COMPLEX HESSIAN EQUATIONS WITH PRESCRIBED SINGULARITY ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT. Let (X, ω) be a compact Kähler manifold of dimension n and fix $1 \leq m \leq n$. We prove that the total mass of the complex Hessian measure of ω-m-subharmonic functions is non-decreasing with respect to the singularity type. We then solve complex Hessian equations with prescribed singularity, and prove a Hodge index type inequality for positive currents.

CONTENTS

1. INTRODUCTION

Let (X, ω) be a compact Kähler manifold of dimension n and fix an integer m such that $1 \leq m \leq n$. For convenience we normalize ω by $\int_X \omega^n = 1$.

In this paper we study complex Hessian equations of the form

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = \mu, \tag{1.1}$$

where μ is a positive measure, and we want to solve the equation for u in a given singularity class.

The case when $m = n$ (the Monge-Ampère case) has numerous important applications in differential geometry, see [2, 65, 43], to only cite a few. The complex Hessian equation appears in the study of the Fu-Yau equation related to the Strominger system [54, 55, 56]. It is also motivated by the study of the Calabi problem for HKT-manifolds [1]. Its real counterpart, the real Hessian equation, was studied intensively with many interesting applications [10, 61, 13].

After several attempts [42], [39], [41], the existence of smooth solutions in the smooth case (when $\mu = e^f \omega^n$, for some smooth function f) was solved [28] by combining a Liouville type theorem for m-subharmonic functions [28] and a second order a priori estimate [40]. This idea was recently used in [60], [14] to solve the Dirichlet problem for complex Hessian equations on complex manifolds. Weak solutions were studied in [27, 29], [34], [44], [47, 50] and many others.

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In [50], we have developed a global potential theory for ω-m-subharmonic functions, solving (1.1) in the full mass class $\mathcal{E}(X,\omega,m)$. This class consists of functions with very mild singularity, e.g. in case $n=m$, these have zero Lelong number everywhere. In this paper we extend the study of [50] to classes of ω-m-sh functions with heavy singularities, inspired by [20, 22, 23]. To do this, we first need a monotonicity result which is the first main result of this paper.

Theorem 1.1. *Assume that $u_1,...,u_m,v_1,...,v_m$ are ω-m-sh functions on X such that $u_p\leq v_p$, for all $p\in\{1,...,m\}$. Then*

$$\int_X H_m(u_1,...,u_m)\leq \int_X H_m(v_1,...,v_m).$$

Here $H_m(u_1,...,u_m):=(\omega+dd^c u_1)\wedge...\wedge(\omega+dd^c u_m)\wedge\omega^{n-m}$ is the non-m-polar product; the relevant definitions will be given in Section 2.

For $n=m$, the above result was conjectured in [9] in the general context of big cohomology classes, and proved in [62]. The approach of [62] was recently used in [63] to prove an integration by parts formula. Our proof of Theorem 1.1 uses the monotonicity of the Hessian energy avoiding the geodesic notion which is not yet avaliable in the Hessian setting.

Having the monotonicity result and using recent techniques in [20, 22] we study the complex Hessian equation with prescribed singularities. The second main result is the following:

Theorem 1.2. *Assume that ϕ is a ω-m-sh function such that $P[\phi]=\phi$. Let μ be a non-m-polar positive measure such that $\mu(X)=\int_X H_m(\phi)>0$. Then there exists a unique $u\in\mathscr{E}_\phi$ normalized by $\sup_X u=0$, such that $H_m(u)=\mu$.*

The definition of the envelope $P[u]$, and the relative finite energy class $\mathscr{E}_\phi$ will be given in Section 3.2. One can prove the uniqueness of solution by slightly modifying the proof of S. Dinew in the Monge-Ampère case (see [26, 30]), which crucially uses the resolution of the equation. We propose in this paper an alternative proof using the fact that the Hessian measure of the envelope is supported on the contact set. To prove the existence of solutions we use the supersolution method of [35] as in [22]: we take the lower envelope of supersolutions. To do so, we need to bound the supersolutions from below. This was done in [22] by establishing a relative L^∞-estimate which is quite delicate in the Hessian setting due to a lack of integrability of ω-m-subharmonic functions. We overcome this by constructing ω-m-subharmonic subextensions via a complete metric in the space $\mathscr{E}^1$, inspired by [15, 17, 19].

Using the resolution of the complex Hessian equations with prescribed singularity we prove a Hodge-index type inequality for positive closed $(1,1)$-currents.

Theorem 1.3. *Let $u_j, j=1,...,m$ be ω-m-subharmonic functions on X. Then*

$$\int_X H_m(u_1,...,u_m)\geq \prod_{k=1}^m\left(\int_X H_m(u_k)\right)^{1/m}.$$

The above result generalizes that of [22] which considers the case $m=n$, and [64] which considers smooth forms. Other directions can also be explored to extend the above result to the case of big cohomology classes. The proof of Theorem 1.3 is an obvious modification of the Monge-Ampère case (see [20, 22]) given Theorem 1.1 and Theorem 1.2.

Organization of the paper. In Section 2 we recall backgrounds on ω-m-subharmonic functions and the complex Hessian operator. The relative potential theory adapted to the Hessian setting is discussed in Section 3, where we prove Theorem 1.1 in Section 3.1 (Theorem 3.4). We use the metric defined in Section 4 to establish the existence of solutions in Section 5, where Theorem 1.2 is proved (Theorem 5.4). The uniqueness is given a new proof in Section 5.3. Theorem 1.3 is proved in Section 5.5.

2. Backgrounds

Let (X, ω) be a compact Kähler manifold of dimension n, and fix an integer m such that $1 \leq m \leq n$.

2.1. ω-m-subharmonic functions.

In this section, we recall backgrounds on m-subharmonic functions on domains as well as on compact Kähler manifolds. Many properties of the complex Hessian operator can be proved by easy adaptations of the Monge-Ampère case. More details on several classes of m-subharmonic functions can be found in [46, 8, 59, 30, 12, 49, 50, 27, 28, 52, 51, 34, 31, 53, 45] and the references therein.

Fix Ω an open subset of $\mathbb{C}^n$ and $\beta := dd^c\rho$ a Kähler form in Ω with smooth bounded potential.

Definition 2.1. A function $u \in C^2(\Omega, \mathbb{R})$ is called m-subharmonic (m-sh for short) with respect to β if the following inequalities hold in Ω :

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \forall k \in \{1, ..., m\}.$$

Definition 2.2. A function $u \in L^1(\Omega, \mathbb{R})$ is called m-subharmonic with respect to β if

(1) u is upper semicontinuous in Ω,
(2) $dd^c u \wedge dd^c u_2 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m} \geq 0$, for all $u_2, ..., u_m \in C^2(\Omega)$, m-sh with respect to β,
(3) if $v \in L^1(\Omega)$ satisfies the above two conditions and $u = v$ a.e. in Ω then $u \leq v$.

As observed by Błocki [8], Gårding's inequality [33] ensures that the two definitions of m-sh functions above coincide for smooth functions.

Definition 2.3. A function $u \in L^1(X, \omega^n)$ is called ω-m-subharmonic (ω-m-sh for short) if, locally in $\Omega \subset X$ where $\omega = dd^c\rho$, $u + \rho$ is m-subharmonic with respect to ω.

The set of all ω-m-sh functions on X is denoted by $\mathrm{SH}_m(X, \omega)$.

The above definition depends heavily on the Kähler form ω. This makes the smooth approximation of ω-m-subharmonic functions quite complicated unless ω is flat. Nevertheless, it was shown in [50], [44] using the viscosity theory and an approximation scheme of Berman [7], and in [57], [38] using the local smooth resolution, that the smooth approximation of m-subharmonic functions is possible. As mentioned in [38], the global approximation theorem in [50] yields the local one. A direct proof of the local approximation property (which is also valid in the Hermitian setting) was given in [34, Theorem 3.18].

Given $u, v \in \mathrm{SH}_m(X, \omega)$, we say that u is less singular than v if there exists a constant C such that $v \leq u + C$. We say that u has the same singularity as v if there exists a constant C such that $u - C \leq v \leq u + C$.

In the flat case, Błocki proved in [8] that m-sh functions are in L^p for any $p < n/(n-m)$, and conjectured that it holds for $p < nm/(n-m)$. Using the L^∞ estimate due to S. Dinew and Kołodziej, one can prove the same integrability property for ω-m-sh functions, see [27], [50, Corollary 6.7].

2.2. **Complex Hessian operator.** Given bounded ω-m-sh functions u_1, ..., u_m the complex Hessian operator

$$H_m(u_1,...,u_m) := (\omega + dd^c u_1) \wedge ... \wedge (\omega + dd^c u_m) \wedge \omega^{n-m}$$

is defined recursively by following Bedford-Taylor's seminal works [3, 4]. This gives a positive Borel measure and H_m enjoys many nice convergence properties (see [50],[47],[34]). When $u_1 = ... = u_m = u$ we simply denote the m-Hessian measure of u by $H_m(u)$.

By plurifine locality (see [27, 28, 47, 50]) we have the following property:

$$\mathbf{1}_U H_m(\max(u_1,v_1),...,\max(u_m,v_m)) = \mathbf{1}_U H_m(u_1,...,u_m),$$

where $u_1,..,u_m$, $v_1,...,v_m$ are bounded ω-m-sh functions, and $U := \cap_{j=1}^m \{u_j > v_j\}$.

For a Borel set $E \subset X$ we define

$$\mathrm{Cap}_m(E) := \sup \left\{ \int_E H_m(u) \mid u \in \mathrm{SH}_m(X,\omega), \ -1 \leq u \leq 0 \right\}.$$

A sequence of functions u_j converges in capacity to u if for all $\varepsilon > 0$,

$$\lim_{j\to+\infty} \mathrm{Cap}_m(|u_j - u| > \varepsilon) = 0.$$

Given $u_1,...,u_m \in \mathrm{SH}_m(X,\omega)$, not necessarily bounded, and $s > t$ we have

$$\mathbf{1}_{U^s} H_m(u_1^t,...,u_m^t) = \mathbf{1}_{U^s} H_m(u_1^s,...,u_m^s),$$

where $U^s := \cap_{p=1}^m \{u_p > -s\}$ and $u^s := \max(u,-s)$. It thus follows that the family of positive measures $\mathbf{1}_{U^s} H_m(u_1^s,...,u_m^s)$ is increasing in s, allowing to define

$$H_m(u_1,...,u_m) := \lim_{s\to+\infty} \mathbf{1}_{U^s} H_m(u_1^s,...,u_m^s).$$

When $u_1 = ... = u_m = u$ we simply denote the Hessian measure $H_m(u,u,...,u)$ by $H_m(u)$. An application of the Stokes theorem gives

$$0 \leq \int_X H_m(u) \leq 1.$$

A Borel set E is called m-polar (with respect to ω) if there exists $u \in \mathrm{SH}_m(X,\omega)$ such that $E \subset \{u = -\infty\}$.

Lemma 2.4. *The positive measure $H_k(u)$ does not charge m-polar sets.*

Proof. If $v \in \mathrm{SH}_m(X,\omega)$ is bounded then $(2\omega + dd^c v)^m \wedge (2\omega)^{n-m}$ vanishes on m-polar sets (see [47, 50]). Since

$$(2\omega + dd^c v)^m \wedge \omega^{n-m} = \sum_{k=0}^m \binom{m}{k} H_k(v),$$

it follows that $H_k(v)$ also vanishes on m-polar sets for $k = 1,...,m$. Each $H_k(u_j)$ does not charge m-polar sets because $u_j := \max(u,-j)$ is bounded. Since $H_k(u)$ is the strong limit of $\mathbf{1}_{\{u>-j\}} H_k(u_j)$ it follows that $H_k(u)$ vanishes on m-polar sets. □

Definition 2.5. A Borel set $E \subset X$ is called quasi-open (quasi-closed) if for each $\varepsilon > 0$, there exists an open (closed) set U such that

$$\mathrm{Cap}_m((E \setminus U) \cup (U \setminus E)) < \varepsilon.$$

Since ω-m-sh functions are quasi-continuous, see [47], the sets of the form

$$\cap_{j=1}^N \{u_j > v_j\},$$

where u_j, v_j are ω-m-sh functions, are quasi-open, while the corresponding sets with $\geq$ sign are quasi-closed.

