F) \) for \( \int F \, d\mu \).)

The next lemma is [11, Lemma 2.1], (with \( \nu_j(dx) = g_j(x) \, dx, j = 1, \ldots, k \)).
Lemma 2.1 Let \( k \geq 2 \) and \( g_j, j = 1, \ldots, k \) be bounded integrable functions on \( \mathbb{R}^d \). Then the loop measure \( \mu \) defined in (2.11) satisfies
\[
\mu \left( \prod_{j=1}^{k} \int_{0}^{\infty} g_j(X_t) \, dt \right) = \sum_{\pi \in \mathcal{P}_k} \int \cdots \int u(y_{\pi(1)}, y_{\pi(2)}) \cdots u(y_{\pi(k-1)}, y_{\pi(k)}) u(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(k)}) \prod_{j=1}^{k} g_j(y_j) \, dy_j
\]
where \( \mathcal{P}_k \) denotes the set of permutations of \([1, k]\) on the circle. (For example, \( (1, 2, 3) \), \( (3, 1, 2) \) and \( (2, 3, 1) \) are considered to be one permutation \( \pi \in \mathcal{P}_3 \).)

We show in (4.6) that when \( u(0) = \infty \),
\[
\mu \left( \int_{0}^{\infty} g_j(X_t) \, dt \right) = \infty.
\]

2.1 Renormalized intersection local times

Let \( f(y) \) be a positive smooth function supported in the unit ball of \( \mathbb{R}^d \) with \( \int f(x) \, dx = 1 \). Set \( f_r(y) = r^{-d} f(y/r) \), and \( f_{x,r}(y) = f_r(y-x) \). Let
\[
L(x, r) := \int_{0}^{\infty} f_{x,r}(X_t) \, dt.
\]

\( L(x, r) \) can be thought of as the approximate total local time of \( X \) at the point \( x \in \mathbb{R}^d \). When \( u(0) = \infty \), the local time of \( X \) does not exist and we can not take the limit \( \lim_{r \to 0} L(x, r) \). Nevertheless, it is often the case that renormalized intersection local times exist. We proceed to define renormalized intersection local times.

We begin with the definition of the chain functions
\[
ch_k(r) = \int u(r y_1, r y_2) \cdots u(r y_k, r y_{k+1}) \prod_{j=1}^{k+1} f(y_j) \, dy_j, \quad k \geq 1.
\]

Note that \( ch_k(r) \) involves \( k \) factors of the potential density \( u \), but \( k+1 \) variables of integration. For any \( \sigma = (k_1, k_2, \ldots) \) let
\[
|\sigma| = \sum_{i=1}^{\infty} i k_i \quad \text{and} \quad |\sigma|_+ = \sum_{i=1}^{\infty} (i+1) k_i.
\]
We set \( L_1(x, r) = L(x, r) \) and define recursively
\[
L_n(x, r) = L^n(x, r) - \sum_{\{\sigma | 1 \leq |\sigma| < |\sigma|_+ \leq n\}} J_n(\sigma, r),
\]
where
\[
J_n(\sigma, r) = \frac{n!}{\prod_{i=1}^{\infty} k_i!(n - |\sigma|_+)!} \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x, r).
\]
(Note that \( n - |\sigma|_+ \geq 0 \).)

To help in understanding (2.17) we note that
\[
L_2(x, r) = L^2(x, r) - 2 \text{ch}_1(r) L(x, r)
\]
and
\[
L_3(x, r) = L^3(x, r) - 6 \text{ch}_1(r) L_2(x, r) - 6 \text{ch}_2(r) L(x, r)
\]
\[
= L^3(x, r) - 6 \text{ch}_1(r) L^2(x, r) + (12 \text{ch}_1^2(r) - 6 \text{ch}_2(r)) L(x, r).\]

We show in Remark 2.2 that \( L_n(x, r) \) can also be defined by a generating function.

**Proof of Theorems 1.1 and 1.3**

Set
\[
L_{n, \epsilon}(\nu) = \int L_n(x, \epsilon) d\nu(x).
\]

We show that for \( \nu \in B_{2n}(R^d), d = 1, 2 \) and \( n \geq 1 \),
\[
L_n(\nu) = \lim_{\epsilon \to 0} L_{n, \epsilon}(\nu) \quad \text{exists in } L^p(\mu), \text{ for all } p \geq 2.
\]

The techniques used in the proof (2.22) allow us to show that for \( n = n_1 + \cdots + n_k, k \geq 2 \), and \( \nu_i \in B_{2n_1}(R^d) \),
\[
\mu \left( \prod_{i=1}^k L_{n_i}(\nu_i) \right) = \frac{\prod_{i=1}^k (n_i!)}{n} \sum_{\pi \in \mathcal{M}_n} \int \prod_{j=1}^n u(x_{\pi(j)}, x_{\pi(j)+1}) \prod_{i=1}^k d\nu_i(x_i)
\]
\[
\leq \frac{\mathcal{M}_n}{n} \prod_{i=1}^k n_i! \|\nu_i\|_{2, \tau_{2n_i}},
\]
where \( \pi(n+1) = \pi(1) \) and \( \mathcal{M}_n \) is the set of maps \( \pi : [1, n] \to [1, k] \) with \( |\pi^{-1}(i)| = n_i \) for each \( i \) and such that, if \( \pi(j) = i \) then \( \pi(j+1) \neq i \).
We begin by showing that for approximate identities $f_{r,x}$ and $f_i = f_{r_i,y_i}$, $i = 1, \ldots, m$,

\[
\mu \left( L_n(x,r) \prod_{i=1}^{m} \int_{0}^{\infty} f_i(X_t) \, dt \right) = \sum_{\pi \in P_{m,n}^\otimes} \int \prod_{j=1}^{m+n} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{i=1}^{m} f_i(z_i) \, dz_i \prod_{i=m+1}^{m+n} f_{r,x}(z_i) \, dz_i \\
+ \int E_r(x,z) \prod_{i=1}^{m} f_i(z_i) \, dz_i \prod_{i=m+1}^{m+n} f_{r,x}(z_i) \, dz_i,
\]

where $\pi(m+n+1) = \pi(1)$ and $P_{m,n}^\otimes$ is the subset of permutations $\pi \in P_{m+n}^\otimes$ with the property that for all $j$, when $\pi(j) \in [m+1, m+n]$, then $\pi(j+1) \in [1,m]$. That is, under the permutation $\pi$, no two elements of $[m+1,m+n]$ are adjacent, mod $m+n$. When $m < n$, $P_{m,n}^\otimes$ is empty.

The last term in (2.24) is an error term. It is actually the sum of many terms, some of which may depend on some of the $z_1, \ldots, z_{n+m}$. We use $z$ to designate $z_1, \ldots, z_{n+m}$. Since the functions $f_i$ are probability density functions we write last term in (2.24) as an expectation,

\[
E_f(E_r(x,z)) := \int E_r(x,z) \prod_{i=1}^{m} f_i(z_i) \, dz_i \prod_{i=m+1}^{m+n} f_{r,x}(z_i) \, dz_i.
\]

We show in Section 2.2 that for $\nu \in B_{2n}(R^d)$,

\[
\lim_{r \to 0} \sup_{|z_i| \leq M} \int E_r(x,z) \, d\nu(x) = 0,
\]

for some finite number $M$, which implies that

\[
\lim_{r \to 0} \int E_f(E_r(x,z)) \, d\nu(x) = 0.
\]
4, 5). There are \( n! \) internal permutations in \( P_{m,n} \). This accounts for the factor \( n! \) in (2.18).

For a function \( g(x) \), \( x \in \mathbb{R}^d \) we define \( \Delta_h g(x) = g(x+h) - g(x) \). Note that since \( u(x, y) = u(y - x) \)

\[
\Delta_h u(x, y) = u(x, y + h) - u(x, y) = u(x - h, y) - u(x, y).
\] (2.28)

By (2.12) and (2.14)

\[
\mu \left( L^2(x, r) \prod_{i=1}^m \int_0^\infty f_i(X_t) \, dt \right) \]

\[
= \sum_{\pi \in P_{m+2} \setminus P_{m,2}} \int \prod_{j=1}^{m+2} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{i=1}^m f_i(z_i) \, dz_i \prod_{i=m+1}^{m+2} f_{r,x}(z_i) \, dz_i,
\]

where \( \pi(m + 3) = \pi(1) \). Considering (2.19) it is clear that to obtain (2.24) for \( n = 2 \) we need only show that

\[
2 \text{ch}_1(r) \mu \left( L^2(x, r) \prod_{i=1}^m \int_0^\infty f_i(X_t) \, dt \right)
\] (2.30)

is equal to the second line in (2.29), except that the sum is taken over permutations \( \pi \in P_{m+2} \setminus P_{m,2} \), plus an error term that satisfies (2.26). Note that by Lemma 2.1

\[
\mu \left( L(x, r) \prod_{i=1}^m \int_0^\infty f_i(X_t) \, dt \right)
\]

\[
= \sum_{\pi \in P_{m+1}} \int \prod_{j=1}^{m+1} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{i=1}^m f_i(z_i) \, dz_i \prod_{i=m+1}^{m+2} f_{r,x}(z_i) \, dz_i.
\] (2.31)

Let \( \pi \in P_{m+2} \setminus P_{m,2} \), i.e. for some \( j, j+1 \mod m+2 \), we have \( \pi(j) = m+1, \pi(j+1) = m+2 \), and consider the term on the right-hand side of (2.29) for \( \pi \),

\[
\int \prod_{j=1}^{m+2} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{i=1}^m f_i(z_i) \, dz_i \prod_{i=m+1}^{m+2} f_{r,x}(z_i) \, dz_i.
\] (2.32)

Note that there is a sequence in (2.32) of the form

\[
u(z_a, z_{m+1}) u(z_{m+2}, z_b)
\] (2.33)
where \( a, b \neq m + 1 \) or \( m + 2 \).

Consider a portion of (2.32) of the form
\[
\int u(z_a, z_{m+1}) u(z_{m+1}, z_{m+2}) u(z_{m+2}, z_b) \prod_{i=m+1}^{m+2} f_{r,x}(z_i) \, dz_i
\] (2.34)
\[
= \int u(z_a, x + rz_{m+1}) u(rz_{m+1}, rz_{m+2}) u(x + rz_{m+2}, z_b) f(z_{m+1}) f(z_{m+2}) \, dz_{m+1} \, dz_{m+2}.
\]

Note that
\[
u(z_a, x + rz_{m+1}) u(x + rz_{m+2}, z_b)
= u(z_a, x) u(x, z_b) + (\Delta_{rz_{m+1}} u(z_a, x)) u(x, z_b)
+ u(z_a, x) (\Delta_{-rz_{m+2}} u(x, z_b)) + (\Delta_{rz_{m+1}} u(z_a, x)) (\Delta_{-rz_{m+2}}) u(x, z_b).
\] (2.35)

We abbreviate this by
\[
u(z_a, x + rz_{m+1}) u(x + rz_{m+2}, z_b)
= u(z_a, x) u(x, z_b) + C \Delta (z_a, z_b, x, r, z_{m+1}, z_{m+2}).
\] (2.36)

In this notation we can write (2.34) as
\[
\text{ch}_1(r) u(z_a, x) u(x, z_b) + \int C \Delta (z_a, z_b, x, r, z_{m+1}, z_{m+2})
\int u(rz_{m+1}, rz_{m+2}) f(z_{m+1}) f(z_{m+2}) \, dz_{m+1} \, dz_{m+2}.
\] (2.37)

Using (2.36) again with \( z_{m+1} = z_{m+2} = z \) and the fact that the integral of \( f \) is equal to 1, we see that (2.37) is equal to
\[
\text{ch}_1(r) \left( \int u(z_a, x + rz) u(x + rz, z_b) f(z) \, dz \right)
- \text{ch}_1(r) \left( \int C \Delta (z_a, z_b, x, r, z, z) f(z) \, dz \right)
+ \int C \Delta (z_a, z_b, x, r, z_{m+1}, z_{m+2}) u(rz_{m+1}, rz_{m+2}) f(z_{m+1}) f(z_{m+2}) \, dz_{m+1} \, dz_{m+2}.
\] (2.38)

Using (2.38) and the identity
\[
\int u(z_a, x + rz) u(x + rz, z_b) f(z) \, dz = \int u(z_a, z) u(z, z_b) f_{r,x}(z) \, dz
\] (2.39)
we can write the integral in (2.32) as
\[ \text{ch}_1(r) \int \prod_{j=1}^{m+1} u(z_{\pi'(j)}, z_{\pi'(j+1)}) f_{r,x}(z_{m+1}) dz_{m+1} \prod_{i=1}^{m} f_i(z_i) dz_i + E_f \left( H'_r(x, z) \right), \] (2.40)
where \( \pi' \) is the permutation in \( \mathcal{P}_{m+1}^{\circ} \) obtained from \( \bar{\pi} \) by removing \( m+2 \) from the sequence \( (\bar{\pi}(1), \ldots, \bar{\pi}(m+2)) \) and \( H'_r(x, z) \) contains the error terms which are given in the last two lines of (2.38). (All of them have at least one factor of the form \( \Delta \cdot u(\cdot) \)). Moreover we can repeat the argument in the last three paragraphs when \( \bar{\pi}(j) = m+2 \) and \( \bar{\pi}(j+1) = m+1 \), so that there are a total of 2 terms that can rewritten as the integral in (2.40). Using this and (2.31) we establish (2.24) for \( n = 2 \).

Assume that (2.24) is proved for \( L_{n'}(x, r), n' < n \). For any \( \sigma = (k_1, k_2, \ldots) \) let \( \mathcal{P}_{m+n}^{\circ}(\sigma) \) denote the set of permutations \( \bar{\pi} \in \mathcal{P}_{m+n}^{\circ} \) that contain \( k_i \) chains of order \( i = 1, 2, \ldots \) in \([m+1, m+n] \). (A chain of order \( i \geq 1 \) is a sequence \( \bar{\pi}(j), \bar{\pi}(j+1), \ldots, \bar{\pi}(j+i) \) in \([m+1, m+n] \) which is maximal in the sense that \( \bar{\pi}(j-1) \) and \( \bar{\pi}(j+i+1) \) are not in \([m+1, m+n] \). In such a case we refer to \( j, j+1, \ldots, j+i \) as chain integers.) Note that \( \mathcal{P}_{m+n}^{\circ} = \bigcup_{|\sigma| \geq 1} \mathcal{P}_{m+n}^{\circ}(\sigma) \).

In the same way we obtained (2.38) and (2.40), we see that the term for any \( \bar{\pi} \in \mathcal{P}_{m+n}^{\circ}(\sigma) \) in the evaluation of
\[ \mu \left( L^n(x, r) \prod_{i=1}^{m} \int_0^{\infty} f_i(X_t) dt \right), \] (2.41)
the generalization of (2.29), is the same as the term in (2.24) for a particular permutation \( \pi' \in \mathcal{P}_{m,n-|\sigma|}^{\circ} \) in the evaluation of
\[ \mu \left( \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x, r) \prod_{i=1}^{m} \int_0^{\infty} f_i(X_t) dt \right), \] (2.42)
up to error terms \( H'_r(x, z) \). To see this we note that (2.42) can be written as
\[ \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} \mu \left( L_{n-|\sigma|}(x, r) \prod_{i=1}^{m} \int_0^{\infty} f_i(X_t) dt \right). \] (2.43)
We use Lemma 2.1 to write out (2.41), as in (2.31), and (2.24) and the induction hypothesis to write out (2.42) so they can be easily compared. The permutation \( \pi' \) is obtained from \( \bar{\pi} \) by a method we call ‘remove and relabel’,
which is used in the much simpler case considered in (2.40). We illustrate this with an example. Consider the case in which $m = 10$, $n = 8$ and

$$\bar{\pi} = (6, 7, 11, 13, 8, 9, 10, 1, 14, 12, 16, 2, 3, 4, 15, 5, 17, 18). \quad (2.44)$$

There are three chains in this sequence:

$$(11, 13) \quad (14, 12, 16) \quad (17, 18) \quad (2.45)$$

so that $\sigma = (2, 1, 0, \ldots)$ and $\bar{\pi} \in \mathcal{P}_{18}^\odot(2, 1, 0, \ldots, 0)$. We first remove all but the first element in each chain in (2.44) to obtain

$$\bar{\pi}' = (6, 7, 11, 8, 9, 10, 1, 14, 2, 3, 4, 15, 5, 17). \quad (2.46)$$

The permutation $\pi' \in \mathcal{P}_{10,4}^\odot$ is obtained from (2.46) by relabeling the remaining elements in $\{11, \ldots, 18\}$ in increasing order from left to right, i.e.,

$$\pi' = (6, 7, 11, 8, 9, 10, 1, 12, 2, 3, 4, 13, 5, 14). \quad (2.47)$$

Let $|\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)|_{\pi'}$ denote the number of permutations in $\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)$ that give rise to $\pi'$. We compute $|\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)|_{\pi'}$. Clearly, any of the 8! permutations of the elements $\{11, 12, \ldots, 18\}$ in $\bar{\pi}$ give rise to distinct permutations in $\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)_{\pi'}$. We call these internal permutations. Furthermore, we consider the single integer 15 in $\bar{\pi}$ to be a chain of order zero. Adding this chain to the three in (2.45) allows us to consider that $\bar{\pi}$ contains four chains. Clearly, each of the 4! arrangements of these four chains correspond to distinct permutations in $\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)_{\pi'}$. However, we do not want to count the interchanges of the two chains of order one, since they are counted in the internal permutations. Consequently

$$|\mathcal{P}_{18}^\odot(2, 1, 0, \ldots)|_{\pi'} = \frac{8!4!}{2!}. \quad (2.48)$$

For general $\sigma$ and $\pi' \in \mathcal{P}_{m+n-|\sigma|}^\odot$, in which, as in the example above, the integers $m+1, \ldots, m+n-|\sigma|$ appear in increasing order,

$$|\mathcal{P}_{m+n}^\odot(\sigma)|_{\pi'} = \frac{n!}{\prod_{i=1}^{\infty} k_i! (n-|\sigma|)_i!}. \quad (2.49)$$

To see this first note that there are $n!$ internal permutations. Since for any $\bar{\pi} \in \mathcal{P}_{m+n}^\odot(\sigma)_{\pi'}$ there are $|\sigma|_+$ integers from $\{m+1, \ldots, m+n\}$ in the chains of order 1, 2, $\ldots$, there are also $n-|\sigma|_+$ remaining integers in $\{m+1, \ldots, m+n\}$ which, as above, we consider to be chains of order 0. The total number of these
chains, including those of order 0, is \( n - |\sigma| + \sum_{i=1}^{\infty} k_i = n - |\sigma| \). Thus any of the \((n - |\sigma|)!\) permutations of these chains in \( \bar{\pi} \) are in \( \mathcal{P}_{m+n}^\circ(\sigma)_{\pi'} \). However, we do not want to count the \((n - |\sigma|)! \prod_{i=1}^{\infty} k_i!\) interchanges of chains of order 0 and \( k_i \) among themselves, since this has already been counted in the internal permutations. Putting all this together gives (2.49).

Consider (2.42) again and the particular permutation \( \pi' \in \mathcal{P}_{m,n-|\sigma|}^\circ \). We have already pointed out that there are \((n - |\sigma|)!\) different permutations, the internal permutations, in \( \mathcal{P}_{m,n-|\sigma|}^\circ \), for which

\[
\mu \left( \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x,r) \prod_{i=1}^{m} \int_0^\infty f_i(X_t) \, dt \right) \tag{2.50}
\]

is the same as it is for \( \pi' \). Therefore up to the error terms, the contribution to (2.41) from \( \mathcal{P}_{m+n}^\circ(\sigma) \) is equal to

\[
\prod_{i=1}^{\infty} k_i!(n - |\sigma|)! \mu \left( \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x,r) \prod_{i=1}^{m} \int_0^\infty f_i(X_t) \, dt \right) = \mu \left( J_n(\sigma,r) \prod_{i=1}^{m} \int_0^\infty f_i(X_t) \, dt \right). \tag{2.51}
\]

Considering (2.17) and the fact that \( \mathcal{P}_{m+n}^\circ \subset \mathcal{P}_{m,n}^\circ \cup |\sigma| \geq 1 \mathcal{P}_{m+n}^\circ(\sigma) \), we see that the induction step in the proof of (2.24) is proved.

We iterate the steps used in the proof of (2.24), and use the fact that each of the \( L_{n_i}(x_i, r) \) are sums of multiples of \( L(x_i, r) = \int_0^\infty f_{\tau, x_i}(X_t) \, dt \) to obtain

\[
\mu \left( \prod_{i=1}^{k} L_{n_i}(x_i, r_i) \right) \tag{2.52}
\]

\[
= \sum_{\pi \in \mathcal{P}_{n_1, \ldots, n_k}^\circ} \int \prod_{j=1}^{n} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{j=1}^{n} f_{\tau(g(j), x_{g(j)}(z_j))} \, dz_j + \mathcal{E}_\mathcal{L}(E_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)),
\]

where \( \pi(n+1) = \pi(1) \) and \( \mathcal{P}_{n_1, \ldots, n_k}^\circ \) is the set of permutations \( \pi \) of \([1, n]\) on the circle, with the property that for all \( j \), when \( \pi(j) \in [1 + \sum_{p=1}^{i-1} n_p, \sum_{p=1}^{i-1} n_p] : B_i \) then \( \pi(j + 1) \notin B_i \), for all \( i \in [1, k] \), and \( g(j) = i \) when \( j \in B_i \). (In the last term in (2.52) we use the notation introduced in (2.25).)
Note that
\[
\int \prod_{j=1}^{n} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{j=1}^{n} f_{r_g(j), x_g(j)}(z_j) \, dz_j = (2.53)
\]
\[
\int \prod_{j=1}^{n} u(x_g(\pi(j)), x_g(\pi(j)+1)) \prod_{j=1}^{n} f(z_j) \, dz_j.
\]
For each \(j = 1, \ldots, n\) we write
\[
u(x_g(\pi(j)) + r_g(\pi(j))\delta_{\pi(j)}, x_g(\pi(j)+1)) + r_g(\pi(j+1))\delta_{\pi(j+1)})
\]
\[
u(x_g(\pi(j)), x_g(\pi(j)+1)) + \Delta_h u(x_g(\pi(j)), x_g(\pi(j)+1)))
\]
where \(h = r_g(\pi(j+1))\delta_{\pi(j+1)} - r_g(\pi(j))\delta_{\pi(j)}\). Substituting this into (2.53) and putting all the terms with one or more \(\Delta\) into \(E'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)\) we see that (2.53) is equal to
\[
\int \prod_{j=1}^{n} u(x_g(\pi(j)), x_g(\pi(j)+1)) \prod_{j=1}^{n} f(z_j) \, dz_j + \mathcal{E}(E'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)).
\]
We use (2.53) and (2.55) in (2.52) and sum over \(\pi \in M_a\), rather than \(\pi \in \mathcal{P}_{n_1, \ldots, n_k}\), to get
\[
\mu \left( \prod_{i=1}^{k} L_{n_i}(x_i, r_i) \right) = \frac{n_{i_1}}{n} \prod_{i=1}^{k} u(x_{\pi(j)}, x_{\pi(j+1)}) + \mathcal{E}(E'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)).
\]
We use the fact that each term in the sum in (2.56) comes from \(\prod_{i=1}^{k} n_i!\) different terms in (2.52). The factor \(1/n\) comes from the fact that \(\mathcal{P}_{n_1, \ldots, n_k}\) contains permutations of \([1, n]\) on the circle, whereas \(M_a\) does not.

We integrate both sides of (2.56) with respect to \(\prod_{i=1}^{k} \nu_i(x_i)\) to get
\[
\mu \left( \prod_{i=1}^{k} L_{n_i}(x_i, r_i) \right)
\]
\[
eq \frac{n_{i_1}}{n} \prod_{i=1}^{k} \int u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{i=1}^{k} \nu_i(x_i)
\]
\[
+ \mathcal{E}(E'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)) \prod_{i=1}^{k} \nu_i(x_i).
\]
We show in Lemma 2.2 that for fixed \( \{ n_i \} \), and \( \{ \nu_i \} \)
\[
\int \prod_{j=1}^{n} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{i=1}^{k} dv_i(x_i) \leq \prod_{i=1}^{k} \| \nu_i \|_{2, r_2 n_i},
\]  
(2.58)

We show in Section 2.2 that
\[
\lim_{|r| \to 0} \sup_{|z| \leq M} \int E_{r_1, \ldots, r_k} (x_1, \ldots, x_k, z) \prod_{i=1}^{k} dv_i(x_i) = 0,
\]  
(2.59)

where \( r = (r_1, \ldots, r_k) \).

We can now prove Theorem 1.1. By (2.57)-(2.59), for \( \nu \in B_{2n} \),
\[
\mu \left( \left( \int L_n(x, r) \, d\nu(x) \right)^p \right) < \infty
\]  
(2.60)

for all \( 0 < r \leq r_0 \) and all even integers \( p \geq 2 \). By (2.57)
\[
\mu \left( (L_{n,r}(\nu))^j (L_{n,r'}(\nu))^{p-j} \right) = \frac{\prod_{i=1}^{k} (n!)^p}{np} \sum_{\pi \in M_n} \int \prod_{j=1}^{np} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{i=1}^{p} dv_i(x_i)
\]  
\[+ \int E_{r, r', j} (x_1, \ldots, x_p, z) \prod_{i=1}^{p} dv_i(x_i)
\]  
:= A + \mathcal{I}_{r, r', j}(z). 

Therefore
\[
\mu \left( (L_{n,r}(\nu) - L_{n,r'}(\nu))^p \right) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \left( A + \mathcal{I}_{r, r', j}(z) \right)
\]  
(2.62)

\[
= \sum_{j=0}^{p} (-1)^j \binom{p}{j} \mathcal{I}_{r, r', j}(z) \leq 2^p \sum_{j=0}^{p} \mathcal{I}_{r, r', j}(z).
\]

Consequently, by (2.59)
\[
\lim_{r, r' \to 0} \mu \left( (L_{n,r}(\nu) - L_{n,r'}(\nu))^p \right) = 0.
\]  
(2.63)

This gives (1.9) for all even integers \( p \geq 2 \). Furthermore, we can interpolate to see that it holds for all \( p \geq 2 \). This completes the proof of Theorem 1.1.

Now that we have Theorem 1.1, we can return to (2.57) and take the limit as the \( r_i \to 0 \) to complete the proof of Theorem 1.3. \( \square \)
Remark 2.1 Theorem 1.3 does not give the value of $\mu(L_n(\nu))$ for any $n$. When $n = 1$ it follows from (2.13) that $\mu(L_1(x,r)) = \infty$. In general, for $n \geq 2$, $\mu(L_n(x,r)) = \pm \infty$. See, for example, (2.19) and (2.20).

