Arithmetic Geometry - Takeshi 60

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September 10, 2021

Revisiting Deligne-Illusie

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Plan

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1. The old result

Theorem 1 (DI, 1987). k perfect field, char. $p > 0$, X/k smooth. Let $X' = X \otimes_k (k, F_k)$, and $F : X \to X'$ = relative Frobenius.

Smooth liftings of X to $W_2(k)$ correspond to decompositions

(1.1) $$\mathcal{O}_{X'} \oplus \Omega^1_{X'/k}[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}$$

in $D(X', \mathcal{O}_{X'})$, inducing C^{-1} (Cartier isomorphism) on H^i.

Gives an affine bijection on isomorphism classes of objects, inducing identity on translation group $H^1(X', T_{X'})$.

Moreover, any decomposition (1.1) uniquely extends multiplicatively to a decomposition

(1.2) $$\oplus_{i \leqslant p-1} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau^{\leqslant p-1} F_* \Omega^\bullet_{X/k}$$

inducing C^{-1} on H^i.

Idea of proof

• local liftings of X' to $W_2(k)$: a gerbe on X'

$$\mathrm{Lift}(X'/W_2),$$

banded by $T_{X'}$ (sheaf of automorphisms of any object).

• local splittings of $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/k}$ (= local sections of $F_* Z\Omega^1_{X/k} \twoheadrightarrow \Omega^1_{X'/k}$): a gerbe on X'

$$\mathrm{Split}(\tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}),$$

banded again by $T_{X'/k}$.

Using local liftings of X' plus local liftings $\widetilde{F}$ of F (and associated $p^{-1}\widetilde{F}^*$ on Ω^1), can construct an equivalence of gerbes

$$\mathrm{Lift}(X'/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}) \tag{1.3}$$

inducing identity on $T_{X'/k}$. (NB. more general (1.3) holds over bases $/\mathbf{F}_p$ flatly lifted mod p^2.)

2. Another strategy

Local liftings of X' to W_2 controlled by $\tau^{\geqslant -1} L_{X'/W_2}$
(NB. X/k smooth $\Rightarrow L_{X/W} \xrightarrow{\sim} \tau^{\geqslant -1} L_{X/W_2}$).

Goal: directly construct isomorphism in $D(X', \mathcal{O}_{X'})$

$$L_{X'/W}[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_* \Omega^\bullet_{X/k} \tag{2.1}$$

inducing C^{-1} on $H^1 = \Omega^1_{X'/k}$ and $H^0 = \mathcal{O}_{X'}$.

Basics on cotangent complex and deformations show that (2.1) implies the isomorphism

$$\mathrm{Lift}(X'/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}) \tag{1.3}$$

(Proof:

$\mathrm{Lift}(X'/W_2)$ = fiber at $1 \in \mathcal{O}_{X'}$ of map

(Picard stack associated to) $R\mathcal{H}om(L_{X'/W}, \mathcal{O}_{X'})[1] \to H^0$

• $\mathrm{Split}(\tau^{\leqslant 1} F_* \Omega^\bullet_{X/k})$ = fiber at $1 \in \mathcal{O}_{X'}$ of map

(Picard stack associated to) $R\mathcal{H}om(\tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}, \mathcal{O}_{X'}) \to H^0$,

both stacks having $H^0 = \mathcal{O}_{X'}$ and $H^{-1} = T_{X'}$, $H^i = 0$ otherwise.)

Will deduce

$$L_{X'/W}[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_* \Omega^\bullet_{X/k} \tag{2.1}$$

from:

Theorem 2 (I., 2019). There exists a filtered isomorphism (i.e., in $DF(X, W)$), with $\mathcal{O}$-linear associated graded:

$$L\Omega^\bullet_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} W\Omega^\bullet_X/\mathcal{N}^p, \tag{2.2}$$

where

$L\Omega^\bullet$: derived de Rham complex

(if $X = \mathrm{Spec}(R)$, $L\Omega^\bullet_{R/W} := \mathrm{Tot}(\Omega^\bullet_{P_\bullet/W})$,
$P_\bullet \to R$ a simplicial resolution by polynomial algebras over W.)

