Around p-adic cohomologies

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New advances on de Rham cohomology in positive or mixed characteristic, after Bhatt-Lurie, Drinfeld, and Petrov

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Plan

1. The Hodge to de Rham spectral sequence
2. The diffracted Hodge complex
3. Sen classes and obstructions
4. Petrov's example
5. Open problems

1. The Hodge to de Rham spectral sequence

X/k proper smooth; k a field

de Rham cohomology

$$H^n_{\mathrm{dR}}(X/k) := H^n(X, \Omega^\bullet_{X/k})$$

Hodge to de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k) \tag{1}$$

$$h^{ij} := \dim(E_1^{ij}) < \infty \qquad h^n := \dim(H^n_{\mathrm{dR}}(X/k)) < \infty.$$

(1) degenerates at $E_1 \Leftrightarrow \forall n \;\; h^n = \sum_{i+j=n} h^{ij}$.

$\mathrm{char}(k) = 0 \Rightarrow$ (1) degenerates at E_1.

$k = \mathbf{C}$:

dR comparison th. + Serre's GAGA

$$H^n_{\mathrm{dR}}(X/\mathbf{C}) \xrightarrow{\sim} H^n(X(\mathbf{C}), \mathbf{C})$$

Hodge decomposition

$$H^n_{\mathrm{dR}}(X/\mathbf{C}) \xrightarrow{\sim} \oplus_{i+j=n} F^i \cap \overline{F}^j$$

$$H^j(X, \Omega^i_{X/\mathbf{C}}) \xrightarrow{\sim} F^i \cap \overline{F}^j,$$

(F^i: Hodge filtration)

Algebraic proof of degeneration $/k$: [DI] (1987).

$\mathrm{char}(k) = p > 0 \Rightarrow$ (1) may not degenerate at E_1

Smooth proper surfaces: Mumford (1961), I. (1979), Raynaud-Szpiro (1981), Antieau-Bhatt-Mathew (2021)

However:

Theorem [DI]. k perfect field of char. $p > 0$, X/k proper smooth. Assume:

(i) $\dim(X) \leqslant p$

(ii) X/k liftable to $\widetilde{X}/W_2(k)$.

Then (1) degenerates at E_1.

Question (DI 2.6 (iii)) Does there exist X/k proper smooth, of dimension $p+1$, liftable to $W_2(k)$, such that (1) does not degenerate at E_1?

Answer (A. Petrov, 2022). Yes! Can even choose X/k projective and liftable to a smooth projective scheme over $W(k)$.

2. The diffracted Hodge complex

k perfect field, char. $p > 0$.

Change notation: $X/W(k)$ formal smooth, $Y := X_k/k$.

To $X/W(k)$ Bhatt-Lurie associate the diffracted Hodge complex

$$\Omega^{\not{D}}_{X/W(k)} \in D^{\geqslant 0}(X, \mathcal{O}_X)$$

a perfect complex of $\mathcal{O}_X$-modules, of perfect amplitude in $[0, \dim(X/W(k))]$

endowed with:

- a product structure

$$\Omega^{\not{D}}_{X/W(k)} \otimes^L \Omega^{\not{D}}_{X/W(k)} \to \Omega^{\not{D}}_{X/W(k)}$$

underlying a cosimplicial commutative $\mathcal{O}_X$-algebra,

- multiplicative $\mathcal{O}_X$-linear isomorphisms

$$(*) \qquad H^i(\Omega^{\not D}_{X/W(k)}) \xrightarrow{\sim} \Omega^i_{X/W(k)},$$

- an endomorphism

$$\Theta_X \in \mathrm{End}(\Omega^{\not D}_{X/W(k)}),$$

the Sen operator, acting as a derivation, and satisfying

$$\Theta_X | H^i(\Omega^{\not D}_{X/W(k)}) = -i\mathrm{Id},$$

- an isomorphism in $D(Y^{(1)}, \mathcal{O}_{Y^{(1)}})$:

$$\varepsilon_Y : \Omega^{\not{D}}_{Y^{(1)}/k} \xrightarrow{\sim} F_* \Omega^\bullet_{Y/k}$$

where

$$\Omega^{\not{D}}_{Y/k} := \Omega^{\not{D}}_{X/W(k)} \otimes^L k,$$

inducing the Cartier isomorphism on H^i

$$C^{-1} : \Omega^i_{Y^{(1)}/k} \xrightarrow{\sim} F_* H^i(\Omega^\bullet_{Y/k})$$

via the reduction mod p

$$H^i(\Omega^{\not{D}}_{Y/k}) \xrightarrow{\sim} \Omega^i_{Y/k}$$

of the isomorphisms (*).

Construction of $\Omega^{\not{D}}_{X/W(k)}$ relies on Bhatt-Lurie-Drinfeld theory of Cartier-Witt and Hodge-Tate stacks. See Appendix for a sketch.

As an object of $D(X, \mathcal{O}_X)$, $\Omega^{\not{D}}_{X/W(k)}$ is described as

$$\Omega^{\not{D}}_{X/W(k)} = \varphi_{W(k)*}(q\Omega_{X/W(k)[[q-1]]})^{\mathbf{F}_p^*}_{q=\zeta_p}$$

where $q\Omega$ denotes the q-crystalline complex (intrinsic form of the (local) q-de Rham complexes).

