FINITE ENTROPY VS FINITE ENERGY

ELEONORA DI NEZZA, VINCENT GUEDJ, CHINH H. LU

Abstract. Probability measures with either finite Monge-Ampère energy or finite entropy have played a central role in recent developments in Kähler geometry. In this note we make a systematic study of quasi-plurisubharmonic potentials whose Monge-Ampère measures have finite entropy. We show that these potentials belong to the finite energy class $\mathcal{E}^{\frac{n}{n-1}}$, where n denotes the complex dimension, and provide examples showing that this critical exponent is sharp. Our proof relies on refined Moser-Trudinger inequalities for quasi-plurisubharmonic functions.

Introduction

Probability measures with either finite energy [BEGZ10, BBGZ13] or finite entropy [BBEGZ19] have played an important role in recent developments in Kähler geometry (see [BBJ15, BDL20, BDL17, CC17, CC18, Don18, BBEGZ19] and the references therein).

Indeed the search for Kähler-Einstein metrics on Fano manifolds boils down to maximizing the Ding functional whose leading term is a Monge-Ampère energy, while a constant scalar curvature Kähler metric minimizes the Mabuchi functional, whose leading term is an entropy. The purpose of this note is to systematically compare these two notions.

Let (X,ω) be a compact Kähler manifold of complex dimension $n \geq 1$, normalized so that

$$\mathrm{Vol}_\omega(X) := \int_X \omega^n = 1.$$

We consider $\mu = f\omega^n$, $0 \leq f$, a probability measure with finite entropy

$$0 \leq \mathrm{Ent}_{\omega^n}(\mu) := \int_X f \log f \ \omega^n < +\infty.$$

Since μ is absolutely continuous with respect to the volume form ω^n, it is in particular "non-pluripolar" hence it follows from [GZ07, Din09] that there exists a unique full mass potential $\varphi \in \mathcal{E}(X,\omega)$ such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c\varphi)^n = \mu.$$

Here $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2\pi}\left(\partial - \overline{\partial}\right)$ are real operators so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$, and $\mathcal{E}(X,\omega)$ denotes the set of ω-plurisubharmonic functions φ whose non pluripolar Monge-Ampère measure $(\omega+dd^c\varphi)^n$ is a probability measure. We refer the reader to Section 1 for a precise definition. We consider, for $p > 0$,

$$\mathcal{E}^p(X,\omega) := \{\varphi \in \mathcal{E}(X,\omega) \mid E_p(\varphi) < +\infty\},$$

where $E_p(\varphi) := \int_X |\varphi|^p(\omega + dd^c\varphi)^n$.

Date: January 3, 2021.

2010 *Mathematics Subject Classification.* 32W20, 32U05, 32Q15, 35A23.

Key words and phrases. Monge-Ampère energy, entropy, Moser-Trudinger inequality.

The authors are partially supported by the ANR project CRACK.

It has been observed in [BBEGZ19, Theorem 2.17] that

$$\mathrm{Ent}(X,\omega) \subset \mathcal{E}^1(X,\omega),$$

and the injection $\mathrm{Ent}(X,\omega) \hookrightarrow \mathcal{E}^1(X,\omega)$ is compact, where $\mathrm{Ent}(X,\omega)$ is the set of ω-psh functions whose Monge-Ampère measure has finite entropy. However all computable examples suggest that $\mathrm{Ent}(X,\omega)$ is actually contained in a higher energy class $\mathcal{E}^p(X,\omega)$ for some $p > 1$ depending on the dimension. We confirm this experimental observation by showing the following:

Theorem A. *Let $\mu = (\omega + dd^c\varphi)^n = f\omega^n$ be a probability measure with finite entropy $\mathrm{Ent}_{\omega^n}(\mu) = \int_X f \log f \omega^n < +\infty$. Then*

$$\varphi \in \mathcal{E}^{\frac{n}{n-1}}(X,\omega).$$

Moreover the inclusion $\mathrm{Ent}(X,\omega) \hookrightarrow \mathcal{E}^p(X,\omega)$ is compact for any $p < \frac{n}{n-1}$.

This exponent is sharp when $n \geq 2$. If $n = 1$ then φ is continuous, hence it belongs to $\mathcal{E}^p(X,\omega)$ for all $p > 0$.

The case of Riemann surfaces deserves a special treatment: finite entropy potentials turn out to be bounded (and even continuous), but this is no longer the case in higher dimension. The proof of Theorem A relies on a Moser-Trudinger inequality which provides a strong integrability property of finite energy potentials. This is the content of our second main result:

Theorem B. *Fix $p > 0$. There exist positive constants $c, C > 0$ depending on X, ω, n, p such that, for all $\varphi \in \mathcal{E}^p(X,\omega)$ with $\sup_X \varphi = -1$,*

$$\int_X \exp\left(c|E_p(\varphi)|^{-1/n}|\varphi|^{1+\frac{p}{n}}\right)\omega^n \leq C.$$

Theorem B is an interesting variant of Trudinger's inequality on compact Kähler manifolds. The case $p = 1$ settles a conjecture of Aubin (called Hypothèse fondamentale [Aub84]) which is motivated by the search for Kähler-Einstein metrics on Fano manifolds. The conjecture was previously proved by Berman-Berndtsson [BB11] under the assumption that the cohomology class of ω is the first Chern class of an ample holomorphic line bundle.

We also establish local versions of these results, valid in any bounded hyperconvex domain of $\mathbb{C}^n$.

Organization of the paper. We recall the definition of finite energy classes in Section 1 where we also give explicit examples of finite entropy potentials. We then establish a Moser-Trudinger inequality in Section 2, proving Theorem B. We show in Proposition 3.1 that finite entropy measures act continuously on $\mathrm{PSH}(X,\omega)$ endowed with the L^1-topology. In the special case of compact Riemann surfaces the latter is equivalent to the potential being continuous (see Section 3.2). Theorem A will be proved in Section 3.3. We finally treat the case of bounded hyperconvex domains by analyzing the size of the capacity of sublevel sets in Section 4, where we also briefly discuss an extension of the celebrated Moser-Trudinger-Adams inequalities.

Acknowledgement. We thank Sébastien Boucksom and Mattias Jonsson for raising this interesting question. We also thank Bo Berndtsson for useful discussions on this subject. We thank Ahmed Zeriahi for carefully reading the first version of the note and giving numerous fruitful comments. We thank Nicholas McCleerey for pointing out an error in Lemma 4.1 in the previous version.

1. Preliminaries

In the whole paper (X, ω) is a compact Kähler manifold of complex dimension $n \in \mathbb{N}^*$. We assume ω is normalized so that $\int_X \omega^n = 1$.

1.1. Envelopes of quasi-psh functions.
Recall that a function is quasi-plurisubharmonic (*qpsh* for short) if it is locally given as the sum of a smooth and a psh function. Quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ satisfying

$$\omega + dd^c\varphi \geq 0$$

in the weak sense of currents are called ω-psh functions.

In particular quasi-psh functions are upper semi-continuous and Lebesgue-integrable. They are actually in L^p for all $p \geq 1$, and the induced topologies are all equivalent.

Definition 1.1. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-psh functions which are not identically $-\infty$.

The set $\mathrm{PSH}(X, \omega)$ is a closed subset of $L^1(X)$ for the L^1-topology. It was proved by Demailly [Dem92] (see also [BK07]) that any ω-psh function can be approximated by a decreasing sequence of smooth ω-psh functions.

Definition 1.2. A set $P \subset X$ is called pluripolar if $P \subset \{\varphi = -\infty\}$ for some quasi-psh function φ.

Pluripolar sets are the "small sets" of pluripotential theory. In the definition above one can further assume that φ is ω-psh. There are many ways to characterize pluripolar sets, we refer the reader to the book [GZ17].

Definition 1.3. Given a Lebesgue measurable function $h : X \to \mathbb{R}$, its ω-psh envelope is defined by

$$P(h) := \left(\sup \{u \mid u \in \mathrm{PSH}(X, \omega) \text{ and } u \leq h \text{ on X }\}\right)^*.$$

Here * denotes the upper semi-continuous regularization. The function $P(h)$ is either identically $-\infty$ or it belongs to $\mathrm{PSH}(X, \omega)$.

Theorem 1.4. [BD12] *If h is smooth then $P(h)$ has bounded Laplacian, $\omega + dd^c h \geq 0$ on the contact set $\{P(h) = h\}$, and*

$$(\omega + dd^c P(h))^n = 1_{\{P(h)=h\}}(\omega + dd^c h)^n.$$

The fact that $\omega + dd^c h \geq 0$ on the contact set $\{P(h) = h\}$ follows from basic properties of plurisubharmonic functions. We refer the reader to [BD12, Page 46] or [EGZ11, Proof of Proposition 1.3] for a proof of this fact. An alternative proof of this result has been provided by Berman in [Ber19].

1.2. Finite energy classes.
Given $\varphi \in \mathrm{PSH}(X, \omega)$, we consider

$$\varphi_j := \max(\varphi, -j) \in \mathrm{PSH}(X, \omega) \cap L^\infty(X).$$

It follows from the Bedford-Taylor theory [BT76, BT82] that the measures $(\omega + dd^c\varphi_j)^n$ are well defined probability measures and the sequence

$$\mu_j := \mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c\varphi_j)^n$$

is increasing [GZ07, p.445]. Since the μ_j's all have total mass bounded from above by 1, we consider

$$\mu_\varphi := \lim_{j \to +\infty} \mu_j,$$

which is a positive Borel measure on X, with total mass ≤ 1.

Definition 1.5. We set

$$\mathcal{E}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) \mid \mu_\varphi(X) = 1\}.$$

For $\varphi \in \mathcal{E}(X,\omega)$, we set $\omega_\varphi^n = (\omega + dd^c\varphi)^n := \mu_\varphi$.

Every bounded ω-psh function clearly belongs to $\mathcal{E}(X,\omega)$. The class $\mathcal{E}(X,\omega)$ also contains many ω-psh functions which are unbounded:

- when X is a compact Riemann surface, $\mathcal{E}(X,\omega)$ is precisely the set of ω-sh functions whose Laplacian does not charge polar sets.
- if $\varphi \in \mathrm{PSH}(X,\omega)$ is normalized so that $\varphi \le -1$, then $-(-\varphi)^\varepsilon$ belongs to $\mathcal{E}(X,\omega)$ whenever $0 \le \varepsilon < 1$.
- the functions in $\mathcal{E}(X,\omega)$ have relatively mild singularities; in particular they have zero Lelong number at every point.

It is proved in [GZ07] that the complex Monge-Ampère operator $\varphi \mapsto \omega_\varphi^n$ is well defined on the class $\mathcal{E}(X,\omega)$, in the sense that if $\varphi \in \mathcal{E}(X,\omega)$ then for every sequence of bounded ω-psh functions φ_j decreasing to φ, the measures $(\omega + dd^c\varphi_j)^n$ converge weakly on X towards μ_φ.

It follows from [GZ07, Din09] that a probability measure μ does not charge pluripolar sets if and only if there exists a unique $\varphi \in \mathcal{E}(X,\omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ with $\sup_X \varphi = 0$.