Fil^i: Hodge filtration,

$W\Omega^\bullet_X$: de Rham-Witt complex

$\mathcal{N}^i$: Nygaard filtration:

$$\mathrm{Fil}^i L\Omega^\bullet_{R/W} := \mathrm{Tot}(\Omega^{\geqslant i}_{P_\bullet/W})$$

(2.3) $$\mathrm{gr}^i_{\mathrm{Fil}} = L\Omega^i_{X/W}[-i]\,(:= (L\Lambda^i L_{X/W})[-i])$$

(in particular $\mathrm{gr}^1 = L\Omega^1_{X/W}[-1] = L_{X/W}[-1]$).

$$\mathcal{N}^i W\Omega^n_X = p^{i-n-1} VW\Omega^n_X$$

(for $n < i$, and $\mathcal{N}^i W\Omega^n_X = W\Omega^n_X$ for $n \geqslant i$) with

(2.4) $$\mathrm{gr}^i_{\mathcal{N}} W\Omega^\bullet_{X'} = \tau^{\leqslant i} F_* \Omega^\bullet_{X/k}.$$

(in particular $\mathrm{gr}^1 = \tau_{\leqslant 1} F_* \Omega^\bullet_{X/k}$).

- graded piece of degree 1 of

$$L\Omega^{\bullet}_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} W\Omega^{\bullet}_X/\mathcal{N}^p, \tag{2.2}$$

plus formulas for gr^1 imply (2.1), i.e.,

$$L_{X'/W}[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_*\Omega^{\bullet}_{X/k}.$$

- Any smooth lifting $\widetilde{X'}$ of X' to W_2 gives a decomposition

$$L_{X'/W}[-1] = \tau^{\geqslant -1} L_{X'/\widetilde{X'}}[-1] \oplus \Omega^1_{X'/k}[-1] = \mathcal{O}_{X'} \oplus \Omega^1_{X'/k}[-1],$$

hence, by applying $L\Lambda^{p-1}$ and formulas for gr^{p-1}, a decomposition (1.2)

$$\oplus_{i \leqslant p-1} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau^{\leqslant p-1} F_*\Omega^{\bullet}_{X/k}.$$

Remark. Not possible to remove $/Fil^p$ and $/\mathcal{N}^p$ from (2.2), because of

Example (Bhatt):

$$\widehat{L\Omega^\bullet_{k/W}} = \widehat{W\langle x\rangle}/(x-p) = \widehat{W\langle y\rangle}/(y),$$

where $\widehat{(-)}$ means p-adic completion, $W\langle x\rangle$ = divided power envelope of $W[x]$.

Proof of Th. 2

Use (local) embeddings $X \hookrightarrow Z$, ideal J, Z/W smooth.

Gives

$$(*) \qquad L_{X/W}[-1] = (J/J^2 \to \mathcal{O}_X \otimes \Omega^1_{Z/W}),$$

from which one deduces a filtered isomorphism

$$(**) \qquad L\Omega^\bullet_{X/W}/\mathrm{Fil}^p = \Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W},$$

$$J^r\Omega^\bullet_{Z/W} := (J^r \to J^{r-1}\Omega^1_{Z/W} \to \cdots \to \Omega^r_{Z/W} \to \cdots).$$

Key points:

$$\mathrm{gr}^1_J\Omega^\bullet_{Z/W} = L\Omega^1_{X/W}[-1](:= L_{X/W}[-1])$$

by (*), and

$$L\Gamma^r(M[-1]) = L\Lambda^r[M][-r],$$

$\Gamma^r = S^r$ for $r < p$.

Additional Frobenius lift F on Z gives (F,Frobenius)-compatible

$$\mathcal{O}_Z \to W\mathcal{O}_X,$$

sending J to $VW\mathcal{O}_X$, hence filtered (J,$\mathcal{N}$)-map

$$\Omega^\bullet_{Z/W} \to W\Omega^\bullet_X,$$

inducing a filtered quasi-isomorphism

$$\Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W} \xrightarrow{\sim} W\Omega^\bullet_X/\mathcal{N}^p,$$

as checked locally by taking for Z a lifting of X, and applying Nygaard's formula for gr^r.