2.2 Bounds for the error terms

As in (1.12) and (1.16) we define

\[ \|\nu\|_{2,\tau_k} = \left( \int |\hat{\nu}(\lambda_1 + \cdots + \lambda_k)|^2 \prod_{j=1}^{k} |\hat{u}(\lambda_j)| d\lambda_j \right)^{1/2} \tag{2.64} \]

where $u$ is the potential density of a Lévy process in $\mathbb{R}^d$ and $\nu$ is a finite measure on $\mathbb{R}^d$.

Lemma 2.2 For any $n_i, i = 1, \ldots, k$

\[ \int \prod_{j=1}^{n} u(x_{\pi(j+1)} - x_{\pi(j)}) \prod_{i=1}^{k} d\nu_i(x_i) \leq \frac{1}{(2\pi)^{nd}} \prod_{i=1}^{k} \|\nu_i\|_{2,\tau_{2n_i}}, \tag{2.65} \]

where $\pi(n + 1) = \pi(1), n = \sum_{i=1}^{k} n_i$ and $\pi : [1, n] \mapsto [1, k]$ is such that $|\pi^{-1}(i)| = n_i$ for each $i = 1, \ldots, k$, and has the property that when $\pi(j) = i$, $\pi(j+1) \neq i$.

In particular for any $\pi \in \mathcal{P}_n$ and smooth function $f$ with compact support

\[ \int \prod_{j=1}^{n} u(x_{\pi(j+1)} - x_{\pi(j)}) \prod_{i=1}^{n} f(x_i) dx_i \leq \frac{1}{(2\pi)^{nd}} \left( \int |\hat{f}(\lambda)|^2 \tau_2(\lambda) d\lambda \right)^{n/2} < \infty. \tag{2.66} \]

Proof Since the integrand is positive, and the $\nu_i$ are finite measures, we can use Fubini’s theorem to see that

\[ \int \prod_{j=1}^{n} u(x_{\pi(j+1)} - x_{\pi(j)}) \prod_{i=1}^{k} d\nu_i(x_i) \tag{2.67} \]

\[ = \int_{\mathbb{R}_+^k} \left( \int \prod_{j=1}^{n} \nu_j(x_{\pi(j+1)} - x_{\pi(j)}) \prod_{i=1}^{k} d\nu_i(x_i) \right) \prod_{j=1}^{n} e^{-\gamma t_j} dt_j. \]
Considering (2.2) we can use Fubini’s theorem again to see that the inner integral immediately above, is equal to \((2\pi)^n\) times
\[
\int \left( \prod_{j=1}^{n} \int e^{-i(x_{\pi(j+1)} - x_{\pi(j)})^2} e^{-t_j \tilde{\kappa}(\lambda_j)} d\lambda_j \right) \prod_{i=1}^{k} d\nu_i(x_i) \tag{2.68}
\]
\[
= \int \left( \prod_{i=1}^{k} \int e^{i(\sum J_i \pm \lambda_j) - x_i} d\nu_i(x_i) \right) \prod_{j=1}^{n} e^{-t_j \tilde{\kappa}(\lambda_j)} d\lambda_j
\]
\[
= \int \left( \prod_{i=1}^{k} \hat{\nu}_i \left( \sum J_i \pm \lambda_j \right) \right) \prod_{j=1}^{n} e^{-t_j \tilde{\kappa}(\lambda_j)} d\lambda_j.
\]
In this formulation \(J_i = \pi^{-1}(i) \cup \{\pi^{-1}(i) + 1\}\), and the sum is taken over all \(\lambda_j \in J_i\), half of which are multiplied by \(-1\). (The cardinality \(|J_i| = 2n_i\), \(i = 1, \ldots, k\) and each \(\lambda_j, j = 1, \ldots, n\) appears twice, once in each of two distinct \(J_i\), and is multiplied by \(-1\) in one of its appearances. It is not necessary to be more specific.)

Using (2.68) in (2.67) we see that (2.67) is bounded by
\[
\int_{R^n} \int \left( \prod_{i=1}^{k} \hat{\nu}_i \left( \sum J_i \pm \lambda_j \right) \right) \prod_{j=1}^{n} e^{-t_j \tilde{\kappa}(\lambda_j)} d\lambda_j \prod_{j=1}^{n} e^{-\gamma t_j} dt_j \tag{2.69}
\]
\[
= \int_{R^n} \int \left( \prod_{i=1}^{k} \hat{\nu}_i \left( \sum J_i \pm \lambda_j \right) \right) \prod_{j=1}^{n} e^{-t_j} \text{Re} \tilde{\kappa}(\lambda_j) d\lambda_j \prod_{j=1}^{n} e^{-\gamma t_j} dt_j
\]
\[
= \int \left( \prod_{i=1}^{k} \hat{\nu}_i \left( \sum J_i \pm \lambda_j \right) \right) \prod_{j=1}^{n} \frac{1}{\gamma + \text{Re} \tilde{\kappa}(\lambda_j)} d\lambda_j
\]
\[
\leq C^n \int \prod_{i=1}^{k} \left| \hat{\nu}_i \left( \sum J_i \pm \lambda_j \right) \right| \prod_{j=1}^{n} \left| \hat{u}(\lambda_j) \right| d\lambda_j.
\]

Here the second equality uses Fubini’s theorem since the integrand is positive, and the last inequality follows from (2.1).

Repeated applications of the Cauchy-Schwarz inequality to the final line of (2.69) and the eventual recognition that we can change \(\{\pm \lambda_j\}\) to \(\{\lambda_j\}\) gives (2.65). We give some idea of how this goes. Assume for definiteness that \(\lambda_1\)
appears in \( J_l \) and \( J_m \). Then by the Cauchy-Schwarz inequality

\[
\int \prod_{i=1}^{k} |\hat{v}_l (\sum_{j} \pm \lambda_j) | \prod_{j=1}^{n} |\hat{u}(\lambda_j)| \ d\lambda_j
\]

\[
\leq \int \prod_{i \neq l, m} |\hat{v}_l (\sum_{j} \pm \lambda_j) | \left( \int |\hat{v}_l (\sum_{j} \pm \lambda_j) |^2 |\hat{u}(\lambda_1)| \ d\lambda_1 \right)^{1/2}
\]

\[
\left( \int |\hat{v}_m (\sum_{j} \pm \lambda_j) |^2 |\hat{u}(\lambda_1)| \ d\lambda_1 \right)^{1/2} \prod_{j=2}^{n} |\hat{u}(\lambda_j)| \ d\lambda_j.
\]

Next assume now that \( \lambda_2 \) appears in \( J_l \) and \( J_m' \). Using the Cauchy-Schwarz inequality again we see that

\[
\leq \int \prod_{i \neq l, m, m'} |\hat{v}_l (\sum_{j} \pm \lambda_j) | \left( \int |\hat{v}_l (\sum_{j} \pm \lambda_j) |^2 |\hat{u}(\lambda_1)| \ d\lambda_1 \right)^{1/2}
\]

\[
\left( \int |\hat{v}_m (\sum_{j} \pm \lambda_j) |^2 |\hat{u}(\lambda_1)| \ d\lambda_1 \right)^{1/2} \left( \int |\hat{v}_{m'} (\sum_{j} \pm \lambda_j) |^2 |\hat{u}(\lambda_2)| \ d\lambda_2 \right)^{1/2}
\]

\[
\prod_{j=3}^{n} |\hat{u}(\lambda_j)| \ d\lambda_j.
\]

Let us continue and concentrate on the the integrals of \(|\hat{v}_l (\sum_{j} \pm \lambda_j) |^2\). Since \(|J_l| = 2n_i\), the procedure above will ultimately result in the term

\[
\left( \int |\hat{v}_l (\sum_{j} \pm \lambda_j) |^2 \prod_{i=1}^{2n_i} |\hat{u}(\lambda_i)| \ d\lambda_i \right)^{1/2},
\]

where \( \lambda_i \) is an ordering of the \( \lambda_j \in J_l \). Now we can use the fact that \(|\hat{u}(-\lambda)| = |\hat{u}(\lambda)|\) to replace \( |\hat{v}_l (\sum_{j} \pm \lambda_j)|\) by \( |\hat{v}_l (\sum_{j} \lambda_j)|\). With this change (2.72) is equal to \( \|\nu\|_{2,2n_i} \), in which the norm is written as in the first equation in (2.64). The other terms in (2.65) follow similarly.

That (2.66) is finite follows from (1.11), Lemma 10.2 and the fact that \( \hat{f} \) is bounded and rapidly decreasing. \( \square \)

The following is an immediate corollary of Lemma 2.2

**Corollary 2.1**

\[
\prod_{i=1}^{k} \frac{n_i!}{n} \sum_{\pi \in \mathcal{M}_a} \int \prod_{j=1}^{n} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{i=1}^{k} d\nu_i(x_i) \leq \frac{|\mathcal{M}_a|}{n} \prod_{i=1}^{k} n_i! \|\nu_i\|_{2,2n_i},
\]

(2.73)
We now deal with the error terms. It should be clear that these come about in many different ways. We begin by considering a relatively simple way that they occur. Consider \((2.52)\). The error terms represented by \(E_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)\) contain chains or variables, which when the analysis is complete, give rise to chains. More error terms are introduced when we show that

\[
\sum_{\pi \in \mathcal{P}_{n_1, \ldots, n_k}} \prod_{j=1}^{n} u(z_{\pi(j)}, z_{\pi(j+1)}) \prod_{j=1}^{n} f_{r_g(j)}(x_{g(j)}(z_j)) \, dz_j \tag{2.74}
\]

\[
= \prod_{i=1}^{k} \frac{n_i!}{n} \sum_{\pi \in \mathcal{M}, j=1}^{n} u(x_{\pi(j)}, x_{\pi(j+1)}) + E_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)\]

in \((2.52)–(2.56)\). (Note that \(F'\) is not the same as \(E'\) in \((2.56)\) because \(E'\) also contains the error terms in \(E\) in \((2.52)\).)

\section*{Lemma 2.3}

\[
\lim_{|r| \to 0} \sup_{|z| \leq M} \left| \int F'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z) \prod_{i=1}^{k} d\nu_i(x_i) \right| = 0. \tag{2.75}
\]

\section*{Proof}

Consider \((2.53)–(2.55)\) and set

\[
h(g, \pi, j) = r_{g(\pi(j+1))} z_{\pi(j+1)} - r_{g(\pi(j))} z_{\pi(j)}. \tag{2.76}
\]

The expression \(F'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)\) consists of \(2^n - 1\) error terms created in the transition from \((2.53)–(2.55)\). Each of them is of the form

\[
\int \prod_{j=1}^{\ell-1} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{j=\ell}^{n} \Delta_{h(g, \pi, j)} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{j=1}^{n} f(z_j) \, dz_j, \tag{2.77}
\]

for some \(\ell \leq n\).

Let

\[
V(x_1, \ldots, x_k; \pi) := \prod_{j=1}^{\ell-1} u(x_{\pi(j)}, x_{\pi(j+1)}) \prod_{j=\ell}^{n} \Delta_{h(g, \pi, j)} u(x_{\pi(j)}, x_{\pi(j+1)}). \tag{2.78}
\]

We go through the steps in the proof of Lemma 2.2 to see that

\[
\left| \int V(x_1, \ldots, x_k; \pi) \prod_{i=1}^{k} d\nu_i(x_i) \right| \leq \int \prod_{i=1}^{k} \left| \hat{\nu}_i \left( \sum_{J_i} \pm \lambda_j \right) \right| \prod_{j=1}^{n} |T_{h(g, \pi, j)} u(\lambda_j)| \, d\lambda_j, \tag{2.79}
\]

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where $T_{h(g, π, j)}$ is either the identity or $Δ_{h(g, π, j)}$ and at least for one $1 \leq j \leq n$, $T_{h(g, π, j)}$ is of the form $Δ_{h(g, π, j)}$. Note that in general

\[
|Δ_{h(g, π, j)}u(λ_j)| = |1 - e^{iθh(g, π, j)}λ_j| |\hat{u}(λ_j)|
\]

(2.80)

\[
\leq \left( |1 - e^{irg(π(j + 1))zi(j + 1)}λ_j| + |1 - e^{irg(π(j))zi(j)}λ_j| \right) |\hat{u}(λ_j)|
\]

\[
\leq 4|\hat{u}(λ_j)|.
\]

We use the bound $|Δ_{h(g, π, j)}u(λ_j)| \leq 4|\hat{u}(λ_j)|$ for all but one of the terms $|Δ_{h(g, π, j)}u(λ_j)|$ and use the bound in the second line of (2.80) for only one of the terms $|Δ_{h(g, π, j)}u(λ_j)|$.

To simplify the notation let us suppose that for this choice $j = 1$ and we have

\[
|Δ_{h(g, π, 1)}u(λ_1)| \leq \left( |1 - e^{i r' z' λ_1}| + |1 - e^{i r'' z'' λ_1}| \right) |\hat{u}(λ_1)|.
\]

(2.81)

(The reader will see that it doesn’t matter what the specific values of $r', r'', z', z''$ are.)

It follows from this that the expression in (2.79) is

\[
\leq 2^n \int \prod_{i=1}^{k} |\hat{v}_i (\sum J_i \pm λ_j) | |1 - e^{i r' z' λ_1}| \prod_{j=1}^{n} |\hat{u}(λ_j)| dλ_j
\]

(2.82)

\[
+ 2^n \int \prod_{i=1}^{k} |\hat{v}_i (\sum J_i \pm λ_j) | |1 - e^{i r'' z'' λ_1}| \prod_{j=1}^{n} |\hat{u}(λ_j)| dλ_j.
\]

We define $J_l$ and $J_m$ as in (2.70) and apply the Cauchy Schwarz inequality as in (2.70) and get that (2.79)

\[
\leq 2^n \int \prod_{i \neq l, m} |\hat{v}_i (\sum J_i \pm λ_j) | \left( \int |\hat{v}_l (\sum J_l \pm λ_j) |^2 |1 - e^{i r' z' λ_1}|^2 |\hat{u}(λ_1)| dλ_1 \right)^{1/2}
\]

\[
\times \left( \int |\hat{v}_m (\sum J_m \pm λ_j) |^2 |\hat{u}(λ_1)| dλ_1 \right)^{1/2} \prod_{j=2}^{n} |\hat{u}(λ_j)| dλ_j,
\]

(2.83)

plus a similar term but with $r', z'$ replaced by $r'', z''$. Note that in applying the Cauchy Schwarz inequality we take

\[
|1 - e^{i r' z' λ_1}| |\hat{u}(λ_1)| = \left( |1 - e^{i r' z' λ_1}| |\hat{u}(λ_1)|^{1/2} \right) |\hat{u}(λ_1)|^{1/2},
\]

(2.84)
so that we get $|1 - e^{ir'z'\lambda_1}|^2 |\hat{u}(\lambda_1)|$ in one of the terms we integrate with respect to $\nu_1$.

We proceed as in the proof of Lemma 2.2 to get
\[
\left| \int V(x_1, \ldots, x_k; \pi) \prod_{i=1}^k d\nu_i(x_i) \right| \leq 2^n \sup_{|z'| \leq 1} \|\nu_1\|_{2,2n_1, r', \|\hat{u}(\lambda_j)\|_{d\lambda}} \prod_{i=2}^k \|\nu_i\|_{2,2n_i}, \tag{2.85}
\]
plus a similar term but with $r', z'$ replaced by $r'', z''$. Here
\[
\|\nu\|_{2,\tau_k, r', z'} = \left( \int |\hat{\nu}(\lambda_1 + \cdots + \lambda_k)|^2 |1 - e^{ir'z'\lambda_1}|^2 \prod_{j=1}^k |\hat{u}(\lambda_j)| d\lambda_j \right)^{1/2} \tag{2.86}
\]
and
\[
\tau_{k, r', z'}(\lambda) = \int |1 - e^{ir'z'\lambda_1}|^2 |\hat{u}(\lambda_1)| \tau_{k-1}(\lambda - \lambda_1) d\lambda_1. \tag{2.87}
\]
We also use the fact that $|z'| \leq 1$, since $z'$ is in the domain of $f$ which is the unit ball of $R^d$. Since the right-hand side of (2.85) does not depend on $z'$ and the integral over $\prod_{j=1}^n f(z_j) dz_j$ is equal to one, we see that each of the error terms
\[
\leq 4^n \sup_{|z'| \leq M} \|\nu_1\|_{2,2n_1, r', \|\hat{u}(\lambda_j)\|_{d\lambda}} \prod_{i=2}^k \|\nu_i\|_{2,2n_i}. \tag{2.88}
\]
(Here we combine the terms in $r'$, $z'$ and $r''$, $z''$.)

Putting this together we see that there exists a $j \in [1, k]$ such that
\[
\int F_{r'_1, \ldots, r'_k}(x_1, \ldots, x_k, z) \prod_{j=1}^k d\nu(x_i) \leq 8^n \sup_{|z'| \leq M} \|\nu_j\|_{2,2n_j, r_jz'} \prod_{i=1, i \neq j}^k \|\nu_i\|_{2,2n_i}, \tag{2.89}
\]
where $z'$ may be any of the values $z_1, \ldots, z_n$. Note that by (1.11)
\[
\tau_{k, r', z'}(\lambda) \leq \partial_k (r'z', \lambda), \tag{2.90}
\]
but with $h$ replaced by $\theta_n$. (See Lemma 10.4 and the comments in Section 10.) Therefore, by (10.41) and (10.65), for any $2 \leq k \leq 2n_1$, and $1 \leq i \leq k$
\[
\|\nu_i\|_{2,\tau_k, r', z'} \leq o\left( \tilde{H}(1/|r'z'|) \right)^{n_1-k/2}, \text{ as } r' \to 0, \tag{2.91}
\]

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and, in particular,
\[ \sup_{|z'| \leq M} \| \nu_t \|_{2, r, 2n, 1, r'z'} = o(1), \quad \text{as } r' \to 0. \tag{2.92} \]
Consequently
\[ \lim_{|r| \to 0} \int F'_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z) \prod_{j=1}^{k} d\nu(x_i) = 0. \tag{2.93} \]

We now deal with the error terms created when we form the chains. Let us first note that these are similar regardless of the length of the chain. Suppose we have a chain of length \( \ell \). Following the analysis in (2.34)–(2.38) we see that in place of (2.37) we get
\[ \text{ch}_{\ell}(r) \, u(z_a, x) u(x, z_b) \tag{2.94} \]
\[ + \int C_{\Delta}(z_a, z_b, x, r, z_1, z_{\ell+1}) \prod_{i=1}^{\ell} u(r z_i, r z_{i+1}) \prod_{i=1}^{\ell+1} f(z_i) \, dz_i. \]
Here \( z_1 \) and \( z_{\ell+1} \) are the variables at the end of the chain and \( z_a, z_b \) and \( x \) are variables that are not in this chain. Moreover, the variable \( x \) is in the same interval that contains the variables in the chain, i.e. if the chain consists of a sequence of variables \( z_{\pi(j)} \) with \( \pi(j) = i \) then \( x \) is an \( x_i \), that is to be integrated, generally along with other terms, by \( d\nu_i \). Also note that there may be many chains, of various length, that consist of sequences of variables in the \( i \)-th interval. Similarly, in place of (2.38) we get
\[ \text{ch}_{\ell}(r) \left( \int u(z_a, x + rz) u(x + rz, z_b) f(z) \, dz \right) \tag{2.95} \]
\[ - \text{ch}_{\ell}(r) \left( \int C_{\Delta}(z_a, z_b, x, r, z, z) f(z) \, dz \right) \]
\[ + \int C_{\Delta}(z_a, z_b, x, r, z_1, z_{\ell+1}) \prod_{i=1}^{\ell} u(r z_i, r z_{i+1}) \prod_{i=1}^{\ell+1} f(z_i) \, dz_i. \]
The integrals containing the \( C_{\Delta} \) are in the error terms.

Note that we can not extract \( \text{ch}_{\ell}(r) \) from the last line in (2.95) because \( z_1 \) and \( z_{\ell+1} \) are in \( C_{\Delta}(z_a, z_b, x, r, z_1, z_{\ell+1}) \). They give rise to terms like \( \Delta_{rx} u(z_a, x) \) in (2.35) that contain variables that are not in the chains. Therefore, to evaluate the error integrals we put off integrating with respect to any of the \( z \)
variables and first integrate with respect to the measures $\nu_i$. To this end we go back to the definition of the chain functions in (2.15) and write

$$\text{ch}_\ell(r') = \int u(r'z_1', r'z_2') \cdots u(r'z_\ell', r'z_{\ell+1}') \prod_{j=1}^{\ell+1} f(z'_j) \, dz'_j \quad (2.96)$$

We refer to the terms $\prod_{i=1}^\ell u(r'z'_i, r'z'_{i+1})$ as chain integrands.

We arrange the order of integration in the error terms so that we integrate with respect to the $\{z_i\}_{i=1}^n$ first. Doing this we can write a typical error term in the form

$$\int \left( \int \prod_{j=1}^{n'} \tilde{u}_j(x, r', z') \prod_{i=1}^k d\nu_i(x_i) \right) V(r, z) \prod_{j=1}^n f(z_j) \, dz_j, \quad (2.97)$$

where $n' = \sum_{i=1}^k n'_i$, $n'_i = n_i - |\sigma(i)|$, $|\sigma(i)| = \sum_{j=1}^\infty j k_j(i)$. The term $V(r, z)$ is the product of all the chain integrands. (Let $A$ denote the set of all the variables $z_i$ in the chain integrands. The term $z$ in $V(r, z)$ refers to these variable. There are $\sum_{i=1}^k \sum_{j=1}^\infty (j + 1) k_j(i)$ of them. The term $r$ refers to whatever values of $r_1, \ldots, r_k$ are in $V(r, z)$.)

The functions $\tilde{u}_j(x, r', z')$ include all the terms not included in the chain integrands. In particular they include all terms of the form $C_\Delta(z_a, z_b, x, r, z_1, z_k)$ and $C_\Delta(z_a, z_b, x, r, z, z)$, (see (2.38) and its generalization in (2.95)). They also include many terms that are not of the form $C_\Delta(\cdot)$. These are terms of the form

$$u(x_{\pi(j+1)} - x_{\pi(j)}) + r_\ell(z_{j+1} - z_j)). \quad (2.98)$$

They come from the change of variables that has already taken place when we write (2.97) with $f$ rather than $f_{r_t,x}$. (Here $r_\ell$ refers to one one of the $r_1, \ldots, r_k$.) The product $\prod_{j=1}^{n'} \tilde{u}_j(x, r', z')$ contains all the variables $z_1, \ldots z_{n'}$ and also some of the variables $\{z_j, j \in A\}$.

We bound (2.97) by

$$\left( \sup_{|z| \leq M} \int \prod_{j=1}^{n'} \tilde{u}_j(x, r', z') \prod_{i=1}^n d\nu_i(x_i) \right) \int V(r, z) \prod_{j=1}^n f(z_j) \, dz_j \quad (2.99)$$

$$= \left( \sup_{|z| \leq M} \int \prod_{j=1}^{n'} \tilde{u}_j(x, r', z') \prod_{i=1}^k d\nu_i(x_i) \right) \prod_{i=1}^k \prod_{j=1}^\infty (\text{ch}_j(r))^{k_j(i)}.$$
When we perform the integration with respect to \( \prod_{j=1}^{n} f(z_j) \, dz_j \) for the variables \( z_i \notin A \) we just get 1.) We replace the terms in \( \tilde{u} \) of the form given in (2.98) by \( u(x_{\pi(j)}, x_{\pi(j+1)}) + \Delta r(z_{j+1} - z_j) u(x_{\pi(j)}, x_{\pi(j+1)}) \) and write

\[
\prod_{j=1}^{n'} \tilde{u}_j(x, r', z') = \sum_{q} \prod_{j=1}^{n'} \tilde{u}_{j,q}(x, r', z').
\]

(2.100)

Considering the terms relating to the \( C_{\Delta}(\cdot \cdot \cdot) \), we see that \( \tilde{u}_{j,q}(\cdot) \) has one of the following forms:

\[
u(x_{\pi(j)}, x_{\pi(j+1)}), \quad \Delta r' z' - r'' z'' u(x_{\pi(j)}, x_{\pi(j+1)}), \quad \Delta \pm r' z' u(x_{\pi(\ell)}, x_{\pi(m)}),
\]

where \( r', r'' \) take values in \( r_1, \ldots, r_k \), and \( z', z'' \) take values in \( z_1, \ldots, z_n \), \( x_{\pi(j)} \neq x_{\pi(j+1)} \) and \( x_{\pi(\ell)} \neq x_{\pi(m)} \).

Consider a term of the form

\[
\int \prod_{j=1}^{n'} \tilde{u}_{j,q}(x, r', z') \prod_{i=1}^{k} d\nu_i(x_i).
\]

(2.102)

Following the proof of Lemma 2.2 we can bound this by

\[
C \int \prod_{i=1}^{k} |\tilde{\nu}_i (\sum_j \lambda_j) | \prod_{j=1}^{n'} |\tilde{u}_{j,q}(\lambda, r', z')| \, d\lambda_j,
\]

(2.103)

in which the Fourier transform of \( \tilde{u}_{j,q} \) is taken with respect to the \( x \) variable. Note that \( n_i' \) of the functions \( \tilde{u}_{j,q} \) in (2.102) are integrated by \( \nu_i, 1 \leq i \leq k \). Recall that \( n' = \sum_{i=1}^{k} n_i' \), where \( 0 \leq n_i' \leq n_i \). To simplify the notation suppose that for \( i = 1, \ldots, p \), \( 1 \leq p < k \), \( n_i < n_i \), and for \( i = p + 1, \ldots, k \), \( n_i = n_i \). These are the intervals that contain chains and do not contain chains, respectively. We now proceed as in (2.82)-(2.88) to see that (2.103) is bounded by

\[
\leq 4^p \prod_{i=1}^{p} \| \nu_i \|_{2, r_{2n_i}, r' r''} \prod_{i=p+1}^{k} \| \nu_i \|_{2, r_{2n_i}},
\]

(2.104)

in which we set \( \| \nu_i \|_{2, r_0} = 1 \). By (2.91), for \( i = 1, \ldots, p \)

\[
\sup_{|z'| \leq M} \| \nu_i \|_{2, r_{2n_i}, r' r''} \leq o \left( H(1/|r'|) \right)^{n_i - n_i'} \quad \text{as} \ r' \to 0.
\]

(2.105)
By Lemma 10.6 applied to the chains formed by variables in the \(i\)-th interval,
\[
\prod_{j=1}^{\infty} (ch_j(r))^{k_j(i)} \leq O \left((\tilde{H}(1/|r|))^{-(n_i-n'_i)}\right). \tag{2.106}
\]
Therefore, the limit of (2.99), as \(r = r' \to 0\), is zero.