But Θ usually invisible! We'll show: Θ controls deep cohomological invariants of $\Omega^\bullet_{Y/k} = \Omega^\bullet_{X/W(k)} \otimes k$ (key input in Petrov's construction).

Application: new structure on the de Rham complex

$d := \dim(X/W(k))$

$\Theta | H^i \Omega^{\not D}_{X/W(k)} = -i \Rightarrow \prod_{0 \leqslant i \leqslant d} (\Theta + i) \in \mathrm{End}(\Omega^{\not D}_{X/W(k)})$ nilpotent,

gives a decomposition of $\Omega^{\not D}_{Y/k} = \Omega^{\not D}_{X/W(k)} \otimes k$ into generalized eigenspaces:

$$\Omega^{\not D}_{Y/k} = \oplus_{0 \leqslant i < p} (\Omega^{\not D}_{Y/k})_i$$

with $(\Omega^{\not D}_{Y/k})_i$ cohomologically concentrated in degrees $\equiv i \bmod p$, and

$$\Theta | (\Omega^{\not D}_{Y/k})_i = -i\mathrm{Id} + \Theta_i$$

with Θ_i nilpotent.

NB. $\Omega^{\not D}_{Y/k} \in D(B(\mathbf{G}_m^\sharp)_k)$, where $(\mathbf{G}_m^\sharp)_k = (\mu_p \times \mathbf{G}_a^\sharp)_k$.
$(\Omega^{\not D}_{Y/k})_i$ = summand of weight i in the $\mathbf{Z}/p\mathbf{Z}$-grading associated with the μ_p-action.

By the isomorphism

$$\varepsilon_Y : \Omega^{\not{D}}_{Y^{(1)}/k} \xrightarrow{\sim} F_*\Omega^\bullet_{Y/k},$$

get $\Theta \in \mathrm{End}_{\mathcal{O}_{Y^{(1)}}}(F_*\Omega^\bullet_{Y/k})$ and Θ-stable decomposition

$$F_*\Omega^\bullet_{Y/k} = \oplus_{0 \leqslant i < p}(F_*\Omega^\bullet_{Y/k})_i,$$

with $\Theta = -i\mathrm{Id} + \Theta_i$ on the summand of weight i.

In particular, get decompositions for all $a \in \mathbf{Z}$,

$$\tau^{[a,a+p-1]}F_*\Omega^\bullet_{Y/k} = \oplus_{a \leqslant i < a+p-1} H^i(F_*\Omega^\bullet_{Y/k})[-i]$$

generalizing those of [DI] and Achinger-Suh [AS].
(NB. depend only on $X \otimes W_2(k)$, and for $a = 0$ coincide with those of [DI].)

3. Sen classes and obstructions

Fix $X/W(k)$ formal smooth, $Y := X \otimes k$. To get classes on Y rather than on $Y^{(1)}$, use Petrov's notation:

$$\begin{array}{ccccc} Y^{(-1)} & \longleftarrow & Y & \xleftarrow{F} & Y^{(-1)} \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(k) & \xleftarrow{F_k} & \mathrm{Spec}(k) & & \end{array}$$

($F : Y^{(-1)} \to Y$ the relative Frobenius).

In particular, the Cartier isomorphism

$$C^{-1} : \Omega^i_{Y/k} \xrightarrow{\sim} H^i F_* \Omega^{\bullet}_{Y^{(-1)}/k}$$

is induced on H^i by the basic isomorphism ε_Y of diffraction theory, which reads

$$\varepsilon_Y : \Omega^{\not D}_{Y/k} \xrightarrow{\sim} F_* \Omega^{\bullet}_{Y^{(-1)}/k}.$$

3.1. The first obstruction class:

$$e_{Y,X} \in \mathrm{Ext}^{p+1}_{\mathcal{O}_Y}(\Omega^p_{Y/k}, \mathcal{O}_Y)$$

is defined as follows.

Because $\tau^{[0,p-1]}$ (resp. $\tau^{[1,p]}$) of $F_*\Omega^\bullet_{Y^{(-1)}/k}$ is decomposable by [DI] (resp. diffraction), the obstruction to decomposing $\tau^{[0,p]}F_*\Omega^\bullet_{Y^{(-1)}/k}$, i.e., the map of degree 1 of the triangle

$$\tau^{<p}F_*\Omega^\bullet_{Y^{(-1)}/k} \to \tau^{[0,p]}F_*\Omega^\bullet_{Y^{(-1)}/k} \to H^pF_*\Omega^\bullet_{Y^{(-1)}/k}[-p] \to,$$

is a class

$$e_{Y,X} \in \mathrm{Ext}^{p+1}_{\mathcal{O}_Y}(H^pF_*\Omega^\bullet_{Y^{(-1)}/k}, H^0F_*\Omega^\bullet_{Y^{(-1)}/k}) = \mathrm{Ext}^{p+1}_{\mathcal{O}_Y}(\Omega^p_{Y/k}, \mathcal{O}_Y).$$