Theorem 2.6. *Assume that $u_1^j, ..., u_m^j$ are sequences of ω-m-sh functions which are uniformly bounded. If u_p^j converges in m-capacity to $u_p \in \mathrm{SH}_m(X, \omega)$, for all $p = 1, ..., m$, then*

$$\liminf_j \int_E H_m(u_1^j, ..., u_m^j) \geq \int_E H_m(u_1, ..., u_m),$$

for all quasi-open set E, and

$$\limsup_j \int_K H_m(u_1^j, ..., u_m^j) \leq \int_K H_m(u_1, ..., u_m),$$

for all quasi-closed set K.

The proof of the above theorem is an obvious modification of the Monge-Ampère case, see [37], [21, Corollary 2.9].

The following result, called the plurifine locality, will be used several times in this paper.

Lemma 2.7. *Assume that $u_1, ..., u_m, v_1, ..., v_m$ are ω-m-sh functions on X and $\Omega \subset X$ is a quasi-open set such that $u_p = v_p$ on Ω, for $p = 1, ..., m$. Then*

$$\mathbf{1}_\Omega H_m(u_1, ..., u_m) = \mathbf{1}_\Omega H_m(v_1, ..., v_m).$$

Proof. The proof for bounded functions is classical, see [5, Corollary 4.3] and the discussion in [9, Section 1.2]. For convenience we repeat it here. For $\varepsilon > 0$ set $w_p^\varepsilon := \max(u_p + \varepsilon, v_p)$, $w_p := \max(u_p, v_p)$. Then $\Omega \subset \cap_{p=1}^m \{u_p + \varepsilon > v_p\}$, hence by the pluripotential maximum principle (see [47, Theorem 3.14], [37, Theorem 3.27]),

$$\mathbf{1}_\Omega H_m(w_1^\varepsilon, ..., w_m^\varepsilon) = \mathbf{1}_\Omega H_m(u_1, ..., u_m).$$

Since Ω is quasi open and the functions u_p, v_p are uniformly bounded, letting $\varepsilon \to 0^+$ we obtain

$$\mathbf{1}_\Omega H_m(w_1, ..., w_m) \leq \mathbf{1}_\Omega H_m(u_1, ..., u_m).$$

For a fixed compact subset $K \Subset \Omega$ we have

$$\mathbf{1}_K H_m(w_1^\varepsilon, ..., w_m^\varepsilon) = \mathbf{1}_K H_m(u_1, ..., u_m).$$

Letting $\varepsilon \to 0^+$ we arrive at

$$\mathbf{1}_K H_m(w_1, ..., w_m) \geq \mathbf{1}_K H_m(u_1, ..., u_m).$$

Since the Hessian measure $H_m(u_1, ..., u_m)$ is inner regular, we can conclude that

$$\mathbf{1}_\Omega H_m(w_1, ..., w_m) = \mathbf{1}_\Omega H_m(u_1, ..., u_m).$$

Changing the role of u_p and v_p we obtain the result for bounded functions.

For the general case we set $u_p^t := \max(u_p, -t)$, for $t > 0$. From the previous step we have

$$\mathbf{1}_\Omega \mathbf{1}_{U^t} H_m(u_1^t, ..., u_m^t) = \mathbf{1}_\Omega \mathbf{1}_{V^t} H_m(v_1^t, ..., v_m^t),$$

where $U^t := \cap_{p=1}^m \{u_p > -t\}$, $V^t := \cap_{p=1}^m \{v_p > -t\}$. Now, we let $t \to +\infty$ to conclude the proof. □

Corollary 2.8. *Assume that $u_1, ..., u_m, v_1, ..., v_m$ are ω-m-sh on X. Then*

$$\mathbf{1}_\Omega H_m(\max(u_1, v_1)..., \max(u_m, v_m)) = \mathbf{1}_\Omega H_m(u_1, ..., u_m),$$

where $\Omega := \cap_{p=1}^m \{u_p > v_p\}$.

Lemma 2.9. *If $u, v \in \mathrm{SH}_m(X, \omega)$ then*

$$H_m(\max(u, v)) \geq \mathbf{1}_{\{u>v\}} H_m(u) + \mathbf{1}_{\{u \leq v\}} H_m(v).$$

Proof. For $t > 0$ set $u^t := \max(u, -t), v^t := \max(v, -t), \phi^t := \max(u^t, v^t)$. Then

$$H_m(\phi^t) \geq \mathbf{1}_{\{u^t > v^t\}} H_m(u^t) + \mathbf{1}_{\{u^t \leq v^t\}} H_m(v^t).$$

Multiplying both sides with $\mathbf{1}_{U^t}$, where $U^t := \{\min(u, v) > -t\}$, and using Lemma 2.7, we obtain

$$\mathbf{1}_{U^t} H_m(\phi) = \mathbf{1}_{U^t} H_m(\phi^t) \geq \mathbf{1}_{U^t} \mathbf{1}_{\{u>v\}} H_m(u) + \mathbf{1}_{U^t} \mathbf{1}_{\{u \leq v\}} H_m(v).$$

Letting $t \to +\infty$ we arrive at the conclusion. □

Proposition 2.10. *If $u, v \in \mathrm{SH}_m(X, \omega)$ and $u \leq v$, then*

$$\mathbf{1}_{\{u=v\}} H_m(u) \leq \mathbf{1}_{\{u=v\}} H_m(v).$$

Intuitively, v can be thought of as an upper test function for u on the contact set $\{u = v\}$, see [32, 48] for more details on the viscosity theory.

Proof. We first assume that u, v are bounded. For $\varepsilon > 0$ set $u_\varepsilon := \max(u, v - \varepsilon)$. By Lemma 2.9 we have

$$\mathbf{1}_{\{u=v\}} H_m(u_\varepsilon) \geq \mathbf{1}_{\{u=v\}} \mathbf{1}_{\{u \geq v-\varepsilon\}} H_m(u) \geq \mathbf{1}_{\{u=v\}} H_m(u).$$

Since the set $\{u = v\}$ is quasi-closed, and u_ε is uniformly bounded, we can invoke Theorem 2.6 to get

$$\mathbf{1}_{\{u=v\}} H_m(v) \geq \limsup_{\varepsilon \to 0} \mathbf{1}_{\{u=v\}} H_m(u_\varepsilon) \geq \mathbf{1}_{\{u=v\}} H_m(u).$$

To treat the general case we set

$$u^t := \max(u, -t),\ v^t := \max(v, -t),\ U^t := \{u > -t\}.$$

The first step gives $\mathbf{1}_{U^t} \mathbf{1}_{\{u^t = v^t\}} H_m(v_t) \geq \mathbf{1}_{U^t} \mathbf{1}_{\{u^t = v^t\}} H_m(u^t)$. Using Lemma 2.7 we then have that $\mathbf{1}_{U^t} \mathbf{1}_{\{u=v\}} H_m(v) \geq \mathbf{1}_{U^t} \mathbf{1}_{\{u=v\}} H_m(u)$. We finally let $t \to +\infty$ to arrive at the conclusion. □

Lemma 2.11. *Assume that $u_1, ..., u_m$ are ω-m-sh on X and $t_1, ..., t_m \in [0, 1]$ with $\sum_{p=1}^m t_p = 1$. Then*

$$H_m\left(\sum_{p=1}^m t_p u_p\right) = \sum_{\sigma \in \Sigma} t_{\sigma(1)} ... t_{\sigma(m)} H_m(u_{\sigma(1)}, ..., u_{\sigma(m)}),$$

where Σ is the set of all maps $\sigma : \{1, ..., m\} \to \{1, ..., m\}$.

Proof. Fix $C > 0$ and set

$$U^C := \cap_{p=1}^m \{u_p > -C\},\ \phi := \sum_{p=1}^m t_p u_p,\ \phi^C := \max(\phi, -C).$$

Then $\phi > -C$ on U^C, hence by Lemma 2.7 we have

$$\begin{aligned}\mathbf{1}_{U^C} H_m(\phi) &= \mathbf{1}_{U^C} H_m\left(\sum_{p=1}^m t_p u_p^C\right)\\ &= \mathbf{1}_{U^C}\sum_{\sigma\in\Sigma} t_{\sigma(1)}...t_{\sigma(m)} H_m(u^C_{\sigma(1)},...,u^C_{\sigma(m)})\\ &= \mathbf{1}_{U^C}\sum_{\sigma\in\Sigma} t_{\sigma(1)}...t_{\sigma(m)} H_m(u_{\sigma(1)},...,u_{\sigma(m)}).\end{aligned}$$

Letting $C \to +\infty$ we arrive at the conclusion. □

Lemma 2.12 (Mixed Hessian inequality)**.** *Assume that μ is a non-m-polar positive measure and $f_1, ..., f_m$ are in $L^1(X,\mu)$. If $u_1,...,u_m \in \mathrm{SH}_m(X,\omega)$ satisfy $H_m(u_p) \geq f_p\mu$, $p = 1,...,m$ then*

$$H_m(u_1,...,u_m) \geq (f_1...f_m)^{1/m}\mu.$$

Proof. Having the mixed Hessian inequality for bounded ω-m-sh functions [30], the proof of the lemma is identical to that of [9, Proposition 1.11]. □

2.3. **Finite energy classes.** The class $\mathcal{E}(X,\omega,m)$ consists of functions $u \in \mathrm{SH}_m(X,\omega)$ such that $\int_X H_m(u) = 1$. The class $\mathcal{E}^1(X,\omega,m)$ consists of $u \in \mathcal{E}(X,\omega,m)$ such that $\int_X |u|H_m(u) < +\infty$.

To ease the notations, we will occasionally denote these classes by $\mathcal{E}$, $\mathcal{E}^1$.

The Hessian energy of $u \in \mathrm{SH}_m(X,\omega)\cap L^\infty(X)$ is defined by:

$$E_m(u) := \frac{1}{m+1}\sum_{k=0}^m \int_X uH_k(u).$$

When (ω, m) is fixed we will simply denote this functional by E.

The following result is well-known in the Monge-Ampère case and the proof can be adapted in an obvious way to the Hessian setting, see [50].

Proposition 2.13. *Suppose $u, v \in \mathrm{SH}_m(X,\omega)\cap L^\infty(X)$. The following hold:*
(i) $E(u) - E(v) = \frac{1}{m+1}\sum_{k=0}^n \int_X (u-v)\omega_u^k \wedge \omega_v^{m-k}\wedge\omega^{n-m}$.
(ii) E is non-decreasing and concave along affine curves. Additionally, the following estimates hold: $\int_X (u-v)H_m(u) \leq E(u) - E(v) \leq \int_X (u-v)H_m(v)$.
(iii) If $v \leq u$ then, $\frac{1}{m+1}\int_X (u-v)H_m(v) \leq E(u)-E(v) \leq \int_X (u-v)H_m(v)$. In particular, $E(v) \leq E(u)$.

One can thus extend E to $\mathrm{SH}_m(X,\omega)$ by

$$E(u) := \inf\{E(v) \mid v \in \mathrm{SH}_m(X,\omega)\cap L^\infty,\ v \geq u\}.$$

A function $u \in \mathrm{SH}_m(X,\omega)$ belongs to $\mathcal{E}^1$ iff $E(u) > -\infty$.

Following [16, 15] we introduce the functional I_1

$$I_1(u,v) := \int_X |u-v|\left(H_m(u) + H_m(v)\right).$$

Proposition 2.14. *Assume that $u_j \in \mathcal{E}^1$ is a monotone sequence converging to $u \in \mathcal{E}^1$. Then $I_1(u_j,u) \to 0$ and $E(u_j) \to E(u)$.*

Proof. The proof is an obvious modification of the Monge-Ampère case, see e.g. [9], [19, Proposition 2.7]. □

3. Relative Potential Theory

3.1. **Monotonicity of the complex Hessian mass.** In this section we extend the monotonicity results of [62], [20] to the Hessian cases $m < n$. The proof is new in the Monge-Ampère case.