Conclude by applying

$$(**) \qquad L\Omega^\bullet_{X/W}/\mathrm{Fil}^p = \Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W}.$$

The singular case

By left Kan extension from finite polynomial rings over k, (2.2) extends to any scheme X/k, provided that $W\Omega^\bullet_X$ is replaced by its derived variant $LW\Omega^\bullet_X$:

$$LL\Omega^\bullet_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} LW\Omega^\bullet_X/\mathcal{N}^p \tag{2.5}$$

Again,

$$\mathrm{gr}^i_{\mathrm{Fil}} L\Omega^\bullet_{X/W} = L\Omega^i_{X/W}[-i],$$

but

$$\mathrm{gr}^i_{\mathcal{N}} LW\Omega^\bullet_{X'} = \mathrm{Fil}^{\mathrm{conj}}_i F_* L\Omega^\bullet_{X/k}, \tag{2.6}$$

where $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ is the (increasing) conjugate filtration, with

$$L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_i F_* L\Omega^\bullet_{X/k}. \tag{2.7}$$

By left Kan extension from finite polynomial rings over W_2, any (flat) lifting $\widetilde{X'}$ of X' over W_2 gives a decomposition

$$(2.8) \qquad \oplus_{i \leqslant p-1} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_{p-1} F_* L\Omega^\bullet_{X/k}$$

generalizing (1.2). Indeed:

Key observation (A. Mathew): though, for X/k not lci, $L_{X/W} \to \tau^{\geqslant -1} L_{X/W_2}$ no longer an isomorphism, still the datum of $\widetilde{X'}$ gives functorial map $L_{X'/W} \to L_{X'/W_2} \to L_{X'/\widetilde{X'}} \to \mathcal{O}_{X'}[1]$ splitting the triangle

$$\mathcal{O}_{X'}[1] \to L_{X'/W} \to L_{X'/k} \to .$$

For X/k lci, (2.8) yields partial degeneration and vanishing theorems, both in char. p and in char. 0, see section 4.

3. A prismatic generalization (after Bhatt et al.)

The RHS of the isomorphism

$$(3.0) = (2.2) \qquad L\Omega^\bullet_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} W\Omega^\bullet_X/\mathcal{N}^p,$$

can be re-written in terms of prismatic cohomology relative to the prism $(W = W(k), (p))$: one has a canonical isomorphism (special case of the prismatic-crystalline comparison theorem)
(Bhatt-Scholze, Li-Liu)

$$(3.1) \qquad W\Omega^\bullet_X \xrightarrow{\sim} \varphi_W^* \mathbb{\Delta}_{X/W},$$

where φ_W = Frobenius of W,

$$\mathbb{\Delta}_{X/W} := R\nu_* \mathcal{O}_{(X/W)_{\mathbb{\Delta}}},$$

$$\nu : (X/W)_{\mathbb{\Delta}} \to X_{\mathrm{et}}$$

the canonical map from the prismatic site to the étale one.

Isomorphism (3.1) is compatible with Nygaard filtrations on both sides.

More generally:

Let (A, I) be a prism (examples: $(W, (p))$, $(W[[u]], (E(u))$ (with $W[[u]]/(E(u)) = \mathcal{O}_K$), $(A_{\inf}, \xi)$ (with $A_{\inf}/(\xi) = \mathcal{O}_C$), $\varphi_A : A \to A$ given by

$$\varphi_A(x) = x^p + p\delta_A(x)$$

Assume (A, I) bounded (i.e. $(A/I)[p^\infty] = (A/I)[p^n]$ for some n).

Let $X/(A/I)$ a smooth formal scheme.

Define

$$\Delta_{X/A} := R\nu_* \mathcal{O}_{(X/A)_\Delta},$$

$$\nu : (X/A)_\Delta \to X_{\text{et}}$$

the canonical map from the prismatic site to the étale one.