Remark 2.2 The process \(L_n(x, r) = B_n(L(x, r)), n \geq 1\), where the polynomials \(B_n(u)\) satisfy
\[
\sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} ch_i(r) s^i\right)^n B_n(u) = e^{su}, \tag{2.107}
\]
with \(ch_0(r) = 1\). To see that (2.107) agrees with (2.17) and (2.18) we expand
\[
\left(\sum_{i=0}^{\infty} ch_i(r) s^i\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=0}^{\infty} k_i \prod_{i=0}^{\infty} (ch_i(r))^{k_i} s^{(i+1)k_i}\right) n!.
\]

Fix \(N\) and consider all \(k_0, k_1, \ldots\) with
\[
N = \sum_{i=0}^{\infty} (i+1)k_i = k_0 + |\sigma|_+.
\]
Consequently we can replace \(k_0\) with \(N - |\sigma|_+\) in (2.108).
In addition when \(\sum_{i=0}^{\infty} k_i = n\), we have \(n = k_0 + |\sigma|_+ - |\sigma| = N - |\sigma|\).
We use this observation and (2.108) to equate coefficients of \(s^N\) in (2.107) to obtain
\[
\sum_{\{\sigma \mid |\sigma|_+ \leq N\}} \frac{1}{k_0! \prod_{i=1}^{\infty} k_i!} \prod_{i=1}^{\infty} (ch_i(r))^{k_i} B_{N-|\sigma|}(u) = \frac{u^N}{N!}. \tag{2.110}
\]
Setting \(u = L(x, r)\) and \(L_n(x, r) = B_n(L(x, r))\) we get (2.17) and (2.18).
We begin by reviewing some conditions for the continuity of stochastic processes with increments in an exponential Orlicz space. For a proof of Theorem 3.1 see [14, Section 3]. Let

$$\rho_q(x) = \exp(x^q) - 1 \quad (3.1)$$

for $1 \leq q < \infty$, and for $0 < q < 1$, we define

$$\rho_q(x) = \begin{cases} K_q x & 0 \leq x < \left(\frac{1}{q}\right)^{1/q} \\ \exp(x^q) - 1 & x \geq \left(\frac{1}{q}\right)^{1/q} \end{cases} \quad (3.2)$$

where

$$K_q = \frac{\exp(x_0^q) - 1}{x_0} \quad \text{and} \quad x_0 := x_0(q) = \left(\frac{1}{q}\right)^{1/q}, \quad (3.3)$$

so that $\rho_q(x)$ is continuous.

Let $L^{\rho_q}(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \to \mathbb{R}$ such that $E\rho_q(|\xi|/c) < \infty$ for some $c > 0$. $L^{\rho_q}(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$\|\xi\|_{\rho_q} = \inf \{c > 0 : E\rho_q(|\xi|/c) \leq 1\}. \quad (3.4)$$

Let $(T, \tilde{d})$ be a metric or pseudo-metric space. Let $B_{\tilde{d}}(t, u)$ denote the closed ball in $(T, \tilde{d})$ with radius $u$ and center $t$. For any probability measure $\sigma$ on $(T, \tilde{d})$ we define

$$J_{T, \tilde{d}, \sigma, n}(a) = \sup_{t \in T} \int_0^a \left(\frac{1}{\sigma(B_{\tilde{d}}(t, u))}\right)^n du. \quad (3.5)$$

We use the following basic continuity theorem to obtain sufficient conditions for continuity of permanental fields.

**Theorem 3.1** Let $Y = \{Y(t) : t \in T\}$ be a stochastic process such that $Y(t, \omega) : T \times \Omega \mapsto [-\infty, \infty]$ is $\mathcal{A} \times \mathcal{F}$ measurable for some $\sigma$-algebra $\mathcal{A}$ on $T$. Suppose $Y(t) \in L^{\rho_1/n}(\Omega, \mathcal{F}, P)$, where $n \geq 2$ is an integer, and there exists a metric $\tilde{d}$ on $T$ such that

$$\|Y(s) - Y(t)\|_{\rho_1/n} \leq \tilde{d}(s, t). \quad (3.6)$$

(Note that the balls $B_{\tilde{d}}(s, u)$ are $\mathcal{A}$ measurable.)
Suppose that \((T, \bar{d})\) has finite diameter \(D\), and that there exists a probability measure \(\sigma\) on \((T, A)\) such that

\[
J_{T,\bar{d},\sigma,n}(D) < \infty.
\] (3.7)

Then there exists a version \(Y' = \{Y'(t), t \in T\}\) of \(Y\) such that

\[
E \sup_{t \in T} Y'(t) \leq C J_{T,\bar{d},\sigma,n}(D),
\] (3.8)

for some \(C < \infty\). Furthermore for all \(0 < \delta \leq D\),

\[
\sup_{s, t \in T} \bar{d}(s, t) \leq \delta 
\]

almost surely, where

\[
Z(\omega) := \inf \left\{ \alpha > 0 : \int_T \rho_{1/\alpha}(\alpha^{-1}|Y(t, \omega)|) \sigma(dt) \leq 1 \right\}
\] (3.10)

and \(\|Z\|_{\rho_{1/\alpha}} \leq K\), where \(K\) is a constant.

In particular, if

\[
\lim_{\delta \to 0} J_{T,\bar{d},\sigma,n}(\delta) = 0,
\] (3.11)

\(Y'\) is uniformly continuous on \((T, \bar{d})\) almost surely.

For any positive measures \(\phi, \chi \in \mathcal{B}_2(\mathbb{R}^d)\) set

\[
P^{\phi, \chi}(A) = \frac{\mu \left( L_1(\phi) L_1(\chi)^{1_A} \right)}{\mu \left( L_1(\phi) L_1(\chi) \right)}.
\] (3.12)

In the next lemma we show that \(L_n(\nu) \in L^{\rho_{1/n}}(\Omega, \mathcal{F}, P^{\phi, \chi})\) for a probability measure \(P^{\phi, \chi}\). Consequently, we can use Theorem 3.1 to obtain continuity conditions for \(\{L_n(\nu), \nu \in \mathcal{V}'\}\), where \(\mathcal{V}' \in \mathcal{B}_{2n}(\mathbb{R}^d), n \geq 2\).

**Lemma 3.1** For \(\nu, \mu \in \mathcal{B}_{2n}(\mathbb{R}^d)\)

\[
\|L_n(\nu) - L_n(\mu)\|_{\rho_{1/n}, P^{\phi, \chi}} \leq C_{\phi, \rho, n} \|\nu - \mu\|_{2, \tau_{2n}},
\] (3.13)

where \(\|\cdot\|_{\rho_{1/n}, P^{\phi, \chi}}\) denotes the Orlicz space norm with respect to the probability measure \(P^{\phi, \chi}\) and \(C_{\phi, \chi, n}\) is a constant depending on \(\phi, \chi\) and \(n\).
Proof It is obvious from (1.7) that \( \alpha_{n,\epsilon}(\nu_{s_1}, t) - \alpha_{n,\epsilon}(\nu_{s_1}, t) = \alpha_{n,\epsilon}(\nu_{s_1} - \nu_{s_2}, t) \) and since \( L_{1,\epsilon}(\nu) = \alpha_{1,\epsilon}(\nu) \), we see from (1.8) and (1.9) that \( L_n(\nu) - L_n(\mu) = L_n(\nu - \mu) \). Therefore it suffices to show that \( \|L_n(\nu)\|_{\mathcal{L}^1_{\epsilon,\phi,\chi}} \leq C \|\nu\|_{2,\tau_n} \).

We first note that for even integers \( m \geq 2 \)
\[
\mu(|L_n(\nu)|^m) \leq (mn)! \|\nu\|^m_{2,\tau_n}. \tag{3.14}
\]
To see this we use (1.20) to get
\[
\mu(|L_n(\nu)|) \leq (n!)^m |\mathcal{M}_a| \|\nu\|^m_{2,\tau_n}. \tag{3.15}
\]
Since \( |\mathcal{M}| = \binom{mn}{n-n} \) we get (3.14).

Next we note that since
\[
\mu(|L_n(\nu)|^p) \leq (\mu(L_1^2(\phi)L_1^2(\chi)))^{1/2} + (\mu(L_1^2(\phi)L_1^2(\chi)))^{1/2} + (\mu(L_1^2(\phi)L_1^2(\chi)))^{1/2}, \tag{3.16}
\]
and for all integers \( k \geq 1 \) and \( r = 2n/2n - 1 \),
\[
E_{\phi,\chi} \left( \left( \frac{L_n(\nu)}{(2n)^{n/2,\tau_n}} \right)^{k/n} \right) \leq C_{\phi,\chi} C^{k!}, \tag{3.19}
\]
for some constant \( C \). This gives (3.13).

Theorem 3.2 Let \( \{L_n(\nu), \nu \in \mathcal{V}\} \) be an \( n \)-fold intersection local time process, where \( \mathcal{V} \in \mathcal{B}_{2n}(R^d) \), \( n \geq 2 \), and let \( \bar{d}((\nu, \mu)) = \|\nu - \mu\|_{2,\tau_n} \). If (3.11) holds \( \{L_n(\nu), \nu \in \mathcal{V}\} \) is continuous on \( (\mathcal{V}, \bar{d}) \), \( P^{\phi,\chi} \) almost surely.

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Proof This is an immediate application of Theorem 3.1 and Lemma 3.1.

When $\mathcal{V} = \{\nu_x, x \in R^d\}$, the set of translates of a fixed measure $\nu$, the simple concrete condition in (1.17) implies that (3.11) holds.

Proof of Theorem 1.2 We show that

$$\int_1^\infty \left( \int_{|\xi| \geq x} \tau_{2n}(\xi) |\hat{\nu}(\xi)|^2 d\xi \right)^{1/2} \frac{(\log x)^{n-1}}{x} dx < \infty, \quad (3.20)$$

is a sufficient condition for $\{L_n(\nu_x), x \in R^d\}$ to be continuous $P^y$ almost surely, for all $y \in R^d$.

For all $s, h \in R^d$

$$\|\nu_{s+h} - \nu_{s}\|_{2,\tau_{2n}} = \|\nu_{h} - \nu\|_{2,\tau_{2n}} \quad (3.21)$$

$$= \left( \int \left( \int \left( |1 - e^{ixh}|^2 |\hat{\nu}(x)|^2 \tau_{2n}(x) dx \right)^{1/2} \right)^{1/2} \leq \left( 3 \int \left( (|x||h|^2 \wedge 1) |\hat{\nu}(x)|^2 \tau_{2n}(x) dx \right)^{1/2} \right).$$

Using this bound, the fact that (3.20) implies (3.11) with $\tilde{d} = \|\nu_{s+h} - \nu_{s}\|_{2,\tau_{2n}}$ is routine. (See the proof of [12, Theorem 1.6], where this is proved in a slightly different context). Consequently we have that $\{L_n(\nu_x), x \in R^d\}$ is continuous $P^\phi,\chi$ almost surely. As explained in the proof of [11, Theorem 5.1], this implies that $\{L_n(\nu_x), x \in R^d\}$ is continuous $P^y$ almost surely, for all $y \in R^d$.

Example 3.1 Using Theorem 1.2 we give some examples of Lévy processes and measures $\nu$ for which $\{L_n(\nu_x), x \in R^d\}$ is continuous almost surely. As usual let $u$ denote the potential density of the Lévy process and assume that (1.11) holds. Let

$$(\tilde{H}(|\xi|))^{-1} := \int_{|\eta| \leq |\xi|} (\varrho_\alpha(|\eta|))^{-1} d\eta. \quad (3.22)$$

It follows from (10.5) that when $\alpha < d$

$$(\tilde{H}(|\xi|))^{-1} \leq C|\xi|^d(\varrho_\alpha(|\xi|))^{-1} \quad (3.23)$$

and, for the functions $\varrho_\alpha$ that we consider, (1.5) holds.
By Theorem 1.2 and (10.23), \( \{L_n(\nu_x), x \in \mathbb{R}^d\} \) is continuous almost surely when
\[
|\hat{\nu}(\xi)|^2 = O \left( \frac{\varrho(\|\xi\|) H^{2n-1}(\|\xi\|)}{\|\xi\|^d (\log \|\xi\|)^{2n+1+\delta}} \right) \quad \text{as } |\xi| \to \infty, \quad (3.24)
\]
for any \( \delta > 0 \).

When \( \varrho(\|\xi\|) = |\xi|^\alpha \), for \( d(1 - \frac{1}{2n}) < \alpha < d \), using (3.23) the right hand-side of (3.24) is
\[
O \left( \frac{1}{|\xi|^{2n(d-\alpha)} (\log |\xi|)^{2n+1+\delta}} \right). \quad (3.25)
\]
When \( 1/\varrho(\|\xi\|) = O(|\xi|^{-d}) \) as \( |\xi| \to \infty \), the right hand-side of (3.24) is
\[
O \left( \frac{1}{(\log |\xi|)^{4n+\delta}} \right). \quad (3.26)
\]

Considering (1.2) we can replace condition (1.11) by
\[
\tilde{\kappa}(\xi) \geq C \varrho(\|\xi\|), \quad (3.27)
\]
for all \( |\xi| \) sufficiently large, where \( \tilde{\kappa} \) is the Lévy exponent of \( X \). Since \( \tilde{\kappa}(\xi) = o(|\xi|^2) \) as \( |\xi| \to \infty \), it follows that the possible values of \( \alpha \) must be less than or equal to 2 in any dimension. When \( d = 1 \), the condition that \( \alpha \leq 1 \) in (1.3) initially appears to be very restrictive. However, when \( \alpha > 1 \) the Lévy process \( X \) has local times, and self intersection local times can be studied without resorting to the complicated process of renormalization.

4 Loop soup and permanental chaos

Let \( \mathcal{L}_\alpha \) be the Poisson point process on \( \Omega_\Delta \) with intensity measure \( \alpha \mu \). Note that \( \mathcal{L}_\alpha \) is a random variable; each realization of \( \mathcal{L}_\alpha \) is a countable subset of \( \Omega_\Delta \). To be more specific, let
\[
N(A) := |\{\mathcal{L}_\alpha \cap A\}|, \quad A \subseteq \Omega_\Delta. \quad (4.1)
\]
Then for any disjoint measurable subsets \( A_1, \ldots, A_n \) of \( \Omega_\Delta \), the random variables \( N(A_1), \ldots, N(A_n) \), are independent, and \( N(A) \) is a Poisson random variable with parameter \( \alpha \mu(A) \), i.e.
\[
P_{\mathcal{L}_\alpha} (N(A) = k) = \frac{(\alpha \mu(A))^k}{k!} e^{-\alpha \mu(A)}. \quad (4.2)
\]
(When \( \mu(A) = \infty \), this means that \( P(N(A) = \infty) = 1 \).) We call the Poisson point process \( L_{\alpha} \) the ‘loop soup’ of the Markov process \( X \). See [6, 7, 8, 10].

Let \( D_m \subseteq D_{m+1} \) be a sequence of sets in \( \Omega_{\Delta} \) of finite \( \mu \) measure, such that

\[
\Omega_{\Delta} = \bigcup_{m=1}^{\infty} D_m. \tag{4.3}
\]

For \( \nu \in \mathcal{B}_{2n}(R^d) \) we define the ‘loop soup \( n \)-fold self-intersection local time’, \( \psi_n(\nu) \), by

\[
\psi_n(\nu) = \lim_{m \to \infty} \psi_{n,m}(\nu), \tag{4.4}
\]

where

\[
\psi_{n,m}(\nu) = \left( \sum_{\omega \in L_{\alpha}} 1_{D_m} L_n(\nu)(\omega) \right) - \alpha \mu(1_{D_m} L_n(\nu)). \tag{4.5}
\]

(Each realization of \( L_{\alpha} \) is a countable set of elements of \( \Omega_{\Delta} \). The expression \( \sum_{\omega \in L_{\alpha}} \) refers to the sum over this set. Because \( L_{\alpha} \) itself is a random variable, \( \psi_{n,m}(\nu) \) is a random variable.)

The factor \( 1_{D_m} \) is needed to make \( \mu(1_{D_m} L_n(\nu)) \) finite. To understand this recall that \( L_n(\nu) \in L^2(\mu) \) by Theorem 1.1, hence \( \mu(1_{D_m} L_n(\nu)) \) is finite. On the other hand, using (2.11) we see that

\[
\mu \left( \int_0^{\infty} f(X_s) \, ds \right) \tag{4.6}
\]

\[
= \int_0^{\infty} \frac{1}{t} \int Q_t^{-x} \left( \int_0^t f(X_s) \, ds \right) \, dm(x) \, dt
\]

\[
= \int_0^{\infty} \frac{1}{t} \int \int_0^t Q_t^{-x} (f(X_s)) \, ds \, dm(x) \, dt
\]

\[
= \int_0^{\infty} \frac{1}{t} \int \int_0^t \left( \int p_t(x,y) p_{t-s}(y,x) f(y) \, dy \right) \, ds \, dm(x) \, dt
\]

\[
= \int_0^{\infty} \frac{1}{t} \int \int_0^t \left( \int p_t(y,y) f(y) \, dy \right) \, ds \, dt
\]

\[
= \left( \int_0^{\infty} p_t(0) \, dt \right) \int f(y) \, dm(y) = \infty,
\]

whenever \( u(0) = \infty \).

The next lemma gives a formula for the joint moments of \( \{ \psi_{n_j,m_j}(\nu_j) \} \). When all the \( n_j = 1 \) it is essentially the same as [11 (2.37)].
Lemma 4.1 For all $\nu_j \in \mathcal{B}_{2n_j}$, $j = 1, \ldots, k$,

$$E_{\mathcal{L}_\alpha} \left( \prod_{j=1}^{k} \psi_{n_j,m_j}(\nu_j) \right) = \sum_{\cup_i B_i = [1,k], |B_i| \geq 2} \prod_i \alpha \mu \left( \prod_{j \in B_i} 1_{D_{m_j}} L_{n_j}(\nu_j) \right),$$

(4.7)

where the sum is over all partitions $B_1, \ldots, B_k$ of $[1,k]$ with parts $|B_i| \geq 2$.

**Proof** For $j = 1, \ldots, k$, let

$$Y_j := 1_{D_{m_j}} \text{sign} \left( L_{n_j}(\nu_j) \right) (|L_{n_j}(\nu_j)| \wedge M),$$

(4.8)

for some constant $M > 0$. By the master formula for Poisson processes, [5, (3.6)]

$$E_{\mathcal{L}_\alpha} \left( e^{\sum_{j=1}^{k} y_j (\omega) \left( \sum_{j=1}^{k} Y_j - \alpha \mu(Y_j) \right)} \right)$$

(4.9)

$$= \exp \left( \alpha \left( \int_{\Omega \Delta} \left( e^{\sum_{j=1}^{k} y_j Y_j} \right)^{z_j} \right)^{z_j} \right).$$

Differentiating each side of (4.9) with respect to $z_1, \ldots, z_k$ and then setting $z_1, \ldots, z_k$ equal to zero, we obtain

$$E_{\mathcal{L}_\alpha} \left( \prod_{j=1}^{k} \left( \sum_{\omega} Y_j(\omega) - \alpha \mu(Y_j) \right) \right) = \sum_{\cup_i B_i = [1,k], |B_i| \geq 2} \prod_i \alpha \mu \left( \prod_{j \in B_i} Y_j \right),$$

(4.10)

where the sum is over all partitions $B_1, \ldots, B_k$ of $[1,k]$ with parts $|B_i| \geq 2$. Taking the limit as $M \to \infty$ gives (4.7).

(We did not initially define $Y_j = 1_{D_{m_j}} L_{n_j} (\nu_j)$ because it is not clear that the right hand side of (4.9) is finite without the truncation at $M.$)

It follows from Lemma 4.1 that

$$E_{\mathcal{L}_\alpha} (\psi_{n,d}(\nu)) = 0.$$  

(4.11)

The next Theorem asserts that we can take the limit in (4.7) and consequently that (4.4) exists in $L^p(\mu)$ for all $p \geq 1$.

**Theorem 4.1** Let $X$ be as in Theorem 1.1. If $\nu \in \mathcal{B}_{2n}(\mathbb{R}^d)$, the limit in (4.4) exists in $L^p(\mu)$ for all $p \geq 1$. 

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In addition, let $n = n_1 + \cdots + n_k$, and $\nu_j \in B_{2n_j}(R^d)$, $j = 1, \ldots, k$. Then

$$E_{\mathcal{L}_{n}} \left( \prod_{j=1}^{k} \psi_{n_j}(\nu_j) \right) = \sum_{\cup_i B_i = [1,k], |B_i| \geq 2} \prod_{i} \alpha \mu \left( \prod_{j \in B_i} L_{n_j}(\nu_j) \right), \quad (4.12)$$

where the sum is over all partitions $B_1, \ldots, B_k$ of $[1,k]$ with parts $|B_i| \geq 2$.

**Proof** We take the limit as the $m_j \to \infty$ in (4.7) and use Theorem 1.1 to see that the right hand side of (4.7) converges to the right hand side of (4.12). Applying this with $\prod_{j=1}^{k} \psi_{k_j,\delta_j}(\nu_j)$ replaced by $(\psi_{n,m}(\nu_j) - \psi_{n,m'}(\nu_j))$ shows that the limit in (4.4) exists in $L^p(\mu)$ for all $p \geq 1$.

**Proof of Theorem 1.4** We show that for $n = n_1 + \cdots + n_k$ and $c(\pi)$ equal to the number of cycles in the permutation $\pi$,

$$sE_{\mathcal{L}_{n}} \left( \prod_{i=1}^{k} \psi_{n_i}(\nu_i) \right) = \sum_{\pi \in \mathcal{P}_0} \alpha^{c(\pi)} \int \prod_{j=1}^{n} u(z_j, z_{\pi(j)}) \prod_{i=1}^{k} dv_{i}(x_i), \quad (4.13)$$

where $z_1, \ldots, z_{n_1}$ are all equal to $x_1$, the next $n_2$ of the $\{z_j\}$ are all equal to $x_2$, and so on, so that the last $n_k$ of the $\{z_j\}$ are all equal to $x_k$ and $\mathcal{P}_0$ is the set of permutations $\pi$ of $[1,n]$ with cycles that alternate the variables $\{x_i\}$; (i.e., for all $j$, if $z_j = x_i$ then $z_{\pi(j)} \neq x_i$), and in addition, for each $i = 1, \ldots, k$, all the $\{z_j\}$ that are equal to $x_i$ appear in the same cycle.

The relationship in (4.13) is simply a restatement of Theorem 4.1 that allows it to be easily compared to Theorem 1.3. To begin we use Theorem 1.3 on the right-hand side of (4.12). One can see from Theorem 1.3 that the measure acts on cycles; (we have $\pi(n+1) = \pi(1)$). Therefore, the term $\mu \left( \prod_{j \in B_i} L_{n_j}(\nu_j) \right)$ requires that all $n_j$ terms, say $x_j$, that are in the variables $L_{n_j}(\nu_j)$, $j \in B_i$, are in the same cycle. The sum over $\cup_i B_i = [1,k]$, $|B_i| \geq 2$ gives all the terms one gets from permutations listed according to their cycles. Thus there are $c(\pi)$ factors of $\alpha$. The fact that $|B_i| \geq 2$ means that the permutations must alternate the variables $x_i$, a property that is required in both Theorems 1.3 and 1.4.

**Remark 4.1** It follows from Theorem 4.1 and Lemma 2.2 that

$$E_{\mathcal{L}_{n}} \left( |\psi_{n}^{2k}(\nu)|^{1/n} \right) \leq (2k)! C_{2k}^{2k} \|\nu\|_{2,\tau_{2n}}^{2k/n}, \quad (4.14)$$

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where $C_{\alpha,n}$ is a constant, depending on $\alpha$ and $n$. Then, as in the proof of Lemma $3.1$ for $\nu, \mu \in \mathcal{B}_{2n}(\mathbb{R}^d)$

$$
\|\psi_n(\nu) - \psi_n(\mu)\|_{\rho_{1/n,\rho_{2n}}} \leq C'_{\alpha,n} \|\nu - \mu\|_{2,\rho_{2n}}.
$$

(4.15)

Consequently all the continuity results given in Section $3$ for intersection local times, including Theorem $1.2$, also hold for the loop soup intersection local times.

## 5 Isomorphism Theorem I

Theorem $1.5$, Isomorphism Theorem I, contains the measure $Q^{x,y}$ defined by

$$
Q^{x,y}(1_{\{s > \tau\}} F_s) = P^x(F_s u(X_s, y)), \quad \forall F_s \in b\mathcal{F}_s^0,
$$

(5.1)

where $u$ is the potential density of the Lévy process $X_s$. It is related to the measure $Q_t^{x,y}$, introduced in (2.10), by the equation

$$
Q^{x,y}(F) = \int_0^\infty Q_t^{x,y}(F \circ k_t) \, dt, \quad F \in b\mathcal{F}_t^0;
$$

(5.2)

see [11, Lemma 4.1].

The next lemma is proved by a slight modification of the proof of [11, (4.10)].

**Lemma 5.1** For any positive measures $\rho, \phi \in \mathcal{B}_2(\mathbb{R}^d)$, and all bounded measurable functions $f_j$, $j = 1, \ldots, k$,

$$
\int Q^{x,x} \left( L_1(\phi) F \left( \int_0^\infty f_1(X_t) \, dt, \ldots, \int_0^\infty f_k(X_t) \, dt \right) \right) \, d\rho(x) \quad (5.3)
$$

$$
= \mu \left( L_1(\rho) L_1(\phi) \right) F \left( \int_0^\infty f_1(X_t) \, dt, \ldots, \int_0^\infty f_k(X_t) \, dt \right) ,
$$

for any bounded measurable function $F$ on $\mathbb{R}^k$.