Recall that we normalize ω such that $\int_X \omega^n = 1$. We first establish the following slope formula:

Lemma 3.1. *For any $u \in \mathrm{SH}_m(X,\omega)$ we have*

$$\lim_{s\to+\infty} \frac{E(\max(u,-s))}{s} = -1 + \frac{1}{m+1}\sum_{k=0}^{m}\int_X H_k(u).$$

Proof. We set $u^s := \max(u,-s)$ and compute

$$\frac{(m+1)E(u^s)}{s} = \sum_{k=0}^{m}\int_{\{u>-s\}} \frac{u}{s} H_k(u^s) - \sum_{k=0}^{m}\int_{\{u\leq -s\}} H_k(u^s).$$

We note that, by the Lemma 2.7, $\mathbf{1}_{\{u>-s\}}H_k(u^s) = \mathbf{1}_{\{u>-s\}}H_k(u)$. Thus we can continue the above computation to write

$$\frac{(m+1)E(u^s)}{s} = \sum_{k=0}^{m}\int_{\{u>-s\}} \frac{u}{s} H_k(u) - \sum_{k=0}^{m}\int_{\{u\leq -s\}} H_k(u^s). \tag{3.1}$$

The functions $\mathbf{1}_{\{u>-s\}}\frac{u}{s}$ are uniformly bounded and converge to 0 outside the m-polar set $\{u=-\infty\}$. Since $H_k(u)$ does not charge m-polar sets, we see that

$$\lim_{s\to+\infty}\sum_{k=0}^{m}\int_{\{u>-s\}} \frac{u}{s} H_k(u) = 0. \tag{3.2}$$

Since u^s is bounded, by Stokes theorem we have $\int_X H_k(u^s) = \int_X \omega^n = 1$. It thus follows from Lemma 2.7 that

$$\begin{aligned} 1 = \int_X H_k(u^s) &= \int_{\{u>-s\}} H_k(u^s) + \int_{\{u\leq -s\}} H_k(u^s) \\ &= \int_{\{u>-s\}} H_k(u) + \int_{\{u\leq -s\}} H_k(u^s). \end{aligned}$$

Summing up the above equalities for $k=0,...,m$ we arrive at

$$m+1 = \sum_{k=0}^{m}\int_{\{u>-s\}} H_k(u) + \sum_{k=0}^{m}\int_{\{u\leq -s\}} H_k(u^s).$$

Letting $s\to+\infty$ we obtain

$$m+1 = \sum_{k=0}^{m}\int_X H_k(u) + \lim_{s\to+\infty}\sum_{k=0}^{m}\int_{\{u\leq -s\}} H_k(u^s).$$

From this, (3.1), and (3.2) we obtain the result. □

Proposition 3.2. *Let $u,v \in \mathrm{SH}_m(X,\omega)$, and assume that there exists a constant $C\in\mathbb{R}$ such that $v - C \leq u \leq v + C$ on X. Then*

$$\int_X H_k(u) = \int_X H_k(v),\ \forall k\in\{0,...,m\}.$$

Proof. Fix $1 \leq l \leq m$, and observe that $\mathrm{SH}_m(X,\omega) \subset \mathrm{SH}_l(X,\omega)$. For each $s > 0$ set $u^s := \max(u, -s)$. By assumption we have

$$v^s - C \leq u^s \leq v^s + C.$$

Hence, the monotonicity of the energy E_l [50, Lemma 6.3] gives, for all $s > 0$,

$$\frac{E_l(v^s) - C}{s} \leq \frac{E_l(u^s)}{s} \leq \frac{E_l(v^s) + C}{s}.$$

Letting $s \to +\infty$ and using Lemma 3.1 we obtain the following equalities

$$\sum_{k=0}^{l} \int_X H_k(u) = \sum_{k=0}^{l} \int_X H_k(v), \ l = 1, ..., m,$$

which imply the result. □

Theorem 3.3. *Assume that $u_1^j, ..., u_m^j$ are sequences of ω-m-sh functions converging in m-capacity to ω-m-sh functions $u_1, ..., u_m$. Let χ_j be a sequence of positive uniformly bounded quasi-continuous functions which converges in capacity to χ. Then,*

$$\liminf_{j\to+\infty} \int_X \chi_j H_m(u_1^j, ..., u_m^j) \geq \int_X \chi H_m(u_1, ..., u_m).$$

In particular, if $\Omega \subset X$ is a quasi-open set then

$$\liminf_{j\to+\infty} \int_\Omega H_m(u_1^j, ..., u_m^j) \geq \int_\Omega H_m(u_1, ..., u_m).$$

Proof. We borrow the ideas of [20]. Fix $C > 0$, $\varepsilon > 0$, and set

$$U_C^j := \cap_{p=1}^m \{u_p^j > -C\}, \ f_{C,\varepsilon}^j := \prod_{p=1}^m \frac{\max(u_p^j + C, 0)}{\max(u_p^j + C, 0) + \varepsilon}.$$

Observe that $0 \leq f_{C,\varepsilon}^j \leq 1$ and $f_{C,\varepsilon}^j$ vanishes outside U_C^j. We thus have

$$\begin{aligned}
\liminf_{j\to+\infty} \int_X \chi_j H_m(u_1^j, ..., u_m^j) &\geq \liminf_{j\to+\infty} \int_{U_C^j} \chi_j H_m(u_1^j, ..., u_m^j) \\
&= \liminf_{j\to+\infty} \int_{U_C^j} \chi_j H_m(\max(u_1^j, -C), ..., \max(u_m^j, -C)) \\
&\geq \liminf_{j\to+\infty} \int_X \chi_j f_{C,\varepsilon}^j H_m(\max(u_1^j, -C), ..., \max(u_m^j, -C)),
\end{aligned}$$

where in the second line we have used the plurifine locality. For fixed $C > 0$ the functions $\max(u_p^j, -C)$ are uniformly bounded, hence we can use [47, Proposition 3.12], which is a direct adaptation of the case $m = n$, to continue the above inequality in the following way

$$\begin{aligned}
\liminf_{j\to+\infty} \int_X \chi_j H_m(u_1^j, ..., u_m^j) &\geq \int_X \chi f_{C,\varepsilon} H_m(\max(u_1, -C), ..., \max(u_m, -C)) \\
&\geq \int_{U_C} \chi f_{C,\varepsilon} H_m(\max(u_1, -C), ..., \max(u_m, -C)) \\
&\geq \int_{U_C} \chi f_{C,\varepsilon} H_m(u_1, ..., u_m).
\end{aligned}$$

In the last line above we have used Lemma 2.7. We now let $\varepsilon \to 0$ and then $C \to +\infty$ to conclude the proof of the first statement.

To prove the last statement we follow the lines above with $\chi_j = 1$, X replaced by Ω, and we use Theorem 2.6. □

We are now in the position to prove the main result of this section.

Theorem 3.4. *Let $u_1, ..., u_m, v_1, ..., v_m \in \mathrm{SH}_m(X, \omega)$ and assume that u_j is more singular than v_j for all j. Then*

$$\int_X H_m(u_1, ..., u_m) \le \int_X H_m(v_1, ..., v_m).$$

Proof. We first assume that u_p has the same singularity as v_p for all $p = 1, ..., m$. For $t = (t_1, ..., t_m) \in [0, 1]^m$ with $\sum_{p=1}^m t_p = 1$, we set

$$\phi_t := \sum_{p=1}^m t_p u_p, \quad \psi_t := \sum_{p=1}^m t_p v_p.$$

Then $\phi_t, \psi_t \in \mathrm{SH}_m(X, \omega)$ have the same singularity. It thus follows from Proposition 3.2 that

$$\int_X H_m(\phi_t) = \int_X H_m(\psi_t).$$

From this and Lemma 2.11 we obtain an equality between two polynomials in $(t_1, ..., t_m)$. Identifying the coefficients we obtain

$$\int_X H_m(u_1, ..., u_m) = \int_X H_m(v_1, ..., v_m).$$

To treat the general case we define, for $C > 0$, $w_p^C := \max(u_p, v_p - C)$. Then the previous step yields

$$\int_X H_m(w_1^C, ..., w_m^C) = \int_X H_m(v_1, ..., v_m).$$

Letting $C \to +\infty$ and using Theorem 3.3 we arrive at the conclusion. □

As shown in Theorem 3.3, the (non-m-polar) Hessian measure is lower semi-continuous along sequences converging in m-capacity. We give below sufficient conditions for the convergence.

Corollary 3.5. *Assume that $u_1^j, ..., u_m^j$ are sequences of ω-m-sh functions which increase a.e. to ω-m-sh functions $u_1, ..., u_m$. Then*

$$H_m(u_1^j, ..., u_m^j) \to H_m(u_1, ..., u_m)$$

weakly in the sense of measures.

Proof. It is a direct consequence of Theorem 3.4 and Theorem 3.3. Indeed, from Theorem 3.4 we have that $\limsup_j \int_X H_m(u_1^j, ..., u_m^j) \le \int_X H_m(u_1, ..., u_m)$, and from Theorem 3.3 we have that any cluster point ν of the sequence of positive measures $H_m(u_1^j, ..., u_m^j)$ is greater than $H_m(u_1, ..., u_m)$. But $\nu(X) \le \int_X H_m(u_1, ..., u_m)$, hence we have equality $\nu = H_m(u_1, ..., u_m)$. □

Lemma 3.6. *Let μ be a positive measure vanishing on m-polar sets. Then there exists a continuous function $f : [0, +\infty) \to [0, +\infty)$ such that, for all Borel set E,*

$$\mu(E) \le f(\mathrm{Cap}_m(E)).$$

Proof. The proof is an easy adaptation of [35]. We repeat this argument here for the reader's convenience. It follows from [50, Theorem 1.3] that there exists $\psi \in \mathscr{E}$ such that $\sup_X \psi = 0$ and $\mu = CH_m(\psi)$, for some positive constant C.

Let $E \subset X$ be a Borel set such that $\mathrm{Cap}_m(E) > 0$. For $t > 1$ we have

$$\mu(E \cap \{\psi > -t\}) = C \int_E H_m(\max(\psi, -t)) \leq Ct^m \mathrm{Cap}_m(E).$$

Let $\chi : (-\infty, 0) \to (-\infty, 0)$ be a convex increasing function such that $\chi(-\infty) = -\infty$ and $C_1 := \int_X |\chi(\psi)| d\mu < +\infty$. For $t > 1$ we have

$$\mu(\psi \leq -t) \leq \frac{1}{|\chi(-t)|} \int_X |\chi(\psi)| d\mu = \frac{C_1}{|\chi(-t)|}.$$

Choosing t such that $t^{m+1} = \max(\mathrm{Cap}_m(E)^{-1}, 1)$, we finish the proof of the lemma. □

Theorem 3.7. *Assume that $u_j \in \mathrm{SH}_m(X, \omega)$ decreases to $u \in \mathrm{SH}_m(X, \omega)$. If there exists a non-m-polar positive measure μ such that*

$$H_m(u_j) \leq \mu, \forall j,$$

then $H_m(u_j)$ weakly converges to $H_m(u)$.

Proof. By Theorem 3.3 we have that $H_m(u) \leq \mu$ and it remains to prove the convergence of the total mass.

We can assume that $\sup_X u_j = \sup_X u = 0$. For a function v and a constant t we set $v^t := \max(v, -t)$. For all $t > 0$ we have

$$\mu(u \leq -t) \leq f\left(\mathrm{Cap}_m(u \leq -t)\right),$$

where f is the continuous function in Lemma 3.6. By continuity of f we have

$$\lim_{t \to +\infty} f\left(\mathrm{Cap}_m(u \leq -t)\right) = 0.$$

Therefore, fixing $\varepsilon > 0$, for $t > 0$ large enough we have

$$\int_{\{u \leq -t\}} H_m(u_j) \leq \mu(u \leq -t) \leq f\left(\mathrm{Cap}_m(u \leq -t)\right) \leq \varepsilon, \ \forall j.$$

Thus, for fixed $s > t$ we have

$$\int_X H_m(u_j) \leq \int_{\{u \geq -t\}} H_m(u_j) + \varepsilon \leq \int_{\{u \geq -t\}} H_m(u_j^s) + \varepsilon.$$

Here, we use Lemma 2.7 and the assumption that $u_j \geq u$ to have that

$$\mathbf{1}_{\{u > -s\}} H_m(u_j) = \mathbf{1}_{\{u > -s\}} H_m(u_j^s),$$

hence

$$\int_{\{u \geq -t\}} H_m(u_j) = \int_{\{u \geq -t\}} \mathbf{1}_{\{u > -s\}} H_m(u_j) = \int_{\{u \geq -t\}} \mathbf{1}_{\{u > -s\}} H_m(u_j^s)$$
$$= \int_{\{u \geq -t\}} H_m(u_j^s).$$

Since $\{u \geq -t\}$ is quasi compact and u_j^s are uniformly bounded, letting $j \to +\infty$ we obtain

$$\limsup_j \int_X H_m(u_j) \leq \int_{\{u \geq -t\}} H_m(u^s) + \varepsilon = \int_{\{u \geq -t\}} H_m(u) + \varepsilon.$$

Letting $t \to +\infty$, and then $\varepsilon \to 0$ we arrive at the conclusion. □

Having the monotonicity theorem in hand most of the pluripotential tools in [20, 22] can be adapted directly to the Hessian setting. Since the references [20, 22] are quite recent, we give the full details.