Then:

Theorem 3 (Li-Liu [LL, Th. 4.24]). There exists a (canonical) filtered isomorphism in the derived ∞-category $\mathcal{D}F(X, A)$:

$$\varphi_A^* \Delta_{X/A} \otimes_A^L L\Omega^\bullet_{(A/I)/A} \xrightarrow{\sim} L\Omega^\bullet_{X/A} \tag{3.2}$$

(derived tensor product and derived de Rham complexes are p-completed).

The filtrations are the I-adic filtration on A, the Nygaard filtration $\mathcal{N}$ on $\varphi_A^* \Delta_{X/A}$, and the Hodge filtration Fil on derived de Rham complexes. The associated graded of (3.2) is an isomorphism in $\mathcal{D}(X, \mathcal{O}_X)$.

Examples

- For (A, I) transversal (i.e., A/I p torsion free),

$$L\Omega^{\bullet}_{(A/I)/A} = \widehat{D_A(I)}$$

(p-completed PD-envelope of I in A), and (3.2) is rewritten

$$\varphi_A^* \Delta_{X/A} \otimes_A \widehat{D_A(I)} \xrightarrow{\sim} L\Omega^{\bullet}_{X/A}, \tag{3.2.1}$$

which, in this case, due to a classical result on $L\Omega^{\bullet}_{X/A}$, is a form of the prismatic-crystalline comparison theorem.

- Take $(A, I) = (W(k), (p))$, k perfect. It is not transversal. Then, by (3.1), (3.2) reads

$$W\Omega^{\bullet}_X \otimes^L_W L\Omega^{\bullet}_{k/W(k)} \xrightarrow{\sim} L\Omega^{\bullet}_{X/W(k)}. \tag{3.2.2}$$

For $X = \mathrm{Spec}(k)$, $W\Omega^{\bullet}_X = W(k)$, and (3.2.2) is tautologically the identity. Recall (Bhatt)

$$\widehat{L\Omega^{\bullet}_{k/W}} = \widehat{W\langle y\rangle}/(y),$$

and Hodge filtration = filtration on $\widehat{W\langle y\rangle}/(y)$ by the $(y)^{[n]}$.

Application

- Dividing (3.2) by p-th steps of the filtrations gives

$$(3.3) \qquad \varphi_A^* \Delta_{X/A}/\mathcal{N}^p \xrightarrow{\sim} L\Omega^\bullet_{X/A}/\mathrm{Fil}^p,$$

which, for $(A, I) = (W, (p))$ is the inverse of the isomorphism (3.0).

- Applying gr^1 to (3.3) gives Bhatt-Scholze [BS, 15.6]:

$$(3.4) \qquad L\Omega^1_{X/A}[-1]\{-1\} \xrightarrow{\sim} \tau^{\leqslant 1}(\Delta_{X'/A} \otimes^L_A A/I)$$

$(\{-1\} := \otimes(I/I^2)^{-1}$, $X' := X \otimes_{A,\varphi_A} A)$.

- (3.4) generalized by Anschütz-Le Bras [AL, 3.2.1) to any formal $X/(A/I)$, with Δ denoting derived prismatic cohomology, and $\tau^{\leqslant 1}$ replaced by first step of conjugate filtration.

- (3.3) proposed by Bhatt (email to I., 21 Feb. 2019) with sketch of proof.

Techniques of proof

Same as in the Bhatt-Scholze prismatic-crystalline comparison theorem and construction of the Nygaard filtration.

• Use (corrected) Čech-Alexander complex calculating prismatic cohomology to define the map (3.2)(in the other direction of (3.0)).

• To analyze compatibility of (3.5) with filtrations, use quasisyntomic descent and large quasisyntomic (A/I)-algebras. Work first in the transversal case.

R large quasisyntomic (A/I)-algebra: $A/I \to R$ quasisyntomic (i.e. p-completely flat and $\mathrm{tor.amp}(L_{R/(A/I)}) \subset [-1, 0]$), and we have a surjection of a Tate algebra $(A/I)\langle X_s^{1/p^\infty} \rangle_{s \in \Sigma} \twoheadrightarrow R$, Σ a set.

