Chronology and posterity of SGA 5

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The acronym SGA stands for "Séminaire de Géométrie Algébrique" ("Seminar on Algebraic Geometry"). With the exception of SGA 4 1/2 [17], which mostly consists[1] of texts written by Deligne between 1974 and 1977, it refers to volumes, numbered from 1 to 7, of a seminar held by Grothendieck at the *Institut des Hautes Études Scientifiques* (IHÉS) from 1960 through 1969, published in *Lecture Notes in Mathematics* nos. 151, 152, 153, 224, 225, 269, 270, 288, 305, 340, 589 (Springer-Verlag, Berlin-New York); Advanced Studies in Pure Math. (North Holland), 1968; with partial re-editions in *Documents Mathématiques* (SMF) nos. 3, 4, 7, 8. The seminar was labeled *Séminaire de Géométrie Algébrique du Bois-Marie* (= "Seminar on Algebraic Geometry of Bois-Marie"), "Bois-Marie" being the name of the small wood of the estate in Bures-sur-Yvette, some twenty-eight kilometers south from Paris, where the IHÉS was located.[2]

Together with the *Éléments de Géométrie Algébrique* (= "Elements of Algebraic Geometry"), written with the collaboration of Dieudonné, and published between 1960 and 1967 in *Publications Mathématiques de l'IHÉS*, 4, 8, 11, 17, 20, 24, 28, and 32, with a re-edition of EGA 1 in *Grundlehren des mathematischen Wissenschaften* 166 (Springer-Verlag, Berlin-New York), 1971, these seminars introduced and worked out in great detail a new, powerful approach to algebraic geometry, and initiated fundamental, seminal developments. The language and the results of SGA and EGA were to become universally used by algebraic geometers in the world.

In part I I review the story of SGA 5 from its beginning in 1965 until its belated publication in 1977. In part II I briefly discuss related developments that occurred afterwards.

I Chronology of SGA 5

Cohomologie ℓ-adique et fonctions L [30], later[3] labeled SGA 5, extended itself over two periods: January 1965 – June 1965, January 1966 – June 1966.

January 1965 – June 1965

[1] Deligne's exposés on étale cohomology [Arcata] were written up by Boutot, [Th. finitude] contains an appendix by Illusie, and [C. D.] is Verdier's initial text on derived categories.

[2] The first IHÉS buildings opened in 1962. From 1960 to 1962 the seminar was held at *Fondation Thiers*, in the 16th arrondissement of Paris.

[3] During the course of the oral seminar, the previous seminars had no numbering: SGA 3 was referred to as SGAD (D for Demazure), and SGA 4 as SGAA (A for Artin). Grothendieck chose the numbering SGA 1, etc., when SGA 6 started.

In his first exposés, Grothendieck reviewed the formalism of derived categories, the basic theorems in étale cohomology (proper base change, smooth base change, ...) and the global duality theorem, which had been established in SGA 4, following the lines of the sketch given by Verdier in [67]. The proof presented in exposés XVII and XVIII of the published version of SGA 4 [4], which is based on a different approach to the functor $f^!$ and the trace morphism, is due to Deligne and was written up by him several years later.

Then Grothendieck moved on to a part that he considered to make the real beginning of the seminar, namely, local duality. He introduced the notion of dualizing complexes, discussed their uniqueness and basic properties, formulated, for the first time, the *conjecture of absolute purity*, and proved that modulo this conjecture plus resolution of singularities, on a good regular scheme X the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, for $n \in \mathbb{Z}$ invertible on X, is dualizing. It is to be noted that Grothendieck did not use the word "operation" to denote a functor, and, contrary to what is asserted at several places of [31], did not give any talk on the "formalism of six operations." But he proved various remarkable formulas concerning the interaction of these functors, such as the fact that the dualizing functor[4] exchanges ! and $*$.[5]

The next topic was the Lefschetz–Verdier trace formula. He discussed cohomological correspondences, and explained the construction of the so-called *Verdier pairing*, and the definition of the *Verdier local terms*, which, except in low dimensional cases, were conditional on the validity of the resolution of singularities. He then stated the Lefschetz–Verdier trace formula, i.e., the compatibility of the formation of these local terms with proper maps, but did not prove it. He simply said that checking the required compatibilities was a routine exercise, which should probably be rather long. He did not discuss its application to transversal endomorphisms of curves, due to Verdier, for which the hypotheses are satisfied, but requires additional arguments [68]. The reason is that he had a full proof of this application, independent of the Lefschetz–Verdier formula and free of any resolution assumption, based on the theory of Nielsen–Wecken traces, that he would explain at length later in the seminar. At the end, he suggested the possibility of a variant of the Lefschetz–Verdier formula in the context of coherent sheaves, leading to a generalization of the so-called *Woods Hole formula* (([15], p. 150), [5]), but he did not elaborate.

Grothendieck then gave a series of exposés on the construction of cycle

[4]Now usually called *Verdier dual*, though its construction is due to Grothendieck.