3.2. **Envelopes.** Let f be a function on X. We define

$$P_{(\omega,m)}(f) := \left(\sup\{u \mid u \in \mathrm{SH}_m(X,\omega),\ u \leq f\}\right)^*,$$

where the $*$ operator means the upper semicontinuous regularization. Following [58], [20, 22] we define

$$P_{(\omega,m)}[f] := \left(\lim_{C\to+\infty} P_{(\omega,m)}(\min(f+C,0))\right)^*.$$

If (ω,m) is fixed we will simply denote these envelopes by $P(f)$ and $P[f]$. For $u_1,...,u_N \in \mathrm{SH}_m(X,\omega)$ we denote $P(u_1,...,u_N) := P(\min(u_1,...,u_N))$.

Lemma 3.8. *If $u_1,...,u_m,v_1,...,v_m \in \mathrm{SH}_m(X,\omega)$ satisfy $P[u_p] = P[v_p]$, for all p, then*

$$\int_X H_m(u_1,...,u_m) = \int_X H_m(v_1,...,v_m).$$

Proof. For each $C > 0$ $P(u_j + C, 0)$ has the same singularity as u_j, hence by Theorem 3.4,

$$\int_X H_m(u_1,...,u_m) = \int_X H_m(P(u_1+C,0),...,P(u_m+C,0)).$$

Letting $C \to +\infty$, Corollary 3.5 ensures that

$$\int_X H_m(u_1,...,u_m) = \int_X H_m(P[u_1],...,P[u_m]).$$

The same arguments apply for $v_1,...,v_m$, yielding the result. □

Lemma 3.9. *If $u,v \in \mathrm{SH}_m(X,\omega)$ and $t \in (0,1)$ then*

$$P[tu+(1-t)v] \geq tP[u] + (1-t)P[v].$$

Proof. For each $C > 0$ we have that $tP(u+C,0)+(1-t)P(v+C,0)$ is ω-m-sh and it is smaller than $\min(tu+(1-t)v+C,0)$. Thus

$$P(tu+(1-t)v+C,0) \geq tP(u+C,0)+(1-t)P(v+C,0),$$

hence letting $C \to +\infty$ we obtain the result. □

Proposition 3.10. *Assume that $f = a\varphi - b\psi$, where $\varphi,\psi \in \mathrm{SH}_m(X,\omega)$, and a,b are positive constants. If $P(f) \not\equiv -\infty$ then*

$$\int_{\{P(f)<f\}} H_m(P(f)) = 0.$$

Here, the function $f = a\varphi - b\psi$ is well-defined in the complement of a pluripolar set and the inequality $u \leq a\varphi - b\psi$, for $u \in \mathrm{SH}_m(X,\omega)$, means $u + b\psi \leq a\varphi$ on X.

Proof. We first assume that φ is continuous. Then $P(f)$ is bounded. Let ψ_j be a sequence of continuous ω-m-sh functions decreasing to ψ and set $f_j := a\varphi - b\psi_j$, $u_j = P(f_j)$. By [50] we have

$$\int_X \min(f_j - u_j, 1) H_m(u_j) = 0,\ \forall j.$$

Let $u := (\lim_{j\to+\infty} u_j)^*$. It follows from [47, Proposition 3.12] that

$$\int_X (\min(f-u,1)H_m(u) = 0,$$

hence $\int_{\{u<P(f)\}} H_m(u) = 0$ and the domination principle [30, Lemma 3.5] gives $u = P(f)$. By the above equality we also have that $H_m(u)$ vanishes in $\{u < f\}$.

We now treat the general case. Let φ_j be a sequence of continuous ω-m-sh functions decreasing to φ and set $f_j := a\varphi_j - b\psi$. Then $P(f_j) \searrow P(f)$. From the first step we have

$$\int_X \min(f_j - P(f_j), 1) H_m(P(f_j)) = 0, \ \forall j.$$

Letting $j \to +\infty$ and using Theorem 3.3 we arrive at the conclusion. □

From Proposition 3.10 and Proposition 2.10 we obtain the following :

Corollary 3.11. *Let $u, v \in \mathrm{SH}_m(X,\omega)$ be such that $P(u,v) \in \mathrm{SH}_m(X,\omega)$. Then*

$$H_m(P(u,v)) \leq \mathbf{1}_{\{P(u,v)=u\}} H_m(u) + \mathbf{1}_{\{P(u,v)=v\}} H_m(v).$$

In particular, $H_m(P[u]) \leq \mathbf{1}_{\{P[u]=0\}}\omega^n$. Finally, if $H_m(u) \leq \mu$ and $H_m(v) \leq \mu$, for a non-m-polar positive measure μ, then $H_m(P[u,v]) \leq \mu$.

Definition 3.12. A function $\phi \in \mathrm{SH}_m(X,\omega)$ is a model potential if $\int_X H_m(\phi) > 0$ and $P[\phi] = \phi$.

Given a model potential ϕ, the class $\mathcal{E}_\phi := \mathcal{E}_\phi(X,\omega,m)$ consists of functions $u \in \mathrm{SH}_m(X,\omega)$ such that u is more singular than ϕ and $\int_X H_m(u) = \int_X H_m(\phi)$.

3.3. Comparison principle.

Theorem 3.13. *Let $\phi_2, ..., \phi_m, u, v \in \mathrm{SH}_m(X,\omega)$ and assume that $P[u] \geq P[v]$. Then*

$$\int_{\{u<v\}} H_m(v, \phi_2, ..., \phi_m) \leq \int_{\{u<v\}} H_m(u, \phi_2, ..., \phi_m).$$

Proof. Fix $\varepsilon > 0$ and set $v_\varepsilon := \max(v - \varepsilon, u)$. Then $P[v^\varepsilon] = P[u]$, hence by Lemma 3.8 we have

$$\int_X H_m(v^\varepsilon, \phi_2, ..., \phi_m) = \int_X H_m(u, \phi_2, ..., \phi_m).$$

By Lemma 2.7 we also have

$$\int_X H_m(v^\varepsilon, \phi_2, ..., \phi_m) \geq \int_{\{u>v-\varepsilon\}} H_m(u, \phi_2, ..., \phi_m) + \int_{\{u<v-\varepsilon\}} H_m(v, \phi_2, ..., \phi_m).$$

Comparing these we arrive at

$$\int_{\{u<v-\varepsilon\}} H_m(v, \phi_2, ..., \phi_m) \leq \int_{\{u\leq v-\varepsilon\}} H_m(u, \phi_2, ..., \phi_m).$$

Letting $\varepsilon \to 0^+$ we obtain the result. □

3.4. Domination principle.

Lemma 3.14. *Assume that $u \in \mathrm{SH}_m(X,\omega)$ and $\int_X H_m(u) > 0$. If $E \subset X$ is a Borel set such that $\int_E \omega^n > 0$ then there exists $v \in \mathrm{SH}_m(X,\omega)$ such that v has the same singularity as u and*

$$\int_E H_m(v) > 0.$$

Proof. Let $\phi \in \mathrm{SH}_m(X,\omega) \cap L^\infty(X)$ be such that $H_m(\phi) = c\mathbf{1}_E\omega^n$, where $c > 0$ is a normalization constant. For $t > 0$ set $u_t := P(\min(u+t,\phi))$. Corollary 3.11 gives

$$\int_{X\setminus E} H_m(u_t) \le \int_{X\setminus E} \mathbf{1}_{\{u_t=u+t\}} H_m(u) \le \int_{\{u\le \phi-t\}} H_m(u).$$

Thus, for $t > 0$ large enough we have $\int_{X\setminus E} H_m(u_t) < \int_X H_m(u) = \int_X H_m(u_t)$, where the last equality follows from Theorem 3.4 since u_t has the same singularity as u. For such t we thus have $\int_E H_m(u_t) > 0$, finishing the proof. □

Theorem 3.15. *Assume that $u, v \in \mathrm{SH}_m(X,\omega)$ and u is less singular than v. If $\int_{\{u<v\}} H_m(u) = 0$ and $\int_X H_m(u) > 0$ then $u \ge v$.*

Proof. Assume by contradiction that $E := \{u < v\}$ is not empty. Then $\int_E \omega^n > 0$ and hence Lemma 3.14 provides us with $h \in \mathrm{SH}_m(X,\omega)$ having the same singularity as u such that $\int_E H_m(h) > 0$. We can assume that $h \le u$. For $t \in (0,1)$ set $v_t := th + (1-t)v$. Then $E_t := \{u < v_t\} \subset E$ and $\cup E_t = E$. Hence for t small enough we have $\int_{E_t} H_m(h) > 0$. But the comparison principle gives

$$t^m \int_{E_t} H_m(h) \le \int_{E_t} H_m(v_t) \le \int_{E_t} H_m(u) = 0,$$

which is a contradiction. □

Corollary 3.16. *If ϕ is a model potential then $u \in \mathscr{E}_\phi$ iff $P[u] = \phi$.*

Proof. If $u \in \mathscr{E}_\phi$ then the domination principle, Theorem 3.15, gives $P[u] = \phi$. Assume now that $P[u] = \phi$. Since $P[u]$ is the increasing limit of $P(\min(u+t,0))$ as $t \to +\infty$, Theorem 3.3 gives $\int_X H_m(u) = \int_X H_m(P[u])$, hence $u \in \mathscr{E}_\phi$. □

Corollary 3.17. *If ϕ is a model potential and $u \in \mathscr{E}_\phi$ then $u - \sup_X u \le \phi$.*

Lemma 3.18. *If $u, v \in \mathrm{SH}_m(X,\omega)$ and $P(u,v) \in \mathrm{SH}_m(X,\omega)$ then $P[\min(u,v)] = P[P(u,v)]$.*

Proof. By definition we have

$$\begin{aligned} P[\min(u,v)] &= \left(\lim_{C\to+\infty} P(\min(u+C, v+C, 0))\right)^* \\ &\le \left(\lim_{C\to+\infty} P(\min(P(u,v)+C, 0)\right)^* = P[P(u,v)]. \end{aligned}$$

The reverse inequality follows directly from the definition. □

3.5. **Strongly m-positive currents.** We borrow the idea in [23].

Theorem 3.19. *Assume that $b > 1$, $u, v \in \mathrm{SH}_m(X,\omega)$, $u \le v$, and*

$$\int_X H_m(v) > b^m \left(\int_X H_m(v) - \int_X H_m(u)\right).$$

Then $P(bu - (b-1)v) \in \mathrm{SH}_m(X,\omega)$.

If $v = 0$ and $\int_X H_m(u) > 0$ then by the above result there exists $b > 1$ such that $P(bu) \in \mathrm{SH}_m(X,\omega)$. Therefore $b^{-1}P(bu)$ is a strongly ω-m-sh function lying below u. This will be used in proving the existence of solutions to complex Hessian equations with prescribed singularity.

Proof. We can assume that $P[v] = v$.

For $t > 0$ set $u_t := \max(u, v - t)$, $\varphi_t := P(bu_t - (b-1)v) \in \mathrm{SH}_m(X, \omega)$, and $D := \{\varphi_t = bu_t - (b-1)v\}$. Then $b^{-1}\varphi_t + (1 - b^{-1})v \le u_t$ with equality on D, hence Proposition 2.10 gives

$$\mathbf{1}_D b^{-m} H_m(\varphi_t) \le \mathbf{1}_D H_m(b^{-1}\varphi_t + (1-b^{-1})v) \le \mathbf{1}_D H_m(u_t).$$

Fix $s < t$. By the above inequality and Proposition 3.10 we have

$$\begin{aligned}\int_{\{\varphi_t \le v - bs\}} H_m(\varphi_t) &\le b^m \int_{\{bu_t \le bv - bs\}} H_m(u_t) = b^m \int_{\{u \le v - s\}} H_m(u_t)\\ &= b^m \left(\int_X H_m(v) - \int_{\{u > v - s\}} H_m(u_t)\right)\\ &= b^m \left(\int_X H_m(v) - \int_{\{u > v - s\}} H_m(u)\right),\end{aligned}$$

where in the last line we use Lemma 2.7.