4. The lci case: partial degeneration and vanishing theorems (after Bhatt)

Back to k = perfect field of char. p, $W = W(k)$, $W_n := W_n(k)$.

Recall that, for any X/k, a flat lifting $\widetilde{X'}/W_2$ of X' gives a decomposition in $D(X', \mathcal{O}_{X'})$

$$\oplus_{i \leqslant p-1} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_{p-1} F_* L\Omega^{\bullet}_{X/k} \tag{2.8}$$

This decomposition is compatible with the obvious filtrations on both sides, and induces the (generalized) Cartier isomorphism

$$C^{-1} : L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_i F_* L\Omega^{\bullet}_{X/k}$$

on gr^i.

From (2.8) Bhatt deduced the following theorem:

Theorem 4.1 (Bhatt). X/k proper, lci, of pure dimension $d < p$, liftable to W_2.

s:= dimension of singular locus of X.

$L\widehat{\Omega}^{\bullet}_{X/k} := R\varprojlim_r L\Omega^{\bullet}_{X/k}/\mathrm{Fil}^r$

(Hodge completed derived de Rham complex).

Then, for $n < d - s - 1$,

$$\dim_k H^n(X, L\widehat{\Omega}^{\bullet}_{X/k}) = \sum_{0 \leqslant i \leqslant d} \dim_k H^n(X, L\Omega^i_{X/k}[-i]),$$

and

$$H^n(X, L\Omega^i_{X/k}[-i]) = 0$$

for $i > d$.

Remarks. 1. If X/k smooth, then $\mathrm{Sing}(X) = \emptyset$, $s = -\infty$, $d - s - 1 = +\infty$, $L\widehat{\Omega}^\bullet_{X/k} = \Omega^\bullet_{X/k}$, so for all n,

$$\dim_k H^n_{\mathrm{dR}}(X/k) = \sum_i \dim_k H^{n-i}(X, \Omega^i_{X/k}),$$

i.e., one recovers [DI]'s result that Hodge to de Rham spectral sequence degenerates at E_1.

2. In [DI], degeneration (even, decomposition of the de Rham complex with no properness assumption) holds for $d = p$. Unknown if conclusion of Th. 4.1 holds assuming only $d \leqslant p$.

By standard spreading out arguments Th. 4.1 implies:

Theorem 4.2 (Bhatt). K: field of char. 0, X/K proper, lci, of (arbitrary) pure dimension d.

s: dimension of singular locus of X

Then, for $n < d - s - 1$,

$$\dim_K H^n(X, L\widehat{\Omega}^{\bullet}_{X/K}) = \sum_{0 \leqslant i \leqslant d} \dim_k H^n(X, L\Omega^i_{X/K}[-i]),$$

and

$$H^n(X, L\Omega^i_{X/K}[-i]) = 0$$

for $i > d$.

Remarks. 1. As above, for X/K smooth, Th. 4.2 recovers the classical E_1-degeneration of Hodge to de Rham spectral sequence in char. 0.

2. For $K = \mathbf{C}$, and any (separated and of finite type) X/K, one has (Bhatt, 2012):

$$R\Gamma(X, L\widehat{\Omega}^{\bullet}_{X/\mathbf{C}}) \stackrel{\sim}{\to} R\Gamma(X(\mathbf{C}), \mathbf{C})$$

(Betti cohomology).

Proof of Th. 4.1

Main ingredient: cohomological amplitude estimates on $L\Omega^i_{X/k}$, $LZ\Omega^i_{X/k}$, $LB\Omega^i_{X/k}$ (derived cycles, boundaries), using $\mathrm{perf.amp}(L_{X/k}) \subset [-1, 0]$ and Cartier isomorphism.

Key Lemma (Bhatt). X/k lci, purely of dimension d, $s := \dim(\mathrm{Sing}(X))$. Then all complexes

$$L\Omega^i_{X/k}[-i],\ F_* L\Omega^i_{X/k}[-i],\ LZ\Omega^i_{X/k}[-i],\ LB\Omega^i_{X/k}[-i]$$

live in $D^{\geqslant}(X')$, and, for $i > d$, live in $D^{\geqslant d-s}(X')$.

