# COMPARISON OF MONGE-AMPÈRE CAPACITIES

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Abstract. Let $(X, \omega)$ be a compact Kähler manifold. We prove that all Monge-Ampère capacities are comparable. Using this we give a direct proof of the integration by parts formula for non-pluripolar products recently proved by M. Xia.

## 1. Introduction

Since Yau's solution to Calabi's conjecture [28] geometric pluripotential theory has found its important place in the development of differential geometry. An important tool in the theory is the Monge-Ampère capacity introduced by Bedford and Taylor [2]. By analyzing capacities of sublevel sets, Kołodziej [22] has established a fundamental $L^\infty$-estimate for complex Monge-Ampère equations. Several capacities have been studied in the literature with interesting applications, see [19, 5, 15, 16, 9, 2, 22, 13]. The goal of this note is to quantitatively compare these capacities.

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. Fix a smooth closed real $(1,1)$-form $\theta$ such that the De Rham cohomology class $\{\theta\}$ is big. Given $\psi \in \mathrm{PSH}(X, \theta)$ and a Borel subset $E \subset X$ we define

$$\mathrm{Cap}_{\theta,\psi}(E) = \sup\left\{ \int_E \theta_u^n \ : \ u \in \mathrm{PSH}(X,\theta),\ \psi - 1 \leq u \leq \psi \right\}.$$

Here $\theta_u^n$ is the non-pluripolar Monge-Ampère measure of $u$ (see Section 2).

The fact that these capacities characterize pluripolar sets suggests that they are all comparable. This is the content of our main result:

**Theorem 1.1.** *Let $\theta_1, \theta_2$ be smooth closed real $(1,1)$-forms on $X$ which represent big cohomology classes. Assume that $\psi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\psi_2 \in \mathrm{PSH}(X, \theta_2)$ are such that $\int_X (\theta_1 + dd^c\psi_1)^n > 0$ and $\int_X (\theta_2 + dd^c\psi_2)^n > 0$. Then there exist continuous functions $f, g : [0, +\infty) \to [0, +\infty)$ with $f(0) = g(0) = 0$ such that, for any Borel set $E \subset X$,*

$$\mathrm{Cap}_{\theta_1,\psi_1}(E) \leq f(\mathrm{Cap}_{\theta_2,\psi_2}(E)),\ \mathrm{Cap}_{\theta_2,\psi_2}(E) \leq g(\mathrm{Cap}_{\theta_1,\psi_1}(E)).$$

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Using the comparison of capacities we provide a new proof of the integration by parts formula, a result recently proved in [27] by M. Xia. His proof relies on a construction of D. Witt-Nyström [26]. Our proof uses a direct approximation method partially inspired by [17].

**Theorem 1.2.** [27] *Let $u, v \in L^\infty(X)$ be differences of quasi-plurisubharmonic functions. Fix $\phi_j \in \mathrm{PSH}(X, \theta_j)$, $j = 2, ..., n$ where $\{\theta_j\}$ is big. Then*

$$\int_X u dd^c v \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} = \int_X v dd^c u \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n}.$$

Here, if $u = \varphi - \psi$ with $\varphi, \psi \in \mathrm{PSH}(X, \eta)$ then, by definition,

$$dd^c u \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} := \eta_\varphi \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} - \eta_\psi \wedge \theta_{n,\phi_2} \wedge ... \wedge \theta_{n,\phi_n}$$

is a difference of non-pluripolar products (see Section 2 for the definition).

The integration by parts formula is a key ingredient in the variational approach to solve complex Monge-Ampère equations (see [3], [9]). When the potentials have small unbounded locus, i.e. these are locally bounded outside a closed complete pluripolar set, the above result was proved in [5].

The main idea of our proof of Theorem 1.2 is as follows. We first start with the simple case where $u = \varphi_1 - \varphi_2$ (with $\varphi_1, \varphi_2 \in \mathrm{PSH}(X, \omega)$) vanishes in some open neighborhood of the pluripolar set $\{\varphi_1 = -\infty\}$. In this case the result is a simple consequence of the plurifine locality of non-pluripolar products. For the general case we apply the first step with $\varphi_1$ and $\varphi_{2,\lambda} = \max(\varphi_1, \lambda\varphi_2)$ for $\lambda > 1$. We next use the comparison of capacities above to pass to the limit as $\lambda \searrow 1$.

**Organization of the note.** After preparing necessary background materials in Section 2 we systematically compare Monge-Ampère capacities in Section 3, proving Theorem 1.1. A new proof of Theorem 1.2 is given in Section 4.

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## 2. Preliminaries

In this section we recall necessary notions and tools in pluripotential theory. Several proofs more or less available in the literature will be included in this preliminary section for the convenience of the reader. For more details we refer to [5], [3], [10, 9, 8, 11, 12].

2.1. **Quasi-plurisubharmonic functions.** Let $(X,\omega)$ be a compact Kähler manifold of dimension $n$. Fix a closed smooth real $(1,1)$-form $\theta$. A function $u : X \to \mathbb{R}\cup\{-\infty\}$ is quasi-plurisubharmonic (qpsh) if locally $u = \rho + \varphi$ where $\rho$ is smooth and $\varphi$ is plurisubharmonic (psh). If additionally $\theta_u := \theta + dd^c u \geq 0$ in the weak sense of currents then $u$ is called $\theta$-psh. We let $\mathrm{PSH}(X,\theta)$ denote the set of all $\theta$-psh functions which are not identically $-\infty$. By elementary properties of psh functions one has that $\mathrm{PSH}(X,\theta) \subset L^1(X)$. Here, if nothing is stated $L^1(X)$ is $L^1(X,\omega^n)$. The De Rham cohomology class $\{\theta\}$ is big if $\mathrm{PSH}(X,\theta-\varepsilon\omega)$ is non-empty for some $\varepsilon > 0$.

Given $u, v \in \mathrm{PSH}(X,\theta)$ we say that $u$ is more singular than $v$, and denote by $u \preceq v$, if there exists a constant $C$ such that $u \leq v + C$ on $X$. We say that $u$ and $v$ have the same singularity type, and denote by $u \simeq v$, if $u \preceq v$ and $v \preceq u$. There is a natural least singular potential in $\mathrm{PSH}(X,\theta)$ given by

$$V_\theta(x) := \sup\{u(x) \ : \ u \in \mathrm{PSH}(X,\theta),\ u \leq 0 \text{ on } X\}.$$

As is well-known $V_\theta$ is locally bounded in the ample locus of $\{\theta\}$ which is a Zariski open set. A potential $u \in \mathrm{PSH}(X,\theta)$ has minimal singularity type if it has the same singularity type as $V_\theta$. Note that $V_\theta \equiv 0$ if and only if $\theta \geq 0$.

Let $\theta_1, ..., \theta_n$ be smooth closed $(1,1)$-forms representing big cohomology classes. Given $u_j \in \mathrm{PSH}(X,\theta_j)$, $j = 1, ..., n$, with minimal singularity type the Monge-Ampère measure

$$(\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_n + dd^c u_n)$$

is well defined, by Bedford and Taylor [1, 2], as a positive Borel measure on the intersection of the ample locus of $\{\theta_j\}$ with finite total mass. One extends this measure trivially over $X$, and the resulting measure is called the non-pluripolar Monge-Ampère product of $u_1, ..., u_n$. For general $u_j \in \mathrm{PSH}(X,\theta_j)$ one can consider the canonical approximants $u_{j,t} := \max(u_j, V_{\theta_j} - t)$, $t > 0$, $j = 1, ..., n$. The sequence of measures

$$\mathbf{1}_{\cap\{u_j > V_{\theta_j} - t\}}(\theta_1 + dd^c u_{1,t}) \wedge ... \wedge (\theta_n + dd^c u_{n,t})$$

is increasing in $t$. Its limit, denoted by $(\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_n + dd^c u_n)$, is a positive Borel measure on $X$. To simplify the notation we also denote the latter by $\theta_{1,u_1} \wedge .... \wedge \theta_{n,u_n}$. When $u_1 = ... = u_n$ and $\theta_1 = ... = \theta_n = \theta$ we obtain the non-pluripolar Monge-Ampère measure of $u$, denoted by $(\theta + dd^c u)^n$ or simply by $\theta_u^n$.

We let $\mathcal{E}(X,\theta)$ denote the set of all $\theta$-psh functions $u$ with full Monge-Ampère mass, i.e. such that $\int_X (\theta + dd^c u)^n = \int_X (\theta + dd^c V_\theta)^n = \mathrm{Vol}(\theta)$.

Let $u \in L^\infty(X)$ be a difference of quasi-plurisubharmonic functions, and fix $\phi_j \in \mathrm{PSH}(X, \theta_j)$, $j = 2, ..., n$. We define

$$dd^c u \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} := \eta_\varphi \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} - \eta_\psi \wedge \theta_{n,\phi_2} \wedge ... \wedge \theta_{n,\phi_n},$$

if $u = \varphi - \psi$ and $\varphi, \psi \in \mathrm{PSH}(X, \eta)$ for some Kähler form $\eta$. As shown in Lemma 2.1 below $dd^c u \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n}$ is well-defined as a signed measure on $X$.