We want to prove that φ_t decreases to some ω-m-subharmonic function on X. Assume by contradiction that it is not the case. Then $\sup_X \varphi_t$ decreases to $-\infty$. Since $v = P[v]$, by Corollary 3.17 we have $\varphi_t \le v + \sup_X \varphi_t$. Thus, for $s > 0$ fixed and for t large enough $\{\varphi_t \le v - s\} = X$. Fixing $s > 0$ and letting $t \to +\infty$ we obtain

$$\int_X H_m(v) \le b^m \left(\int_X H_m(v) - \int_{\{u > -s\}} H_m(u)\right).$$

Now, letting $s \to +\infty$ we obtain a contradiction with the assumption. □

Corollary 3.20. *Assume that $u, v \in \mathrm{SH}_m(X, \omega)$, $P[u] = P[v]$ and $\int_X H_m(v) > 0$. Then for all $b > 1$, $P(bu - (b-1)v) \in \mathcal{E}_{P[v]}$.*

Proof. We can assume that $u, v \le 0$. Then $u \le P[u] = P[v]$. Fix $b > 1$. We first observe that $P(bu - (b-1)P[v]) \in \mathrm{SH}_m(X, \omega)$ as follows from Theorem 3.19. Hence $P(bu - (b-1)v) \in \mathrm{SH}_m(X, \omega)$. For $t > b$ we have

$$u \ge P(bu - (b-1)P[v]) \ge bt^{-1} P(tu - (t-1)P[v]) + (1 - bt^{-1})P[v].$$

By monotonicity of mass, see Theorem 3.4, we have

$$\int_X H_m(P(bu - (b-1)P[v])) \ge (1 - bt^{-1})^m \int_X H_m(P[v]).$$

Letting $t \to +\infty$ we obtain $P(bu - (b-1)P[v]) \in \mathcal{E}_{P[v]}$. We also have

$$b^{-1} P(bu - (b-1)v) + (1 - b^{-1})v \le u,$$

hence, by Lemma 3.9 we have $b^{-1}P[P(bu-(b-1)v)] + (1-b^{-1})P[v] \le P[u] = P[v]$, which implies $P[P(bu - (b-1)v)] \le P[v]$. But we have already proved that

$$P[P(bu - (b-1)v)] \ge P[P(bu - (b-1)P[v])] = P[v].$$

We thus have equality. □

Corollary 3.21. *Assume that $u, v \in \mathrm{SH}_m(X, \omega)$ are such that $P[u] \ge P[v]$ and $\int_X H_m(v) > 0$. Then, for all $b > 1$, $P(bu - bv) \in \mathcal{E}$.*

Proof. We can assume that $u, v \le 0$. Then $v \le P[v] \le P[u]$, hence $u \le \max(u, v) \le P[u]$. It thus follows that $\max(u, v) \in \mathcal{E}_{P[u]}$. Hence by Corollary 3.20 we have, for all $b > 1$, $P(bu - bv) \geq P(bu - (b-1)\max(u, v)) \in \mathrm{SH}_m(X, \omega)$. For $t > b > 1$, we have

$$P(bu - bv) \geq bt^{-1} P(tu - (t-1)v) + (1 - bt^{-1})v.$$

Comparing total mass and letting $t \to +\infty$ we obtain the result. □

Proposition 3.22. *If ϕ is a model potential and $u, v \in \mathcal{E}_\phi$ then $P(u, v) \in \mathcal{E}_\phi$.*

Proof. The proof is similar to that of Theorem 3.19. We first prove that $P(u, v) \in \mathrm{SH}_m(X, \omega)$. For $t > 0$ set $u_t := \max(u, \phi - t)$, $v_t := \max(v, \phi - t)$, and $\varphi_t := P(u_t, v_t)) \in \mathcal{E}_\phi$. We want to prove that φ_t decreases to some ω-m-subharmonic function on X. Assume by contradiction that it is not the case. Then $\sup_X \varphi_t$ decreases to $-\infty$. Since $\phi = P[\phi]$, by Corollary 3.17 we have $\varphi_t \leq \phi + \sup_X \varphi_t$. Thus, for $s > 0$ fixed and for t large enough we have $\{\varphi_t \leq \phi - s\} = X$. Using this and Corollary 3.11 we obtain

$$\begin{aligned}\int_X H_m(\phi) = \int_{\{\varphi_t \leq \phi - s\}} H_m(\varphi_t) &\leq \int_{\{u \leq \phi - s\}} H_m(u_t) + \int_{\{v \leq \phi - s\}} H_m(v_t) \\ &= 2\int_X H_m(\phi) - \int_{\{u > \phi - s\}} H_m(u) - \int_{\{v > \phi - s\}} H_m(v).\end{aligned}$$

Letting $s \to +\infty$ we obtain $\int_X H_m(\phi) \leq 0$, a contradiction. Thus $P(u, v) \in \mathrm{SH}_m(X, \omega)$.

Now, by Corollary 3.20 we have that, for all $b > 1$, $u_b := P(bu - (b-1)\phi) \in \mathcal{E}_\phi$ and $v_b := P(bv - (b-1)\phi) \in \mathcal{E}_\phi$. Hence by the previous step we have $P(u_b, v_b) \in \mathrm{SH}_m(X, \omega)$. We also have that $P(u, v)$ is more singular than ϕ and

$$P(u, v) \geq b^{-1} P(u_b, v_b) + (1 - b^{-1})\phi.$$

Thus $\int_X H_m(P(u, v)) \geq (1 - b^{-1})^m \int_X H_m(\phi)$. Letting $b \to +\infty$ we arrive at the conclusion. □

4. A metric on $\mathcal{E}^1$

Following [19], we introduce a metric on $\mathcal{E}^1(X, \omega, m)$ and use it to construct subextensions of a family of ω-m-subharmonic functions. Most of this section are taken from [19] but we recall them for completeness, since we will crucially use Theorem 4.11.

4.1. Define a metric on $\mathcal{E}^1$.

Given $u, v \in \mathcal{E}^1$ we define

$$d(u, v) := E(u) + E(v) - 2E(P(u, v)).$$

Here $P(u, v) := P(\min(u, v))$ is the largest ω-m-sh function lying below $\min(u, v)$. This is called the rooftop envelope [24] which plays a crucial role in the recent developments in Geometric Pluripotential Theory (see [18]). The proof of [16, Theorem 3.6], applied to the Hessian setting, shows that $P(u, v) \in \mathcal{E}^1$. Arguing as in [19] we can show that d is a metric and $(\mathcal{E}^1, d)$ is compete, along with many useful properties.

Lemma 4.1. *Let $u, v, w \in \mathcal{E}^1$. Then the following hold:*
(i) If $u \leq v$ then $d(u, v) = E(v) - E(u)$.
(ii) If $u \leq v \leq w$ then $d(u, v) + d(v, w) = d(u, w)$.
(iii) (Pythagorean formula) $d(u, v) = d(u, P(u, v)) + d(v, P(u, v))$.

Proposition 4.2. *Let u, v be bounded ω-m-sh functions, and set*

$$\varphi_t := P((1-t)u + tv, v),\ \ t \in [0,1].$$

Then

$$\frac{d}{dt}E(\varphi_t) = \int_X (v - \min(u,v))H_m(\varphi_t),\ \ \forall t \in [0,1].$$

Proof. We will only prove the formula for the right derivative as the same argument can be applied to treat the left derivative. Fix $t \in [0,1)$ and let $s > 0$ be small. For notational convenience we set

$$f_t(x) := \min((1-t)u(x) + tv(x), v(x)),\ \ x \in X,\ \ t \in [0,1].$$

It follows from [50, Theorem 3.2] that $H_m(\varphi_t)$ is supported on the set $\{\varphi_t = f_t\}$. Combining this with the concavity of the energy E, see Proposition 2.13, we obtain

$$\begin{aligned} E(\varphi_{t+s}) - E(\varphi_t) &\leq \int_X (\varphi_{t+s} - \varphi_t)H_m(\varphi_t) \\ &= \int_X (\varphi_{t+s} - f_t)H_m(\varphi_t) \leq \int_X (f_{t+s} - f_t)H_m(\varphi_t). \end{aligned}$$

On the other hand we have that $f_{t+s} - f_t = s(v - \min(u,v))$. It thus follows that

$$\lim_{s\to 0^+} \frac{E(\varphi_{t+s}) - E(\varphi_t)}{s} \leq \int_X (v - \min(u,v))H_m(\varphi_t).$$

We use the same argument to prove the reverse inequality:

$$\begin{aligned} E(\varphi_{t+s}) - E(\varphi_t) &\geq \int_X (\varphi_{t+s} - \varphi_t)H_m(\varphi_{t+s}) = \int_X (f_{t+s} - \varphi_t)H_m(\varphi_{t+s}) \\ &\geq \int_X (f_{t+s} - f_t)H_m(\varphi_{t+s}) = s\int_X (v - \min(u,v))H_m(\varphi_{t+s}). \end{aligned}$$

As $s \to 0^+$ we have that φ_{t+s} converges uniformly to φ_t. Moreover, $v - \min(u,v)$ is a bounded quasi continuous function on X, hence [47, Proposition 3.12] gives

$$\lim_{s\to 0^+} \frac{E(\varphi_{t+s}) - E(\varphi_t)}{s} \geq \int_X (v - \min(u,v))H_m(\varphi_t).$$

This completes the proof. ☐

Corollary 4.3. *Let u, v, φ_t be as in Proposition 4.2. Then*

$$E(v) - E(P(u,v)) = \int_0^1 \int_X (v - \min(u,v))H_m(\varphi_t)dt.$$

Proposition 4.4. *If $u, v \in \mathscr{E}^1$ then $d(\max(u,v), u) \geq d(v, P(u,v))$.*

Proof. Set $\varphi = \max(u,v)$, $\psi = P(u,v)$. Observe that since $v \geq \psi$ and $\varphi \geq u$, the inequality to be proved is equivalent to $E(v) - E(\psi) \leq E(\varphi) - E(u)$.

Recall that for any $w \in \mathscr{E}^1$ the sequence of bounded potentials $w_k := \max(w, -k)$ decreases to w. Consequently, using approximation, we can assume that both u and v (hence also φ and ψ) are bounded. Using the formula for the derivative of $t \mapsto E((1-t)u + t\varphi)$, see [50, Lemma 6.3], [6, Eq. (2.2)], we can write

$$E(\varphi) - E(u) = \int_0^1 \int_X (\varphi - u)H_m((1-t)u + t\varphi)\,dt. \tag{4.1}$$

Set $w_t := (1-t)u + tv$, for $t \in [0,1]$, and observe that

$$(1-t)u + t\varphi = \max(w_t, u) \ \text{ and } \ \mathbf{1}_{\{w_t > u\}} = \mathbf{1}_{\{v > u\}},\ \forall t \in (0,1].$$

It then follows from the plurifine locality that

$$\mathbf{1}_{\{v>u\}}H_m(\max(w_t,u)) = \mathbf{1}_{\{w_t>u\}}H_m(\max(w_t,u)) = \mathbf{1}_{\{v>u\}}H_m(w_t).$$

Using this, (4.1), and the equality $\varphi - u = \mathbf{1}_{\{v>u\}}(v-u)$, we can write

$$E(\varphi) - E(u) = \int_0^1 \int_{\{v>u\}} (v-u)H_m(w_t)\, dt.$$

On the other hand, it follows from Corollary 3.11 that

$$H_m(P(w_t,v)) \le \mathbf{1}_{\{w_t\le v\}}H_m(w_t) + \mathbf{1}_{\{w_t\ge v\}}H_m(v).$$

Using this, Corollary 4.3 and the fact that $\{w_t \le v\} = \{u \le v\}$, for $t \in [0,1)$, we get

$$\begin{aligned} E(v) - E(\psi) &= \int_0^1 \int_X (v - \min(u,v))H_m(P(w_t,v))\, dt \\ &\le \int_0^1 \int_{\{u<v\}} (v-u)H_m(w_t)\, dt, \end{aligned}$$

hence the conclusion. □

Lemma 4.5. *For all $u,v,w \in \mathscr{E}^1$ we have $d(u,v) \ge d(P(u,w),P(v,w))$.*

Proof. We first assume that $v \le u$. It follows that $v \le \max(v,P(u,w)) \le u$, hence by Lemma 4.1(*iii*) and Proposition 4.4 we have

$$\begin{aligned} d(v,u) &\ge d(v,\max(v,P(u,w))) \ge d(P(u,w),P(P(u,w),v)) \\ &= d(P(u,w),P(v,w)). \end{aligned}$$

Observe that the last identity follows from the fact that $P(P(u,w),v) = P(u,w,v)$ and $P(u,w,v) = P(w,v)$ since $v \le u$. Now, we remove the assumption $u \ge v$. Since $\min(u,v) \ge P(u,v)$ we can use the first step to write $d(u,P(u,v)) \ge d(P(u,w),P(u,v,w))$, and $d(v,P(u,v)) \ge d(P(v,w),P(u,v,w))$. To finish the proof, it suffices to use Lemma 4.1(iii) and to note that $P(P(u,w),P(v,w)) = P(u,v,w)$. □

Theorem 4.6. *d is a distance on $\mathscr{E}^1$.*

Proof. The quantity d is non-negative, symmetric and finite by definition. The fact that d is non degenerate is a simple consequence of the domination principle. Suppose $d(u,v)=0$. Lemma 4.1(iii) implies that $d(u,P(u,v)) = d(v,P(u,v)) = 0$. Moreover, Lemma 4.1(iii) gives that $P(u,v) \ge u$ a.e. with respect to $H_m(P(u,v))$. By the domination principle, see [30] (or Theorem 3.15), we obtain that $P(u,v) \ge u$, hence trivially $u = P(u,v)$. By symmetry $v = P(u,v)$, implying that $u = v$.