Proof of key lemma relies on (easy points) of a theory of Cohen-Macaulayness for complexes developed by Bhatt and used by him in his proof of Cohen-Macaulayness (modulo powers of a prime p) of absolute integral closures of excellent noetherian domains.

Combining decomposition (2.8) with Bhatt's estimates and Raynaud's trick [DI, 2.9] one gets Kodaira type vanishing theorems:

Theorem 4.3. X/k as in Th. 4.1: proper, lci, liftable to W_2, dim. $d < p$, singular locus of dim. s. Let L be an ample invertible sheaf on X. Then:

For $n < \min(d, d - s - 1)$ and all i,

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{-1}) = 0.$$

Remarks. 1. For X/k smooth (i.e., $s = -\infty$), one gets

$$H^n(X, \Omega^i_{X/k}[-i] \otimes L^{-1}) = 0$$

for $n < d$ and all i, i.e., [DI, (2.8.2)].

2. For X/k smooth, the vanishing

$$H^n(X, \Omega^i_{X/k}[-i] \otimes L^{-1}) = 0$$

for $n < d$ and all i is, by Serre duality, equivalent to

$$(*) \qquad H^n(X, \Omega^i_{X/k}[-i] \otimes L) = 0$$

for $n > d$ and all i.

However, for X/k singular, the analogue of (*) fails ($H^0(X, L\Omega^{d+1}_{X/k} \otimes L) \neq 0$ if X has a single isolated singularity) (observed by Bhatt-Blickle-Lyubeznik-Singh-Zhang [BBLSZ, 3.4]).

3. For X/k not lci, conclusion of 4.3 fails (by Avramov's solution of Quillen's conjecture).

Again, by standard spreading out arguments, Th. 4.3 implies a (slightly weaker form of) [BBLSZ, Th. 3.2] (with $d - s$ replaced by $d - s - 1$):

Theorem 4.4. K: field of char. 0, X/K proper, lci, of pure dimension d.

s: dimension of singular locus of X

L: an ample invertible sheaf on X.

Then, for $n < \min(d, d - s - 1)$ and all i,

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{-1}) = 0.$$

Remark. I don't know how to get $d - s$ instead of $d - s - 1$ by mod p^2 techniques.

5. A new perspective: the stacky approach (after Bhatt - Lurie, Drinfeld)

For X/k smooth, liftable to W_2, of dimension $d = p$, the whole complex $F_*\Omega^\bullet_{X/k}$ is decomposable, not just $\tau^{<p} F_*\Omega^\bullet_{X/k}$ ([DI], 2.3]). Raises the question:

Question ([DI, 2.6 (iii)], still open): X/k smooth, liftable to W_2, of dimension $d > p$, is $F_*\Omega^\bullet_{X/k}$ decomposable (in $D(X', \mathcal{O}_{X'})$) (i.e., $\xrightarrow{\sim} \oplus \mathcal{H}^i[-i]$)? For X/k assumed moreover proper, does Hodge to dR degenerate at E_1?

Partial results:

- (Suh, 2006, unpublished). For X/k smooth, liftable to W_2, all truncations $\tau^{[a,a+1]} F_*\Omega^\bullet_{X/k}$ are decomposable.
- (Achinger, 2020). For X/k smooth, liftable to W_2, all truncations $\tau^{[a,a+p-2]} F_*\Omega^\bullet_{X/k}$ ($a > 0$) are decomposable.

Recent improvement by Drinfeld (and, independently, Bhatt-Lurie):

Theorem 5.1. (Drinfeld, Bhatt-Lurie, 2020) Let X/k be smooth, liftable to $W_2 = W_2(k)$.