Equivalently, $e_{Y,X}$ is the map of degree 1 of the triangle

$$H^0(\tau^{[0,p]}(\Omega^{\not{D}}_{Y/k})_0) \to \tau^{[0,p]}(\Omega^{\not{D}}_{Y/k})_0 \to H^p(\tau^{[0,p]}(\Omega^{\not{D}}_{Y/k})_0)[-p] \to,$$

where $(\Omega^{\not{D}}_{Y/k})_0$ is the weight zero summand of $\Omega^{\not{D}}_{Y/k}$, as

$$H^0(\tau^{[0,p]}(\Omega^{\not{D}}_{Y/k})_0) = \mathcal{O}_Y,\ \ H^p(\tau^{[0,p]}(\Omega^{\not{D}}_{Y/k})_0) = \Omega^p_{Y/k}.$$

Relation with the Hodge to de Rham spectral sequence

In addition to the Hodge to de Rham spectral sequence, we have the conjugate spectral sequence, deduced via the Cartier isomorphism from the conjugate filtration of $F_*\Omega^\bullet_{Y^{(-1)}/k}$ (i.e., the canonical filtration $\tau^{\leqslant i}$):

$$E_2^{ij} = H^i(Y, \Omega^j_{Y/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(Y^{(-1)}/k).$$

For Y/k proper, we have:

(Hodge to de Rham ss degenerates at E_1) $\Leftrightarrow$ ($\forall n, \sum_{i+j} h^{ij} = h^n$) $\Leftrightarrow$ (Conjugate ss degenerates at E_2)).

As $\tau^{[0,p-1]}F_*\Omega^\bullet_{Y^{(-1)}/k}$ and $\tau^{[1,p]}F_*\Omega^\bullet_{Y^{(-1)}/k}$ are decomposable, we have

$$H^0(Y,\Omega^p_{Y/k})(=E_2^{0,p})=E_{p+1}^{0,p},\ H^{p+1}(Y,\mathcal{O}_Y)(=E_2^{p+1,0})=E_{p+1}^{p+1,0}$$

and

$$d_{p+1}^{0,p}:H^0(Y,\Omega^p_{Y/k})\to H^{p+1}(Y,\mathcal{O}_Y)$$

is the composition with $e_{Y,X}:\Omega^p_{Y/k}\to\mathcal{O}_Y[p+1]$. Moreover,

$$(h^p=\sum_{i+j=p}h^{ij})\Leftrightarrow(H^0(Y,\Omega^p_{Y/k})=E_\infty^{0,p})\Leftrightarrow(d_{p+1}^{0,p}=0)$$

Petrov constructs an example of a projective and smooth $X/W(k)$, of relative dimension $p+1$ for which, not only $e_{Y,X}\neq 0$ (hence $\tau^{[0,p]}F_*\Omega^\bullet_{Y^{(-1)}/k}$ not decomposable), but $d_{p+1}^{0,p}\neq 0$, hence Hodge to de Rham does not degenerate at E_1). The diffracted Hodge complex is crucial in his construction.

3.2. The first Sen class.

Back to the hypotheses and notation of 3.1: $X/W(k)$ formal smooth, and $Y = X \otimes k$.

The restriction $\Theta = \Theta_0$ of the Sen operator to $\tau^{\leqslant p}(\Omega^{\not{D}}_{Y/k})_0$ sits in an endomorphism of the exact triangle

$$\begin{array}{ccccccc}
\mathcal{O}_Y & \longrightarrow & \tau^{\leqslant p}(\Omega^{\not{D}}_{Y/k})_0 & \longrightarrow & \Omega^p_{Y/k}[-p] & \longrightarrow & \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Theta_0} & & \downarrow{\scriptstyle 0} & & \\
\mathcal{O}_Y & \longrightarrow & \tau^{\leqslant p}(\Omega^{\not{D}}_{Y/k})_0 & \longrightarrow & \Omega^p_{Y/k}[-p] & \longrightarrow &
\end{array}$$

hence is induced by composition (with the right upper arrow and the left lower one) from a class (easily seen to be unique)

$$c_{Y,X} \in \mathrm{Ext}^p(\Omega^p_{Y/k}, \mathcal{O}_Y),$$

called the first Sen class.

Petrov has the following useful interpretation of $c_{Y,X}$.

Consider the diffracted Hodge complex $\Omega^{\not{D}}_{X/W(k)}$ as an object of the ∞-derived category $D_{\mathbf{N}}(\mathcal{O}_X)$ of pairs (K, u) of a quasi-coherent complex K and an endomorphism u. In particular, the extension defined by the canonical filtration

$$0 \to (\mathcal{O}_X, 0) \to (\tau^{[0,p]}(\Omega^{\not{D}}_{X/W(k)})_0, \Theta_X) \to (\Omega^p_{X/W(k)}[-p], p) \to 0$$

is a map

$$c \in \mathrm{Hom}_{D_{\mathbf{N}}(\mathcal{O}_X)}((\Omega^p_{X/W(k)}[-p], p), (\mathcal{O}_X[1], 0)).$$

By the exact sequence (where $i : Y \hookrightarrow X$)

$$0 \to \mathcal{O}_X \xrightarrow{p} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0,$$