[5]The first appearance of the functor $f^!$ was in the coherent sheaves context, in the notes he wrote in the summer of 1963 [32].

classes, and their compatibility with intersection and Gysin maps. In particular, he introduced, for the first time, the notion of *homology* of a complex K as cohomology (with a change of signs in the degrees) with values in its dual DK, where D is the dualizing functor.[6]

The next talks were given by Jean-Pierre Jouanolou on his ongoing work, which was to be part of his thesis. The largest part of them was devoted to the definition of (constructible) $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves and the construction of certain cohomological functors on them, such as $R^if_!$, which, at the end of the second part of the seminar, would enable the formulation and proof of Grothendieck's theorem of rationality of L-functions. A derived category formalism for ℓ-adic complexes was to be defined much later, by Deligne and others. His last talks were on an independent topic, namely the calculation of the étale cohomology (for torsion coefficients) of certain classical schemes, such as projective bundles, the construction of Chern classes, and the proof of the so-called self-intersection formula in the Chow ring, a formula due to Mumford, and its application to the calculation of étale cohomology of certain blow-ups.

January 1966 – June 1966.

This part started by two exposés of Grothendieck (Jan. 4 and 7, 1966)[7] where he briefly recalled the Lefschetz–Verdier formula, and its application to the case of curves and transversal endomorphisms (which he said would be treated by another method later in the seminar), and moved on to Euler–Poincaré characteristics of schemes with finite group actions, announcing the Grothendieck–Ogg–Shafarevitch formula, and discussing variants and conjectures for analytic or topological spaces.

Then Grothendieck proceeded to the proofs of the two major theorems of the seminar, namely:

(i) the Grothendieck–Ogg–Shafarevitch formula, that Raynaud presented in 1966 at the Bourbaki seminar [53];

(ii) a Lefschetz trace formula on curves.

The local terms of (i) involve Swan conductors. Serre gave lectures on the Swan module, published independently [61]. Grothendieck had presented (ii) in 1966 at the Bourbaki seminar [28] shortly before Raynaud, using the method of the Lefschetz–Verdier trace formula. Not waiting for Verdier to check the compatibilities of his formula and write up the details of the ap-

[6]Later, this construction was referred to as *Borel–Moore homology*, though no dualizing complex appears in the original article of Borel–Moore [10].

[7]Called *exposé introductif* in the introduction of [30].

plication to curves, Grothendieck gave his own proof, alluded to above. For this he developed a formalism of non-commutative traces generalizing that of Stallings [62], and by a method inspired by the (much older) work of Nielsen and Wecken proved the desired Lefschetz trace formula on curves.

The last part consisted of exposés by Christian Houzel. After preliminaries on the Frobenius correspondence in étale cohomology, he used the formalism of ℓ-adic cohomology previously constructed by Jouanolou to define the L-functions of ℓ-adic sheaves on schemes over finite fields, proved their main formal properties, and eventually deduced from the trace formula for curves Grothendieck's cohomological expression for L-functions, which was the culminating point of the whole seminar. Grothendieck must have given the last talk but, unfortunately, I have no memory nor any document about its date and its contents.

The writing up

Exposés I, II, III

I wrote up I and III in the first semester of 1966. For this, I used the handwritten notes I had taken. Grothendieck did not give me any personal notes. He made many comments on my first drafts, that we discussed at length at his place. He was satisfied with the final versions. For Exposé I, this is the version in [30]. Both I and III were faithful transcriptions of Grothendieck's talks. In particular, the Lefschetz–Verdier local terms were defined modulo resolution assumptions, the formula itself was admitted, and no application to transversal endomorphisms of curves was given.

At the same time I also wrote up notes that Grothendieck handed me on Künneth formulas, generic cohomological properness and local acyclicity. They did not correspond to any oral exposé, and Grothendieck labeled them II. The main statements were conditional on resolution of singularities. Again, he was satisfied with the final drafts. These versions of I, II, III were typed by the IHÉS, mimeographed, and distributed the same year.

I will explain further below the story of the publication of II and III.

Exposé IV

Grothendieck asked Jouanolou to write up his exposés on the cycle class and homology. Jouanolou made a preliminary draft (Exposé IV), of which Grothendieck was not satisfied. A full revision was needed, and Grothendieck told me that he was afraid of having to do it himself.[8] One serious obstacle to an immediate re-writing was that the construction of cycle classes heavily

[8]This happened from time to time. For example, Grothendieck was not happy with Verdier's first draft of SGA 4, Exp. IV [4], that he eventually totally re-wrote (with Verdier's collaboration).

depended on the global duality theory of SGA 4, namely, the properties of the functors $f^!$ and $f_!$, and the trace map. Grothendieck had asked Deligne to write it up. Deligne used the Verdier approach[9] that he had just successfully applied in his appendix to Hartshorne's seminar [33]. Because he wanted to write solid foundations on the formalism of derived categories, especially on the question of signs, and that on his way he was discovering new results,[10] the writing took him much longer than expected. He had also to use at certain places his theory of cohomological descent, which was written up by Saint-Donat in ([4], Vbis) and was not immediately available. Grothendieck wrote the introduction to [4] in November, 1969. At that time he was interested in other mathematical topics (crystalline cohomology and Dieudonné theory), and was gradually absorbed by new political preoccupations. The revision of IV was never made.

Exposés V, VI, VII

Jouanolou wrote up his exposés on the ℓ-adic formalism and Chern classes. He finished by 1970.

Exposé XIV

Houzel wrote up his exposé in 1966. Grothendieck was satisfied, and the mimeographed text was then distributed by the IHÉS.