Then a lifting of X to W_2 defines a μ_p-action on $F_*\Omega^\bullet_{X/k}$ in $D(X', \mathcal{O}_{X'})$,

i.e., a $\mathbf{Z}/p$-grading

$$F_*\Omega^\bullet_{X/k} = \oplus_{\alpha \in \mathbf{Z}/p} (F_*\Omega^\bullet_{X/k})_\alpha,$$

with nonzero $H^i F_*\Omega^\bullet_{X/k}$ of weight the class of $-i$ in $\mathbf{Z}/p$.

Corollary 5.2. Under the assumption of Th. 5.1, all truncations $\tau^{[a,a+p-1]} F_*\Omega^\bullet_{X/k}$ are decomposable.

Glimpses on the proof.

Details haven't yet appeared. Work in progress.

Main idea (Bhatt-Lurie, 2019): cohomology of prismatic sites underlies richer structure: cohomology of prismatic stacks, giving rise to objects in $D(BG)$, for certain group schemes G/S.

The stacks $X^{\mathbb{\Delta}}$.

To any (formal scheme) $X/W(k)$ is associated a ringed, (formal) stack

$$X^{\mathbb{\Delta}}/W(k),$$

called the prismatic stack or prismatization of X, functorial in X: $f : X \to Y$ gives map of ringed stacks

$$f^{\mathbb{\Delta}} : X^{\mathbb{\Delta}} \to Y^{\mathbb{\Delta}}.$$

Fundamental Example. (Drinfeld's Σ [Dr])

$$\mathrm{Spf}(\mathbf{Z}_p)^{\triangle} := [W_{\mathrm{prim}}/W^\times](=\Sigma)$$

where:

W:= (p-typical) Witt scheme over $\mathbf{Z}_p$

W_{prim}:= formal completion of W along locally closed subscheme defined by $p = x_0 = 0$, $x_1 \neq 0$, the formal scheme of primitive Witt vectors.

$W^\times \subset W$:= $\mathbf{Z}_p$-group scheme of units in W, acting on W_{prim} by multiplication.

Definition 5.3. (Bhatt-Lurie)

For $X/W(k)$ formal, R a p-nilpotent $W(k)$-algebra,

$$X^{\Delta}(R)$$

is the groupoid of pairs

$$((I,a), f : \mathrm{Spf}([W(R)/I]) \to X)$$

where

• I: an invertible $W(R)$-module, $a : I \to W(R)$: $W(R)$-linear map landing into $W_{\mathrm{prim}}(R)$ ("Cartier-Witt divisor")

• $\mathrm{Spf}([W(R)/I])$: formal derived scheme such that for $X = \mathrm{Spf}(A)$
$\mathrm{Mor}(\mathrm{Spf}([W(R)/I]), \mathrm{Spf}(A)) := \mathrm{Mor}_{\mathrm{Ani}}(A, [I \to W(R)])$

Ani = category of animated $W(k)$-algebras (= derived category of simplicial $W(k)$-algebras) ($[I \to W(R)]$ is a 1-truncated animated algebra)

In particular,
$\mathrm{Spf}(\mathbf{Z}_p)^{\Delta} = \Sigma$, $\mathrm{Spec}(\mathbf{F}_p)^{\Delta} = \mathbf{Z}_p$.

Another key example (Bhatt-Lurie)

$\mathrm{Spec}(W_2(k))^{\mathbb{\Delta}}$:= stack associated to prestack with values

$$[\mathcal{F}_{W(R)}(p^2)/W(R)^\times],$$

on p-nilpotent $W(k)$-algebras R, where

$\mathcal{F}_{W(R)}(p^2)$:= set of factorizations $p^2 = db$ in $W(R)$, with $d \in W_{\mathrm{prim}}(R)$, $b \in W(R)$,

and $u \in W(R)^\times$ acts by $(d, b) \mapsto (du, u^{-1}b)$.