When μ is absolutely continuous with respect to the Lebesgue measure, one expects φ to belong to a weighted finite energy class: we let $\mathcal{W}$ denote the set of all functions $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that χ is increasing and $\chi(-\infty) = -\infty$.

Definition 1.6. Fix $\chi \in \mathcal{W}$. We let $\mathcal{E}_\chi(X,\omega)$ be the set of ω-psh functions with finite χ-energy,

$$\mathcal{E}_\chi(X,\omega) := \left\{\varphi \in \mathcal{E}(X,\omega) \mid \chi(-|\varphi|) \in L^1(X,\omega_\varphi^n)\right\}.$$

When $\chi(t) = -(-t)^p$, $p > 0$, we set $\mathcal{E}^p(X,\omega) = \mathcal{E}_\chi(X,\omega)$.

For $u \in \mathcal{E}^p(X,\omega)$ the E_p energy is defined as:

$$E_p(u) = \int_X |u|^p (\omega + dd^c u)^n.$$

It follows from [GZ07, Theorem C] that a probability measure μ is the Monge-Ampère measure of a potential in $\mathcal{E}^p(X,\omega)$ if and only if $\mathcal{E}^p(X,\omega) \subset L^p(X,\mu)$.

The class $\mathcal{E}^p(X,\omega)$, $p \ge 1$, can be equipped with a Finsler metric d_p making it a complete geodesic metric space. We refer the reader to [Dar15] for more details. We stress here that the distance d_p can be uniformly estimated by pluripotential terms ([Dar15, Theorem 5.5]):

$$C^{-1} d_p^p(u,v) \le \int_X |u-v|^p (\omega_u^n + \omega_v)^n \le C d_p^p(u,v),\ u,v \in \mathcal{E}^p(X,\omega), \tag{1.1}$$

for a constant $C = C(p,n) > 0$.

1.3. **Examples.** In this section we present some explicit computations suggesting that ω-psh functions with finite entropy Monge-Ampère measure belong to an energy class $\mathcal{E}^p(X,\omega)$ for some $p > 1$ depending on n.

We recall that the measure $(\omega + dd^c\varphi)^n = f\omega^n$ has finite entropy iff

$$\int_X f \log f\, \omega^n < +\infty.$$

1.3.1. *Radial singularities.* We assume here that the ω-psh functions φ under consideration are locally bounded in $X \setminus \{p\}$. We choose a local chart biholomorphic to the unit ball $\mathbb{B}$ of $\mathbb{C}^n$, with p corresponding to the origin. We further assume that φ has a *radial singularity* at p, i.e. it is invariant under the group $O(2n,\mathbb{R})$ near p, and so is ω. The singularity type of φ only depends on its local behavior near p.

We thus consider the local situation of a psh function $v = \chi \circ L$, where $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is a convex increasing function and $L : z \in \mathbb{B} \to \log|z| \in \mathbb{R}^-$. Since we only consider Monge-Ampère measures $(dd^c v)^n$ which are absolutely continuous with respect to Lebesgue measure (having finite entropy), we can assume w.l.o.g. that χ is $\mathcal{C}^2$ and $\chi'(-\infty) = 0$. Then a standard computation shows that

$$(dd^c v)^n = c_n \frac{(\chi' \circ L)^{n-1} \chi'' \circ L}{|z|^{2n}} dV = f_\chi dV.$$

By definition the entropy of the measure $(dd^c v)^n$ is defined as $\int_{\mathbb{B}} f_\chi \log f_\chi dV$. Hence, after passing in polar coordinates $r = |z|$, we see that $(dd^c v)^n$ has finite entropy if and only if

$$\int_0^1 \alpha(r) \log(\alpha(r)) r^{2n-1} dr < +\infty,$$

where $\alpha(r) := (\chi'(\log(r))^{n-1} \chi''(\log r) r^{-2n}$. Since $\chi'(-\infty) = 0$, the above condition is equivalent to

$$\int_0^1 (-\log(r))(\chi'(\log(r))^{n-1} \chi''(\log r) r^{-1} dr < +\infty.$$

Making the change of variable $t = \log r$, this is equivalent to

$$\int_{-\infty}^0 (-t) \chi'(t)^{n-1} \chi''(t) dt < +\infty.$$

By the same computations we see that v belongs to the finite energy class $\mathcal{E}^p(X,\omega)$ if and only if

$$\int_{-\infty}^0 (-\chi(t))^p \chi'(t)^{n-1} \chi''(t) dt < +\infty.$$

Integrating by parts, the finite entropy condition is thus equivalent to

$$\lim_{t \to -\infty} (-t\chi'(t)^n) + \int_{-\infty}^0 \chi'(t)^n dt < +\infty.$$

The latter condition implies (since χ' increases) that $0 \le \chi'(t) \le C(-t)^{-1/n}$, hence $(dd^c v)^n$ has finite entropy iff $\int_{-\infty}^0 \chi'(t)^n dt < +\infty$. Note that

$$0 \le \chi'(t) \le C(-t)^{-1/n} \Longrightarrow 0 \le |\chi(t)| \le C'|t|^{1-1/n},$$

thus if $(dd^c v)^n$ has finite entropy then

$$\int_{-\infty}^0 (-\chi(t))^p \chi'(t)^{n-1} \chi''(t) dt \le C' \int_{-\infty}^0 |t|^{p(1-1/n)} \chi'(t)^{n-1} \chi''(t) dt < +\infty$$

if $p(1-1/n) \le 1$, i.e. precisely when $p \le n/(n-1)$.

1.3.2. *Explicit radial examples.* We let $X = \mathbb{CP}^n$ denote the complex projective space, $n \geq 2$, and $\omega = \omega_{FS}$ be the Fubini-Study Kähler form. We let $\mathbb{C}^n \subset X$ denote an affine chart and fix $0 < \alpha < 1$. Consider

$$\chi_\alpha(t) = \begin{cases} -(-t)^\alpha/\alpha & \text{for} \quad t \leq -1 \\ t & \text{for} \quad t \geq 1 \end{cases}$$

and extend χ_α smoothly in $[-1, 1]$, so that the resulting weight χ_α is convex increasing. Hence $\psi_\alpha(z) = \chi_\alpha \circ \log |z|$ is plurisubharmonic in $\mathbb{C}^n$ and extends at infinity as a ω-psh function φ_α s.t.

$$(\omega + dd^c\varphi_\alpha)^n = (dd^c\psi_\alpha)^n = f_\alpha\, dV,$$

where the density f_α is concentrated in the euclidean ball $\{|z| \leq e\}$. On $\{e^{-1} < |z| \leq e\}$ it is smooth while on $\{|z| < e^{-1}\}$ it can be written as

$$f_\alpha(z) = \frac{c'}{(-\log |z|)^{n(1-\alpha)+1}|z|^{2n}}.$$

Using the general computations above one can check that the entropy $\int_{\{|z|\leq e^{-1}\}} f_\alpha \log f_\alpha\, dV$ is finite if and only if $\alpha < \frac{n-1}{n}$. On the other hand, φ_α belongs to $\mathcal{E}^p(X, \omega)$ if and only if

$$\int_{\mathbb{B}} |\chi_\alpha|^p \chi_\alpha'' \chi_\alpha'^{n-1}\, dV \simeq \int_{-\infty}^0 \frac{|t|^{p\alpha}}{|t|^{n(1-\alpha)+1}} < +\infty \Longleftrightarrow p < n\frac{1-\alpha}{\alpha}.$$

Fix $\varepsilon > 0$ very small. If we choose $\alpha = \frac{n-1}{n+\varepsilon}$ then the probability measure $(\omega + dd^c\varphi_\alpha)^n$ has finite entropy and φ_α belongs to the finite energy class $\mathcal{E}^{\frac{n}{n-1}}(X, \omega)$, but it does not belong to $\mathcal{E}^{\frac{n}{n-1}(1+\varepsilon)}(X, \omega)$.

1.3.3. *Divisorial singularities.* We assume here X is projective, $\omega = i\Theta_h$ is the (positive) curvature of a smooth hermitian metric h of an ample holomorphic line bundle L on X (i.e. ω is a Hodge form), and $D = (s = 0)$ is a divisor defined as the zero set of a holomorphic section $s \in H^0(X, L)$.

We assume w.l.o.g. that the ω-psh function $\psi = \log |s|_h$ is normalized by $\sup_X \psi \leq -1$. By construction $\omega + dd^c\psi = [D]$ is the current of integration along D. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a smooth convex increasing function such that $\chi' \leq 1/2$. The function $\varphi := \chi \circ \psi$ then satisfies

$$\omega_\varphi := \omega + dd^c\varphi = [1 - \chi' \circ \psi]\omega + \chi' \circ \psi\, \omega_\psi + \chi'' \circ \psi\, d\psi \wedge d^c\psi \geq 0$$

It is thus ω-psh and has *divisorial singularities* along D.

We assume $\chi'(-\infty) = 0$. Thus $\chi' \circ \psi\omega_\psi = \chi' \circ \psi[D] = 0$ and

$$\omega_\varphi = [1 - \chi' \circ \psi]\omega + \chi'' \circ \psi d\psi \wedge d^c\psi \geq 0.$$

We let the reader then check that

$$(\omega + dd^c\varphi)^n = f\omega^n$$

where the density f is comparable to $1 + \frac{\chi''\circ\psi}{|s|^2}$. Then the finite entropy condition in a neighborhood, U_ε, of a smooth point of $D = (z_1 = 0)$ reads as

$$\begin{aligned} \int_{U_\varepsilon} \frac{(-\log |z_1|)\chi'' \circ \log |z_1|}{|z_1|^2} dV(z_1) &= C \int_0^\varepsilon \frac{(-\log \rho)\chi'' \circ \log \rho}{\rho} d\rho \\ &= C \int_{-\infty}^{\log \varepsilon} (-t)\chi''(t) dt < +\infty, \end{aligned}$$

where $\varepsilon > 0$ and C is a positive constant. Observe that in the first equality we passed in polar coordinates. An integration by parts yields

$$\begin{aligned}\int_{-\infty}^{\log\varepsilon}(-t)\chi''(t)dt &= \lim_{t\to-\infty}(-t)\chi'(t)+\int_{-\infty}^{\log\varepsilon}\chi'(t)dt\\ &= \lim_{t\to-\infty}(-t)\chi'(t)+(-\chi)(-\infty)+O(1).\end{aligned}$$

This shows that $\varphi = \chi\circ\log|s|_h$ is a finite entropy potential if and only if $\chi(-\infty) > -\infty$, i.e. when φ is bounded. Said differently, there is no finite entropy potential with divisorial singularities.

2. Moser-Trudinger inequalities

Theorem 2.1. *Fix $p > 0$. Then there exist $c > 0, C > 0$ depending on X, ω, n, p such that, for all $\varphi \in \mathcal{E}^p(X,\omega)$ with $\sup_X \varphi = -1$, we have*

$$\int_X \exp\left(c|E_p(\varphi)|^{-\frac{1}{n}}|\varphi|^{1+\frac{p}{n}}\right)\omega^n \le C.$$

Theorem 2.1 generalizes an important result of Berman-Berndtsson [BB11, Theorem 1.1] who established the above inequality in the case when $\{\omega\} = c_1(L)$ for some holomorphic line bundle L and $p = 1$.