Exposés VIII, X, XI, XII

Ionel Bucur was in charge of writing up Grothendieck's exposés on the Grothendieck–Ogg–Shafarevich formula and the Lefschetz trace formula on curves. Except for a couple of short visits to France he was in Romania, working in very difficult conditions, and his writing was unfortunately not finished until 1972.

I have no information on the precise date at which the writing of Exposés VIII and X was finished, but it must have been before 1972.

In Dec. 1972, Deligne, who was at Harvard, received Bucur's write-up of Exposé XII and sent it to the IHÉS to be typed. On Feb. 4, 1973, Bucur wrote me that he was concerned about the draft of his Exposé XI, that he had seen for the last time in Grothendieck's room at IHÉS, and had not received any news from him about it. I asked Bucur for a copy, but to no avail. It seems that his text was lost when Grothendieck moved from the IHÉS.[11]

[9] Definition of $f^!$ as a right adjoint to $f_!$.

[10] Such as the symmetric Künneth formula ([4], XVII Th. 5.5.21). See the introduction of [4] for a list of them.

[11] I thank Leila Schneps for kindly informing me that in a letter to Bucur's widow, dated Feb. 27, 1986, Grothendieck wrote that Bucur had sent him his write-up of XI in 1969, and that he must have lost it when he left the IHÉS.

On Jan. 28, 1974 Bucur wrote me that he was still thinking about the local terms of the trace formula. I wrote him back asking him to tell me more about this, and informing him that his Exposé XII had been distributed by the IHÉS, but that his Exposé XI had probably been lost in Grothendieck's moving. Bucur was already ill, and our correspondence stopped. He died on Sept. 6, 1976.

The introductory and closing exposés

The introductory exposé consisted of the two talks given by Grothendieck at the beginning of 1966, that I have mentioned above. Grothendieck had not assigned the writing up of these talks to any participant of the seminar, and had not distributed any personal notes. It was tacitly assumed that he would write them up himself. He did so for the introductory and closing exposés of SGA 6 [8].

Finalization?

In 1974 the question was whether the existing write-ups of the exposés could be assembled into a volume.

A critical point was that the mere statement of the Lefschetz formula needed for proving Grothendieck's trace formula for Frobenius and the cohomological interpretation of L-functions in Exposé XIV could not be found in the existing write-up of XII.[12] It might have been possible to deduce it from the contents of XII (as probably Bucur was trying to do in 1974), but the proof would have been incomplete, as XII relied on the formalism of the lost exposé XI. Even if XI had been recovered, XI and XII needed to be carefully revised by Bucur in close coordination with Grothendieck. That would not have been possible, as at the time Grothendieck was campaigning for stopping mathematical research and had other occupations and interests. On the other hand, as explained above, the Lefschetz–Verdier formula of III had not been checked and its application to curves not given, hence was of no help.

Also, the absence of Exposé IV (not to mention that of the introductory and closing exposés) posed problem.

What to do?

Two events

In 1973–1974 two unrelated events happened, which were to have a crucial impact on the edition of the seminar.

[12] For the local terms to have the simple form as the trace of the endomorphism on the stalks of the sheaf at the fixed points, transversality of the endomorphism of the curve with respect to the diagonal is essential, and this was nowhere discussed in Exposé XII.

(a) In June 1973, Deligne announced he had proven the Weil conjecture about the eigenvalues of Frobenius on ℓ-adic cohomology of projective, smooth varieties over finite fields. He explained his proof in six talks at a conference held in July, 1973, in Cambridge in honor of Hodge, and quickly wrote it up. It was published in [16]. The proof relied on Grothendieck's Lefschetz formula recalled in ([16], (1.5.1)). Concern started growing on the fact that no written account of the proof of this formula was available.

(b) In 1973–74 Deligne was mostly working on a generalization of [16], which was to become *Weil II* [18]. But, quite unrelated to this, on Jan. 7, 1974, he wrote a letter to Mike Artin, in which he proved unconditionally the stability of constructibility by direct images for morphisms of finite type over a field, and sketched important complements in generic situations, and similar finiteness theorems for nearby cycles and dualizing complexes. Soon afterwards, he wrote up the details in what was to become ([17], Théorèmes de finitude).

The genesis of SGA 4 1/2

The results in (b) made it possible to re-write Exposés II and III without hypotheses of resolution, and desirable to check the compatibilities needed for the proof of the Lefschetz–Verdier formula. On May 20, 1974, Deligne wrote me a letter suggesting such a re-writing of II, using the contents of his letter to Artin, and giving a proof of a conjecture Grothendieck had made in II, using the notion of cospecialization map. I did not work on it until Oct. 1976.

On May 28, 1974, Deligne wrote me again, about III this time, sketching a strategy for the verification of the Lefschetz–Verdier formula. I worked about this during the winter of 1974–75, and I completed the verification by the spring of 1975. He proposed that as an application I wrote a proof of a statement Langlands had made in ([41], Proposition 7.12) (without proof). This statement was a far reaching generalization of Verdier's formula ([68], 4.1). And it contained, as a special case, Grothendieck's trace formula. It was unclear how to prove Langlands' statement by Grothendieck's Nielsen–Wecken method, but it looked feasible to apply the (now established) Lefschetz–Verdier formula to deduce it by a suitable adaptation of Verdier's arguments in [68]. In the summer of 1975, I succeeded in doing this, and, at the same time, I showed the coincidence of Lefschetz–Verdier local terms with those defined by Grothendieck by means of the Nielsen–Wecken method, developing for this a sheafified version of the theory of non-commutative traces of the (missing) XI.