It remains to prove the triangle inequality: for $u,v,\varphi \in \mathscr{E}^1$ we want to prove that

$$d(u,v) \le d(u,\varphi) + d(v,\varphi).$$

Using the definition of d this amounts to showing that

$$E(P(\varphi,u)) - E(P(u,v)) \le E(\varphi) - E(P(\varphi,v)).$$

But this follows from Lemma 4.5, as we have the following sequence of inequalities:

$$\begin{aligned} E(\varphi) - E(P(\varphi,v)) &= d(\varphi,P(\varphi,v)) \ge d(P(\varphi,u),P(P(\varphi,v),u)) \\ &= E(P(\varphi,u)) - E(P(\varphi,v,u)) \ge E(P(\varphi,u)) - E(P(u,v)), \end{aligned}$$

where in the last line we have used the monotonicity of E, Lemma 4.1. □

4.2. Comparison with I_1.

Lemma 4.7. *For all $u, v \in \mathscr{E}^1$ we have $d\left(u, \frac{u+v}{2}\right) \leq \frac{3(m+1)}{2} d(u,v)$.*

Proof. We have the following estimates:

$$\begin{aligned}
d\Big(u, \frac{u+v}{2}\Big) &= d\Big(u, P\Big(u, \frac{u+v}{2}\Big)\Big) + d\Big(\frac{u+v}{2}, P\Big(u, \frac{u+v}{2}\Big)\Big) \\
&\leq d(u, P(u,v)) + d\Big(\frac{u+v}{2}, P(u,v)\Big) \\
&\leq \int_X (u - P(u,v)) H_m(P(u,v)) + \int_X \Big(\frac{u+v}{2} - P(u,v)\Big) H_m(P(u,v)) \\
&\leq \frac{3}{2} \int_X (u - P(u,v)) H_m(P(u,v)) + \frac{1}{2} \int_X (v - P(u,v)) H_m(P(u,v)) \\
&\leq \frac{3(m+1)}{2} d(u, P(u,v)) + \frac{m+1}{2} d(v, P(u,v)) \\
&\leq \frac{3(m+1)}{2} d(u,v),
\end{aligned}$$

where in the second line we have additionally used that $P(u,v) \leq P(u,(u+v)/2)$. □

Theorem 4.8. *For all $u, v \in \mathscr{E}^1$ we have*

$$d(u,v) \leq \int_X |u-v| (H_m(u) + H_m(v)) \leq 3(m+1) 2^{m+2} d(u,v).$$

Proof. It follows from Lemma 4.1 that $d(u,v) = d(u, P(u,v)) + d(v, P(u,v))$. Since the energy E is concave along affine curves, Proposition 2.13, we have

$$\begin{aligned}
d(u, P(u,v)) &= E(u) - E(P(u,v)) \leq \int_X (u - P(u,v)) H_m(P(u,v)) \\
&\leq \int_{\{v = P(u,v)\}} (u-v) H_m(v) \leq \int_X |u-v| H_m(v).
\end{aligned}$$

Similarly we get $d(v, P(u,v)) \leq \int_X |u-v| H_m(u)$. Putting these two inequalities together we get the first inequality.

Next we establish the lower bound for d. By Lemma 4.7 and the Pythagorean formula we have

$$\begin{aligned}
\frac{3(m+1)}{2} d(u,v) &\geq d\Big(u, \frac{u+v}{2}\Big) \geq d\Big(u, P\Big(u, \frac{u+v}{2}\Big)\Big) \\
&\geq \int_X \Big(u - P\Big(u, \frac{u+v}{2}\Big)\Big) H_m(u).
\end{aligned}$$

By a similar reasoning as above, and the fact that $2^m H_m((u+v)/2) \geq H_m(u)$ we can write:

$$\begin{aligned}
\frac{3(m+1)}{2} d(u,v) &\geq d\Big(u, \frac{u+v}{2}\Big) \geq d\Big(\frac{u+v}{2}, P\Big(u, \frac{u+v}{2}\Big)\Big) \\
&\geq \int_X \Big(\frac{u+v}{2} - P\Big(u, \frac{u+v}{2}\Big)\Big) H_m((u+v)/2) \\
&\geq \frac{1}{2^m} \int_X \Big(\frac{u+v}{2} - P\Big(u, \frac{u+v}{2}\Big)\Big) H_m(u).
\end{aligned}$$

Adding the last two estimates we obtain

$$\begin{aligned}&3(m+1)2^m d(u,v)\\&\quad\geq \int_X \left(\left(u-P\left(u,\frac{u+v}{2}\right)\right)+\left(\frac{u+v}{2}-P\left(u,\frac{u+v}{2}\right)\right)\right)H_m(u)\\&\quad\geq \frac{1}{2}\int_X |u-v|H_m(u).\end{aligned}$$

By symmetry we also have $3(m+1)2^{m+1}d(u,v)\geq \int_X |u-v|H_m(v)$, and adding these last two estimates together the lower bound for d is established. □

Lemma 4.9. *There exists $A,B\geq 1$ such that for any $\varphi\in\mathcal{E}^1$*

$$-d(0,\varphi)\leq \sup_X\varphi\leq Ad(0,\varphi)+B.$$

Proof. If $\sup_X\varphi\leq 0$, then the right-hand side inequality is trivial, while

$$-d(0,\varphi)=E(\varphi)\leq\sup_X\varphi.$$

We therefore assume that $\sup_X\varphi\geq 0$. In this case the left-hand inequality is trivial. By compactness property of the set of normalized ω-m-sh functions [47, Lemma 2.13] we have

$$\int_X |\varphi-\sup_X\varphi|\omega^n\leq C_1,$$

where $C_1>0$ is a uniform constant. Using Theorem 4.8 the result then follows in the following manner:

$$\begin{aligned}d(0,\varphi)&\geq C_2 I_1(0,\varphi)\geq C_2\int_X|\varphi|\omega^n\geq C_2\sup_X\varphi-C_2\int_X|\varphi-\sup_X\varphi|\omega^n\\&\geq C_2\sup_X\varphi-C_1C_2.\end{aligned}$$

□

4.3. d is complete.

Theorem 4.10. *Assume that u_j is a Cauchy sequence in $(\mathcal{E}^1,d)$. Then u_j d-converges to $u\in\mathcal{E}^1$. In particular, we can extract a subsequence, still denoted by u_j, such that*

$$\lim_{l\to+\infty}P(u_k,u_{k+1},....,u_{k+l})\in\mathcal{E}^1.$$

Proof. The argument is due to Darvas [15, 16], see also [19, Theorem 3.10]. We can assume that

$$d(u_j,u_{j+1})\leq 2^{-j}, j\geq 1.$$

As in the proof of [16, Theorem 9.2] we introduce the following sequences

$$\psi_{j,k}:=P(u_j,u_{j+1},\ldots,u_k),\ j\in\mathbb{N},k\geq j.$$

Observe that, for $k\geq j+1$, $\psi_{j,k}=P(u_j,\psi_{j+1,k})$ and hence it follows from Lemma 4.1(iii) and the triangle inequality that

$$\begin{aligned}d(u_j,\psi_{j,k})&\leq d(u_j,\psi_{j+1,k})\leq d(u_j,u_{j+1})+d(u_{j+1},\psi_{j+1,k})\\&\leq 2^{-j}+d(u_{j+1},\psi_{j+1,k}).\end{aligned}$$

Repeating this argument several times we arrive at

$$d(u_j,\psi_{j,k})\leq 2^{-j+1},\ \forall k\geq j+1. \tag{4.2}$$

Using the triangle inequality for d and the above we see that

$$d(0,\psi_{j,k}) \leq d(0,u_j) + d(u_j,\psi_{j,k}) \leq d(0,u_1) + 2 + 2^{-j+1}$$

is uniformly bounded. It follows from Theorem 4.8 and Lemma 4.9 that $I_1(0,\psi_{j,k})$, as well as $\sup_X \psi_{j,k}$, is uniformly bounded. We then infer, using the triangle inequality for d, that $d(0,\psi_{j,k} - \sup_X \psi_{j,k})$ is uniformly bounded hence so is $E(\psi_{j,k})$. Therefore, Proposition 2.14 ensures that $\psi_j := \lim_k \psi_{j,k}$ belongs to $\mathcal{E}^1$. From (4.2) we obtain that $d(u_j,\psi_j) \leq 2^{-j+1}$, hence we only need to show that the d-limit of the increasing sequence $\{\psi_j\}_j \subset \mathcal{E}^1$ is in $\mathcal{E}^1$.

Lemma 4.9 implies that $\sup_X \psi_j$ is uniformly bounded, hence $\psi := \lim_j \psi_j \in \mathrm{SH}_m(X,\omega)$. Now ψ_j increases a.e. towards ψ, hence $\psi \in \mathcal{E}^1$. Therefore by Proposition 2.14 we have $I_1(\psi_j,\psi) \to 0$. It thus follows from Theorem 4.8 that $d(\psi_j,\psi) \to 0$. □

4.4. **ω-m-subharmonic subextension.** In the previous sections, we easily adapted the arguments in [19]. These are necessary to derive the following result which is important in the sequel.

Theorem 4.11. *Assume that $u_j \in \mathcal{E}$ satisfies $\sup_X u_j = 0$ and $H_m(u_j) \leq AH_m(\psi)$, for some positive constant A and some $\psi \in \mathrm{SH}_m(X,\omega) \cap L^\infty(X)$. Then $u_j \in \mathcal{E}^1$, and a subsequence of u_j d-converges to some $u \in \mathcal{E}^1$. In particular, we can extract a subsequence of u_j, still denoted by u_j, such that*

$$\lim_{l\to+\infty} P(u_k,...,u_{k+l}) \in \mathcal{E}^1, \ \forall k.$$

The result above is also new in the Monge-Ampère case. It produces in particular a ω-m-sh function lying below a suitably chosen subsequence of (u_j).

Proof. We will use $C_1, C_2, ...$ to denote uniform constants.

We can assume that $-1 \leq \psi \leq 0$ and u_j converges in L^1 to $u \in \mathrm{SH}_m(X,\omega)$. By the Chern-Levine-Nirenberg inequality [47, Corollary 3.18] we have that

$$\int_X |u_j| H_m(u_j) \leq A\int_X |u_j| H_m(\psi) \leq C_1, \forall j.$$

It thus follows from Proposition 2.13 that $u_j \in \mathcal{E}^1$ and $|E(u_j)| \leq C_1$. Thus by [50, Lemma 6.8] we have

$$\int_X u_j^2 H_m(\psi) \leq 2\int_0^{+\infty} t\mathrm{Cap}_m(u_j < -t)dt \leq C_2$$

is also uniformly bounded. Therefore, by the proof of [37, Lemma 11.5] we have $\int_X (u_j - u)H_m(\psi) \to 0$. Define $\tilde{u}_k := (\sup(u_l, l \geq k))^*$. Then

$$|u_k - u| = 2\max(u,u_k) - u - u_k \leq 2(\tilde{u}_k - u) + u - u_k.$$

Since $\tilde{u}_k$ decreases to u, it follows that

$$\int_X |u_j - u| H_m(u_j) \leq A\int_X |u_j - u| H_m(\psi) \to 0. \tag{4.3}$$

We next claim that $H_m(u) \leq AH_m(\psi)$. The proof of this part is taken from [11], [36]. After extracting a subsequence we can assume that

$$\int_X |u_j - u| H_m(u_j) \leq 2^{-j}.$$

We define $v_j := \max(u_j, u - 1/j)$. Then v_j converges in m-capacity to u. Hence by [47, Theorem 3.9] $H_m(v_j)$ weakly converges to $H_m(u)$. On the other hand we have

$$\int_{\{u_j \leq u-1/j\}} H_m(u_j) \leq j \int_X |u_j - u| H_m(u_j) \leq j 2^{-j} \to 0.$$

We thus have, for any positive continuous function χ,

$$\begin{aligned} \int_X \chi H_m(u) &= \lim_{j\to+\infty} \int_X \chi H_m(v_j) \geq \limsup_{j\to+\infty} \int_{\{u_j > u-1/j\}} \chi H_m(u_j) \\ &\geq \limsup_{j\to+\infty} \int_X \chi H_m(u_j), \end{aligned}$$

where in the first inequality we have used Lemma 2.7. But $H_m(u_j)$ and $H_m(u)$ have the same total mass, hence $H_m(u_j)$ weakly converges to $H_m(u)$ and therefore $H_m(u) \leq A H_m(\psi)$ as claimed. This together with (4.3) yields $I_1(u_j, u) \to 0$, hence by Theorem 4.8 we have $d(u_j, u) \to 0$. The last statement follows from Theorem 4.10. □

5. Complex Hessian equations with prescribed singularity

Given a non-pluripolar positive measure μ and a model potential ϕ such that $\mu(X) = \int_X H_m(\phi) > 0$, we want to find $u \in \mathcal{E}_\phi$ such that $H_m(u) = \mu$.