Proof. By approximation we can assume that φ is smooth. For notational convenience we set $q = \frac{n+p}{n} > 1$, $\psi := -a(-\varphi)^q$ and $u := P(\psi)$, where $a > 0$ is a small constant suitably chosen below. Note also that $E_p(\varphi) = \int_X(-\varphi)^p\omega_\varphi^n \ge 1$ because $\varphi \le -1$. A direct computation shows that

$$\begin{aligned}\omega + dd^c\psi &= \omega + aq(-\varphi)^{q-1}dd^c\varphi - aq(q-1)(-\varphi)^{q-2}d\varphi\wedge d^c\varphi\\ &\le [1-aq(-\varphi)^{q-1}]\omega + aq(-\varphi)^{q-1}(\omega+dd^c\varphi).\end{aligned}$$

We set

$$F := \left\{aq|\varphi|^{q-1}\ge 1\right\} = \left\{qa^{\frac{n}{n+p}}|\psi|^{\frac{p}{n+p}}\ge 1\right\},\ \ G := \left\{|u| \ge a^{-\frac{n}{p}}q^{-\frac{n+p}{p}}\right\}.$$

Observe that

$$(\omega + dd^c\psi) \le aq|\varphi|^{q-1}(\omega+dd^c\varphi) \quad \text{on } F$$

and $G\cap\{\psi = P(\psi)\}\subseteq F$. Hence on $G\cap\{\psi = P(\psi)\}$ we have

$$0 \le (\omega+dd^c\psi) \le aq|\varphi|^{q-1}(\omega+dd^c\varphi),$$

where the first inequality follows from Theorem 1.4. By Theorem 1.4 again,

$$\begin{aligned}\mathbf{1}_F(\omega+dd^cu)^n &= 1_{\{\psi=P(\psi)\}\cap F}(\omega+dd^c\psi)^n\\ &\le 1_{\{\psi=P(\psi)\}\cap F}\, a^nq^n|\varphi|^p(\omega+dd^c\varphi)^n. \end{aligned}\tag{2.1}$$

We now choose $a \in (0,1)$ so that

$$2a^nq^n\int_X|\varphi|^p(\omega+dd^c\varphi)^n = 2a^nq^nE_p(\varphi) = 1.$$

Integrating over X both sides of (2.1), we obtain

$$\int_G(\omega+dd^cu)^n \le \frac{1}{2}.$$

It thus follows that $G \neq X$, or equivalently that $G^c \neq \emptyset$. In particular $\sup_X u \geq -a^{-\frac{n}{p}} q^{-\frac{n+p}{p}} =: -b$. Recall that we have chosen $c_0, C_0 > 0$ so that, for all $\phi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \phi = 0$ we have

$$\int_X e^{c_0|\phi|}\omega^n \leq C_0.$$

In particular, for $\phi = u + b$ we have $\sup_X \phi \geq 0$, hence

$$\int_X e^{c_0(a(-\varphi)^q - b)}\omega^n \leq \int_X e^{-c_0(u+b)}\omega^n = \int_X e^{-c_0(\phi - \sup_X \phi) - c_0 \sup_X \phi} \leq C_0.$$

Set $K = \{a|\varphi|^{q-1} \leq q^{-1}2^{1-1/q}\}$ and note that $a|\varphi|^q \leq q^{-1}2^{1-1/q}|\varphi| \leq 2|\varphi|$ on K. On $X \setminus K$ we have that $a^{\frac{q}{q-1}}|\varphi|^q > 2q^{-\frac{q}{q-1}}$ or, equivalently,

$$\frac{a|\varphi|^q}{2} < a|\varphi|^q - b.$$

Thus

$$\begin{aligned}\int_X e^{c_0\frac{a|\varphi|^q}{2}}\omega^n &\leq \int_K e^{c_0|\varphi|}\omega^n + \int_{X\setminus K} e^{c_0(a|\varphi|^q - b)}\omega^n \\ &= \int_K e^{-c_0(\varphi+1)+c_0}\omega^n + \int_{X\setminus K} e^{c_0(a|\varphi|^q-b)}\omega^n \leq C_0(e^{c_0}+1).\end{aligned}$$

The result follows with $c = 2^{-1-1/n}q^{-1}c_0$, and $C = C_0(e^{c_0}+1)$. □

An interesting consequence of Theorem 2.1 is the following result which was conjectured by Aubin [Aub84, page 148] when $p = 1$:

Corollary 2.2. *Fix $p > 0$. There exist uniform constants $A, B > 0$ such that for all $k \in \mathbb{N}$ and $\varphi \in \mathcal{E}^p(X,\omega)$ normalized by $\sup_X \varphi = -1$,*

$$\log \int_X e^{-k\varphi}\omega^n \leq Ak^{1+\frac{n}{p}} E_p(\varphi)^{1/p} + B.$$

We follow [BB14, page 359]. As explained in [BB11, page 2] Corollary 2.2 is actually equivalent to Theorem 2.1. A similar estimate has been observed in [AC19, Remark page 1461] for the case when $p \geq 1$ (and $\{\omega\}$ is integral) by a direct application of [BB11] and Hölder inequality.

Proof. We set $\alpha = 1 + \frac{p}{n}$, $\beta = 1 + \frac{n}{p}$ so that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. For $\eta > 0$, $\xi > 0$, Young's inequality yields

$$\eta\xi \leq \frac{\eta^\beta}{\beta} + \frac{\xi^\alpha}{\alpha} = \frac{p\eta^{1+\frac{n}{p}}}{n+p} + \frac{n\xi^{1+\frac{p}{n}}}{n+p}.$$

Dividing by η and replacing η by $\tau^{1+\frac{p}{n}}$ we obtain

$$\xi \leq \frac{p\tau^{1+\frac{n}{p}}}{n+p} + \frac{n\xi^{1+\frac{p}{n}}}{(n+p)\tau^{1+\frac{p}{n}}}.$$

We now choose $\xi = -k\varphi(x)$, $x \in X$, and $\tau = aE_p(\varphi)^{\frac{1}{p+n}}$, $a > 0$, to obtain

$$-k\varphi(x) \leq \frac{pa^{1+\frac{n}{p}}E_p(\varphi)^{\frac{1}{p}}}{n+p} + \frac{n(-k\varphi(x))^{1+\frac{p}{n}}}{(n+p)a^{1+\frac{p}{n}}E_p(\varphi)^{\frac{1}{n}}}. \tag{2.2}$$

We choose a so that $\frac{nk^{1+\frac{p}{n}}}{(n+p)a^{1+\frac{p}{n}}} = c$, where c is the constant in Theorem 2.1, i.e. $a = k\left(\frac{n}{(n+p)c}\right)^{\frac{n}{n+p}}$. By inserting this choice of a in (2.2) we get

$$-k\varphi(x) \leq \frac{p}{(n+p)}\left(\frac{n}{(n+p)c}\right)^{n/p} k^{1+\frac{n}{p}} E_p(\varphi)^{\frac{1}{p}} + c|\varphi(x)|^{1+\frac{p}{n}} E_p(\varphi)^{-\frac{1}{n}}.$$

Exponentiating and integrating, we thus obtain the desired inequality with

$$A = \frac{pn^{n/p}}{c^{n/p}(n+p)^{1+n/p}}, \quad \text{and} \quad B = \log C,$$

where C is the constant in Theorem 2.1. □

We now connect integrability properties of a given qpsh function v to the finite energy of its envelopes $P(-(-v)^q)$, $q > 1$. Since $q > 1$, the function $-(-v)^q$ is not qpsh and it is more singular than v. For an arbitrary qpsh function v, it could happen that $P(-(-v)^q)$ is identically $-\infty$.

Proposition 2.3. *Fix $v \in \mathrm{PSH}(X,\omega)$ with $v \leq -1$ and $p > 0$.*

- *If $v \in \mathcal{E}^p(X,\omega)$ then $P(-(-v)^{1+\frac{p}{n}}) \in \mathcal{E}(X,\omega)$.*
- *If $P(-(-v)^s) \in \mathrm{PSH}(X,\omega)$ for some $s > 1+p$ then $v \in \mathcal{E}^p(X,\omega)$.*

Proof. Assume that $v \in \mathcal{E}^p(X,\omega)$. By approximation we can assume that v is smooth. Set $q = 1 + \frac{p}{n}$, $u = -(-v)^q$, and $\psi = P(u)$. We compute

$$\begin{aligned} \omega + dd^c u &= \omega + q(-v)^{q-1}dd^c v - q(q-1)(-v)^{q-2}dv \wedge d^c v \\ &\leq [1 - q(-v)^{q-1}]\omega + q(-v)^{q-1}(\omega + dd^c v) \leq q(-v)^{q-1}(\omega + dd^c v). \end{aligned}$$

It then follows from Theorem 1.4 that

$$(\omega + dd^c\psi)^n = 1_{\{u=P(u)\}}(\omega + dd^c u)^n = 1_{\{u=P(u)\}} q^n |v|^{n(q-1)}(\omega + dd^c v)^n.$$

Since $v \in \mathcal{E}^p(X,\omega)$, it follows from [GZ07] that there exists $C > 0$ and an increasing function $\chi_1 : \mathbb{R}^- \to \mathbb{R}^-$ with $\chi_1(-\infty) = -\infty$, such that for all j,

$$\int_X |\chi_1 \circ v||v|^p(\omega + dd^c v)^n \leq C.$$

Consider $\chi_2 : \mathbb{R}^- \to \mathbb{R}^-$ defined such that $\chi_1(t) = \chi_2(-(-t)^q))$. Observe that χ_2 is again an increasing function and $\chi_2(-\infty) = -\infty$. Now

$$\begin{aligned} \int_X |\chi_2 \circ \psi|(\omega + dd^c\psi)^n &= \int_{\{u=P(u)\}} |\chi_2 \circ u|(\omega + dd^c u)^n \\ &\leq q^n \int_X |\chi_2 \circ u|(-v)^{n(q-1)}(\omega + dd^c v)^n \\ &= q^n \int_X |\chi_1 \circ v||v|^p(\omega + dd^c v)^n \leq Cq^n. \end{aligned}$$

This shows that $\psi \in \mathcal{E}_{\chi_2}(X,\omega)$ and in particular $\psi \in \mathcal{E}(X,\omega)$.

For the reverse implication we assume that $\psi := P(-(-v)^s) \in \mathrm{PSH}(X,\omega)$ for some $s > 1+p$ and set $w := -(-\psi)^{1/s}$. It follows from Lemma 2.4 below that $w \in \mathcal{E}^p(X,\omega)$, hence v also belongs to $\mathcal{E}^p(X,\omega)$ since $v \geq w$. □

Lemma 2.4. *Fix $p > 0$ and $r < \frac{1}{1+p}$. If $u \in \mathrm{PSH}(X,\omega)$ and $u \leq -1$ then $-(-u)^r \in \mathcal{E}^p(X,\omega)$.*

The exponent r is sharp: the function $\varphi := -(-\log|s|)^r$ (introduced in Section 1.3.3) belongs to $\mathcal{E}^1(X,\omega)$ iff $r < 1/2$ [Dn15, Proposition 2.8].