(and definition of Hom in $D_{\mathbf{N}}(\mathcal{O}_X)$) we have

$$\mathrm{Hom}_{D_{\mathbf{N}}(\mathcal{O}_X)}((\Omega^p_{X/W(k)}[-p], p), (\mathcal{O}_X[1], 0)) = \mathrm{Ext}^p(\Omega^p_{X/W(k)}, i_*\mathcal{O}_Y)$$

and

$$\mathrm{Ext}^p(\Omega^p_{X/W(k)}, i_*\mathcal{O}_Y) = \mathrm{Ext}^p(\Omega^p_{Y/k}, \mathcal{O}_Y)$$

by adjunction. We have

$$c = c_{Y,X} \in \mathrm{Ext}^p(\Omega^p_{Y/k}, \mathcal{O}_Y).$$

Relation between the obstruction class and the first Sen class

Consider the Bockstein class

$$\beta_{Y,X} \in \mathrm{Ext}^1_{\mathcal{O}_X/p^2\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$$

defined by the exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{p} \mathcal{O}_X/p^2\mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Theorem 1 (Petrov). We have

$$e_{Y,X} = \beta_{Y,X} \circ c_{Y,X} \in \mathrm{Ext}^{p+1}_{\mathcal{O}_Y}(\Omega^p_{Y/k}, \mathcal{O}_Y)$$

In particular:

Corollary. $\tau^{\leqslant p} F_* \Omega^\bullet_{Y^{(-1)}/k}$ not decomposable $\Rightarrow c_{Y,X} \neq 0$.

Proof of Corollary. Assume $c_{Y,X} = 0$. Then $\Theta \in \mathrm{End}(\tau^{\leqslant p}(\Omega^{\not{D}}_{X/k})_0)$ is (exactly) divisible by p, say $\Theta = p\Theta'$. Then $\Theta' \otimes \mathcal{O}_Y$ gives an endomorphism of the triangle

$$(*) \qquad \mathcal{O}_Y \to \tau^{\leqslant p}(\Omega^{\not{D}}_{Y/k})_0 \to \Omega^p_{Y/k}[p] \to$$

which is zero on $\mathcal{O}_Y$ and an isomorphism on $\Omega^p_{Y/k}[p]$, hence (*) splits.

To unravel the Sen class $c_{Y,X}$, Petrov constructs a new characteristic class:

3.3. The class alpha

Let R be an $\mathbf{F}_p$-algebra. For M an R-module, consider the sequence

$$0 \to M^{(1)} \to \mathrm{Sym}^p M \to \Gamma^p M \to M^{(1)} \to 0,$$

where $M^{(1)} = F_R^* M$, and the first (resp. last map) is given by $x \mapsto x^p$ (resp. induced by $x \mapsto F^* x$, which is homogeneous polynomial of degree p, cf. [Ro, Th. IV.1]), and the middle one is the canonical one, in particular, sending x^p to $p! x^{[p]}$. It is exact for M flat.

Left deriving, and using Petrov's notation

$$T(M) := \mathrm{Cofib}(\mathrm{Sym}^p M \to \Gamma^p M)$$

for $M \in D(R)$), get exact triangles

$$T(M)[-1] \to \mathrm{Sym}^p M \to \Gamma^p M \to,$$

and

$$M^{(1)} \to T(M)[-1] \to M^{(1)}[-1] \to .$$

For E flat, using Quillen's décalage formula

$$\Gamma^p(E[-1]) \xrightarrow{\sim} (\Lambda^p E)[-p],$$

get exact triangles

$$T(E[-1])[-1] \to \mathrm{Sym}^p(E[-1]) \to (\Lambda^p E)[-p] \to,$$

$$E^{(1)}[-2] \to \tau^{\geqslant 2}\mathrm{Sym}^p(E[-1]) \to (\Lambda^p E)[-p] \to,$$

and a class

$$\alpha(E) \in \mathrm{Ext}^{p-1}(\Lambda^p E, E^{(1)}).$$

Remark. For $p = 2$, $\alpha(E)$ is the class of the canonical extension

$$0 \to E^{(1)} \to \mathrm{Sym}^2 E \to \Lambda^2 E \to 0.$$

3.4. The obstruction to lifting Frobenius

Let $X_2 := X \otimes W_2(k)$. The obstruction to lifting $F : Y^{(-1)} \to Y$ to a $W_2(k)$-map $X_2^{(-1)} \to X_2$ is a class

$$\mathrm{ob}_{F,X} \in \mathrm{Ext}^1(F^*\Omega^1_{Y/k}, \mathcal{O}_{Y^{(-1)}}) = \mathrm{Ext}^1(F^*_{\mathrm{abs}}\Omega^1_{Y/k}, \mathcal{O}_Y)$$

where $F_{\mathrm{abs}} : Y \to Y$ is the absolute Frobenius.

Petrov's key result is the following description of the first Sen class:

Theorem 2 (Petrov). The Sen class $c_{Y,X}$ is the composition

$$\Omega^p_{Y/k} \stackrel{\alpha(\Omega^1_{Y/k})}{\to} F^*_{\mathrm{abs}}\Omega^1_{Y/k}[p-1] \stackrel{\mathrm{ob}_{F,X}}{\to} \mathcal{O}_Y[p].$$

Main ingredients of proof of Theorem 2:

• The description of $\Omega^{\not D}_{X/W(k)}$ as a cosimplicial commutative algebra, in particular, enabling the definition of a map

$$\Omega^{\not D}_{Y/k} \to \Omega^{\not D}_{Y/k}, \ a \mapsto a^p$$

inserting itself in a map of exact triangles (with $B = \Omega^{\not D}_{X/W(k)}$, $A = \Omega^{\not D}_{Y/k}$),

$$\begin{array}{ccccccc}
\mathrm{Sym}^p B & \xrightarrow{N} & \Gamma^p B & \longrightarrow & i_* F^*_{\mathrm{abs}} A & \longrightarrow & \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle a \mapsto a^p} & & \\
B & \xrightarrow{p!} & B & \longrightarrow & i_* A & \longrightarrow &
\end{array}$$

where N is the norm map, and m the multiplication map.