We proceed to the proof of Theorem $1.5$. We use a special case of the Palm formula for a Poisson process $\mathcal{L}$ with intensity measure $\vartheta$ on a measurable space $\mathcal{S}$, see [11, Lemma 2.3], that states that for any positive function $f$ on $\mathcal{S}$ and any measurable functional $G$ of $\mathcal{L}$

$$
E_{\mathcal{L}} \left( \sum_{\omega \in \mathcal{L}} f(\omega) G(\mathcal{L}) \right) = \int E_{\mathcal{L}} \left( G(\omega' \cup \mathcal{L}) \right) f(\omega') \, d\vartheta(\omega').
$$

(5.4)

$40$
**Proof of Theorem 1.5**  We show that for any positive measures \( \rho, \phi \in B_2(\mathbb{R}^d) \) there exists \( \theta^{\rho, \phi} \) such that for any finite measures \( \nu_j \in B_{2n_j}(\mathbb{R}^d) \), \( j = 1, 2, \ldots \), and bounded measurable functions \( F \) on \( \mathbb{R}^\infty \),

\[
E_{\mathcal{L}_\alpha} \int Q_x^x \left( L_1(\phi) F \left( \psi_{n_j}(\nu_j) + L_{n_j}(\nu_j) \right) \right) \, d\rho(x) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha} \left( \theta^{\rho, \phi} F \left( \psi_{n_j}(\nu_j) \right) \right).
\]

(5.5)

Note that (1.24) follows from (5.5) with \( F \equiv 1 \). Set

\[
\theta^{\rho, \phi} = \sum_{\omega \in \mathcal{L}_\alpha} L_1(\rho)(\omega)L_1(\phi)(\omega).
\]

(5.6)

By Theorem 1.3 we see that \( L_1(\rho)L_1(\phi) \) is integrable with respect to \( \mu \). We use (5.4) with \( \vartheta = \mu \),

\[
f(\omega) = L_1(\rho)(\omega)L_1(\phi)(\omega)
\]

(5.7)

and

\[
G(\mathcal{L}_\alpha) = F \left( \psi_{n_j, \delta}(\nu_j)(\mathcal{L}_\alpha) \right),
\]

(5.8)

see (4.5), where \( F \) is a bounded continuous function on \( \mathbb{R}^k \). Note that since \( \mu \) is non-atomic, for any fixed \( \omega' \) it follows that \( \omega' \notin \mathcal{L}_\alpha \) a.s. Hence

\[
\psi_{n, \delta}(\nu)(\omega' \cup \mathcal{L}_\alpha) = \left( \sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} L_n(\nu)(\omega) \right) - \alpha \mu(1_{\{\zeta > \delta\}} L_n(\nu))
\]

\[
= 1_{\{\zeta(\omega') > \delta\}} L_n(\nu)(\omega') + \left( \sum_{\omega \in \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} L_n(\nu)(\omega) \right) - \alpha \mu(1_{\{\zeta > \delta\}} L_n(\nu))
\]

\[
= 1_{\{\zeta(\omega') > \delta\}} L_n(\nu)(\omega') + \psi_{n, \delta}(\nu),
\]

(5.9)

so that

\[
G(\omega' \cup \mathcal{L}_\alpha) = F \left( \psi_{n_j, \delta}(\nu_j) + 1_{\{\zeta(\omega') > \delta\}} L_{n_j}(\nu_j)(\omega') \right).
\]

(5.10)

By (5.4), and the fact that \( \alpha \mu \) is the intensity measure of \( \mathcal{L}_\alpha \), we see that

\[
E_{\mathcal{L}_\alpha} \left( \theta^{\rho, \phi} F \left( \psi_{n_j, \delta}(\nu_j) \right) \right)
\]

(5.11)

\[
= \alpha \int E_{\mathcal{L}_\alpha} \left( F \left( \psi_{n_j, \delta}(\nu_j) + 1_{\{\zeta(\omega') > \delta\}} L_{n_j}(\nu_j)(\omega') \right) \right) L_1(\rho)(\omega')L_1(\phi)(\omega') \, d\mu(\omega').
\]

We now use (5.3) and take the limit as \( \delta \to 0 \) to obtain (5.5) when \( F \) is a bounded continuous function \( \mathbb{R}^k \). The extension to general bounded measurable functions \( F \) on \( \mathbb{R}^\infty \) is routine.
6 Permanental Wick powers

In (2.14) we define the approximate local time of $X$ as

$$L_1(x, r) = \int_0^\infty f_{x,r}(X_t) \, dt.$$  \hfill (6.1)

Let $\{D_m\}$ be as defined in (4.3). Set

$$\psi(x, r) = \lim_{m \to \infty} \sum_{\omega \in L_n} \alpha_1 D_m L_1(x, r)(\omega) - \alpha \mu(D_m L_1(x, r)).$$  \hfill (6.2)

Since $L_1(x, r) \in L^2(\mu)$ by (2.12), $\mu(1 D_m L_1(x, r)) < \infty$. (The factor $1 D_m$ is needed since $\mu(L_1(x, r)) = \infty$, see (2.13).)

In this section we construct $\tilde{\psi}_n(x, r)$, the analogue of the higher order Wick powers, $2^{-n} : G^{2^n} : (\nu)$, from the $\alpha$-permanental field $\psi(\nu)$. We call $\tilde{\psi}_n(\nu)$ a permanental Wick power. In Section 8 we obtain a loop soup interpretation of $\tilde{\psi}_n(\nu)$.

The definition of permanental Wick powers is similar to the definition of intersection local times in Section 2. We use chain functions, defined in (2.15), and a similar quantity which we call circuit functions

$$c_{k_1}(r) := \int u(ry_1, ry_2) \cdots u(ry_{k-1}, ry_k) u(ry_k, ry_1) \prod_{j=1}^k f(y_j) \, dy_j.$$  \hfill (6.3)

For any $\sigma = (k_1, k_2, \ldots; m_2, m_3, \ldots)$ we extend the definitions in (2.16) and set

$$|\sigma| = \sum_{i=1}^\infty i k_i + \sum_{j=2}^\infty j m_j, \quad |\sigma|_+ = \sum_{i=1}^\infty (i+1) k_i + \sum_{j=2}^\infty j m_j.$$  \hfill (6.4)

We abbreviate the notation by not indicating the values of $k_{i+1}, i \geq 1$, when it and all subsequent $k_i$ are equal to 0, and similarly for $m_{i+2}$. For example, $\sigma = (2; 0, 1)$ indicates that $k_1 = 2$, $m_2 = 0$ and $m_3 = 1$ and all other $k_i$ and $m_j = 0$. In this case $|\sigma| = 5$ and $|\sigma|_+ = 7$.

In what follows $k_i$ indicates the number of chains $\text{ch}_i$ and $m_j$ indicates the number of circuits $\text{ci}_j$.

Set $\tilde{\psi}_1(x, r) = \psi(x, r)$ and $\tilde{\psi}_0(x, r) = 1$. Analogous to (2.17) and (2.18), in the construction of $L_n(x, r)$, we define recursively

$$\tilde{\psi}_n(x, r) = \psi^n(x, r) - \sum_{\{\sigma : 1 \leq |\sigma| \leq |\sigma|_+ \leq n\}} I_n(\sigma, r).$$  \hfill (6.5)
where

\[ I_n(\sigma, r) = \frac{n!}{\prod_{i=1}^{\infty} k_i! \prod_{j=2}^{\infty} m_j!} \prod_{i=1}^{\infty} (\chi_i(r))^{k_i} \prod_{j=2}^{\infty} \left( \frac{\alpha c_{ij}(r)}{j} \right)^{m_j} \frac{\psi_{n-|\sigma|}(x, r)}{(n - |\sigma| + 1)!}. \]  

(6.6)

(Here we use the notation under the summation sign in (6.5) to indicate that we sum over all \( k_1, k_2, \ldots; m_2, m_3, \ldots \) such that \( \{ |\sigma| \leq |\sigma| \leq |\sigma| + n \} \).

It is easy to check that

\[ \tilde{\psi}_2(x, r) = \psi^2(x, r) - I_2((1; 0), r) - I_2((0; 1), r) \]  

(6.7)

and

\[ \tilde{\psi}_3(x, r) = \psi^3(x, r) - I_3((1; 0), r) \]  

(6.8)

\[ = \psi^3(x, r) - 6\chi_1(r)\psi_2(x, r) \]  

\[ - \{ 6\chi_2(r) + 3\alpha c_2(r) \} \psi(x, r) - 2\alpha c_3(r). \]

Using (6.7) we can write \( \tilde{\psi}_3(x, r) \) as a third degree polynomial in \( \psi(x, r) \),

\[ \tilde{\psi}_3(x, r) = \psi^3(x, r) - 6\chi_1(r)\psi^2(x, r) \]  

(6.9)

\[ - \{ 6\chi_2(r) + 3\alpha c_2(r) - 12\chi^2_1(r) \} \psi(x, r) - 2\alpha c_3(r) + 6\alpha c_1(r)\alpha c_2(r). \]

We show in Remark 6.2 that \( \tilde{\psi}_n(x, r) \) can also be defined by a generating function.

Analogous to Theorems 1.1, 1.3 and 1.4 we have the following results about \( \tilde{\psi}_n \) and its joint moments.

**Theorem 6.1** Let \( X = \{ X(t), t \in R^+ \} \) be a Lévy process in \( R^d \), \( d = 1, 2 \), that is killed at the end of an independent exponential time, with potential density \( u \) that satisfies (1.11) and (1.3) and let \( \nu \in B_{2n}(R^d) \), \( n \geq 1 \). Then

\[ \tilde{\psi}_n(\nu) = \lim_{r \to 0} \int \tilde{\psi}_n(x, r) d\nu(x) \]  

(6.10)

exists in \( L^p(PL_\alpha) \) for all \( p \geq 1 \).

**Theorem 6.2** Let \( X \) be as in Theorem 6.1 and let \( n = n_1 + \cdots + n_k \), and \( \nu_i \in B_{2n_i}(R^d) \). Then

\[ E_{\mathcal{L}_\alpha} \left( \prod_{i=1}^{k} \tilde{\psi}_{n_i}(\nu_i) \right) = \sum_{\pi \in \mathcal{P}_{n,a}} \alpha^{\mathcal{C}(\pi)} \int \prod_{j=1}^{n} u(z_j, z_{\pi(j)}) \prod_{i=1}^{k} d\nu_i(x_i), \]  

(6.11)
where \( z_1, \ldots, z_{n_1} \) are all equal to \( x_1 \), the next \( n_2 \) of the \( \{z_j\} \) are all equal to \( x_2 \), and so on, so that the last \( n_k \) of the \( \{z_j\} \) are all equal to \( x_k \) and \( \mathcal{P}_{n,a} \) is the set of permutations \( \pi \) of \([1, n]\) with cycles that alternate the variables \( \{x_i\} \); (i.e., for all \( j \), if \( z_j = x_i \) then \( z_{\pi(j)} \neq x_i \)).

It is interesting to note that a simple version of (6.11) appears in [11, (1.1)] for permanental fields and in [16, Proposition 4.2] for permanental random variables, (where they are referred to as multivariate gamma distributions).

The following corollary shows that the processes \( \{\tilde{\psi}_n(\nu)\}_{n=1}^{\infty} \) are orthogonal with respect to \( P_{\lambda} \).

**Corollary 6.1** Let \( X \) be as in Theorem 6.1. If \( \nu, \nu' \in B_{2n}(R^d) \), then for any \( k \leq n \),

\[
E_{\mathcal{L}} \left( \tilde{\psi}_n(\nu) \tilde{\psi}_k(\nu') \right) = \delta_{n,k} \ K(\alpha, n) \int (u(x,y)u(y,x))^n \ d\nu(x) \ d\nu'(y) \tag{6.12}
\]

where

\[
K(\alpha, n) = n! \alpha(\alpha + 1) \cdots (\alpha + n - 1). \tag{6.13}
\]

**Remark 6.1** Theorem 6.2 holds when \( k = 1 \), in which case the right-hand side of (6.11) is empty. Therefore,

\[
E \left( \tilde{\psi}_n(\nu_i) \right) = 0. \tag{6.14}
\]

When \( k = n \), the right-hand side of (6.12), with \( \delta_{n,k} = 1 \), is the covariance of \( \{\psi_n(\nu), \nu \in \mathcal{V}\} \). To see that (6.12) has the same form as many of our other results we note that

\[
\int (u(x,y)u(y,x))^n \ d\nu(x) \ d\nu(y) = \frac{1}{(2\pi)^{nd}} \int \theta(\xi) |\hat{\nu}(\xi)|^2 \ d\xi \tag{6.15}
\]

where \( \theta(\xi) \) is the Fourier transform of \((u(x,y)u(y,x))^n\). Also note that \( u(x,y) \) \( u(y,x) \geq 0 \) and is symmetric.

**Proof of Theorems 6.1 and 6.2** The proof is very similar to the proof of Theorem 1.3 so we shall not go through all the details. The main difference is the possibility of circuits. They introduce the factors \( c_i_k(r) \).
In analogy with (2.24) in the proof of Theorem 1.3 we first show that for any \( n \geq 2 \) and \( m \geq 1 \),

\[
E_{\mathcal{L}_a} \left( \tilde{\psi}_n(x, r) \prod_{i=1}^m \psi(f_i) \right) = \sum_{\pi \in \mathcal{P}_{m,n}} \alpha^{c(\pi)} \int \prod_{j=1}^{m+n} u(z_j, z_{\pi(j)}) \prod_{i=1}^m f_i(z_i) \prod_{i=m+1}^{m+n} f_{r,x}(z_i) \, dz_i \]

\[
+ \mathcal{E}_f(E_r(x, z)),
\]

where, \( \mathcal{P}_{m,n} \) is the set of permutations \( \pi \) of \([1, m+n]\) such that \( \pi : [m+1, m+n] \mapsto [1, m] \), (when \( m < n \) there are no such permutations), and the last term is as given in (2.25), although the terms in \( E_r(x, z) \) are not the same. We show below that for \( \nu \in \mathcal{B}_{2n}(R^d) \),

\[
\lim_{r \to 0} \sup_{|z_i| \leq M} \int E_r(x, z) \, d\nu(x) = 0, \tag{6.17}
\]

which implies that

\[
\lim_{r \to 0} \int \mathcal{E}_f(E_r(x, z)) \, d\nu(x) = 0. \tag{6.18}
\]

The proof of (6.16) when \( n = 1 \) follows from (4.12) and (1.20), and in this case the error term \( \mathcal{E}_f(E_r(x, z)) = 0 \).

For the proof of (6.16) when \( n = 2 \) we note that by Theorem 1.4

\[
E_{\mathcal{L}_a} \left( \psi^2(x, r) \prod_{i=1}^m \psi(f_i) \right) = \sum_{\pi \in \mathcal{P}_0} \alpha^{c(\pi)} \int \prod_{j=1}^{m+2} u(z_j, z_{\pi(j)}) \prod_{i=1}^m f_i(z_i) \prod_{i=m+1}^{m+2} f_{r,x}(z_i) \, dz_i,
\]

where, because all the \( n_i = 1 \), \( \mathcal{P}_0 \) is the set of permutations on \([1, m+2]\).

Consider the permutations in \( \mathcal{P}_0 \) that do not take \( \pi : [m+1, m+2] \mapsto [1, m] \). They can be divided into three sets. Those that have the cycle \((m+1, m+2)\), those that take only \( m+1 \mapsto [1, m] \) and those that take only \( m+2 \mapsto [1, m] \). Taking these into consideration and considering (6.7), we get (6.16) when \( n = 2 \). This is obvious for the terms involving the cycle. To understand this for the terms involving the two chains consider (2.32)–(2.38) and note that an extra density function is introduced. It is this that gives the factor \( \psi(x, r) \).

Assume that (6.16) is proved for \( \psi_{n'}(x, r) \), \( n' < n \). For any \( \sigma = (k_1, k_2, \ldots; m_2, m_3, \ldots) \) let \( \mathcal{P}_0(\sigma) \) denote the set of permutations \( \tilde{\pi} \in \mathcal{P}_0 \) that contain
m_j circuits of order j = 2, 3, ..., and k_i chains of order i = 1, 2, 3, ..., in [m + 1, m + n]. Note that P_0 - P_{m,n} = \bigcup_{|\sigma| \geq 1} P_0(\sigma). In this rest of this proof we make these definitions more explicit by writing P_0(\sigma) as P_{0,m+n}(\sigma).

Any term \tilde{\pi} \in P_{0,m+n}(\sigma) in the evaluation of
\[ E_{\mathcal{L}_{\alpha}} \left( \psi^n(x, r) \prod_{i=1}^{m} \psi(f_i) \right) \]
is the same as the term in (6.16) for a particular permutation \pi' \in P_{m,n-|\sigma|} in the evaluation of
\[ E_{\mathcal{L}_{\alpha}} \left( \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} \prod_{j=2}^{\infty} (\alpha \circ_i \beta_j(r))^{m_j} \tilde{\psi}_{n-|\sigma|}(x, r) \prod_{i=1}^{m} \psi(f_i) \right). \]
This follows by the same argument given in (2.41)–(2.43) except that here we use Theorem 1.4 and (6.16).

Similar to the analysis on page 17 the permutation \pi' is obtained from \tilde{\pi} by the remove and relabel technique. However there is a significant difference in this case. In Theorem 1.3 we work with the loop measure \mu so that the permutation \tilde{\pi} consists of a single cycle. (We have \pi(n + 1) = \pi(1).) In this theorem we take the expectation with respect to \E_{\mathcal{L}_{\alpha}}. As one can see in (1.12), the permutation \tilde{\pi} generally has many cycles, each of order greater than or equal to 2.

We illustrate this with an example similar to (2.44). Consider the case when m = 10, n = 14 and the permutation on [1, 24] given by
\[ \tilde{\pi} = (6, 7, 11, 13, 8, 9, 10)(1, 14, 12, 16, 2, 3, 4, 15, 5, 17, 18)(19, 20, 21)(22, 23, 24). \]
in which we use standard cycle notation. \tilde{\pi} has 4 cycles. We are primarily concerned with the effects of \tilde{\pi} on [11, 24]. There are three chains
\[ (11, 13) \quad (14, 12, 16) \quad (17, 18), \]
and two circuits
\[ (19, 20, 21) \quad (22, 23, 24), \]
so that \sigma = (2, 1; 0, 2), |\sigma| = 10 and |\sigma|_+ = 13.

We first remove all circuits and all but the first element in each chain in (6.22) to obtain
\[ (6, 7, 11, 8, 9, 10)(1, 14, 2, 3, 4, 15, 5, 17). \]
The permutation $\pi'$ is obtained from (6.25) by relabeling the remaining elements in $\{11, \ldots, 24\}$ in increasing order from left to right, i.e.,

$$\pi' = (6, 7, 11, 8, 9, 10)(1, 12, 2, 3, 4, 13, 5, 14).$$

(6.26)

Let $P_{0.24}(\sigma)\pi'$ denote the permutations in $P_{0.24}$ that have the number of chains and circuits designated by $\sigma$ and that give rise to $\pi'$ by the above procedure. We compute $|P_{0.24}((2, 1; 0, 2))\pi'|$. Each of the 14! permutations of the elements $(1, 2, \ldots, 24)$ give rise to distinct permutations in $P_{0.24}((2, 1; 0, 2))\pi'$, except for the 3 rotations in each circuit and the interchange of the two circuits. We call these internal permutations. There are 14! permutations in $P_{0.24}((2, 1; 0, 2))\pi'$, since they have already been counted in the internal permutations. Consequently

$$|P_{0.24}((2, 1; 0, 2))\pi'| = \frac{14!4!}{2!2!3^2}. \quad (6.27)$$

For general $\sigma$ and $\pi' \in P_{0,m+n-|\sigma|}$, in which, as in the example above, the integers $m + 1, \ldots, m + n - |\sigma|$ appear in increasing order,

$$|P_{0,m+n}(\sigma)\pi'| = \frac{n!}{\prod_{j=2}^{\infty} (m_j!j^{m_j}) \prod_{i=1}^{\infty} k_i! (n-|\sigma|)!}. \quad (6.28)$$

To see this first note that there are $n!/\prod_{j=2}^{\infty} (m_j!j^{m_j})$ internal permutations. Here we divide by the $m_j!$ interchanges of circuits of order $j$, and the $j$ rotations in each circuit of order $j$, for each $j$. Note that for any $\pi \in P_{0,m+n}(\sigma)\pi'$, there are $|\sigma|_+$ integers from $\{m + 1, \ldots, m + n\}$ in the circuits and chains of order 1, 2, ..., Consequently, there are $n - |\sigma|_+$ remaining integers in $\{m + 1, \ldots, m + n\}$ which, as in the example above, we consider to be chains of order 0. Therefore, total number of chains, including those of order 0, in $\{m + 1, \ldots, m + n\}$ is $n - |\sigma|_+ + \sum_{i=1}^{\infty} k_i = n - |\sigma|$. Thus any of the $(n-|\sigma|)!$ permutations of these chains in $\pi$ are in $P_{0,m+n}(\sigma)\pi'$. However, we do not want to count the $(n-|\sigma|_+)!\prod_{i=1}^{\infty} k_i!$ interchanges of chains of the same order, since this is counted in the internal permutations. Putting all this together gives (6.28).

Consider (6.21) again and the particular permutation $\pi' \in P_{m+n-|\sigma|}$. We have pointed out before that there are $(n - |\sigma|)!$ different permutations, the
internal permutations, in \( P_{m+n-|\sigma|} \), whose contribution to (6.21) is the same as it is for \( \pi' \). Therefore up to the error terms, the contribution to (6.16) from \( P_{m+n}(\sigma) \) is equal to

\[
\frac{n!}{\prod_{i=1}^{\infty} k_i! \prod_{j=2}^{\infty} m_j! j^{m_j}(n - |\sigma|)!} \times E_{\mathcal{L}_a} \left( \prod_{i=1}^{\infty} (c_i(r))^{k_i} \prod_{j=2}^{\infty} (\alpha c_j(r))^{m_j} \tilde{\psi}_{n-|\sigma|}(x, r) \prod_{i=1}^{m} \psi(f_i) \right) = E_{\mathcal{L}_a} \left( I_n(\sigma, r) \prod_{i=1}^{m} \psi(f_i) \right). \tag{6.29}
\]

The rest of the proof follows as in the proof of Theorems 1.1 and 1.3. (In controlling the error terms we also use Lemma 10.6.) \( \square \)

**Proof of Corollary 6.1** It follows easily from the proof of Theorem 6.1 that \( E_{\mathcal{L}_a} \left( \tilde{\psi}_n(\nu) \tilde{\psi}_k(\nu) \right) = 0 \) when \( n \neq k \). When \( n = k \) we have

\[
E_{\mathcal{L}_a} \left( \tilde{\psi}_n(\nu) \tilde{\psi}_n(\nu) \right) = \sum_{\pi \in P_{2n,a}} \alpha^{c(\pi)} \int \prod_{j=1}^{2n} u(z_j, z_{\pi(j)}) d\nu(x) d\nu(y). \tag{6.30}
\]

To evaluate this, we first note that since the \( x \) and \( y \) terms alternate in (6.30) it is equal to

\[
\sum_{\pi \in P_{2n,a}} \alpha^{c(\pi)} \int (u(x, y)u(y, x))^n d\nu(x) d\nu(y). \tag{6.31}
\]

It remains to show that

\[
\sum_{\pi \in P_{2n,a}} \alpha^{c(\pi)} = n! \alpha(\alpha + 1) \cdots (\alpha + n - 1). \tag{6.32}
\]

To see this, consider an arrangement of

\[\{x_1, \ldots, x_n, y_1, \ldots, y_n\}\]

into (oriented) cycles, such that each cycle contains an equal number of \( x \) and \( y \) terms in an alternating arrangement. For each such arrangement we define a permutation \( \sigma \) of \([1, n]\) by setting \( \sigma(i) = j \) if \( x_i \) is followed by \( y_j \). We refer to the \( n \) ordered pairs \((x_1, y_{\sigma(1)}), \ldots, (x_n, y_{\sigma(n)})\) as the pairs generated by \( \sigma \).
Let \( \phi(\sigma, l) \) denote the number of permutations in \( P_{2n,a} \) with \( l \) cycles that are obtained by a rearrangement of the pairs generated by \( \sigma \). In the next paragraph we show that

\[
\sum_l \alpha^l \phi(\sigma, l) = \alpha(\alpha + 1) \cdots (\alpha + n - 1).
\]  

(6.33)

Since each of the \( n! \) permutations of \([1, n]\) gives a different \( \sigma \), we get (6.32).

To prove (6.33) we construct the rearrangements of the pairs generated by \( \sigma \) and consider how many cycles each one contains. We begin with the ordered pair \((x_1, y_{\sigma(1)})\) which we consider as an incipient cycle, \(x_1 \rightarrow y_{\sigma(1)} \rightarrow x_1\). (We say incipient because as we construct the rearrangements of the pairs generated by \( \sigma \), \((x_1, y_{\sigma(1)})\) is sometimes a cycle and sometimes part of a larger cycle.)

We next take the ordered pair \((x_2, y_{\sigma(2)})\) and use it to write

\[
(x_1, y_{\sigma(1)})(x_2, y_{\sigma(2)}) \quad \text{and} \quad (x_1, y_{\sigma(1)}, x_2, y_{\sigma(2)}).
\]  

(6.34)

We consider that the first of these terms contains two cycles and the second one cycle. Therefore, so far, we have accumulated an \( \alpha^2 \) and an \( \alpha \) towards the factor \( \alpha^{c(\pi)} \) for the cycles in the rearrangements of the pairs generated by \( \sigma \) that we are constructing and write

\[
\alpha^2 + \alpha = \alpha(\alpha + 1).
\]

Similarly, we use \((x_3, y_{\sigma(3)})\) to extend the terms in (6.34) as follows:

\[
(x_1, y_{\sigma(1)})(x_2, y_{\sigma(2)})(x_3, y_{\sigma(3)}) \quad (y_{\sigma(1)}, x_1, x_2, y_{\sigma(2)})(x_3, y_{\sigma(3)})
\]  

(6.35)

\[
(x_1, y_{\sigma(1)}, x_3, y_{\sigma(3)})(x_2, y_{\sigma(2)}) \quad (x_1, y_{\sigma(1)})(x_2, y_{\sigma(2)}, x_3, y_{\sigma(3)})
\]

\[
(x_1, y_{\sigma(1)}, x_3, y_{\sigma(3)}, x_2, y_{\sigma(2)}) \quad (x_1, y_{\sigma(1)}, x_2, y_{\sigma(2)}, x_3, y_{\sigma(3)})
\]

The accumulated \( \alpha \) factors are now equal to \( \alpha(\alpha + 1)(\alpha + 2) = \alpha^3 + 3\alpha^2 + 2\alpha \). The \( \alpha^3 \) comes from the first term in (6.35) which has 3 cycles, the \( 3\alpha^2 \) comes from the next three terms which have have two cycles each, and the \( 2\alpha \) comes from the last two terms each of which has a single cycle. It should be clear now that when we add the term \((x_4, y_{\sigma(4)})\) to this we multiply the previous accumulated \( \alpha \) factor by \((\alpha + 3)\). The \( \alpha \) in \((\alpha + 3)\) because we can add \((x_4, y_{\sigma(4)})\) to each of the terms in (6.35) as a separate cycle. The factor 3 because we can place \((x_4, y_{\sigma(4)})\) to the right of each \((x_1, y_{\sigma(i)})\), \(i = 1, 2, 3\), in each of the other terms in (6.35), without changing the number of cycles they contain. Proceeding in this way gives (6.33).  

\[\Box\]

**Corollary 6.2** Let \( \nu \in \mathcal{B}_{2n} \), then for any \( k \) and any \( \alpha > 0 \),

\[
|E_{\mathcal{L}} (\tilde{\psi}^k_n(\nu))| \leq (kn)!C_{\alpha}^{kn} \|\nu\|_{2,\tau_{2n}}^k.
\]  

(6.36)
Proof In this case the integral in (6.11) is equal to
\[
\int \prod_{j=1}^{kn} u(z_j, z_{\pi(j)}) \prod_{i=1}^k d\nu(x_i)
\] (6.37)
in which there are exactly 2n factors of u containing \(x_i\), for each \(i = 1, \ldots, k\). It follows from Lemma 2.2 that
\[
\left| \int \prod_{j=1}^{kn} u(z_j, z_{\pi(j)}) \prod_{i=1}^k d\nu(x_i) \right| \leq \|\nu\|_{2,\tau_{2n}}^k.
\] (6.38)
Since there are \((kn)!\) unrestricted permutations of \([1, kn]\) we get (6.36).