Proof. Let (u_j) be a decreasing sequence of smooth ω-psh functions such that $u_j \searrow u$ and $u_j \leq -1$. Set $w_j := -(-u_j)^r$. A direct computation shows

$$\begin{aligned} \omega + dd^c w_j &= \omega + r|u_j|^{r-1} dd^c u_j + r(1-r)|u_j|^{r-2} du_j \wedge d^c u_j \\ &= \left(1 - r|u_j|^{r-1}\right)\omega + r|u_j|^{r-1}(\omega + dd^c u_j) \\ &\quad + r(1-r)|u_j|^{r-2} du_j \wedge d^c u_j \\ &\leq \left(1 - r|u_j|^{r-1}\right)\omega + |u_j|^{r-1}(\omega + dd^c u_j) + |u_j|^{r-2} du_j \wedge d^c u_j, \end{aligned}$$

using that $u_j \leq -1$ and $r < 1$. We now integrate on X to obtain

$$\begin{aligned} \int_X (-w_j)^p (\omega + dd^c w_j)^n &\leq \int_X |u_j|^{pr} (\omega + |u_j|^{r-1}(\omega + dd^c u_j))^n \\ &\quad + n \int_X |u_j|^{pr+r-2} (\omega + |u_j|^{r-1}(\omega + dd^c u_j))^{n-1} \wedge du_j \wedge d^c u_j. \end{aligned}$$

We bound the first term by

$$\begin{aligned} &\int_X |u_j|^{pr} (\omega + |u_j|^{r-1}(\omega + dd^c u_j))^n \\ &\qquad \leq 2^n \sum_{k=0}^{n} \int_X |u_j|^{pr} (|u_j|^{r-1}(\omega + dd^c u_j))^k \wedge \omega^{n-k} \leq 2^n (n+1), \end{aligned}$$

noticing that $r(p+1) - 1 < 0$, hence $|u_j|^{r(p+1)-1} \leq 1$. For the second term we write:

$$\begin{aligned} &|u_j|^{pr+r-2} (\omega + |u_j|^{r-1}(\omega + dd^c u_j))^{n-1} \wedge du_j \wedge d^c u_j \\ &\qquad \leq 2^{n-1} |u_j|^q du_j \wedge d^c u_j \wedge \sum_{k=0}^{n-1} (\omega + dd^c u_j)^k \wedge \omega^{n-k-1} \\ &\qquad = \frac{2^{n-1}}{(q+1)} d(-(-u_j)^{q+1}) \wedge d^c u_j \wedge \sum_{k=0}^{n-1} (\omega + dd^c u_j)^k \wedge \omega^{n-k-1}, \end{aligned}$$

where $q := pr + r - 2 < -1$. Integrating by parts we obtain

$$\begin{aligned} &\int_X |u_j|^{pr+r-2} (\omega + |u_j|^{r-1}(\omega + dd^c u_j))^{n-1} \wedge du_j \wedge d^c u_j \\ &\qquad \leq \frac{2^{n-1}}{(q+1)} \int_X |u_j|^{q+1} dd^c u_j \wedge \sum_{k=0}^{n-1} (\omega + dd^c u_j)^k \wedge \omega^{n-k-1} \\ &\qquad \leq \frac{2^{n-1}}{(q+1)} \sum_{k=0}^{n-1} \int_X (\omega + dd^c u_j)^{k+1} \wedge \omega^{n-k-1} = \frac{n 2^{n-1}}{(q+1)}. \end{aligned}$$

Thus $\int_X (-w_j)^p (\omega + dd^c w_j)^n \leq C(n, p, r)$, proving that $w \in \mathcal{E}^p(X, \omega)$. □

3. Finite entropy potentials

In this Section we systematically study finite entropy potentials. For each $B > 0$ we set

$$\mathrm{Ent}_B := \{u \in \mathcal{E}^1(X, \omega) \mid \sup_X u = -1, \mathrm{Ent}_{\omega^n}(\omega_u^n) \leq B\}. \tag{3.1}$$

Recall that $\mathrm{Ent}_{\omega^n}(\omega_u^n) := \int_X f \log f \, \omega^n$ if $\omega_u^n = f \omega^n$ is absolutely continuous with respect to ω^n and $\mathrm{Ent}_{\omega^n}(\omega_u^n) = +\infty$ otherwise.

3.1. **Continuity of the mean-value.** Given a probability measure μ with finite entropy we denote by L_μ the map

$$u \in \mathrm{PSH}(X,\omega) \mapsto \int_X u \, d\mu.$$

We consider the convex function $\chi : s \in \mathbb{R}^+ \mapsto (s+1)\log(s+1) - s \in \mathbb{R}^+$. Its conjugate convex function is

$$\chi^* : t \in \mathbb{R}^+ \mapsto \sup_{s>0}\{st - \chi(s)\} = e^t - t - 1 \in \mathbb{R}^+.$$

By definition these functions satisfy, for all $s, t > 0$,

$$st \leq \chi(s) + \chi^*(t). \tag{3.2}$$

Such simple inequality will be used to prove the following:

Proposition 3.1. *Let $\mu = f\omega^n$ be a probability measure on X with finite entropy $\int_X f \log f \omega^n < +\infty$. Then L_μ is well-defined and continuous.*

Here $\mathrm{PSH}(X,\omega)$ is endowed with the L^1-topology.

Proof. Thanks to [GZ17, Theorem 2.50] we can ensure that there exist $c, C > 0$ such that $\int_X e^{4c|\psi|}\omega^n \leq C$ for all $\psi \in \mathrm{PSH}(X,\omega)$ normalized with $\sup_X \psi = 0$. Using (3.2) with $s = f(x)$ and $t = -c\psi(x)$, $x \in X$, we have

$$c|\psi| f \leq e^{c|\psi|} + \chi(f).$$

Integrating over X we see that L_μ is well-defined (since f has finite entropy). We now prove that L_μ is continuous. Assume that ψ_j is a sequence in $\mathrm{PSH}(X,\omega)$ such that $\|\psi_j - \psi\|_{L^1(X)} \to 0$ for some $\psi \in \mathrm{PSH}(X,\omega)$. Since $\sup_X \psi_j \to \sup_X \psi$, we can assume that $\sup_X \psi_j = \sup_X \psi = 0$. Up to extracting a subsequence, we can also assume that ψ_j converges a.e. to ψ. By the choice of c and elementary inequalities we can show that $\|e^{-2c\psi_j} - e^{-2c\psi}\|_{L^1(X)} \to 0$. Thus, passing to a subsequence we can assume that $e^{-2c\psi_j} \leq g$ for some $g \in L^1(X)$. Applying (3.2) with $s = f(x)$ and $t = c|\psi_j - \psi|(x)$, $x \in X$, we obtain

$$c|\psi_j - \psi| f \leq \chi(f) + e^{c|\psi_j - \psi|} \leq \chi(f) + e^{c|\psi_j|}e^{c|\psi|} \leq \chi(f) + g + e^{2c|\psi|} \in L^1(X).$$

The dominated convergence theorem thus yields $\int_X |\psi_j - \psi| f\omega^n \to 0$. ☐

3.2. **Compact Riemann surfaces.** In this section we treat the case of compact Riemann surfaces. We thus assume that $n = 1$ and the set $\mathrm{PSH}(X,\omega)$ will be denoted by $\mathrm{SH}(X,\omega)$, where the latter is endowed with the L^1-topology. The results in this section are probably well known to experts. We include them as a warm-up for the reader (this is Exercise 12.1, page 339 in [GZ17]).

Lemma 3.2. *Let $\mu = \omega + dd^c\varphi$, $\varphi \in \mathrm{SH}(X,\omega)$ be a probability measure. Then*

1. *φ is bounded if and only if $\mathrm{SH}(X,\omega) \subset L^1(\mu)$;*
2. *φ is continuous if and only if L_μ is continuous on $\mathrm{SH}(X,\omega)$.*

This characterization does not hold when $n \geq 2$: if $\mu = (\omega + dd^c\varphi)^n$ has finite entropy then $\mathrm{PSH}(X,\omega) \subset L^1(\mu)$ and L_μ is continuous on $\mathrm{PSH}(X,\omega)$ but this does not necessarily imply that φ is bounded (see Section 1.3).

Proof. We prove (1). Assume first that φ is bounded and fix $\psi \in \mathrm{SH}(X,\omega)$. We want to show that $\int_X |\psi| \, d\mu < +\infty$. Since ψ is bounded from above, we can assume without loss of generality that $\psi \le 0$. Since φ is bounded from below, we can assume $\varphi \ge 0$. Now by Stokes theorem,

$$\int (-\psi)\, d\mu = \int (-\psi)\, \omega + \int \varphi(-dd^c\psi) \le \int (\varphi - \psi)\, \omega < +\infty,$$

since $\varphi \ge 0$ and $-dd^c\psi \le \omega$. This shows $\mathrm{SH}(X,\omega) \subset L^1(\mu)$.

Assume now that $\mathrm{SH}(X,\omega) \subset L^1(\mu)$. We need to show that φ is bounded from below. Let $\delta_a = \omega + dd^c\psi_a$, with $\psi_a \in \mathrm{SH}(X,\omega)$ and $\sup_X \psi_a = 0$, be the Dirac mass at a point $a \in X$. By [GZ05, Proposition 2.7], there there exists $C_\mu > 0$ such that

$$\forall a \in X, \int_X \psi_a d\mu \ge -C_\mu.$$

Integrating by parts we therefore obtain that for all $a \in X$,

$$\varphi(a) = \int_X \varphi\, \delta_a = \int_X (\varphi - \psi_a)\omega + \int_X \psi_a\, d\mu \ge -C_\mu + \int_X \varphi\omega,$$

hence φ is bounded.

We next prove (2). Let $(a_j) \in X^{\mathbb{N}}$ be a sequence of points converging to $a \in X$. With the same notations as above, it follows from Stokes theorem that

$$\int_X \psi_{a_j}\, d\mu - \int_X \psi_a\, d\mu = \int_X (\psi_{a_j} - \psi_a)\, \omega + [\varphi(a_j) - \varphi(a)].$$

We claim that ψ_{a_j} converges to ψ_a in L^1. Indeed, by construction we have that $\delta_{a_j} = \omega + dd^c\psi_{a_j}$ weakly converges to $\delta_a = \omega + dd^c\psi_a$. On the other hand by compactness [GZ05, Proposition 2.7], ψ_{a_j} converges in L^1 to a potential $\tilde{\psi}_a$ solving $\delta_a = \omega + dd^c\tilde{\psi}_a$. By uniqueness we get $\tilde{\psi}_a = \psi_a$. This proves the claim. Hence, if the mapping $u \in \mathrm{SH}(X,\omega) \mapsto \int_X u\, d\mu \in \mathbb{R}$ is continuous, so is φ.