**Lemma 2.1.** *The above definition does not depend on $\eta, \varphi, \psi$.*

*Proof.* Assume that $u = \varphi' - \psi'$ and $\varphi', \psi' \in \mathrm{PSH}(X, \eta')$ for some Kähler form $\eta'$. Then $\varphi + \psi' = \varphi' + \psi$ is $(\eta + \eta')$-psh. Since non-pluripolar products are multilinear (see [5, Proposition 1.4]), we infer that

$$(\eta_\varphi + \eta'_{\psi'}) \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} = \eta_\varphi \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} + \eta'_{\psi'} \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n},$$

and

$$(\eta'_{\varphi'} + \eta_\psi) \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} = \eta_\psi \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n} + \eta'_{\varphi'} \wedge \theta_{2,\phi_2} \wedge ... \wedge \theta_{n,\phi_n}.$$

From these equalities we get the result. □

2.2. **Monotonicity of Monge-Ampère mass.**

**Lemma 2.2.** *Let $u, v \in \mathrm{PSH}(X, \theta)$. If $u$ and $v$ have the same singularity type, then $\int_X \theta_u^n = \int_X \theta_v^n$.*

The above result was first proved by D. Witt-Nyström [26]. A different proof has been recently given in [23] using the monotonicity of the energy functional. We give below a direct proof using a standard approximation process. Another different proof has been recently given in [25] where generalized non-pluripolar products of positive currents are studied.

We also stress that our proof only uses the invariance of the Monge-Ampère mass of bounded $\omega$-psh functions. It is thus valid on non-Kähler manifolds $(X, \omega)$ satisfying, for all $u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$,

$$\int_X (\omega + dd^c u)^n = \int_X \omega^n.$$

As shown in [6] the above condition is equivalent to $i\partial\bar{\partial}\omega^k = 0$, for all $k = 1, ..., n-1$.

*Proof.* **Step 1.** Assume that $\theta$ is a Kähler form.

We first prove the following claim: if there exists a constant $C > 0$ such that $u = v$ on the open set $U := \{\min(u, v) < -C\}$ then $\int_X \theta_u^n = \int_X \theta_v^n$.

Fix $t > C$. Since $u = v$ on $U$, we have $\max(u, -t) = \max(v, -t)$ on $U$. Since $U$ is open, we have

$$\mathbf{1}_U (\theta + dd^c \max(u, -t))^n = \mathbf{1}_U (\theta + dd^c \max(v, -t))^n.$$

We also have $\{u \leq -t\} = \{v \leq -t\} \subset U$. Indeed, if $u(x) \leq -t$ then $x \in U$ because $-t < -C$. But on $U$ we have $u = v$, hence $v(x) = u(x) \leq -t$.

By plurifine locality we thus have

$$\begin{aligned}\theta^n_{\max(u,-t)} &= \mathbf{1}_{\{u>-t\}}\theta^n_{\max(u,-t)} + \mathbf{1}_{\{u\leq -t\}}\theta^n_{\max(u,-t)}\\ &= \mathbf{1}_{\{u>-t\}}\theta^n_u + \mathbf{1}_{\{v\leq -t\}}\theta^n_{\max(v,-t)},\end{aligned}$$

and

$$\theta^n_{\max(v,-t)} = \mathbf{1}_{\{v>-t\}}\theta^n_v + \mathbf{1}_{\{v\leq -t\}}\theta^n_{\max(v,-t)}.$$

Integrating over $X$, since $\int_X \theta^n_{\max(u,-t)} = \int_X \theta^n_{\max(v,-t)} = \mathrm{Vol}(\theta)$, we get

$$\int_{\{u>-t\}} \theta^n_u = \int_{\{v>-t\}} \theta^n_v.$$

Letting $t \to +\infty$ we prove the claim.

We now come back to the proof of the lemma in the Kähler case. We can assume that $v \leq u \leq v + B \leq 0$, for some positive constant $B$. For each $a \in (0, 1)$ we set $v_a := av$, $u_a := \max(u, v_a)$ and $C := Ba(1-a)^{-1}$. To use the claim above we need to check that $u_a = v_a$ on the open set $U_a := \{\min(u_a, v_a) < -C\}$. Observe that $\min(u_a, v_a) = v_a$ because $v_a \leq u_a$. If $x \in U_a$ then $av(x) < -C$ hence $(1-a)v(x) < -B$, which implies (recall that $v + B \geq u$)

$$av(x) \geq v(x) + B \geq u(x).$$

We infer that $v_a(x) \geq u(x)$, hence $u_a(x) = v_a(x)$. We can thus apply the claim above to get $\int_X \theta^n_{u_a} = \int_X \theta^n_{v_a}$. Since non-pluripolar products are multilinear, see [5, Proposition 1.4], we have that

$$\int_X \theta^n_{v_a} = a^n \int_X \theta^n_v + \sum_{k=0}^{n-1} a^k(1-a)^{n-k} \int_X \theta^k_v \wedge \theta^{n-k} \to \int_X \theta^n_v$$

as $a \nearrow 1$. Since $u_a \searrow u$ as $a \nearrow 1$, by [9, Theorem 2.3] we have

$$\liminf_{a\to 1^-} \int_X \theta^n_{u_a} \geq \int_X \theta^n_u.$$

We thus have $\int_X \theta^n_u \leq \int_X \theta^n_v$. For the reverse inequality we repeat the same arguments with $\tilde{u}_a = au$ and $\tilde{v}_a := \max(v+B, u_a)$. We finally have $\int_X \theta^n_u = \int_X \theta^n_v$, finishing the proof of Step 1.

**Step 2.** We treat the general case: $\{\theta\}$ is merely big. We use an idea in [17]. Fix $s > 0$ so large that $\theta + s\omega$ is Kähler. For $t > s$ we have, by the first step,

$$\int_X (\theta + t\omega + dd^c u)^n = \int_X (\theta + t\omega + dd^c v)^n.$$

Since non-pluripolar products are multilinear, see [5, Proposition 1.4], we have for all $t > s$,

$$\sum_{k=0}^{n} \binom{n}{k} \int_X \theta^k_u \wedge \omega^{n-k} t^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \int_X \theta^k_v \wedge \omega^{n-k} t^{n-k}.$$

We thus obtain an equality between two polynomials and identifying the coefficients we infer the desired equality. □

2.3. **Quasi-psh envelopes and relative full mass classes.** Let $f = f_1 - f_2$ be a difference of two quasi-psh functions. We let $P_\theta(f)$ denote the largest $\theta$-psh function on $X$ lying below $f$:

$$P_\theta(f)(x) := \left(\sup\{u(x) \ : \ u \in \mathrm{PSH}(X,\theta), \ u \leq f \text{ on } X\}\right)^* .$$

Here, the inequality $u \leq f$ is understood as $u + f_2 \leq f_1$ on $X$. A potential $\phi \in \mathrm{PSH}(X,\theta)$ is called a model potential if $\int_X (\theta + dd^c\phi)^n > 0$ and $P_\theta[\phi] = \phi$, where $P_\theta[\phi]$ is the envelope of singularity type of $\phi$, introduced by J. Ross and D. Witt-Nyström [24]:

$$P_\theta[\phi] := \left(\lim_{C\to+\infty} P_\theta(\min(\phi + C, 0))\right)^* .$$

**Lemma 2.3.** *The operator $P_\theta[\cdot]$ is concave: if $u, v \in \mathrm{PSH}(X,\theta)$ and $t \in (0,1)$ then*

$$P_\theta[tu + (1-t)v] \geq tP_\theta[u] + (1-t)P_\theta[v].$$

*Proof.* Assume $u, v \in \mathrm{PSH}(X,\theta)$ and $t \in (0,1)$. Then, for all $C > 0$,

$$P_\theta(\min(tu + (1-t)v + C, 0)) \geq tP_\theta(\min(u + C, 0)) + (1-t)P_\theta(\min(v + C, 0)),$$

because the right-hand side is a $\theta$-psh function lying below $\min(tu+(1-t)v+C, 0)$. Letting $C \nearrow +\infty$ we arrive at the result. □

When $u \in \mathcal{E}(X,\theta)$ we have $P_\theta[u] = V_\theta$. Given a model potential $\phi$ we define

$$\mathrm{PSH}(X,\theta,\phi) := \{u \in \mathrm{PSH}(X,\theta), \ u \preceq \phi\},$$

and

$$\mathcal{E}(X,\theta,\phi) := \left\{u \in \mathrm{PSH}(X,\theta,\phi), \ \int_X \theta_u^n = \int_X \theta_\phi^n\right\}.$$

The following characterization of the class $\mathcal{E}(X,\theta,\phi)$ was proved in [9, Theorem 1.3], [12, Lemma 2.7].

**Theorem 2.4.** *Let $\phi \in \mathrm{PSH}(X,\theta)$ be a model potential and $u \in \mathrm{PSH}(X,\theta)$. Then*

$$u \in \mathcal{E}(X,\theta,\phi) \Longleftrightarrow u \preceq \phi \text{ and } P_\theta[u] = \phi,$$

$$u \in \mathrm{PSH}(X,\theta,\phi) \Longleftrightarrow u - \sup_X u \leq \phi.$$

One of the main tools in relative pluripotential theory is the comparison principle, proved in [26], [9].