- The section

$$s : \Omega^1_{Y/k}[-1] \to \tau^{[0,1]} F_* \Omega^\bullet_{Y^{(-1)}/k}$$

given by the $\mathbf{Z}/p$-grading, whose composition with the projection to $F_* \mathcal{O}_{Y^{(-1)}}$ is the obstruction to lifting Frobenius.

- The interpretation of $c_{Y,X}$ as a map in $D_{\mathbf{N}}(\mathcal{O}_X)$:

$$c_{Y,X} : (\Omega^p_{X/W(k)}[-p], p) \to (\mathcal{O}_X[1], 0).$$

4. Petrov's example

Recall Petrov's main result:

Theorem 3 (Petrov). There exists a projective, smooth scheme $X/W(k)$, of relative dimension $p+1$, such that

$$h^p_{\mathrm{dR}}(X_k) < \sum_{i+j=p} h^{ij}(X_k),$$

where $X_k = X \otimes k$, $h^n_{\mathrm{dR}} = \dim H^n_{\mathrm{dR}}(-)$, $h^{ij} = \dim H^j(-, \Omega^i)$.
In particular, both the Hodge to de Rham spectral sequence and the conjugate spectral sequence for X_k do not degenerate at their first page.

Construction is in 2 steps.

A. Construction of a finite, flat group scheme $G/W(k)$ such that, in the conjugate spectral sequence for $G_k = G \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k)$

$$E_2^{ij} = H^i(BG_k, \Omega^j_{BG_k/k}) \Rightarrow H^{i+j}_{dR}(BG_k^{(-1)}/k),$$

for which

$$d_{p+1}^{0,p} : H^0(BG_k, \Omega^p_{BG_k/k}) \to H^{p+1}(BG_k, \mathcal{O}))$$

(see slide 17) does not vanish (and in particular, the obstruction

$$e_{BG_k,BG} : \Omega^p_{BG_k/k} \to \mathcal{O}_{BG_k}[p+1]$$

to splitting the pth step of the conjugate filtration does not vanish). Here differentials are taken in the derived sense, for the stack BG_k over k. Implicit is a generalization of diffraction theory to smooth Artin stacks (see [KP1], [KP2]).

B. Approximation of BG.

By a variant of the method of Godeaux-Serre-Raynaud (cf. [ABM]), construct a projective smooth scheme $X/W(k)$, of relative dimension $p+1$, and a morphism $f : X \to BG$, such that the map

$$f^* : H^{p+1}(BG_k, \mathcal{O}) \to H^{p+1}(X_k, \mathcal{O})$$

is injective.

A. Definition of G.

Le $E/W(k)$ be an elliptic curve whose reduction E_k is supersingular.) Fix $q = p^r$, with $r \geqslant 2$, and consider the flat commutative group scheme over $W(k)$

$$E[p] \otimes_{\mathbf{F}_p} \mathbf{F}_q^{\oplus p}$$

(a sum of $p[\mathbf{F}_q : \mathbf{F}_p]$ copies of $E[p]$). The discrete group $SL_p(\mathbf{F}_q)$ acts on it via its action on the second factor. Petrov defines

$$G := SL_p(\mathbf{F}_q) \ltimes (E[p] \otimes_{\mathbf{F}_p} \mathbf{F}_q^{\oplus p}).$$

This is a finite, flat, non-commutative group scheme over $W(k)$.

Theorem 4 (Petrov). The differential

$$d_{p+1}^{0,p} : H^0(BG_k, \Omega^p_{BG_k/k}) \to H^{p+1}(BG_k, \mathcal{O}))$$

in the conjugate spectral sequence of BG_k does not vanish.

Glimpses on proof.

The difficulty is that the extension class

$$e = e_{BG_k, BG} : \Omega^p_{BG_k/k} \to \mathcal{O}_{BG_k}[p+1]$$

is a product of 3 classes

$$e = \mathrm{Bockstein} \circ \mathrm{ob}_F \circ \alpha(\Omega^1).$$

Not only each of them must not vanish, but the product must not vanish either, nor the map $d^{0,p}_{p+1}$ it induces on $H^0(BG_k, -)$.

• As E_k is supersingular, the obstruction ob_F to lifting F doesn't vanish.

• The non-vanishing of $\alpha(\Omega^1)$, which uses the action of SL_p, is more difficult (and the non-vanishing of $d^{0,p}_{p+1}$ requires further delicate arguments).

The non-vanishing of $\alpha(\Omega^1)$ relies on the following key lemma:

Lemma. (Petrov) Let V be a k-vector space of dimension p, viewed as a vector bundle on $BSL(V)$ via the standard representation. Then the map

$$k(\det) = H^0(BSL(V), \Lambda^p V) \to H^{p-1}(BSL(V), V^{(1)})$$

induced by the class

$$\alpha(V) : \Lambda^p V \to V^{(1)}[p-1]$$

of 3.3 is an isomorphism.