Conversely assume φ is continuous. Let (ψ_j) be a sequence of ω-psh functions that converges in $L^1(X)$ to $\psi \in \mathrm{SH}(X,\omega)$. Then

$$\int_X \psi_j\, d\mu - \int_X \psi\, d\mu = \int_X (\psi_j - \psi)\, \omega + \left[\int_X \varphi(\omega + dd^c\psi_j) - \int_X \varphi(\omega + dd^c\psi)\right].$$

Now $\omega + dd^c\psi_j$ converges to $\omega + dd^c\psi$ in the sense of distributions hence in the sense of Radon measures, since these are probability measures. We infer that $\int_X \varphi(\omega + dd^c\psi_j) \to \int_X \varphi(\omega + dd^c\psi)$ hence $\psi \mapsto \int \psi\, d\mu$ is continuous. □

As a direct consequence of Lemma 3.2 and Proposition 3.1 we obtain:

Corollary 3.3. *If $\mu = \omega + dd^c\varphi$ has finite entropy then φ is continuous.*

It is also worth to mention that Example 3.6 below shows that the injection $\mathrm{Ent}_B(X,\omega) \hookrightarrow \mathcal{C}^0(X)$, when $n = 1$, is not compact.

3.3. Higher dimensional compact setting.

Theorem 3.4. *Fix $B > 0$ and set $p = \frac{n}{n-1}$. There exist $c, C > 0$ depending on B, X, ω, n such that, for all $\varphi \in \mathrm{Ent}_B$ we have*

$$\int_X e^{c(-\varphi)^p}\omega^n \le C \text{ and } E_p(\varphi) \le C.$$

In particular, $\mathrm{Ent}(X,\omega) \subset \mathcal{E}^{\frac{n}{n-1}}(X,\omega)$.

We refer to (3.1) for a definition of Ent_B.

Proof. Fix $B > 0$ and $\varphi \in \mathrm{Ent}_B$ with $\omega_\varphi^n = f\omega^n$.

Step 1: the function φ belongs to $\mathcal{E}^p(X,\omega)$.
We claim that $\mathcal{E}^p(X,\omega) \subset L^p(X,\mu)$. Indeed, fix $v \in \mathcal{E}^p(X,\omega)$ with $\sup_X v = -1$ and observe that $1 + \frac{p}{n} = p$. It follows from Theorem 2.1 that for some $c > 0$ small enough,

$$\int_X e^{c|v|^p}\omega^n < +\infty. \tag{3.3}$$

We apply (3.2) with $s = f(x)$ and $t = c|v(x)|$. This yields

$$\int_X c|v|^p\omega_\varphi^n = \int_X c|v|^p f\omega^n \leq \int_X \chi\circ f\omega^n + \int_X (e^{c|v|^p} - c|v|^p - 1)\omega^n,$$

where the first integral is finite since f has finite entropy, while the second is bounded thanks to (3.3) and the integrability properties of qpsh functions. This means that $\int_X |v|^p\omega_\varphi^n < +\infty$, proving the claim. By [GZ07, Theorem C] we can thus infer that $\varphi \in \mathcal{E}^p(X,\omega)$.

Step 2: Integrability of $e^{|\varphi|^p}$ and energy bound.
Using the Hölder-Young inequality again we see that

$$\begin{aligned}\int_X c\frac{|\varphi|^p}{E_p(\varphi)^{1/n}} f\omega^n &\leq \int_X \left(e^{c\frac{|\varphi|^p}{E_p(\varphi)^{1/n}}} - c\frac{|\varphi|^p}{E_p(\varphi)^{1/n}} - 1\right)\omega^n + \int_X \chi\circ f\omega^n \\ &\leq C_1.\end{aligned}$$

Theorem 2.1 insures that $C_1 > 0$ depends only on X, ω, n, and $\mathrm{Ent}(f)$. From the above inequality we get $cE_p(\varphi)^{1-1/n} \leq C_1$, which yields $E_p(\varphi) \leq C_2$. Invoking Theorem 2.1 again we obtain

$$\int_X e^{\gamma|\varphi|^p}\omega^n \leq C_3,$$

where $\gamma = c\,C_2^{-1/n} > 0$ is a uniform constant. □

The examples from Section 1.3 show that the exponent $\frac{n}{n-1}$ is sharp.

Theorem 3.5. *The set* Ent_B *is compact in* $(\mathcal{E}^p(X,\omega), d_p)$*, for any* $p < \frac{n}{n-1}$.

Proof. Fix $p < r := \frac{n}{n-1}$, and $B > 0$. Let φ_j be a sequence in Ent_B. By [BBEGZ19, Theorem 2.17] and (1.1) there exists a subsequence of φ_j, still denoted by φ_j, such that $d_1(\varphi_j,\varphi) \to 0$ as $j \to +\infty$ for some $\varphi \in \mathcal{E}^1(X,\omega)$ with $\sup_X \varphi = -1$. If we write $g_j\omega^n = (\omega + dd^c\varphi_j)^n + (\omega + dd^c\varphi)^n$ then

$$\int_X |\varphi_j - \varphi| g_j\omega^n \to 0.$$

By lower semicontinuity of the entropy [BBEGZ19, Proposition 2.10] we also have that $\mathrm{Ent}_{\omega^n}(\omega_\varphi^n) \leq B$, hence by Theorem 3.4, $\varphi \in \mathcal{E}^{\frac{n}{n-1}}(X,\omega)$. We want to prove that $d_p(\varphi_j,\varphi) \to 0$ which by (1.1) is equivalent to showing that

$$\lim_{j\to+\infty}\int_X |\varphi_j - \varphi|^p g_j\omega^n = 0.$$

Fix $r \in (0,1)$ such that $(1-r)\frac{n}{n-1}+r=p$, that is $p=\frac{n-r}{n-1}$. Using the Hölder inequality with exponent $\frac{1}{r}$ and $\frac{1}{1-r}$ we obtain

$$\int_X |\varphi_j-\varphi|^p g_j\omega^n = \int_X |\varphi_j-\varphi|^{r+(1-r)\frac{n}{n-1}} g_j\omega^n$$
$$\leq \left(\int_X |\varphi_j-\varphi| g_j\omega^n\right)^r \left(\int_X |\varphi_j-\varphi|^{\frac{n}{n-1}} g_j\omega^n\right)^{1-r}. \tag{3.4}$$

Now, $|\varphi_j-\varphi|^{\frac{n}{n-1}} \leq C_1(|\varphi_j|^{\frac{n}{n-1}}+|\varphi|^{\frac{n}{n-1}})$ and $g_j\omega^n$ has finite entropy. By Hölder-Young inequality and Theorem 3.4 we infer

$$\int_X c|\varphi_j|^{\frac{n}{n-1}} g_j\omega^n \leq \int_X \chi\circ g_j\omega^n + \int_X (e^{c|\varphi_j|^{\frac{n}{n-1}}} - c|\varphi_j|^{\frac{n}{n-1}}-1)\omega^n \leq C.$$

Similarly we have $\int_X |\varphi|^{\frac{n}{n-1}} g_j\omega^n \leq C$. The second factor in (3.4) is thus bounded while the first one converges to 0 as $j\to+\infty$. □

The injection $\mathrm{Ent}(X,\omega) \hookrightarrow \mathcal{E}^{\frac{n}{n-1}}(X,\omega)$ is however not compact:

Example 3.6. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth increasing convex function such that $\chi(t)=0$ for $t\leq -\log 2$, $\chi(t)=t$ for $t\geq \log 2$, and χ is strictly convex in $(-\log 2, \log 2)$. We fix $\varepsilon_j>0$ a sequence decreasing to zero, C_j a sequence increasing to $+\infty$, and we set

$$\psi_j(z) := \varepsilon_j\chi\circ L_j(z) - \varepsilon_j C_j,$$

where $L_j(z) := \log|e^{C_j}z| = C_j + L(z)$. The functions ψ_j are psh in $\mathbb{C}^n$, since

$$dd^c\psi_j = \varepsilon_j\chi''\circ L_j\, dL\wedge d^cL + \varepsilon_j\chi'\circ L_j\, dd^cL.$$

Thus

$$(dd^c\psi_j)^k = k\varepsilon_j^k\,\chi''\circ L_j(z)(\chi'\circ L_j)^{k-1}dL\wedge d^cL\wedge(dd^cL)^{k-1} + \varepsilon_j^k\,(\chi'\circ L)^k(dd^cL)^k.$$

Since $\chi''\circ L_j=0$ outside $\mathcal{C}_j := \{\frac{e^{-C_j}}{2}<|z|<2e^{-C_j}\}$, we obtain

$$(dd^c\psi_j)^k\wedge\omega^{n-k} = F_{j,k}(z)dV(z),\ k=1,2,...,n,$$

where $\omega = dd^c\log\sqrt{1+|z|^2}$ denotes the Fubini-Study form on $\mathbb{C}^n$ and

$$F_{j,k} = \varepsilon_j^k\chi''\circ L_j(\chi'\circ L_j)^{k-1}\frac{g_k(z)}{|z|^{2k}} + \varepsilon_j^k(\chi'\circ L_j)^k\frac{h_k(z)}{|z|^{2k}}$$

with $g_k, h_k>0$ bounded functions. We then see that for $k=0,\cdots,n-1$ we can ensure that $\|F_{j,k}\|_{L^q(\mathbb{C}^n)}\leq A$ for some $q>1$ and $A>0$.

We note that $F_{j,n} = \varepsilon_j^n\chi''\circ L_j(\chi'\circ L_j)^{n-1}\frac{g_n(z)}{|z|^{2n}}$ is supported on $\mathcal{C}_j$ with

$$F_{j,n}(z)\leq C\varepsilon_j^n e^{2nC_j}f_n(e^{C_j}z), \quad \text{where} \quad f_n(z) := |z|^{-2n}\chi''\circ L(\chi'\circ L)^{n-1}.$$

We take $C_j=\varepsilon_j^{-n}$ so that the entropy of $F_{j,n}$ is uniformly bounded. Indeed we see that $\int_{\mathbb{C}^n}F_{j,n}(z)\log F_{j,n}(z)dV(z)$ is comparable to

$$\int_{\mathcal{C}}(2n\varepsilon_j^nC_jf_n(w) + \varepsilon_j^nf_n(w)(\log f_n(w)+n\log\varepsilon_j))dV(w),$$

where $\mathcal{C} := \{1/2<|w|<2\}$ (use the change of variables $w=e^{C_j}z$). Now

$$\int_{\mathcal{C}}\varepsilon_j^nf_n(w)(\log f_n(w)+n\log\varepsilon_j))dV(w)\to 0$$

as $j \to +\infty$, while

$$\int_{\mathcal{C}} 2n f_n(w) dV(w) \sim \int_{\mathcal{C}} \frac{1}{|w|^{2n}} dV(w) \sim \int_{1/2}^{2} \frac{1}{\rho} d\rho$$

is uniformly bounded. The same type of computations yields

$$\begin{aligned}
\int_{\mathbb{C}^n} |\psi_j|^p (dd^c \psi_j)^n &\sim \int_{\mathcal{C}_j} \varepsilon_j^p |\chi \circ L_j(z)|^p F_{j,n}(z) dV(z) + \int_{\mathcal{C}_j} \varepsilon_j^p C_j^p F_{j,n}(z) dV(z) \\
&\sim \varepsilon_j^{p+n} \int_{\mathcal{C}} f_n(w) dV(w) + \varepsilon_j^{n+p} C_j^p \int_{\mathcal{C}} f_n(w) dV(w) \\
&\sim \varepsilon_j^{n+p} C_j^p \sim \varepsilon_j^{n+p-np} \sim 1, \quad \text{iff} \quad p = \frac{n}{n-1}.
\end{aligned}$$

We finally consider the induced ω-psh functions

$$\varphi_j = \psi_j(z) - \varepsilon_j \log \sqrt{1 + |z|^2}$$

on $(\mathbb{P}^n, \omega)$, and conclude that

- $\varphi_j \in \mathrm{Ent}_B$ and $C^{-1} \leq E_{\frac{n}{n-1}}(\varphi_j) \leq C$;
- $d_p(\varphi_j, 0) \to 0$ for all $p < \frac{n}{n-1}$ but $d_{\frac{n}{n-1}}(\varphi_j, 0) \not\to 0$.