Theorem 6.3 Let \(\{\tilde{\psi}_n(\nu), \nu \in V\}\) be an n-th order permanental Wick power, where \(V \in B_{2n}(R^d)\), \(n \geq 2\), and let \(d((\nu, \mu)) = \|\nu - \mu\|_{2,\tau_{2n}}\). If (3.11) holds \(\{\tilde{\psi}(\nu), \nu \in V\}\) is continuous on \((V, \bar{d})\), \(P_{\mathcal{L}_\alpha}\) almost surely.

Proof We use (6.36) and the inequalities given in the transition from (3.18) to (3.19) to see that
\[
\|\tilde{\psi}_n(\nu) - \tilde{\psi}_n(\mu)\|_{\rho_{1/n}, P_{\mathcal{L}_\alpha}} \leq C\|\nu - \mu\|_{2,\tau_{2n}}.
\] (6.39)
The theorem now follows from Theorem 3.1.

In Corollary 6.1 we give a formula for \(E_{\mathcal{L}_\alpha}((\tilde{\psi}_n(\nu))^2)\). We now give an alternate expression for this expectation which we use in the proof of Theorem 8.2. Let \(\{A_p\}_{p=1}^k\) be a partition of \([1, n]\). Let \(m_i(\{A_p\}_{p=1}^k)\) be the number of sets in this partition with \(|A_p| = i\). We define the degree of the partition to be
\[
d(\{A_p\}_{p=1}^k) := \left( m_1 \left( \{A_p\}_{p=1}^k \right), \ldots, m_k \left( \{A_p\}_{p=1}^k \right) \right).
\] (6.40)

Lemma 6.1 For \(\nu \in B_{2n}(R^d)\),
\[
E_{\mathcal{L}_\alpha}((\tilde{\psi}_n(\nu))^2)
\] (6.41)
\[
= \sum_{\{A_1 \cup \ldots \cup A_k = [1, n]\}} \sum_{\{B_1 \cup \ldots \cup B_{k'} = [1, n]\}} \delta_d(\{A_p\}_{p=1}^k, \{B_{p'}\}_{p'=1}^{k'}) \prod_{i=1}^k m_i(\{A_p\}_{p=1}^k)!
\]
\[
\lim_{r \to 0} \int \prod_{l=1}^k \alpha \mu(L_{|A_l|}(x, r)L_{|A_l|}(y, r)) \ d\nu(x) \ d\nu(y),
\]
where \( \delta \{d(\{A_p\}_{p=1}^k), d(\{B_p\}_{p=1}^{k'})\} \) is equal to one when the vectors \( d(\{A_p\}_{p=1}^k) = d(\{B_p\}_{p=1}^{k'}) \), (so that \( k = k' \)), and is equal to zero otherwise.

Moreover, the third line of (6.41) is equal to
\[
\alpha^k \prod_{l=1}^k |A_l|!^{|A_l| - 1} \left( \int (u(x, y)u(y, x))^{\frac{|A_l|}{|A_l|}} d\nu(x) d\nu(y) \right). \tag{6.42}
\]

**Proof** To understand (6.41) consider Theorem 6.2 with \( n_1 = n_2 = n \), and replace the products of the potentials on the right-hand side of (6.11) by the limit, as \( r \) goes to zero, of (2.57). The sum in (6.11) is a sum over permutations with cycles that alternate two pairs of \( n \) variables, which we denote by \( x \) and \( y \). These permutations divide \([1, n]\) and \([n+1, 2n]\), which we identify with another copy of \([1, n]\), into two partitions \( \mathcal{A} \) and \( \mathcal{B} \) of \([1, n]\) which must have the same degree. Moreover each set \( A_l \) in \( \mathcal{A} \) must be paired with a set \( B_{l'} \) of \( \mathcal{B} \) of the same cardinality. There are \( e(\mathcal{A}) \) ways to make such a pairing. If \( \mathcal{A} \) consists of \( k \) sets then the permutation \( \pi \) has \( k \) cycles, with the \( l \)-th cycle alternating the elements of \( A_l \) and \( B_{l'} \), for some \( 1 \leq l' \leq k' \).

We use (2.57) to get (6.42). \( \square \)

**Remark 6.2** The process \( \tilde{\psi}_n(x, r) = A_n(\psi(x, r)) \), \( n \geq 1 \), where the polynomials \( A_n(u) \) satisfy

\[
\sum_{n=0}^{\infty} \frac{\left( \sum_{i=0}^{\infty} \text{ch}_i(r) s^i + \sum_{j=2}^{\infty} \alpha (ci_j(r)/j) s^{j-1} \right)^n}{n!} s^n A_n(u) = e^{su}, \tag{6.43}
\]

with \( \text{ch}_0(r) = 1 \). To prove that this this agrees with (6.5) and (6.6) we need only minor modifications of the proof in Remark 2.2. As in (2.108) we have

\[
\left( \sum_{i=0}^{\infty} \text{ch}_i(r) s^i + \sum_{j=2}^{\infty} \frac{\alpha ci_j(r)}{j} s^{j-1} \right)^n \frac{s^n}{n!} = \prod_{i=0}^{\infty} \left( \text{ch}_i(r) s^i \right)^{k_i} \prod_{j=2}^{\infty} \left( \frac{\alpha ci_j(r)}{j} s^{j-1} \right)^{m_j} s^n, \tag{6.44}
\]
\[
\sum_{i=0}^{\infty} \sum_{j=2}^{\infty} m_j = n - \sigma_+ - |\sigma| = N - |\sigma|.
\]

Consequently we can replace \( k_0 \) with \( N - |\sigma|_+ \) in (6.44).

In addition, when \( \sum_{i=0}^{\infty} k_i + \sum_{j=2}^{\infty} m_j = n \), we have \( n = k_0 + |\sigma| - |\sigma| = N - |\sigma| \). We use this observation and (6.44) to equate the coefficients of \( s^N \) in (6.43) to obtain

\[
\sum_{\{\sigma : |\sigma|_+ \leq N\}} \frac{1}{\prod_{i=1}^{\infty} k_i! \prod_{j=2}^{\infty} m_j! (N - |\sigma|_+)!} \prod_{i=1}^{\infty} (ch_i(r))^{k_i} \prod_{j=2}^{\infty} (\alpha (c_{i,j}(r)/j))^{m_j} A_{N-|\sigma|}(u) = \frac{u^N}{N!},
\]

in which we use the expanded definitions of \( |\sigma| \) and \( |\sigma|_+ \) in (6.44). Setting \( u = \psi(x, r) \) and \( \tilde{\psi}_n(x, r) = A_n (\psi(x, r)) \) we get (6.5) and (6.6).

### 7 Poisson chaos decomposition, I

We continue our study of \( L_{\alpha} \), a Poisson point process on \( \Omega_\Delta \) with intensity measure \( \alpha \mu \), and obtain a decomposition of \( L^2(P_{L_{\alpha}}) \) into orthogonal function spaces (in (7.36)) which we refer to as the Poisson chaos decomposition of \( L^2(P_{L_{\alpha}}) \). This is used in Section 8 to obtain a definition of \( \tilde{\psi}_n(\nu) \) in terms of loop soups. Some of the results in this section generalize results in [10, Chapter 5.4] which considers processes with finite state spaces.

As explained at the beginning of Section 4, each realization of \( L_{\alpha} \) is a countable set of elements of \( \Omega_\Delta \). The expression \( \sum_{\omega \in L_{\alpha}} \) refers to the sum over this set. Because \( L_{\alpha} \) itself is a random variable such a sum is also a random variable.

For any set \( A \), we use \( S_n(A) \subset A^n \) to denote the subset of \( A^n \) with distinct entries. That is, if \( (a_{i_1}, \ldots, a_{i_n}) \in S_n(A) \) then \( a_{i_l} \neq a_{i_k} \) for \( i_l \neq i_k \).
Let $D_k(\mu) = \cap_{p \geq k} L^p(\mu)$. Note that $1_B \in D_1(\mu)$ for any $B \subseteq \Omega_\Delta$ with finite $\mu$ measure. For $g_j \in D_1(\mu)$, $j = 1, \ldots, n$, we define

$$\phi_n(g_1, \ldots, g_n) = \sum_{(\omega_1, \ldots, \omega_n) \in S_n(L_\alpha)} \prod_{j=1}^n g_j(\omega_{i_j}). \tag{7.1}$$

In particular

$$\phi_1(g_1) = \sum_{\omega \in L_\alpha} g(\omega), \tag{7.2}$$

and $\phi_1(1_B) = N(B)$; (see (4.1)).

For any subset $A \subseteq \mathbb{Z}_+$ set $g_A(\omega) = \prod_{j \in A} g_j(\omega)$. In this notation the factors $g_j, j \in A$, are functions of the same variable. Thus, in contradistinction to (7.1) we have,

$$\phi_1(g_{[1,n]}) = \sum_{\omega \in L_\alpha} \prod_{j=1}^n g_j(\omega). \tag{7.3}$$

Note that

$$\prod_{j=1}^n \phi_1(g_j) = \sum_{(\omega_1, \ldots, \omega_n) \in L_\alpha^n} \prod_{j=1}^n g_j(\omega_{i_j}) \tag{7.4}$$

$$= \sum_{k=1}^n \sum_{\bigcup_{l=1}^k A_l = [1,n]} \sum_{(\omega_1, \ldots, \omega_k) \in S_k(L_\alpha)} \prod_{l=1}^k g_{A_l}(\omega_{i_l})$$

$$= \sum_{k=1}^n \sum_{\bigcup_{l=1}^k A_l = [1,n]} \phi_k(g_{A_1}, \ldots, g_{A_k}),$$

in which all $\{A_l\}_{l=1}^k$, $1 \leq k \leq n$, are partitions of $[1, n]$. For example,

$$\prod_{j=1}^2 \phi_1(g_j) = \phi_2(g_1, g_2) + \phi_1(g_{[1,2]}). \tag{7.5}$$

The set of random variables $\phi_1(1_B) = N(B)$, for $B \subseteq \Omega_\Delta$ with finite $\mu$ measure, generate the $\sigma$-algebra for $P_{L_\alpha}$. By the master formula for Poisson processes, [5 (3.6)], the random variables $N(B)$ are exponentially integrable. Hence the polynomials in $\phi_1$ are dense in $L^2(P_{L_\alpha})$. Let $\phi_0 \equiv 1$ and let $\mathcal{H}_n$ denote the closure in $L^2(P_{L_\alpha})$ of all linear combinations of $\phi_j, j = 0, 1, \ldots, n$. Then it follows from (7.4) that

$$L^2(P_{L_\alpha}) = \bigcup_{n=0}^{\infty} \mathcal{H}_n. \tag{7.6}$$

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For integers $n \leq p$, let $\{A_l\}$ be a partition of $\{1, \ldots, p\}$ with the property that no two integers in $\{1, 2, \ldots, n\}$ are in the same set $A_l$. That is, the partition $\{A_l\}$ separates $\{1, 2, \ldots, n\}$. We denote such a partition by $\bigcup A_l = \{1, p\} \perp \{1, 2, \ldots, n\}$. We use the notation $\sum_{\bigcup A_l = \{1, p\} \perp \{1, 2, \ldots, n\}}$ to indicate the sum over all such partitions.

**Lemma 7.1** Let $\mathcal{L}_\alpha$ be a Poisson point process on $\Omega_\Delta$ with intensity measure $\alpha \mu$. Let $\{n_j\}_{j=1}^N$ be a set of integers and define $s_m = \sum_{j=1}^{m-1} n_j$, $m = 1, \ldots, N+1$. Then for any functions $g_j \in \mathcal{D}_1(\mu)$, $j = 1, \ldots, s_{N+1}$,

$$
E_{\mathcal{L}_\alpha} \left( \prod_{m=1}^{N} \phi_{n_m} (g_{s_m+1}, \ldots, g_{s_m+n_m}) \right)
= \sum_{\bigcup A_l = \{1, s_{N+1}\} \perp \{s_m+1, s_m+2, \ldots, s_m+n_m\}, m=1, \ldots, N} \prod_l \alpha \mu (g_{A_l}).
$$

(7.7)

where the notation under the summation sign indicates that no sets in the partitions can contain more than one element from each of the sets $\{s_m+1, s_m+2, \ldots, s_m+n_m\}$, $m=1, \ldots, N$.

**Proof** As in (4.9), by the master formula for Poisson processes, [5, (3.6)], for any finite set of positive integers $B$ and random variables $h_j$,

$$
E_{\mathcal{L}_\alpha} \left( e^{\sum_{j \in B} z_j \sum_\omega \mathcal{L}_\alpha h_j(\omega)} \right)
= \exp \left( \alpha \left( \int_{\Omega_\Delta} \left( e^{\sum_{j \in B} z_j h_j(\omega)} - 1 \right) d\mu(\omega) \right) \right).
$$

(7.8)

Differentiating each side of (7.8) with respect to each $z_j$, $j \in B$, and then setting all the $z_j$ equal to zero, we obtain

$$
E_{\mathcal{L}_\alpha} \left( \prod_{j \in B} \phi_{\sum_\omega \mathcal{L}_\alpha h_j(\omega)} \right)
= \sum_{\bigcup A_l = B} \prod_l \alpha \mu (\prod_{j \in A_l} h_j),
$$

(7.9)

in which we sum over all partitions of $B$. We can write this as

$$
E_{\mathcal{L}_\alpha} \left( \prod_{j \in B} \phi_{h_j} \right)
= \sum_{\bigcup A_l = B} \prod_l \alpha \mu (h_{A_l}).
$$

(7.10)

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The lemma follows from (7.10). To see this we first consider first the case \( N = 1 \) and show that

\[
E_{\mathcal{L}_\alpha}(\phi_1(g_1, \ldots, g_n)) = \prod_{j=1}^{n} \alpha\mu(g_j).
\]

(7.11)

This equation is stated in [5, (3.14)]. We present a detailed proof. When \( n = 1 \), (7.11) follows immediately from (7.8) and is well known. It is obviously the same as (7.10). The proof for general \( n \) proceeds by induction. By (7.10)

\[
E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n} \phi_1(g_j)\right) = \sum_{k=1}^{n} \sum_{\cup_{i=1}^{k} A_i = [1, n]} \prod_{l=1}^{k} \alpha\mu(g_{A_l}).
\]

(7.12)

Using (7.4) and (7.12) we have

\[
E_{\mathcal{L}_\alpha}(\phi_n(g_1, \ldots, g_n)) = E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n} \phi_1(g_j)\right) - \sum_{k=1}^{n-1} \sum_{\cup_{i=1}^{k} A_i = [1, n]} E_{\mathcal{L}_\alpha}(\phi_k(g_{A_1}, \ldots, g_{A_k}))
\]

\[
= \sum_{k=1}^{n} \sum_{\cup_{i=1}^{k} A_i = [1, n]} \prod_{l=1}^{k} \alpha\mu(g_{A_l}) - \sum_{k=1}^{n-1} \sum_{\cup_{i=1}^{k} A_i = [1, n]} E_{\mathcal{L}_\alpha}(\phi_k(g_{A_1}, \ldots, g_{A_k})).
\]

(7.13)

Assuming that (7.11) holds for \( k \leq n - 1 \) we get

\[
E_{\mathcal{L}_\alpha}(\phi_n(g_1, \ldots, g_n)) = \sum_{k=1}^{n} \sum_{\cup_{i=1}^{k} A_i = [1, n]} \prod_{l=1}^{k} \alpha\mu(g_{A_l}) - \sum_{k=1}^{n-1} \sum_{\cup_{i=1}^{k} A_i = [1, n]} \prod_{l=1}^{k} \alpha\mu(g_{A_l}).
\]

(7.14)

Since the only partition of \([1, n]\) with \( n \) parts is given by \( A_j = \{j\}, j = 1, \ldots, n \), we obtain (7.11) for \( n \).
We show below that
\[ \prod_{m=1}^{N} \phi_{n_m} \left( g_{s_m+1}, \ldots, g_{s_m+n_m} \right) \]  
\[ = \sum_k \sum_{\cup_{l=1}^{k} A_l = \{1, p\}} \prod_{m=1}^{N} \phi_k \left( g_{A_1}, \ldots, g_{A_k} \right). \]  
Taking the expectation of (7.15) and using (7.11), we get (7.7). This follows because
\[ \sum_k \sum_{\cup_{l=1}^{k} A_l = \{1, p\}} \prod_{m=1}^{N} \alpha \mu(g_{A_l}) \]  
(7.16)
is the same as the right-hand side of (7.7).
To obtain (7.15) we note that by (7.1)
\[ \prod_{m=1}^{N} \phi_{n_m} \left( g_{s_m+1}, \ldots, g_{s_m+n_m} \right) \]  
\[ = \prod_{m=1}^{N} \sum_{(\omega_{i_{s_m+1}}, \ldots, \omega_{i_{s_m+n_m}}) \in S_{n_m}(\mathcal{L}_\alpha)} \prod_{l=1}^{n_m} g_{s_m+l}(\omega_{i_{s_m+l}}). \]  
We rearrange this product of sums, taking into account the points that occur in \( S_{n_m} \) for more than one \( m \), to see that the second line in (7.17) is the same as
\[ \sum_k \sum_{\cup_{l=1}^{k} A_l = \{1, p\}} \sum_{(\omega_{i_1}, \ldots, \omega_{i_k}) \in S_k(\mathcal{L}_\alpha)} \prod_{l=1}^{k} g_{A_l}(\omega_{i_l}) \]  
(7.18)
By the definition (7.1) this is the same as the second line of (7.15).

The relationship in (7.15) is a Poissonian analogue of the Wick expansion formula in [13, Lemma 2.3]. We restate it as a lemma.
Lemma 7.2 Let $\phi_n$ be as defined in (7.1), $s_m = \sum_{j=1}^{m-1} n_j$, $m = 1, \ldots, N + 1$ and $p = s_{N+1}$. Then for any functions $g_j$, $j = 1, \ldots, p$,

$$\prod_{m=1}^{N} \phi_n (s_{m+n}, \ldots, s_{m+n}) = \sum_k \sum_{\cup_{i=1}^{k} A_i = [1, p]} \phi_k (g_{A_1}, \ldots, g_{A_k}).$$

Example 7.1 When all $n_m = 1$, $m = 1, \ldots, N$, in (7.19) it gives the equation in (7.4). Here are some other examples,

$$\phi_2 (g_1, g_2) \phi_1 (g_3) = \phi_3 (g_1, g_2, g_3) + \phi_2 (g_{[1,3]}, g_2) + \phi_2 (g_1, g_{[2,3]}),$$

(7.20)

$$\phi_2 (g_1, g_2) \phi_2 (g_3, g_4) = \phi_4 (g_1, g_2, g_3, g_4)$$

+ $\phi_3 (g_{[1,3]}, g_2, g_4) + \phi_3 (g_{[1,4]}, g_2, g_3) + \phi_3 (g_1, g_{[2,3]}, g_4) + \phi_3 (g_1, g_3, g_{[2,4]})$

+ $\phi_2 (g_{[1,3]}, g_{[2,4]}) + \phi_2 (g_{[1,4]}, g_{[2,3]}).$

In addition when all $n_m = 1$, and $g_m = g$, $m = 1, \ldots, n$

$$\phi_n (g) = \sum_{k=1}^{n} \sum_{\cup_{i=1}^{k} A_i = [1, n]} \phi_k (g_{A_1}, \ldots, g_{A_k})$$

(7.22)

$$= \frac{1}{n!} \sum_{n_1 + \cdots + n_k = n} \binom{n}{n_1 \ldots n_k} \phi_k (g_{[1,n_1]}, \ldots, g_{[1,n_k]}).$$

This is a special case of (7.4).

The random variables $\{\phi_n (g_1, \ldots, g_n)\}_{n=1}^{\infty}$ are not necessarily orthogonal with respect to $E_{\alpha}$. Let

$$I_n (g_1, \ldots, g_n) := \sum_{D = \{i_1, \ldots, i_D\} \subseteq [1, n]} (-1)^{|D^c|} \phi_D (g_{i_1}, \ldots, g_{i_D}) \prod_{j \in D^c} \alpha (g_j).$$

(7.23)

We refer to the random variables $I_n (g_1, \ldots, g_n)$ as Poisson Wick products and show in Corollary 7.1 that they are orthogonal with respect to $E_{\alpha}$.

Considering (7.23) we note that $I_n (g_1, \ldots, g_n) - \phi_n (g_1, \ldots, g_n) \in H_{n-1}$. 

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Let $p \geq k$ and let $\{A_i\}$ be a partition of $[1, p]$. Let $\{\cup A_i = [1, p] \setminus [1, k]_{ns}\}$ be the subset of partitions of $[1, p]$ in which all partitions that contain a set consisting of only one member of $[1, k]$ are eliminated. (The symbol $ns$ indicates no singleton.)

**Lemma 7.3** Let $\mathcal{L}_\alpha$ be a Poisson point process on $\Omega_\Delta$ with intensity measure $\alpha \mu$. Let $s_m = \sum_{j=1}^{m-1} n_j$, $m = 1, \ldots, N + 1$ and let $p \geq s_{N+1}$. Then for any functions $g_j \in D_1(\mu)$, $j = 1, \ldots, p$,

$$
E_{\mathcal{L}_\alpha} \left( \prod_{m=1}^{N} I_{n_m} (g_{s_m+1}, \ldots, g_{s_m+n_m}) \prod_{j=s_{N+1}+1}^{p} \phi_1(g_j) \right)
$$

(7.24)

\[ = \sum_{\cup A_l = [1, p], \cup [s_{N+1}+1, p]} \prod_l \alpha \mu(g_{A_l}). \]

(7.25)

(The following proof is direct, but involves a good deal of combinatorics. In Section 9 we give an alternate proof using Poissonian exponentials.)

**Proof** Consider first the case in which $N = 1$. Using (7.23), with $n_1 = n$, we see that the left-hand side of (7.24) is equal to

$$
\sum_{D = \{i_1, \ldots, i_{|D|}\} \subseteq [1, n]} (-1)^{|D^c|} E_{\mathcal{L}_\alpha} \left( \phi_{|D|}(g_{i_1}, \ldots, g_{i_{|D|}}) \prod_{j=n+1}^{p} \phi_1(g_j) \right) \prod_{j \in D^c} \alpha \mu(g_j).
$$

(7.25)

By (7.7) this is equal to

$$
\sum_{D = \{i_1, \ldots, i_{|D|}\} \subseteq [1, n]} (-1)^{|D^c|} \sum_{(\cup A_l = D \cup [n+1, p]) \perp D} \prod_l \alpha \mu(g_{A_l}) \prod_{j \in D^c} \alpha \mu(g_j), \quad (7.26)
$$

which we write as

$$
\sum_{D \subseteq [1, n]} (-1)^{|D^c|} \sum_{(\cup A_l = [1, p] \setminus \{1, 2, \ldots, n\}) \perp D^c} \prod_l \alpha \mu(g_{A_l}), \quad (7.27)
$$

where $(D^c)_s$ indicates that each element in $D^c$ is a singleton of the partition $\cup A_l = [1, p]$. (This is simple, we just take a partition in $(\cup A_l = D \cup [n+1, p]) \perp D$ and add the singletons in $D^c$.)

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Consider a partition with singletons. Let \( C \subseteq [1, n] \) denote the singletons in such a partition. The terms in (7.27) which give rise to such a partition are precisely those with \( D^c \subseteq C \). Therefore, if \( |C| = m \),

\[
\sum_{D^c \subseteq C} (-1)^{|D^c|} = \sum_{k=0}^{m} \binom{m}{k} (-1)^k = 0. 
\]

(7.28)

This shows that when \( N = 1 \), (7.27) equals the right hand side of (7.24).

The proof for general \( N \) consists of a straightforward, albeit tedious, generalization of the arguments used for \( N = 1 \). Using (7.23) and expanding the left-hand side of (7.24) we get that it is equal to

\[
\sum_{D_m = \{s_m, 1, \ldots, s_m, |D_m|\}} (-1)^{|\cup D_m^c|} \prod_{j \in \cup D_m^c} \alpha \mu(g_j) 
\]

(7.29)

By (7.7) this is

\[
\sum_{D_m \subseteq [s_m + 1, sm + n_m]} (-1)^{|\cup D_m^c|} \prod_{j \in \cup D_m^c} \alpha \mu(g_j) 
\]

(7.30)

As in (7.27) this is

\[
\sum_{D_m \subseteq [s_m + 1, sm + n_m]} (-1)^{|\cup D_m^c|} \sum_{(\cup A_l = [1, p]) \cup \{s_m + 1, \ldots, sm + n_m\}} \prod_l \alpha \mu(g_{A_l}) .
\]

(7.31)

Consider a partition with singletons, and let \( C \) denote the collection of singletons in such a partition. We write \( C = \cup C_m \) where \( C_m \subseteq [s_m + 1, sm + n_m] \). The terms in (7.31) which give rise to such a partition are precisely those with \( D^c_m \subseteq C_m \) for all \( m \). Note then if \( |C_m| = s_m \),

\[
\sum_{D_m^c \subseteq C_m} (-1)^{|D_m^c|} = \prod_{m=1}^{N} \left( \sum_{D_m^c \subseteq C_m} (-1)^{|D_m^c|} \right) = \prod_{m=1}^{N} \left( \sum_{k=0}^{s_m} \binom{s_m}{k} (-1)^k \right) = 0.
\]

(7.32)
This shows that (7.31) equals the right hand side of (7.24). □

An important consequence of Lemma 7.3 is that $I_n(g_1, \ldots, g_n)$ extends to a continuous multilinear map from $(D_2(\mu))^n$ to $D_1(P_{\alpha})$, and that (7.24) holds for the extension. This observation is critical in the proofs in Section 8. We state it as a lemma.

Lemma 7.4 Let $\mathcal{L}_\alpha$ be a Poisson point process on $\Omega_\Delta$ with intensity measure $\alpha \mu$. Let $s_m = \sum_{j=1}^{m-1} n_j$, $m = 1, \ldots, N+1$. Then for any functions $g_j \in D_2(\mu)$, $j = 1, \ldots, s_{N+1}$,

$$E_{\mathcal{L}_\alpha} \left( \prod_{m=1}^{N} I_{n_m}(g_{s_m+1}, \ldots, g_{s_m+n_m}) \right) = \sum_{\cup_{l} A_l = [1, s_{N+1}], |A_l| \geq 2, \downarrow \{s_m+1, \ldots, s_m+n_m\}, m=1, \ldots, N} \prod_l \alpha \mu \left( g_{A_l} \right). \quad (7.33)$$

Proof Let $\{D_m\}$ be as defined in (4.3). If $g \in D_2(\mu)$ then $1_{\{D_m\}} g \in D_1(\mu)$ and $1_{\{D_m\}} g \to g$ as $m \to \infty$ in all $L^p(\mu)$, $p \geq 2$. By Lemma 7.3 we see that (7.33) holds with $g_j$ replaced by $1_{\{D_m\}} g_j$. Because of the condition that all the sets $A_l$ have $|A_l| \geq 2$,

$$\lim_{m \to \infty} \prod_l \alpha \mu \left( \prod_{j \in A_l} 1_{\{D_m\}} g_j \right) = \prod_l \alpha \mu \left( \prod_{j \in A_l} g_j \right). \quad (7.34)$$

In this way we can extend the left-hand side of (7.33) to random variables $g_j \in D_2(\mu)$, and get that (7.33) holds for the extension. □

Corollary 7.1 The random variables $\{I_n(g_1, \ldots, g_n)\}_{n=0}^{\infty}$, with $I_0 \equiv 1$, are orthogonal in $L^2(P_{\alpha})$.