**Theorem 2.5** (Comparison principle)**.** *If $u, v \in \mathcal{E}(X,\theta,\phi)$ then*

$$\int_{\{u<v\}} \theta_v^n \leq \int_{\{u<v\}} \theta_u^n \ \text{ and } \int_{\{u\leq v\}} \theta_v^n \leq \int_{\{u\leq v\}} \theta_u^n.$$

*Proof.* The first inequality was proved in [9, Proposition 3.5]. The second inequality is a simple consequence of the first one as we now explain:

$$\int_{\{u\leq v\}} \theta_v^n = \int_X \theta_v^n - \int_{\{v<u\}} \theta_v^n.$$

By the first inequality of the theorem we also have:

$$\int_{\{v<u\}} \theta_v^n \geq \int_{\{v<u\}} \theta_u^n.$$

Combining these we obtain

$$\int_{\{u\leq v\}} \theta_v^n \leq \int_X \theta_u^n - \int_{\{v<u\}} \theta_u^n = \int_{\{u\leq v\}} \theta_u^n.$$

□

**Lemma 2.6.** *Assume $u, v \in \mathrm{PSH}(X,\theta)$ and $u \leq v$. Assume also that, for all $t > 1$, $P_\theta(tu-(t-1)v) \in \mathrm{PSH}(X,\theta)$. Then, for all $b > 1$,*

$$\int_X (\theta + dd^c P_\theta(bu-(b-1)v))^n = \int_X \theta_v^n.$$

*Proof.* Fix $t > b$ and observe that

$$\varphi := P_\theta(bu-(b-1)v) \geq t^{-1}bP_\theta(tu-(t-1)v) + (1-t^{-1}b)v.$$

By Lemma 2.3,

$$P_\theta[\varphi] \geq t^{-1}bP_\theta[P_\theta(tu-(t-1)v)] + (1-t^{-1}b)P_\theta[v].$$

Set $\psi_t := P_\theta[P_\theta(tu-(t-1)v)]$. Then $\sup_X \psi_t = 0$ and $\psi_t \in \mathrm{PSH}(X, A\omega)$ for some fixed constant $A > 0$. It follows from [19, Proposition 2.7] that the functions $\psi_t$ stay in a compact set of $L^1(X)$. Letting $t \to +\infty$ we thus have

$$P_\theta[\varphi] \geq P_\theta[v].$$

Since $u \leq v$, we also have $\varphi \leq v$. Combining this and the above inequality we see that $P_\theta[\varphi] = P_\theta[v]$, and using Theorem 2.4 we arrive at the result. □

**Lemma 2.7.** *If $u \in \mathcal{E}(X,\omega)$ then $P_\theta(u) \in \mathcal{E}(X,\theta)$.*

*Proof.* Since $u \in \mathcal{E}(X, A\omega)$ for any $A \geq 1$, we can assume that $\omega \geq \theta$. By Lemma 2.6 it suffices to prove that, for all $b \geq 1$,

$$P_\theta(bu + (1-b)V_\theta) \in \mathrm{PSH}(X,\theta).$$

Set $u_j = \max(u, -j)$, $v_j := P_\theta(bu_j + (1-b)V_\theta)$, $\varphi_j := b^{-1}v_j + (1-b^{-1})V_\theta$, and

$$D := \{v_j = bu_j + (1-b)V_\theta\}.$$

Observe that $bu_j + (1-b)V_\theta$ is bounded from below by $-bj + V_\theta$, hence $v_j$ has minimal singularities. Since $0 \leq \theta + dd^c\varphi_j \leq \omega + dd^c\varphi_j$, we have

$$(\theta + dd^c\varphi_j)^n \leq (\omega + dd^c\varphi_j)^n.$$

On the other hand, since $\varphi_j \leq u_j$ and $\{\varphi_j = u_j\} = D$, it follows from the pluripotential maximum principle (see e.g. [21, Corollary 3.28]) that

$$\mathbf{1}_D(\omega + dd^c\varphi_j)^n \leq \mathbf{1}_D(\omega + dd^c u_j)^n.$$

Since non-pluripolar products are multilinear, we have

$$(\theta + dd^c\varphi_j)^n \geq b^{-n}(\theta + dd^c v_j)^n.$$

By [12, Lemma 4.4], $(\theta + dd^c v_j)^n$ is supported on $D$. It thus follows that

$$(\theta + dd^c v_j)^n \leq \mathbf{1}_D b^n(\omega + dd^c u_j)^n. \tag{2.1}$$

We choose $t > 0$ so large that $b^n \int_{\{u \leq -b^{-1}t\}}(\omega + dd^c u)^n < \mathrm{Vol}(\theta)$. Note that for $j > b^{-1}t$ we have

$$\{u > -b^{-1}t\} = \{u_j > -b^{-1}t\} \subset \{u_j > -j\} = \{u > -j\}.$$

Note also that $\int_X \omega_{u_j}^n = \int_X \omega^n$ because $u_j$ is bounded, and $\int_X \omega_u^n = \int_X \omega^n$ because $u \in \mathcal{E}(X,\omega)$. By plurifine locality, we then have

$$\begin{aligned} b^n \int_{\{u_j \leq -b^{-1}t\}} \omega_{u_j}^n &= b^n \int_X \omega_{u_j}^n - b^n \int_{\{u > -b^{-1}t\}} \omega_{u_j}^n \\ &= b^n \int_X \omega_u^n - b^n \int_{\{u > -b^{-1}t\}} \omega_u^n \\ &= b^n \int_{\{u \leq -b^{-1}t\}} \omega_u^n < \mathrm{Vol}(\theta). \end{aligned}$$

Combining this with (2.1) we arrive at

$$\int_{\{v_j \leq -t\}} (\theta + dd^c v_j)^n \leq b^n \int_{\{u_j \leq -b^{-1}t\}} (\omega + dd^c u_j)^n < \mathrm{Vol}(\theta),$$

where in the above inequalities we used that $\{v_j \leq -t\} \cap D \subseteq \{u_j \leq -b^{-1}t\}$. It thus follows that $\{v_j \leq -t\} \neq X$ and so $\sup_X v_j > -t$. It then follows that $v_j \searrow v \in \mathrm{PSH}(X,\theta)$ and $v = P_\theta(bu + (1-b)V_\theta)$, finishing the proof. □

**Lemma 2.8.** *Assume $\phi \in \mathrm{PSH}(X,\theta)$, $P_\theta[\phi] = \phi$ and $\int_X \theta_\phi^n > 0$. If $b > 1$ and $u, v \in \mathcal{E}(X,\theta,\phi)$ then $P_\theta(bu - (b-1)v) \in \mathcal{E}(X,\theta,\phi)$.*

*Proof.* The proof is similar to that of [23, Corollary 3.20] which is inspired by [12]. Since $u, v \in \mathcal{E}(X,\theta,\phi)$, both $u$ and $v$ are more singular than $\phi$. We can assume that $\max(u,v) \leq \phi$. It follows from [12, Lemma 4.3] that $P_\theta(bu-(b-1)\phi) \in \mathrm{PSH}(X,\theta)$ for all $b > 1$. Lemma 2.6 ensures that $P_\theta(bu - (b-1)\phi) \in \mathcal{E}(X,\theta,\phi)$ for all $b > 1$. Set $\psi := P_\theta(bu - (b-1)v)$ and $\varphi := P_\theta(bu - (b-1)\phi)$. Then $\psi \geq \varphi$ and $\varphi \in \mathcal{E}(X,\theta,\phi)$. It follows that $P_\theta[\psi] \geq P_\theta[\varphi] = \phi$. We also have

$$b^{-1}\psi + (1-b^{-1})v \leq u,$$

which, by Lemma 2.3, gives

$$b^{-1}P_\theta[\psi] + (1-b^{-1})P_\theta[v] \leq P_\theta[b^{-1}\psi + (1-b^{-1})v] \leq P_\theta[u] = \phi.$$

Since $P_\theta[v]=\phi$, we can thus conclude that $P_\theta[\psi]=P_\theta[\varphi]=\phi$, and $\psi\in\mathcal{E}(X,\theta,\phi)$. □

2.4. **Monge-Ampère capacities.** Fix a $\theta$-psh function $\psi\le 0$. We define, for each Borel set $E\subset X$,

$$\mathrm{Cap}_{\theta,\psi}(E):=\sup\left\{\int_E\theta_u^n\ :\ u\in \mathrm{PSH}(X,\theta),\ \psi-1\le u\le\psi\right\}.$$

When $\theta=\omega$ and $\psi=0$ we recover the classical Monge-Ampère capacity $\mathrm{Cap}_\omega:=\mathrm{Cap}_{\omega,0}$ (see [2], [22], [19]). The Monge-Ampère capacity $\mathrm{Cap}_{\theta,\psi}$ is inner regular: for any Borel set $E$ we have

$$\mathrm{Cap}_{\theta,\psi}(E)=\sup\{\mathrm{Cap}_{\theta,\psi}(K)\ :\ K\subset E \text{ is compact}\}.$$

Given a Borel set $E\subset X$, the global $\phi$-extremal function is defined by

$$V_{E,\theta,\phi}(x):=\sup\{v(x)\ :\ v\in\mathrm{PSH}(X,\theta),\ v\preceq\phi,\ v\le\phi \text{ on } E\},\ x\in X.$$

It was shown in [9, 11], when $\phi$ is a model potential and $E$ is non-pluripolar, that $V^*_{E,\theta,\phi}$ is a $\theta$-psh function having the same singularity type as $\phi$. Moreover $V^*_{E,\theta,\phi}=\phi$ on $E$ modulo a pluripolar set. We set

$$M_{E,\theta,\phi}:=\sup_X V^*_{E,\theta,\phi}.$$

When $\phi=V_\theta$ we will simply write $V_{E,\theta}:=V_{E,\theta,V_\theta}$ and $M_{E,\theta}:=M_{E,\theta,V_\theta}$. Observe that $V_{E,\theta,\phi}\le V_{E,\theta}$, hence $M_{E,\theta,\phi}\le M_{E,\theta}$.