5.4. Hodge-Tate point

$$V(1) = (0, 1, 0, 0, \cdots) : \mathrm{Spf}(\mathbf{Z}_p) \to \mathrm{Spf}(\mathbf{Z}_p)^{\mathbb{\Delta}}$$

induces (unique) "physical" point:

$$i : \mathrm{Spec}(k) \to \mathrm{Spec}(W_2(k))^{\mathbb{\Delta}},$$

corresponding to the factorization $p^2 = pp$, with stabilizer

$$\mathrm{Stab}(i) = G := W_k^{\times}[F]$$

(kernel of F on the k-group scheme $W_k^{\times}$) (also denoted $\mathbf{G}_m^{\sharp}$)

Basic formula (Drinfeld, Li-Mondal):

$$W_k^{\times}[F] = (\mu_p)_k \times W_k[F],$$

where $\mathbf{W}_k[F]$ = kernel of F on W_k = PD-envelope of 0 in $\mathbf{A}_k^1$ ($= \mathbf{G}_a^{\sharp}$)

Back to Drinfeld's theorem 5.1

Data of $Y/W_2(k)$ lifting X/k gives maps

$$Y^{\Delta} \xrightarrow{\nu} X_{\mathrm{Zar}} \times \mathrm{Spec}(W_2(k))^{\Delta} \xleftarrow{i} X_{\mathrm{Zar}} \times BG \leftarrow X_{\mathrm{Zar}} \times B\mu_p \leftarrow X_{\mathrm{Zar}}.$$

Prismatic Hodge-Tate comparison theorem implies:

$$(\varphi^* R\nu_*(\mathcal{O}/p))|X' \xrightarrow{\sim} F_*\Omega^{\bullet}_{X/k} \in D(X', \mathcal{O})$$

(for $\varphi =$ Frobenius on the base $W_2(k)$).

Therefore, $F_*\Omega^{\bullet}_{X/k}$ underlies an object of $D(X' \times B\mu_p, \mathcal{O})$, and one checks that H^i is of weight $-i$.

In fact, $F_*\Omega^\bullet_{X/k}$ underlies an object of $D(X' \times BG, \mathcal{O})$.

As quasi-coherent sheaves on $BW_k[F]$ correspond to comodules over the Hopf algebra $k\langle x\rangle$ (PD-envelope of (x) in $k[x]$), i.e. pairs (E, N), where E is a k-vector space, and N a nilpotent endomorphism of E,

the basic formula $G = (\mu_p)_k \times W_k[F]$ gives that:

Each summand $(F_*\Omega^\bullet_{X/k})_\alpha$ $(\alpha \in \mathbf{Z}/p)$ is endowed with an $\mathcal{O}_{X'}$-linear, nilpotent endomorphism N_α.

Remark (Bhatt-Lurie). Datum

$$(F_*\Omega^\bullet_{X/k} = \oplus_{\alpha\in\mathbf{Z}/p}(F_*\Omega^\bullet_{X/k})_\alpha), (N_\alpha : (F_*\Omega^\bullet_{X/k})_\alpha \to (F_*\Omega^\bullet_{X/k})_\alpha)$$

equivalent to datum of

$$\theta \in \mathrm{End}_{D(X',\mathcal{O}_{X'})}(F_*\Omega^\bullet_{X/k})$$

with "generalized eigenvalues" in $\mathbf{Z}/p$. Analogous to a Sen operator. For X proper smooth over $\mathrm{Spec}(W(k))$ (instead of $\mathrm{Spec}(k)$), analogy is upgraded into a comparison theorem, involving a new theory of diffraction (ongoing work by Bhatt and Lurie).

Remarks. 1. Alexander Petrov recently gave examples where $N_0 \neq 0$.

2. New approach to action of $W_k^{\times}[F]$ by Shizhang Li and Shubhodip Mondal [LM], based on study of endomorphisms of the de Rham cohomology functor. In particular:

Theorem ([LM, Th. 4.23])

$$\mathrm{Aut}(\widetilde{R}\,\mathrm{smooth}/W_2(k) \mapsto \Omega^{\bullet}_{\widetilde{R}\otimes k/k} \in \mathrm{CAlg}(D(k))) = \mathbf{G}^{\sharp}_{m,k}$$

Corollary (LM, Mathew). There is no functorial splitting for $\widetilde{X}$ smooth over $W_2(k)$ of $\Omega^{\bullet}_{\widetilde{X}\otimes k/k}$.

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