A Summer Institute in Algebraic Geometry, organized by the AMS, had been held at Arcata, California, in July and August, 1974. An important part of it was a seminar, chaired by Artin, on Deligne's proof of the Weil conjectures and of the Hard Lefschetz theorem (which was to be part of [18]). As a preparation, Deligne gave seven lectures on the basics of étale cohomology. However, they did not include the formalism of ℓ-adic cohomology, that he recalled in [16], nor Grothendieck's trace formula.

In the fall of 1974, Deligne had no idea how long it would take me to check the Lefschetz–Verdier formula and give the required application to Grothendieck's trace formula, nor even if I would eventually succeed. It was becoming more and more urgent to make a proof of it available. That's why he decided to quickly write up a self-contained, neat proof of Grothendieck's trace formula for Frobenius, independent of Bucur's write-up of XI and XII, with the simplifications brought by the use of the notion of *perfect complex*, which was not available at the time of the oral seminar.[13] In fact, more was needed, namely the notion of filtered derived category, and the corresponding additivity of traces ([17], Rapport, (4.4.1)).[14]

In the course of this writing, Deligne realized that he could prove (and he quickly wrote it up) a souped up version ([17], fonctions L modulo ℓ^n et modulo p, Th. 2.2) of the trace formula of ([17], Rapport, 4.10), for torsion coefficients. The key new ingredient was the symmetric Künneth formula he had established in ([4], XVII 5.5).

Deligne was still concerned with the absence of Exposé IV. He therefore decided to do what he had done for the trace formula (and, for nearby cycles, in SGA 7 ([29], Exp. I)), i.e., quickly write up a self-contained account of the main points of Grothendieck's construction. He probably used his own notes and the memories he had of Grothendieck's talks that he attended in the first semester of 1965, but mostly reconstructed the theory by himself, with the help of the duality formalism he had developed in ([4], XVII, XVIII). However, he did not prove the compatibility of cycle classes with Gysin maps, nor did he discuss the formalism of homology constructed by Grothendieck.

In 1974–75, A. Douady and J.-L. Verdier ran a seminar at the ENS around the Baum–Fulton–MacPherson's version of the Riemann–Roch theorem and various questions in étale or singular cohomology. Bernard Angéniol gave a talk on Deligne's finiteness theorems [3], Verdier gave talks on constructibility and homology in topological or complex analytic set-ups [69], and Gérard

[13]It was to be developed in SGA 6 [8] and became standard afterwards.

[14]Daniel Ferrand discovered in 1968 that, in general, an endomorphism of a distinguished triangle of perfect complexes does not imply an additivity for the traces [22]. Soon afterwards, a satisfactory formalism (filtered derived categories), where additivity was restored was constructed in ([34], V). However, this (wrong) addivity is implicitly used in Bucur's XII, (5.3), referring to the (missing) XI, 4. This should have been fixed in the expected revision.

Laumon on the construction of homology classes in étale cohomology, parallel to Deligne's write-up, but using Grothendieck's homology formalism[15] and proving the compatibility with Gysin maps.

It is probably in the course of 1975 that Deligne conceived the idea of assembling Boutot's notes on his exposés at Arcata plus the various pieces he had just written up (proof of the trace formula and of its mod ℓ^n and mod p variants, finiteness theorems, cycle class, plus complements to global duality[16]) into a separate publication. In his spirit it was related both to SGA 4, as Boutot's notes were a gentle introduction to étale cohomology, and to SGA 5 by the trace formula. That led him to choose the title SGA 4 1/2.

The final steps

In 1975–76 Deligne had obtained beautiful applications of Grothendieck's trace formula and of his work "Weil II" [18] (which was still in preparation) to estimates of exponential sums. He decided to include them in the future SGA 4 1/2. Verdier's thesis on derived categories and derived functors had not been published. The summary he had written up in 1963 had been superseded by other expositions (the beginning of [33] and the first part of ([4] XVII)). However, Deligne thought that it was still interesting, and that it was a good idea to include it as well, which he did with the permission of Verdier. On Sept. 20, 1976, Deligne wrote the introduction to SGA 4 1/2. In Oct. 1976, thinking again about SGA 5 II, he invited me to write up the (unconditional) results on cohomological properness and local acyclicity he had sketched in his letter to me of May 20, 1974, as they would constitute a natural complement to his write-up of his finiteness theorems in SGA 4 1/2. I did it quickly, and in Dec. 1976, he submitted the volume[17] to the Springer Lecture Notes. He also told me that in his letter to A. Dold, he had said that SGA 5 should be ready by March, 1977. That left little time.

I hurried to return to the writing up of the results I had obtained in 1974–75, namely:

(i) the checking of the compatibilities in the Lefschetz–Verdier formula;

(ii) at the suggestion of Deligne, the same verification for the generalized Woods-Hole formula mentioned above ([15], p. 150);

(iii) the proof of the Langlands formula ([41] Proposition 7.12);

[15]Laumon told me that, at the time, he was unaware that this formalism was due to Grothendieck, and he was not instructed to give proper credit for what he was reporting on.

[16]Including a crucial compatibility that had been admitted in ([4], XVIII, 3.1.10.3).

[17]This last text was included as an appendix to Théorèmes de finitude.