**Lemma 2.9.** *Let $\phi\in\mathrm{PSH}(X,\theta)$ be such that $\int_X\theta_\phi^n>0$. If $E\subset X$ is a Borel set and $P\subset X$ is a pluripolar set then $V^*_{E\cup P,\theta,\phi}=V^*_{E,\theta,\phi}$.*

*Proof.* It follows from the definition that $V_{E,\theta,\phi}\ge V_{E\cup P,\theta,\phi}$ since $E\subset E\cup P$. Let now $u\in\mathrm{PSH}(X,\theta)$ be a candidate defining $V_{E,\theta,\phi}$. We claim that there exists $v\in\mathrm{PSH}(X,\theta)$ such that $v\le\phi$ and $P\subset\{v=-\infty\}$. Indeed, it follows from [19, 20] that there exists $v_0\in\mathrm{PSH}(X,\omega)$ such that $P\subset\{v_0=-\infty\}$. Moreover, up to replacing $v_0$ with $-(-v_0)^\alpha$, $0<\alpha<1$, we can assume that $v_0\in\mathcal{E}(X,\omega)$ (see [7, Corollary 2.6]). By Lemma 2.7 we have $P_\theta(v_0)\in\mathcal{E}(X,\theta)$, in particular

$$\int_X\theta_\phi^n+\int_X\theta^n_{P_\theta(v_0)}=\int_X\theta_\phi^n+\mathrm{Vol}(\theta)>\int_X\theta^n_{\max(\phi,P_\theta(v_0))}=\mathrm{Vol}(\theta).$$

It then follows from [12, Lemma 5.1] that

$$v:=P_\theta(\min(\phi,v_0))=P_\theta(\min(\phi,P_\theta(v_0)))\in\mathrm{PSH}(X,\theta).$$

Note also that $P\subset\{v_0=-\infty\}\subset\{v=-\infty\}$ and $v\le\phi$. This proves the claim.

For each $\lambda\in(0,1)$ the function $u_\lambda:=\lambda v+(1-\lambda)u$ is $\theta$-psh and satisfies $u_\lambda\preceq\phi$, $u_\lambda\le\phi$ on $E\cup P$. We thus have $u_\lambda\le V^*_{E\cup P,\theta,\phi}$. Letting $\lambda\to 0^+$ we obtain $u\le V^*_{E\cup P,\theta,\phi}$ on $X$ modulo a pluripolar set, hence on the whole $X$. This finally gives $V^*_{E,\theta,\phi}\le V^*_{E\cup P,\theta,\phi}$. □

**Proposition 2.10.** *If* $\psi \in \mathrm{PSH}(X,\theta)$ *satisfies* $\int_X (\theta + dd^c\psi)^n > 0$ *then* $\mathrm{Cap}_{\theta,\psi}$ *characterizes pluripolar sets: for all Borel sets* $E$ *we have*

$$\mathrm{Cap}_{\theta,\psi}(E) = 0 \Longleftrightarrow E \textit{ is pluripolar.}$$

The proof below is similar to that of [9, Lemma 4.3].

*Proof.* If $E$ is pluripolar then by definition $\mathrm{Cap}_{\theta,\psi}(E) = 0$.

Conversely, assume that $E$ is non-pluripolar. Then there exists a compact set $K \subset E$ such that $K$ is non-pluripolar. Let $V_{K,\theta}$ be the global extremal $\theta$-psh function of $K$. Then $V^*_{K,\theta} \in \mathrm{PSH}(X,\theta)$ has minimal singularity type. In particular there exists $C > 0$ such that $\psi - C \leq V^*_{K,\theta}$. For $t > 0$ we set

$$u_t := P_\theta(\min(\psi + t, V^*_{K,\theta})).$$

Observe that $\psi - C \leq u_t \leq \psi + t$, i.e. $u_t$ and $\psi$ have the same singularity type. It follows from [11, Lemma 3.6] that $\theta^n_{V^*_{K,\theta}}$ is supported on $K$. By [9, Lemma 3.7], the measure $\theta^n_{u_t}$ is supported on $\{u_t = \min(\psi + t, V^*_{K,\theta})\}$. By [12, Lemma 4.5],

$$\mathbf{1}_{\{u_t=\psi+t\}}\theta^n_{u_t} \leq \mathbf{1}_{\{u_t=\psi+t\}}\theta^n_{\psi},$$

and

$$\mathbf{1}_{\{u_t=V^*_{K,\theta}\}\setminus K}\theta^n_{u_t} \leq \mathbf{1}_{\{u_t=V^*_{K,\theta}\}\setminus K}(\theta + dd^c V^*_{K,\theta})^n = 0.$$

By [26] (see also [9] and Lemma 2.2), $\int_X \theta^n_\psi = \int_X \theta^n_{u_t}$. We thus get

$$0 < \int_X \theta^n_\psi = \int_X \theta^n_{u_t} \leq \int_{\{u_t=\psi+t\}} \theta^n_\psi + \int_K \theta^n_{u_t}.$$

The first term on the right-hand side converges to 0 as $t \to +\infty$. Thus for $t > 1$ large enough we have that $\int_K \theta^n_{u_t} > 0$, hence $\mathrm{Cap}_{\theta,\psi}(K) > 0$. □

A sequence of functions $u_j$ converges in capacity to $u$ if, for any $\delta > 0$,

$$\lim_{j\to+\infty} \mathrm{Cap}_\omega(\{x \in X \ : \ |u_j(x) - u(x)| > \delta\}) = 0.$$

We will also need the following convergence result whose proof is similar to that of [10, Corollary 2.9]:

**Theorem 2.11.** *Assume that* $\mu_j$ *is a sequence of positive Borel measures converging weakly to* $\mu$. *Assume that there exists a continuous function* $f : [0,+\infty) \to [0,+\infty)$ *with* $f(0) = 0$ *such that, for any Borel set* $E$,

$$\mu_j(E) + \mu(E) \leq f\left(\mathrm{Cap}_\omega(E)\right).$$

*Let* $u_j$ *be a sequence of uniformly bounded quasi-continuous functions which converges in capacity to a bounded quasi-continuous function* $u$. *Then* $u_j\mu_j \to u\mu$ *in the sense of Radon measures on* $X$.

*Proof.* Fixing $\varepsilon > 0$ there exist $v, v_j$ continuous functions on $X$ such that

$$\mathrm{Cap}_\omega(\{x \in X \ : \ u_j(x) \neq v_j(x) \text{ or } u(x) \neq v(x)\}) < \varepsilon.$$

Let $A > 0$ be a constant such that $|u_j| + |v_j| + |u| + |v| \leq A$ on $X$. Fix $\delta > 0$. For $j > N$ large enough we have, by the assumption that $u_j$ converges in capacity to $u$, that

$$\mathrm{Cap}_\omega(\{x \in X \ : \ |u_j(x) - u(x)| > \delta) < \varepsilon.$$

Fixing a continuous function $\chi$, it follows from the above that

$$\begin{aligned}
&|\int_X (\chi u_j \mu_j - \chi u d\mu)| \leq \int_X |\chi(u_j - u)|\mu_j + |\int_X \chi u(\mu_j - \mu)| \\
&\leq \delta \int_X |\chi|\mu_j + A \sup_X |\chi| \mu_j(\{|u_j - u| > \delta\}) \\
&+ |\int_X \chi(u - v)(\mu_j - \mu)| + |\int_X \chi v(\mu_j - \mu)| \\
&\leq \delta \int_X |\chi|\mu_j + A \sup_X |\chi| f(\varepsilon) + |\int_X \chi(u - v)(\mu_j - \mu)| + |\int_X \chi v(\mu_j - \mu)| \\
&\leq \delta \int_X |\chi|\mu_j + 2A \sup_X |\chi| f(\varepsilon) + |\int_X \chi v(\mu_j - \mu)|.
\end{aligned}$$

Since $v$ is continuous on $X$ the last term converges to $0$ as $j \to +\infty$. This completes the proof. □

## 3. Comparison of Monge-Ampère capacities

In this section we establish a comparison between Monge-Ampère capacities. We first prove a version of the Chern-Levine-Nirenberg inequality.

**Lemma 3.1.** *Let $u, v, \psi \in \mathrm{PSH}(X, \omega)$. Assume that $v \leq u \leq v + B$ for some positive constant $B$. Then*

$$\int_X \psi \omega_u^n \geq \int_X \psi \omega_v^n - nB \int_X \omega^n.$$

*Proof.* By subtracting a constant we can assume that $u \leq 0$. We first prove the lemma under the assumption that $u = v$ on the open set

$$U := \{\min(u, v) = v < -C\},$$

for some positive constant $C$.

We approximate $u$ and $v$ by $u^t := \max(u, -t)$ and $v^t := \max(v, -t)$. For $t > 0$ we apply the integration by parts formula for bounded $\omega$-psh functions, which is a consequence of Stokes theorem, to get

$$\int_X \psi(\omega_{u^t}^n - \omega_{v^t}^n) = \int_X (u^t - v^t) dd^c \psi \wedge S^t,$$

where $S^t := \sum_{k=0}^{n-1} \omega_{u^t}^k \wedge \omega_{v^t}^{n-1-k}$. Note that $\int_X \omega \wedge S^t = n\int_X \omega^n$ as there are $n$ terms in the sum and each of them is equal to $\int_X \omega^n$ by Stokes' theorem. Since $u^t \geq v^t$ we can continue the above estimate and obtain

$$\int_X \psi(\omega_{u^t}^n - \omega_{v^t}^n) = \int_X (u^t - v^t)(\omega_\psi \wedge S^t - \omega \wedge S^t) \geq -Bn\int_X \omega^n.$$

For $t > B+C$ we have that $u^t = v^t$ on the open set $U$ which contains $\{u \leq -t\} = \{v \leq -t\}$. It thus follows that $\mathbf{1}_U\omega_{u^t}^n = \mathbf{1}_U\omega_{v^t}^n$. Thus, for $t > B+C$ we have

$$\int_{\{v>-t\}} \psi(\omega_u^n - \omega_v^n) \quad = \quad \int_X \psi(\omega_{u^t}^n - \omega_{v^t}^n) \geq -Bn\int_X \omega^n.$$

Letting $t \to +\infty$ we finish the first step.