Remark. For $p = 2$, the statement of the lemma boils down to the following: the canonical extension

$$0 \to V^{(1)} \to \mathrm{Sym}^2 V \to \Lambda^2 V \to 0$$

admits no $SL(V)$-invariant splitting (this is elementary).

Proof of lemma. Delicate analysis of the map $(M)^{\otimes p}_{S_p} \to \mathrm{Sym}^p M$ (S_p the symmetric group), using, in addition to the non-vanishing of certain Steenrod operations, that $V^{\otimes p}$ has a good filtration (i.e., with quotients of the form $F(\lambda) = H^0(SL(V)/B, \mathcal{L}(-\lambda))$ for λ dominant weights of $SL(V)$) (Jantzen, Mathieu), Kempf vanishing theorem ($H^i(SL(V), F(\lambda)) = 0$ for $i > 0$), and an additional vanishing (Petrov), namely $H^i(BSL(V), V^{(1)}) = 0$ for $i \neq p-1$ (and $H^{p-1}(BSL(V), V^{(1)}) = k$).

Scheme-theoretic vs discrete cohomology. Petrov's group G involves not the group scheme $SL(V)$ but the discrete group of its $\mathbf{F}_q$-points, $SL_p(\mathbf{F}_q)$. A result of Cline-Parshall-Scott-van der Kallen [CPSvdK] ensures that the map

$$H^{p-1}(BSL(V), V^{(1)}) \to H^{p-1}(BSL_p(\mathbf{F}_q), V^{(1)})$$

induced by $SL_p(\mathbf{F}_q) \to SL(V)$ is injective. This is a key ingredient in the proof of Th. 4.

B. Approximation of BG.

Recall the statement:

Theorem 5 (Godeaux, Serre, Raynaud, Antieau-Bhatt-Mathew). There exists a projective smooth scheme $X/W(k)$, of relative dimension $p+1$, and a morphism $f : X \to BG$, such that the map

$$f^* : H^{p+1}(BG_k, \mathcal{O}) \to H^{p+1}(X_k, \mathcal{O})$$

is injective.

Proof of Th. 3. Commutative diagram:

$$\begin{array}{ccc} H^0(BG_k, \Omega^p_{BG_k/k}) & \xrightarrow{d_{p+1}^{0,p}(BG_k)} & H^{p+1}(BG_k, \mathcal{O}) \\ \downarrow & & \downarrow f^* \\ H^0(X_k, \Omega^p_{X_k/k}) & \xrightarrow{d_{p+1}^{0,p}(X_k)} & H^{p+1}(X_k, \mathcal{O}) \end{array}$$

$(d_{p+1}^{0,p}(BG_k) \neq 0$ (Th. 4) + f^* injective (Th.5)) $\Rightarrow d_{p+1}^{0,p}(X_k) \neq 0$.

Sketch of proof of Th. 5.

Lemma 1. (Godeaux-Serre-Raynaud) H: a finite, flat group scheme over a local scheme S.

For any integer $d \geqslant 0$ there exists

(i) a projective space $P = \mathbf{P}_S^N$, equipped with a linear action of H, such that if U_P is the largest open over which H acts freely, and $Z_P = P - U_P$, Z_P has codimension $\geqslant d+1$ on each fiber.

(ii) a relative complete intersection

$$\widetilde{X} = V(f_1, \cdots, f_{N-d}) \subset U_P,$$

of relative dimension d, stable under H, which is an H-torsor, and such that

$$X := \widetilde{X}/H \ (= [\widetilde{X}/H])$$

is smooth over S.

Proof. [R, 4.2.3], [BMS, Lemma 2.7] (plus [G] or [Po] if the residue field is finite).

Lemma 2 ([ABM, Th. 2.1]). In Lemma 1, assume $S = \mathrm{Spec}(k)$. Let $f : X \to BH$ be the map defined by the H-torsor $\widetilde{X} \to X$. Then, for $i + j \leqslant d$, the map

$$f^* : H^j(BH, \Omega^i_{BH/k}) \to H^j(X, \Omega^i_{X/k})$$

is injective ($\Omega^i_{BH/k}$ taken in the derived, stack-theoretic sense). In particular, for $j \leqslant d$, the map

$$f^* : H^j(BH, \mathcal{O}_{BH}) \to H^j(X, \mathcal{O}_X)$$

is injective.

Remark. This is a weak Lefschetz type property. Main difficulty in *loc. cit.*: $\widetilde{X}$ may be singular. Same result for $S = \mathrm{Spec}(W(k))$, or even $S = \mathrm{Spec}(\mathcal{O}_K)$ $(K : \mathrm{Frac}(W(k)] < \infty)$, [Li, 4.13].

Proof of Th. 5. In Lemma 1, take $S = \mathrm{Spec}(W(k))$, $H = G$, $d = p + 1$. Choose X as in (ii).
Lemma 2 (for $H = G_k$) $\Rightarrow$

$$f^* : H^{p+1}(BG_k, \mathcal{O}) \to H^{p+1}(X_k, \mathcal{O})$$

is injective.

5. Open problems

5.1. Higher Sen and extension classes.

Let $X/W(k)$ be formal smooth, and $Y = X_k$.