4. The local setting

We fix $\Omega \subset \mathbb{C}^n$ a bounded hyperconvex domain, i.e. there exists a continuous psh function $\rho : \Omega \to [-1, 0)$ such that $\{\rho < -c\} \Subset \Omega$ for all $c > 0$.

4.1. **Cegrell's classes.** Let $\mathcal{T}(\Omega)$ denote the set of bounded non-positive psh functions u defined on Ω such that $\lim_{z \to \zeta} u(z) = 0$, for every $\zeta \in \partial\Omega$, and $\int_\Omega (dd^c u)^n < +\infty$. Cegrell [Ceg98, Ceg04] has studied the complex Monge-Ampère operator $(dd^c \cdot)^n$ and introduced different classes of plurisubharmonic functions on which the latter is well defined:

- $\mathrm{DMA}(\Omega)$ is the set of psh functions u such that for all $z_0 \in \Omega$, there exists a neighborhood V_{z_0} of z_0 and $u_j \in \mathcal{T}(\Omega)$ a decreasing sequence which converges to u in V_{z_0} and satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$.
- the class $\mathcal{F}(\Omega)$ is the "global version" of $\mathrm{DMA}(\Omega)$: a function u belongs to $\mathcal{F}(\Omega)$ iff there exists $u_j \in \mathcal{T}(\Omega)$ a sequence decreasing towards u *in all of* Ω, which satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$;
- the class $\mathcal{E}^p(\Omega)$ (respectively $\mathcal{F}^p(\Omega)$) is the set of psh functions u for which there exists a sequence of functions $u_j \in \mathcal{T}(\Omega)$ decreasing towards u in all of Ω, and so that $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty$ (respectively $\sup_j \int_\Omega [1 + (-u_j)^p] (dd^c u_j)^n < +\infty$).

Given $u \in \mathcal{E}^p(\Omega)$ we define the weighted energy of u by

$$E_p(u) := \int_\Omega (-u)^p (dd^c u)^n < +\infty.$$

The operator $(dd^c \cdot)^n$ is well defined on these sets, and continuous under decreasing limits. If $u \in \mathcal{E}^p(\Omega)$ for some $p > 0$ then $(dd^c u)^n$ vanishes on all pluripolar sets [BGZ09, Theorem 2.1]. If $u \in \mathcal{E}^p(\Omega)$ and $\int_\Omega (dd^c u)^n < +\infty$ then $u \in \mathcal{F}^p(\Omega)$. Also, note that

$$\mathcal{T}(\Omega) \subset \mathcal{F}^p(\Omega) \subset \mathcal{F}(\Omega) \subset \mathrm{DMA}(\Omega) \quad \text{and} \quad \mathcal{T}(\Omega) \subset \mathcal{E}^p(\Omega) \subset \mathrm{DMA}(\Omega).$$

It has been established in [BGZ09] that

$$\mathcal{E}^p(\Omega) = \left\{ \varphi \in \mathrm{PSH}^-(\Omega) \mid \int_0^{+\infty} t^{n+p-1}\mathrm{Cap}(\varphi < -t)dt < +\infty \right\}.$$

Here $\mathrm{Cap}(\cdot) := \mathrm{Cap}(\cdot, \Omega)$ denotes the Monge-Ampère capacity [BT82]:

$$\mathrm{Cap}(E, \Omega) := \sup \left\{ \int_E (dd^c u)^n \mid u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0 \right\}.$$

Given a Borel function h defined in Ω we let $P(h)$ denote the psh envelope:

$$P(h) := \left(\sup\{u \in \mathrm{PSH}(\Omega) \mid u \le h \text{ in } \Omega\}\right)^*.$$

Lemma 4.1. *Assume that h is bounded and there exists a decreasing sequence (h_j) of continuous functions on $\bar{\Omega}$ such that $h_j \to h$ in capacity. Then $P(h)$ is a bounded plurisubharmonic function whose Monge-Ampère measure $(dd^c P(h))^n$ is supported on the contact set $\{z \in \Omega \mid P(h)(z) = h(z)\}$.*

Proof. By assumption there exists $C > 0$ such that $h \ge -C$, hence $P(h) \ge -C$. In particular $P(h) \in \mathrm{PSH}(\Omega) \cap L^\infty$. The fact that $(dd^c P(h))^n$ is supported on the contact set follows from a standard balayage argument if h is continuous. By assumption there exists a decreasing sequence (h_j) of continuous functions on $\bar{\Omega}$ such that $h_j \to h$ in capacity in Ω. Then $P(h_j) \searrow P(h)$ and

$$\int_\Omega (h_j - P(h_j))(dd^c P(h_j))^n = 0.$$

Also, we note that $(h_j - P(h_j))$ converges in capacity to $(h - P(h))$. It follows from [GZ17, Theorem 4.26] that $(h_j - P(h_j))(dd^c P(h_j))^n$ weakly converges to $(h - P(h))(dd^c P(h))^n$. Letting $j \to +\infty$ we thus obtain

$$\int_\Omega (h - P(h))(dd^c P(h))^n \le \liminf_{j \to +\infty} \int_\Omega (h_j - P(h_j))(dd^c P(h_j))^n = 0$$

which yields the desired result. □

We shall need the following maximum principle in the sequel:

Lemma 4.2. *Assume $u \le v$ are bounded psh functions on Ω. Then*

$$\mathbf{1}_{\{u=v\}}(dd^c u)^n \le \mathbf{1}_{\{u=v\}}(dd^c v)^n.$$

Proof. It follows from [GZ17, Corollary 3.28] that

$$(dd^c \max(u, v))^n \ge \mathbf{1}_{\{u \ge v\}}(dd^c u)^n + \mathbf{1}_{\{u<v\}}(dd^c v)^n,$$

Multiplying by $\mathbf{1}_{\{u=v\}}$ yields the desired inequality. □

Lemma 4.3. *Fix $u \in \mathcal{T}(\Omega)$, $p > 0$ and set $q = \frac{n+p}{n+1}$. Then for all $s > 0$,*

$$s^{n+p}\mathrm{Cap}(u \le -s) \le q^n \int_\Omega (-u)^p (dd^c u)^n.$$

Proof. Consider $v := P(-(-u)^q)$. Since u is bounded and $q > 1$, we have $Cu \le -(-u)^q$ for some constant $C > 0$. This yields $v \ge Cu$ and since $Cu \in \mathcal{T}(\Omega)$ we also have that $v \in \mathcal{T}(\Omega) \subset \mathcal{E}^1(\Omega)$. We then set $w = -(-v)^{1/q}$ and $D := \{x \in \Omega \mid v = -(-u)^q\}$. Since $w \le u$ with equality on D it follows from Lemma 4.2 that

$$\mathbf{1}_D (dd^c w)^n \le \mathbf{1}_D (dd^c u)^n.$$

A direct computation shows that $dd^c w \geq q^{-1}(-v)^{1/q-1} dd^c v$, hence

$$\mathbf{1}_D q^{-n}(-v)^{\frac{n(1-q)}{q}}(dd^c v)^n \leq \mathbf{1}_D (dd^c u)^n.$$

By [Ceg04, Theorem 2.1] there exists a sequence (u_j) of psh functions which are continuous in $\bar{\Omega}$ and $u_j \searrow u$. In particular, $-(-u_j)^q \searrow -(-u)^q$ and $-(-u_j)^q$ converges in capacity to $-(-u)^q$ (see [GZ17, Proposition 4.25]). We can thus apply Lemma 4.1 to deduce that $(dd^c v)^n$ is supported on D and we infer that

$$(-v)(dd^c v)^n \leq \mathbf{1}_D q^n (-v)^{1+\frac{n(q-1)}{q}}(dd^c u)^n = \mathbf{1}_D q^n (-u)^p (dd^c u)^n.$$

Integrating on Ω gives $E_1(v) \leq q^n E_p(u)$.

We now use the simple fact that $\{u \leq -s\} \subseteq \{v \leq -s^q\}$ together with [ACKPZ09, Lemma 2.2] applied to the function $v \in \mathcal{E}^1(\Omega)$ to obtain

$$\mathrm{Cap}(u \leq -s) \leq \mathrm{Cap}(v \leq -s^q) \leq s^{-q(n+1)} E_1(v) \leq s^{-n-p} q^n E_p(u),$$

finishing the proof. □

The following is a local analogue of Proposition 2.3:

Proposition 4.4. *If $p > 0$ and $u \in \mathcal{E}^p(\Omega)$ then $P(-(-u)^{1+p/n}) \in \mathcal{F}(\Omega)$.*

Proof. Let u_j be a sequence in $\mathcal{T}(\Omega)$ such that $u_j \searrow u$ and

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty.$$

Set $v_j := P(-(-u_j)^q)$, $D := \{v_j = -(-u_j)^q\}$, where $q = 1 + p/n$. Since

$$dd^c(-(-u_j)^q) = q|u_j|^{q-1} dd^c u_j - q(q-1)|u_j|^{q-2} du_j \wedge d^c u_j \leq q|u_j|^{q-1} dd^c u_j,$$

it follows from Lemma 4.1 that

$$(dd^c v_j)^n \leq \mathbf{1}_D q^n (-u_j)^{n(q-1)} (dd^c u_j)^n \leq q^n (-u_j)^p (dd^c u_j)^n.$$

Thus

$$\sup_j \int_\Omega (dd^c v_j)^n \leq q^n \sup_j E_p(u_j) < +\infty.$$

Now $v_j \in \mathcal{T}(\Omega)$ and $v_j \searrow v := P(-(-u)^q)$, hence v belongs to $\mathcal{F}(\Omega)$. □

We will also need the following energy estimate:

Lemma 4.5. *Fix $p \geq 1$. If $u, v \in \mathcal{F}^p(\Omega)$ and $u \leq v$ then $E_p(v) \leq 2^{n+p} E_p(u)$.*

Proof. Observe that $\{v < -2t\} \subseteq \{2u < v - 2t\} \subseteq \{u < -t\}$.Therefore

$$\begin{aligned}
E_p(v) &= p\int_0^{+\infty} t^{p-1}(dd^c v)^n(v < -t)dt = p2^p \int_0^{+\infty} t^{p-1}(dd^c v)^n (v < -2t)dt \\
&\leq p2^p \int_0^{+\infty} t^{p-1}(dd^c v)^n (2u < v - 2t)dt \\
&\leq p2^{p+n} \int_0^{+\infty} t^{p-1}(dd^c u)^n (2u < v - 2t)dt \\
&\leq p2^{n+p} \int_0^{+\infty} t^{p-1}(dd^c u)^n (u < -t)dt = 2^{n+p} E_p(u),
\end{aligned}$$

where in the second inequality we have used the comparison principle [Ceg98, Lemma 4.4]. □

4.2. **Moser-Trudinger-Adams inequality.**

Let $H_0^{1,2}$ be the Sobolev space of L^2-functions with gradient in L^2, completion of the space $\mathcal{D}(\Omega)$ of smooth functions with compact support in Ω.