(iv) the sheafified version of non-commutative traces and the coincidence of Lefschetz–Verdier and Grothendieck Nielsen–Wecken local terms.

I put (iii) and (iv) together in a package that I called III B.

The manuscript was ready by February 1977.

Because of the original work I had done on the new version of III, Deligne proposed to me to be the editor of SGA 5, which I accepted. I wrote the introduction[18] on Feb. 19, 1977. I sent a copy of the whole volume to Grothendieck, asking for his observations. In a letter dated March 17, 1977, he answered: "Tout semble parfait." ("Everything looks perfect."). I then made the submission.

II Glimpses on the posterity of SGA 5

Purity

As I recalled above, in ([30], I) Grothendieck introduced the notion of *dualizing complex* on locally noetherian schemes in étale cohomology for torsion coefficients (prime to the characteristics). He proved the uniqueness (up to shift and Tate twist) of such complexes (*loc. cit.*, Theorem 2.1). In the discussion of their existence, he was led to make expectations amounting to the following conjectures:

Conjecture 1 (*absolute purity conjecture*). Let X be a regular, locally noetherian scheme, and let $i\colon Y \hookrightarrow X$ be a regular closed subscheme, of pure relative codimension d. Let $n \in \mathbb{Z}$ be an integer invertible on X. Then:

$$R^q i^!(\mathbb{Z}/n\mathbb{Z}) = 0$$

for $q \neq 2d$, and $R^{2d} i^!(\mathbb{Z}/n\mathbb{Z})$ is canonically isomorphic to $(\mathbb{Z}/n\mathbb{Z})_Y(-d)$, by the cycle class isomorphism ([17], [Cycle], 2.2.6).

In (*loc. cit.*, 3.1.4) Grothendieck defined a pair (X, Y) verifying these properties as "satisfying the theorem of absolute purity." He noted that the theorem held for X of dimension $\leqslant 1$ (elementary), and (by results in ([4], XVI, XIX)) for X and Y smooth over a field of characteristic prime to n, or X excellent of characteristic zero. In footnote (1) of the introduction he expressed the hope that the theorem would hold in general.

Conjecture 2 (*local duality conjecture*). With X and n as in Conjecture 1, the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on X is a dualizing complex.

In (*loc. cit.*, Theorem 3.4.1) Grothendieck proved the statement of Conjecture 2 for X of dimension $\leqslant 1$ (elementary), and, in general, under additional

[18] p. v, l. −9, "VII" should be replaced by "VIII". p. vi, l. 2: the "déménagement" was that of Grothendieck, as explained above. p. vi, l. 3, "commutative" should be replaced by "non-commutative".

hypotheses, including resolution of singularities and the truth of conjecture 1, satisfied in particular for X excellent of characteristic zero. In the same footnote, he expected that Conjecture 2 would hold in general.

In ([17], [Finitude], Th. 4.3) Deligne directly proved the existence of dualizing complexes over schemes X of finite type over a regular base S of dimension $\leqslant 1$, namely, for such an $f\colon X \to S$, and n invertible on S, he showed that $f^!(\mathbb{Z}/n\mathbb{Z})_S$ is dualizing. However, Conjectures 1 and 2 remained open.

In 1976, Gabber (unpublished)[19] proved Conjecture 1 for X of dimension 2, using a certain Zariski–Riemann space. With tools of algebraic K-theory Thomason ([64], Theorem 3.5) proved Conjecture 1 for $\mathbb{Z}_\ell$-coefficients (ℓ invertible on X) up to bounded torsion. Gabber, in 1994, at a colloquium in Toulouse, announced a proof of Conjecture 1 in general. His proof, elaborating on Thomason's arguments (and using original new inputs), was written up by Fujiwara [24].

A new proof of Conjecture 1, independent of K-theory, was given by Gabber in [36] (see Exp. XIV, Th. 3.1.1). The proof relies on deep refinements of de Jong's alterations and delicate ingredients of logarithmic geometry. In the same volume (*loc. cit.*, Exp. XVII, Th. 0.2) Gabber also proved Conjecture 2, which had remained open until then. The techniques introduced in *loc. cit.* had many more applications. In particular, they provided proofs for longstanding finiteness conjectures in étale cohomology.

That was not the end of the story. In ([13], Theorem 3.1.3, Remark 3.1.4) a totally different proof of Conjecture 1 was given, relying on perfectoid and tilt techniques of Scholze. This proof takes less than 10 pages. The methods of [13] have far reaching applications to *semipurity* questions in flat cohomology. In particular, they yield a proof of the Grothendieck–Auslander–Goldman conjecture for the Brauer group[20] and conjectures of Gabber extending the Grothendieck–Lefschetz theorem.

The ℓ-adic formalism

The formalism of ℓ-adic cohomology constructed by Jouanolou in ([30], V, VI) sufficed for Deligne's first paper on the Weil conjecture [16]. However, in [18] Deligne needed derived categories of ℓ-adic sheaves and corresponding functors, an extension which was not to be found in Jouanolou's exposés. In

[19] See footnote (*) in ([30], I, p. 23), where O. Gabber is misspelled G. Ofer. Gabber announced the results at the conference *Journées de Géométrie Algébrique d'Angers* (1979). A different proof is given in ([54], Remark 5.6).

[20] [12], simplified in [13].