We now treat the general case. By approximating $\psi$ from above by smooth $\omega$-psh functions, see [14], [4], we can assume that $\psi$ is smooth (in fact, we only need the continuity of $\psi$). We fix $a \in (0,1)$ and set $v_a := av$, $u_a := \max(u, v_a)$. Setting $C := a(1-a)^{-1}B$ we have that $u_a = v_a$ on $\{v_a < -C\}$ (see Step 1 in the proof of Lemma 2.2). We can thus apply the first step to get

$$\int_X \psi\omega_{u_a}^n \geq \int_X \psi\omega_{v_a}^n - nB\int_X \omega^n.$$

Observe that $v_a \searrow v$ and $u_a \searrow u$ as $a \nearrow 1$. Also, by the multilinearity of non-pluripolar products, we have

$$\lim_{a\to 1^-} \int_X (\omega + dd^c v_a)^n = \int_X (\omega + dd^c v)^n.$$

Recalling that $v \leq u \leq 0$, we have $u \leq u_a = \max(u, av) \leq \max(u, au) = au$. By [9, Theorem 2.4] we thus have

$$\int_X \omega_u^n \leq \int_X \omega_{u_a}^n \leq \int_X (\omega + add^c u)^n.$$

Using the multilinearity of non-pluripolar products, we then have

$$\lim_{a\to 1^-} \int_X (\omega + dd^c u_a)^n = \int_X (\omega + dd^c u)^n.$$

Hence it follows from [9, Theorem 2.3] that the positive measures $\omega_{u_a}^n$ converge to $\omega_u^n$ in the weak sense of Radon measures as $a \nearrow 1$. Since $\psi$ is continuous on $X$ we thus obtain

$$\int_X \psi\omega_u^n \geq \int_X \psi\omega_v^n - nB\int_X \omega^n.$$

□

**Lemma 3.2.** *Let $\phi \in \mathrm{PSH}(X,\omega)$ be such that $P_\omega[\phi] = \phi$ and $\int_X \omega_\phi^n > 0$. Then for any Borel set $E \subset X$ we have*

$$\mathrm{Cap}_{\omega,\phi}(E) \leq \frac{C}{M_{E,\omega,\phi}},$$

*where $C > 0$ is a uniform constant independent of $\phi$.*

Note that the above estimate holds for a big class $\{\theta\}$ as well but to prove this we need to invoke the integration by parts formula in Section 4.

*Proof.* Fix $C_0$ a positive constant such that for all $v \in \mathrm{PSH}(X,\omega)$ with $\sup_X v = 0$ we have $\int_X |v|\omega^n \leq C_0$. The existence of $C_0$ follows from [19, Proposition 2.7].

Without loss of generality, we can assume that $0 < M_{E,\omega,\phi} < +\infty$. Indeed, if $M_{E,\omega,\phi} = 0$ there is nothing to prove, while if $M_{E,\omega,\phi} = +\infty$ then $E$ is a pluripolar set, hence $\mathrm{Cap}_{\omega,\phi}(E) = 0$. Let $u \in \mathrm{PSH}(X,\omega)$ be such that $\phi - 1 \leq u \leq \phi$. Observe that the function $V^*_{E,\omega,\phi} - M_{E,\omega,\phi}$ is $\omega$-psh satisfying $\sup_X(V^*_{E,\omega,\phi} - M_{E,\omega,\phi}) = 0$. As recalled above we thus have

$$\int_X |V^*_{E,\omega,\phi} - M_{E,\omega,\phi}|\omega^n \leq C_0.$$

Since $\phi \leq 0 \leq M_{E,\omega,\phi}$, we also have that

$$|V^*_{E,\omega,\phi} - M_{E,\omega,\phi}| = |\phi - M_{E,\omega,\phi}| \geq M_{E,\omega,\phi}$$

on $E$ modulo a pluripolar set. Applying Lemma 3.1 (for $u = u$ and $v = \phi$ hence $B = 1$) we have that, for all negative $v \in \mathrm{PSH}(X,\omega)$,

$$\int_X |v|(\omega_u^n - \omega_\phi^n) \leq n \int_X \omega^n.$$

By [9, Theorem 3.8] we also have that $\omega_\phi^n \leq \omega^n$. It thus follows that, for all $v \in \mathrm{PSH}(X,\omega)$ normalized by $\sup_X v = 0$,

$$\int_X |v|\omega_u^n \leq n \int_X \omega^n + C_0.$$

Applying the above inequality for $v = V^*_{E,\omega,\phi} - M_{E,\omega,\phi}$ we obtain

$$\int_E \omega_u^n \leq \frac{1}{M_{E,\omega,\phi}} \int_X |V^*_{E,\omega,\phi} - M_{E,\omega,\phi}|\omega_u^n \leq \frac{n\int_X \omega^n + C_0}{M_{E,\omega,\phi}}.$$

Taking the supremum over all candidates $u$ we obtain the desired inequality. □

**Lemma 3.3.** *Fix $\varphi, \psi \in \mathrm{PSH}(\mathrm{X},\theta)$ such that $\psi \leq \varphi$ and $\int_X \theta_\varphi^n = \int_X \theta_\psi^n$. Then there exists a continuous function $g : [0,+\infty) \to [0,+\infty)$ with $g(0) = 0$ such that, for all Borel sets $E$,*

$$\mathrm{Cap}_{\theta,\psi}(E) \leq g\left(\mathrm{Cap}_{\theta,\varphi}(E)\right).$$

Our proof uses an idea in [18].

*Proof.* We can assume that $\varphi \leq 0$. Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous increasing function such that $\chi(-\infty) = -\infty$ and

$$A := \int_X |\chi(\psi - 1 - \varphi)|\theta_\psi^n < +\infty.$$

The existence of $\chi$ is an elementary fact in measure theory as we now explain. Write $\mu = \theta_\psi^n$ and $f = \varphi - \psi + 1 \geq 1$ and note that

$$\lim_{t\to+\infty} \mu(f > t) = 0,$$

since $\mu$ does not charge pluripolar sets. We can thus find a sequence $(t_j)$ such that $t_j \nearrow +\infty$ and, for all $j \in \mathbb{N}$,

$$\mu(f > t_j) \leq 2^{-j}.$$

Define $h : \mathbb{R}^+ \to \mathbb{R}^+$ to be a piecewise affine function such that $h(t_j) = j$, $j \in \mathbb{N}$. Then $h$ is continuous, increasing and $h(+\infty) = +\infty$. Moreover,

$$\int_X h(f) d\mu \leq \sum_j j 2^{-j} < +\infty.$$

The function $\chi$ defined by $\chi(t) = -h(-t)$ satisfies the requirements above. If the sequence $(t_j)$ satisfies $t_{j+2} + t_j \geq t_{j+1}$ for all $j$, then $h$ is concave. Indeed, to prove concavity of $h$ it suffices to check it locally, but the choice of $(t_j)$ as above ensures that $h'(t)$ is decreasing in $t$ which ensures that $h$ is concave.

We next claim that if $v \in \mathrm{PSH}(X, \theta)$ with $\varphi - t \leq v \leq \varphi$ (for $t \geq 0$) then for any Borel set $E$ we have

$$\int_E \theta_v^n \leq \max(t, 1)^n \mathrm{Cap}_{\theta,\varphi}(E).$$

If $t \in [0, 1]$ then $v$ is a candidate defining the capacity $\mathrm{Cap}_{\theta,\varphi}$, hence the desired inequality holds. For $t > 1$, the function $v_t := t^{-1} v + (1 - t^{-1})\varphi$ is $\theta$-psh and $\varphi - 1 \leq v_t \leq \varphi$. Since non-pluripolar products are multilinear, we thus have

$$t^{-n} \int_E \theta_v^n \leq \int_E \theta_{v_t}^n \leq \mathrm{Cap}_{\theta,\varphi}(E),$$

yielding the claim.