Proof By (7.24)

$$E_{\mathcal{L}_\alpha} (I_n(g_1, \ldots, g_n)I_m(g_1, \ldots, g_m)) = \sum_{\cup_{l} A_l = [1, n+m], \{1, n+m\} \neq \emptyset, \downarrow \{1, \ldots, n\}, \{n+1, \ldots, n+m\}} \prod_l \alpha \mu \left( \prod_{j \in A_l} g_j \right). \quad (7.35)$$
The restrictions on the partitions that are summed over require that all the sets \( \{ A_l \} \) contain two elements, one from \([1, n]\) and one from \([1, m]\). Therefore we must have \( m = n \) to get any contribution from the right-hand side of (7.35).

In the paragraph containing (7.6) we define \( \mathcal{H}_n \) to be the closure in \( L^2(\mathbb{P}_\alpha) \) of all linear combinations of \( \phi_j, j = 0, 1, \ldots, n \). By Corollary 7.1 and the remarks in the paragraph containing (7.6) we see that \( I_n \in \mathcal{H}_n \ominus \mathcal{H}_{n-1} = \mathcal{H}_n \cap \mathcal{H}_{n-1}^\perp \) and that linear combinations of the \( \{ I_n \} \) are dense in \( H_n := \mathcal{H}_n \ominus \mathcal{H}_{n-1} \).

Using (7.6) we then have the orthogonal decomposition

\[
L^2(\mathbb{P}_\alpha) = \oplus_{n=0}^\infty H_n. \tag{7.36}
\]

We call this the Poisson chaos decomposition of \( L^2(\mathbb{P}_\alpha) \).

Let \( \{ A'_l \} \) be a partition of \([1, s_{N+1}]\), we define

\[
J_k(g_{A'_1}, \ldots, g_{A'_k}) \tag{7.37}
\]

\[
:= \sum_{D^c \subseteq \{ l \mid |A'_l| = 1 \}} (-1)^{|D^c|} \phi_{|D|}(g_{A'_j}; j \in D) \prod_{j \in D^c} \alpha \mu(g_{A'_j}).
\]

Using Lemma 7.3 we have the following analogue of Lemma 7.2.

**Lemma 7.5** Let \( I_n \) be as defined in (7.23), \( s_m = \sum_{j=1}^{m-1} n_j, m = 1, \ldots, N+1 \).

Then for any functions \( g_j, j = 1, \ldots, s_{N+1}, \)

\[
\prod_{m=1}^N I_{s_m}(g_{s_m+1}, \ldots, g_{s_m+n_m}) \tag{7.38}
\]

\[
= \sum_k \sum_{\bigcup_{l=1}^k A'_l = [1, s_{N+1}], \downarrow \{ s_m+1, \ldots, s_m+n_m \}, m=1, \ldots, N} J_k(g_{A'_1}, \ldots, g_{A'_k}).
\]

**Proof** Since polynomials in \( \phi_1 \) are dense in \( L^2(\mathbb{P}_\alpha) \), it suffices to show that both sides of (7.38) have the same \( L^2(\mathbb{P}_\alpha) \) inner product with expressions of the form \( \prod_{j=s_{N+1}+1}^N \phi_1(g_j) \). For the left hand side we use Lemma 7.3 and for the right hand side we use Lemma 7.2 and follow the proof of Lemma 7.3. The subtractions in the definition of \( J_k \) are needed to insure the condition \([1, s_{N+1}]_{ns} \). □
8 Loop soup decomposition of permanental Wick powers

The \( n \)-th order permanental Wick power, \( \tilde{\psi}_n(\nu) \) is defined in Section 6 as a renormalized sum of powers of a permanental field, \( \psi(\nu) \). In this section we define \( \tilde{\psi}_n(\nu) \) using loop soups. This enables us to easily obtain an isomorphism theorem involving \( \tilde{\psi}_n(\nu) \) and \( n \)-fold intersection local times \( L_n(\nu) \).

The loop soup description of \( \tilde{\psi}_n(\nu) \) involves the permanental chaos \( \psi_n(\nu) \), which we think of as involving self-intersections of each path in the loop soup, and new random variables, \( I_{l_1}, \ldots, I_{l_j}(\nu) \), which we think of as involving intersections of different paths. The representation of \( \tilde{\psi}_n(\nu) \) is actually its Poisson chaos decomposition into random variables \( I_{l_1}, \ldots, I_{l_j}(\nu) \), for different indices \( l_1, \ldots, l_j \).

We proceed to develop the material needed to define \( I_{l_1}, \ldots, I_{l_n}(\nu) \). Note that for fixed \( r > 0 \), \( L_j(x, r) \) is a polynomial in \( L(x, r) \) without a constant term. Therefore by (2.12), with \( g_j = f_{x_j, r} \) we see that \( L_j(x, r) \in D_2(\mu) \) for each \( j \geq 1 \). It follows from this and Corollary 7.4 that

\[
I_{l_1, \ldots, l_n}(x, r) := \lim_{r \to 0} \int I_n(L_{l_1}(x, r), \ldots, L_{l_n}(x, r)) \, d\nu(x)
\]

is well defined.

**Theorem 8.1** Let \( l = l_1 + \cdots + l_n \) and \( \nu \in B_{2l}(R^d) \). Then

\[
I_{l_1, \ldots, l_n}(\nu) := \lim_{r \to 0} \int I_n(L_{l_1}(x, r), \ldots, L_{l_n}(x, r)) \, d\nu(x)
\]

exists in all \( L^p(P_{L_\alpha}) \).

Note that \( I_{l_1, \ldots, l_n}(\nu) \in H_n \).

**Proof** It follows from (7.33) that for integers \( v \geq 1 \),

\[
E_{L_\alpha} \left( \prod_{i=1}^{v} I_{l_1, \ldots, l_n}(x_i, r_i) \right) = \sum_{\cup A_i = [1, v], |A_i| \geq 2} \prod_{\{i \in A_i\}} g_j \prod_{m=1}^{v} \alpha_{\mu} \left( \prod_{j \in A_i} g_j \right),
\]

in which each \( g_j \) is one of the random variables \( L_{l_k}(x_i, r_i) \). The lemma follows from (2.57) and the work that follows it in Section 2.\( \square \)
It should be clear from the definitions that \( \mathcal{I}_{l_1, \ldots, l_n}(\nu) = \mathcal{I}_{\sigma(l_1), \ldots, \sigma(l_n)}(\nu) \) for any permutation \( \sigma \) of \([1, n]\).

Let \( \ell = (l_1, \ldots, l_k) \) and let \( m_i(\ell) \) be the number of indices \( j \) with \( l_j = i \). We define the degree of \( \ell \) to be

\[
d(\ell) = (m_1(\ell), m_2(\ell), \ldots, m_k(\ell)),
\]

and set

\[
e(\ell) = \prod_{i=1}^{k} m_i(\ell_k)!
\] (8.5)

The next lemma shows that the processes \( \mathcal{I}_{l_1, \ldots, l_k}(\nu) \) with different degrees \( d(\ell) \) are orthogonal.

**Lemma 8.1** Let \( \ell = (l_1, \ldots, l_k) \) and \( \ell' = (l'_1, \ldots, l'_{k'}) \). Then

\[
E_{\mathcal{L}_{\alpha}} \left( \mathcal{I}_{l_1, \ldots, l_k}(\nu) \mathcal{I}_{l'_1, \ldots, l'_{k'}}(\nu) \right) = \delta_{d(\ell), d(\ell')} e(\ell) \lim_{r \to 0} \int \prod_{j=1}^{k} \alpha \mu \left( L_{l_j}(x, r) L_{l'_j}(y, r) \right) d\nu(x) d\nu(y).
\] (8.6)

**Proof** This proof is similar to the proof of Lemma 6.1. By (7.24),

\[
E_{\mathcal{L}_{\alpha}} \left( \mathcal{I}_{l_1, \ldots, l_k}(\nu) \mathcal{I}_{l'_1, \ldots, l'_{k'}}(\nu) \right)
= \sum_{\cup_m A_m = [1, k+k'], |A_m| \geq 2} \lim_{r \to 0} \int \prod_{m} \alpha \mu \left( \prod_{j \in A_m} \tilde{L}_{a_j}(\bar{x}, r) \right) d\nu(x) d\nu(y),
\]

where \( \tilde{L}_{a_j}(\bar{x}, r) = L_{l_j}(x, r) \) if \( j \in [1, k] \) and \( \tilde{L}_{a_j}(\bar{x}, r) = L_{l'_{j-k}}(y, r) \) if \( j \in [k+1, k+k'] \). By the separation condition, \( \perp [1, k], [k+1, k+k'] \), we see that we must have \( |A_m| = 2 \) for each \( m \), and each \( A_m \) consists of one element \( j_m \) from \([1, k]\) and one element \( j'_m \) from \([k+1, k+k']\). However, by the alternating condition in (2.55), we see that in the limit \( \mu \left( L_{l_{j_m}}(x, r) L_{l'_{j'_m-k}}(y, r) \right) \) would give a contribution of 0 unless \( l_{j_m} = l'_{j_m-k} \). This forces \( d(\ell) = d(\ell') \) and (8.6) follows since there are \( e(\ell) \) ways to pair the elements of \( \ell \) with those of \( \ell' \) (with identical integers in each pair).
Lemma 8.2 Let \( l_m = \sum_{j=1}^{n_m} l_m(j) \) and \( l = \sum_{m=1}^{N} l_m \). Furthermore, let \( v_m \in B_{2m} \), \( m = 1, \ldots, N \). Then

\[
E_{\mu} \left( \prod_{m=1}^{N} \mathcal{I}_{l_m} \cdots \mathcal{I}_{l_m(n_m)} (v_m) \psi^p(x,r) \right)
= \sum_{\pi \in P_{l+p,a-}} \alpha^\pi \int \prod_{j=1}^{l+p} u(z_j, z_{\pi(j)}) \prod_{m=1}^{N} \sum_{j=1}^{l+p} f_{x,r}(x_i) dx_i,
\]

where \( z_1, \ldots, z_l \) are all equal to \( x_1 \), the next \( l_2 \) of the \( \{z_j\} \) are all equal to \( x_2 \), and so on, so that the last \( l_N \) of the \( \{z_j\} \) are all equal to \( x_N \).

In addition \( P_{l+p,a-} \) is the set of permutations \( \pi \) of \([1, l+p] \) with cycles that alternate the variables \( \{x_i\} \); (i.e., for all \( j \), if \( z_j = x_i \) then \( z_{\pi(j)} \neq x_i \)), and, for each \( m = 1, \ldots, N \), \( \pi \) contains distinct cycles \( C_m(1), \ldots, C_m(n_m) \), such that \( C_m(1) \) contains the first \( l_m(1) \) of the \( x_m \)’s, \( C_m(2) \) contains the next \( l_m(2) \) of the \( x_m \)’s, etc. (If \( z_{a+1} = \cdots = z_{a+l_m} = x_m \), then the first \( l_m(1) \) of the \( x_m \)’s are \( z_{a+1}, \ldots, z_{a+l_m(1)} \), the next \( l_m(2) \) of the \( x_m \)’s are \( z_{a+l_m(1)+1}, \ldots, z_{a+l_m(1)+l_m(2)} \), etc.)

**Proof** Let \( s_m = \sum_{j=1}^{n_m-1} n_j \), \( m = 1, \ldots, N+1 \). Then for any functions \( g_j \in \mathcal{D}_2(\mu) \), \( j = 1, \ldots, s_{N+1} + p \) it follows from (7.33) that

\[
E_{\mu} \left( \prod_{m=1}^{N} \mathcal{I}_{l_m} (g_{s_m+1}, \ldots, g_{s_m+n_m}) \prod_{j=1}^{p} \mathcal{I}_{l_1} (g_{s_{N+1}+j}) \right)
= \sum_{\cup_{j=A_l} \mid s_m+1, \ldots, s_m+n_m, m=1, \ldots, N} \prod_{l} \alpha^\mu \left( \prod_{j \in A_l} g_{j} \right).
\]

Since \( 1_{D_m}L_1(x,r) \rightarrow L_1(x,r) \) in \( \mathcal{D}_2(\mu) \) it follows from (6.2) that

\[
\psi(x,r) = I_1(L_1(x,r)).
\]

We use (8.9) with \( g_{s_m+j} = L_m(j)(x_m,r) \), \( 1 \leq j \leq n_m \) and \( g_j = L_1(x,r) \), \( s_{N+1} + 1 \leq j \leq p \) and integrate both sides with respect to the measures \( \nu_m(x_m) \), \( 1 \leq m \leq N \) and use Theorem 8.1 to get

\[
E_{\mu} \left( \prod_{m=1}^{N} \mathcal{I}_{l_m} \cdots \mathcal{I}_{l_m(n_m)} (v_m) \psi^p(x,r) \right)
= \sum_{\cup_{j=A_l} \mid s_m+1, \ldots, s_m+n_m, m=1, \ldots, N} \prod_{l} \alpha^\mu \left( \prod_{j \in A_l} \hat{I}_{j} \right).
\]
where $\hat{L}_{s_m+j} = L_{m(j)}(\nu_m)$, $1 \leq j \leq n_m$ and $\hat{L}_j = L_1(x, r)$, $l + 1 \leq j \leq l + p$.

The right-hand side of (8.11) is the same as the right-hand side of (8.8). To see this we use Theorem 1.3 to analyze each factor $\mu \left( \prod_{j \in A_l} \hat{L}_j \right)$ as in the proof of Theorem 1.4 on page 39. Each of these factors corresponds to a cycle in a permutation. The condition in the second line under the summation sign in (8.11) requires that in each factor $\mu \left( \prod_{j \in A_l} \hat{L}_j \right)$, for each $m = 1, \ldots, N$, there is at most one term $L_{m(j)}(\nu_m), 1 \leq j \leq n_m$. Therefore distinct terms $L_{m(j)}(\nu_m), 1 \leq j \leq n_m$, are contained in distinct cycles.

\textbf{Theorem 8.2} For $\nu \in \mathcal{B}_{2n}(\mathbb{R}^d)$

$$\tilde{\psi}_n(\nu) = \sum_{D_1 \cup \cdots \cup D_l = [1, n]} I_{|D_1|, \ldots, |D_l|}(\nu),$$

where the sum is over all partitions of $[1, n]$.

\textbf{Proof} Set

$$\phi_n(\nu) = \sum_{D_1 \cup \cdots \cup D_l = [1, n]} I_{|D_1|, \ldots, |D_l|}(\nu).$$

It follows from (8.6) and (6.41) and then from (6.30) and (6.31), that

$$E_{\mathcal{L}_a} \left( \phi_n^2(\nu) \right) = E_{\mathcal{L}_a} \left( \tilde{\psi}_n \psi_n(\nu) \right) = \sum_{\pi \in \mathcal{P}_{2n,a}} \alpha^{c(\pi)} \int \int (u(x, y)u(y, x))^n \, d\nu(x) \, d\nu(y),$$

where, as above, $\mathcal{P}_{2n,a}$ is the set of permutations of $[1, 2n]$ which alternate $1, \ldots, n$ and $n + 1, \ldots, 2n$.

It follows from (8.8) with $N = 1$ and $k = n$, that

$$E_{\mathcal{L}_a} \left( I_{|D_1|, \ldots, |D_l|}(\nu)\psi^n(x, r) \right) = \sum_{\pi \in \mathcal{P}_{2n,a}^-} \alpha^{c(\pi)} \int \prod_{j=1}^{2n} u(z_j, z_{\pi(j)}) \, d\nu(x_1) \prod_{i=2}^{n+1} f_{x,r}(x_i) \, dx_i.$$
give rise to a disjoint set of partitions and adding over all of them as in (8.13) we get
\[
E_{\mathcal{L}_\alpha} \left( \phi_n(\nu) \psi^n(x, r) \right) = \sum_{\pi \in \mathcal{P}_{2n,a}} \alpha^{c(\pi)} \int \prod_{j=1}^{2n} u(z_j, z_{\pi(j)}) \, d\nu(x_1) \prod_{i=2}^{n+1} f_{x,r}(x_i) \, dx_i.
\]

We write the left-hand side of (8.16) as
\[
E_{\mathcal{L}_\alpha} \left( \phi_n(\nu) \left( \tilde{\psi}_n(x, r) + H(x, r) \right) \right)
\]
and use (8.15) to estimate \( E_{\mathcal{L}_\alpha}(\phi_n(\nu)H(x, r)) \). Exactly as in the proof of Theorem 6.1 and 6.2 we see that these are what we called error terms and that after integrating with respect to \( \nu \) and taking the limit as \( r \) goes to zero we get
\[
E_{\mathcal{L}_\alpha} \left( \phi_n(\nu) \tilde{\psi}_n(\nu) \right) = \sum_{\pi \in \mathcal{P}_{2n,a}} \alpha^{c(\pi)} \int \prod_{j=1}^{2n} u(z_j, z_{\pi(j)}) \, d\nu(x) \, d\nu(y).
\]

It follows from (8.14) and (8.18) that
\[
E_{\mathcal{L}_\alpha} \left( \left( \phi_n(\nu) - \tilde{\psi}_n(\nu) \right)^2 \right) = 0. \tag{8.19}
\]

\[\square\]

**Corollary 8.1** For \( \nu \in \mathcal{B}^{2n} \)
\[
\mathcal{I}_n(\nu) = \psi_n(\nu) \tag{8.20}
\]
in \( L^p(\mathcal{P}_{\mathcal{L}_\alpha}) \).

**Proof** As in the proof of (8.10) we have
\[
\psi_n(\nu) = I_1(L_n(\nu)). \tag{8.21}
\]

By Theorem 8.1
\[
\mathcal{I}_n(\nu) = \lim_{r \to 0} \int I_1(L_n(x, r)) \, d\nu(x). \tag{8.22}
\]
We show below that
\[ \int I_1(L_n(x,r)) \, d\nu(x) = I_1 \left( \int L_n(x,r) \, d\nu(x) \right). \] (8.23)

Since \( \int L_n(x,r) \, d\nu(x) \to L_n(\nu) \) in \( D_2(\mu) \) by (2.22) and \( I_1 \) is continuous, we get (8.20).

To prove (8.23) we show that
\[ E_{\mathcal{L}_\alpha} \left( \left( \int I_1(L_n(x,r)) \, d\nu(x) - I_1 \left( \int L_n(x,r) \, d\nu(x) \right) \right)^2 \right) = 0. \] (8.24)

Using Fubini’s theorem and (7.33) we get
\[ E_{\mathcal{L}_\alpha} \left( \left( \int I_1(L_n(x,r)) \, d\nu(x) \right)^2 \right) = \int \int E_{\mathcal{L}_\alpha} \left( I_1(L_n(x,r))I_1(L_n(y,r)) \right) \, d\nu(x) \, d\nu(y) \]
\[ = \alpha \int \int \mu \left( L_n(x,r)L_n(y,r) \right) \, d\nu(x) \, d\nu(y), \]
and
\[ E_{\mathcal{L}_\alpha} \left( \left( \int L_n(x,r) \, d\nu(x) \right)^2 \right) = \alpha \mu \left( \int L_n(x,r) \, d\nu(x) \right) \int L_n(y,r) \, d\nu(y). \] (8.26)

Also
\[ E_{\mathcal{L}_\alpha} \left( \int I_1(L_n(x,r)) \, d\nu(x) \, I_1 \left( \int L_n(x,r) \, d\nu(x) \right) \right) = \int \int E_{\mathcal{L}_\alpha} \left( I_1(L_n(x,r))I_1 \left( \int L_n(y,r) \, d\nu(y) \right) \right) \, d\nu(x) \]
\[ = \alpha \int \mu \left( L_n(x,r) \int L_n(y,r) \, d\nu(y) \right) \, d\nu(x). \] (8.27)

The statement in (8.23) follows by Fubini’s theorem. This completes the proof of the lemma. \( \square \)
Remark 8.1 To illustrate (8.12), by Theorem 8.2 and Corollary 8.1
\[
\tilde{\psi}_2(\nu) = \psi_2(\nu) + \mathcal{I}_{1,1}(\nu)
\]  
(8.28)
and
\[
\tilde{\psi}_3(\nu) = \psi_3(\nu) + 3\mathcal{I}_{2,1}(\nu) + \mathcal{I}_{1,1,1}(\nu).
\]  
(8.29)
Consider
\[
\mathcal{I}_{1,1}(\nu) = \lim_{r \to 0} \int I_2(L_1(x,r), L_1(x,r)) \, d\nu(x).
\]  
(8.30)
Ignoring all limits and renormalization terms, this involves the process
\[
\int L_1(x,r)(\omega)L_1(x,r)(\omega') \, d\nu(x)
\]  
(8.31)
\[
= \int \int_{0}^{\infty} \int_{0}^{\infty} f_r(\omega_s - x) f_r(\omega'_t - x) \, ds \, dt \, d\nu(x).
\]
In the limit, as \( r \) goes to zero, this is an intersection local time for the two paths \( \omega \) and \( \omega' \). Thus we see that \( \mathcal{I}_{1,1}(\nu) \) involves intersections between different paths in the loop soup. Similar remarks apply to all \( \mathcal{I}_{n_1,\ldots,n_l}(\nu) \). This should be investigated further.

Remark 8.2 We have often commented that \( \tilde{\psi}_{2n}(\nu) \) is a Gaussian chaos, the \( 2n \)-th Wick power, when the potential of the underlying Lévy process is symmetric and \( \alpha = 1/2 \). A similar characterization for \( \psi_{2n}(\nu) \) would be interesting.

We next obtain an isomorphism theorem for \( \tilde{\psi}_n \). To recall the meaning of the notation it may be useful to look at Section 5.

Lemma 8.3 For \( \nu \in \mathcal{B}_{2(j+k)}(R^d) \)
\[
\left( \tilde{\psi}_j \times L_k \right)(\nu) = \lim_{r \to 0} \int \tilde{\psi}_j(x,r)L_k(x,r) \, d\nu(x)
\]  
(8.32)
events in \( L^p(\mu) \) for all \( p \geq 2 \).

Proof This follows as in the proofs of similar results in Theorems 1.3 and Theorem 6.1.

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Lemma 8.4 For $\nu \in B_{2n}(R^d)$

$$\tilde{\psi}_n(\nu)(L_{\alpha} \cup \bar{\omega}) = \sum_{l=0}^{n} \binom{n}{l} \left( \tilde{\psi}_{n-l} \times L_l(\nu)(L_{\alpha}, \bar{\omega}) \right). \quad (8.33)$$

Proof By (7.23), for $g_1, \ldots, g_l \in D_1(\mu)$

$$I_l(g_1, \ldots, g_l)(L_{\alpha} \cup \bar{\omega}) = \sum_{D=\{i_1, \ldots, i_{|D|}\} \subseteq [1,l]} (-1)^{|D^c|} \left( \sum_{(\omega_{i_1}, \ldots, \omega_{i_{|D|}}) \in S_{|D|}(L_{\alpha} \cup \bar{\omega})} \prod_{j \in D} g_j(\omega_{i_j}) \prod_{j \in D^c} \alpha(\mu(g_j)). \right) \quad (8.34)$$

We have

$$\sum_{(\omega_{i_1}, \ldots, \omega_{i_{|D|}}) \in S_{|D|}(L_{\alpha} \cup \bar{\omega})} \prod_{j \in D} g_j(\omega_{i_j}) = \sum_{(\omega_{i_1}, \ldots, \omega_{i_{|D|}}) \in S_{|D|}(L_{\alpha})} \prod_{j \in D} g_j(\omega_{i_j})$$

$$+ \sum_{m=1}^{|D|} \left( \sum_{(\omega_{i_1}, \ldots, \omega_{i_m}, \ldots, \omega_{i_{|D|}}) \in S_{|D|-1}(L_{\alpha})} \prod_{j \in D-\{m\}} g_j(\omega_{i_j}) \right) g_m(\bar{\omega}), \quad (8.35)$$

where the notation $(a_1, \ldots, \tilde{a}_m, \ldots, a_k)$ means that we remove the $m$-th entry $a_m$ from the vector $(a_1, \ldots, a_k)$.

Using (8.35) in (8.34) and rearranging we obtain

$$I_l(g_1, \ldots, g_l)(L_{\alpha} \cup \bar{\omega}) = I_l(g_1, \ldots, g_l)(L_{\alpha})$$

$$+ \sum_{j=1}^l I_l(g_1, \ldots, \tilde{g}_j, \ldots, g_l)(L_{\alpha}) g_j(\bar{\omega}). \quad (8.36)$$

The first term on the right-hand side comes from the first term on the right hand side of (8.35). The term $I_l(g_1, \ldots, \tilde{g}_j, \ldots, g_l)(L_{\alpha}) g_j(\bar{\omega})$, on the second line of (8.36), contains all terms with $g_j(\bar{\omega})$ which arise after substituting (8.35) in (8.34).

Proceeding as in the proof of Theorem 8.1 this implies that

$$I_{n_1, \ldots, n_l}(\nu)(L_{\alpha} \cup \bar{\omega}) = I_{n_1, \ldots, n_l}(\nu)(L_{\alpha}) + \sum_{j=1}^l \left( I_{n_1, \ldots, \tilde{n}_j, \ldots, n_l} \times L_{n_j}(\nu)(L_{\alpha}, \bar{\omega}) \right), \quad (8.37)$$
where \( (I_{n_1, \ldots, n_l} \times L_{n_j})(\nu) \) is defined in a manner similar to (8.32). The lemma now follows from (8.12) and the fact that there \( \binom{n}{n_j} \) ways to choose a part of size \( n_j \).

\[ \square \]

**Theorem 8.3 (Isomorphism Theorem II)** For any positive measures \( \rho, \phi \in B_2(\mathbb{R}^d) \) and all finite measures \( \nu_j \in B_2(\mathbb{R}^d) \), \( j = 1, \ldots, n \), and bounded measurable functions \( F \) on \( \mathbb{R}^\infty_+ \),

\[
E_{\mathcal{L}_a} \int Q^{x,x} \left( L_1(\phi) F \left( \sum_{k=0}^{n} \binom{n}{k} \left( \tilde{\psi}_{n-k} \times L_k \right)(\nu_i) \right) \right) \, d\rho(x) \\
= \frac{1}{\alpha} E_{\mathcal{L}_a} \left( \theta^{\rho,\phi} F \left( \tilde{\psi}_n(\nu_i) \right) \right). \tag{8.38}
\]

(The notation \( F(f(x)) \) is explained in the statement of Theorem 1.5.)