([18], 1.1.2) he briefly explained how to get such a formalism. For example, on a scheme X where the prime ℓ is invertible, he defined $D^b_c(X, \mathbb{Z}_\ell)$ as the inverse 2-limit of the categories $D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z})$ for $n \geqslant 1$, and for an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, the category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ was defined as $D^b_c(X, \mathbb{Z}_\ell) \otimes \overline{\mathbb{Q}}_\ell$. He showed that under certain suitable finiteness conditions, these categories are triangulated and satisfy a 6-functor formalism. Along the same lines, Ekedahl [20] developed a more general theory, working under weaker assumptions (but he did not discuss the case of $\overline{\mathbb{Q}}_\ell$-coefficients).

In [9] Bhatt and Scholze proposed a totally new approach to the problem. Adapting to the setup of schemes a construction made in the context of rigid analytic spaces ([59], [60]), they defined for any scheme X a refinement $X_{\text{proét}}$ of its étale site $X_{\text{ét}}$, called the *proétale* site of X, with the miraculous property that, with $\overline{\mathbb{Q}}_\ell$ as above, if the underlying space of X is noetherian, the (naive) cohomology of $X_{\text{proét}}$ with values in the 'constant' sheaf $\overline{\mathbb{Q}}_\ell$[21] calculates the (continuous)[22] cohomology of $X_{\text{ét}}$ with values in $\overline{\mathbb{Q}}_\ell$, in other words, we have a canonical isomorphism

$$R\Gamma(X_{\text{proét}}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} (R \varprojlim R\Gamma(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell.$$

In this setup there is a good notion of constructibility on schemes whose underlying space is noetherian, yielding triangulated categories $D_{\text{cons}}(X, E)$, $D_{\text{cons}}(X, \mathcal{O}_E)$ with coefficients in an algebraic extension E of $\mathbb{Q}_\ell$ and its ring of integers $\mathcal{O}_E$. Under suitable additional assumptions on the schemes and the morphisms, similar to those needed for torsion coefficients in Gabber's theorems of [36], it leads to a 6-functor formalism superseding that of Ekedahl [20].

In the 2000s developments arising from the geometric Langlands program demanded an extension of the ℓ-adic 6-functor formalism to Artin stacks. Because of the non-functoriality[23] of the lisse-étale topos of Laumon and Moret-Bailly [46] this extension turned out to be highly nontrivial, already for *torsion* coefficients. The problem was solved by Laszlo–Olsson ([42], [43], [44]). In addition to the use of simplicial techniques to circumvent the non-functoriality problem, a key ingredient was to exploit the local nature of the dualizing complex, which globalizes well on stacks, and to define the $f^!$ functor for $f\colon X \to Y$ by "biduality," i.e., by $f^! := D_X f^* D_Y$, a trick that seems to have been used for the first time by Ramis–Ruget–Verdier [52] in the context of coherent sheaves and complex analytic geometry.

[21] See ([9], Lemma 4.2.12) for the precise definition of this sheaf.
[22] In the sense of Jannsen [38].
[23] See ([50], 1.1), ([6], 5.3.12).

Despite their generality the constructions made by Laszlo–Olsson had certain drawbacks which limited their range of application. One main problem was that the proper base change isomorphism was constructed only on the level of cohomology sheaves, not as an isomorphism in a derived category, which is usually needed in practice, for example to deal with perverse cohomology. Solving this problem and getting rid of unnecessary finiteness or constructibility restrictions in [42] and [43] was a difficult task. It is only recently that a fully satisfactory formalism has been constructed, by Liu and Zheng [47]. There the theory is developed in the setup of ∞-categories. This is not only its natural framework, but also a crucial tool to perform all the necessary gluing in a systematic and uniform way. In particular, it is used in an essential way to define *adic* coefficients. A partial extension to stacks of the pro-étale formalism discussed above was made in [14]. It does not seem that the question of a common generalization of the two approaches has been considered yet. But the pro-étale approach of [9] has led Clausen and Scholze to develop a new theory, named *condensed mathematics*, the goal of which is to unify algebraic, p-adic, and complex analytic geometries, see Peter Scholze's home page for references.

The Lefschetz–Verdier formula

As I explained in part I, the Lefschetz–Verdier formula of SGA 5 ([30], III) was unnecessary for the proof of Grothendieck's trace formula for Frobenius, giving the cohomological ℓ-adic expression for L-functions of $\mathbb{Q}_\ell$-sheaves on schemes of finite type over finite fields ([30], XV), ([17], [Rapport]). Aside from the few cases treated in ([30], III B), the calculation of the local terms at the fixed points looked intractable.