The strategy is described in [22] which is inspired by the supersolution method of [35]. One constructs supersolutions of a well chosen family of equations and takes the lower envelope of supersolutions to get a solution. The main issue is to bound the supersolutions from below. To make the arguments of [22] work in Hessian setting we need a volume-capacity comparison of the form :

$$\int_E f\omega^n \leq \left(\mathrm{Cap}_\phi(E)\right)^{1+\varepsilon},\ E \subset X,$$

for some $\varepsilon > 0$. Here

$$\mathrm{Cap}_\phi(E) = \sup\left\{ \int_E H_m(u) \mid u \in \mathrm{SH}_m(X,\omega),\ \phi - 1 \leq u \leq \phi \right\}.$$

In the flat case where $\omega = dd^c\|z\|^2$ and $X = \Omega \subset \mathbb{C}^n$, it was conjectured by Błocki [8] that $\mathrm{SH}_m(\Omega) \subset L^q(\Omega)$, for all $q < nm/(n-m)$. If the compact manifold version of Błocki's conjecture holds then the L^∞ estimate in [22] can be adapted in the Hessian setting giving solution for L^p densities $p > n/m$. In the general case of non-m-polar measures the approach in [22] using Cegrell's method [11] also breaks down in the Hessian setting.

Below, we will follow the main lines of [22] with several modifications. One of this is the use of the complete metric d in $\mathcal{E}^1$ to construct subextensions of a d-converging sequence in $\mathcal{E}^1$. This procedure not only replaces the relative L^∞ estimate in [22] but also allows us to solve the complex Hessian equation directly without regularizing the measure μ by taking local convolution.

5.1. **Existence of solutions for bounded densities.** To explain the main ideas of the proof we first start with the case where $\mu = f\omega^n$ for some $0 \leq f \in L^\infty(X, \omega^n)$, and $\phi = P[\alpha\phi_0]$, for some $\alpha \in (0,1)$ and $\phi_0 \in \mathrm{SH}_m(X,\omega)$. The general case, which is more involved and requires extra work, will be treated later.

Theorem 5.1. *Assume that $\phi = P[\alpha\phi_0]$, where $\alpha \in (0,1)$, $\phi_0 \in \mathrm{SH}_m(X,\omega)$, $0 \leq f \in L^\infty(X)$ and $\int_X H_m(\phi) = \int_X f\omega^n$. Then there exists $u \in \mathcal{E}_\phi(X,\omega,m)$ such that $H_m(u) = f\omega^n$.*

As shown in [20, 25], in this case one can use the ϕ-capacity to establish a L^∞-estimate. We propose, however, in this section a different approach using the envelope which is interesting in its own right.

Lemma 5.2. *Fix $\alpha \in (0,1)$ and let ϕ_0 be a ω-m-sh function on X, normalized by $\sup_X \phi_0 = 0$. Assume that $u \in \mathrm{SH}_m(X,\omega)$ is less singular than $\alpha\phi_0$ and*

$$H_m(u) = f\omega^n, \ \sup_X u = 0,$$

where $f \in L^p(X,\omega^n)$, $p > n/m$. Then, for a constant C depending on p, n, m, X, ω, α, $\|f\|_p$, we have

$$u \geq \alpha\phi_0 - C.$$

Proof. Set $b := (1-\alpha)^{-1}$ and $v_b := P(bu - \alpha b\phi_0) \in \mathrm{SH}_m(X,\omega)$. From the assumption that u is less singular than $\alpha\phi_0$ we deduce that $bu - b\alpha\phi_0$ is bounded from below, hence v_b is bounded. Then $b^{-1}v_b + \alpha\phi_0 \leq u$ with equality on $D := \{v_b = bu - \alpha b\phi_0\}$. Hence by Proposition 2.10, Lemma 2.11 and Proposition 3.10 we have

$$b^{-m}H_m(v_b) = \mathbf{1}_D b^{-m} H_m(v_b) \leq \mathbf{1}_D H_m(b^{-1}v_b + \alpha\phi_0) \leq \mathbf{1}_D H_m(u).$$

Next, we want to bound $\sup_X v_b$. Let q be the conjugate of p: $\frac{1}{p} + \frac{1}{q} = 1$. By Proposition 3.10 we have

$$\begin{aligned} \int_X |v_b|^{1/q} H_m(v_b) &= \int_D |bu - (b-1)\phi_0|^{1/q} H_m(v_b) \\ &\leq \int_X (|bu| + |(b-1)\phi_0|)^{1/q} b^m f\omega^n. \end{aligned}$$

Using the Hölder inequality we see that the above term is uniformly bounded. Since $\int_X H_m(v_b) = 1$ we infer that $\sup_X v_b$ is uniformly bounded. We thus can invoke [27], [47] to obtain a uniform bound for v_b, hence $bu \geq \alpha b\phi_0 - C$. This completes the proof. □

Using the same idea we obtain the following estimate :

Lemma 5.3. *Fix $a \in (0,1)$, $\phi_0 \in \mathrm{SH}_m(X,\omega)$, $\sup_X \phi_0 = 0$. Assume that $u \in \mathscr{E}$ satisfies*

$$H_m(u) \leq f\omega^n + aH_m(\phi_0), \ \sup_X u = 0,$$

where $f \in L^p(X,\omega^n), p > n/m$. Then, for a constant C depending on p, n, m, X, ω, a, $\|f\|_p$, we have

$$u \geq a^{1/m}\phi_0 - C.$$

Proof. Fix a constant $b > 1$ such that $(1-b^{-1})^m = a$, and set

$$v_b := P(bu - (b-1)\phi_0), \quad D := \{v_b = bu - (b-1)\phi_0\}.$$

It follows from Theorem 3.19 that $v_b \in \mathscr{E}$. Since $b^{-1}v_b + (1-b^{-1})\phi_0 \leq u$ with equality on D, by Proposition 2.10 and Lemma 2.11, we have

$$\mathbf{1}_D \left(b^{-m}H_m(v_b) + (1-b^{-1})^m H_m(\phi_0)\right) \leq \mathbf{1}_D H_m(u).$$

Using the above inequality, the assumption, and Proposition 3.10 we deduce that

$$H_m(v_b) = \mathbf{1}_D H_m(v_b) \leq b^m f\omega^n.$$

Having this, we can proceed as in the proof of Lemma 5.2. The details are left to the interested readers. □

Proof of Theorem 5.1. We use the supersolution method of [35, 22].

Construction of supersolutions. Fix $a \in (0,1)$ and solve, for each $k > 0$

$$H_m(u_k) = a\mathbf{1}_{\{\phi \leq -k\}} H_m(\max(\phi, -k)) + c_k f\omega^n,$$

with $u_k \in \mathscr{E}$, $\sup_X u_k = 0$. Here $c_k > 0$ is a constant ensuring that the two sides have the same total mass. The existence of the solution was proved in [50]. Computing the total mass we see that $c_k \searrow c(a) \geq 1$ defined by

$$a\left(1 - \int_X H_m(\phi)\right) + c(a)\int_X H_m(\phi) = 1.$$

It follows from Lemma 5.3 that, for a uniform constant C_1 depending on the fixed parameters (and also on a),

$$u_k \geq \phi - C_1.$$

For each $l > 0$ we define $\tilde{u}_{k,l} := P(\min(u_k, u_{k+1}, ..., u_{k+l}))$. Then by Corollary 3.11, for $t > 0$ fixed and $k > t$ we have

$$\mathbf{1}_{\{\phi > -t\}} H_m(\tilde{u}_{k,l}) \leq c_k f\omega^n.$$

As $l \to +\infty$, $\tilde{u}_{k,l}$ decreases to a function $\tilde{u}_k \in \mathrm{SH}_m(X,\omega)$ such that $\phi - C_1 \leq \tilde{u}_k \leq 0$. Thus by Theorem 3.3 we have

$$\mathbf{1}_{\{\phi > -t\}} H_m(\tilde{u}_k) \leq c_k f\omega^n.$$

As $k \to +\infty$, $\tilde{u}_k$ increases a.e. to a function $\tilde{u} \in \mathrm{SH}_m(X,\omega)$ such that $\phi - C_1 \leq \tilde{u} \leq 0$ and by Theorem 3.3 we have

$$\mathbf{1}_{\{\phi > -t\}} H_m(\tilde{u}) \leq c f\omega^n.$$

Letting $t \to +\infty$ we arrive at $H_m(\tilde{u}) \leq c f\omega^n$.

Envelope of supersolutions is a solution. The above analysis shows that for each $j \in \mathbb{N}$, there exists $w_j \in \mathrm{SH}_m(X,\omega)$ such that $\phi - C_j \leq w_j \leq 0$ and

$$H_m(w_j) \leq (1 + 2^{-j}) f\omega^n.$$

Adding a constant we can assume that $\sup_X w_j = 0$. By Lemma 5.2 we have

$$w_j \geq \alpha\phi_0 - C,$$

for a uniform constant C. For $k, l \in \mathbb{N}$, we set as above

$$\tilde{w}_{k,l} := P(\min(w_k, ..., w_{k+l})).$$

Then, $\tilde{w}_{k,l} \geq \alpha\phi_0 - C$, for all k, l, hence $\tilde{w}_k := \lim_l \tilde{w}_{k,l} \in \mathrm{SH}_m(X,\omega)$. Since $H_m(\tilde{w}_{k,l}) \leq (1 + 2^{-k}) f\omega^n$ it follows from Theorem 3.3 and Theorem 3.7 that $H_m(\tilde{w}_{k,l})$ weakly converges to $H_m(\tilde{w}_k)$. We thus have

$$H_m(\tilde{w}_k) \leq (1 + 2^{-k}) f\omega^n, \quad \tilde{w}_k \geq \alpha\phi_0 - C.$$

As $k \to +\infty$, $\tilde{w}_k$ increases a.e. to $\tilde{w}$. Again, it follows from Theorem 3.3 that $H_m(\tilde{w}_k)$ weakly converges to $H_m(\tilde{w})$, hence $H_m(\tilde{w}) \leq f\omega^n$. Since $\tilde{w} \geq \alpha\phi_0 - C$, it follows from Theorem 3.4 that $\int_X H_m(\tilde{w}) \geq \int_X f\omega^n$. We thus have equality, finishing the proof. □

5.2. **Existence of solutions for non-m-polar measures.**

Theorem 5.4. *Assume that μ is a positive measure vanishing on m-polar sets, and ϕ is a model potential such that $\mu(X) = \int_X H_m(\phi) > 0$. Then there exists a unique $u \in \mathcal{E}_\phi$ such that $H_m(u) = \mu$.*

Proof. It suffices to treat the case when $\mu \leq AH_m(\psi_0)$, for some constant $A > 0$ and some $\psi_0 \in \mathrm{SH}_m(X, \omega)$, with $-1 \leq \psi_0 \leq 0$. The general case will follow by a well-known projection argument due to Cegrell as shown in [37, 20].

In the arguments below we use C to denote various uniform constants.