The classical Moser-Trudinger inequality asserts that if $\Omega \subset \mathbb{R}^2$ has bounded area and u belongs to the unit ball of $H_0^{1,2}$, then

$$\int_\Omega \exp(2\pi u^2) dV_{eucl} \leq C_1 \text{Area}(\Omega),$$

for some absolute constant $C_1 > 0$ (see [Tru67, Mos71]).

This famous inequality has been generalized in various ways. Adams [Ada88] notably showed that if $\Omega \subset \mathbb{R}^4$ has finite volume and $u \in \mathcal{D}(\Omega)$ satisfies $\int_\Omega (\Delta u)^2 dV \leq 1$, then

$$\int_\Omega \exp(32\pi^2 u^2)\, dV_{eucl} \leq C_2 \text{Vol}(\Omega).$$

Identifying $\mathbb{R}^2 \simeq \mathbb{C}$ and $\mathbb{R}^4 \simeq \mathbb{C}^2$, we propose here yet another type of generalization of these inequalities, replacing the average bound on Δu by a plurisubharmonicity condition. Let us emphasize that the psh condition can be seen as a one-sided bound $dd^c u \geq 0$, that is exactly $\Delta u \geq 0$ in complex dimension 1.

Theorem 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Fix $p > 0$ and $0 < \gamma < \frac{2n(n+1)}{n+p}$. There exists $C_\gamma > 0$ such that*

$$\int_\Omega \exp\left(\gamma E_p(u)^{-1/n} |u|^{1+\frac{p}{n}}\right) dV \leq C_\gamma \tag{4.1}$$

for any non-constant $u \in \mathcal{E}^p(\Omega)$.

Theorem 4.6 generalizes an important result of Berman-Berndtsson [BB11, Theorem 1.5] which treats the case $p = 1$ and provides the sharp constant $\gamma = 2n$ (beware of the different normalization for d^c in [BB11]). The proof of [BB11, Theorem 1.5] uses induction on dimension and the "thermodynamical formalism" introduced in [Ber13]. This approach does not seem to work for other energies E_p, $p \neq 1$. Similar estimates have been established by many authors using various techniques (see [Ceg19], [AC19], [BB14], [WWZ20, Theorem 1.1] and the references therein).

Remark 4.7. Note that $E_p(u) > 0$ since $u \leq 0$ is non-constant. Indeed if $E_p(u) = 0$ then $(dd^c u)^n = 0$ in Ω. By uniqueness of solutions to the complex Monge-Ampère equation [Ceg98, Theorem 4.5], we would then get $u \equiv 0$.

Remark 4.8. A scaling argument insures that the exponent $-1/n$ of the energy E_p in (4.1) is optimal. Indeed $E_p(su) = s^{p+n} E_p(u)$ hence

$$E_p(su)^{-1/n} |su|^{1+\frac{p}{n}} = E_p(u)^{-1/n} |u|^{1+\frac{p}{n}}.$$

Proof. We first assume that $u \in \mathcal{T}(\Omega)$. For convenience we set $A := E_p(u)$, $q = \frac{n+p}{n+1}$ and $r = 1 + \frac{p}{n}$. It follows from [ACKPZ09, Proposition 6.1] that for all $\beta < 2n$, there exists $C_\beta > 0$ such that

$$\text{Vol}(u < -t) \leq C_\beta \exp\left(-\frac{\beta}{\text{Cap}(u < -t)^{1/n}}\right) \leq C_\beta \exp\left(-\frac{\beta t^r}{q A^{1/n}}\right),$$

where the last inequality follows from Lemma 4.3. We infer

$$\begin{aligned}\int_\Omega \exp(\gamma'|u|^r)dV &= \int_0^{+\infty} r\gamma' t^{r-1} e^{\gamma' t^r} \mathrm{Vol}(u < -t)dt \\ &\leq C'_\beta \int_0^{+\infty} t^{r-1} \exp\left(\left[\gamma' - \frac{\beta}{qA^{1/n}}\right] t^r\right) dt < +\infty\end{aligned}$$

as long as $\gamma' := \gamma A^{-1/n} < \beta q^{-1} A^{-1/n}$, i.e. $\gamma < \frac{2n(n+1)}{n+p}$.

This proves the statement for $u \in \mathcal{T}(\Omega)$. We next use an approximation argument to treat the general case of $u \in \mathcal{E}^p(\Omega)$. By definition there exists a sequence $u_j \in \mathcal{T}(\Omega)$ such that $u_j \searrow u$ and

$$\sup_j E_p(u_j) := \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty.$$

By [BGZ09, Theorem 3.4], $E_p(u_j) \to E_p(u)$. Using the first step and Fatou's lemma we conclude the proof. □

In the case $p = n$ we obtain the following higher dimensional complex version of Adam's inequality:

Corollary 4.9. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Fix $0 < \gamma < n+1$. There is $C_\gamma > 0$ such that for all $u \in \mathcal{E}^n(\Omega)$ with $E_n(u) \leq 1$,*

$$\int_\Omega e^{\gamma u^2}\, dV \leq C_\gamma.$$

Arguing as in the proof of Corollary 2.2, we obtain the following version of the Moser-Trudinger inequality:

Corollary 4.10. *Fix $p > 0$, $\varepsilon > 0$ and $A = \frac{p}{n+p}\left(\frac{1}{2(n+1)}\right)^{\frac{n}{p}}$. There exists a uniform constant $B > 0$ depending on ε such that for all $u \in \mathcal{E}^p(\Omega)$,*

$$\log \int_\Omega e^{-u} dV \leq (A+\varepsilon) E_p(u)^{\frac{1}{p}} + B.$$

When $p = 1$ the constant A is sharp as shown in [BB11, Theorem 1.5]. A similar result (with a less precise constant) has been established in [AC19, Theorem 4.1] by a different method. It was also observed in [AC19] that the constant A can not be smaller than $\frac{p}{n+p}\left(\frac{1}{2(n+p)}\right)^{\frac{n}{p}}$ (beware of the different normalizations of dd^c in [BB11, AC19] and in the present article !).

Proof. The proof is the same as that of Corollary 2.2. In particular (2.2) with $k = 1$ gives

$$-u(x) \leq \frac{pa^{1+\frac{n}{p}} E_p(u)^{\frac{1}{p}}}{n+p} + \frac{n(-u(x))^{1+\frac{p}{n}}}{(n+p)a^{1+\frac{p}{n}} E_p(u)^{\frac{1}{n}}}. \tag{4.2}$$

Fix $0 < c < c_0 := \frac{2n(n+1)}{n+p}$. Then

$$a := \left(\frac{n}{(n+p)c}\right)^{\frac{n}{n+p}} > \left(\frac{1}{2(n+1)}\right)^{\frac{n}{n+p}},$$

and

$$A' := \frac{pa^{1+\frac{n}{p}}}{n+p} > \frac{p}{n+p}\left(\frac{1}{2(n+1)}\right)^{\frac{n}{p}} = A,$$

with equalities when $c = c_0$. The inequality (4.2) can then be rewritten as

$$-u(x) \leq A' E_p(u)^{\frac{1}{p}} + cE_p(u)^{-\frac{1}{n}}(-u(x))^{1+\frac{p}{n}}.$$

Now Theorem 4.6 ensures that

$$\log \int_\Omega e^{-u} dV \leq A' E_p(u)^{\frac{1}{p}} + B.$$

For any $\varepsilon > 0$ we can choose c so close to $\frac{2n(n+1)}{n+p}$ that $A' \leq A + \varepsilon$. □

4.3. **Finite entropy potentials.** Let $\mu = f dV \geq 0$ be a probability measure on Ω with finite entropy, i.e. $\mathrm{Ent}(f) := \int_\Omega f \log f dV < +\infty$. Cegrell [Ceg04, lemma 5.14] has shown that there exists a unique psh function $\varphi \in \mathcal{F}(\Omega)$ such that $(dd^c\varphi)^n = \mu$. We show here that φ belongs to an appropriate finite energy class:

Theorem 4.11. *The function φ belongs to $\mathcal{E}^p(\Omega)$ for all $0 < p \leq \frac{n}{n-1}$. Moreover, there exist $c, C > 0$ depending on n, p, Ω and $\mathrm{Ent}(f)$ such that*

$$E_p(\varphi) \leq C \quad \textit{and} \quad \int_\Omega e^{c|\varphi|^p} \omega^n \leq C.$$

Let us stress that the RHS integral estimate has been obtained with completely different methods by Wang-Wang-Zhou [WWZ20, Theorem 3.2].

Proof. By Hölder inequality it suffices to prove the result for $p = \frac{n}{n-1}$. We approximate f by $f_j := \min(f, j)$ and observe that $f_j dV \leq dd^c(bj^{1/n}|z|^2)^n$, where $b > 0$ is a normalization constant such that $dV = b^n \, (dd^c|z|^2)^n$.

Note that for each $j \in \mathbb{N}$, f_j still has finite entropy and $\int_\Omega f_j dV < +\infty$. By [Ceg98, Proposition 6.1] there exists a unique $\varphi_j \in \mathcal{F}^1(\Omega) \cap L^\infty(\Omega)$ such that $(dd^c\varphi_j)^n = f_j dV$. The comparison principle [Ceg98, Theorem 4.5] insures that $j \mapsto \varphi_j$ is decreasing and the same argument as in the proof of Theorem 4.6 shows that φ_j decreases to φ. Now Theorem 4.6 yields

$$\int_\Omega \exp(\gamma E_p(\varphi_j)^{-1/n} |\varphi_j|^p) dV \leq C_1,$$

where C_1 depends on γ. Applying Hölder-Young inequality (3.2) we obtain

$$\begin{aligned}
&\int_\Omega \gamma E_p(\varphi_j)^{-1/n} |\varphi_j|^p f_j dV \\
&\leq \int_\Omega \left(e^{\gamma E_p(\varphi_j)^{-1/n}|\varphi|^p} - \gamma E_p(\varphi_j)^{-1/n} |\varphi_j|^p - 1 \right) dV \\
&\quad + \int_\Omega (f_j + 1) \log(f_j + 1) dV - \int_\Omega f_j dV \leq C_2,
\end{aligned}$$

where C_2 depends on C_1 and $\mathrm{Ent}(f)$. Thus $\gamma E_p(\varphi_j)^{1-1/n} \leq C_2$, hence $E_p(\varphi_j) \leq C_3$ where C_3 depends on C_1 and $\mathrm{Ent}(f)$. Using [BGZ09, Theorem 3.4] it thus follows that $E_p(\varphi) \leq C_3$. Combining the above upper bound with Theorem 4.6 yields

$$\int_\Omega \exp\left(\gamma C_2^{-1/n} |\varphi|^p\right) dV \leq C,$$

as desired. □

We finally establish a local analogue of Theorem 3.5:

Theorem 4.12. *Let $0 \leq f_j$ be a sequence of densities with uniformly bounded entropy. Let $u_j \in \mathcal{F}^{\frac{n}{n-1}}(\Omega)$ be the unique solutions to $(dd^c u_j)^n = f_j dV$ and fix $0 < p < \frac{n}{n-1}$. There exist $u \in \mathcal{F}^p(\Omega)$ and a subsequence, still denoted by u_j, such that $\|u_j - u\|_{L^1(\Omega)} \to 0$ and*

$$\lim_{j\to+\infty} \int_\Omega |u_j - u|^p((dd^c u_j)^n + (dd^c u)^n) = 0. \tag{4.3}$$

In particular $(dd^c u_j)^n$ weakly converges to $(dd^c u)^n$.