**Proof** The proof is the same as the proof of Theorem 1.5 except we use (8.33) in place of (5.9). \( \square \)

An alternate, combinatorial proof of this isomorphism theorem can be obtained following the techniques in [13]. See also [2].

**Example 8.1** Consider (8.38). In the simplest case, \( n = 2 \), it is

\[
E_{\mathcal{L}_a} \int Q^{x,x} \left( L_1(\phi) F \left( \tilde{\psi}_2(\nu_i) + 2 \left( \tilde{\psi}_1 \times L_1 \right)(\nu_i) \right) \right) \, d\rho(x) \\
= \frac{1}{\alpha} E_{\mathcal{L}_a} \left( \theta^{\rho,\phi} F \left( \tilde{\psi}_2(\nu_i) \right) \right). \tag{8.39}
\]

**Remark 8.3** We present here a variant of Lemma 7.4 that does not use Lemma 7.3, but which is adequate for the needs of this section.

**Lemma 8.5** Let \( \mathcal{L}_a \) be a Poisson point process on \( \Omega_\Delta \) with intensity measure \( \alpha \mu \). Then for each \( n \) there exists a continuous multilinear map \( \tilde{I}_n(g_1, \ldots, g_n) \) from \( D_2^n(\mu) \) to \( D_1(\mathcal{P}_{\mathcal{L}_a}) \) such that for any \( N \) and \( n_j, j = 1, \ldots, N \), and any functions \( g_j \in D_2(\mu) \), \( j = 1, \ldots, s_{N+1} \), where \( s_m = \sum_{j=1}^{m-1} n_j, m = 1, \ldots, N + 1 \),

\[
E_{\mathcal{L}_a} \left( \prod_{m=1}^{N} \tilde{I}_{n_m}(g_{s_m+1}, \ldots, g_{s_m+n_m}) \right) \tag{8.40}
= \sum_{\cup_i A_i = [1, s_{N+1}], |A_i| \geq 2} \prod_{l} \alpha \mu(g_{A_l}).
\]

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Proof Let
\[ \mathcal{D}_{1,0} = \{ f \in \mathcal{D}_1 | \mu(f) = 0 \}. \] (8.41)

We obtain \( \tilde{I}_n(g_1, \ldots, g_n) \) as an extension of \( \phi_n(g_1, \ldots, g_n) \) from \( \mathcal{D}_{1,0}^n(\mu) \) to \( \mathcal{D}_2^n(\mu) \). Lemma 7.1 implies (8.40) for \( g_1, \ldots, g_n \in \mathcal{D}_{1,0} \). We get that \( |A_l| \geq 2 \) because \( g_j \in \mathcal{D}_{1,0} \).

The key observation is that \( \mathcal{D}_{1,0} \) is dense in \( \mathcal{D}_2 \). We note in the proof of Lemma 7.4 that \( \mathcal{D}_1 \) is dense in \( \mathcal{D}_2 \). Therefore, to show that \( \mathcal{D}_{1,0} \) is dense in \( \mathcal{D}_2 \), it suffices to show that for any \( f \in \mathcal{D}_1 \) we can find a sequence \( f_n \in \mathcal{D}_{1,0} \) which converges in \( L^p(\mu) \) for any \( p \geq 2 \).

To see this, let \( \{D_n\} \) be as defined in (4.3) and let
\[ f_n = f - \frac{\mu(f)}{\mu(D_n)} 1_{D_n}. \] (8.42)

Clearly, \( f_n \in \mathcal{D}_{1,0} \). Then for any \( p > 1 \)
\[ \int |f_n - f|^p d\mu = \frac{\mu^p(f)}{\mu^p(D_n)} \int 1_{D_n} d\mu = \frac{\mu^p(f)}{\mu^{p-1}(D_n)} \to 0 \] (8.43)
since \( \mu(D_n) \to \infty \) as \( n \to \infty \).

We can use \( \tilde{I}_n \) in place of \( I_n \) in the definition (8.1) of \( I_{l_1, \ldots, l_n}(x, r) \). To prove Lemma 8.2, we would go back to the definition of \( \psi(x, r) \) and instead of (8.10), set
\[ \psi(x, r) = I_1(L_1(x, r)). \] (8.44)

Finally, the proof of Lemma 8.4 will be even easier since we can work with \( \phi_n \) instead of \( I_n \).

One can think of the extension obtained in this remark as proceeding in two steps. The first step extends \( \phi_n(g_1, \ldots, g_n) \) from \( \mathcal{D}_{1,0}^n(\mu) \) to \( \mathcal{D}_1^n(\mu) \). By Lemma 7.3, we know that \( I_n(g_1, \ldots, g_n) \) is a continuous multilinear map from \( \mathcal{D}_2^n(\mu) \) to \( \mathcal{D}_1(\mathcal{P}_{\mu}) \). Since \( I_n(g_1, \ldots, g_n) = \phi_n(g_1, \ldots, g_n) \) for \( g_1, \ldots, g_n \in \mathcal{D}_{1,0} \), it follows that the extension of \( \phi_n(g_1, \ldots, g_n) \) from \( \mathcal{D}_{1,0}^n(\mu) \) to \( \mathcal{D}_1^n(\mu) \) is \( I_n(g_1, \ldots, g_n) \).

Thus the \( \tilde{I}_n \) obtained in this Remark is the same as the extension of \( I_n(g_1, \ldots, g_n) \) obtained in Lemma 7.4, although this fact is not needed for the applications of this section, as we point out in the preceding paragraph.
9 Poisson chaos decomposition, II

We introduce several new processes because we find them interesting and think that they point out a direction for future research. We continue the development of the Poisson chaos decomposition considered in Section 7. We define an exponential version of the Poisson chaos and an enlargement of it that gives Poissonian chaos martingales. We give an alternate proof of Lemma 7.3 using the exponential Poisson chaos.

9.1 Exponential Poisson chaos

Let

$$\phi_{\exp}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_n(g, \ldots, g),$$

(9.1)

where, as before, $\phi_0 \equiv 1$. Using (7.11) we see that

$$E_{\mathcal{L}_\alpha} (|\phi_{\exp}(g)|) \leq E_{\mathcal{L}_\alpha} (\phi_{\exp}(|g|)) = \sum_{n=0}^{\infty} \frac{(\alpha \mu(|g|))^n}{n!} = e^{\alpha \mu(|g|)},$$

(9.2)

so that $\phi_{\exp}(g)$ is well defined for all $g \in \mathcal{D}_1(\mu)$ and a similar calculation without absolute values shows that

$$E_{\mathcal{L}_\alpha} (\phi_{\exp}(g)) = \sum_{n=0}^{\infty} \frac{(\alpha \mu(g))^n}{n!} = e^{\alpha \mu(g)}.$$  

(9.3)

Let $g \in \mathcal{D}_1(\mu)$. By (7.11)

$$E_{\mathcal{L}} (\phi_1(|g|)) = \alpha \int |g(\omega)| d\mu(\omega) < \infty.$$  

(9.4)

Therefore,

$$\phi_1(|g|) = \sum_{\omega \in \mathcal{L}_\alpha} |g(\omega)|$$  

(9.5)

converges almost surely. Consequently

$$\prod_{\omega \in \mathcal{L}_\alpha} \left(1 + g(\omega)\right)$$  

(9.6)

also converges almost surely.
Lemma 9.1 For \( f, g \in \mathcal{D}_1(\mu) \)

\[
\phi_{\exp}(g) = \prod_{\omega \in \mathcal{L}_\alpha} (1 + g(\omega)) \tag{9.7}
\]

and

\[
\phi_{\exp}(f)\phi_{\exp}(g) = \phi_{\exp}(f + g + fg). \tag{9.8}
\]

Proof By distinguishing between ordered and unordered \( n \)-tuples,

\[
\phi_{\exp}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g, \ldots, g) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1, \ldots, \omega_n) \in S_n(\mathcal{L}_\alpha)} \prod_{j=1}^{n} g(\omega_j) \tag{9.9}
\]

which is exactly what one obtains by expanding the product on the right hand side of (9.7).

The relationship in (9.8) follows from (9.7) and the fact that \((1+f)(1+g) = (1 + f + g + fg)\). \(\square\)

For random variables \( g \in \mathcal{D}_1(\mu) \) we define the exponential Poisson chaos

\[
\mathcal{E}(g) := I_{\exp}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g, \ldots, g). \tag{9.10}
\]

Lemma 9.2 Let \( g \in \mathcal{D}_1(\mu) \), then

\[
\mathcal{E}(g) = \phi_{\exp}(g) e^{-\alpha\mu(g)} = \prod_{\omega \in \mathcal{L}_\alpha} (1 + g(\omega)) e^{-\alpha\mu(g)}. \tag{9.11}
\]

When \( g > -1 \),

\[
\mathcal{E}(g) = e^{\sum_{\omega \in \mathcal{L}_\alpha} \log(1+g(\omega)) - \alpha\mu(g)}. \tag{9.12}
\]

In addition

\[
\mathcal{E}(f) \mathcal{E}(g) = \mathcal{E}(f + g + fg) e^{\alpha\mu(fg)} \tag{9.13}
\]

and

\[
E_{\mathcal{L}_\alpha}(\mathcal{E}(f)) = 1. \tag{9.14}
\]
Proof  By the definition (7.23),

\[ I_n(g, \ldots, g) = \sum_{k=0}^{n} \binom{n}{k} \phi_k(g, \ldots, g)(-\alpha \mu(g))^{n-k}. \]  \hspace{1cm} (9.15)

Therefore

\[ \mathcal{E}(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \phi_k(g, \ldots, g)(-\alpha \mu(g))^{n-k} \]  \hspace{1cm} (9.16)

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\phi_k(g, \ldots, g)}{k!} \frac{(-\alpha \mu(g))^{n-k}}{(n-k)!} = \phi_{\exp}(g)e^{-\alpha \mu(g)}. \]

The second equality in (9.11) follows from this and (9.7).

Note that when \( g > -1 \), this can be written as

\[ \phi_{\exp}(g) = e^{\sum_{\omega \in \mathcal{L}_{\alpha}} \log(1+g)(\omega)}. \]  \hspace{1cm} (9.17)

Using this and (9.16) we get (9.12). The product formula in (9.13) follows from (9.8) and (9.14) follows from (9.3).

Remark 9.1 We give a proof of Lemma 7.4 using exponential Poisson chaoses. As in the proof of Lemma 9.1, for \( h_1, \ldots, h_N \in \mathcal{D}_1(\mu) \),

\[ \prod_{m=1}^{N} \phi_{\exp}(h_m) = \phi_{\exp} \left( \prod_{m=1}^{N} (1 + h_m) - 1 \right) = \phi_{\exp} \left( \sum_{B \subseteq [1,N], |B| \geq 1} \prod_{m \in B} h_m \right). \]  \hspace{1cm} (9.18)

Therefore, by (9.3)

\[ E_{\mathcal{L}_{\alpha}} \left( \prod_{m=1}^{N} \phi_{\exp}(h_m) \right) = e^{\sum_{B \subseteq [1,N], |B| \geq 1} \alpha \mu(\prod_{m \in B} h_m)}. \]  \hspace{1cm} (9.19)

Using (9.16)

\[ E_{\mathcal{L}_{\alpha}} \left( \prod_{m=1}^{N} \mathcal{E}(h_m) \right) = e^{\sum_{B \subseteq [1,N], |B| \geq 2} \alpha \mu(h_B)} = \prod_{B \subseteq [1,N], |B| \geq 2} e^{\alpha \mu(\prod_{m \in B} h_m)}. \]  \hspace{1cm} (9.20)
Set \( h_m = \sum_{j=1}^{n_m} z_{s_m+j} g_{s_m+j} \) where, as in Lemma 7.4, \( s_m := \sum_{j=1}^{m-1} n_j \). We consider the coefficients of \( \prod_{m=1}^{N} \prod_{j=1}^{n_m} z_{s_m+j} \) in the expansion of the first and third term in (9.20). Equating them we obtain

\[
E_{\mathcal{L}_\alpha} \left( \prod_{m=1}^{N} I_{n_m} (g_{s_m+1}, \ldots, g_{s_m+n_m}) \right) = \sum_{\cup_{i} A_i = [1,s_{N+1}], |A_i| \geq 2} \prod_{i} \alpha\mu \left( \prod_{j \in A_i} g_j \right). \tag{9.21}
\]

This is fairly easy to see for the third term and the right-hand side of (9.20). The coefficient of \( \prod_{m=1}^{N} \prod_{j=1}^{n_m} z_{s_m+j} \) is a sum of products of \( \prod_{l} \alpha\mu \left( \prod_{j \in A_l} g_j \right) \) where \( \{B_l\} \) is a partition of \([1,s_{N+1}]\). The fact that each \( h_k \) appears at most once in each term \( \mu \left( \prod_{m \in B_l} h_m \right) \) gives the condition denoted by \( \perp \{s_m+1, \ldots, s_m+n_m\}, m = 1, \ldots, N \).

Consider the expansion of first term in (9.20). It follows from the fact that the functions \( I_{n_m} \) are multilinear, that the coefficients of \( \prod_{m=1}^{N} \prod_{j=1}^{n_m} z_{s_m+j} \) are of the form

\[
\prod_{m=1}^{N} I_{n_m} (g_{s_m+1}, \ldots, g_{s_m+n_m}) \frac{1}{n_m!}. \tag{9.22}
\]

We add up all the terms of this form taking into account the fact that the variables \( g_{s_m+1}, \ldots, g_{s_m+n_m} \) can be arranged in \( n_m! \) and that (9.22) remains the same in all these arrangements, we get the first term in (9.21).

Note that Lemma 7.4 is Lemma 7.3 for \( p = s_{N+1} \). This is all we need in Section 8. The argument just given can be extended to give a proof of Lemma 7.3 for \( p > s_{N+1} \).

### 9.2 Extensions to martingales

Until this point we have considered the variable \( \alpha \) in the Poisson intensity measure \( \alpha\mu \) to be fixed. We show here that many of the processes considered in this monograph can be extended in such a way that they become martingales in \( \alpha \). Consider the Poisson process \( \mathcal{L} \) with values in \( R_1^+ \times \Omega_\Delta \) and intensity measure \( \lambda \times \mu \), where \( \lambda \) is Lebesgue measure. Clearly

\[
\mathcal{L} \cap ([0,\alpha] \times \Omega_\Delta) \quad \text{and} \quad \mathcal{L} \cap ((\alpha, \alpha+\alpha') \times \Omega_\Delta)
\]

are independent Poisson processes with intensity measures \( \alpha\mu \) and \( \alpha'\mu \) respectively.
We define  
\[ \mathcal{L}_\alpha = \mathcal{L} \cap ([0, \alpha] \times \Omega_\Delta), \quad (9.23) \]
and  
\[ \tilde{\mathcal{L}}_{\alpha'} = \mathcal{L} \cap ((\alpha, \alpha + \alpha'] \times \Omega_\Delta). \quad (9.24) \]
Clearly \( \tilde{\mathcal{L}}_{\alpha'} \) law \( \alpha \) and \( \mathcal{L}_\alpha \) and \( \tilde{\mathcal{L}}_{\alpha'} \) are independent. We consider the definition in (7.1) to depend on \( \alpha \) and write  
\[ \phi_n^{(\alpha)}(g_1, \ldots, g_n) = \sum_{(\omega_1, \ldots, \omega_n) \in S_n(\mathcal{L}_\alpha)} \prod_{j=1}^{n} g_j(\omega_j). \quad (9.25) \]
and define  
\[ \tilde{\phi}_n^{(\alpha')}(g_1, \ldots, g_n) = \sum_{(\omega_1, \ldots, \omega_n) \in S_n(\tilde{\mathcal{L}}_{\alpha'})} \prod_{j=1}^{n} g_j(\omega_j). \quad (9.26) \]
Since \( \mu \) is non-atomic, \( \mathcal{L}_\alpha \cap \tilde{\mathcal{L}}_{\alpha'} = \emptyset \). Therefore  
\[ \phi_n^{(\alpha+\alpha')}(g_1, \ldots, g_n) = \sum_{A \subseteq [1, n]} \phi_n^{(\alpha)}(g_j, j \in A) \tilde{\phi}_n^{(\alpha')}(g_j, j \in A^c). \quad (9.27) \]
We define \( I_n^{(\alpha)} \) and \( \tilde{I}_n^{(\alpha')} \) with respect to \( \phi_n^{(\alpha)} \) and \( \tilde{\phi}_n^{(\alpha')} \), \( j = 1, \ldots, n \) as in (7.23).

**Lemma 9.3** Let \( g_j \in D_1(\mu), \ j = 1, \ldots, n \). Then  
\[ I_n^{(\alpha+\alpha'})(g_1, \ldots, g_n) = \sum_{A \subseteq [1, n]} I_n^{(\alpha)}(g_j, j \in A) \tilde{I}_n^{(\alpha')}(g_j, j \in A^c). \quad (9.28) \]

For example  
\[ I_2^{(\alpha+\alpha'})(g_1, g_2) = I_2^{(\alpha)}(g_1, g_2) + I_1^{(\alpha)}(g_1) \tilde{I}_1^{(\alpha'})(g_2) + I_1^{(\alpha)}(g_2) \tilde{I}_1^{(\alpha'})(g_1) + \tilde{I}_2^{(\alpha'})(g_1, g_2). \quad (9.29) \]

**Proof** We define \( \mathcal{E}_n^{(\alpha)} \) with respect to \( I_n^{(\alpha)} \) \( n = 1, \ldots \) as in (9.10). By (9.11)  
\[ \mathcal{E}_n^{(\alpha+\alpha')}(f) = \prod_{\omega \in \mathcal{L}_{\alpha+\alpha'}} (1 + f(\omega)) e^{-(\alpha+\alpha')\mu(f)} \quad (9.30) \]
\[ = \prod_{\omega \in \mathcal{L}_\alpha} (1 + f(\omega)) \prod_{\omega \in \mathcal{L}_{\alpha'}} (1 + f(\omega)) e^{-(\alpha+\alpha')\mu(f)} \]
\[ = \mathcal{E}_n^{(\alpha)}(f) \mathcal{E}_n^{(\alpha')}(f). \]
We get (9.28) by writing \( f = \sum_{j=1}^{n} z_j g_j \) and matching coefficients of \( \prod_{j=1}^{n} z_j \); (see Remark 9.1).

We also give a direct proof of Lemma 9.3 that only uses the material developed in Section 7. Using (9.27) and the definition of \( I_n \) we have

\[
I_n^{(\alpha+\alpha')}(g_1, \ldots, g_n) = \sum_{D \subseteq [1, n]} \sum_{\substack{A \subseteq D \cap (g_j, j \in D)}} \phi_{\mid A\mid}^{(\alpha)}(g_j, j \in A) \phi_{\mid D-A\mid}^{(\alpha')}\left(\prod_{j \in D-A} (\alpha + \alpha') \mu(g_j)\right) 
\]

Writing

\[
(-1)^{|D^c|} \prod_{j \in D^c} (\alpha + \alpha') \mu(g_j) 
\]

we have

\[
\sum_{D \subseteq [1, n]} \sum_{\substack{A \subseteq D \cap (g_j, j \in D)}} \phi_{\mid A\mid}^{(\alpha)}(g_j, j \in A) \phi_{\mid D-A\mid}^{(\alpha')}\left(\prod_{j \in D-A} (\alpha + \alpha') \mu(g_j)\right) 
\]

Consider \( T \subseteq [1, n] \) and \( A \subseteq T \) and \( B \subseteq T^c \). Let \( D = A \cup B \) and \( R = T - A \). Therefore \( T^c - B = D^c - R \) and obviously, \( B = D - A \). Using this we can
rewrite the terms on the right hand side of (9.33) following the double sum as

\[
\left( \phi_{\alpha|A|} (g_j, j \in A) (-1)^{|T-A|} \prod_{j \in R=T-A} \alpha \mu(g_j) \right) \left( \bar{\phi}_{\alpha' -1 |B|} (g_j, j \in B) (-1)^{|T^c-B|} \prod_{j \in T^c-B} \alpha' \mu(g_j) \right). 
\] (9.34)

Consequently (9.33) is equal to

\[
\sum_{T \subseteq [1,n]} I_{|T|}^{(\alpha)}(g_j, j \in T) \bar{I}_{|T^c|}^{(\alpha')} (g_j, j \in T^c). 
\] (9.35)

These equalities prove the lemma.

Since \( I_{|A|}^{(\alpha)} \) and \( \bar{I}_{|A^c|}^{(\alpha')} \) are independent and have zero mean, the next theorem follows immediately from Lemma 9.3:

**Theorem 9.1** For \( g_i \in D_1(\mu), i = 1, \ldots, n \) the stochastic process \( I_n^{(\alpha)}(g_1, \ldots, g_n) \) is an \((E_L, F_{\alpha})\) martingale.

The same analysis applies to many of the processes we have studied. Using the same notation as above and Corollary 9.1, Theorems 8.1 and 8.2 and Corollary 8.1 we get:

**Theorem 9.2** For \( \nu \in B_2(R^d), \bar{\psi}_n^{(\alpha)}(\nu) \) and \( \psi_n^{(\alpha)}(\nu) \) are \((E_L, F_{\alpha})\) martingales and for \( \nu \in B_2(R^d), l = \sum_{i=1}^n l_i, I_{l_1,...,l_n}^{(\alpha)}(\nu) \) is an \((E_L, F_{\alpha})\) martingale.

### 10 Convolutions of regularly varying functions

Let \( f \) and \( \bar{f} \) be functions on \( R^d_+ \). We write \( f \approx \bar{f} \) if there exists a \( C < \infty \), and \( 0 < C_1, C_2 < \infty \), such that

\[
C_1 f(x) \leq \bar{f}(x) \leq C_2 f(x), \quad \forall x \geq C. 
\] (10.1)

We also express this as \( f(|x|) \approx \bar{f}(|x|) \), with \( x \in R^d \). We say that \( f \) is approximately regularly varying at infinity with index \( \alpha \) if \( f \approx \bar{f} \) for some
function \( f \) which is regularly varying at infinity with index \( \alpha \). We say that a function \( f \geq 0 \) on \( \mathbb{R}^d_+ \) is controllable if
\[
\int_K (f(|\xi|))^{-1} \, d\xi < \infty, \tag{10.2}
\]
for all compact sets \( K \subset \mathbb{R}^d \) and
\[
\int_{\mathbb{R}^d} (f(|\xi|))^{-1} \, d\xi = \infty. \tag{10.3}
\]

**Lemma 10.1** Let \( h \) and \( g \) be controllable functions on \( \mathbb{R}^d_+ \) that are approximately regularly varying at infinity with indices \( \alpha \) and \( \beta \) respectively, where \( \alpha + \beta > d \) and \( \max(\alpha, \beta) \leq d \), \( d = 1, 2 \). Then
\[
(h)^{-1} * (g)^{-1}(|\xi|) =: \int_{\mathbb{R}^d} (h(|\xi - \eta|))^{-1} (g(|\eta|))^{-1} \, d\eta \tag{10.4}
\]
\[
\approx (h(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} (g(|\eta|))^{-1} \, d\eta + (g(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1} \, d\eta,
\]
Furthermore, \( 1/((h)^{-1} * (g)^{-1}) \), is a controllable function which is approximately regularly varying at infinity with index \( \alpha + \beta - d \).

It is useful to recall some facts about the integrals of regularly varying functions. If \( f \) is a regularly varying function at infinity with index \( 0 < \alpha < d \) and \( (f)^{-1} \) is locally integrable, then
\[
\int_{|\eta| \leq |\xi|} (f(|\eta|))^{-1} \, d\eta \approx |\xi|^d (f(|\xi|))^{-1} \quad \text{as} \quad \xi \to \infty. \tag{10.5}
\]
If \( f \) is a regularly varying function at infinity with index \( d \) and \( (f)^{-1} \) is locally integrable, then
\[
\int_{|\eta| \leq |\xi|} (f(|\eta|))^{-1} \, d\eta \quad \text{is slowly varying at infinity} \tag{10.6}
\]
and
\[
|\xi|^d (f(|\xi|))^{-1} = o \left( \int_{|\eta| \leq |\xi|} (f(|\eta|))^{-1} \, d\eta \right) \quad \text{as} \quad \xi \to \infty. \tag{10.7}
\]
If \( f \) is a regularly varying function at infinity with index \( \alpha > d \), then
\[
\int_{|\eta| \geq |\xi|} (f(|\eta|))^{-1} \, d\eta \approx |\xi|^d (f(|\xi|))^{-1} \quad \text{as} \quad \xi \to \infty. \tag{10.8}
\]
It is well known, (see e.g. [15, Lemma 7.2.4]), that when $\alpha > 0$, $(h(|\xi|))^{-1}$ is asymptotic to a decreasing function at infinity and similarly for $(g(|\xi|))^{-1}$. Therefore, since we do not specify the constant in the asymptotic bounds that we give, we assume that $(h(|\xi|))^{-1}$ and $(g(|\xi|))^{-1}$ are decreasing for all $|\xi| \geq |\xi_0|$ for some $|\xi_0|$ sufficiently large.

**Proof of Lemma 10.1** We first assume that $h$ and $g$ are themselves regularly varying at infinity.

Let $|\xi| \geq 2|\xi_0|$. To obtain (10.4) we take

$$\int_{R^d} (h(|\xi - \eta|))^{-1}(g(|\eta|))^{-1} d\eta$$

(10.9)

$$= \left( \int_{|\eta| \leq |\xi|/2} + \int_{|\xi|/2 \leq |\eta| \leq (3/2)|\xi|} + \int_{|\eta| \geq 3/2|\xi|} \right) \cdots = I + II + III.$$ Using the fact that $|\xi| \geq 2|\xi_0|$, we see that for $|\eta| \leq |\xi|/2$,

$$(h(3|\xi|/2))^{-1} \leq (h(|\xi - \eta|))^{-1} \leq (h(|\xi|/2))^{-1}.$$ (10.10)

Therefore, since $h^{-1}$ is regularly varying at infinity,

$$(h(|\xi - \eta|))^{-1} \approx (h(|\eta|))^{-1}.$$ (10.11)

Consequently for $|\xi| \geq 2|\xi_0|$

$$I \approx (h(|\xi|))^{-1} \int_{|\eta| \leq |\xi|/2} (g(|\eta|))^{-1} d\eta \approx (h(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} (g(|\eta|))^{-1} d\eta$$

(10.12)

since the integral is also is regularly varying at infinity.