Construction of supersolutions. For each $c > 1$, we claim that there exists $u_c \in \mathrm{SH}_m(X, \omega)$ such that

$$P[u_c] \geq \phi, \text{ and } H_m(u_c) \leq c\mu.$$

To prove the claim, we fix $a \in (0, 1)$ and solve, using [50, Theorem 1.3], for each $k > 0$

$$H_m(u_k) := a\mathbf{1}_{\{\phi \leq -k\}} H_m(\max(\phi, -k)) + c_k\mu,$$

with $u_k \in \mathcal{E}$, $\sup_X u_k = 0$. Recall that $\mathcal{E} := \mathcal{E}(X, \omega, m)$ is the class of ω-m-sh functions u with full mass, $\int_X H_m(u) = 1$. Here $c_k > 0$ is a constant ensuring that the two sides have the same total mass. Computing the total mass we see that $c_k \to c(a) \geq 1$ defined by

$$a\left(1 - \int_X H_m(\phi)\right) + c(a)\int_X H_m(\phi) = 1. \tag{5.1}$$

Fix $b > 1$ such that $(1 - b^{-1})^m = a$ and set

$$v_k := P(bu_k - (b-1)\max(\phi, -k)).$$

Since $0 = P[u_k]$, it follows from Corollary 3.20 (with $u, v \in \mathcal{E}$ hence $P[u] = P[v] = 0$) that $v_k \in \mathcal{E}$. Setting $D_k := \{v_k = bu_k - (b-1)\max(\phi, -k)\}$, it follows from Proposition 2.10 that

$$\mathbf{1}_{D_k}\left(b^{-m}H_m(v_k) + (1 - b^{-1})^m H_m(\max(\phi, -k))\right) \leq H_m(u_k).$$

By the choice of b and by Proposition 3.10 we have $H_m(v_k) \leq c_k b^m \mu$. By Proposition 3.10 again we have

$$\int_X |v_k| H_m(v_k) \leq \int_{D_k} |bu_k - (b-1)\max(\phi, -k)| b^m c_k \mu \leq C,$$

where the last estimate follows from [47, Corollary 3.18]. It thus follows that $\sup_X v_k$ is uniformly bounded. We can invoke Theorem 4.11 to construct a subsequence, still denoted by v_j, such that for all k,

$$\tilde{v}_k := \lim_{l \to +\infty} P(v_k, v_{k+1}, \ldots v_{k+l}) \in \mathcal{E}^1.$$

For each k, l we define

$$\tilde{u}_{k,l} := P(u_k, \ldots, u_{k+l});\ \tilde{u}_k := \lim_{l \to +\infty} \tilde{u}_{k,l},\ \tilde{u} := \left(\lim_{k \to +\infty} \tilde{u}_k\right)^*.$$

By the above construction we have that $u_k \geq b^{-1}v_k + (1 - b^{-1})\phi$, hence

$$\tilde{u}_k \geq b^{-1}\tilde{v}_k + (1 - b^{-1})\phi.$$

It thus follows from Lemma 3.9 that $P[\tilde{u}_k] \geq \phi$, hence $P[\tilde{u}] \geq \phi$. Fixing $t > 0$, by Corollary 3.11 we have that, for all $k > t$,

$$\mathbf{1}_{\{\phi > -t\}} H_m(\tilde{u}_{k,l}) \leq c_k\mu.$$

Since $\{\phi > -t\}$ is quasi-open, we can invoke Theorem 3.3 to obtain, letting $l \to +\infty$ and then $k \to +\infty$,

$$\mathbf{1}_{\{\phi>-t\}} H_m(\tilde{u}) \leq c(a)\mu,$$

Letting $t \to +\infty$ we obtain $H_m(\tilde{u}) \leq c(a)\mu$. From (5.1) we see that $c(a) \to 1$ as $a \to 1$, hence $c(a)$ can be made arbitrarily near 1. This proves the claim.

Envelope of supersolutions is a solution. The first step shows that for each $j \in \mathbb{N}$, there exists $w_j \in \mathrm{SH}_m(X,\omega)$ such that

$$\sup_X w_j = 0, \ P[w_j] \geq \phi, \text{ and } H_m(w_j) \leq (1+2^{-j})\mu.$$

It follows from Theorem 3.19 that there exists a constant $\lambda > 1$ such that $P(\lambda\phi) \in \mathrm{SH}_m(X,\omega)$. Fix $b > 1$ such that $b = (b-1)\lambda$. It follows from

$$P[w_j] \geq (1-b^{-1})\lambda\phi \geq (1-b^{-1})P(\lambda\phi)$$

and Corollary 3.21 that

$$h_j := P(bw_j - (b-1)P(\lambda\phi)) \in \mathscr{E}.$$

Moreover, it follows from Proposition 3.10 that

$$\int_X |h_j| H_m(h_j) \leq 2\int_X (|bw_j| + (b-1)|P(\lambda\phi)|) b^m \mu \leq C,$$

where the last estimate follows from the Chern-Levine-Nirenberg inequality [47, Corollary 3.18]. It thus follows that $\sup_X h_j$ is uniformly bounded, as well as $E(h_j)$. As in the proof of the claim we can find a subsequence, still denoted by h_j, such that

$$\tilde{h}_k := \lim_{l\to+\infty} P(h_k, ..., h_{k+l}) \in \mathscr{E}^1.$$

As in the first step we set

$$\tilde{w}_{k,l} := P(w_k, ..., w_{k+l}), \ \tilde{w}_k := \lim_{l\to+\infty} \tilde{w}_{k,l}, \ \tilde{w} := \left(\lim_{k\to+\infty} \tilde{w}_k\right)^*.$$

By construction we have

$$\tilde{w}_k \geq b^{-1}\tilde{h}_k + (1-b^{-1})P(\lambda\phi),$$

hence $\tilde{w}_k \in \mathrm{SH}_m(X,\omega)$. It follows from Proposition 3.22 that $\tilde{w}_{k,l} \in \mathscr{E}_\phi$. By Corollary 3.11 we have

$$H_m(\tilde{w}_{k,l}) \leq (1+2^{-k})\mu, \ \int_X H_m(\tilde{w}_{k,l}) \geq \mu(X).$$

By Theorem 3.7 we have that $H_m(\tilde{w}_{k,l})$ weakly converges to $H_m(\tilde{w}_k)$, hence

$$H_m(\tilde{w}_k) \leq (1+2^{-k})\mu, \ \int_X H_m(\tilde{w}_k) \geq \mu(X).$$

By Theorem 3.7 again we have $H_m(\tilde{w}) \leq \mu$ and $\int_X H_m(\tilde{w}) \geq \mu(X)$, hence equality. □

5.3. **Uniqueness.** To prove uniqueness, as shown in [22], one can follow closely the argument of S. Dinew [26]. We provide here a new proof using the orthogonal property of the envelopes. We hope that this proof, which is also new in the Monge-Ampère case, will be useful in studying Monge-Ampère type equations on non-Kähler manifolds.

Theorem 5.5. *Let ϕ be a model potential and let $u, v \in \mathcal{E}_\phi$. If $H_m(u) = H_m(v)$ then $u - v$ is constant.*

Proof. **Step 1.** We first assume that μ is concentrated on $\{u = v\}$ [1]. Fix $b > 1$ and set

$$\varphi_b := P(bu - (b-1)v), \; D := \{\varphi_b = bu - (b-1)v\}.$$

It follows from Theorem 3.19 that $\varphi_b \in \mathcal{E}_\phi$. Since $b^{-1}\varphi_b + (1-b^{-1})v \leq u$, with equality on D, it follows from Proposition 2.10 that

$$\mathbf{1}_D H_m(b^{-1}\varphi_b + (1-b^{-1})v) \leq \mathbf{1}_D H_m(u). \tag{5.2}$$

Combining this with the fact that $H_m(\varphi_b)$ is concentrated on D, and Lemma 2.11, we arrive at

$$b^{-m} H_m(\varphi_b) = b^{-m}\mathbf{1}_D H_m(\varphi_b) \leq \mathbf{1}_D H_m(u).$$

Writing $H_m(\varphi_b) = f_b\mu$, for some $0 \leq f_b \in L^1(\mu)$, and using the mixed Hessian inequality (Lemma 2.12), and multilinearity of the Hessian measure (Lemma 2.11) we obtain

$$\begin{aligned}
H_m(b^{-1}\varphi_b + (1-b^{-1})v) \;&\geq\; \sum_{k=0}^{m} \binom{m}{k} b^{-k}(1-b^{-1})^{m-k} \omega_{\varphi_b}^k \wedge \omega_v^{m-k} \wedge \omega^{n-m} \\
&\geq\; \sum_{k=0}^{m} \binom{m}{k} b^{-k}(1-b^{-1})^{m-k} f_b^{k/m} \mu \\
&=\; \left(b^{-1} f_b^{1/m} + 1 - b^{-1}\right)^m \mu. \qquad (5.3)
\end{aligned}$$

From (5.2) and (5.3) we have

$$\mathbf{1}_D \left(b^{-1} f_b^{1/m} + 1 - b^{-1}\right)^m \mu \leq \mathbf{1}_D H_m(b^{-1}\varphi_b + (1-b^{-1})v) \leq \mathbf{1}_D \mu.$$

We thus have $f_b \leq 1$, hence $f_b = 1$, μ-a.e. because $\int_X f_b\mu = \mu(X)$. It thus follows from (5.3) that, for $\psi_b := b^{-1}\varphi_b + (1-b^{-1})v$, we have $H_m(\psi_b) \geq \mu$ with the same total mass, hence $H_m(\psi_b) = \mu$. Thus, we have $\mu = H_m(\psi_b) = H_m(\varphi_b)$, therefore

$$\mu(\psi_b < u) = \int_{\{\psi_b < u\}} H_m(\psi_b) = \int_{\{\varphi_b < bu - (b-1)v\}} H_m(\varphi_b) = 0, \tag{5.4}$$

where in the last equality we use the fact that $H_m(\varphi_b)$ is concentrated in the contact set $\{\varphi_b = bu - (b-1)v\}$, thanks to Proposition 3.10. Now, we use the assumption that μ is concentrated on $\{u = v\}$ to deduce, using (5.4), that μ is concentrated on the set $\{\varphi_b = u = v\}$. Therefore

$$\mu(X) = \mu(u = \varphi_b) \leq \mu(u \leq \sup_X \varphi_b). \tag{5.5}$$

From (5.5) and the assumption that μ vanishes on m-polar sets, we infer that $\sup_X \varphi_b$ is uniformly bounded. Now, letting $b \to +\infty$ we see that the function $\lim_{b\to+\infty}(\varphi_b - \sup_X \varphi_b)$ is a ω-m-sh function which takes value $-\infty$ in the set $\{u < v\}$. This forces $\{u < v\}$ to be m-polar, hence $u = v$.

[1] One can also invoke the domination principle.

Step 2. We treat the general case. We normalize u, v by $\sup_X u = 0$, $\sup_X v = 0$. Set $\mu := H_m(u) = H_m(v)$. It follows from Lemma 2.9 that $w := \max(u, v)$ satisfies $H_m(w) \geq \mu$, and since $u \leq w \leq \phi$ we have by Theorem 3.4, $\int_X H_m(w) = \mu(X)$, hence $H_m(w) = \mu$. Thus, we can assume that $u \leq v$

We use the same notations and repeat the same arguments as above to arrive at (5.4). We then get

$$H_m(\psi_b) = H_m(u) = \mu,\ \psi_b \leq u, \text{ and } \mu(\psi_b < u) = 0.$$

Using the first step we have that $u = \psi_b$, hence $\varphi_b = bu-(b-1)v$, and $\sup_X \varphi_b = 0$ since $u \leq v \leq 0$ and $\sup_X u = 0$. Letting $b \to +\infty$ we obtain $u = v$. □

5.4. **Aubin-Yau equation.** Having the solutions to the complex Hessian equation $H_m(u) = \mu$, one can follow [20, 22] to prove the following result:

Theorem 5.6. *Assume that μ is a non-m-polar positive measure on X and ϕ is a model potential. Then there exists a unique $u \in \mathcal{E}_\phi$ such that $H_m(u) = e^u\mu$.*

We omit the proof of the above theorem and refer the interested readers to [20, 22].

5.5. **A Hodge index type inequality.** The proof of Theorem 1.3 is very similar to that of [22, Theorem 5.1] given Theorem 1.1, Theorem 1.2 and the mixed Hessian inequality (Lemma 2.12). For the reader's convenience we give the details below.

Proof of Theorem 1.3. For each $j = 1, ..., m$, let $v_j \in \mathcal{E}_{P[u_j]}$ solve $H_m(v_j) = c_j\omega^n$, where $c_j = \int_X H_m(v_j) = \int_X H_m(u_j)$. The existence of v_j follows from Theorem 1.2. The mixed Hessian inequality, Lemma 2.12, gives

$$H_m(v_1, ..., v_m) \geq (c_1...c_m)^{1/m}\omega^n.$$

By Lemma 3.8 we have that $\int_X H_m(v_1, ..., v_m) = \int_X H_m(u_1, ..., u_m)$, hence integrating the above inequality over X, we obtain the result. □

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