(a) By the decomposition into weights

$$\Omega^{\not{D}}_{Y/k} = \oplus_{0 \leqslant i \leqslant p-1} (\Omega^{\not{D}}_{Y/k})_i$$

the Sen operator Θ induces nilpotent operators

$$\Theta_i = \Theta + i \in \mathrm{End}((\Omega^{\not{D}}_{Y/k})_i).$$

In particular, we have classes

$$c_{i,j} \in \mathrm{Ext}^p(H^{i+p(j+1)}(\Omega^{\not{D}}_{Y/k})_i, H^{i+pj}(\Omega^{\not{D}}_{Y/k})_i) = \mathrm{Ext}^p(\Omega^{i+p(j+1)}_{Y/k}, \Omega^{i+pj}_{Y/k})$$

induced by Θ_i on $\tau^{[i+pj, i+p(j+1)]}(\Omega^{\not{D}}_{Y/k})_i$.

Questions. (i) Can one recover $c_{i,j}$ from $c_{0,0}$, at least for $j+1$ not divisible by p? (Plausible, according to Petrov.)

(ii) When some $c_{i,j}$ (resp. $e_{i,j}$) vanish, higher Sen (resp. extension) classes appear. How are they related?

(b) Let $d = \dim(Y)$. As $\Omega^{\not{D}}_{Y/k} \in D^{[0,d]}(Y, \mathcal{O})$, one has, for all i,

$$\Theta_i^{[d/p]+1} = 0,$$

i.e., the exponent of nilpotence of Θ_i is $\leqslant [d/p]+1$. Can one improve that bound?

5.2. Sen and Kodaira-Spencer classes.

Assume $p = 2$. Let $S \to \mathrm{Spec}(W(k))$ be formally smooth of relative dimension 1, and $f : X \to S$ formally smooth of relative dimension 1. Consider the Sen class for $Y = X_k$,

$$c_{Y,X} = \mathrm{ob}_{F,X} \circ \alpha(\Omega^1_{Y/k}) : \Omega^2_{Y/k} \to \mathcal{O}_Y[2],$$

where

$$\alpha(\Omega^1_{Y/k}) : \Omega^2_{Y/k} \to F^*_{\mathrm{abs}}\Omega^1_{Y/k}[1],$$

and

$$\mathrm{ob}_{F,X} : F^*_{\mathrm{abs}}\Omega^1_{Y/k} \to \mathcal{O}_Y[1]$$

is the obstruction to lifting F to $W_2(k)$.

On the other end, consider the Kodaira-Spencer class,

$$\mathrm{KS}_{f_k} : \Omega^1_{Y/S_k} \to f_k^* \Omega^1_{S_k/k}[1].$$

and the map deduced from the functoriality map $f_k^* \Omega^1_{S_k/k} \to \Omega^1_{Y/k}$ by applying F^*_{abs}:

$$\gamma : (f_k^* \Omega^1_{S_k/k})^{\otimes 2} = F^*_{\mathrm{abs}} f_k^* \Omega^1_{S_k/k} \to F^*_{\mathrm{abs}} \Omega^1_{Y/k}.$$

Proposition (Petrov). $\alpha(\Omega^1_{Y/k})$ is the composition

$$\Omega^2_{Y/k} = f_k^* \Omega^1_{S_k/k} \otimes \Omega^1_{Y/S_k} \xrightarrow{u} (f_k^* \Omega^1_{S_k/k})^{\otimes 2}[1] \xrightarrow{v} F^*_{\mathrm{abs}} \Omega^1_{Y/k}[1],$$

with $u = f_k^* \Omega^1_{S_k/k} \otimes \mathrm{KS}_{f_k}$ and $v = \gamma[1]$.

Application. Using this Petrov constructs, for $p = 2$, examples of fibered relative surfaces $X/S/W(k)$ for which Θ on $X \otimes k$ does not vanish.

Problem. Investigate more generally relations between Sen operators and Kodaira-Spencer classes.

5.3. Relative variants.

(a) Smooth bases over $W(k)$.

Let $S/W(k)$ be formal smooth. For $f : X \to S$ formal smooth, with special fiber $f_k : X_k \to S_k$ it was shown in [I] (in a slightly more general form) that:

(i) locally on S_k, the choice of a lifting of Frobenius to S produces a decomposition of $\tau^{<p} F_* Rf_{k*} \Omega^\bullet_{X_k/S_k}$ in $D(S_k^{(1)}, \mathcal{O})$;

(ii) if moreover f is proper, and

$$H := \oplus R^i f_{k*}\Omega^{\bullet}_{X_k/S_k}$$

denotes the relative de Rham cohomology of f, endowed with its Gauss-Manin connection $\nabla : H \to \Omega^1_{S_k/k} \otimes H$, and $\Omega^{\bullet}_{S_k/k}(H)$ the associated de Rham complex, with its Hodge filtration Fil, then, if $\dim(X_k) < p$, there is a canonical decomposition

$$\oplus_i \mathrm{gr}^i_{\mathrm{Fil}} \Omega^{\bullet}_{S_k^{(1)}/k}(H^{(1)}) \xrightarrow{\sim} F_* \Omega^{\bullet}_{S_k/k}(H)$$

in $D(S_k, \mathcal{O})$ (with the left-hand side the Kodaira-Spencer Higgs field).

Generalizations of various kinds obtained later by Kato, Ogus, Ogus-Vologodsky, and many others.