Similarly

$$II \approx ((g(|\xi|))^{-1} \int_{|\xi|/2 \leq |\eta| \leq 3|\xi|/2} (h(|\xi - \eta|))^{-1} d\eta$$

(10.13)

and

$$\int_{0 \leq |\eta| \leq |\xi|/2} (h(|\eta|))^{-1} d\eta \leq \int_{|\xi|/2 \leq |\eta| \leq 3|\xi|/2} (h(|\xi - \eta|))^{-1} d\eta$$

(10.14)

$$\leq \int_{0 \leq |\eta| \leq 3|\xi|} (h(|\eta|))^{-1} d\eta.$$ (10.15)

Consequently

$$II \approx ((g(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1} d\eta.$$ (10.16)
Using the fact that $|\xi| \geq 2|\xi_0|$, we see that when $|\eta| \geq 3|\xi|/2$, $(h(|\xi - \eta|))^{-1} \approx (h(|\eta|))^{-1}$. Therefore, by (10.8),

$$III \approx \int_{|\eta| \geq 3|\xi|/2} (h(|\eta|))^{-1}(g(|\eta|))^{-1}d\eta \approx |\xi|^d(h(|\xi|))^{-1}(g(|\xi|))^{-1},$$

(10.17)

as $\xi \to \infty$ because $\alpha + \beta > d$.

The approximate equivalence in (10.4) follows from (10.12), (10.16), and (10.17).

We next show that (10.2) holds for $1/(h)^{-1}$ and $(g)^{-1}$, that is

$$\int_K \int_{R^d} (h(|\xi - \eta|))^{-1}(g(|\eta|))^{-1} d\eta d\xi < \infty \text{ for all compact sets } K \subset R^d.$$ 

(10.18)

To get (10.18) we first note that

$$\int_{|\xi| \leq N} \int_{|\eta| \leq M} (h(|\xi - \eta|))^{-1}(g(|\eta|))^{-1} d\eta d\xi \leq \int_{|\xi| \leq N + M} (h(|\xi|))^{-1} d\xi \int_{|\eta| \leq M} (g(|\eta|))^{-1} d\eta < \infty,$$

(10.19)

by (10.2). In addition by (10.17) if $M \geq 3N/2$, for $N$ sufficiently large

$$\int_{|\eta| \geq M} |\eta|^{-d}h(|\eta|)^{-1}(g(|\eta|))^{-1} d|\eta| < \infty.$$ 

(10.20)

That (10.3) holds for $1/(h)^{-1}$ and $(g)^{-1}$ follows easily from (10.4) and (10.3) applied to either $(h)^{-1}$ or $(g)^{-1}$. This concludes the proof of our Lemma when $h$ and $g$ are themselves regularly varying at infinity.

Suppose that $h$ and $g$ are controllable and approximately regularly varying functions at infinity with respect to functions $\hat{h}$ and $\hat{g}$ that regularly varying at infinity, and let $C$ in (10.1) be large enough so that (10.1) holds for both pairs $h, \hat{h}$ and $g, \hat{g}$. It is clear that for $|x| \geq C$ we can pass freely between $h$ and $\hat{h}$ and $g$ and $\hat{g}$.

In (10.12), (10.14) and (10.19) the integrals are over compact sets. These integrals remain bounded by the hypothesis (10.2).

Let $h$ be a controllable function that is approximately regularly varying at infinity with index $\alpha$. We define

$$(H(|\xi|))^{-1} = \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1}d\eta.$$ 

(10.21)
When $\alpha = d$, $(H(|\xi|))^{-1}$ is a controllable function that is approximately slowly varying, and by hypothesis we have that $\lim_{|\xi| \to \infty} H^{-1}(|\xi|) = \infty$.

**Remark 10.1** Note that by (10.5) when $d/2 < \alpha < d$, $(H(|\xi|))^{-1} \approx |\xi|^d(h(|\xi|))^{-1}$ as $|\xi| \to \infty$. Therefore, if $h$ and $g$ are both controllable functions that are approximately regularly varying at infinity with indices less than $d$, the right-hand side of (10.4) is asymptotic to $C|\xi|^d(h(|\xi|))^{-1}((g(|\xi|))^{-1}$ as $|\xi| \to \infty$. If $h$ is a controllable function that is approximately regularly varying at infinity with index $d$, and $g$ is a controllable function that is approximately regularly varying at infinity with index less than $d$, then the right hand side of (10.4) is asymptotic to $C'((g(|\xi|))^{-1}(H(|\xi|))^{-1}$ as $|\xi| \to \infty$, and similarly with $h$ and $g$ interchanged. If $h$ and $g$ are both controllable functions that are approximately regularly varying at infinity with index $d$ the situation is less clear. However, whenever we are in this situation in what follows, one of the terms on the right-hand side of (10.4) will be larger than the other, as $|\xi| \to \infty$, so, obviously, the right-hand side of (10.4) is asymptotic to the larger term.

For all $n \geq 2$ let

$$ \widetilde{\theta}_n(|\xi|) := \underbrace{(h)^{-1} \ast (h)^{-1} \ast \cdots \ast (h)^{-1}}_{\text{n-times}}(|\xi|). $$  \hspace{1cm} (10.22)

**Lemma 10.2** Let $h$ be a controllable function which is approximately regularly varying at infinity with index $d(1 - \frac{1}{k}) < \alpha \leq d$, $k \geq 2$. Then

$$ \widetilde{\theta}_k(|\xi|) \approx (h(|\xi|))^{-1}(H(|\xi|))^{-(k-1)}, $$  \hspace{1cm} (10.23)

and $1/\widetilde{\theta}_k$, is a controllable function which is approximately regularly varying at infinity with index $k\alpha - (k-1)d$.

**Proof** For $k = 2$ this lemma is just Lemma 10.1 with $g^{-1} = h^{-1}$. Assume that this lemma holds $j - 1$, where $j \leq k$. Consider

$$ \widetilde{\theta}_j(|\xi|) = \int_{\mathbb{R}^d} (h(|\xi - \eta|))^{-1} \widetilde{\theta}_{j-1}(|\eta|) \, d\eta. $$  \hspace{1cm} (10.24)

To check that the hypotheses of Lemma 10.1 are satisfied with $g = \widetilde{\theta}_j$, we consider the indices of regular variation of $h$ and $1/\widetilde{\theta}_{j-1}$. By (10.23), which we assume holds for $j - 1$ and Remark 10.1, the index of regular variation of
\( hH^{j-2} \) is \( \tilde{\beta} := \alpha - (j - 2)(d - \alpha) \). Using (1.3) it is easy to see that \( \alpha \) and \( \tilde{\beta} \) satisfy the hypotheses of Lemma 10.1. Therefore, by Lemma 10.1

\[
\tilde{\vartheta}_j(|\xi|) \approx (h(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} \tilde{\vartheta}_{j-1}(|\eta|) d\eta + (h(|\eta|))^{-1} \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1} \tilde{\vartheta}_{j-1}(|\eta|) d\eta
\]

\[
\approx (h(|\xi|))^{-1} \int_{|\eta| \leq |\xi|} \tilde{\vartheta}_{j-1}(|\eta|) d\eta + (h(|\xi|))^{-1} (H(|\xi|))^{-1} (j-1). \tag{10.25}
\]

Furthermore

\[
\int_{|\eta| \leq |\xi|} \tilde{\vartheta}_{j-1}(|\eta|) d\eta \approx \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1} (H^{-1}(|\eta|))^{j-2} d\eta \tag{10.26}
\]

\[
\leq (H(|\xi|))^{-j} \int_{|\eta| \leq |\xi|} (h(|\eta|))^{-1} d\eta,
\]

since \( H^{-1}(|\xi|) \) is increasing. Using this in (10.25) gives (10.23). \( \square \)

**Lemma 10.3** Let \( h \) be a controllable function which is approximately regularly varying at infinity with index \( d(1 - \frac{1}{2n}) < \alpha \leq d \), and assume that

\[
\int \tilde{\vartheta}_{2n}(|\lambda|) |\hat{\nu}(|\lambda|)|^2 d\lambda < \infty, \tag{10.27}
\]

for some finite measure \( \nu \). Then for any \( 1 \leq k \leq 2n \)

\[
\int \tilde{\vartheta}_k(|\lambda|) |\hat{\nu}(|\lambda|)|^2 d\lambda < \infty, \tag{10.28}
\]

and

\[
\int |1 - e^{iz \cdot \lambda}|^2 \tilde{\vartheta}_k(|\lambda|) |\hat{\nu}(|\lambda|)|^2 d\lambda = o \left( (H(1/|z|))^{2n-k} \right), \tag{10.29}
\]

as \( |z| \to 0 \).

**Proof** By Lemma 10.2 (10.2) holds with \( f^{-1} = \tilde{\vartheta}_k \). Using this and the fact that \( |\hat{\nu}(\lambda)| \leq |\hat{\nu}(0)| < \infty \), we see that for any \( N \),

\[
\int_{|\lambda| \leq N} \tilde{\vartheta}_k(|\lambda|) |\hat{\nu}(|\lambda|)|^2 d\lambda < \infty. \tag{10.30}
\]

By (10.23), for \( N \) sufficiently large, \( \tilde{\vartheta}_k(|\lambda|) \leq \tilde{\vartheta}_{2n}(|\lambda|) \), for all \( |\lambda| \geq N \). Using (10.30) and (10.27) we get (10.28).
Now since (10.29) is obvious when $k = 2^n$ we assume that $1 \leq k \leq 2^n - 1$. For any positive number $M$ and $1/|z| > M$ we write

$$
\int |1 - e^{iz \cdot \lambda}|^2 \bar{\theta}_k(|\lambda|) |\hat{\nu}(\lambda)|^2 d\lambda
= \int_{|\lambda| \leq M} + \int_{M \leq |\lambda| \leq 1/|z|} + \int_{|\lambda| \geq 1/|z|}
= I + II + III.
$$

It follows from (10.28) that

$$
I \leq C_M |z|^2
$$

for some constant depending on $M$. By (10.23)

$$
\frac{\bar{\theta}_k(|\lambda|)}{\bar{\theta}_{2n}(|\lambda|)} \approx (H(|\lambda|))^{2n-k}.
$$

Since this is decreasing for all $|\lambda|$ sufficiently large, we have that for all $|z|$ sufficiently small,

$$
III \leq 4 \int_{|\lambda| \geq 1/|z|} \bar{\theta}_k(|\lambda|) |\hat{\nu}(\lambda)|^2 d\lambda
\leq \frac{4 \hat{\theta}_k(1/|z|)}{\hat{\theta}_{2n}(1/|z|)} \int_{|\lambda| \geq 1/|z|} \bar{\theta}_{2n}(|\lambda|) |\hat{\nu}(\lambda)|^2 d\lambda = o((H(1/|z|))^{2n-k}).
$$

For $0 < \delta < 2$ we write

$$
|1 - e^{iz \cdot \lambda}|^2 \leq C |1 - e^{iz \cdot \lambda}|^2 - |z|^{\delta} |\lambda|^{\delta}.
$$

We choose $\delta$ so that

$$
\frac{|\lambda|^{\delta} \bar{\theta}_k(|\lambda|)}{\bar{\theta}_{2n}(|\lambda|)}
$$

is regularly varying with a strictly positive index, i.e. $\delta > (d - \alpha)(2n - k)$. That this is possible follows from the bounds on $\alpha$ which imply that

$$
(d - \alpha)(2n - k) < d(1 - \frac{k}{2n}).
$$
We choose $M$ so that $\bar{\theta}_{2n}(\lambda) > 0$ for $\lambda \geq M$. Using (10.36) we see that for all $|z|$ sufficiently small

$$II \leq \int_{M \leq |\lambda| \leq 1/|z|} |1 - e^{iz \lambda^2} - \frac{z^2 |\lambda|^5 \bar{\theta}_k(|\lambda|)}{\bar{\theta}_{2n}(|\lambda|)}| \hat{\nu}(\lambda)|^2 d\lambda$$

(10.38)

$$\leq \frac{|z|^5 (1/|z|) \delta \bar{\theta}_k(1/|z|)}{\bar{\theta}_{2n}(1/|z|)} \int |1 - e^{iz \lambda^2} - \delta \bar{\theta}_{2n}(|\lambda|)| \hat{\nu}(\lambda)|^2 d\lambda$$

$$= o \left( H(1/|z|)^{2n-k} \right),$$

by the dominated convergence theorem.

It follows from (10.37) that

$$|z|^2 = o \left( H(1/|z|)^{2n-k} \right).$$

(10.39)

Combining (10.32), (10.34), (10.38) and (10.39) the proof is complete. □

The next lemma is a variation of Lemma 10.3.

**Lemma 10.4** Let $h$ be as in Lemma 10.3 and let

$$\vartheta_k(z, \lambda) := \int |1 - e^{iz \lambda^1}|^2 h^{-1}(|\lambda_1|) \bar{\theta}_{k-1}(|\lambda - \lambda_1|) d\lambda_1.$$  

(10.40)

If (10.27) holds, then for any $2 \leq k \leq 2n$

$$\int \vartheta_k(z, \lambda) |\hat{\nu}(\lambda)|^2 d\lambda = o \left( H(1/|z|)^{2n-k} \right).$$

(10.41)

**Proof** Consider

$$\int \int |\lambda_1| \leq M h^{-1}(|\lambda_1|) \bar{\theta}_{k-1}(|\lambda - \lambda_1|) d\lambda_1 |\hat{\nu}(\lambda)|^2 d\lambda$$

(10.42)

$$= \int |\lambda_1| \leq M h^{-1}(|\lambda_1|) \left( \int \bar{\theta}_{k-1}(|\lambda - \lambda_1|) |\hat{\nu}(\lambda)|^2 d\lambda \right) d\lambda_1$$

For $|\lambda_1| \leq M$

$$\int |\lambda| \leq 2M \bar{\theta}_{k-1}(|\lambda - \lambda_1|) |\hat{\nu}(\lambda)|^2 d\lambda \leq |\hat{\nu}(0)|^2 \int |\lambda| \leq 3M \bar{\theta}_{k-1}(|\lambda|) d\lambda = C_M < \infty$$

(10.43)
because $1/\tilde{\theta}_{k-1}$ is a controllable function. In addition by (10.28),

$$
\int_{|\lambda| \geq 2M} \tilde{\theta}_{k-1}(|\lambda - \lambda_1|) |\hat{\nu}(\lambda)|^2 d\lambda \leq \int \tilde{\theta}_{k-1}(|\lambda|)|\hat{\nu}(\lambda)|^2 d\lambda = C'_M < \infty. \quad (10.44)
$$

(Here we use the fact that $|\lambda - \lambda_1| \geq |\lambda|/2$ and choose $M$ sufficiently large so that $\tilde{\theta}_{k-1}(|\lambda|)$ is decreasing and $\tilde{\theta}_{k-1}(|\lambda|/2) \leq c \tilde{\theta}_{k-1}(|\lambda|)$.) Therefore, by (10.2)

$$
\int \int_{|\lambda_1| \leq M} |1 - e^{iz\cdot \lambda_1}|^2 (h(|\lambda_1|))^{-1} \tilde{\theta}_{k-1}(|\lambda - \lambda_1|) d\lambda_1 |\hat{\nu}(\lambda)|^2 d\lambda \leq C''_M |z|^2.
$$

(10.45)

For $z < 1/M$ we write

$$
\int_{|\lambda_1| \geq M} |1 - e^{iz\cdot \lambda_1}|^2 (h(|\lambda_1|))^{-1} \tilde{\theta}_{k-1}(\lambda - \lambda_1) d\lambda_1 = \int_{M \leq |\lambda_1| \leq 1/|z|} + \int_{|\lambda_1| \geq 1/|z|} := I(z, \lambda) + II(z, \lambda).
$$

Choose $(d - \alpha)(2n - k) < \delta < 2$ as in (10.36), and note that, as $|\lambda_1| \to \infty$,

$$
\frac{|\lambda_1|^\delta (h(|\lambda_1|))^{-1}}{\tilde{\theta}_{2n-k+1}(\lambda_1)} \sim |\lambda_1|^\delta (H(|\lambda_1|))^{2n-k} =: F(|\lambda_1|).
$$

(10.47)

We have

$$
I(z, \lambda) \leq C|z|^\delta \int_{M \leq |\lambda_1| \leq 1/|z|} |1 - e^{iz\cdot \lambda_1}|^{2-\delta} \hat{F}(|\lambda_1|) \tilde{\theta}_{2n-k+1}(\lambda_1) \tilde{\theta}_{k-1}(\lambda - \lambda_1) d\lambda_1.
$$

$F(|\lambda_1|)$ is a regularly varying function at infinity with a strictly positive index, and we choose $M$ above so that we can take $F(|\lambda_1|)$ to be increasing for $|\lambda_1| \geq M$. Consequently

$$
I(z, \lambda) \leq (H(|z|/|z|))^{2n-k} \int |1 - e^{iz\cdot \lambda_1}|^{2-\delta} \tilde{\theta}_{2n-k+1}(\lambda_1) \tilde{\theta}_{k-1}(\lambda - \lambda_1) d\lambda_1.
$$

(10.48)

Consider

$$
\int I(z, \lambda) |\hat{\nu}(\lambda)|^2 d\lambda.
$$

(10.49)

Since

$$
\int \tilde{\theta}_{2n-k+1}(\lambda_1) \tilde{\theta}_{k-1}(\lambda - \lambda_1) d\lambda_1 = \tilde{\theta}_{2n}(\lambda)
$$

(10.50)
it follows from (10.27) and dominated convergence theorem that
\[ \int I(z, \lambda) |\hat{\nu}(\lambda)|^2 d\lambda = o \left( |z|^d h(1/|z|) \right) \]
(10.51)
Also
\[ II(z, \lambda) \leq \frac{1}{4} \int_{|\lambda_1| > 1/|z|} (h(|\lambda_1|))^{-1} \tilde{\theta}_{n-k+1}(\lambda - \lambda_1) d\lambda_1. \]
(10.52)
By (10.23)
\[ (h(|\lambda_1|))^{-1} \sim (H(|\lambda_1|))^{2n-k} \tilde{\theta}_{2n-k+1}(\lambda_1). \]
(10.53)
Since \( H(|\lambda_1|) \) is deceasing in \(|\lambda_1|\) we see that
\[ II(z, \lambda) \leq \left( H(1/|z|) \right)^{2n-k} \int_{|\lambda_1| > 1/|z|} \tilde{\theta}_{2n-k+1}(\lambda - \lambda_1) d\lambda_1. \]
(10.54)
The estimates in (10.51) and (10.54) give (10.41).

The next lemma gives inequalities that are used to calculate the rate of growth of chain and cycle functions.

**Lemma 10.5** Let \( h \) be controllable function that is regularly varying at infinity with index \( d/2 < \alpha \leq d \) and let \( f \) be a smooth function of compact support. Then for all \( r > 0 \) sufficiently small
\[ \int (h(|x|))^{-1} |\hat{f}(rx)| dx \leq C(H(1/r))^{-1} \]
(10.55)
and
\[ \int \int (h(|s - y|))^{-1} (h(|y|))^{-1} dy |\hat{f}(rs)|^2 ds \leq C(H(1/r))^{-2}. \]
(10.56)

**Proof** The function \( \hat{f} \) is rapidly decreasing. We choose \( K \) so that for \(|x| > K\), \(|\hat{f}(x)| \leq |x|^{-p}\), for some \( p > d \). We write
\[ \int (h(|x|))^{-1} |\hat{f}(rx)| dx = \int_{|x| \leq K} + \int_{K < |x| \leq K/r} + \int_{|x| > K/r} \]
\[ = I + II + III. \]

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Since $h^{-1}$ is locally integrable and $|\hat{f}(x)| \leq \hat{f}(0) = 1$, $I \leq C$, for some constant $C$ depending on $K$. In addition

$$II \leq \int_{K <|x| \leq K/r} (h(|x|)^{-1}) \, dx \leq C_1 (H(1/r))^{-1}$$  

(10.58)

for all $r \leq r_0$ sufficiently small. Lastly

$$III \leq \frac{1}{r^p} \int_{|x| > K/r} \frac{(h(|x|)^{-1})}{|x|^p} \, dx \leq C_2 \frac{(h(1/r))^{-1}}{r^d} \leq C_3 (H(1/r))^{-1}$$  

(10.59)

by (10.7). Thus we get (10.55).

To obtain (10.56) we first write

$$\int \int (h(|s - y|))^{-1} (h(|y|))^{-1} \, dy |\hat{f}(rs)|^2 \, ds$$

(10.60)

$$= \int_{|u| \leq K} \mathcal{H}(u) \, du + \int_{K <|u| \leq K/r} \mathcal{H}(u) \, du \int_{|u| \geq K/r} \mathcal{H}(u) \, du,$$

where

$$\mathcal{H}(u) = (h(|u| - w|))^{-1} (h(|w|))^{-1} \, dw |\hat{f}(ru)|^2.$$  

(10.61)

By Lemma 10.1

$$\int_{|u| \leq K} \mathcal{H}(u) \, du \leq C.$$  

(10.62)

Also, by (10.23) and the fact that $(H(|u|))^{-1}$ is increasing and regularly varying

$$\int_{K <|u| \leq K/r} \mathcal{H}(u) \, du \leq C \int_{K <|u| \leq K/r} (h(|u|))^{-1} (H(|u|))^{-1} \, du$$

$$\leq C' (H(1/r))^{-1} \int_{K <|u| \leq K/r} (h(|u|))^{-1} \, du$$

$$\leq C' (H(1/r))^{-2}.$$  

(10.63)

Substituting the upper bound for $|\hat{f}(ru)|^2$, and using the fact that $2\alpha > d$, which implies that $(h(|u|))^{-1} (H(|u|))^{-1}$ is decreasing for large $|u|$, we obtain

$$\int_{|u| \geq K/r} \mathcal{H}(u) \, du \leq \frac{1}{r^p} \int_{|u| \geq K/r} (h(|u|))^{-1} (H(|u|))^{-1} \frac{1}{|u|^p} \, du$$

(10.64)

$$\leq C' \frac{(h(1/r))^{-1} (H(1/r))^{-1}}{r^d} \int_{|u| \geq K/r} \frac{1}{|u|^p} \, du$$

$$\leq C' \frac{(h(1/r))^{-1} (H(1/r))^{-1}}{r^d} \leq C' (H(1/r))^{-2}.$$  

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Combining (10.60)–(10.64) we get (10.56). □

**Remark 10.2** In $R^1$ the condition in Lemma 10.1 that $\alpha, \beta \leq 1$ seems unnatural and it is. We use it for our convenience since (10.3) does not hold when $\alpha > 1$. We can handle these cases but it requires rewriting the proofs in this section and it doesn’t seem worthwhile for reasons discussed at the end of Example 3.1.

We now can provide the bounds used in Sections 2 and 6. Consider (1.10) and set

$$
(\tilde{H}(|\xi|))^{-1} := \int_{|\eta| \leq |\xi|} (g_\alpha(|\eta|))^{-1} d\eta. 
$$

(10.65)

Note that when $\alpha < d$

$$
(\tilde{H}(1/r))^{-1} \approx \frac{1}{r^d g_\alpha(1/r)} 
$$

(10.66)

and, for the functions $g_\alpha$ that we consider, (1.5) holds.

Finally, we obtain bounds for the chain and cycle functions.

**Lemma 10.6** When $\hat{u}$ satisfies (1.11)

$$
ch_k(r) = O\left((\tilde{H}(1/r))^{-k}\right) \quad \text{as } r \to 0 
$$

(10.67)

and

$$
ci_k(r) = O\left((\tilde{H}(1/r))^{-k}\right) \quad \text{as } r \to 0. 
$$

(10.68)

**Proof** By Lemma 2.2 (1.11) and (1.12)

$$
\begin{align*}
\text{ci}_k(r) & = \frac{1}{r^{dk}} \int \prod_{j=1}^n u(x_{\pi(j+1)} - x_{\pi(j)}) \prod_{i=1}^n f(x_i/r) \, dx_i \\
& \leq \frac{1}{(2\pi)^{dk}} \left( \int |\hat{f}(r\lambda)|^2 \, \tilde{\tau}_2(\lambda) \, d\lambda \right)^{k/2} = O\left((\tilde{H}(1/r))^{-k}\right),
\end{align*}
$$

where, for the last inequality we use (10.56). This proves (10.68).
The proof of (10.67) follows the proof of Lemma 2.2, however, there are enough differences that it seems necessary to give details. We have

\[ \text{ch}_k(r) = \int u(ry_1, ry_2) \cdots u(ry_k, ry_{k+1}) \prod_{j=1}^{k+1} f(y_j) \, dy_j, \quad (10.70) \]

\[ = \int \left( \prod_{l=1}^{k} e^{i r (y_{l+1} - y_l) \cdot \lambda_l} \hat{u}^{(\lambda_l)} \, d\lambda_l \right) \prod_{j=1}^{k+1} f(y_j) \, dy_j \]

\[ = \int \left( \prod_{l=1}^{k} \hat{u}^{(\lambda_l)} \right) \left( \hat{f}(r \lambda_1) \hat{f}(r (\lambda_2 - \lambda_1)) \cdots \hat{f}(r (\lambda_k - \lambda_{k-1})) \hat{f}(r \lambda_k) \right) \, d\lambda_l. \]

Here we take the Fourier transform of \( u \) considered as a distribution in \( S' \); (see the paragraph containing (2.8)).

To estimate this last integral we note that for functions \( v, u \) and \( w_j \), by repeated use of the Cauchy-Schwarz Inequality,

\[ \int \ldots \int |v(\lambda_1)|^{1/2} \left( \prod_{j=2}^{k} |w_j(\lambda_j, \lambda_{j-1})||u(\lambda_n)| \right)^{1/2} \prod_{j=1}^{k} d\lambda_j \]

\[ \leq \left( \int |v(\lambda)|^2 \, d\lambda \right)^{1/2} \left( \int |u(\lambda)|^2 \, d\lambda \right)^{1/2} \left( \prod_{j=2}^{k} \int \int |w_j(\lambda, \lambda')|^2 \, d\lambda \, d\lambda' \right)^{1/2}. \]

Take

\[ v = \hat{u}^{1/2}(\lambda_1) f(r \lambda_1), \quad u = \hat{u}^{1/2}(\lambda_k) f(r \lambda_k) \quad (10.72) \]

and

\[ w_j = \hat{u}^{1/2}(\lambda_{j-1}) \hat{u}^{1/2}(\lambda_j) f(r (\lambda_j - \lambda_{j-1})), \quad j = 2, \ldots, k. \quad (10.73) \]

Using (10.71)–(10.73) and (1.11) and (1.12) we see that the last integral in (10.70)

\[ \leq \left( \int (\Phi_{\alpha}(|x|))^{-1} \hat{f}(r x) \, dx \right) \]

\[ \left( \int \int (\Phi_{\alpha}(|s - y|))^{-1}(\Phi_{\alpha}(|y|))^{-1} d y |\hat{f}(r s)|^2 \, ds \right)^{(k-1)/2}. \]

Using (10.55) and (10.56) we get (10.68). \( \square \)
References


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