The topic remained dormant until in the late 1980s Deligne made a conjecture which, because of its relation with modular problems arising from the Langlands program, attracted much attention. The conjecture says, roughly, that given a scheme X separated and of finite type over a finite field $k = \mathbb{F}_q$, a proper correspondence $a = (a_1, a_2)\colon Y \to X \times X$, with a_1 proper and a_2 quasi-finite, and a cohomological correspondence $c : a_1^* K \to a_2^! K$ over a, for $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ (ℓ prime to q), there exists an integer $N > 0$ such that, over an algebraic closure $\overline{k}$ of k, the fixed points of the correspondence $\mathrm{Fr}^N a_{\overline{k}}$, where Fr is the relative Frobenius, are *isolated*, and, at each one of them, the Verdier local term of the twisted correspondence $\mathrm{Fr}^N c_{\overline{k}}$ is the *naive* local term, given by the trace of the correspondence induced at the point. The case of curves followed from ([30], III B). After partial results by Zink [71], Shpiz [63], Pink [51], the conjecture was proven in general by Fujiwara [23]. The proof relied on a trace formula for contracting correspondences in rigid-étale cohomology suggested by Gabber. A different, simpler proof of a more general result was given by Varshavsky [65]. The conjecture had numerous applications to the Langlands program. In particular, it was a key ingredient in Lafforgue's construction of the Langlands correspondence for GL_n over function fields over finite fields [40]. A common generalization of the transversal case for curves ([30], III B) and the contracting case [65] was recently given by Varshavsky [66], proving another conjecture of Deligne, with an application to a generalization of the Deligne–Lusztig trace formula.

Let $f : X \to \mathrm{Spec}(k)$ be a separated scheme of finite type over an algebraically closed field k, and let $\mathcal{F}$ be an object of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. The Lefschetz–Verdier pairing for the identical cohomological correspondence on $\mathcal{F}$ produces an interesting invariant $C_{X/k}(\mathcal{F})$ in the group $H^0(X, K_X)$ of global sections of the dualizing complex $K_X = f^!\overline{\mathbb{Q}}_\ell$ of X, defined and studied by Abbes–Saito [1], called the *characteristic class* of $\mathcal{F}$. When f is proper, by the Lefschetz–Verdier formula, the trace map $Rf_*K_X \to \overline{\mathbb{Q}}_\ell$ sends this class to the trace of the identity map on $R\Gamma(X, \mathcal{F})$, i.e., the Euler characteristic of $\mathcal{F}$. I will come back to this in the next section.

The verification of the compatibilities required for proving the Lefschetz–Verdier formula in ([30], III Th. 4.4) were lengthy and tedious. A few years ago, a much shorter and more conceptual proof was found by Lu–Zheng [48]. Their result is more general, as it works in a relative situation (assuming certain local acyclicity conditions on the coefficients, which are automatically statisfied when the base is the spectrum of an algebraically closed field as in ([30], III)). The method of proof is totally different from that of *loc. cit.*. It exploits the formalism of *categorical traces* and *dualizable* objects in symmetric monoidal categories, which can be traced back to Dold–Puppe [19]. In particular, cohomological correspondences are viewed as morphisms in a certain symmetric monoidal 2-category, and traces as certain functors. A crucial point is that in this category dualizability turns out to be equivalent to universal local acyclicity. Fargues and Scholze adapted the arguments given there to prove similar results in the étale cohomology of diamonds in their study of the geometrization of the local Langlands correspondence [21]. A critical use of the categorical Lefschetz–Verdier formula of [48] was made by Abe in his proof of the Serre conjecture for Artin characters in the equal characteristic case [2]. In a different direction, a theory of categorical traces for (true) local Lefschetz–Verdier terms, in the context of Artin stacks, has just been worked out by Gaitsgory and Varshavsky [25], providing proofs for key lemmas in basic papers on the geometric Langlands program (and generalizing earlier results of Varshavsky mentioned above).

The Grothendieck–Ogg–Shafarevich formula

Let X be a proper, smooth, connected curve of genus g over an algebraically closed field k, and let $j : U \hookrightarrow X$ be a dense open subscheme. Let ℓ be a prime number invertible in k, and let $\mathcal{F}$ be a lisse $\mathbb{Q}_\ell$-sheaf of rank r on U. The Grothendieck–Ogg–Shafarevich formula ([53], ([30], X)) describes the Euler number of $j_!\mathcal{F}$ on X, i.e., $\sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i(X, j_!\mathcal{F})$, as

$$\chi(X, j_! F) = r\chi(U, \mathbb{Q}_\ell) - \sum_{x \in X-U} \mathrm{Sw}_x(\mathcal{F}),$$

where $\chi(U, \mathbb{Q}_\ell) = 2-2g-|(X-U)(k)|$ is the Euler number of U, and $\mathrm{Sw}_x(\mathcal{F})$ is the *Swan conductor* of $\mathcal{F}$ at the closed point x, an integer which measures the wild ramification of $\mathcal{F}$ at x. In particular, if $\mathcal{F}$ is tamely ramified at x, $\mathrm{Sw}_x(\mathcal{F}) = 0$.

Finding generalizations of this formula in higher dimension, for constructible coefficients, and in relative situations is the *discrete Riemann–Roch problem.* It has been the focus of extensive work up to now. Here are a few highlights.

When X/k is proper and smooth, purely of dimension d, $\chi(X, \mathbb{Q}_\ell)$ has a topological interpretation, as the degree of the top Chern class $c_d(T_X) \in H^{2d}(X, \mathbb{Q}_\ell(d))$, where T_X is the tangent bundle. This is also the degree of the self-intersection of the diagonal $\Delta : X \hookrightarrow X \times_k X$ in $X \hookrightarrow X \times_k X$ (see ([30], VII 4.5), ([17], [Cycle]), [45]). For the constant sheaf $\mathbb{Q}_\ell$ replaced by a constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ (or an object of $D^b_c(X, \mathbb{Q}_\ell)$), it is only very recently that a formula for $\chi(X, \mathcal{F})$ in terms of intersection numbers has been given, see the end of this report.