Let $u$ be a $\theta$-psh function such that $\psi - 1 \leq u \leq \psi$. Fix $t > 1$ and set $u_t := \max(u, \varphi - 2t)$, $E_t := E \cap \{u > \varphi - 2t\}$, $F_t := E \cap \{u \leq \varphi - 2t\}$. Observe that $\varphi - 2t \leq u_t \leq \varphi$. By plurifine locality and the claim we have that

$$\int_{E_t} \theta_u^n = \int_{E_t} \theta_{u_t}^n \leq (2t)^n \mathrm{Cap}_{\theta,\varphi}(E_t) \leq (2t)^n \mathrm{Cap}_{\theta,\varphi}(E).$$

On the other hand, using the inclusions

$$F_t \subset \left\{ \psi - 1 \leq \frac{u + \varphi}{2} - t \right\} \subset \{\psi - 1 \leq \varphi - t\}$$

and the comparison principle, Theorem 2.5, we infer

$$\begin{aligned}\int_{F_t} \theta_u^n \leq \int_{\{\psi - 1 \leq \frac{u+\varphi}{2} - t\}} \theta_u^n &\leq 2^n \int_{\{\psi - 1 \leq \frac{u+\varphi}{2} - t\}} \theta_{\frac{u+\varphi}{2}}^n \\ &\leq 2^n \int_{\{\psi \leq \varphi - t + 1\}} \theta_\psi^n \leq \frac{2^n}{|\chi(-t)|} \int_X |\chi(\psi - 1 - \varphi)| \theta_\psi^n.\end{aligned}$$

Taking the supremum over all candidates $u$ we obtain

$$\mathrm{Cap}_{\theta,\psi}(E) \leq (2t)^n \mathrm{Cap}_{\theta,\varphi}(E) + \frac{2^n A}{|\chi(-t)|}.$$

Set $t := (\mathrm{Cap}_{\theta,\varphi}(E))^{-1/2n}$. If $t > 1$ we get $\mathrm{Cap}_{\theta,\psi}(E) \leq g\left(\mathrm{Cap}_{\theta,\varphi}(E)\right)$, where $g$ is defined on $[0, +\infty)$ by

$$g(s) := (2^n + \mathrm{Vol}(\theta))s^{1/2} + \frac{2^n A}{|\chi(-s^{-1/2n})|}.$$

If $t \leq 1$ then by the choice of $g$ above we have $\mathrm{Cap}_{\theta,\psi}(E) \leq \mathrm{Vol}(\theta) \leq g\left(\mathrm{Cap}_{\theta,\varphi}(E)\right)$, finishing the proof. □

**Lemma 3.4.** *Assume that $\phi \in \mathrm{PSH}(X, \omega)$, $\int_X \omega_\phi^n > 0$ and $P_\omega[\phi] = \phi$. Then there exists a constant $A > 0$ such that for any Borel set $E$ we have*

$$A^{-1} \left(\mathrm{Cap}_\omega(E)\right)^n \leq \mathrm{Cap}_{\omega,\phi}(E) \leq A \left(\mathrm{Cap}_\omega(E)\right)^{1/n}.$$

The proof uses an idea in [10]. We shall use $C_1, C_2, \ldots$ to denote various uniform constants.

*Proof.* By inner regularity of the capacity we can assume that $E$ is compact. If $M_{E,\omega,\phi} \geq 1$ then, by Lemma 3.2 and [9, Lemma 4.9], we have

$$\mathrm{Cap}_\omega(E)^n \leq CM_{E,\omega}^{-n} \leq CM_{E,\omega,\phi}^{-n} \leq C'\mathrm{Cap}_{\omega,\phi}(E).$$

Note that $V^*_{E,\omega,\phi} \geq \phi$ because $\phi$ is a candidate defining $V_{E,\omega,\phi}$. By Theorem 2.4, we also have $V^*_{E,\omega,\phi} - M_{E,\omega,\phi} \leq \phi$ because $\phi$ is a model potential. If $M_{E,\omega,\phi} < 1$ then $V^*_{E,\omega,\phi} - 1 \leq \phi$ is a candidate defining the capacity $\mathrm{Cap}_{\omega,\phi}$, hence

$$\int_X \omega_\phi^n = \int_X (\omega + dd^c V^*_{E,\omega,\phi})^n = \int_E (\omega + dd^c V^*_{E,\omega,\phi})^n \leq \mathrm{Cap}_{\omega,\phi}(E),$$

where the second equality follows from the fact that $(\omega + dd^c V^*_{E,\omega,\phi})^n$ is supported on $K$, see [11, Lemma 3.5]. Since $\mathrm{Cap}_\omega(E) \leq \int_X \omega^n$, we have

$$\mathrm{Cap}_\omega(E) \leq \frac{\int_X \omega^n}{\int_X \omega_\phi^n} \mathrm{Cap}_{\omega,\phi}(E) \leq \frac{\int_X \omega^n}{\left(\int_X \omega_\phi^n\right)^{1/n}} \mathrm{Cap}_{\omega,\phi}(E)^{1/n},$$

where the last inequality follows from the fact that $\mathrm{Cap}_{\omega,\phi}(E) \leq \int_X \omega_\phi^n$. We thus obtain the left-hand side inequality in the lemma.

We next prove the right-hand side inequality. By [12, Lemma 4.3] there exists a constant $b > 1$ such that $P_\omega(b\phi) \in \mathrm{PSH}(X, \omega)$. Set

$$v := (1 - b^{-1})V^*_{E,\omega} + b^{-1} P_\omega(b\phi).$$

Recall that $V_{E,\omega} = V_{E,\omega,0}$ is the global extremal function of $E$ which takes values 0 on $E$ modulo a pluripolar set. As $V^*_{E,\omega}$ is bounded we have that $v \in \mathrm{PSH}(X, \omega)$, $v \preceq \phi$, and $v \leq \phi$ on $E$ modulo a pluripolar set. By Lemma 2.9 we thus have $v \leq V^*_{E,\omega,\phi}$. Set

$$C_0 := -\sup_X P_\omega(b\phi) \geq 0 \text{ and } G := \{P_\omega(b\phi) \geq -C_0 - 1\}.$$

Note that $G$ has positive Lebesgue measure, hence $G$ is non-pluripolar. In particular $M_{G,\omega} < +\infty$. We have

$$\sup_X V^*_{E,\omega,\phi} \geq \sup_X v \geq \sup_G v \geq (1-b^{-1}) \sup_G V^*_{E,\omega} - b^{-1}(C_0+1).$$

On the other hand we have that $u := V^*_{E,\omega} - \sup_G V^*_{E,\omega}$ is $\omega$-psh and $u \leq 0$ on $G$. It thus follows that $u \leq M_{G,\omega} < +\infty$, hence

$$\sup_G V^*_{E,\omega} \geq V^*_{E,\omega} - M_{G,\omega}.$$

Taking the supremum over $X$ we get $\sup_G V^*_{E,\omega} \geq M_{E,\omega} - M_{G,\omega}$. Therefore

$$M_{E,\omega,\phi} \geq (1-b^{-1}) M_{E,\omega} - C_1. \tag{3.1}$$

It follows from [19, Proposition 7.1] that $\mathrm{Cap}_\omega(E) \geq C_2 M_{E,\omega}^{-n}$. Set

$$a = C_2(1-b^{-1})^n (2C_1)^{-n}.$$

If $\mathrm{Cap}_\omega(E) \leq a$ then $M_{E,\omega} \geq (a^{-1}C_2)^{1/n}$, hence

$$(1-b^{-1}) M_{E,\omega} \geq (1-b^{-1})(a^{-1}C_2)^{1/n} = 2C_1.$$

Equivalently, $C_1 \leq \frac{1}{2}(1-b^{-1})M_{E,\omega}$. Combining with (3.1), we thus have

$$M_{E,\omega,\phi} \geq \frac{1}{2}(1-b^{-1}) M_{E,\omega}.$$

Then by Lemma 3.2 and [19, Proposition 7.1] we have

$$\mathrm{Cap}_{\omega,\phi}(E) \leq C_3 \mathrm{Cap}_\omega(E)^{1/n}.$$

Assume now $\mathrm{Cap}_\omega(E) > a$. Observe that $\mathrm{Cap}_{\omega,\phi}(E) \leq \int_X \omega^n$. Let $C_4$ be a positive constant such that $C_4 \geq C_3$ and $C_4 a^{1/n} \geq \int_X \omega^n$. We then have

$$\mathrm{Cap}_{\omega,\phi}(E) \leq C_4 \mathrm{Cap}_\omega(E)^{1/n}.$$

□

The main result of this note is a direct consequence of the following:

**Theorem 3.5.** *Assume that $\psi \in \mathrm{PSH}(X,\theta)$ and $\int_X \theta_\psi^n > 0$. Then there exist continuous functions $f, g : [0,+\infty) \to [0,+\infty)$ with $f(0) = g(0) = 0$ such that, for any Borel set $E$,*

$$\mathrm{Cap}_{\theta,\psi}(E) \leq f\left(\mathrm{Cap}_\omega(E)\right) \text{ and } \mathrm{Cap}_\omega(E) \leq g\left(\mathrm{Cap}_{\theta,\psi}(E)\right).$$

*Proof.* If $E$ is pluripolar then all quantities are 0 and the result follows. We can thus assume that $E$ is non-pluripolar.

By inner regularity of the capacities we can assume that $E$ is compact. By scaling we can assume that $\theta \leq \omega$, hence $\mathrm{Cap}_{\theta,\psi}(E) \leq \mathrm{Cap}_{\omega,\psi}(E)$ and $V_{E,\theta,\phi} \leq V_{E,\omega,\phi} \leq V_{E,\omega}$. Set $\phi := P_\omega[\psi]$. It follows from Lemma 3.3 that

$$\mathrm{Cap}_{\omega,\psi} \leq f\left(\mathrm{Cap}_{\omega,\phi}\right),$$

for some continuous function $f$ with $f(0) = 0$, while Lemma 3.4 gives

$$\mathrm{Cap}_{\omega,\phi} \le A\, \mathrm{Cap}_{\omega}^{1/n},$$

for some positive constant $A$. Combining these two inequalities we obtain the first inequality in the theorem.

We next prove the second one. We observe that

$$\mathrm{Cap}_{\theta,\phi}(E) \le \mathrm{Cap}_{\theta,\phi}(X) = \int_X \theta_\phi^n,$$

where in the last equality we have used [26] (or Lemma 2.2).