Question. Can these results be explained - and viewed as a special case of a richer structure - by a suitable relative variant of the diffracted Hodge complex?

By [BL1, 9.1] the Hodge-Tate stack

$$\mathrm{WCart}_S^{\mathrm{HT}} \to S$$

is a gerbe for the group-scheme $T^\sharp_{S/W(k)} \rtimes \mathbf{G}_m^\sharp$, and the choice of a local lifting of Frobenius of S_k to S (equivalently, a δ-structure on S) splits it into

$$\mathrm{WCart}_S^{\mathrm{HT}} = B(T^\sharp_{S/W(k)} \rtimes \mathbf{G}_m^\sharp).$$

The relative diffracted Hodge complex

$$\Omega^{\not D}_{X/S} \in D(X \times_S \mathrm{WCart}^{\mathrm{HT}}_S, \mathcal{O})$$

(defined by $\mathrm{WCart}^{\mathrm{HT}}_X \to X \times \mathrm{WCart}^{\mathrm{HT}}_S$), in the case S is endowed with a δ-structure has a concrete description [BL1,9.2] in terms of a triple

$$(\Omega^{\not D}_{X/S} \in D(\mathcal{O}_X), \Theta, \psi : \Omega^{\not D}_{X/S} \to \Omega^{\not D}_{X/S} \otimes \Omega^1_{S/W(k)}\{-1\}),$$

where $\Theta \in \mathrm{End}(\Omega^{\not D}_{X/S})$ is a Sen operator and ψ a Higgs field.

As (hopefully)

$$\Omega^{\not D}_{X/S} \otimes \mathcal{O}_{S_k} \xrightarrow{\sim} (F_{X_k})_* \Omega^\bullet_{X_k/S_k},$$

could that explain (i) above?

(b) Non-crystalline prisms.

Prismatic variants of [DI] ([Li], [Li-Liu]) suggest to investigate analogues of diffracted Hodge complexes, with $(W(k), (p))$ replaced by non-crystalline prisms (A, I). For $X/(A/I)$ formal smooth, lifted to $\widetilde{X}/A$, which extra structure would one get on $\Omega^\bullet_{X/A}$?

Work in progress by Bhatt.

Appendix: The diffracted Hodge complex (sketch of construction)

Ingredients:

- The absolute prismatic site

$$\Delta_{W(k)} = \{\mathrm{Spf}(W(k)) \leftarrow \mathrm{Spf}(A/I) \to \mathrm{Spf}(A)\}$$

- The associated Bhatt-Lurie-Drinfeld Cartier-Witt and Hodge-Tate stacks

$$\mathrm{WCart}^{\mathrm{HT}}_{W(k)} \hookrightarrow \mathrm{WCart}_{W(k)}$$

- The description of $\mathrm{WCart}^{\mathrm{HT}}_{W(k)}$ as a classifying stack:

$$\mathrm{WCart}^{\mathrm{HT}}_{W(k)} \xrightarrow{\sim} B(\mathbf{G}^{\sharp}_m)_{W(k)},$$

where $(\mathbf{G}^{\sharp}_m)_{W(k)}$ = PD-envelope of $(\mathbf{G}_m)_{W(k)}$ at 1

• The identification of the category of p-complete Hodge-Tate crystals $A \mapsto E(A/I) \in \widehat{D}(A/I)$ on $\Delta_{W(k)}$ with that of quasi-coherent complexes on $\mathrm{WCart}^{\mathrm{HT}}_{W(k)}$,

$$D(\mathrm{WCart}^{\mathrm{HT}}_{W(k)}) = D(B(\mathbf{G}_m^{\sharp})_{W(k)})$$

• Description of $D(B(\mathbf{G}_m^{\sharp})_{W(k)})$ as category of pairs $(M \in \widehat{D}(W(k)), \Theta \in \mathrm{End}(M))$ (Θ the Sen operator), such that $\Theta^p - \Theta$ is locally nilpotent on $H^*(M \otimes^L k)$.

• The basic Hodge-Tate and de Rham prismatic comparison theorems of [BS].

The diffracted Hodge complex $\Omega^{\not D}_{X/W(k)}$ is defined as the object of $D(\mathrm{WCart}^{\mathrm{HT}}_{W(k)})$ associated with the Hodge-Tate crystal

$$(A \in \mathrm{Spf}(W(k))_{\Delta}) \mapsto \overline{\Delta}_{X_{(A/I)}/A},$$

where $X_{(A/I)}$ is the pull-back of $X/W(k)$ by $\mathrm{Spf}(A/I) \to \mathrm{Spf}(W(k))$, and $\overline{\Delta}_{X_{(A/I)}/A}$ is the (relative) Hodge prismatic cohomology of $X_{(A/I)}$ over A.

Upshot:

$$\Omega^{\not D}_{X/W(k)} = \overline{\Delta}_{X/P} = \varphi_*(q\Omega_{X/W(k)[[q-1]]})^{\mathbf{F}_p^*}_{q=\zeta_p} \in D(X, \mathcal{O}_X),$$

where $(P, I) = (W(k)[[q-1]], ([p]_q))^{\mathbf{F}_p^*}$ is the $\mathbf{F}_p^*$-invariant q-de Rham prism ($P/I = W(k)$), and $q\Omega_{X/W(k)[[q-1]]}$ the q-crystalline complex ([BS], 16.18).

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