If k is of characteristic zero, then $\chi(X, \mathcal{F})$ depends only on the rank function $x \mapsto \mathrm{rk}_{\mathbb{Q}_\ell}(\mathcal{F}_x)$, which is a constructible function on X. This must have been known in the 1960s though it is hard to find a reference. A stronger result, valid in characteristic $p > 0$ different from ℓ, but for $\mathcal{F}$ *tamely ramified* in a suitable sense, was later proved by Deligne ([35], 2.9). For $k = \mathbb{C}$, MacPherson [49][24] expressed $\chi(X, \mathcal{F})$ in terms of a *total Chern class* of $\mathcal{F}$ (or, rather, of the rank function $\mathrm{rk}(\mathcal{F})$), and more generally gave a Riemann–Roch type formula, in terms of a natural transformation from the functor associating to a smooth compact complex variety X the group of constructible functions on it to the functor associating to X its (total) singular homology group $H_*(X)$, sending the constant function 1 on X to the Poincaré dual of the total Chern class of X, thus solving a conjecture of Deligne and Grothendieck.

[24] See also [27] for a complementary formula for MacPherson's Euler obstruction due to Gonzalez–Sprinberg and Verdier, and [11], [26], [39] for an expression of $\chi(X, \mathcal{F})$ as the intersection number, in the cotangent bundle T^*X, of the zero section and a *characteristic cycle* associated to $\mathcal{F}$, a formula which was to cast a long shadow on the subject.

In the 1970's, for k of characteristic $p > 0$, in view of the almost total absence of an understanding of ramification in dimension > 1, the generalization of the Grothendieck–Ogg–Shafarevich formula to proper smooth schemes X/k of higher dimension looked like an impossible mission. In 1976, Deligne wrote several letters to me, mostly for the case of surfaces, which had a considerable influence. See [37] for a discussion of them, and the developments they generated until 2013.

Finally, let me briefly discuss some of the progress realized since then in this domain. The main achievement is the solution to the *characteristic cycle* problem in positive characteristic. Let X be a smooth scheme of pure dimension n over an algebraically closed field k of characteristic $p > 0$, and let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X (or an object of $D^b_c(X, \mathbb{Q}_\ell)$)[25]. Using a Radon transform, Beilinson [7] constructed the *singular support* of $\mathcal{F}$, a certain conical[26] closed subset $SS(\mathcal{F})$ of the cotangent bundle T^*X, which is the minimal one such that local pencils whose differentials do not meet it are *locally acyclic* for $\mathcal{F}$, i.e., produce no vanishing cycles. This is an analogue of the singular support of holonomic $\mathcal{D}$-modules ([26], [39]). Its irreducible components are all of dimension n, but, contrary to the complex case, it is not necessarily Lagrangian. Let I be the set of irreducible components of $SS(\mathcal{F})$, and C_i the component of index i. In ([55], [56]) T. Saito constructs an n-dimensional cycle with $\mathbb{Z}$-coefficients

$$CC(\mathcal{F}) := \sum_{i \in I} m_i [C_i] \in Z_n(T^*X),$$

characterized by a formula of Deligne–Milnor type for local pencils.[27] For X/k projective, it gives rise to an *index formula*

$$\chi(X, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X)_{T^*X},$$

calculating the Euler number of $\mathcal{F}$ as the intersection number, in the cotangent bundle T^*X, of the characteristic cycle $CC(\mathcal{F})$ and the zero section $T^*_X X$, just as in the characteristic zero case.

Concrete descriptions of $CC(\mathcal{F})$ in terms of the ramification of $\mathcal{F}$ and functoriality properties of $CC(\mathcal{F})$ have been the subject of extensive work

[25] Other coefficients can be considered, e.g., $\overline{\mathbb{Q}}_\ell$, or a finite extension of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$.

[26] I.e., stable under the action of $\mathbf{G}_m$ by homotheties.

[27] Actually, T. Saito defined $CC(\mathcal{F})$ with coefficients $m_i \in \mathbb{Z}[1/p]$, and Beilinson proved their integrality. His proof is given in ([55], section 5.4). According to T. Saito (private communication), Deligne asked Beilinson why the integrality was not established in the first version of the paper, and Beilinson, having found a proof, insisted that Theorem 5.18 should be attributed to Deligne.

since then. However, the question of the relation of the characteristic cycle with the Abbes–Saito characteristic class mentioned in the section on the Lefschetz–Verdier formula remained open until quite recently. This relation was expressed as a conjecture in T. Saito's article ([55], Conjecture 6.8.1). It has just been solved (in the quasi-projective case) by Yang and Zhao [70]: for $\mathcal{F}$ supported on a closed subscheme Y of X (assumed quasi-projective), the characteristic cycle $CC(\mathcal{F})$ determines (by an explicit formula) the characteristic class $C_{Y/k}(\mathcal{F})$.

The construction of the characteristic cycle in [55] was an essential tool in the work of Sawin *et al.* [58] on the study of the behavior of the *complexity* of ℓ-adic complexes under the 6-functor formalism, with applications to equidistribution results for exponential sums.

The topic is, however, far from being exhausted. Many challenging problems remain, such as the behavior of characteristic cycles in families, and the relations between the constructions in characteristic zero and in characteristic $p > 0$. The case of a base of mixed characteristic is currently studied by T. Saito [57].

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