If $M_{E,\theta,\phi} \le 1$ then $V^*_{E,\theta,\phi} - 1$ is a $\theta$-psh function which is more singular than $\phi$ and takes negative values, hence $V^*_{E,\theta,\phi} - 1 \le \phi$ because $\phi$ is a model potential (see Theorem 2.4). Since $V^*_{E,\theta,\phi} \ge \phi$, we thus have that that $V^*_{E,\theta,\phi} - 1$ is a candidate defining the capacity $\mathrm{Cap}_{\theta,\phi}$, hence

$$\int_E (\theta + dd^c V^*_{E,\theta,\phi})^n \le \mathrm{Cap}_{\theta,\phi}(E).$$

It follows from [11, Lemma 3.6] that $(\theta + dd^c V^*_{E,\theta,\phi})^n$ is supported on $E$ (recall that $E$ is compact). We thus get

$$\int_X \theta_\phi^n = \int_X (\theta + dd^c V^*_{E,\theta,\phi})^n = \int_E (\theta + dd^c V^*_{E,\theta,\phi})^n \le \mathrm{Cap}_{\theta,\phi}(E),$$

where in the first equality we have used the monotonicity of mass (see [26] or Lemma 2.2). We thus have $\mathrm{Cap}_{\theta,\phi}(E) = \int_X \theta_\phi^n$, and

$$\mathrm{Cap}_\omega(E) \le \frac{\int_X \omega^n}{\int_X \theta_\phi^n} \mathrm{Cap}_{\theta,\phi}(E) = \frac{\int_X \omega^n}{\left(\int_X \theta_\phi^n\right)^{1/n}} \mathrm{Cap}_{\theta,\phi}(E)^{1/n}.$$

If $M_{E,\omega,\phi} > 1$ then, by [19, Proposition 7.1] and [9, Lemma 4.9], we have

$$\mathrm{Cap}_\omega(E) \le C M_{E,\omega}^{-1} \le C M_{E,\theta,\phi}^{-1} \le C' \mathrm{Cap}_{\theta,\phi}(E)^{1/n}.$$

We thus have that, in both cases,

$$\mathrm{Cap}_\omega(E) \le C' \mathrm{Cap}_{\theta,\phi}(E)^{1/n}.$$

Since $\int_X \theta_\psi^n > 0$, by Lemma 2.8, $P_\theta(2\psi - \phi) \in \mathcal{E}(\mathrm{X}, \theta, \phi)$. Set

$$u := \frac{P_\theta(2\psi - \phi) + \phi}{2}.$$

Observe that $u \le \psi$, and by [9, Corollary 3.15], $\int_X \theta_u^n = \int_X \theta_\psi^n$. Hence by Lemma 3.3 we have $\mathrm{Cap}_{\theta,u} \le g(\mathrm{Cap}_{\theta,\psi})$, for some continuous function $g$ with $g(0) = 0$. The proof is finished if we can show that $\mathrm{Cap}_{\theta,\phi} \le 2^n \mathrm{Cap}_{\theta,u}$. Take $v \in \mathrm{PSH}(X, \theta)$ such that $\phi - 1 \le v \le \phi$. Then

$$u - 1 \le h := \frac{v + P_\theta(2\psi - \phi)}{2} \le u,$$

and hence

$$\int_E \theta_v^n \le 2^n \int_E \theta_h^n \le 2^n \mathrm{Cap}_{\theta,u}(E).$$

Taking the supremum over all $v$ we obtain $\mathrm{Cap}_{\theta,\phi} \leq 2^n \mathrm{Cap}_{\theta,u}$, finishing the proof. □

## 4. Integration by parts

The integration by parts formula was recently studied in [27] using Witt-Nyström's construction. In this section we give a direct proof which also applies to the setting of complex $m$-Hessian equations considered in [23]. We first start with the following key lemma.

**Lemma 4.1.** *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathrm{PSH}(X, \theta)$ be such that $\varphi_1 \simeq \varphi_2$ and $\psi_1 \simeq \psi_2$. Then*

$$\int_X (\varphi_1 - \varphi_2)\left(\theta_{\psi_1}^n - \theta_{\psi_2}^n\right) = \int_X (\psi_1 - \psi_2)(S_1 - S_2),$$

*where $S_j := \sum_{k=0}^{n-1} \theta_{\varphi_j} \wedge \theta_{\psi_1}^k \wedge \theta_{\psi_2}^{n-k-1}$, $j = 1, 2$.*

*Proof.* It follows from [9, Theorem 2.4] that

$$\int_X (\theta_{\psi_1}^n - \theta_{\psi_2}^n) = \int_X (S_1 - S_2) = 0.$$

By adding a constant we can assume that $\varphi_1, \varphi_2, \psi_1, \psi_2$ are negative.

**Step 1.** We assume $\theta$ is Kähler and $\psi_1, \psi_2, \varphi_1, \varphi_2$ are $\lambda\theta$-psh for some $\lambda \in (0, 1)$.

**Step 1.1.** We also assume that there exists $C > 0$ such that $\psi_1 = \psi_2$ on $U := \{\min(\psi_1, \psi_2) < -C\}$ and $\varphi_1 = \varphi_2$ on $V := \{\min(\varphi_1, \varphi_2) < -C\}$.

For a function $u$ we consider its canonical approximant $u^t := \max(u, -t)$, $t > 0$. It follows from Stokes' theorem that

$$\int_X (\varphi_1^t - \varphi_2^t)\left(\theta_{\psi_1^t}^n - \theta_{\psi_2^t}^n\right) = \int_X (\psi_1^t - \psi_2^t)(S_1^t - S_2^t),$$

where $S_j^t := \sum_{k=0}^{n-1} \theta_{\varphi_j^t} \wedge \theta_{\psi_1^t}^k \wedge \theta_{\psi_2^t}^{n-k-1}$, $j = 1, 2$. We now consider the limit as $t \to +\infty$ in the above equality.

Fix $t > C$. By assumption we have $\{\varphi_1 \leq -t\} = \{\varphi_2 \leq -t\} \subset V$ and $\varphi_1^t = \varphi_2^t$ on $V$. The same properties for $\psi_1, \psi_2$ also hold: $\psi_1^t = \psi_2^t$ in the open set $U$ and $\{\psi_1 \leq -t\} = \{\psi_2 \leq -t\} \subset U$. It thus follows that

$$\mathbf{1}_{\{\psi_1 \leq -t\}} \theta_{\psi_1^t}^n = \mathbf{1}_{\{\psi_1 \leq -t\}} \theta_{\psi_2^t}^n \text{ and } \mathbf{1}_{\{\varphi_1 \leq -t\}} S_1^t = \mathbf{1}_{\{\varphi_1 \leq -t\}} S_2^t.$$

By plurifine locality of the non-pluripolar product we thus have

$$\begin{aligned}\int_X (\varphi_1^t - \varphi_2^t)\left(\theta_{\psi_1^t}^n - \theta_{\psi_2^t}^n\right) &= \int_{\{\psi_1 > -t\} \cap \{\varphi_1 > -t\}} (\varphi_1^t - \varphi_2^t)(\theta_{\psi_1^t}^n - \theta_{\psi_2^t}^n) \\ &= \int_{\{\psi_1 > -t\} \cap \{\varphi_1 > -t\}} (\varphi_1 - \varphi_2)(\theta_{\psi_1}^n - \theta_{\psi_2}^n),\end{aligned}$$

and

$$\int_X (\psi_1^t - \psi_2^t)(S_1^t - S_2^t) = \int_{\{\psi_1 > -t\} \cap \{\varphi_1 > -t\}} (\psi_1 - \psi_2)(S_1 - S_2).$$

Since $\varphi_1 - \varphi_2$ and $\psi_1 - \psi_2$ are bounded, using the dominated convergence theorem we finish Step 1.1.

**Step 1.2.** We remove the assumptions made in Step 1.1.

It follows from [9, Theorem 2.4] that

$$\int_X (\theta_{\psi_1}^n - \theta_{\psi_2}^n) = \int_X (S_1 - S_2) = 0.$$

Thus adding a constant we can assume that $\varphi_1 \leq \varphi_2$ and $\psi_1 \leq \psi_2$. Let $B > 0$ be a constant such that

$$\varphi_2 \leq \varphi_1 + B \ ; \ \psi_2 \leq \psi_1 + B.$$

Recall that $\psi_1, \psi_2, \varphi_1, \varphi_2$ are $\lambda\theta$-psh for some $\lambda \in (0,1)$. For each $\varepsilon \in (0, \lambda^{-1} - 1)$ we define

$$\psi_{2,\varepsilon} := \max(\psi_1, (1+\varepsilon)\psi_2) \ ; \ \varphi_{2,\varepsilon} := \max(\varphi_1, (1+\varepsilon)\varphi_2).$$

Since $\varepsilon \in (0, \lambda^{-1} - 1)$, the functions $(1+\varepsilon)\psi_2$ and $(1+\varepsilon)\varphi_2$ are still $\theta$-psh. Also, $\psi_1 \leq \psi_{2,\varepsilon} \leq \psi_1 + B$ and $\varphi_1 \leq \varphi_{2,\varepsilon} \leq \varphi_1 + B$. These are $\theta$-psh functions satisfying the assumptions in Step 1.1 with $C = B + B\varepsilon^{-1}$. Indeed, if $\varphi_1(x) < -C$ then

$$(1+\varepsilon)\varphi_2(x) = \varphi_2(x) + \varepsilon\varphi_2(x) \leq \varphi_1(x) + B + \varepsilon(B - C) = \varphi_1(x).$$