Example 3.6 shows that (4.3) does not hold for $p = \frac{n}{n-1}$.

Proof. Fix $0 < p < r := \frac{n}{n-1}$. It follows from Theorem 4.11 that $E_r(u_j)$ and $\int_\Omega e^{c|u_j|^r}$ are uniformly bounded. Thus up to extracting and relabelling, u_j converges in $L^1_{\rm loc}$ and a.e. to $u \in \mathrm{PSH}(\Omega)$. If we set $v_k := (\sup_{j\geq k} u_j)^*$ then $v_k \searrow u$ and by Lemma 4.5 $E_r(v_k) \leq 2^{n+r}E_r(u_k)$ is uniformly bounded. It then follows from [BGZ09, Theorem 3.4] that $u \in \mathcal{F}^r(\Omega)$. The Hölder inequality then ensures that $u \in \mathcal{F}^p(\Omega)$ for any $p < r$. It follows from Theorem 4.6 that, for some constant $\gamma > 0$, $\int_\Omega (e^{\gamma|u_j|^r} + e^{\gamma|u|^r})dV$ is uniformly bounded. It then follows that $\int_\Omega |u_j - u|dV \to 0$.

It follows from [Per99, Theorem 3.4] that for all $\varphi, \psi \in \mathcal{T}(\Omega)$,

$$\int_\Omega |\varphi|^r (dd^c\psi)^n \leq D_{n,r} E_r(\varphi)^{r/(n+r)} E_r(\psi)^{n/(n+r)}.$$

By an approximation procedure one can show that the above inequality holds for $\varphi, \psi \in \mathcal{F}^r(\Omega)$ as well. Using this for u_j and u we conclude that $\int_\Omega |u_j - u|^r((dd^c u_j)^n + (dd^c u)^n)$ is uniformly bounded.

We next prove that $\int_\Omega |u_j - u|^p (dd^c u_j)^n \to 0$. Fixing $\varepsilon > 0$, by Egorov's theorem there exists a Borel subset $G \subset \Omega$ with $\mathrm{Vol}(G) < \varepsilon$ such that u_j converges uniformly to u in $\Omega \setminus G$. We then have

$$\int_{\Omega\setminus G} |u_j - u|^p f_j dV \to 0$$

since $\int_\Omega (dd^c u_j)^n dV$ is uniformly bounded. By Hölder's inequality we have

$$\int_G |u_j - u|^p f_j dV \leq \left(\int_G |u_j - u|^r f_j dV\right)^{p/r} \left(\int_G f_j dV\right)^q$$

with $q := \frac{r}{r-p}$. The first factor is uniformly bounded thanks to the above. Fix now $t > 1$. We estimate the second factor as follows:

$$\int_{G\cap\{f_j\leq t\}} f_j dV \leq t\mathrm{Vol}(G) \leq t\varepsilon,$$

$$\int_{G\cap\{f_j> t\}} f_j dV \leq \frac{1}{\log t}\int_\Omega f_j \log f_j dV \leq \frac{C}{\log t}.$$

Letting $\varepsilon \to 0$ and then $t \to +\infty$ we see that $\int_\Omega |u_j - u|^p (dd^c u_j)^n \to 0$. From this and [Ceg98, Lemma 5.3] we deduce that $(dd^c u_j)^n$ weakly converges to $(dd^c u)^n$. Note that in [Ceg98, Lemma 5.3] it was assumed that $u_j \in \mathcal{T}(\Omega)$ is continuous for all j but the proof does apply to our more general setting. This together with lower semicontinuity of the entropy reveal that $(dd^c u)^n$ also has finite entropy. We can thus repeat the same arguments as above to conclude that $\int_\Omega |u_j - u|^p (dd^c u)^n \to 0$, finishing the proof.

□

References

[Ada88] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. (2) **128** (1988), no. 2, 385–398.

[ACKPZ09] P. Åhag, U. Cegrell, S. Kołodziej, H. H. Phạm, and A. Zeriahi, *Partial pluricomplex energy and integrability exponents of plurisubharmonic functions*, Adv. Math. **222** (2009), no. 6, 2036–2058.

[AC19] P. Åhag, R. Czyż, *On the Moser-Trudinger inequality in complex space.* J. Math. Anal. Appl. **479** (2019), no. 2, 1456–1474.

[Aub84] T. Aubin, *Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kahlériennes compactes à la démonstration d'une inégalité*, J.F.A. **57** (1984), 143–153.

[BT76] E. Bedford, B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation.* Invent. Math. **37** (1976), no. 1, 1–44.

[BT82] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40.

[BGZ09] S. Benelkourchi, V. Guedj, A. Zeriahi, *Plurisubharmonic functions with weak singularities*, Complex analysis and digital geometry, Acta Univ. Upsaliensis, Skr. Uppsala Univ. C Organ. Hist., vol. 86, Uppsala Universitet, Uppsala, 2009, 57–74.

[Ber13] R.J. Berman, *A thermodynamical formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kähler-Einstein metrics*. Adv. Math. **248** (2013), 1254–1297.

[Ber19] R. J. Berman, *From Monge-Ampère equations to envelopes and geodesic rays in the zero temperature limit*, Math. Z. **291** (2019), no. 1-2, 365–394.

[BB11] R. J. Berman, B. Berndtsson, *Moser-Trudinger type inequalities for complex Monge-Ampère operators and Aubin's hypothèse fondamentale.* arXiv:1109.1263

[BB14] R. J. Berman and B. Berndtsson, *Symmetrization of plurisubharmonic and convex functions.* Indiana Univ. Math. J. **63** (2014), no. 2, 345-365.

[BBEGZ19] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties*, J. Reine Angew. Math. **751** (2019), 27–89.

[BBGZ13] R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi, *A variational approach to complex Monge-Ampère equations*, Pub.Math. I.H.E.S. **117** (2013), 179–245.

[BBJ15] R. J. Berman, S. Boucksom, and M. Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, arXiv:1509.04561 (2015).

[BDL17] R. J. Berman, T. Darvas, and C. H. Lu, *Convexity of the extended K-energy and the large time behavior of the weak Calabi flow*, Geom. Topol. **21** (2017), no. 5, 2945–2988.

[BDL20] R. J. Berman, T. Darvas, and C. H. Lu, *Regularity of weak minimizers of the K-energy and applications to properness and K-stability.*, Ann. Sci. École Norm. Sup. **53**, 2020, 267–289.

[BD12] R. J. Berman and J.-P. Demailly, *Regularity of plurisubharmonic upper envelopes in big cohomology classes*, Perspectives in analysis, geometry, and topology, Progr. Math., 296, Birkhäuser/Springer, New York, 2012, 39–66.

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. **205** (2010), no. 2, 199–262.

[BK07] Z. Blocki, S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds.* Proc. Amer. Math. Soc. **135** (2007), no. 7, 2089–2093.

[CC17] X.X. Chen and J. Cheng, *On the constant scalar curvature Kähler metrics, apriori estimates*, arXiv:1712.06697 (2017).

[CC18] X.X. Chen and J. Cheng, *On the constant scalar curvature Kähler metrics, existence results*, arXiv:1801.00656 (2018).

[Ceg98] U. Cegrell, *Pluricomplex energy*, Acta Math. **180** (1998), no. 2, 187–217.

[Ceg04] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159–179.

[Ceg19] U. Cegrell, *Measures of finite pluricomplex energy.* Ann. Polon. Math. **123** (2019), no. 1, 203-213.

[Dar15] T. Darvas, *The Mabuchi geometry of finite energy classes*, Adv. Math. **285** (2015), 182–219.

[Dem92] J.P. Demailly, *Regularization of closed positive currents and interSection theory.* J. Algebraic Geom. **1** (1992), no. 3, 361–409.

[Dn15] E. Di Nezza, *Finite Pluricomplex energy measures*, Potential Analysis, Vol. 44 (2015), no. 1, 155-167.

[Din09] S. Dinew, *Uniqueness in* $\mathcal{E}(X,\omega)$, J. Funct. Anal. **256** (2009), no. 7, 2113–2122.

[Don18] S. Donaldson, *Some recent developments in Kähler geometry and exceptional holonomy*, Proc. Int. Congress of Mathematicians, Rio de Janeiro 2018. Vol. I. Plenary lectures, 425-451, World Sci. Publ., Hackensack, NJ, 2018.

[EGZ11] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Viscosity Solutions to Degenerate Complex Monge-Ampère Equations* , Com.Pure Appl.Math. **64** (2011), no. 8, 1059–1094.

[GZ05] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. **15** (2005), no. 4, 607–639.

[GZ07] V. Guedj and A. Zeriahi, *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. **250** (2007), no. 2, 442–482.

[GZ17] V. Guedj and A. Zeriahi, *Degenerate complex Monge-Ampère equations*, EMS Tracts in Mathematics, vol. 26, European Mathematical Society (EMS), Zürich, 2017.

[Mos71] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[Per99] L. Persson, *A Dirichlet principle for the complex Monge-Ampère operator*. Ark. Mat. **37** (1999), no. 2, 345–356.

[Tru67] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[WWZ20] J. Wang, X.-J. Wang, B. Zhou, *Moser-Trudinger inequality for the complex Monge-Ampère equation.* Preprint arXiv:2003.06056.

Institut de Mathématiques de Jussieu, Sorbonne Université, 4 place Jussieu, 75005 Paris, France

Email address: eleonora.dinezza@imj-prg.fr

URL: https://sites.google.com/site/edinezza/home

Institut de Mathématiques de Toulouse, Université de Toulouse, 118 route de Narbonne, 31400 Toulouse, France,

Email address: vincent.guedj@math.univ-toulouse.fr

URL: https://www.math.univ-toulouse.fr/~guedj/

Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405, Orsay, France.

Email address: hoang-chinh.lu@universite-paris-saclay.fr

URL: https://www.imo.universite-paris-saclay.fr/~lu/