It then follows that $\varphi_{2,\varepsilon} = \varphi_1$ on $V = \{\varphi_1 < -C\}$. We can thus apply Step 1.1 to $\psi_1$, $\psi_{2,\varepsilon}$, $\varphi_1$,$\varphi_{2,\varepsilon}$ to obtain

$$\int_X (\varphi_1 - \varphi_{2,\varepsilon}) \left(\theta_{\psi_1}^n - \theta_{\psi_{2,\varepsilon}}^n\right) = \int_X (\psi_1 - \psi_{2,\varepsilon}) \left(S_{1,\varepsilon} - S_{2,\varepsilon}\right),$$

where $S_{1,\varepsilon} := \sum_{k=0}^{n-1} \theta_{\varphi_1} \wedge \theta_{\psi_1}^k \wedge \theta_{\psi_{2,\varepsilon}}^{n-k-1}$ and $S_{2,\varepsilon} := \sum_{k=0}^{n-1} \theta_{\varphi_{2,\varepsilon}} \wedge \theta_{\psi_1}^k \wedge \theta_{\psi_{2,\varepsilon}}^{n-k-1}$. By Theorem 3.5 there exists a continuous function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ such that for every Borel set $E$,

$$\text{Cap}_{\theta,\psi}(E) \leq f(\text{Cap}_\theta(E)),$$

where

$$\psi := \frac{\varphi_1 + \varphi_2 + \psi_1 + \psi_2}{5} - B.$$

Note that $\psi$ is $\theta$-psh and $\int_X \theta_\psi^n > 0$. Indeed, recalling that in this step $\theta$ is Kähler, we have

$$\int_X \theta_\psi^n \geq 5^{-n} \int_X (\theta + \theta_{\varphi_1} + \theta_{\varphi_2} + \theta_{\psi_1} + \theta_{\psi_2})^n \geq 5^{-n}\theta^n > 0.$$

Since we have assumed that $\psi_2 - B \leq \psi_1 \leq \psi_2 \leq 0$ and $\varphi_2 - B \leq \varphi_1 \leq \varphi_2 \leq 0$, we get

$$\varphi_2 - B \leq \varphi_1 \leq \varphi_{2,\varepsilon} = \max(\varphi_1, (1+\varepsilon)\varphi_2) \leq \max(\varphi_1, \varphi_2) = \varphi_2,$$

and

$$\psi_2 - B \leq \psi_1 \leq \psi_{2,\varepsilon} = \max(\psi_1, (1+\varepsilon\psi_2) \leq \max(\psi_1, \psi_2) = \psi_2.$$

Building on these estimates and a direct computation we have

$$\begin{aligned}\psi &\leq \frac{\varphi_1+\psi_1+\varphi_2+\psi_2}{5} - \frac{2B}{5} \leq \frac{\varphi_1+\psi_1+\varphi_{2,\varepsilon}+\psi_{2,\varepsilon}}{5}\\ &\leq \frac{\varphi_1+\psi_1+\varphi_2+\psi_2}{5} = \psi + B.\end{aligned}$$

Using this and $S_{j,\varepsilon} \leq C(5\theta + dd^c(\varphi_1+\varphi_{2,\varepsilon}+\psi_{2,\varepsilon}+\psi_1))^n$ we obtain that, for any Borel set $E$ and any $\varepsilon \in (0, \lambda^{-1}-1)$, $j=1,2$,

$$\int_E S_{j,\varepsilon} \leq C5^n B^n \mathrm{Cap}_{\theta,\psi}(E) \leq C' f(\mathrm{Cap}_\theta(E)).$$

For each $j \in \{1,2\}$ we also have that $S_{j,\varepsilon} \to S_j$, $\theta^n_{\psi_{2,\varepsilon}} \to \theta^n_{\psi_2}$ as $\varepsilon \to 0$ in the weak sense of Radon measures (see [9, Theorem 2.3]). These measures are uniformly dominated by $\mathrm{Cap}_\theta$. Note also that $\varphi_1-\varphi_{2,\varepsilon}, \varphi_1-\varphi_2, \psi_{2,\varepsilon}-\psi_1, \psi_2-\psi_1$ are uniformly bounded, quasi-continuous. Moreover, $\psi_{2,\varepsilon}-\psi_1 \to \psi_2-\psi_1$, and $\varphi_1-\varphi_{2,\varepsilon} \to \varphi_1-\varphi_2$ in capacity as $\varepsilon \to 0$. It thus follows from Theorem 2.11 that

$$\lim_{\varepsilon\to 0} \int_X (\varphi_1-\varphi_{2,\varepsilon}) \left(\theta^n_{\psi_1} - \theta^n_{\psi_{2,\varepsilon}}\right) = \int_X (\varphi_1-\varphi_2)\left(\theta^n_{\psi_1} - \theta^n_{\psi_2}\right)$$

and

$$\lim_{\varepsilon\to 0} \int_X (\psi_1-\psi_{2,\varepsilon})(S_{1,\varepsilon}-S_{2,\varepsilon}) = \int_X (\psi_1-\psi_2)(S_1-S_2),$$

finishing the proof of Step 1.2.

**Step 2.** We merely assume that $\{\theta\}$ is big. We can assume that $\theta+\omega$ is a Kähler form. For $s>2$ we consider $\theta_s := \theta + s\omega$, which is also Kähler, and we observe that $\varphi_1,\varphi_2,\psi_1,\psi_2$ are $\lambda\theta_s$-psh for some $\lambda \in (0,1)$. We can thus apply the first step to get

$$\int_X u\left((\theta_s+dd^c\psi_1)^n - (\theta_s+dd^c\psi_2)^n\right) = \int_X vT_s,$$

where $u=\varphi_1-\varphi_2$, $v=\psi_1-\psi_2$ and

$$\begin{aligned}T_s =& \sum_{k=0}^{n-1} (\theta_s+dd^c\varphi_1)\wedge(\theta_s+dd^c\psi_1)^k \wedge (\theta_s+dd^c\psi_2)^{n-k-1}\\ &- \sum_{k=0}^{n-1} (\theta_s+dd^c\varphi_2)\wedge(\theta_s+dd^c\psi_1)^k \wedge (\theta_s+dd^c\psi_2)^{n-k-1}.\end{aligned}$$

We thus obtain an equality between two polynomials in $s$. Identifying the coefficients we arrive at the conclusion. □

**Proof of Theorem 1.2.** We first assume that $\theta$ is Kähler, $u=\varphi_1-\varphi_2$ and $v=\psi_1-\psi_2$ where $\psi_1,\psi_2,\varphi_1,\varphi_2$ are $\theta$-psh. Fix $\phi \in \mathrm{PSH}(X,\theta)$ and for each $s\in[0,1]$, $j=1,2$, we set $\psi_{j,s} := s\psi_j + (1-s)\phi$. Note that $\psi_{1,s} \simeq \psi_{2,s}$. It follows from Lemma 4.1 that for any $s \in [0,1]$,

$$\int_X u\left(\theta^n_{s\psi_1+(1-s)\phi} - \theta^n_{s\psi_2+(1-s)\phi}\right) = \int_X (\psi_{1,s}-\psi_{2,s})T_s = \int_X svT_s,$$

where

$$T_s := \sum_{k=0}^{n-1} \theta_{\varphi_1} \wedge \theta_{\psi_{1,s}}^k \wedge \theta_{\psi_{2,s}}^{n-k-1} - \sum_{k=0}^{n-1} \theta_{\varphi_2} \wedge \theta_{\psi_{1,s}}^k \wedge \theta_{\psi_{2,s}}^{n-k-1}.$$

We thus have an identity between two polynomials in $s$. Taking the first derivative in $s=0$ and noting that $T_0 = ndd^c u \wedge \theta_\phi^{n-1}$, we obtain

$$\int_X udd^c v \wedge \theta_\phi^{n-1} = \int_X vdd^c u \wedge \theta_\phi^{n-1}.$$

For the general case, i.e. $\{\theta\}$ is merely big, we can write $u = \varphi_1 - \varphi_2$ and $v = \psi_1 - \psi_2$ where $\psi_1, \psi_2, \varphi_1, \varphi_2$ are $A\omega$-psh, for some $A>0$ large enough. We apply the first step with $\theta$ replaced by $\theta + t\omega$, for $t > A$ to get

$$\int_X udd^c v \wedge (t\omega + \theta_\phi)^{n-1} = \int_X vdd^c u \wedge (t\omega + \theta_\phi)^{n-1}.$$

Identifying the coefficients of these two polynomials in $t$ we obtain

$$\int_X udd^c v \wedge \theta_\phi^{n-1} = \int_X vdd^c u \wedge \theta_\phi^{n-1}.$$

We now consider $\theta = s_2\theta_2 + .... + s_n\theta_n$, $\phi := s_2\phi_2 + ... + s_n\phi_n$ with $s_2, ..., s_n \in [0,1]$ and $\sum_{j=2}^n s_j = 1$. We obtain an identity between two polynomials in $(s_2, ..., s_n)$, and identifying the coefficients we arrive at the result.

Using the integration by parts formula we can repeat the proof of Lemma 3.2 to show that it holds for big cohomology classes as well.

**Corollary 4.2.** *Fix a big class $\{\theta\}$ and a model potential $\phi \in \mathrm{PSH}(X,\theta)$. Then for any Borel set $E \subset X$ we have*

$$\mathrm{Cap}_{\theta,\phi}(E) \leq \frac{C}{M_{E,\theta,\phi}},$$

*where $C>0$ is a uniform constant independent of $\phi$.*

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